The Baire category theorem

xuascaler

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Metric Topology

Let (X, d) be a metric space, and $A \subset X$ a subset.

Definition.

1. *A point $x \in A$ is called an interior point of A if $\exists \epsilon > 0$ s.t. $B(x, \epsilon) = \{y \in X \mid d(y, x) < \epsilon\}$ lies in A.(*ϵ-neigborhood of x*).*

2. *A is open if any point $x \in A$ is an interior point of A.*

Example. *For any $x \in X$ and any $r > 0$, $B(x, r)$ is open.*

Proof. For any $y \in B(x, r)$, we have $d(x, y) < r$. Take any $0 < \epsilon < r - d(x, y)$. Then for any $z \in B(y, \epsilon)$, $d(z, x) \leq d(z, y) + d(y, x) < \epsilon + d(x, y) < r$. □

Property.

1. *$\emptyset, X$ are open sets.*

2. *If $\{A_\alpha\}$ are a collection (could be infinite, or even incountable) of open sets in X, so is $\cup_\alpha A_\alpha$*

3. *If A, B are open sets in X, so is $A \cap B$.(If $A_1, A_2, ..., A_n$ are open, so is $\cap_{i=1} n A_i$)*

Proof.

1. Obvious.

2. Suppose $x \in \cup_\alpha A_\alpha$, then $\exists \alpha, x \in A_\alpha$. Since A_α is open, $\exists \epsilon > 0, B(x, \epsilon) \subset A_\alpha$. It follows $B(x, \epsilon) \subset \cup_\alpha A_\alpha$. So $\cup_\alpha A_\alpha$ is open.

3. Suppose $x \in A \cap B$, then $\exists \epsilon 1, \epsilon 2$ s.t. $B(x, \epsilon 1) \subset A, B(x, \epsilon 2) \subset B$. Take $\epsilon = \min \epsilon 1, \epsilon 2$, Then $B(x, \epsilon) \subset A \cap B$. So $A \cap B$ is open.

   Remark. *It could happen that the intersection of countable open sets is no longer open. The simplest example is $A_n = (-\frac{1}{n}, \frac{1}{n} + 1)$, $\cap_{n=1}^{\infty} A_N$ is not open.*

$\square$

One can characterize convergence using open sets.

Property. $\lim_{i\to\infty} x_i = x_0$ *if and only if for any open set* A *containing* x_0*, there exists* N *s.t.* $\forall i > N, x_i \in A$

Proof.

- Suppose $x_i \to x_0$ and A is an open set containing x_0. Then $\exists \epsilon$ s.t. $\forall i > N, B(x_i, \epsilon) \subset A \Rightarrow x_i \in B(x_0, \epsilon) \forall i > N$. So $\forall i > N, x_i \in A$.
- Suppose for any open set A containing x_0, we can find N s.t. $x_i \in A \forall i > N$, Then in particular $\forall \epsilon > 0, A = B(x_0, \epsilon) \forall i > N$, In other words, $\forall i > N, d(x_i, x_0) < \epsilon.$. So $x_i \to x_0$.

$\square$

Definition. *A subset* $A \subset X$ *is closed if* $X \backslash A$ *is open.*

One can easily convert properties of open sets to properties of closed sets.

Example. *For any* $x \in A$ *and any* $r > 0$*,* $\overline{B}(x, r) = \{y \in X \mid d(y, x) \leq r\}$ *is closed.*

Proof. we prove the set $X \backslash \overline{B}(x, r)$ is open.

$$X \backslash \overline{B}(x, r) = \{y \in X \mid d(y, x) > r\}$$

We claim that $\forall y \in X \backslash \overline{B}, \exists \epsilon > 0, B(y, \epsilon) \subset X \backslash \overline{B}$. We set $\epsilon = d(y, x) - r$. $\forall z \in B(y, \epsilon), d(x, z) + d(z, y) \geq d(x, y) \Leftrightarrow d(x, z) > d(x, y) - d(z, y) > d(x, y) - \epsilon = r$. So $d(x, z) > r, B(y, \epsilon) \subset X \backslash \overline{B}$, $X \backslash \overline{B}$ is open set. $\square$

A characterization of closed sets.

Property. *A is closed iff for any sequence* $x_n \in A, x_n \to x \in X, x \in A$.

Proof.

- Suppose A is closed. $x_n \in A, x_n \to x \in X$. We want to show $x \in A$. By contradiction: If $x \in X \backslash A$, one can find $\epsilon > 0, B(x, \epsilon) \subset X \backslash A$. i.e. $B(x, \epsilon) \cap A = \emptyset$. But $x_n \to x \Rightarrow \exists N, \forall n > N, x_n \in B(x, \epsilon)$. So $\forall n > N, x_n \notin A$. Contradiction!
- Suppose for any sequence $x_n \in A, x_n \to x \in X, x \in A$. We want to show A is closed $\Leftrightarrow$ we want to show $X \backslash A$ is open. $\Leftrightarrow$ we want to show $\forall y \in X \backslash A$, $\exists \epsilon > 0, B(y, \epsilon) \subset X \backslash A \Leftrightarrow B(y, \epsilon) \cap A = \emptyset$. Again by contradiction, suppose $\forall \epsilon > 0, B(y, \epsilon) \cap A \neq \emptyset$. Then choose $x_n \in B(y, \frac{1}{n}) \cap A$. Then $x_n \in A$ and $x_n \to y$. so $y \in A$, contradicts with the fact $y \in X \backslash A$.

$\square$

Definition. *For any $A \subset X$, we define its closure to be the set $\overline{A} = \{x \in X \mid \exists x_n \in A, x_n \to x\}$.*

Example.

- $\mathbb{Q} \subset \mathbb{R}, \overline{\mathbb{Q}} = \mathbb{R}$. *Since any real number is the limit of a sequence of rationals.*

- $P([0,1]) =$ polynomials for $x \in [0,1]$. *The $P([0,1]) \subset C([0,1])$. In mathmatical analysis we learned that any continuous function is approximated uniformly by polynomials(e.g. Bernstain Polynomials), So if we use d_0 metric, the $\overline{P[0,1]} = C([0,1])$.*

Property. *$\overline{A}$ is closed.*

Proof. Suppose $x_n \in \overline{A}, x_n \to x \in X$. We want to show $x \in \overline{A}$. For any n, we choose $x_n \in \overline{A}$ s.t. $d(x_N, x) < \frac{1}{n}$. Since $x_N \in \overline{A}$, we can find an element in A, which we denoted by y_n, s.t. $d(y_n, x_N) < \frac{1}{n}$. Then $y_n \in A$ and $d(y_n, x) < d(y_n, x_N) + d(x_N, x) < \frac{2}{n}$. So $y_n \to x$, i.e. $x \in \overline{A}$.

Remark. *If B is closed, $A \subset B$, then $\overline{A} \subset B$. As a consequence, $\overline{A}$ is the smallest closed subset of X which contains A.*

□

2. The Baire category theorem

Definition. *A subset $A \subset (X, d)$ is dense if $\overline{A} = X$. Equivalently, $\forall x \in X, \exists x_n \in A$ s.t. $x_n \to x$.*

Example.

- *$\mathbb{Q}$ is dense in $\mathbb{R}$, $\mathbb{R} \backslash \mathbb{Q}$ is also dense in $\mathbb{R}$.*

- *$P([0,1])$ is also dense in $C([0,1])$.*

Property. *$A \subset X$ is dense iff for any nonempty open subset $B \subset X$, $A \cap B \neq \emptyset$.*

Proof.

- Suppose A is dense, and $B \neq \emptyset$ is open. We choose $x \in B, \exists \epsilon > 0, B(x, \epsilon) \subset B$. Since A is dense, one can find $\{y_n\}$ s.t. $y_n \to x$. So $\exists y \in A, d(y, x) < \epsilon \Leftrightarrow y \in A \cap B(x, \epsilon) \Leftrightarrow y \in A \cap (B, \epsilon)$. So $y \in A \cap B \Rightarrow A \cap B \neq \emptyset$.

- Suppose for any $B \neq \emptyset$ open, we have $A \cap B \neq \emptyset$. Then $\forall x \in X$, we can find $x_n \in B(x, \frac{1}{n}) \cap A$. We get a sequence $\{x_n\} \in A, x_n \to x$. So $X = \overline{A}$, A is dense.

□

Definition. *A subset $A \subset (X, d)$ is nowhere dense $\overline{A}$ contains no interior point.*

Example.

- $\mathbb{Z}$ *is no where dense in* $\mathbb{R}$.
- *The Cantor set is nowhere dense in* $\mathbb{R}$
- $A = \{f \in C([0,1]) \mid f(0) = 0\}$ *is nowhere dense in* $C([0,1])$.

Definition. *A subset* $A \subset X$ *is of first category if it is the union of countably many nowhere dense subsets. A subset* $A \subset X$ *is of second category if it is not of first cagtegory.*

Example.

- $\mathbb{Q}$ *is of first category.*
- *By the next theorem and it's corollary,* $\mathbb{R}\backslash\mathbb{Q}$ *is* 2^{nd} *category.*

Theorem. *Let* (X, d) *be a complete metric space. Then the intersection of any countable collection of dense open subsets of* X *is still dense in* X*, but not necessary open.*

Proof. Let $A_1, A_2, ..., A_n$ be a sequence of dense open subsets of X. Take any nonempty open set $B \subset X$, we want to show $(\cap_{i=1}^{\infty} A_i) \cap B \neq \emptyset$.

- $A_1 \subset X$ is dense open, we see $A_1 \cap B \neq \emptyset$ and $A_1 \cap B$ is open. So in particular, we can find $x_1 \in X, r_1 > 0, \overline{B(x_1, y_1)} \subset A_1 \cap B$.
- We continue by induction. Suppose we have chosen x_{n-1}, r_{n-1} s.t.

  $$\overline{B(x_{n-1}, r_{n-1})} \subset A_{n-1} \cap B(x_{n-2}, r_{n-2})$$

  Since A_n is dense open, $A_n \cap B(x_{n-1}, y_{n-1})$ is non-empty and open. So one can find $x_n \in X$, $r_n > 0$ s.t.

  $$\overline{B(x_n, r_n)} \subset A_n \cap B(x_{n-1}, r_{n-1})$$

  Note that we can always take $r_n < \frac{1}{n}$.
- We claim that $\{x_n\}$ is a Cauchy sequence. In fact, $\forall N$ and $n, m > N$, $B(x_n, y_n) \subset B(x_N, y_N)$ and $B(x_m, y_m) \subset B(x_N, y_N)$. In particular, $x_n, x_m \in B(x_N, y_N)$. So $d(x_n, x_m) \leq d(x_n, x_N) + d(x_N, x_m) < 2r_N < \frac{2}{N}$, So $\{x_n\}$ is Cauchy.
- By completeness, we can find $x_0 \in X$ s.t. $x_n \to x_0$. Since $\overline{B(x_n, y_n)} \subset B(x_{n-1}, r_{n-1})$, we have

  $$x_0 \in \overline{B(x_n, r_n)} \subset A_n \cap B(x_{n-1}, r_{n-1}) \subset ... \subset A_n \cap B.$$

  It follows that $x_0 \in \cap_{n=1}^{\infty}(A_n \cap B) = (\cap_{n=1}^{\infty} A_n) \cap B$. So $(\cap_{n=1}^{\infty} A_n) \cap B \neq \emptyset$.

$\square$

Corollary. *Any complete metric space is of $2^n d$ category.*

Proof. Assume X is of $1^n d$ category which means $X = \cap_{i=1}^{\infty}$ where A_i is nowhere dense. Then $X = \cap_{i=1}^{\infty} \overline{A_i}$, and thus

$$\cap_{i=1}^{\infty} X \backslash \overline{A_i} = X \backslash \overline{A_0} \backslash \overline{A_1} \backslash ... \backslash \overline{A_n} = \emptyset$$

Each $X \backslash \overline{A_i}$ is open because $\overline{A_i}$ is closed. Each $X \backslash \overline{A_i}$ is dense because for all open set B, if $B \cap (X \backslash \overline{A_i}) = \emptyset$, then $B \subset \overline{A_i}$, $\overline{A_i}$ has no interior point, $B \subset \overline{A_i}$ can never happen. So $B \cap (X \backslash \overline{A_i}) \neq \emptyset$, $X \backslash \overline{A_i}$ is dense. By the previous theorem, $\cap_{i=1}^{\infty} (X \backslash \overline{A_i})$ is dense, and thus not $\emptyset$. Contradiction! $\square$