# Fréchet Spaces

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## 1 Topology defined by semi-norms

Let's start with the example we mentioned at the begining of Lecture4.

$$X = C_0^0((R)) = \{\text{compactly supported continous functions on } (R)\}$$

Then as we have seen, $X = \cup_{n=1}^{\infty} X_n$, where

$$X_n = C_0^0([-n,n]) = \{\text{compactly supported continous functions on } (-n,n), f(n) = f(-n) = 0\}$$

The topology of $X_n$, as a closed subspace of $C^0([-n,n])$, is determined by open sets

$$B(f,r) = \{g \in C_0^0([-n,n]) \mid \sup_{x\in[-n,n]} |f(x) - g(x)| < r\}$$

In particular, if we let $p_n(f) = sup_{x\in[-n,n]}|f(x)|$, then a local base for this topology is $\{f \mid p_n(f) < \frac{1}{k}\}$. We would like to assign a topology to $X$, so that as a subspace of $X$, the space $X_n$ has the same topology mentioned above. The easiest way is to let

$$U_n^k = \{f \in C_0^0(\mathbb{R}) \mid p_n(f) = \sup_{x\in[-n,n]} |f(x)| < \frac{1}{k}\}$$

and take $\mathcal{F}$ be the topology generated by $U_{n,k}$.

**Remark.**

- *As a function on $C_0^0((R))$, $p_n$ is not a norm, although it is a norm on $C_0^0[-n,n]$. Since one can easily find a function $f \in C_0^0(\mathbb{R}), f \neq 0, p_n(f) = 0$.*
- *under this topology, $(X, \mathcal{F})$ is a locally convex TVS.*
- *There are many other spaces of this type.*

**Definition.** *A semi-norm on a vector space $X$ is a function $p : X \to [0, +\infty)$ s.t.*

1. *$p(x + y) \leq p(x) + p(y)$ (subadditivity)*
2. *$p(\alpha x) = |\alpha| p(x)$ (positive homogenerity)*

**Definition.** *A family of semi-norms, $\{p_\lambda\}$ on $X$ is called seperating if $\forall x \neq 0, \exists \lambda, p_\lambda(x) \neq 0$.*

**Example.** $p_n(f) = \sup_{x\in[-n,n]} |f(x)|$ *defines a countable family of seperating semi-norms on $C_0^0(\mathbb{R})$.*

**Theorem.** *Let $\mathcal{P} = \{p_\lambda\}$ be a seperating family of semi-norms on a vector space $X$. For each $p_\lambda \in \mathcal{P}$ and each $k \in \mathbb{N}$ we let $U_{\lambda,k} = \{x \in X \mid p_\lambda x < \frac{1}{k}\}$*
*Let $\mathcal{B} = \{$the collection of all finite intersection of sets of the form $U_{\lambda,k}\}$.*
*Let $\mathcal{F} = \{$the translation invariant topology on $X$ that has $\mathcal{B}$ as a local base at $0\}$.*
*Then*

1. *$(X, \mathcal{F})$ is a topological vector space.*
2. *$\mathcal{B}$ is a convex balanced local base.*
3. *Each $p_\lambda \in \mathcal{P}$ is continuous.*
4. *A set $E \subset X$ is bounded iff each $p_{lambda} \in \mathcal{P}$ is bounded on $E$.*

*Proof.*

1.
   - $\mathcal{F}$ is Hausedoff. It is enough to seperate 0 and $x \neq 0$. One just choose $p_\lambda \in \mathcal{P}$ s.t. $p_\lambda(x) \neq 0$. We denote $p_\lambda(x) = \epsilon$. Take $k$ large s.t. $\frac{1}{k} < \frac{\epsilon}{2}$. Then $U_{\lambda,k}$ is a neigborhood of 0. $x + U_{\lambda,k}$ is a neigborhood of $x$, and $U_{\lambda,k} \cap (x + U_{\lambda,k}) = \emptyset$. So $\mathcal{F}$ is Hausedoff. If $y \in U_{\lambda,k} \cap (x + U_{\lambda,k})$, then $\exists z \in U_{\lambda,k}$ s.t. $y = x + z \Rightarrow x = y - z$. So $p_\lambda(x) = p_\lambda(y - z) \leq p_\lambda(y) + p_\lambda(z) < \frac{1}{k} + \frac{1}{k} = \frac{2}{k} < \epsilon$ Contrdiction! we know $p_\lambda(x) = \epsilon$!
   - Vector addition is continuous. Let $U$ be any open neigborhood of $x+y$. By definition of $\mathcal{F}$, one can choose $p_{\lambda_1}, ..., p_{\lambda_l}$ and $k_1, ..., k_l$ s.t. $x + y + (U_{\lambda_1,k_1} \cap ... \cap U_{\lambda_l,k_l}) \subset U$. Now let
   $$U_1 = x + U_{\lambda_1,2k_1} \cap ... \cap U_{\lambda_l,2k_l} (\text{open neigborhood of x})$$
   $$U_2 = y + U_{\lambda_1,2k_1} \cap ... \cap U_{\lambda_l,2k_l} (\text{open neigborhood of y})$$
   Then $U_1 + U_2 \subset U$. If $z_1, z_2 \in U_{\lambda_1,2k_1} \cap ... \cap U_{\lambda_l,2k_l}$, i.e. $\forall 1 \leq i \leq l, p_{\lambda_i}(z_1) < \frac{1}{2k_i}, p_{\lambda_i}(z_2) < \frac{1}{2k_i}$, then $\forall 1 \leq i \leq l, p_{\lambda_i}(z_1 + z_2) < \frac{1}{k_i}$.
   - Scalar multiplication is continuous. Let $U$ be a neigborhood of $\alpha x$, so as above,
   $$\alpha x + U_{\lambda_1,k_1} \cap ... \cap U_{\lambda_l,k_l} \subset U$$
   for some $p_{\lambda_1}, ..., p_{\lambda_l} \in \mathcal{P}$ and $k_1, ..., k_l \in \mathbb{N}$.
     - case 1. $\alpha = 0$. We choose $A > max(p_{\lambda_1}(x), ..., p_{\lambda_l}(x))$, then for $\delta < \min\{\frac{1}{2Ak_1}, ..., \frac{1}{2Ak_l}, 1\}$, $(-\delta, \delta) \cdot (x + U_{\lambda_1,2k_1} \cap ... \cap U_{\lambda_l,2k_l}) \subset U$
     $$\forall e \in (-\delta, \delta) \cdot x, \forall 1 \leq i \leq l, p_{\lambda_i}(e) < \frac{1}{2k_i}$$

So

$$(-\delta,\delta)\cdot x \subset U_{\lambda_1,2k_1}\cap ...\cap U_{\lambda_l,2k_l}$$

$$|\delta|\le 1 \Rightarrow (-\delta,\delta)\cdot U_{\lambda_1,2k_1}\cap ...\cap U_{\lambda_l,2k_l} \subset U_{\lambda_1,2k_1}\cap ...\cap U_{\lambda_l,2k_l}$$

$$(-\delta,\delta)\cdot (x+U_{\lambda_1,2k_1}\cap ...\cap U_{\lambda_l,2k_l}) \subset U_{\lambda_1,k_1}\cap ...\cap U_{\lambda_l,k_l}\subset U$$

– case 2. $\alpha \neq 0$. We choose

$$A > \max\{p_{\lambda_1}(x),...,p_{\lambda_l}(x),\frac{1}{3k_1|\alpha|},...,\frac{1}{3k_l|\alpha|}\}$$

$$\delta < \min\{\frac{1}{3k_1A},...,\frac{1}{3k_nA},1\}$$

Then

$$(\alpha-\delta,\alpha+\delta)\cdot(x+U_{\lambda_1,3k_1|\alpha|}\cap ...\cap U_{\lambda_l,2k_l|\alpha|})\subset U$$

2. By definition $\mathcal{B}$ is a local base for $\mathcal{F}$. To prove $U_{\lambda_1,k_1}\cap ...\cap U_{\lambda_l,k_l}$ is convex and balanced, it is enough to prove $U_{\lambda,k}$ is convex and balanced. It is balanced by positive homogenerity. It is convex since $\forall x,y\in U_{\lambda,k}$. $\forall 0\le\alpha\le 1$,

$$p_\lambda(\alpha x+(1-\alpha)y)\le \alpha p_\lambda(x)+(1-\alpha)p_\lambda(y)<\alpha\frac{1}{k}+(1-\alpha)\frac{1}{k}=\frac{1}{k}$$

3. By definition each $p_\lambda$ is continuous at 0. The continuity of $p_\lambda$ at $x$ follows from $p_\lambda(x+U_{\lambda,\frac{1}{k}})\subset(p_\lambda(x)-\frac{1}{k},p_\lambda(x)+\frac{1}{k})$. If $y\in U_{\lambda,\frac{1}{k}}$, then

$$p_\lambda(x+y)-p_\lambda(x)\le p_\lambda(y)<\frac{1}{k}$$

$$p_\lambda(x)-p_\lambda(x+y)\le p_\lambda(-y)<\frac{1}{k}$$

4. Suppose $E$ is bounded, and $p_\lambda\in\mathcal{P}$. Then $\exists t>0$ s.t. $E\subset tU_{\lambda,1}$ i.e. $\frac{1}{t}E\subset U_{\lambda,1}$ So $\forall x\in E, p_\lambda(x)<t$. Suppose each $p_\lambda$ is bounded on $E$. Then for any neigborhood $U_{\lambda_1,k_1}\cap ...\cap U_{\lambda_l,k_l}\subset U$ of 0, one pick $t_1,...,t_l$ s.t. $\forall x\in E, p_{\lambda_i}(x)<t_i$. Then for $t>max(t_1k_1,...,t_lk_l)$, we have $E\subset t(U_{\lambda_1,k_1}\cap ...\cap U_{\lambda_l,k_l})$. since $p_{\lambda_i}\frac{x}{t}=\frac{1}{t}p_{\lambda_i}(x)<\frac{1}{t}t_i<\frac{1}{k_i}$.

□