Direct Causal Argument for Penrose 1973

Assuming Weak Cosmic Censorship

Working Document

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Abstract

We attempt to prove Penrose 1973 directly using causal structure, assuming weak cosmic censorship (WCC). We carefully analyze where the argument succeeds and where gaps remain.

1 Setup and Assumptions

1.1 The Physical Setup

Given:

1. Asymptotically flat initial data (M^3, g, k) satisfying DEC
2. A closed trapped surface $\Sigma_0 \subset M$
3. The data embeds into a maximal globally hyperbolic development $(N^4, \bar{g})$

Assumption (WCC): The spacetime $(N, \bar{g})$ can be extended to a spacetime $(\tilde{N}, \tilde{g})$ that is weakly asymptotically predictable from $\mathscr{I}^+$.

This means:

- Future null infinity $\mathscr{I}^+$ exists and is complete
- The domain of outer communications $\langle\langle \mathscr{I}^+ \rangle\rangle$ is globally hyperbolic
- $J^-(\mathscr{I}^+) \cap M$ is the "exterior region"

1.2 Key Objects

Event Horizon: $\mathcal{H}^+ := \partial J^-(\mathscr{I}^+)$ (boundary of causal past of $\mathscr{I}^+$)

Black Hole Region: $B := N \setminus J^-(\mathscr{I}^+)$ (points that cannot send signals to infinity)

Horizon Cross-Section: $S := \mathcal{H}^+ \cap M$ (intersection with initial slice)

2 Penrose's Original Argument

2.1 Step 1: Trapped Surfaces Are in Black Holes

Lemma 2.1 (Penrose, 1965). *If Σ_0 is a trapped surface and NEC holds, then $\Sigma_0 \subset B$.*

Proof. Suppose $p \in \Sigma_0 \cap J^-(\mathscr{I}^+)$. Then there exists a future-directed causal curve from p to $\mathscr{I}^+$.

Consider the outgoing null geodesic congruence from Σ_0. By assumption, $\theta^+ < 0$.

The Raychaudhuri equation under NEC:

$$\frac{d\theta^+}{d\lambda} \leq -\frac{(\theta^+)^2}{2}$$

Starting with $\theta^+(0) < 0$, this implies $\theta^+(\lambda) \to -\infty$ at some finite λ^*.

But $\theta^+ \to -\infty$ means the congruence focuses to a point (conjugate point), beyond which the geodesics are no longer the boundary of $J^+(\Sigma_0)$.

Therefore, no point on the outgoing null geodesics from Σ_0 can reach $\mathscr{I}^+$ along these geodesics. By global hyperbolicity arguments, this extends to all future-directed causal curves from Σ_0.

Hence $\Sigma_0 \cap J^-(\mathscr{I}^+) = \emptyset$, so $\Sigma_0 \subset B$. □

2.2 Step 2: Area Comparison

Goal: Prove $A(\Sigma_0) \leq A(S)$ where $S = \mathcal{H}^+ \cap M$.

Penrose's Intuition:

> "The trapped surface Σ_0 is 'inside' the horizon S. By some monotonicity, the area of the horizon should be at least as large."

The Gap: This intuition needs to be made rigorous. Just because $\Sigma_0 \subset B$ doesn't immediately imply $A(\Sigma_0) \leq A(S)$.

3 Rigorous Analysis of the Area Comparison

3.1 Approach via Null Generators

The event horizon $\mathcal{H}^+$ is ruled by null geodesic generators.

Let $\gamma : [0, \infty) \to \mathcal{H}^+$ be a generator, parametrized so that $\gamma(0) \in S$ (the initial slice).

Lemma 3.1 (Hawking Area Theorem)**.** *The area $A(\mathcal{H}^+ \cap \Sigma_t)$ is non-decreasing in t for any foliation $\{\Sigma_t\}$ by Cauchy surfaces.*

Proof. On $\mathcal{H}^+$, the null expansion $\theta^{\mathcal{H}} \geq 0$ (outward expansion along $\mathcal{H}^+$ is non-negative by definition of event horizon).

Under NEC, Raychaudhuri gives $d\theta^{\mathcal{H}}/d\lambda \leq 0$, so if $\theta^{\mathcal{H}} \geq 0$ at one point, it remains ≥ 0 to the future.

Area evolution: $dA/d\lambda = \int \theta^{\mathcal{H}}\, dA \geq 0$. □

3.2 The Problematic Direction

To compare Σ_0 with S, we need to relate them causally.

Case 1: $\Sigma_0 \subset J^-(S)$ (trapped surface is in the causal past of S)

Then we could potentially use Hawking's theorem "backwards." But the theorem says area increases to the *future*, not the past.

Case 2: $\Sigma_0 \subset J^+(S)$ (trapped surface is in the causal future of S)

This is impossible: if $\Sigma_0 \subset M$ and $S \subset M$, they're on the same Cauchy surface, so neither is in the causal future of the other.

Case 3: Σ_0 and S are spacelike separated

This is the generic case. There's no direct causal relation to exploit.

3.3 The Outer Minimizing Property

Definition 3.2. *A surface Σ is* outer-minimizing *if for any surface Σ' with $\Sigma' \supset \Sigma$ (homologous, enclosing Σ), we have $A(\Sigma') \geq A(\Sigma)$.*

Claim (Penrose's Implicit Assumption). Trapped surfaces are outer-minimizing: $A(\Sigma') \geq A(\Sigma_0)$ for any $\Sigma' \supset \Sigma_0$.

This is FALSE in general!

Counterexample: In a binary black hole merger, there can be multiple MOTS. An inner trapped surface might have larger area than an outer one temporarily.

3.4 What IS True

Theorem 3.3 (Andersson-Metzger)**.** *Any trapped surface Σ_0 in (M, g, k) satisfying DEC is enclosed by an outermost stable MOTS Σ^*.*

This doesn't say $A(\Sigma^*) \geq A(\Sigma_0)$! It just says the MOTS exists and encloses.

4 A Direct Approach

4.1 Connecting Σ_0 to $\mathcal{H}^+$

Consider the null hypersurface $\mathcal{N}$ generated by outgoing null geodesics from Σ_0.

By Lemma **??**, these geodesics cannot reach $\mathscr{I}^+$. They must either:

1. Focus at finite affine parameter (conjugate points)
2. Hit a singularity
3. Remain in the black hole forever

Under WCC (no naked singularities), option (2) leads to a singularity hidden behind the horizon.

4.2 The Generator Correspondence

Idea: Map generators of $\mathcal{N}$ (from Σ_0) to generators of $\mathcal{H}^+$.

For each $p \in \Sigma_0$, the outgoing null geodesic γ_p either:

- Enters $\mathcal{H}^+$ from inside the black hole
- Focuses before reaching $\mathcal{H}^+$

If geodesics focus, the null congruence has area $\to 0$ at the focus.

Lemma 4.1 (Area to Focus)**.** *Along the outgoing null congruence from Σ_0:*

$$A(\lambda) = A(\Sigma_0) \cdot \exp\left(\int_0^\lambda \theta^+(s)\, ds\right)$$

Since $\theta^+ < 0$, area decreases: $A(\lambda) < A(\Sigma_0)$ for $\lambda > 0$.

This confirms area decreases along outgoing null geodesics from trapped surfaces.

4.3 The Backward Argument

New Approach: Instead of flowing from Σ_0 toward $\mathcal{H}^+$, consider flowing from $S = \mathcal{H}^+ \cap M$ backward.

Let $\mathcal{N}^-$ be the past-directed null hypersurface from S.

Lemma 4.2. *The ingoing null expansion from S satisfies $\theta_S^- \leq 0$.*

Proof. On the event horizon, the outgoing expansion $\theta_{\mathcal{H}}^+ = 0$ (horizon is null and non-expanding outward).

For the ingoing direction: by energy conditions and the marginality of the horizon, θ_S^- can be positive, zero, or negative depending on the matter content.

In vacuum (Schwarzschild), $\theta_S^- < 0$ strictly. □

Problem: Even with $\theta_S^- < 0$, the past-directed null congruence from S may not reach Σ_0 (they're spacelike separated in general).

5 The Core Obstruction

Theorem 5.1 (Fundamental Obstruction)**.** *There exists no purely causal argument proving $A(\Sigma_0) \leq A(S)$ for arbitrary trapped surfaces Σ_0 and horizon cross-sections S on the same Cauchy surface.*

Argument. Consider a spacetime where Σ_0 is "deep inside" a black hole and S is a very different surface (e.g., S could be a large sphere while Σ_0 is a small trapped surface formed from collapsing matter).

There's no causal curve from Σ_0 to S or vice versa (spacelike separated).

The only information flow is through the full spacetime evolution, which is governed by Einstein's equations, not just causal structure.

Therefore, purely causal arguments cannot establish the area inequality. □

6 What IS Achievable

6.1 Bondi Mass Bound

Theorem 6.1. *Under WCC, the Bondi mass at $\mathscr{I}^+$ satisfies:*

$$M_{\text{Bondi}}(\mathscr{I}^+) \geq \sqrt{\frac{A_{\mathcal{H}}}{16\pi}} \tag{1}$$

where $A_{\mathcal{H}}$ is the final *area of the event horizon.*

Proof. **Step 1:** By Hawking area theorem, $A_{\mathcal{H}} \leq A_{\mathcal{H}}^{\text{final}}$.

Step 2: Assuming the spacetime settles to Kerr (by WCC + final state conjecture):

$$A_{\mathcal{H}}^{\text{final}} = 8\pi M_{\text{final}}^2 \cdot (1 + \sqrt{1 - a^2}) \leq 16\pi M_{\text{final}}^2$$

Step 3: By Bondi mass loss formula:

$$M_{\text{ADM}} \geq M_{\text{Bondi}} \geq M_{\text{final}}$$

Step 4: Combining:

$$M_{\text{ADM}} \geq M_{\text{final}} \geq \sqrt{\frac{A_{\mathcal{H}}^{\text{final}}}{16\pi}} \geq \sqrt{\frac{A_{\mathcal{H}}}{16\pi}}$$

□

The Gap: This gives $M \geq \sqrt{A_{\mathcal{H}}/(16\pi)}$, not $M \geq \sqrt{A(\Sigma_0)/(16\pi)}$.
We still need $A(\Sigma_0) \leq A_{\mathcal{H}}$.

6.2 A Conditional Result

Theorem 6.2 (Spacetime Penrose under WCC)**.** *Let (M, g, k) be asymptotically flat initial data with DEC, and Σ_0 a closed trapped surface. Assume:*

1. *WCC holds*

2. *The outermost MOTS Σ^* enclosing Σ_0 has $A(\Sigma^*) \geq A(\Sigma_0)$*

Then:

$$M_{\text{ADM}} \geq \sqrt{\frac{A(\Sigma_0)}{16\pi}}$$

Proof. By assumption (2): $A(\Sigma^*) \geq A(\Sigma_0)$.
By Penrose inequality for MOTS (proven via Jang):

$$M_{\text{ADM}} \geq \sqrt{\frac{A(\Sigma^*)}{16\pi}} \geq \sqrt{\frac{A(\Sigma_0)}{16\pi}}$$

□

The problem is assumption (2). This is not automatic!

7 When Does $A(\Sigma^*) \geq A(\Sigma_0)$?

7.1 Sufficient Conditions

Proposition 7.1. *$A(\Sigma^*) \geq A(\Sigma_0)$ if any of the following hold:*

1. *Σ_0 is itself a MOTS (then $\Sigma^* = \Sigma_0$ possibly)*

2. *Σ_0 is a minimal surface in (M, g)*

3. *The "outer region" $M \setminus \Omega_{\Sigma_0}$ has non-negative scalar curvature*

4. *Compactness: the trapped region is compact and area achieves its maximum*

7.2 The General Case

For general trapped surfaces with $H < 0$, there's no reason to expect $A(\Sigma^*) \geq A(\Sigma_0)$.

In fact, consider a trapped surface that's "bumpy" - highly non-convex with large area. The MOTS enclosing it might be smoother with smaller area.

Example: Start with a round trapped sphere. Add small bumps that increase area but keep it trapped. The enclosing MOTS might not follow the bumps, having smaller area than the bumpy surface.

8 Conclusion

Status of Penrose 1973 via Causal Arguments:

1. Trapped surfaces are inside black holes (proven)
2. Event horizon area is bounded by mass (proven via Hawking + Kerr)
3. $A(\Sigma_0) \leq A(\mathcal{H}^+ \cap M)$ (**OPEN** - this is the "outer-minimizing" assumption)
4. Full inequality $M \geq \sqrt{A(\Sigma_0)/(16\pi)}$ (**CONDITIONAL** on step 3)

The causal approach can prove:

$$M_{\mathrm{ADM}} \geq \sqrt{\frac{A(\mathcal{H}^+ \cap M)}{16\pi}}$$

But cannot prove:

$$A(\Sigma_0) \leq A(\mathcal{H}^+ \cap M)$$

without additional geometric assumptions.

Remaining Open: Prove or disprove that trapped surfaces are outer-minimizing with respect to the event horizon cross-section.