

Topological Vector Space

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1. Local geometry of topological vector space

Let (X, \mathcal{F}) be a topological space.

Remark.

- A base for \mathcal{F} is a subcollection $\mathcal{F}' \subset \mathcal{F}$ s.t. $\forall U \in \mathcal{F}, \exists V \in \mathcal{F}', V \subset U$. A base \mathcal{F}' determines $\mathcal{F} \Leftrightarrow \exists S, \forall U \subset \mathcal{F}, U = \{\cup e_i \mid i \in S, e_i \in \mathcal{F}'\}$.
- A local base of x is a subcollection $\mathcal{F}'_x \subset \mathcal{F}_x$ s.t. $\forall U \subset \mathcal{F}_x, \exists V \in \mathcal{F}'_x$ s.t. $V \subset U$. However, elements in \mathcal{F}_x may be not union of elements in \mathcal{F}'_x .

Example. (X, d) is a metric space.

- $\mathcal{F}' = \{B(x, r) \mid x \in X, r > 0\}$ is a base.
- $\mathcal{F}' = \{B(x, r), |\}$ is a local at x .
- $\mathcal{F}' = \{B(x, \frac{1}{n}) \mid n \in \mathbb{N}\}$ another local base at x , countable elements.

Now let X be topological vector space. Last time we showed that $\forall a \in X, \forall \alpha \neq 0$, the maps

- $T_a : X \rightarrow X, x \rightarrow x + a$
- $M_a : X \rightarrow X, x \rightarrow \alpha x$

are both homeomorphism. As a consequence, we see

Corollary. A set $A \subset X$ is open $\Leftrightarrow a + A$ is open, $\forall a \in X \Leftrightarrow \alpha A$ is open, $\forall \alpha \neq 0$.

So the topological \mathcal{F} is determined by any local base at 0 whose elements have special geometric properties for topological vector space.

Definition. X is locally convex if there is a local base whose elements is convex.

Example. Normed Vector Space are locally convex since $\{B(0, r) \mid r > 0\}$ are convex.

Proof. $x, y \in B(0, r) \Leftrightarrow \|x\| < r, \|y\| < r \Leftrightarrow \|\alpha x + (1 - \alpha)y\| \leq \alpha\|x\| + (1 - \alpha)\|y\| \leq r$ \square

A set $E \subset X$ is absorbing if $\forall x \in X, \exists \delta > 0$ s.t. $\delta x \in E \forall |\alpha| < \delta$. (Obviously $0 = 0 * x \in E$)

Property. *In a topological vector space, any neighborhood of 0 is absorbing.*

Proof. Let U be a neighborhood of 0. $\forall x \in X$, the map $\mathbb{R} \leftarrow X$ is continuous. \square