Fréchet Spaces

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May 27, 2025

1 Topology defined by semi-norms

Let's start with the example we mentioned at the begining of Lecture4.

$$X = C_0^0((R)) = \{\text{compactly supported continous functions on } (R)\}$$

Then as we have seen, $X = \cup_{n=1}^{\infty} X_n$, where

$$X_n = C_0^0([-n,n]) = \{\text{compactly supported continous functions on } (-n,n), f(n) = f(-n) = 0\}$$

The topology of X_n, as a closed subspace of $C^0([-n,n])$, is determined by open sets

$$B(f,r) = \{g \in C_0^0([-n,n]) \mid \sup_{x\in[-n,n]} |f(x) - g(x)| < r\}$$

In particular, if we let $p_n(f) = sup_{x\in[-n,n]}|f(x)|$, then a local base for this topology is $\{f \mid p_n(f) < \frac{1}{k}\}$. We would like to assign a topology to X, so that as a subspace of X, the space X_n has the same topology mentioned above. The easiest way is to let

$$U_n^k = \{f \in C_0^0(\mathbb{R}) \mid p_n(f) = \sup_{x\in[-n,n]} |f(x)| < \frac{1}{k}\}$$

and take $\mathcal{F}$ be the topology generated by $U_{n,k}$.

Remark.

- *As a function on $C_0^0((R))$, p_n is not a norm, although it is a norm on $C_0^0[-n,n]$. Since one can easily find a function $f \in C_0^0(\mathbb{R}), f \neq 0, p_n(f) = 0$.*
- *under this topology, $(X, \mathcal{F})$ is a locally convex TVS.*
- *There are many other spaces of this type.*

Definition. *A semi-norm on a vector space X is a function $p : X \to [0, +\infty)$ s.t.*

1. *$p(x+y) \leq p(x) + p(y)$ (subadditivity)*

2. $p(\alpha x) = |\alpha| p(x)$ *(positive homogenerity)*

Definition. *A family of semi-norms, $\{p_\lambda\}$ on X is called seperating if $\forall x \neq 0, \exists \lambda, p_\lambda(x) \neq 0$.*

Example. *$p_n(f) = \sup_{x \in [-n,n]} |f(x)|$ defines a countable family of seperating semi-norms on $C_0^0(\mathbb{R})$.*

Theorem. *Let $\mathcal{P} = \{p_\lambda\}$ be a seperating family of semi-norms on a vector space X. For each $p_\lambda \in \mathcal{P}$ and each $k \in \mathbb{N}$ we let $U_{\lambda,k} = \{x \in X \mid p_\lambda x < \frac{1}{k}\}$*
Let $\mathcal{B} = \{$the collection of all finite intersection of sets of the form $U_{\lambda,k}\}$.
Let $\mathcal{F} = \{$the translation invariant topology on X that has $\mathcal{B}$ as a local base at $0\}$.
Then

1. *$(X, \mathcal{F})$ is a topological vector space.*
2. *$\mathcal{B}$ is a convex balanced local base.*
3. *Each $p_\lambda \in \mathcal{P}$ is continuous.*
4. *A set $E \subset X$ is bounded iff each $p_\lambda \in \mathcal{P}$ is bounded on E.*

Proof.

1. - $\mathcal{F}$ is Hausedoff. It is enough to seperate 0 and $x \neq 0$. One just choose $p_\lambda \in \mathcal{P}$ s.t. $p_\lambda(x) \neq 0$. We denote $p_\lambda(x) = \epsilon$. Take k large s.t. $\frac{1}{k} < \frac{\epsilon}{2}$. Then $U_{\lambda,k}$ is a neigborhood of 0. $x + U_{\lambda,k}$ is a neigborhood of x, and $U_{\lambda,k} \cap (x + U_{\lambda,k}) = \emptyset$. So $\mathcal{F}$ is Hausedoff. If $y \in U_{\lambda,k} \cap (x + U_{\lambda,k})$, then $\exists z \in U_{\lambda,k}$ s.t. $y = x + z \Rightarrow x = y - z$. So $p_\lambda(x) = p_\lambda(y - z) \leq p_\lambda(y) + p_\lambda(z) < \frac{1}{k} + \frac{1}{k} = \frac{2}{k} < \epsilon$ Contrdiction! we know $p_\lambda(x) = \epsilon$!
   - Vector addition is continuous. Let U be any open neigborhood of $x+y$. By definition of $\mathcal{F}$, one can choose $p_{\lambda_1}, ..., p_{\lambda_l}$ and $k_1, ..., k_l$ s.t. $x + y + (U_{\lambda_1,k_1} \cap ... \cap U_{\lambda_l,k_l}) \subset U$. Now let

     $$U_1 = x + U_{\lambda_1,2k_1} \cap ... \cap U_{\lambda_l,2k_l} \text{(open neigborhood of x)}$$

     $$U_2 = y + U_{\lambda_1,2k_1} \cap ... \cap U_{\lambda_l,2k_l} \text{(open neigborhood of y)}$$

     Then $U_1 + U_2 \subset U$. If $z_1, z_2 \in U_{\lambda_1,2k_1} \cap ... \cap U_{\lambda_l,2k_l}$, i.e. $\forall 1 \leq i \leq l, p_{\lambda_i}(z_1) < \frac{1}{2k_i}, p_{\lambda_i}(z_2) < \frac{1}{2k_i}$, then $\forall 1 \leq i \leq l, p_{\lambda_i}(z_1 + z_2) < \frac{1}{k_i}$.
   - Scalar multiplication is continuous. Let U be a neigborhood of αx, so as above,

     $$\alpha x + U_{\lambda_1,k_1} \cap ... \cap U_{\lambda_l,k_l} \subset U$$

     for some $p_{\lambda_1}, ..., p_{\lambda_l} \in \mathcal{P}$ and $k_1, ..., k_l \in \mathbb{N}$.
     - case 1. $\alpha = 0$. We choose $A > max(p_{\lambda_1}(x), ..., p_{\lambda_l}(x))$, then for $\delta < \min\{\frac{1}{2Ak_1}, ..., \frac{1}{2Ak_l}, 1\}$, $(-\delta, \delta) \cdot (x + U_{\lambda_1,2k_1} \cap ... \cap U_{\lambda_l,2k_l}) \subset U$

       $$\forall e \in (-\delta, \delta) \cdot x, \forall 1 \leq i \leq l, p_{\lambda_i}(e) < \frac{1}{2k_i}$$

So

$$(-\delta, \delta) \cdot x \subset U_{\lambda_1, 2k_1} \cap ... \cap U_{\lambda_l, 2k_l}$$

$$|\delta| \le 1 \Rightarrow (-\delta, \delta) \cdot U_{\lambda_1, 2k_1} \cap ... \cap U_{\lambda_l, 2k_l} \subset U_{\lambda_1, 2k_1} \cap ... \cap U_{\lambda_l, 2k_l}$$

$$(-\delta, \delta) \cdot (x + U_{\lambda_1, 2k_1} \cap ... \cap U_{\lambda_l, 2k_l}) \subset U_{\lambda_1, k_1} \cap ... \cap U_{\lambda_l, k_l} \subset U$$

– case 2. $\alpha \neq 0$. We choose

$$A > \max\{p_{\lambda_1}(x), ..., p_{\lambda_l}(x), \frac{1}{3k_1|\alpha|}, ..., \frac{1}{3k_l|\alpha|}\}$$

$$\delta < \min\{\frac{1}{3k_1 A}, ..., \frac{1}{3k_n A}, 1\}$$

Then

$$(\alpha - \delta, \alpha + \delta) \cdot (x + U_{\lambda_1, 3k_1|\alpha|} \cap ... \cap U_{\lambda_l, 2k_l|\alpha|}) \subset U$$

2. By definition $\mathcal{B}$ is a local base for $\mathcal{F}$. To prove $U_{\lambda_1, k_1} \cap ... \cap U_{\lambda_l, k_l}$ is convex and balanced, it is enough to prove $U_{\lambda, k}$ is convex and balanced. It is balanced by positive homogenerity. It is convex since $\forall x, y \in U_{\lambda, k}$. $\forall 0 \le \alpha \le 1$,

$$p_\lambda(\alpha x + (1-\alpha)y) \le \alpha p_\lambda(x) + (1-\alpha)p_\lambda(y) < \alpha\frac{1}{k} + (1-\alpha)\frac{1}{k} = \frac{1}{k}$$

3. By definition each p_λ is continuous at 0. The continuity of p_λ at x follows from $p_\lambda(x + U_{\lambda,k}) \subset (p_\lambda(x) - \frac{1}{k}, p_\lambda(x) + \frac{1}{k})$. If $y \in U_{\lambda,k}$, then

$$p_\lambda(x+y) \le p_\lambda(x) + p_\lambda(y) < p_\lambda(x) + \frac{1}{k}$$

$$p_\lambda(x+y) + p_\lambda(-y) \ge p_\lambda(x) \Rightarrow p_\lambda(x+y) \ge p_\lambda(x) - p_\lambda(-y) > p_\lambda(x) - \frac{1}{k}$$

4. Suppose E is bounded, and $p_\lambda \in \mathcal{P}$. Then $\exists t > 0$ s.t. $E \subset tU_{\lambda,1}$ i.e. $\frac{1}{t}E \subset U_{\lambda,1}$ So $\forall x \in E, p_\lambda(x) < t$. Suppose each p_λ is bounded on E. Then for any neigborhood $U_{\lambda_1,k_1} \cap ... \cap U_{\lambda_l,k_l} \subset U$ of 0, one pick $t_1, ..., t_l$ s.t. $\forall x \in E, p_{\lambda_i}(x) < t_i$. Then for $t > max(t_1k_1, ..., t_lk_l)$, we have $E \subset t(U_{\lambda_1,k_1} \cap ... \cap U_{\lambda_l,k_l})$, since $p_{\lambda_i}(\frac{x}{t}) = \frac{1}{t}p_{\lambda_i}(x) < \frac{1}{t}t_i < \frac{1}{k_i}$.

□

2 Frećhet Spaces

In many applications, as in the case of $C_0^0((R))$, the topology is defined by a countable sequence of semi-norms. As a result, the local base $\mathcal{B}$ contains only countable elements, and thus $\mathcal{F}$ is metrizable. In fact, we can explicitly write down a translation-invariant metric in this case.

Property. *If the topology on X is defined by a seperating sequence $\{p_n\}$ of semi-norms, then for any sequence of positive members $\{c_i\}$ that tends to* 0,

$$d(x,y) = \max_i \frac{c_i p_i(x-y)}{1+p_i(x-y)}$$

is a compatible translation-invariant metric on X.