

Complete metric spaces

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1 Convergence

Let (X, d) be a metric space(not necessary to be a vector space).

Definition. We say that a sequence of vectors, $\{x_1, x_2, \dots, x_n\}$, in X converges to $x_0 \in X$ (under the metric d) if $\forall \epsilon > 0, \exists N > 0$ s.t. $\forall i > N, d(x_i, x_0) < \epsilon$. We will introduce Convergence in more general spaces later.

Example. $X = C([0, 1])$

1. $d_1(f, g) = \int_0^1 |f(x) - g(x)| dx$
Let $f_n(x) = x_n, f_0(x) = 0$. Then $d_1(f_n, f_0) = \int_0^1 x_n dx = \frac{1}{n+1} \rightarrow 0$ as $n \rightarrow \infty$.
so $f_n \rightarrow f_0$ in (X, d_1)
2. $d_0(f, g) = \max_{0 \leq x \leq 1} |f(x) - g(x)|$ Still take $f_n(x) = x_n, f_0(x) = 0$, Then $d_0(f_n, f_0) = \sup_{0 \leq x \leq 1} x_n = 1$, so $f_n \not\rightarrow f_0$ in (X, d_0) .

Remark. Obvious one always have $d_1(f, g) \leq d_0(f, g)$ in X . We say the metric d_0 is stronger than d_1 . If a sequence converges to an element in d_0 , then it converges to that element in d_1 .

3. Still take $d_1(f, g) = \int_0^1 |f(x) - g(x)| dx$.

$$f_n(x) = \begin{cases} 0, & x < \frac{1}{2} - \frac{1}{n}, \\ \frac{1}{2} + \frac{n}{2}(x - \frac{1}{2}), & \frac{1}{2} - \frac{1}{n} \leq x \leq \frac{1}{2} + \frac{1}{n}, \\ 1, & x > \frac{1}{2} + \frac{1}{n} \end{cases}$$

and let

$$\begin{cases} 0, & x \leq \frac{1}{2}, \\ 1, & x \geq \frac{1}{2} \end{cases}$$

Then $d_1(f_n, f_0) = \int_0^1 |f_n(x) - f_0(x)| dx = \frac{1}{2n} \rightarrow 0$.

Property. Suppose a sequence $\{x_n\}$ converges in (X, d) , then

1. $\{x_n\}$ is bounded.

2. the limit is unique.

Proof.

1. Suppose $x_n \rightarrow x_0$, then for $\epsilon = 1, \exists N$, s.t. $\forall i > N, d(x_n, x_0) < 1$.
Let $C = \max(d(x_0, x_1), d(x_0, x_2), \dots, d(x_0, x_n)) + 1$, Then $\forall 1 \leq i < \infty, d(x_n, x_0) < C$ So $\{x_i\}$ is bounded.
2. Suppose $x_i \rightarrow x_0$ and $x_i \rightarrow x'_0$. $\forall \epsilon > 0, \exists N, N'$ s.t. $\forall i > N, d(x_i, x_0) < \epsilon, \forall i' > N, d(x'_i, x'_0) < \epsilon \Rightarrow d(x_0, x'_0) \leq d(x_0, x_i) + d(x_i, x'_0) \leq 2\epsilon \Leftrightarrow d(x_0, x'_0) = 0 \Rightarrow x_0 = x'_0$

□

2 Completeness

As we have seen in part 3 of previous example, we have a sequence in X which converges under d_1 to , an element outside X . So as $\mathcal{Q}, (X, d)$ is NOT complete. To do better analysis, we would like to work on complete spaces. As in mathematical analysis, we define

Definition. A sequence $\{x_i\}$ in (X, d) is a Cauchy sequence if $\forall \epsilon > 0, \exists N > 0$ s.t. $\forall i, j > N, d(x_i, x_j) < \epsilon$

Definition. A metric space (X, d) is complete if any Cauchy sequence in (X, d) converges to an element in X .

Definition. A complete normed vector space is called a Banach space.

- Here, the metric is the induced metric from the norm: $d(x, y) = \|x - y\|$
- Banach space will be one of the main object in this course.

Example. $d(x, y) = (\sum_{i=1}^n (x_i - y_i)^2)^{\frac{1}{2}}$ is a complete metric on \mathbb{R}^n .
 $\|x\| = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}, (\mathbb{R}^n, \|\cdot\|)$ is a Banach space.

Example. $X = C([0, 1])$.

1. $d_1(f, g) = \int_0^1 |f(x) - g(x)| dx$. We have seen $f_n(x) = x^n \rightarrow f_0(x) = 0$.
In fact, $\{f_n\}$ is a Cauchy sequence, since $d(f_n, f_m) = \int_0^1 |x^n - x^m| dx < \int_0^1 x^n dx + \int_0^1 x^m dx = \frac{1}{n+1} + \frac{1}{m+1}$ In general, we have

Property. Any converged sequence in a metric space is a Cauchy sequence.

Proof. Suppose $x_i \rightarrow x_0$. i.e. $\forall \epsilon > 0, \exists N$ s.t. $\forall i > N, d(x_i, x_0) < \epsilon$. So for $\forall i, j > N$, we have $d(x_i, x_j) \leq d(x_i, x_0) + d(x_0, x_j) < 2\epsilon$. So $\{x_i\}$ is Cauchy. □

2. $d_0(f, g) = \max_{0 \leq x \leq 1} |f(x) - g(x)| dx$.
 We have seen $f_n \not\rightarrow f_0$ in (X, d_0) . In fact, $\{f_n\}$ is not a Cauchy sequence since we fix n and let $m \rightarrow \infty$, $d(f_n, f_m) = \max_{0 \leq x \leq 1} |x_n - x_m| dx \rightarrow 1$. In fact, we have

Property. $(C([0, 1]), d_0)$ is complete. As a consequence,
 $\|f\|_0 = \sup_{0 \leq x \leq 1} |f(x)|$, $(C([0, 1]), \|f_0\|)$ is a Banach space.

Proof. Let $\{f_n\}$ be a Cauchy sequence, $\forall \epsilon > 0, \exists N > 0$, s.t. $d_0(f_n, f_m) = \sup_{0 \leq x \leq 1} |f_n(x) - f_m(x)| < \epsilon$. Then for any fixed $x \in [0, 1]$, the sequence (of scalars) $\{f_n(x)\}$ is a Cauchy sequence in \mathbb{R} . It follows that there exists $f_0(x)$ s.t. $f_n(x) \rightarrow f_0(x)$ (use Completeness of \mathbb{R}). Since $|f_n(x) - f_m(x)| < \epsilon$, letting $m \rightarrow \infty$ we get $|f_n(x) - f_0(x)| \leq \epsilon, \forall n > N, \forall x \in X$. So the sequence of functions $\{f_n(x)\}$ converges uniformly to $f_0(x)$, because $\forall x \in X$. By results in mathematical analysis, f_0 is continuous and $f_n \rightarrow f_0$ ($\sup_{0 \leq x \leq 1} |f_n(x) - f_0(x)| < \epsilon, \forall n > N$) in $(C([0, 1]), d_0)$ (Finally, each f_n is continuous, then the uniform limit of continuous functions is continuous, $f \in C([0, 1])$). We have thus found a limit $f \in C([0, 1])$ of the Cauchy sequence $\{f_n\}$ in the metric d_0 . This shows $(C([0, 1]), d_0)$ is complete. See the proof in appendix) \square

3.

$$d_1(f, g) = \int_0^1 |f(x) - g(x)| dx$$

$$f_n(x) = \begin{cases} 0, & x < \frac{1}{2} - \frac{1}{n}, \\ \frac{1}{2} + \frac{n}{2}(x - \frac{1}{2}), & \frac{1}{2} - \frac{1}{n} \leq x \leq \frac{1}{2} + \frac{1}{n} \\ 1, & x > \frac{1}{2} + \frac{1}{n} \end{cases}$$

$$f_0(x) = \begin{cases} 0, & x \leq \frac{1}{2} \\ 1, & x \geq \frac{1}{2} \end{cases}$$

f_n is actually a Cauchy sequence in (X, d_1) , since
 $d_1(f_n, f_m) \leq d_1(f_n, f_0) + d_1(f_0, f_m) \leq \frac{1}{2n} + \frac{1}{2m}$. Conclusion: $(C([0, 1]), d_1)$ is NOT complete.

Remark. $C([0, 1]) \subset L^1([0, 1])$ and we will see (L^1, d_1) is complete.

In general, for any incomplete metric space (X, d) , it is possible to construct a complete metric space (\bar{X}, \bar{d}) so that X is dense (we will define this next time) in \bar{X} and $\bar{d}|_X = d$. The procedure is the same as $\mathbb{Q} \rightarrow \mathbb{R}$. See HW next time.

4. $X = l_1 = \mathbf{x} = (a_1, a_2, \dots) \mid \sum_{i=1}^{\infty} |a_i| < \infty$. $\|\mathbf{x}\| = \sum_{i=1}^{\infty} |a_i|$. Then $(X, \|\cdot\|)$ is a Banach space.

Proof.

- X is a vector space because for $\mathbf{x} = (a_1, a_2, a_3, \dots), \mathbf{y} = (b_1, b_2, b_3, \dots)$,

$$x + y \in X : \left| \sum_{i=1}^{\infty} a_i + b_i \right| \leq \sum_{i=1}^{\infty} |a_i| + \sum_{i=1}^{\infty} |b_i| < \infty$$

$$\alpha x \in X : \left| \sum_{i=1}^{\infty} \alpha a_i \right| = |\alpha| \sum_{i=1}^{\infty} |a_i| < \infty$$

The axioms hold in an obvious way.

- $\|\cdot\|$ is norm since
 - $\|x + y\| \leq \|x\| + \|y\|$
 - $\|\alpha x\| = |\alpha| \|x\|$
 - if $x \neq 0$, then $\exists a_i \neq 0$. So $\|x\| = \sum_{i=1}^{\infty} |a_i| > 0$
- Completeness: Let $x^j = (a_i^j)$ be a Cauchy sequence in l^1 , i.e. $\forall \epsilon > 0, \exists N$ s.t. $\forall i, k \geq N, \|x^j - x^k\| = \sum_l |a_l^j - a_l^k| < \epsilon$. So $\forall l$ are fixed, $\forall j, k > N, |a_l^j - a_l^k| < \epsilon$

$\Rightarrow \forall l$ are fixed, $\{a_l^j\}$ is a Cauchy sequence in \mathbb{R}

$$\Rightarrow \exists a_l^0 \in \mathbb{R} \text{ s.t. } a_l^j \xrightarrow{j \rightarrow \infty} a_l^0.$$

We want to show that $x^0 = (a_l^0) \in l^1$, and $x^j \rightarrow x^0$ in (X, d) with $d(x, y) = \|x - y\|$. To prove this, we choose M large so that $\sum_{i=M}^{\infty} |a_i^N| < \epsilon$ then for $j > N$, we have

$$\sum_{i=M}^{\infty} |a_i^j| \leq \sum_{i=M}^{\infty} |a_i^j - a_i^N| + \sum_{i=M}^{\infty} |a_i^N| < 2\epsilon$$

letting $j \rightarrow \infty$, we get $\sum_{i=M}^{\infty} a_i^0 \leq 2\epsilon$. so $x^0 \in l^1$. Moreover, choose $j > N$ large enough, we can get

$$\begin{aligned} \sum_{i=1}^{M-1} |a_i^j - a_i^0| &< \epsilon, \sum_{i=M}^{\infty} |a_i^j| < 2\epsilon, \sum_{i=M}^{\infty} |a_i^0| < 2\epsilon \\ \Rightarrow \sum_{i=1}^{\infty} |a_i^j - a_i^0| &\leq \sum_{i=1}^{M-1} |a_i^j - a_i^0| + \sum_{i=M}^{\infty} |a_i^0| + \sum_{i=M}^{\infty} |a_i^j| < 5\epsilon. \end{aligned}$$

So $x^j \rightarrow x^0$ in (l^1, d) .

□