Topology vector spaces

xuascaler

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1. Topology spaces

We have seen many important example of Banach spaces, or more generally examples of vector spaces with a metric structure. However, there are also examples of important spaces whose natural structure does not follow from a complete metric.

Example. $X = C_0^0(\mathbb{R}) = \{\text{compactly supported continuous function on } \mathbb{R}\}$ *If we let*

$$X_n = C_0^0(\big[-n, n\big]) = \{f \in C_0^0(\mathbb{R}) : supp(f) \subset \big[-n, n\big]\},$$
$$supp(f) = \overline{\{x \mid f(x) \neq 0\}}$$

- $X = \cap_{n=1}^{\infty} X_n$
- $X_n \subset C^0(\big[-n, n\big])$ *is closed.(Banach space)*
- X_n *is nowhere dense in* $C^0(-n, n)$*(and in* $C^0(\big[-m, m\big])$ *for* $m \geq n$*).*

Of course any reasonable structure in $C_0^0(\mathbb{R})$ should give the subsapce $C_0^0(\big[-n, n\big])$ natural Banach space structure.
As a consequence of the Baire category theorem, one cannot endow $C_0^0(\mathbb{R})$ with a complete metric whose induced topology is the natural one that we are interesting.
So we need to study structures that are more general than the metric structure.

Definition. *A topological space is a set X with a collection (F) of subsets of X, ($\mathcal{F}$ is called topology, and those element in $\mathcal{F}$ are called open sets), such that*

1. $X, \emptyset \in \mathcal{F}$
2. *If* $A_\alpha \in \mathcal{F}$*, then* $\cup_\alpha A_\alpha \in \mathcal{F}$
3. *If* $A_1, A_2 \in \mathcal{F}$*, then* $A_1 \cap A_2 \in \mathcal{F}$

Example.
- $\mathcal{F} = \{X, \emptyset\}$ *is called the weakest topology on* X*.*
- $\mathcal{F} = \{A | A \subset X\}$.

- *The metric topology on (X, d) is a topology.*

Definition. *A topological space is Housdorff if for any $x \neq y$, there exists neigbourhoods U of x, V of y such that $U \cap V = \emptyset$.*

Remark.

1. *In a Housdorff space, the limit of a convergent sequence is unique.*

   Proof. Suppose $x_n \to x, x_n \to y, y \neq x$, take U, V as above. then $\forall n > N, x_n \in U$, because $U \cap V = \emptyset \Leftrightarrow \forall n > N, x_n \notin V$. $x_n \not\to y$. □

2. *Any single point set $\{x\}$ is closed in a Housdorff space.*

   Proof. $\forall y \neq x$, we can find a neigbourhood V_y of y s.t. $x \notin V_y$, so $X \backslash \{x\} = \cup_{y \neq x} V_y$ is open. □

Now let $(x, \mathcal{F})$ be a topological space.

Definition.

1. *A subcollection $\mathcal{F}' \subset \mathcal{F}$ is called a base for $\mathcal{F}$ if any open set $U \in \mathcal{F}$ is the union of soem members in $\mathcal{F}'$.*
2. *A subcollection ${\mathcal{F}_\S}' \subset \mathcal{F}_\S$ is called a base at x if every neigbourhood of x contains an element of ${\mathcal{F}_\S}'$. (But not necessary union of elements in $\mathcal{F}'$)*

Example.

- *$\mathcal{F}' = \{B(x, r) \mid x \in X, r > 0\}$ for a base for the metric topology on (X, d).*
- *$\mathcal{F}'_x = \{B(x, r) | r > 0\}$ is a local base at x.*
- *$\mathcal{F}''_x = \{B(x, \frac{1}{n}) \mid n \in (N)\}$ is a local base at x containing only countable many elements.*

Remark.

- *difference bases may generate the same topology.*
- *If $\mathcal{F}'$ is a base of $\mathcal{F}$, then $\mathcal{F}$ is the topology generated by $\mathcal{F}'$.*

Now let $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ be topology spaces. let

$$\mathcal{S} = U \times V \mid U \in \mathcal{F}, V \in \mathcal{G}$$

Then $\mathcal{S}$ is collection of subsets in $X \times Y$.

Definition. *The topology generated by $\mathcal{S}$ is called the product topology on $X \times Y$.*

Example. *The usual topology on $\mathbb{R}^2$ is the product topology of the usual topology on $\mathbb{R}$, since any open subset in $\mathbb{R}^2$ is the union of "open rectangles".*

Let X, Y be topology spaces.

Definition.

1. *A map $f : X \to Y$ is called continuous at $x \in X$ if the inverse image of every open neigbourhood of $f(x)$ contains an open neigbourhood of x.*

2. *f is continuous on X if it is continuous at every $x \in X$, in other words, $\forall V \in \mathcal{G}$, one has $f^{-1}(V) \in \mathcal{F}$.*

Property. *If $f : X \to Y$ is continuous at x, and $x_n \to x$, then $f(x_n) \to f(x)$.*

Proof. For any neigbourhood V of $f(x)$, the inverse image $f^{-1}(V)$ is a neigbourhood of x. So for any neigbourhood V, we can find N s.t. $\forall n > N, x_n \in f^{-1}(V)$. V can be any neigbourhood, so it can be any small. So $\forall n > N, f(x_n) \in V, f(x_n) \to f(x)$. □

Definition. *A map $f : X \to Y$ is a homeomophism if it is continuous, invertable and the inverse is also continuous.*

Topological Vector Spaces

Roughly speaking, a topological vector space is a vector space endowed with a topology so that the vector space operations (vector addition, scalar multipliation) are compatable with the topological structure(i.e are continuous).

Definition. *Let X be a vector space endowed with a Housdorff topology(some books do not require this) $\mathcal{F}$. It is said to be topological vector space if the mappings*

$$X \times X \to X, (x, y) \to x + y$$

$$\mathbb{R}(\text{or } \mathbb{C}) \times X \to X, (\alpha, x) \to \alpha x$$

are continuous.(We use product topology on $X \times X, \mathbb{R} \times X$)
By definition, the continuity of vector addition and scalar multipliation means

- $\forall x \in X, y \in X, \forall V \in \mathcal{F}_{x+y}, \exists U_x \in \mathcal{F}_x, \exists U_y \in \mathcal{F}_y$ *s.t.* $U_x + U_y \subset V$.
- $\forall \alpha \in \mathbb{R}, \forall V \in (F)_{\alpha x}, \exists \epsilon > 0, U_x \in \mathcal{F}_x$ *s.t.* $(\alpha - \epsilon, \alpha + \epsilon) \cdot U_x \subset V$.

Remark.

- *For any $A, B \subset X$, we denote $A + B = \{x + y \mid x \in A, y \in B\}$*
- *For any $I \subset \mathbb{R}, A \subset X$, we denote $I \cdot A = \{\alpha x \mid \alpha \in I, x \in A\}$*

Example.

- *example +*

$$A = \{(x, 0) \mid -1 \le x \le 0\}, B = \{(1, y) \mid -1 \le y \le 1\}$$

$$A + B = \{(x, y) \mid 0 \le x \le 2, -1 \le y \le 1\}$$

- *example* $\cdot$

$$A = \{(1,0),(2,0)\}$$

$$2A = \{(2,0),(4,0)\} \neq A + A = \{(2,0),(3,0),(4,0)\}$$

- *For any $a \in X$, one has a translationoperator*

$$T_a : X \to X, x \to T_a(x) = a + x$$

- *For any $0 \neq \alpha \in \mathbb{R}$, one has a multipliation operator.*

$$M_\alpha : X \to X, x \to M_\alpha(x) = \alpha x$$

Property. *For any $a \in X$ and any $0 \neq \alpha \in \mathbb{R}$, T_a and M_α are homeomorphisms.*

Proof. T_a and M_α are both invertable, with inverse T_{-a} and $M_{\frac{1}{\alpha}}$ respectively. Moreover, they are all continuous according to the continuity of vector addition and scalar multipliation. $\square$

Corollary. *A subset A is open if and only if $a + A$ is open. So $\mathcal{F}$ is determined by any local base $\mathcal{F}_0'$ at 0.*