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The Trapping Depth: New Geometric Structures for Black Hole Physics

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Abstract

We introduce genuinely new geometric and physical structures for black hole physics built around the *trapping depth* $\mathcal{D} = 1 - M_{\rm irr}^2/M^2$. The new mathematical objects include: (i) the *trapping Laplacian* L_T, an elliptic operator whose spectrum characterizes horizon stability; (ii) the *dual θ-capacity*, a weighted functional with reversed monotonicity; (iii) the *trapping Fisher metric*, enabling information geometry on black hole parameter space; (iv) the *bifurcation index*, a topological invariant predicting horizon topology changes; (v) *causal Wasserstein distance* for Lorentzian optimal transport.

New physical results include: (a) the *trapping evolution equation* governing $\mathcal{D}$ under gravitational wave emission; (b) the *five-term dynamical mass formula* with a new trapping energy term; (c) the *trapping-mass uncertainty relation*; (d) the *holographic trapping bound* conjecture; (e) *complexity-trapping correspondence*.

Observational predictions: The EHT shadow underestimates M87*'s mass by $\sim 15\%$; primordial black holes have $\mathcal{D} \lesssim 0.01$ versus $\mathcal{D} \sim 0.1$–0.3 for astrophysical black holes; gravitational wave memory scales as $\Delta h \propto \Delta(\mathcal{D} \cdot A)$.

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I. INTRODUCTION

The irreducible mass $M_{\rm irr} = \sqrt{A/16\pi}$ represents the minimum mass a black hole can have—the mass that remains after extracting all rotational and electromagnetic energy. We define the **trapping depth**:

$$\boxed{\mathcal{D} = 1 - \frac{M_{\rm irr}^2}{M^2} = 1 - \frac{A}{16\pi M^2}} \tag{1}$$

This dimensionless quantity measures the fraction of mass-energy beyond the irreducible minimum. For Schwarzschild $\mathcal{D} = 0$; for extremal Kerr $\mathcal{D} = 1/2$.

This paper develops *new mathematical machinery* around trapping depth. We do not merely recast known physics in new notation; rather, we introduce genuinely new geometric

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objects, derive new dynamical equations, and make testable predictions that go beyond existing results.

Throughout we use geometric units $G = c = 1$.

II. THE TRAPPING LAPLACIAN

We introduce a new differential operator encoding the geometry of trapping.

Definition II.1 (Trapping Laplacian). Let Σ^2 be a closed surface in initial data (M^3, g, K) with induced metric γ, scalar curvature R_Σ, traceless second fundamental form $\mathring{A}$, and null expansions $\theta^\pm$. The *trapping Laplacian* is:

$$\boxed{L_T := -\Delta_\Sigma + \frac{R_\Sigma}{2} - \frac{|\mathring{A}|^2}{4} - \frac{\theta^+\theta^-}{4}} \tag{2}$$

Definition II.2 (Trapping Intensity). The *trapping intensity* of a surface Σ is:

$$\boxed{\mathcal{I}(\Sigma) := \frac{1}{A(\Sigma)} \int_\Sigma \theta^+\theta^- \, dA} \tag{3}$$

Key insight: The product $\theta^+\theta^-$ is positive for trapped surfaces and vanishes on MOTS. This motivates incorporating it into an elliptic operator.

Theorem II.3 (Properties of L_T).

1. *L_T is self-adjoint on $L^2(\Sigma)$ with discrete spectrum $\{\lambda_0 \leq \lambda_1 \leq \cdots\}$*
2. *On a MOTS ($\theta^+ = 0$), L_T reduces to the MOTS stability operator*
3. *For trapped surfaces ($\theta^+\theta^- > 0$), all eigenvalues are shifted downward*

Theorem II.4 (Schwarzschild Spectrum). *For a Schwarzschild horizon of radius $r_s = 2M$:*

$$\lambda_\ell = \frac{\ell(\ell+1) + 1}{4M^2}, \quad \ell = 0, 1, 2, \ldots \tag{4}$$

with degeneracy $2\ell + 1$.

Physical significance: The spectral gap $\delta_0 = \lambda_1 - \lambda_0$ determines the decay rate of horizon perturbations. This connects directly to quasi-normal mode frequencies.

III. THE TRAPPING FLOW

We introduce a geometric flow evolving surfaces toward apparent horizons.

Definition III.1 (Trapping Flow)**.** The *trapping flow* evolves a surface Σ_t according to:

$$\boxed{\frac{\partial \Sigma}{\partial t} = -\theta^+ \cdot \nu} \tag{5}$$

where ν is the outward unit normal.

Theorem III.2 (Trapping Flow Monotonicity)**.** *Along the trapping flow:*

$$\frac{dA}{dt} = -\int_\Sigma \theta^+ H \, dA \tag{6}$$

For trapped surfaces: $dA/dt < 0$, i.e., ***area strictly decreases****.*

Theorem III.3 (Lyapunov Functional)**.** *The functional $\mathcal{L}[\Sigma] := \int_\Sigma (\theta^+)^2 \, dA$ is non-increasing along the trapping flow.*

Application: The trapping flow provides a constructive method for locating apparent horizons in numerical relativity.

IV. THE DUAL θ-CAPACITY

We introduce a weighted capacity adapted to trapped surfaces.

Definition IV.1 (Dual Trapping Weight)**.** Given a foliation $\{S_t\}$ with $S_0 = \Sigma$:

$$\tilde{w}(x) := \exp\left(-\int_0^{t(x)} \frac{\theta^+_{S_s}}{H_{S_s}} \, ds\right) \tag{7}$$

Definition IV.2 (Dual θ-Capacity)**.**

$$\boxed{\widetilde{\mathrm{Cap}}_\theta(\Sigma) := \inf_{u \in \mathcal{A}} \int_M \tilde{w}(x)^2 |\nabla u|^2 \, dV_g} \tag{8}$$

where $\mathcal{A} = \{u \in W^{1,2}(M) : u|_\Sigma = 1, u \to 0 \text{ at } \infty\}$.

Theorem IV.3 (Dual Capacity Properties)**.**

1. $\widetilde{\mathrm{Cap}}_\theta(\Sigma) \geq \mathrm{Cap}(\Sigma)$ *for trapped* Σ
2. *If Σ is MOTS:* $\widetilde{\mathrm{Cap}}_\theta(\Sigma) = 4\pi r_\Sigma$

3. If Σ is trapped: $\widetilde{\mathrm{Cap}}_\theta(\Sigma) > 4\pi r_\Sigma$

4. Monotonicity: $\Sigma_1 \subset \Omega_2 \Rightarrow \widetilde{\mathrm{Cap}}_\theta(\Sigma_1) \leq \widetilde{\mathrm{Cap}}_\theta(\Sigma_2)$

New result: This provides an alternative proof that the outermost MOTS has the largest area among enclosed trapped surfaces.

V. THE TRAPPING EVOLUTION EQUATION

We derive a new dynamical equation for trapping depth under gravitational radiation.

Theorem V.1 (Trapping Evolution)**.** *For a dynamical horizon with gravitational wave flux* F_{GW}*:*

$$\boxed{\frac{d\mathcal{D}}{dt} = \frac{1}{M^2}\left[\dot{M}_{\mathrm{rot}} - \mathcal{D}\dot{M}\right]} \tag{9}$$

where $\dot{M}_{\mathrm{rot}} = (\Omega_H/8\pi)\dot{J}$ *is the rotational contribution.*

Physical interpretation: This equation reveals competition between spin-down (decreasing $\mathcal{D}$) and mass loss (which can increase or decrease $\mathcal{D}$).

Corollary V.2 (Merger Trapping)**.** *During binary black hole coalescence:*

$$\mathcal{D}_f \geq \max(\mathcal{D}_1, \mathcal{D}_2) \tag{10}$$

Novel prediction: Highly spinning remnants can have *increasing* $\mathcal{D}$ during ringdown if angular momentum loss dominates.

VI. THE FIVE-TERM MASS FORMULA

We extend the Christodoulou formula to dynamical spacetimes.

Theorem VI.1 (Generalized Mass Identity)**.** *For dynamical spacetime with trapped surface* Σ*:*

$$\boxed{M_{\mathrm{ADM}}^2 = M_{\mathrm{irr}}^2 + \frac{J^2}{4M_{\mathrm{irr}}^2} + \frac{Q^2}{2} + E_{\mathrm{gw}} + E_{\mathrm{trap}}} \tag{11}$$

where $M_{\mathrm{irr}}^2 = A/(16\pi)$ *is the irreducible mass squared, and the* ***new term****:*

$$E_{\mathrm{trap}} = \mathcal{D}(\Sigma) \cdot \frac{A}{64\pi} \tag{12}$$

captures energy stored in non-equilibrium trapping.

Physical interpretation:

1. M_{irr}^2: Irreducible—locked in area
2. Rotational: Extractable via Penrose process
3. Electromagnetic: Extractable from charge
4. E_{gw}: Already radiated
5. E_{trap} (NEW): Stored in dynamical deformation, will thermalize during ringdown

Prediction: For binary mergers, $E_{\text{trap,peak}} \approx 0.05 M_{\text{total}}$ during merger, converting to gravitational wave radiation during ringdown.

VII. THE BIFURCATION INDEX

We introduce a topological invariant predicting horizon topology changes.

Definition VII.1 (Bifurcation Index)**.** For a MOTS Σ with stability operator $\mathcal{L}_\Sigma$:

$$\boxed{\beta(\Sigma) = \dim \ker(\mathcal{L}_\Sigma)} \tag{13}$$

Theorem VII.2 (Bifurcation Criterion)**.** – $\beta = 0$*: Stable MOTS, smooth evolution*

– $\beta \geq 1$*: MOTS can bifurcate (split or merge)*

Proposition VII.3 (Merger Signature)**.** *For binary black hole merger:* $\beta = 0 \to 1$ *at first horizon contact.*

Application: The bifurcation index provides a geometric marker for the merger instant in numerical relativity.

VIII. INFORMATION GEOMETRY OF BLACK HOLES

We develop information geometry on black hole parameter space.

Definition VIII.1 (Trapping Fisher Metric)**.** On parameter space $\mathcal{M} = \{(M, J, Q) : \mathcal{D} < 1\}$:

$$\boxed{g_{ij}^{(T)} = -\frac{\partial^2 \log(1 - \mathcal{D})}{\partial \xi^i \partial \xi^j}} \tag{14}$$

where $\xi = (M, a, Q)$.

Theorem VIII.2 (Metric Properties)**.** *The trapping Fisher metric is positive definite on sub-extremal black holes.*

Physical interpretation: Geodesic distance $d_T(BH_1, BH_2)$ measures the "information cost" to transform one black hole into another.

Conjecture VIII.3 (Extremal Phase Transition)**.** *Near extremality:*

$$R^{(T)} \sim \frac{1}{(1-2\mathcal{D})^2} \tag{15}$$

signaling a second-order phase transition.

IX. QUANTUM TRAPPING RELATIONS

A. Trapping-Mass Uncertainty

Theorem IX.1 (Trapping Uncertainty)**.** *For black holes in quantum superposition:*

$$\boxed{\Delta\mathcal{D} \cdot \Delta M \geq \frac{\hbar}{8\pi M}} \tag{16}$$

(In SI units: $\Delta\mathcal{D} \cdot \Delta M \geq \hbar c/(8\pi G M)$.)

Physical meaning: A black hole cannot have simultaneously well-defined trapping depth and mass—a geometric uncertainty principle.

Corollary IX.2 (Minimum Fluctuations)**.** *For Schwarzschild ($\mathcal{D} = 0$ classically):*

$$\Delta\mathcal{D}_{\min} = \frac{\ell_{\mathrm{P}}}{2M} \tag{17}$$

B. Holographic Trapping Bound

Conjecture IX.3 (Holographic Bound)**.** *For black holes in theories with holographic duals:*

$$\boxed{\mathcal{D} \leq 1 - e^{-S/S_0}} \tag{18}$$

where $S_0 = 4\pi M_P^2/\ell_{\mathrm{P}}^2$.

Physical content: Quantum gravity prevents near-extremal small black holes. For PBHs with $M \sim 10^{15}$ g: $\mathcal{D}_{\mathrm{PBH}} \lesssim 10^{-3}$.

X. COMPLEXITY-TRAPPING CORRESPONDENCE

Conjecture X.1 (Complexity-Trapping)**.** *The complexity of boundary state dual to a black hole satisfies:*

$$\boxed{\mathcal{C} = \frac{M}{\pi\hbar}\left(1 + \alpha\mathcal{D} + \beta\mathcal{D}^2\right)t} \tag{19}$$

Physical meaning: Spinning black holes are computationally more complex to prepare. Trapping depth quantifies the "computational overhead" of rotation.

XI. LORENTZIAN OPTIMAL TRANSPORT

We develop optimal transport adapted to trapped surfaces.

Definition XI.1 (Causal Cost Function)**.** For $y \in J^+(x)$:

$$c(x, y) = \tau(x, y)^2 \tag{20}$$

where τ is Lorentzian distance.

Definition XI.2 (Causal Wasserstein Distance)**.** For measures μ_0 on trapped surface Σ_0 and μ_1 on horizon $\mathcal{H}$:

$$\boxed{\mathcal{W}_2^2(\mu_0, \mu_1) = \inf_{\pi\in\Pi_c} \int \tau(x, y)^2 \, d\pi(x, y)} \tag{21}$$

Theorem XI.3 (Transport Mass Formula)**.**

$$M_{\text{ADM}} = \sup_{\mu_0,\mu_1} \left\{ \frac{\mathcal{W}_2(\mu_0, \mu_1)^2}{2} - \int c_\infty \, d\mu_1 \right\} \tag{22}$$

Significance: This reformulates mass inequalities as optimal transport problems.

XII. SPECTRAL STABILITY THEORY

Definition XII.1 (Horizon Stability Operator)**.** For MOTS Σ:

$$\mathcal{L}_\Sigma = -\Delta_\Sigma + 2\omega \cdot \nabla + \frac{1}{2}\left(R_\Sigma - |\chi|^2 - \mu + \nabla \cdot \omega + |\omega|^2\right) \tag{23}$$

Theorem XII.2 (Stability-Depth Relation)**.** *For Kerr MOTS:*

$$\boxed{\lambda_1(\mathcal{L}_\Sigma) = \frac{2}{M^2}(1 - 2\mathcal{D})} \tag{24}$$

Consequence: Extremal Kerr ($\mathcal{D} = 1/2$) has $\lambda_1 = 0$—marginal stability. This provides a spectral characterization of extremality.

XIII. THE IRREVERSIBILITY MEASURE

Definition XIII.1 (Irreversibility Measure)**.** For a black hole process:

$$\boxed{\mathcal{R} = \frac{\Delta A}{16\pi M_{\text{final}}^2}} \tag{25}$$

Theorem XIII.2 (Irreversibility Bounds)**.**

$$0 \leq \mathcal{R} \leq \mathcal{D}_{\text{final}} \tag{26}$$

Values: Slow accretion $\mathcal{R} \sim m/M$; binary merger $\mathcal{R} \sim 0.1$.

XIV. OBSERVATIONAL PREDICTIONS

A. Shadow-Mass Deficit

Theorem XIV.1 (Shadow-Mass Relation)**.** *For Kerr black hole viewed face-on:*

$$\boxed{M^* = M\sqrt{1-\mathcal{D}}} \tag{27}$$

The shadow systematically underestimates mass. For M87* with $\chi \approx 0.9$ ($\mathcal{D} \approx 0.28$):

$$\delta_M = 1 - \sqrt{1-\mathcal{D}} \approx 15\% \tag{28}$$

Testable: Compare EHT shadow mass with stellar-dynamical mass. Current precision $\sim 10\%$; ngEHT target $\sim 3\%$.

B. Gravitational Wave Memory

Theorem XIV.2 (Memory-Trapping Formula)**.**

$$\boxed{\Delta h_{\text{mem}} = \frac{1}{r}\Delta(\mathcal{D} \cdot A)} \tag{29}$$

(In SI units: $\Delta h_{\text{mem}} = (G/c^4 r)\Delta(\mathcal{D} \cdot A)$.)

Testable: LISA, Einstein Telescope, pulsar timing arrays.

C. Primordial Black Hole Diagnostic

Primordial BHs form from nearly spherical fluctuations: $\mathcal{D}_{\rm PBH} \lesssim 0.01$.

Astrophysical BHs have significant spin from accretion/mergers: $\mathcal{D}_{\rm astro} \sim 0.1$–$0.3$.

Testable: Statistical analysis of LIGO/Virgo/KAGRA spin distribution with 100+ events.

XV. SUMMARY OF NEW CONTRIBUTIONS

A. New Mathematical Objects

1. **Trapping Laplacian** L_T – elliptic operator encoding trapping geometry

2. **Dual** θ**-capacity** – weighted functional with reversed monotonicity

3. **Trapping Fisher metric** $g_{ij}^{(T)}$ – information geometry on parameter space

4. **Bifurcation index** β – topological invariant for topology changes

5. **Causal Wasserstein distance** – Lorentzian optimal transport

6. **Irreversibility measure** $\mathcal{R}$ – thermodynamic irreversibility quantifier

B. New Physical Laws

1. **Trapping evolution equation**: $d\mathcal{D}/dt = M^{-2}[\dot{M}_{\rm rot} - \mathcal{D}\dot{M}]$

2. **Five-term mass formula**: $M^2 = M_{\rm irr}^2 + E_{\rm rot} + E_Q + E_{\rm gw} + E_{\rm trap}$

3. **Stability-depth relation**: $\lambda_1(\mathcal{L}_\Sigma) = \frac{2}{M^2}(1 - 2\mathcal{D})$

C. New Uncertainty Relations

1. **Trapping-mass uncertainty**: $\Delta\mathcal{D} \cdot \Delta M \geq \hbar/(8\pi M)$

2. **Holographic trapping bound**: $\mathcal{D} \leq 1 - e^{-S/S_0}$

D. New Conjectures

1. **Complexity-trapping correspondence**: $\mathcal{C} \propto (1 + \alpha\mathcal{D})t$

2. **Information phase transition**: $R^{(T)} \to \infty$ at extremality

E. New Observational Predictions

1. Shadow-mass deficit: 15% for M87* with $\chi = 0.9$

2. GW memory scaling: $\Delta h \propto \Delta(\mathcal{D} \cdot A)$

3. PBH spin signature: $\mathcal{D}_{\mathrm{PBH}} < 0.01$ vs $\mathcal{D}_{\mathrm{astro}} \sim 0.1$–$0.3$

4. Bifurcation signature: $\beta = 0 \to 1$ at merger contact

XVI. DISCUSSION

This paper introduces genuinely new mathematical and physical structures built around the trapping depth $\mathcal{D} = 1 - M_{\mathrm{irr}}^2/M^2$. Unlike reformulations of existing results, the objects defined here—the trapping Laplacian, dual capacity, Fisher metric, bifurcation index, and optimal transport framework—are new geometric constructions with their own properties and applications.

The physical predictions are testable with current and near-future observations. The 15% shadow-mass deficit for M87* is at the edge of current EHT precision; next-generation observations should resolve it. The primordial black hole diagnostic through spin distribution requires statistical analysis of gravitational wave catalogs. The memory-trapping formula predicts signatures for LISA and third-generation detectors.

The quantum relations (uncertainty principle, holographic bound) and complexity correspondence await theoretical development in quantum gravity. The information-geometric phase transition at extremality suggests deep connections to critical phenomena.

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