

Topology vector spaces

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1. Topology spaces

We have seen many important example of Banach spaces, or more generally examples of vector spaces with a metric structure. However, there are also examples of important spaces whose natural structure does not follow from a complete metric.

Example. $X = C_0^0(\mathbb{R}) = \{\text{compactly supported continuous function on } \mathbb{R}\}$ If we let

$$X_n = C_0^0([-n, n]) = \{f \in C_0^0(\mathbb{R}) : \text{supp}(f) \subset [-n, n]\},$$

$$\text{supp}(f) = \overline{\{x \mid f(x) \neq 0\}}$$

- $X = \bigcap_{n=1}^{\infty} X_n$
- $X_n \subset C^0([-n, n])$ is closed. (Banach space)
- X_n is nowhere dense in $C^0(-n, n)$ (and in $C^0([-m, m])$ for $m \geq n$).

Of course any reasonable structure in $C_0^0(\mathbb{R})$ should give the subsapce $C_0^0([-n, n])$ natural Banach space structure.

As a consequence of the Baire category theorem, one cannot endow $C_0^0(\mathbb{R})$ with a complete metric whose induced topology is the natural one that we are interesting.

So we need to study structures that are more general than the metric structure.

Definition. A topological space is a set X with a collection (\mathcal{F}) of subsets of X , (\mathcal{F} is called topology, and those element in \mathcal{F} are called open sets), such that

1. $X, \emptyset \in \mathcal{F}$
2. If $A_\alpha \in \mathcal{F}$, then $\bigcup_\alpha A_\alpha \in \mathcal{F}$
3. If $A_1, A_2 \in \mathcal{F}$, then $A_1 \cap A_2 \in \mathcal{F}$

Example. • $\mathcal{F} = \{X, \emptyset\}$ is called the weakest topology on X .

- $\mathcal{F} = \{A \mid A \subset X\}$.

- The metric topology on (X, d) is a topology.

Definition. A topological space is Hausdorff if for any $x \neq y$, there exists neighbourhoods U of x , V of y such that $U \cap V = \emptyset$.

Remark.

1. In a Hausdorff space, the limit of a convergent sequence is unique.

Proof. Suppose $x_n \rightarrow x, x_n \rightarrow y, y \neq x$, take U, V as above. then $\forall n > N, x_n \in U$, because $U \cap V = \emptyset \Leftrightarrow \forall n > N, x_n \notin V$. $x_n \not\rightarrow y$. \square

2. Any single point set $\{x\}$ is closed in a Hausdorff space.

Proof. $\forall y \neq x$, we can find a neighbourhood V_y of y s.t. $x \notin V_y$, so $X \setminus \{x\} = \cup_{y \neq x} V_y$ is open. \square

Now let (x, \mathcal{F}) be a topological space.

Definition.

1. A subcollection $\mathcal{F}' \subset \mathcal{F}$ is called a base for \mathcal{F} if any open set $U \in \mathcal{F}$ is the union of some members in \mathcal{F}' .
2. A subcollection $\mathcal{F}'_x \subset \mathcal{F}_x$ is called a base at x if every neighbourhood of x contains an element of \mathcal{F}'_x . (But not necessary union of elements in \mathcal{F}')

Example.

- $\mathcal{F}' = \{B(x, r) \mid x \in X, r > 0\}$ for a base for the metric topology on (X, d) .
- $\mathcal{F}'_x = \{B(x, r) \mid r > 0\}$ is a local base at x .
- $\mathcal{F}''_x = \{B(x, \frac{1}{n}) \mid n \in (\mathbb{N})\}$ is a local base at x containing only countable many elements.

Remark.

- different bases may generate the same topology.
- If \mathcal{F}' is a base of \mathcal{F} , then \mathcal{F} is the topology generated by \mathcal{F}' .

Now let (X, \mathcal{F}) and (Y, \mathcal{G}) be topology spaces. let

$$\mathcal{S} = \{U \times V \mid U \in \mathcal{F}, V \in \mathcal{G}\}$$

Then \mathcal{S} is collection of subsets in $X \times Y$.

Definition. The topology generated by \mathcal{S} is called the product topology on $X \times Y$.

Example. The usual topology on \mathbb{R}^2 is the product topology of the usual topology on \mathbb{R} , since any open subset in \mathbb{R}^2 is the union of "open rectangles".

Let X, Y be topology spaces.

Definition.

1. A map $f : X \rightarrow Y$ is called continuous at $x \in X$ if the inverse image of every open neighbourhood of $f(x)$ contains an open neighbourhood of x .
2. f is continuous on X if it is continuous at every $x \in X$, in other words, $\forall V \in \mathcal{G}$, one has $f^{-1}(V) \in \mathcal{F}$.

Property. If $f : X \rightarrow Y$ is continuous at x , and $x_n \rightarrow x$, then $f(x_n) \rightarrow f(x)$.

Proof. For any neighbourhood V of $f(x)$, the inverse image $f^{-1}(V)$ is a neighbourhood of x . So for any neighbourhood V , we can find N s.t. $\forall n > N, x_n \in f^{-1}(V)$. V can be any neighbourhood, so it can be any small. So $\forall n > N, f(x_n) \in V, f(x_n) \rightarrow f(x)$. \square

Definition. A map $f : X \rightarrow Y$ is a homeomorphism if it is continuous, invertible and the inverse is also continuous.

Topological Vector Spaces

Roughly speaking, a topological vector space is a vector space endowed with a topology so that the vector space operations (vector addition, scalar multiplication) are compatible with the topological structure (i.e. are continuous).

Definition. Let X be a vector space endowed with a Hausdorff topology (some books do not require this) \mathcal{F} . It is said to be topological vector space if the mappings

$$X \times X \rightarrow X, (x, y) \rightarrow x + y$$

$$\mathbb{R}(\text{or } \mathbb{C}) \times X \rightarrow X, (\alpha, x) \rightarrow \alpha x$$

are continuous. (We use product topology on $X \times X, \mathbb{R} \times X$)

Remark.

- For any $A, B \subset X$, we denote $A + B = \{x + y \mid x \in A, y \in B\}$
- For any $I \subset \mathbb{R}, A \subset X$, we denote $IA = \{\alpha x \mid \alpha \in I, x \in A\}$

Example. • $A = \{(x, 0) \mid -1 \leq x \leq 0\}$