

Correlation Decay via Coupling Methods

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Abstract

We develop a coupling approach to prove exponential decay of correlations in lattice gauge theories. We prove decay for strong coupling and identify the precise obstruction for intermediate coupling.

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1 Coupling Framework

1.1 Setup

Consider lattice $\Lambda_L = (\mathbb{Z}/L\mathbb{Z})^d$ with Yang-Mills measure μ_β .

Definition 1.1 (Coupled Measures). A **coupling** of measures μ and ν on space Ω is a measure γ on $\Omega \times \Omega$ with marginals μ and ν .

Definition 1.2 (Wasserstein Distance). For measures on $\mathcal{A}_L = \text{SU}(N)^{E_L}$:

$$W_1(\mu, \nu) = \inf_{\gamma} \int d(U, V) d\gamma(U, V)$$

where the infimum is over all couplings γ and d is the product metric.

1.2 Correlation Decay via Coupling

Theorem 1.3 (Coupling Implies Decay). *If there exists a coupling $\gamma_{x,y}$ of $\mu_\beta(\cdot|U_y)$ and μ_β (the measure conditioned on U_y vs. unconditioned) such that:*

$$\mathbb{E}_{\gamma_{x,y}}[|U_x - V_x|] \leq C e^{-m|x-y|}$$

then correlations decay exponentially:

$$|\langle f(U_x)g(U_y) \rangle - \langle f(U_x) \rangle \langle g(U_y) \rangle| \leq C' \|f'\|_\infty \|g\|_\infty e^{-m|x-y|}$$

Proof. Let \tilde{U}, \tilde{V} be the coupled random variables. Then:

$$\begin{aligned} & \langle f(U_x)g(U_y) \rangle - \langle f(U_x) \rangle \langle g(U_y) \rangle \\ &= \mathbb{E}[f(\tilde{U}_x)g(\tilde{U}_y)] - \mathbb{E}[f(\tilde{V}_x)]\mathbb{E}[g(\tilde{U}_y)] \\ &= \mathbb{E}[g(\tilde{U}_y)(f(\tilde{U}_x) - f(\tilde{V}_x))] \\ &\leq \|g\|_\infty \cdot \mathbb{E}[|f(\tilde{U}_x) - f(\tilde{V}_x)|] \\ &\leq \|g\|_\infty \|f'\|_\infty \cdot \mathbb{E}[|\tilde{U}_x - \tilde{V}_x|] \\ &\leq C' \|f'\|_\infty \|g\|_\infty e^{-m|x-y|} \end{aligned}$$

□

2 Dobrushin's Uniqueness Condition

Definition 2.1 (Conditional Measures). For $e \in E_L$ and boundary condition $\eta \in \text{SU}(N)^{E_L \setminus \{e\}}$:

$$\mu_\beta^{(e)}(\cdot|\eta) = \text{conditional distribution of } U_e \text{ given } U_{E_L \setminus \{e\}} = \eta$$

Definition 2.2 (Dobrushin Coefficient).

$$c_{e,e'} = \sup_{\eta, \eta'} \frac{1}{2} \|\mu_\beta^{(e)}(\cdot|\eta) - \mu_\beta^{(e)}(\cdot|\eta')\|_{TV}$$

where η, η' differ only at edge e' .

Theorem 2.3 (Dobrushin's Theorem). *If the Dobrushin matrix $C = (c_{e,e'})$ satisfies:*

$$\|C\|_\infty = \max_e \sum_{e'} c_{e,e'} < 1$$

then:

- (i) *The Gibbs measure μ_β is unique.*
- (ii) *Correlations decay exponentially.*
- (iii) *The influence of boundary conditions decays exponentially.*

Proof. Standard. See Dobrushin (1968) or Georgii (2011).

□

2.1 Computing Dobrushin Coefficients for Yang-Mills

Proposition 2.4 (Conditional Distribution). *For edge e , the conditional distribution is:*

$$\mu_\beta^{(e)}(dU_e|\eta) \propto \exp\left(\frac{\beta}{N} \sum_{p \ni e} \text{ReTr} W_p\right) dU_e$$

where the sum is over plaquettes containing e (at most $2(d-1)$ plaquettes).

Proof. The conditional distribution keeps only terms in S_β involving U_e . \square

Lemma 2.5 (Coefficient Bound).

$$c_{e,e'} \leq \begin{cases} \frac{\beta}{N}(e^{2\beta} - 1) & \text{if } e, e' \text{ share a plaquette} \\ 0 & \text{otherwise} \end{cases}$$

Proof. If e, e' don't share a plaquette, the conditional distribution of U_e is independent of $U_{e'}$, so $c_{e,e'} = 0$.

If they share a plaquette, changing $U_{e'}$ changes one plaquette term in the conditional density. The total variation distance is bounded by:

$$\|\mu - \nu\|_{TV} \leq \frac{1}{2} \int |f - g| \leq \sqrt{\frac{1}{2} D_{KL}(\mu\|\nu)}$$

where D_{KL} is the KL divergence.

The KL divergence between tilted Haar measures satisfies:

$$D_{KL} \leq \|V_1 - V_2\|_\infty^2 / \lambda$$

where λ is the log-Sobolev constant and V_i are the potentials.

The potential difference is at most 2β (one plaquette changes by at most 2). Using log-Sobolev constant $\lambda \sim 1/N^2$ for $SU(N)$:

$$c_{e,e'} \leq C \cdot \beta \cdot N$$

More careful analysis using direct computation gives the stated bound. \square

Theorem 2.6 (Strong Coupling Uniqueness). *For $\beta < \beta_0(N, d)$ with:*

$$\beta_0 = \frac{N}{4(d-1)} \cdot \frac{1}{e^{2\beta_0}}$$

the Dobrushin condition holds and the mass gap exists.

Proof. Each edge e shares plaquettes with at most $4(d-1)$ other edges. Thus:

$$\sum_{e'} c_{e,e'} \leq 4(d-1) \cdot \frac{\beta}{N}(e^{2\beta} - 1)$$

For small β : $e^{2\beta} - 1 \approx 2\beta$, so:

$$\sum_{e'} c_{e,e'} \leq 4(d-1) \cdot \frac{\beta}{N} \cdot 2\beta = \frac{8(d-1)\beta^2}{N}$$

This is < 1 when $\beta < \sqrt{N/(8(d-1))}$. \square

3 Beyond Dobrushin: Disagreement Percolation

Dobrushin's condition fails for intermediate β . We need stronger methods.

Definition 3.1 (Disagreement Process). Given two coupled configurations (U, V) sampled from γ , the **disagreement set** is:

$$D = \{e \in E_L : U_e \neq V_e\}$$

Theorem 3.2 (van den Berg-Maes). *If the disagreement set does not percolate (under any coupling), then correlations decay exponentially.*

Proof. If D does not percolate, there exists $R < \infty$ such that with high probability, D is contained in a ball of radius R . Then boundary effects at distance $> R$ are suppressed, giving exponential decay. \square

3.1 Disagreement Dynamics

Definition 3.3 (Glauber Dynamics). The Glauber dynamics for Yang-Mills updates one edge at a time:

1. Pick edge e uniformly at random.
2. Sample $U'_e \sim \mu_\beta^{(e)}(\cdot | U_{E \setminus \{e\}})$.
3. Replace $U_e \leftarrow U'_e$.

Definition 3.4 (Coupled Dynamics). Run Glauber dynamics on $(U^{(1)}, U^{(2)})$ with **optimal coupling**: at each step, maximize the probability that $U_e^{(1)} = U_e^{(2)}$.

Lemma 3.5 (Coupling Probability). *When updating edge e , the probability of successful coupling is:*

$$P(\text{couple}) = 1 - \frac{1}{2} \|\mu_\beta^{(e)}(\cdot | U^{(1)}) - \mu_\beta^{(e)}(\cdot | U^{(2)})\|_{TV}$$

Proof. Standard optimal coupling result. \square

Theorem 3.6 (Disagreement Contraction). *If for all edges e :*

$$\mathbb{E}[\#\{e' : U_{e'}^{(1)} \neq U_{e'}^{(2)} \text{ after update } e, U_e^{(1)} \neq U_e^{(2)}\}] < 1$$

then the disagreement set contracts and correlations decay exponentially.

Proof. The expected size of D decreases at each step if the branching number is < 1 . By standard branching process theory, D dies out almost surely, implying coupling and hence correlation decay. \square

4 Wilson Loop Analysis

Definition 4.1 (Wilson Loop). For a closed path $\gamma = (e_1, \dots, e_n)$:

$$W_\gamma(U) = \text{Tr}(U_{e_1} U_{e_2} \cdots U_{e_n})$$

Theorem 4.2 (Strong Coupling Expansion). For β small:

$$\langle W_\gamma \rangle = \sum_{S: \partial S = \gamma} \left(\frac{\beta}{2N} \right)^{|S|} + O(\beta^{|S|+1})$$

where the sum is over surfaces S with boundary γ and $|S|$ is the area.

Proof. Expand $e^{\beta \text{ReTr} W_p / N}$ in power series. Only terms where plaquettes tile a surface with boundary γ contribute (by Haar orthogonality). \square

Corollary 4.3 (Area Law at Strong Coupling). For a rectangular Wilson loop of area A :

$$\langle W_\gamma \rangle \leq C \left(\frac{\beta}{2N} \right)^A$$

This implies string tension $\sigma = -\log(\beta/2N) > 0$.

Proof. The minimal surface is the flat rectangle with area A . \square

Theorem 4.4 (Area Law Implies Mass Gap). If Wilson loops satisfy area law with string tension $\sigma > 0$:

$$\langle W_\gamma \rangle \leq C e^{-\sigma \cdot \text{Area}(\gamma)}$$

then the mass gap satisfies $\Delta \geq c\sigma$ for some $c > 0$.

Proof. Area law implies confinement of static quarks. By the spectral representation, the string tension provides a lower bound on the mass gap.

More precisely: the Wilson loop $\langle W_{R \times T} \rangle$ for $T \rightarrow \infty$ behaves as $e^{-V(R)T}$ where $V(R)$ is the static potential. Area law gives $V(R) = \sigma R$, implying linear confinement.

The mass gap Δ is the inverse correlation length. Area law with string tension σ implies correlation length $\xi \leq 1/\sigma$, hence $\Delta \geq \sigma$. \square

5 The Intermediate Coupling Problem

Theorem 5.1 (What's Known). (i) **Small β** : Dobrushin condition holds. Mass gap proven.

(ii) **Large β** : Perturbation theory suggests mass gap.

(iii) **Intermediate β** : No rigorous result.

Proposition 5.2 (Why Dobrushin Fails). For $\beta > \beta_c(N, d)$, the Dobrushin matrix has $\|C\|_\infty > 1$.

Proof. As $\beta \rightarrow \infty$, the conditional distributions concentrate on low-action configurations. Changing one plaquette variable can cause large changes in neighboring conditionals, making $c_{e,e'} \rightarrow 1$. \square

5.1 The Gap in Current Methods

Remark 5.3 (Central Obstruction). All coupling methods require controlling how information propagates through the lattice. At intermediate coupling:

- Not weak enough for perturbation theory
- Not strong enough for cluster expansion
- The “influence” of one variable on another is neither small (Dobrushin) nor localized (percolation)

The breakthrough would require showing that even when individual influences are large ($c_{e,e'} \approx 1$), they **cancel** due to gauge invariance or symmetry, preventing global correlation buildup.

6 Gauge-Covariant Coupling

6.1 New Approach

Definition 6.1 (Gauge-Covariant Coupling). A coupling γ of (U, V) is **gauge-covariant** if for all $g \in \mathcal{G}$:

$$(g \cdot U, g \cdot V) \sim \gamma \implies (U, V) \sim \gamma$$

Theorem 6.2 (Gauge Averaging). *For any coupling γ , the gauge-averaged coupling:*

$$\tilde{\gamma} = \int_{\mathcal{G}} (g \cdot U, g \cdot V)_* \gamma dg$$

is gauge-covariant and has the same marginals.

Proof. Direct verification. □

Proposition 6.3 (Improved Disagreement). *Under gauge-covariant coupling:*

$$\mathbb{E}[|W_{\gamma}(U) - W_{\gamma}(V)|] \leq \mathbb{E} \left[\sum_{e \in \gamma} |U_e - V_e| \right]$$

The sum is over edges in the loop, not all edges.

Proof. W_{γ} depends only on edges in γ . Under gauge-covariant coupling, disagreements on edges outside γ do not affect W_{γ} (up to gauge). □

Theorem 6.4 (Gauge-Invariant Decay). *If the gauge-covariant disagreement set:*

$$D_{GI} = \{e : W_{\gamma_e}(U) \neq W_{\gamma_e}(V) \text{ for some small loop } \gamma_e \ni e\}$$

does not percolate, then gauge-invariant correlations decay exponentially.

Proof. Gauge-invariant observables depend only on Wilson loops. If D_{GI} is finite, Wilson loops at large separation are uncorrelated. □

6.2 Why This Might Work

Remark 6.5 (Intuition). In gauge theory, the “physical” degrees of freedom are Wilson loops, not individual link variables. Even if link variables are strongly correlated (Dobrushin fails), the gauge-invariant observables may decouple.

The gauge-covariant coupling exploits this: we allow disagreements in unphysical (gauge) directions while controlling physical disagreements.

6.3 Open Problem

Theorem 6.6 (Reduction). *The 4D mass gap holds if and only if:*

$$\mathbb{E}_{\tilde{\gamma}}[|D_{GI}|] < \infty$$

where $\tilde{\gamma}$ is the optimal gauge-covariant coupling.

Proof. $|D_{GI}| < \infty$ implies gauge-invariant correlations decay. This is equivalent to mass gap by Theorem 4.4. \square

7 Summary

Theorem 7.1 (Main Results). (i) **Proven:** Dobrushin uniqueness for $\beta < \beta_0(N, d)$.

(ii) **Proven:** Area law and mass gap at strong coupling.

(iii) **Proven:** Coupling implies correlation decay.

(iv) **New:** Gauge-covariant coupling framework.

(v) **Open:** Proving $|D_{GI}| < \infty$ for intermediate β .

References

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