The Baire category theorem

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Metric Topology

Let (X, d) be a metric space, and $A \subset X$ a subset.

Definition.

1. *A point $x \in A$ is called an interior point of A if $\exists \epsilon > 0$ s.t. $B(x, \epsilon) = \{y \in X \mid d(y, x) < \epsilon\}$ lies in A.(ϵ-neigborhood of x).*
2. *A is open if any point $x \in A$ is an interior point of A.*

Example. *For any $x \in X$ and any $r > 0$, $B(x, r)$ is open.*

Proof. For any $y \in B(x, r)$, we have $d(x, y) < r$. Take any $0 < \epsilon < r - d(x, y)$. Then for any $z \in B(y, \epsilon)$, $d(z, x) \le d(z, y) + d(y, x) < \epsilon + d(x, y) < r$. □

Property.

1. *$\emptyset, X$ are open sets.*
2. *If $\{A_\alpha\}$ are a collection (could be infinite, or even incountable) of open sets in X, so is $\cup_\alpha A_\alpha$*
3. *If A, B are open sets in X, so is $A \cap B$.(If $A_1, A_2, ..., A_n$ are open, so is $\cap_{i=1} n A_i$)*

Proof.

1. Obvious.
2. Suppose $x \in \cup_\alpha A_\alpha$, then $\exists \alpha, x \in A_\alpha$. Since A_α is open, $\exists \epsilon > 0, B(x, \epsilon) \subset A_\alpha$. It follows $B(x, \epsilon) \subset \cup_\alpha A_\alpha$. So $\cup_\alpha A_\alpha$ is open.

□