Topological Vector Space

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May 25, 2025

1. Local geometry of topological vector space

Let $(X, \mathcal{F})$ be a topological space.

Remark.

- *A base for $\mathcal{F}$ is a subcollection $\mathcal{F}' \subset \mathcal{F}$ s.t. $\forall U \in \mathcal{F}, \exists V \in \mathcal{F}', V \subset U$. A base $\mathcal{F}'$ determines $\mathcal{F}$* since $U \subset \mathbb{F} \Rightarrow \exists S, \forall U \subset \mathcal{F}, U = \{\cup e_i \mid i \in S, e_i \in \mathcal{F}'\}$.
- *A local base of x is a subcollection $\mathcal{F}'_x \subset \mathcal{F}_x$* s.t $.\forall U \subset \mathcal{F}_x, \exists V \in \mathcal{F}'_x$ s.t $V \subset U$. *However, elements in $\mathcal{F}_x$ may be not union of elements in $\mathcal{F}'_x$.*

Example. (X, d) *is a metric space.*

- $\mathcal{F}' = \{B(x, r) \mid x \in X, r > 0\}$ *is a base.*
- $\mathcal{F}' = \{B(x, r), \mid r > 0\}$ *is a local base at x.*
- $\mathcal{F}' = \{B(x, \frac{1}{n}) \mid n \in \mathbb{N}\}$ *anohter local base at x, countable elements.*

Now let X be topological vector space. Last time we showed that $\forall a \in X, \forall \alpha \neq 0$, the maps

- $T_a : X \to X, x \to x + a$
- $M_a : X \to X, x \to \alpha x$

are both homeomorphism. As a consequence, we see

Corollary. A set $A \subset X$ is open $\Leftrightarrow a + A$ is open, $\forall a \in X \Leftrightarrow \alpha A$ is open, $\forall \alpha \neq 0$.

So the topological $\mathcal{F}$ is determined by any local base at 0 whose elements have special gemometric properties for topological vector space.

Definition. *X is locally convex if there is a local base whose elements are convex.*

Example. *Normed Vector Space are locally convex since* $\{B(0, r) \mid r > 0\}$ *are convex.*

Proof. $x, y \in B(0,r) \Leftrightarrow \|x\| < r, \|y\| < r \Leftrightarrow \|\alpha x + (1-\alpha)y\| \leq \alpha\|x\| + (1-\alpha)\|y\| \leq r$ □

Definition. *A set $E \subset X$ is absorbing if $\forall x \in X, \exists \delta > 0$ s.t. $\alpha x \in E, \forall |\alpha| < \delta$.(Obviously $0 = 0 * x \in E$)*

Property. *In a topological vector space, any neigborhood of 0 is absorbing.*

Proof. Let U be a neigborhood of 0. $\forall x \in X$, the map $\mathbb{C} \to X : \alpha \to \alpha x$ is continuous. since it is the composition $\mathbb{R} \to \mathbb{R} \times X \to X : \alpha \to (\alpha, x) \to \alpha x$ both of which are continuous.

Remark. The function $F : X \to Y$ is continuous $\Leftrightarrow \forall\ Y' \subset Y, Y'$ is open set, *the preimage of Y' is open set.*

So the pre-image of U is an open set in $\mathbb{R}$, which obviously contains 0. So $\exists \delta$ s.t. $\forall |\alpha| < \delta, \alpha x \in U$. □

Corollary. *For any neigborhood U of $0, X = \cup_{k=1}^{\infty}(kU)$*

Proof. $\forall x \in X, \exists k, \frac{1}{k} < \delta, \frac{1}{k}x \in U \Rightarrow x \in kU$. □

Definition. *A set $E \subset X$ is symmetric if $E = -E$.*

Property. *$\forall U, 0 \in U$, one can find a symmetric neigborhood V of 0 s.t. $V + V \subset U$.*

Proof. Since $0 + 0 = 0$, and addition is continuous, for the neigborhood U of 0, one can find neigborhoods U_1, U_2 of 0 s.t. $U_1 + U_2 \subset U$. Take $V = U_1 \cap U_2 \cap (-U_1) \cap (-U_2)$. V is symmetric and $0 \in V$. $V \subset U_1, V \subset U_2, V + V \subset U$. □

Remark. *By iteration, one can find V s.t. $V + V + V + V \subset U$*

Definition. *A neigborhood of 0 in X in balanced if $\alpha E \subset E$ for all α with $|\alpha| \leq 1$.*

Remark.

1. *If E is balanced, then E is symmetric. since $-E \subset E, -(-E) \subset -E$.*
2. *If A, B are balanced, so is $A + B$.*

Property. *In a Topological vector space, any neigborhood of 0 contains a balanced neigborhood of 0.*

Proof. Let U be a neigborhood of 0. By continuity and $0 \cdot 0 = 0$, one can find $\delta > 0$ and neigborhood V_1 of 0 s.t. $\beta V_1 \subset U$ for any β with $|\beta| < \delta$. Let $V = \cup_{0<|\beta|<\delta} \beta V_1$, then

- V is open as union of open sets
- $V \subset U$ since each $\beta V_1 \subset U$.

- V is balanced since $|\alpha| \leq 1$, $|\beta| < \delta \Rightarrow |\alpha\beta| < \delta$.

$\square$

Corollary. *Every topological vector space has a balanced local base.*

Remark. *Simlarly one can prove: any convex neigborhood of* 0 *contains a balanced convex neigborhood of* 0*. So any locally convex topological vector space has a balanced convex local base.*

Definition. *A subset* $E \subset X$ *is bounded if for any neigborhood* U *of* 0 *in* X*,* $\exists s > 0$ *s.t.* $\forall t > s$*, we have* $E \subset tU$*.*

Property. E *is balanced* $\Leftrightarrow$ *For any sequence* $\{x_n\} \subset E$ *and any scalar sequence* $\alpha_n \to 0$*, one has* $\alpha_n x_n \to 0$

Proof.

- $\Rightarrow$, For any neigborhood U of 0, we have $\exists s > 0, \forall t > s, E \subset tU$, Since $\alpha_n \to 0$, $\exists N, \forall n > N, |\alpha_n| < \frac{1}{t}$, we have $|\alpha_n|E \subset U \Rightarrow |\alpha_n|x_n \in U$. we have $|\alpha_n|x_n \to 0$. $\forall \delta > 0, \exists N, \forall n > N, ||\alpha_n|x_n - 0| = |\alpha_n x_n| = |\alpha_n x_n - 0| < \delta$. So $\alpha_n x_n \to 0$.

- $\Leftarrow$, If E is not bounded, then for any neigborhood U of 0, $\forall s > 0, \exists t > s, E \not\subset tU$. We fix U, now we construct $\{\alpha_n\}, \{x_n\}$. We choose $n \in \mathbb{N}, s = n, \exists t_n > n, \frac{E}{t_n} \not\subset U$, so $\alpha_n = \frac{1}{t_n}, x_n \in E, \frac{x_n}{t_n} \notin U$. Now we have $\{\alpha_n \to 0, \alpha_n x_n \notin U \Rightarrow \alpha_n x_n \not\to 0\}$. Contradiction!

$\square$

So in particular, if X *is a metric space, then* E *is bounded iff* $\exists C > 0, E \subset B(0, C)$*. Not every topological vector space admit a bounded open set. In fact,*

Property. *If* V *is bounded neigborhood of* 0*, then for any sequence* $\alpha_k \to 0$*, the collection* $\{\alpha_k V \mid k = 1, 2, 3...\}$ *is a local base of* X*.*

Proof. We construct a set $W = V \cup -V$, $V \subset W$ and W is bounded and symmetric. For any neigborhood U of 0, $\exists s > 0, \forall t > s, W \subset tU$. Since $\alpha_n \to 0 \Rightarrow \exists \alpha_n, \frac{1}{|\alpha_n|} > t$, we have $\exists \alpha_n, |\alpha_n|W \subset U$. W is symmetric, $\alpha_n W \subset U$ $V \subset W \Rightarrow \alpha_n V \subset U$. So $\alpha_n V$ is a local base. $\square$

Definition. X *is locally bounded if* 0 *has a bounded neigborhood.*

So any locally bounded TVS has a countable local base. According the next theorem, it must be metrizable.

2.Metrization

As we mentioned at the begining of this lecture, any metirc space has a countable local base. This gives a necessary condition for topological vector space to be metrizable. In fact this is also sufficient, and we can say more...

Theorem. *Let X be a topological vector space with a countable local base. Then X is metrizable, and one can choose the metric d s.t.*

1. *d is translation invariant.*
2. *The open balls $B(0, r)s$ are balanced.*
3. *Moreover, if X is also locally convex, then d can be chosen so that all open balls are convex.*

Proof.

- One start with a countable local base $\{U_n\}$. Choose a balanced neigborhood V_1 of 0 in U_1. Then choose a neigborhood $\tilde{U_2}$ of 0 s.t. $\tilde{U_2} + \tilde{U_2} + \tilde{U_2} + \tilde{U_2} \subset V_1 \cap U_2$. Choose a balanced neigborhood of V_2 of 0 in $\tilde{U_2}$. Note we have $V_2 + V_2 + V_2 + V_2 \subset V_1$. Continue this way, we get a balanced local base $\{V_n\}$ s.t. $V_{n+1} + V_{n+1} + V_{n+1} + V_{n+1} \subset V_n$. Moreover, if X is locally convex, one can choose V_n s.t. V_n is also convex.

- Let $D = \{r \in \mathbb{Q} \mid r = \sum_{n=1}^{\infty} C_n(r)2^{-n}$, only finitely many $C_n(r) = 1$, other $C_n(r) = 0\}$. So elements of D are $2-$adic rationals with finite digits.

  Remark. *D is dense in* $[0, 1)$.

- For any $r \in D$, we define $A(r)$ to be the subset $A(r) = C_1(r)V_1 + C_2(r)V_2 + C_3(r)V_3 + ...$(finite sum, so it make sense). For $r \geq 1$, set $A(r) = X$.

  Property. $\forall r, s \in D, A(r) + A(s) \subset A(r + s)$.

  Proof.

  $$r = \sum_1^M C_n(r)2^{-n}, s = \sum_1^N C_n(s)2^{-n}$$

  we choose $K = \max\{M, N\}$.

  $$r = \sum_1^K C_n(r)2^{-n}, s = \sum_1^K C_n(s)2^{-n}, r + s = \sum_1^K C_n(r + s)2^{-n}$$

  $V_{n+1} + V_{n+1} + V_{n+1} + V_{n+1} \subset V_n \Rightarrow V_{n+1} + V_{n+1} \subset V_n$, so we have $\sum_1^{2^i} V_{K-i} \subset V_K$. $C_n(r) + C_n(s)$ can be $\{0, 1, 2\}$, if $C_n(r) + C_n(s) = 2$, $V_n + V_n \subset V_{n+1}$, we can have $A(s) + A(r) \subset A(s + t)$. □

Remark.

- *Each $A(r)$ is balanced.* (A, B are balanced $\Rightarrow A + B$ is balanced.)
- $\forall r < s, A(r) \subset A(s).(A(r) + A(s-r) \subset A(s), 0 \in A(s-r))$

- $f(x) = \inf\{r : x \in A(r)\}$. Then $0 \le f(x) \le 1$, and
  - $f(0) = 0$ since $\forall r, 0 \in A(r)$
  - If $x \neq 0$, then $\exists N, \forall n \ge N, x \notin V_n, \Rightarrow f(x) \ge 2^{-N}$, since $V_N = A(2^{-N}), 2^{-N} \in D$.
  - $f(x) = f(-x)$ since $A(r)$ is balanced $\Rightarrow A(r)$ is symmetric.
  - $f(x+y) \le f(x) + f(y)$

    Proof. It is enough to check this for $f(x)+f(y) < 1$. For $\epsilon > 0, \exists r, s \in D$(since D is dense.), $f(x) < r < f(x) + \epsilon, f(y) < s < f(y) + \epsilon$. So $x \in A(r), y \in A(s)$. So $x + y \in A(r + s)$, i.e. $f(x + y) \le r + s < f(x)+f(y)+2\epsilon$ This is true for $\forall \epsilon > 0$, so $f(x+y) \le f(x)+f(y)$. □

- Now it is standard $d(x, y) = f(x-y)$ define a translation-invariant metric on X. Finally, by definition the open balls centered at 0 are $B(0, \delta) = \{x : f(x) < \epsilon\} = \cup_{r<\delta} A(r)$. They are the local basis of the original topology since $\forall r < 2^{-n}, A(r) \subset V_n \Rightarrow \forall \delta < 2^{-n}, B(0, \delta) \subset V_n$. In view of the fact $0 < r < s, A(r) \subset A(s)$, it is easy to see that $B(0, \delta)$ is balanced since each $A(r)$ is balanced and if each V_n is convex, so is $A(r)$, and that so is $B(0, \delta)$, and thus so is $B(x, \delta)$.

□