Topology vector spaces

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1. Topology spaces

We have seen many important example of Banach spaces, or more generally examples of vector spaces with a metric structure. However, there are also examples of important spaces whose natural structure does not follow from a complete metric.

Example. $X = C_0^0(\mathbb{R}) = \{\text{compactly supported continuous function on } \mathbb{R}\}$ *If we let*

$$X_n = C_0^0(\big[-n,n\big]) = \{f \in C_0^0(\mathbb{R}) : supp(f) \subset \big[-n,n\big]\},$$

$$supp(f) = \overline{\{x \mid f(x) \neq 0\}}$$

- $X = \cap_{n=1}^{\infty} X_n$
- $X_n \subset C^0(\big[-n,n\big])$ *is closed.(Banach space)*
- X_n *is nowhere dense in* $C^0(-n,n)$*(and in* $C^0(\big[-m,m\big])$ *for* $m \geq n$*).*

Of course any reasonable structure in $C_0^0(\mathbb{R})$ *should give the subsapce* $C_0^0(\big[-n,n\big])$ *natural Banach space structure.*