Complete metric spaces

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1 Convergence

Let (X, d) be a metric space(not necessary to be a vector space).

Definition. *We say that a sequence of vectors, $\{x_1, x_2, ..., x_n\}$, in X converges to $x_0 \in X$ (under the metric d) if $\forall \epsilon > 0, \exists N > 0$ s.t. $\forall i > N, d(x_i, x_o) < \epsilon$ We will introduce Convergence in more general spaces later.*

Example. $X = C(\big[0, 1\big])$

1. $d_1(f, g) = \int_0^1 |f(x) - g(x)|dx$
   Let $f_n(x) = x_n, f_0(x) = 0$. Then $d_1(f_n, f_0) = \int_0^1 x_n dx = \frac{1}{n+1} \to 0 \text{ as } n \to \infty$.
   so $f_n \to f_0$ in (X, d_1)

2. $d_0(f, g) = \max_{0 \leq x \leq 1} |f(x) - g(x)|$ *Still take $f_n(x) = x_n, f_0(x) = 0$, Then $d_0(f_n, f_0) = \sup_{0 \leq x \leq 1} x_n = 1$, so $f_n \not\to f_0$ in (X, d_0).*

   Remark. *Obvious one always have $d_1(f, g) \leq d_0(f, g) in X$ We say the metric d_0 is stronger than d_1. If a sequence converges to an element in d_0, then it converges to that element in d_1.*

3. *Still take $d_1(f, g) = \int_0^1 |f(x) - g(x)|dx$.*

$$f_n(x) = \begin{cases} 0, & x < \frac{1}{2} - \frac{1}{n}, \\ \frac{1}{2} + \frac{n}{2}(x - \frac{1}{2}), & \frac{1}{2} - \frac{1}{n} \leq x \leq \frac{1}{2} + \frac{1}{n}, \\ 1, & x > \frac{1}{2} + \frac{1}{n} \end{cases}$$

   and let

$$\begin{cases} 0, & x \leq \frac{1}{2}, \\ 1, & x \geq \frac{1}{2} \end{cases}$$

   Then $d_1(f_n, f_0) = \int_0^1 |f_n(x) - f_0(x)|dx = \frac{1}{2n} \to 0$.

Property. *Suppose a sequence $\{x_n\}$ converges in (X, d), then*

1. *$\{x_n\}$ is bounded.*

2. the limit is unique.

Proof.

1. Suppose $x_n \to x_0$, then for $\epsilon = 1, \exists N$, s.t. $\forall i > N$, $d(x_n, x_0) < 1$. Let $C = max(d(x_n, x1), d(x_n, x_1), ..., d(x_n, x_0)) + 1$, Then $d(x_n, x_0) < C \forall 1 \leq i < \infty$ So $\{x_i\}$ is bounded.

2. Suppose $x_i \to x_0$ and $x_i \to x_0'$. $\forall \epsilon > 0, \exists N, N'$ s.t. $\forall i > N, d(x_i, x_0) < \epsilon, \forall i' > N, d(x_i', x_0') < \epsilon \Rightarrow d(x_0, x_0') \leq d(x_0, x_i) + d(x_i, x_0') \leq 2\epsilon \Leftrightarrow d(x_0, x_0') = 0 \Rightarrow x_0 = x_0'$

□

2 Completeness

As we have seen in part 3 of previous example, we have a sequence in X which converges under d_1 to , an element outside X. So as $\mathcal{Q}, (X, d)$ is NOT complete. To do better analysis, we would like to work on complete spaces. As in mathmatical analysis, we define

Definition. *A sequence $\{x_i\}$ in (X, d) is a Cauchy sequence if $\forall \epsilon > 0, \exists N > 0$ s.t. $\forall i, j > N$, $d(x_i, x_j) < \epsilon$*

Definition. *A metric space (X, d) is complete if any Cauchy sequence in (X, d) converges to an element in X.*

Definition. *A complete normed vector space is called a Banach space.*

- *Here, the metric is the induced metric from the norm: $d(x, y) = \|x - y\|$*
- *Banach space will be one of the main object in this course.*

Example. *$d(x, y) = (\sum_{i=1}^{n} (x_i - y_i))^{\frac{1}{2}}$ is a complete metric on $\mathbb{R}^n$.*
$\|x\| = (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}}$, $(\mathbb{R}^n, \|\cdot\|)$ is a Banach space.

Example. *$X = C(\big[0, 1\big])$.*

1. *$d_1(f, g) = \int_0^1 |f(x) - g(x)|dx$. We have seen $f_n(x) = x^n \to f_0(x) = 0$. In fact, $\{f_n\}$ is a Cauchy sequence, since $d(f_n, f_m) = \int_0^1 |x^n - x^m|dx < \int_0^1 x^n dx + \int_0^1 x^m dx = \frac{1}{n+1} + \frac{1}{m+1}$ In general, we have*

   Property. *Any converged sequence in a metric space is a Cauchy sequence.*

   Proof. Suppose $x_i \to x_0$. i.e. $\forall \epsilon > 0, \exists N$ s.t. $\forall i > N, d(x_i, x_0) < \epsilon$. So for $\forall i, j > N$, we have $d(x_i, x_j) \leq d(x_i, x_0) + d(x_0, x_j) < 2\epsilon$. So $\{x_i\}$ is Cauchy. □

2. $d_0(f,g) = \max_{0\le x\le 1} |f(x)-g(x)|dx$.
We have seen $f_n \not\to f_0$ in (X, d_0). In fact, $\{f_n\}$ is not a Cauchy sequence since we fix n and let $m\to\infty$, $d(f_n, f_m) = \max_{0\le x\le 1} |x_n - x_m|dx \to 1$. In fact, we have

Property. *$(C([0,1]), d_0)$ is complete. As a consequence, $\|f\|_0 = \sup_{0\le x\le 1} |f(x)|, (C([0,1]), \|f_0\|)$ is a Banach space.*

Proof. Let $\{f_n\}$ be a Cauchy sequence, $\forall\epsilon > 0, \exists N > 0$, s.t. $d_0(f_n, f_m) = \sup_{0\le x\le 1} |f_n(x) - f_m(x)| < \epsilon$. Then for any fixed $x \in [0,1]$, the sequence (of scalars) $\{f_n(x)\}$ is a Cauchy sequence in R. It follows that there exists $f_0(x)$ s.t. $f_n(x) \to f_0(x)$(use Completeness of $\mathbb{R}$). Since $|f_n(x) - f_m(x)| < \epsilon$. letting $m\to\infty$ we get $|f_n(x) - f_0(x)| \le \epsilon, \forall n > N, \forall x \in X$. So the sequence of functions $\{f_n(x)\}$ converges uniformly to $f_0(x)$, because $\forall x \in X$. By results in mathmatical analysis, f_0 is continuous and $f_n \to f_0(\sup_{0\le x\le 1} |f_n(x) - f_0(x)| < \epsilon, \forall n > N)$ in $(C([0,1]), d_0)$ (Finally, since each f_n is continuous and the uniform limit of continuous functions is continuous, $f \in C([0,1])$. We have thus found a limit $f \in C([0,1])$ of the Cauchy sequence $\{f_n\}$ in the metric d_0. This shows $(C([0,1]), d_0)$ is complete. See the proof in appendix) ☐

3.
$$d_1(f,g) = \int_0^1 |f(x) - g(x)|dx$$
$$f_n(x) = \begin{cases} 0, & x < \frac{1}{2} - \frac{1}{n}, \\ \frac{1}{2} + \frac{n}{2}(x - \frac{1}{2}), & \frac{1}{2} - \frac{1}{n} \le x \le \frac{1}{2} + \frac{1}{n} \\ 1, & x > \frac{1}{2} + \frac{1}{n} \end{cases}$$
$$f_0(x) = \begin{cases} 0, & x \le \frac{1}{2} \\ 1, & x \ge \frac{1}{2} \end{cases}$$

f_n is actually a Cauchy sequence in (X, d_1), since $d_1(f_n, f_m) \le d_1(f_n, f_0) + d_1(f_0, f_m) \le \frac{1}{2n} + \frac{1}{2m}$. Conclusion: $(C([0,1]), d_1)$ is NOT complete.

Remark. *$C([0,1]) \subset L^1([0,1])$ and we will see (L^1, d_1) is complete.*

In general, for any incomplete metric space (X, d), it is possible to construct a complete metric space $(\overline{X}, \overline{d})$ so that X is dense(we will define this next time) in $\overline{X}$ and $\overline{d}|_X = d$. The procedure is the same as $\mathbb{Q}\to\mathbb{R}$. See HW next time.

4. *$X = l_1 = \mathbf{x} = (a_1, a_2, ...) \mid \sum_{i=1}^{\infty} |a_i| < \infty$. $\|\mathbf{x}\| = \sum_{i=1}^{\infty} |a_i|$. Then $(X, \|\cdot\|)$ is a Banach space.*

Proof.

- X is a vector space because for $\mathbf{x} = (a_1, a_2, a_3, ...), \mathbf{y} = (b_1, b_2, b_3, ...)$,

$$x + y \in X : \sum_{i=1}^{\infty} a_i + b_i \leq \sum_{i=1}^{\infty} |a_i| + \sum_{i=1}^{\infty} |b_i| < \infty$$

$$\alpha x \in X : \sum_{i=1}^{\infty} \alpha a_i = |\alpha| \sum_{i=1}^{\infty} |a_i| < \infty$$

  The axioms hold in an obvious way.

- $\| \cdot \|$ is norm since
  - $\|x + y\| \leq \|x\| + \|y\|$
  - $\|\alpha x\| = \alpha \|x\|$
  - if $x \neq 0$, then $\exists a_i \neq 0$. So $\|x\| = \sum_{i=1}^{\infty} |a_i| > 0$

- Completeness: Let $x^j = (a_i^j)$ be a Cauchy sequence in l^1, i.e. $\forall \epsilon > 0, \exists N$ s.t. $\forall i, k \geq N, \|x^j - x^k\| = \sum_l |a_l^j - a_l^k| < \epsilon$. So $\forall l$ are fixed, $\forall j, k > N, |a_l^j - a_l^k| < \epsilon$

$$\Rightarrow \forall l \text{ are fixed }, \{a_l^j\} \text{ is a Cauchy sequence in } \mathbb{R}$$

$$\Rightarrow \exists a_l^0 \in \mathbb{R} \text{ s.t. } a_l^j \xrightarrow{j \to \infty} a_l^0.$$

  We want to show that $x_0 = (a_l^0) \in l^1$, and $x^j \to x^0$ in (X, d) with $d(x, y) = \|x - y\|$, To prove this, we choose M large so that $\sum_{i=M}^{\infty} |a_i^N| < \epsilon$ Then for $j > N$, we have

$$\sum_{i=M}^{\infty} |a_i^j| \leq \sum_{i=M}^{\infty} |a_i^j - a_i^N| + \sum_{i=M}^{\infty} < 2\epsilon$$

  letting $j \to \infty$, we get $\sum_{i=M}^{\infty} a_i^0 < 2\epsilon$. so $x^0 \in l^1$. Moreover, choose $j > N$ large enough, we can get

$$\sum_{i=1}^{M-1} |a_i^j - a_j^0| < \epsilon, \sum_{i=M}^{\infty} |a_i^j| < 2\epsilon, \sum_{i=M}^{M-1} |a_i^0| < 2\epsilon$$

$$\Rightarrow \sum_{1}^{\infty} |a_i^j - a_i^0| \leq \sum_{i=1}^{M} |a_i^j - a_i^0| + \sum_{i=M}^{\infty} a_i^0 < 5\epsilon.$$

  So $x^j \to x^0$ in (l^1, d).

$\square$