A Method of Generating Second-Order Soliton with Specified Time Positions

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***Abstract*—The relationship between the time position and the b-coefficient of a first-order soliton does not apply to a second-order soliton. We propose a method of generating second-order solitons with specified time positions based on nonlinear Fourier transform with modified b-coefficients.**

Keywords—soliton, nonlinear Fourier transform, NFDM

I. Introduction

During the past two decades, fiber optic communication has made a dramatic increase in transmission capacity, thanks to the development of coherent detection and digital signal processing technology. However, fiber nonlinearity hinders an ever-increasing growth of fiber optic capacity and sets the "nonlinear Shannon limit" [1]. Different approaches have been proposed to alleviate the negative effects of fiber nonlinearity. Among them, Nonlinear Frequency Division Multiplexing (NFDM) has been actively investigated in recent years as it can potentially linearize the fiber optic channel [2]. According to NFDM theory, a time domain signal has discrete nonlinear spectrum and continuous nonlinear spectrum, and they are connected in a one-to-one manner by Nonlinear Fourier Transform (NFT). Accordingly, there are discrete-NFDM and continuous-NFDM schemes, which utilize the discrete nonlinear spectrum and continuous spectrum to carry information, respectively.

Discrete-NFDM is essentially a modern version of fiber soliton communication, as both the amplitude, group velocity, time position, and phase of a soliton can be encoded with information, or correspondingly, the imaginary part and real part of eigenvalue, the amplitude and phase of the discrete nonlinear spectrum in the language of NFDM. To reduce the interference of adjacent soliton pulses, the soliton pairs can be designed to have symmetry real parts, and "2-soliton" detection may be carried out at the receiver [3] In this way, the soliton pairs are essentially regarded as one second-order soliton (2-soliton), while at the transmitter side, they are formed by just linearly adding two first-order solitons (1-soliton) together, thus, only an approximation of one 2-soliton.

In this paper, we proposed an effective method to directly generate 2-soliton with specified time positions, by properly modifying the b-coefficients of the soliton. Arbitrary pulse heights, frequencies, and phases can be supported.

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II. NFT of Soliton

The signal propagation in optical fiber is described by the Nonlinear Schrodinger Equation (NLSE). NLSE writes (in the normalized version)

$$i\frac{u(t,z)}{\partial z}+\frac{1}{2}\frac{\partial^2 u(t,z)}{\partial t^2}+|u(t,z)|^2\,u(t,z)=0 \qquad (1)$$

NLSE can be effectively solved by transforming the time-domain signal $u(t,z)$ into the nonlinear frequency domain and evolving the nonlinear spectrum in it. The discrete and continuous spectrum are defined as $Q_d(\lambda_i)=b(\lambda_i)/a'(\lambda_i)$, and $Q_c(\lambda_i)=b(\lambda)/a(\lambda)$, respectively, where λ_i and λ means discrete eigenvalue and continuous nonlinear frequency, respectively [4]. A soliton has no continuous spectrum and a 1-soliton with eigenvalue $\lambda_1=\alpha+i\beta$ can be described by [5]

$$u(t,z)=2\beta\,\mathrm{sech}[2\beta(t-t_0+2\alpha z)]\,e^{i2(\beta^2-\alpha^2)z-i2\alpha t+i\phi_0} \qquad (2)$$

where

$$2\beta t_0=\ln(|Q_d(\lambda_1,z=0)|/2\beta)=\ln(|b(\lambda_1,z=0)|) \qquad (3)$$

$$\phi_0=\pi-\angle Q_d(\lambda,z=0)=\pi/2-\angle b(\lambda_1,z=0) \qquad (4)$$

Note the second equalities in Eq. (3) and (4) are obtained by the fact that in the 1-soliton case, $a'(\lambda_1)$ is simply $1/2i\beta$. The time position of the soliton peak and the phase of the 1-soliton are explicitly determined by the magnitude and phase of the b-coefficient, respectively. Therefore, a 1-soliton with arbitrary time position and phase can be readily generated by inverse NFT with λ_1 and $b(\lambda_1)$.

A 2-soliton whose eigenvalues have different real parts can be asymptotically (i.e. when z approaches $\pm\infty$) expressed as a linear sum of two 1-solitons, but $|b(\lambda_1)|$ and $|b(\lambda_2)|$ no longer directly determine peak positions in the time domain, that is, if $|b(\lambda_1)|$ and $|b(\lambda_2)|$, obtained by Eq. (3) according to two time positions, t_1 and t_2, are used in INFT to generate the time domain waveform, the waveform's peak will not be located at t_1 and t_2.

III. Proposed Method

It is well-known that when two 1-solitons collide, there will be time position shifts and phase shifts for both of them, the amount of which is related to the height and velocity of these two solitons. Mollenauer and Gordon [6] analyzed the shift analytically and modeled the collision of two 1-solitons by the evolution of a 2-soliton. By comparing the time and phase terms in the asymptotical region at $z=\pm\infty$, the value of shifts can be obtained. We found that by offsetting the time shift in the b-coefficients of the 2-soliton, the corresponding time domain waveform can be well located at the desired time position.

In [6], the amplitude and frequency of the two pulses depended on each other, while we extend the idea to arbitrary amplitudes and frequencies by introducing a new variable, p, to the amplitude. Considering a 2-soliton, which can be given by a sum of two components (which are not necessarily 1-soliton), $u(z,t)=u_1(z,t)+u_2(z,t)$ with amplitudes of $A_1=p(1+a)$, $A_2=p(1-a)$, and frequencies[1] of $\Omega_1=\Omega(1-a)$, $\Omega_2=-\Omega(1+a)$, where $p,\ \Omega>0, |a|\le 1$. u_1 and u_2 satisfy

$$\begin{bmatrix} M_{11}(\gamma_1^{-1}+\gamma_1^*) & M_{12}(\gamma_1^{-1}+\gamma_2^*) \\ M_{21}(\gamma_2^{-1}+\gamma_1^*) & M_{22}(\gamma_2^{-1}+\gamma_2^*) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{5}$$

with

$$M_{jk}=\left[A_j+A_k-i(\Omega_j-\Omega_k)\right]^{-1} \tag{6}$$

$$\gamma_j=e^{A_j(t-t_{j0}+\Omega_j z)+i(-\Omega_j t+(A_j^2-\Omega_j^2)z/2+\phi_{j0})} \tag{7}$$

With those parameters, $u(z,t)$ can be analytically given, and when $z\to\pm\infty$, the analytical expression can be well approximated by a sum of two 1-solitons separated from each other, i.e., two humps, as illustrated in Fig.1. They are given by

$$u_{11}(z,t)=p(1+a)\,\mathrm{sech}(pt+py-\eta/2)\,e^{i(\theta+\phi-\delta\phi_1)} \tag{8a}$$

$$u_{12}(z,t)=p(1+a)\,\mathrm{sech}(pt+py+\eta/2)\,e^{i(\theta+\phi+\delta\phi_1)} \tag{8b}$$

$$u_{21}(z,t)=p(1-a)\,\mathrm{sech}(pt-py+\eta/2)\,e^{i(\theta+\phi-\delta\phi_2)} \tag{8c}$$

$$u_{22}(z,t)=p(1-a)\,\mathrm{sech}(pt-py-\eta/2)\,e^{i(\theta+\phi+\delta\phi_2)} \tag{8d}$$

where

$$\begin{gathered} y=at+(1-a^2)(\Omega z-t_0) \\ \theta=a\Omega t+(1+a^2)(p^2-\Omega^2)z/2 \\ \phi=-\Omega t+a(p^2+\Omega^2)z+\phi_0 \\ \eta=\ln[(p^2+\Omega^2)/(p^2a^2+\Omega^2)] \\ \delta\phi_1=\mathrm{atan}(p/\Omega)-\mathrm{atan}\,(pa/\Omega) \\ \delta\phi_2=\mathrm{atan}(p/\Omega)+\mathrm{atan}\,(pa/\Omega) \end{gathered} \tag{9}$$

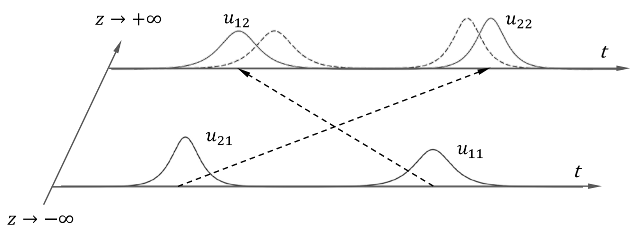


Fig. 1. Illustration of a 2-soliton at $z\to-\infty$ and $+\infty$, which is asymptotically two 1-solitons. The dashed curve at $z\to+\infty$ illustrates the waveform if u_{11}, u_{21} just propagate without the existence of each other, that is, no collision happens.

So the collision-induced time position shifts onto u_{12}, u_{22} are given by (note the time position shift of the waveform at $z\to+\infty$ in Fig. 1)

$$\delta t_1=\frac{\eta}{p(1+a)} \qquad \delta t_2=\frac{\eta}{p(1-a)} \tag{10}$$

where

$$p=\frac{A_1+A_2}{2} \qquad a=\frac{A_1-A_2}{A_1+A_2}$$

Note that the derivation above requires a mean frequency of $-a\Omega$, but it can be omitted as we are free to change the moving reference frame, by which the collision characteristics will not change. Thus, for simplicity, we may choose $\Omega_1=-\Omega_2$ at first, and add a common frequency to the generated 2-soliton, if Ω_1 and Ω_2 are asymmetric.

Suppose we want to generate a 2-soliton with two peaks located at t_1 and t_2, with eigenvalues of $\lambda_1=\alpha_1+i\beta_1$ and $\lambda_2=\alpha_2+i\beta_2$. The method of offsetting the b-coefficients, b_1 and b_2, is described as follows.

Step 1: rewrite Eq. (10) in the language of NFT.

$$\delta t_1=\ln\left[\frac{(\beta_1+\beta_2)^2+(\alpha_1-\alpha_2)^2}{(\beta_1-\beta_2)^2+(\alpha_1-\alpha_2)^2}\right]/(2\beta_1) \tag{11a}$$

$$\delta t_2=\ln\left[\frac{(\beta_1+\beta_2)^2+(\alpha_1-\alpha_2)^2}{(\beta_1-\beta_2)^2+(\alpha_1-\alpha_2)^2}\right]/(2\beta_2) \tag{11b}$$

Step 2: determine $|b_1|$ and $|b_2|$ by Eq. (3), but with offset time shifts $-\delta t_1/2$ and $+\delta t_2/2$, then add phase terms φ_1 and φ_2 to obtain b_1 and b_2, while $\bar{b}_1$ and $\bar{b}_2$ denote the b-coefficient without the time shift offset.

$$b_1=e^{2\beta_1(t_1-\delta t_1/2)+i\varphi_1}, \qquad b_2=e^{2\beta_2(t_2+\delta t_2/2)+i\varphi_2} \tag{12a}$$

$$\bar{b}_1=e^{2\beta_1 t_1+i\varphi_1}, \qquad \bar{b}_2=e^{2\beta_2 t_2+i\varphi_2} \tag{12b}$$

Step 3: generate the 2-soliton waveform by fast INFT [7] with b_1, b_2. Also, for comparison, the waveform of 2-soliton with $\bar{b}_1,\bar{b}_2$ and the waveform of a linear sum of two 1-solitons with $\bar{b}_1,\bar{b}_2$ are denoted by $\bar{u}(t)$ and $\tilde{u}(t)$, respectively.

$$\begin{gathered} u(t)=INFT\{\lambda_1,\lambda_2,b_1,b_2\} \\ \bar{u}(t)=INFT\{\lambda_1,\lambda_2,\bar{b}_1,\bar{b}_2\} \\ \tilde{u}(t)=INFT\{\lambda_1,\bar{b}_1\}+INFT\{\lambda_2,\bar{b}_2\} \end{gathered}$$

IV. Numerical Verification

We compare $u(t)$ and $\bar{u}(t)$ under different situations to verify the performance of the proposed method. Different time positions are specified in the cases below.

A. Same amplitude, symmetric frequency

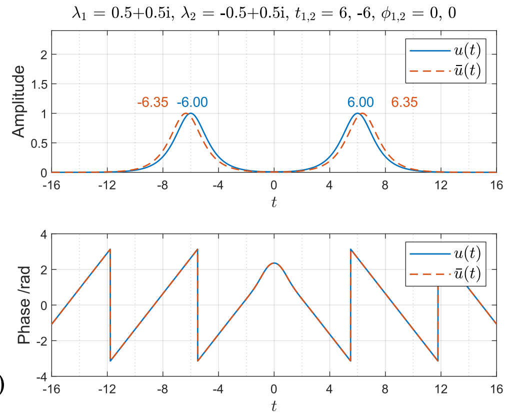
$\lambda_1=0.5+0.5i$, $\lambda_2=-0.5+0.5i$, $t_1=6$, $t_2=-6$.

B. Same amplitude, asymmetric frequency

$\lambda_1=0.5+0.5i$, $\lambda_2=-1.0+0.5i$, $t_1=5$, $t_2=-5$.

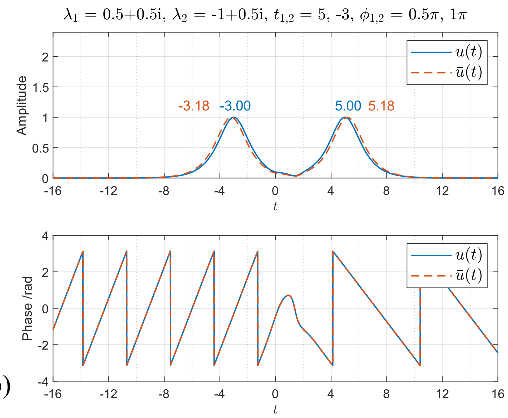
C. Different amplitude, asymmetric frequency

$\lambda_1=0.5+1.0i$, $\lambda_2=-1.0+0.5i$, $t_1=2$, $t_2=-4$.

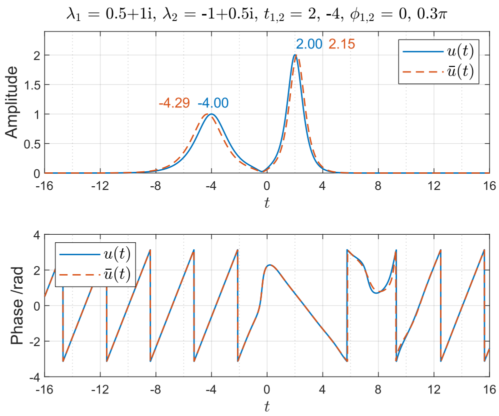


(a)

1. It may be better to express with "real parts" and "imaginary parts" for "amplitudes" and "frequencies", but we follow the language in [6] first, then change it in the end.



(b)

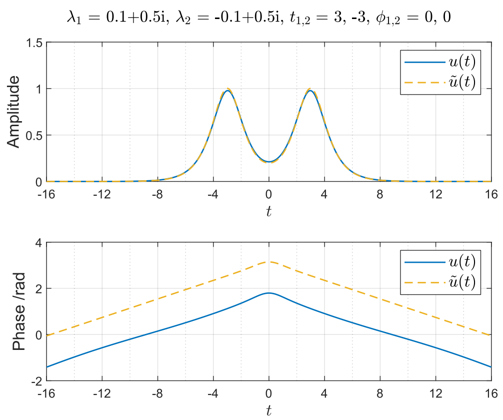


(c)

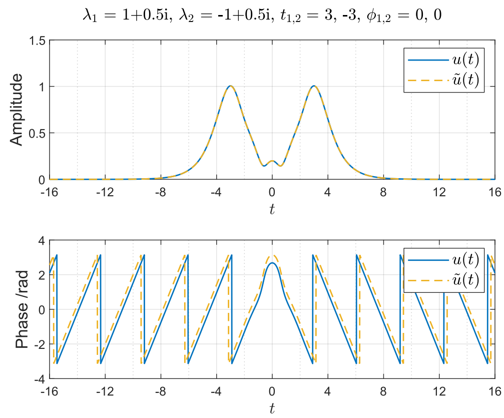
Fig. 2. (a), (b) and (c), comparsion of $u(t)$ and $\bar{u}(t)$ for three different combinations of eigenvalues and specified time positions. The waveforms of $u(t)$ match very well to the specified time positions.

The peak positions of the waveform are labeled with corresponding colors in the figure. In all cases above, the waveform peaks of $u(t)$ match well to the specified time positions, while that of $\bar{u}(t)$ shows unwanted mismatch.

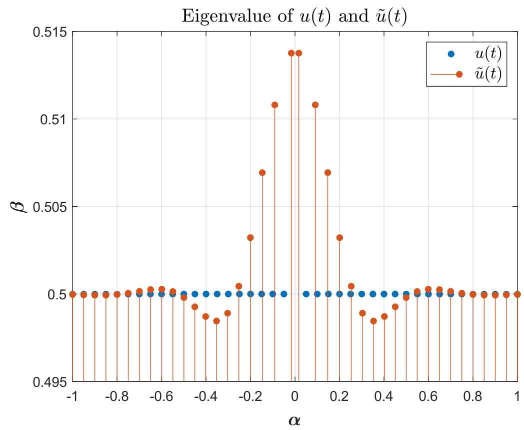
We also compare the waveform of $u(t)$ and $\tilde{u}(t)$ and their eigenvalues, with fixed specified time positions $t_{1,2} = \pm 3$ and symmetric frequencies, whose absolute value varies from 0.05 to 1. Their time domain waveforms match quite well, while both the real and imaginary parts of the eigenvalue of $\tilde{u}(t)$ deviates from the designed value, especially when the real part is small, which will introduce eigenvalue noise at the transmitter in an NFDM system. This is because a linear sum in the time domain does not result in a linear sum in the frequency domain, and the asymptotical expression of a 2-soliton by the sum of two 1-soliton is only accurate when the 1-solitons are well separated from each other. This indicates that when the constituent pulses are close enough, it is preferable to generate 2-soliton by our proposed method.



(a)



(b)



(c)

Fig. 3. Genarating 2-soliton with eigenvalues of $\pm\alpha + 0.5i$ (α = 0.05, 0.1, …, 1) by INFT with modified b-coefficient, $u(t)$, and by directly adding two corresponding 1-solitons, $\tilde{u}(t)$. (a) time domain waveforms of $u(t)$ and $\tilde{u}(t)$, (b) eigenvalues of $u(t)$ and $\tilde{u}(t)$. The stem plot illustrates the deviation of α of $\tilde{u}(t)$ from the designed value.

V. Conclusions

Inspired by the collision-induced time position shift of two 1st-order solitons, we proposed a method of generating second-order solitons with specified time positions based on nonlinear Fourier transform with modified b-coefficients. This method supports different soliton amplitudes, frequencies, and phases the generated pulse has peaks lying exactly at the designed positions, as long as they are not too close.

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