Convexity

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1. The Minkowski Functional

Let X be a vector space and $A \subset X$ be convex set containing 0. Moreover, we assume tha A is absorbing, i.e. $\forall x \in X$, we can find $t > 0$ large s.t. $x \in tA$.

Definition. *The* ***Minkowski Functional*** μ_A *associated to* A *is the function* $\mu_A : X \to R$ *defined by*

$$\mu_A = \inf\{t > 0 : x \in tA\}$$

Remark.

1. *It is not a linear functional in the sense we will define later.*
2. *Since* A *is absorbing,* $0 \in A$.
3. *Since* A *is convex, we have: if* $x \in tA, t > 0$ *and* $s > t$*, then* $x \in sA$.

   Proof. if $x = ty$ for some $y \in A$, then $x = s(\frac{t}{s}y + \frac{s-t}{s}0)$. A is convex, $0 \in A, y \in A \Rightarrow \frac{t}{s}y + \frac{s-t}{s}0 \in A$, so $x \in sA$. □

4. $\{x \in X \mid \mu_A(x) < 1\} \subset A \subset \{x \in X \mid \mu_A(x) \leq 1\}$

   Proof.

   - if $\mu_A(x) < 1$, then $\exists \epsilon > 0$ s.t. $\mu_A(x) \leq 1 - \epsilon$. so $x \in (1 - \frac{\epsilon}{2})A$
   - if $x \in A$, then by definition, $\mu_A(x) \leq 1$.

   □

Property. *if* $A \subset X$ *is convex and absorbing, then*

1. $\mu_A(x + y) \leq \mu_A(x) + \mu_A(y)$
2. $\forall t \geq 0, \mu_A(tx) = t\mu_A(x)$
3. *if* A *is also balanced, then* μ_A *is a semi-norm.*

Proof.

1. for any $\epsilon > 0$, let $t = \mu_A(x) + \epsilon, s = \mu_A(y) + \epsilon$. then $\frac{x}{t} \in A, \frac{y}{s} \in A$. By Convexity, $\frac{x+y}{s+t} = \frac{t}{s+t}\frac{x}{t} + \frac{s}{s+t}\frac{y}{s} \in A$, By definition, $\forall \epsilon > s, \mu_A(x+y) \leq s + t = \mu_A(x) + \mu_A(y) + 2\epsilon$, so $\mu_A(x+y) \leq \mu_A(x) + \mu_A(y)$.

2. By definition, $\forall t > 0, \mu_A(tx) = \inf\{s > 0 \mid tx \in sA\} = \inf\{s > 0 \mid x \in \frac{s}{t}A\} = t \inf\{\frac{s}{t} > 0 \mid x \in \frac{s}{t}A\} = t\mu_A(x)$

3. By definition, A is balanced $\Leftrightarrow \forall |\alpha| \leq 1, \alpha A \subset A$ In particular, $\forall |\alpha| = 1, \alpha A = A$, because $\alpha = -1, -A \subset A, A \subset -A$. Now $\forall \alpha \neq 0$, we get $\mu_A(\alpha x) = \inf\{t > 0 \mid \alpha x \in tA\} = \inf\{t > 0 \mid x \in \frac{t}{\alpha}A\} = \inf\{t > 0 \mid x \in \frac{t}{-\alpha}A\} = \inf\{t > 0 \mid x \in \frac{t}{|\alpha|}A\} = |\alpha| \inf\{\frac{t}{|\alpha|} > 0 \mid x \in \frac{t}{|\alpha|}A\} = |\alpha|\mu_A(x)$

$\square$

As we have seen last time, any Frechét space is locally convex, and the topology is induced by a complete translation-invariant metric d. By using Minkowski functional on can prove the converse.