Convexity

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1. The Minkowski Functional

Let X be a vector space and $A \subset X$ be convex set containing 0. Moreover, we assume tha A is absorbing, i.e. $\forall x \in X$, we can find $t > 0$ large s.t. $x \in tA$.

Definition. *The **Minkowski Functional** μ_A associated to A is the function $\mu_A : X \to R$ defined by*

$$\mu_A = \inf\{t > 0 : x \in tA\}$$

Remark.

1. *It is not a linear functional in the sense we will define later.*
2. *Since A is absorbing, $0 \in A$.*
3. *Since A is convex, we have: if $x \in tA, t > 0$ and $s > t$, then $x \in sA$.*

   Proof. if $x = ty$ for some $y \in A$, then $x = s(\frac{t}{s}y + \frac{s-t}{s}0)$. A is convex, $0 \in A, y \in A \Rightarrow \frac{t}{s}y + \frac{s-t}{s}0 \in A$, so $x \in sA$. □

4. $\{x \in X \mid \mu_A(x) < 1\} \subset A \subset \{x \in X \mid \mu_A(x) \leq 1\}$

   Proof.

   - if $\mu_A(x) < 1$, then $\exists \epsilon > 0$ s.t. $\mu_A(x) \leq 1 - \epsilon$. so $x \in (1 - \frac{\epsilon}{2})A$
   - if $x \in A$, then by definition, $\mu_A(x) \leq 1$.

   □

Property. *if $A \subset X$ is convex and absorbing, then*

1. $\mu_A(x + y) \leq \mu_A(x) + \mu_A(y)$
2. $\forall t \geq 0, \mu_A(tx) = t\mu_A(x)$
3. *if A is also balanced, then μ_A is a semi-norm.*

Proof.

1. for any $\epsilon > 0$, let $t = \mu_A(x) + \epsilon, s = \mu_A(y) + \epsilon$.

□