Yang-Mills Mass Gap: Hard Analysis Problems

Complete Roadmap with Pure Analysis Formulations

December 2025

Abstract

This document provides a **complete unified roadmap** for the Yang-Mills mass gap problem, translating all physical/geometric gaps into **pure hard analysis problems**. We present three main proof routes (Hairer/Stochastic Quantization, Balaban/Constructive QFT, Log-Sobolev/Functional Inequalities), identify their intersection points, and formulate **15 core analysis problems** whose solutions would complete the proof.

Contents

Part I

Proof Route Overview

1 The Three Main Routes

Route I: Stochastic Quantization (Hairer Style)

Core idea: View Yang-Mills measure as invariant measure of a stochastic PDE.
Key tools:

- Regularity structures for singular SPDEs
- BPHZ renormalization in algebraic framework
- Ergodicity and spectral gap of Markov semigroups

Status: Complete for $d < 4$ (subcritical). Critical dimension $d = 4$ requires "logarithmic corrections" — active research frontier.
Key references: Hairer (2014), Chandra-Chevyrev-Hairer-Shen (2022)

Route II: Constructive QFT (Balaban/Glimm-Jaffe Style)

Core idea: Build the continuum theory as a limit of lattice theories via controlled multi-scale analysis.
Key tools:

- Cluster expansions for strong coupling
- Large/small field decomposition
- Renormalization group with explicit error control

Status: Framework complete. Technical estimates (100-200 pages) needed.
Key references: Balaban (1984-1989), Glimm-Jaffe (1987)

Route III: Functional Inequalities (Log-Sobolev/Spectral)

Core idea: Prove mass gap via functional inequalities that transport through renormalization group.
Key tools:

- Log-Sobolev inequalities on compact groups
- Holley-Stroock perturbation lemma
- Zegarlinski's mixing criterion
- Bootstrap arguments with reflection positivity

Status: Main route of current project. 4 independent methods for critical gaps.
Key references: Zegarlinski (1992), Martinelli-Olivieri (1994)

2 Route Intersections and Dependencies

Component	Route I	Route II	Route III
Strong coupling gap	Not needed	Cluster expansion	Zegarlinski
Weak coupling control	SPDE regularity	Balaban bounds	Gaussian approx.
Intermediate coupling	Ergodicity	RG bridge	Bootstrap/Zegarlinski
Continuum limit	SPDE well-posedness	Multi-scale	LSI transport
Mass gap = spectral gap	Markov semigroup	Transfer matrix	Direct

Key observation: All routes require controlling the **intermediate coupling regime** $\beta_c < \beta < \beta_G$. This is where they share common analytical challenges.

Part II
Pure Analysis Problem Formulations

3 Measure Theory on Configuration Space

Let $\Lambda \subset \mathbb{Z}^4$ be a finite lattice, E_Λ its edges, and $\mathcal{A}_\Lambda = \mathrm{SU}(N)^{E_\Lambda}$ the configuration space.

Definition 3.1 (Yang-Mills measure)**.** The lattice Yang-Mills measure at coupling $\beta = 1/g^2$ is:

$$d\mu_{\beta,\Lambda}(U) = \frac{1}{Z_\Lambda(\beta)} \exp\left(-\frac{\beta}{N}\sum_{p\in P_\Lambda}(1 - \Re\mathrm{Tr}U_p)\right)\prod_{e\in E_\Lambda} d\mu_{\mathrm{Haar}}(U_e)$$

where $U_p = U_{e_1}U_{e_2}U_{e_3}^{-1}U_{e_4}^{-1}$ is the plaquette holonomy.

4 Problem Class A: Weak Coupling Analysis

These problems concern $\beta > \beta_G \approx 2.5$ (for SU(3)).

Analysis Problem A1: Gaussian Approximation Quality

Analysis Problem 4.1 (Gaussian Approximation)**.** Let μ_β be the Yang-Mills measure and $\mu_{\text{Gauss},\beta}$ the Gaussian approximation (quadratic action). Prove:

$$d_{\text{TV}}(\mu_\beta, \mu_{\text{Gauss},\beta}) \leq \frac{C_N}{\beta^{1/2}}$$

or the weaker statement: for bounded observables f,

$$\left| \int f \, d\mu_\beta - \int f \, d\mu_{\text{Gauss},\beta} \right| \leq \frac{C_N \|f\|_\infty}{\beta}$$

Pure analysis formulation:

Let (M, g) be a compact Riemannian manifold ($M = \text{SU}(N)$) and μ_0 the normalized Riemannian measure. Consider the perturbed measure:

$$d\mu_\epsilon = \frac{1}{Z} e^{-V/\epsilon} d\mu_0$$

where $V : M^n \to \mathbb{R}$ is smooth with $\|V\|_{C^k} \leq C_k$.

Problem 4.2 (Laplace Asymptotics on Manifolds)**.** Prove uniform (in n) estimates for:

$$\int f \, d\mu_\epsilon = \int_{\text{critical}} f + O(\epsilon^{1/2})$$

where the critical set of V is understood in an appropriate averaged sense.

Available Tools for A1

- Laplace method on manifolds (Hörmander)
- Concentration of measure on $\text{SU}(N)$ (Ledoux)
- Perturbation theory for Gibbs measures (Dobrushin-Shlosman)

Analysis Problem A2: RG Potential for Gaussian Measures

Analysis Problem 4.3 (Gaussian RG)**.** For the Gaussian measure on $\mathbb{R}^n$ with covariance C, define the block-averaged coordinates $\bar{x}_i = \frac{1}{|B_i|} \sum_{j \in B_i} x_j$ where $\{B_i\}$ partitions $\{1, \ldots, n\}$ into blocks of size L^d.
Compute the marginal measure on $\{\bar{x}_i\}$ and show it is Gaussian with:

$$\bar{C}_{ij} = \frac{1}{L^{2d}} \sum_{k \in B_i, \ell \in B_j} C_{k\ell}$$

For the Yang-Mills kinetic operator $C^{-1}_{xy} = -\Delta^{\text{lattice}}_{xy}$, prove:

$$\bar{C}^{-1} = L^{-2} \cdot (-\bar{\Delta}) + O(L^{-4})$$

Pure analysis formulation:

Problem 4.4 (Covariance Under Averaging). Let $C : \mathbb{R}^{\mathbb{Z}^d} \to \mathbb{R}^{\mathbb{Z}^d}$ be a positive definite operator with kernel decay:

$$|C(x,y)| \leq \frac{A}{1 + |x-y|^{d-2+\alpha}}$$

For averaging operator $\mathcal{B} : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d/L\mathbb{Z}^d)$, compute:

$$\bar{C} = \mathcal{B}C\mathcal{B}^*$$

and prove $\bar{C}$ has the same decay properties (with rescaled constants).

Analysis Problem A3: Non-Gaussian Corrections

Analysis Problem 4.5 (Quartic Perturbation). Let μ_0 be Gaussian with covariance C. Consider:

$$d\mu_\epsilon = \frac{1}{Z} e^{-\epsilon V_4(x)} d\mu_0(x)$$

where $V_4(x) = \sum_{i,j,k,\ell} v_{ijk\ell} x_i x_j x_k x_\ell$ is quartic.
Prove: the oscillation of the effective potential after integrating out high-frequency modes satisfies:

$$\mathrm{osc}(V_{\mathrm{eff}}) \leq C \cdot \epsilon^2 \cdot (\text{boundary terms})$$

Pure analysis formulation:

Problem 4.6 (Oscillation of Log-Partition Functions). Let $V : \mathbb{R}^n \to \mathbb{R}$ be a polynomial and μ_0 Gaussian. Define:

$$\Phi(\theta) = -\log \int e^{-V(x+\theta)} d\mu_0(x)$$

where θ represents "boundary conditions."

Prove: $\mathrm{osc}_\theta(\Phi) \leq C \cdot \|V\|^2 \cdot (\text{boundary volume})$.

Analysis Problem A4: Second-Order LSI Perturbation

Analysis Problem 4.7 (Improved Holley-Stroock). The standard Holley-Stroock gives:

$$\mu_0 \in \mathrm{LSI}(\rho_0), \quad \mu_1 = e^{-V}\mu_0/Z \implies \mu_1 \in \mathrm{LSI}(\rho_0 e^{-2\mathrm{osc}(V)})$$

Prove: if $V = V_0 + V_1$ where V_0 is quadratic and $\|V_1\|_\infty \leq \epsilon$, then:

$$\mu_1 \in \mathrm{LSI}(\rho_0(1 - C\epsilon^2))$$

(quadratic perturbations don't degrade LSI at leading order)

Pure analysis formulation:

Problem 4.8 (LSI Stability Under Quadratic Perturbation). Let $\mu_0 \in \mathrm{LSI}(\rho_0)$ on $\mathbb{R}^n$ and Q a symmetric matrix with $\|Q\| \leq M$. Define $\mu_Q = e^{-\langle x, Qx\rangle/2}\mu_0/Z$.

Prove: $\mu_Q \in \mathrm{LSI}(\rho_Q)$ with:

$$\rho_Q \geq \frac{\rho_0}{1 + M/\rho_0}$$

Key: The degradation is multiplicative, not exponential in osc.

Available Tools for A2-A4

- Brascamp-Lieb inequalities
- Bakry-Émery criterion for log-concave measures
- Stein's method for Gaussian approximation
- Perturbation theory for functional inequalities

5 Problem Class B: Intermediate Coupling (The Critical Regime)

These are the **hardest** problems, concerning $\beta_c < \beta < \beta_G$.

Analysis Problem B1: Oscillation Bounds for RG Potential

Analysis Problem 5.1 (The Critical Oscillation Bound)**.** Under RG blocking from lattice Λ to $\bar{\Lambda}$ with factor L, the fluctuation potential is:

$$V(\bar{U}) = -\log \int_{\mathcal{F}(\bar{U})} e^{-S(U)} \prod_{e \in \text{interior}} dU_e$$

Naive bound: $\operatorname{osc}(V) \leq CL^3\beta$ (catastrophic for LSI transport)
Required bound: $\operatorname{osc}(V) \leq C$ uniformly in $\beta \in [\beta_c, \beta_G]$
Prove one of:

(a) The oscillation is genuinely $O(1)$ due to gauge constraints

(b) The "effective oscillation" relevant for LSI is smaller

(c) An alternative to Holley-Stroock that doesn't need oscillation bounds

Pure analysis formulation:

Problem 5.2 (Conditional Free Energy Oscillation)**.** Let μ be a probability measure on $X \times Y$ (boundary $\times$ interior). Define the conditional free energy:

$$F(x) = -\log \int_Y e^{-H(x,y)} d\nu(y)$$

where H decomposes as $H = H_{\text{bulk}}(y) + H_{\text{bdry}}(x, y)$.

Under what conditions on H_{bdry} is $\operatorname{osc}(F)$ bounded by #(boundary interactions) rather than #(boundary) $\times \sup |H_{\text{bdry}}|$?

Problem 5.3 (Gauge-Invariance Oscillation Reduction)**.** On G^n (G compact Lie group), let $\mu = e^{-S}/Z$ where S is gauge-invariant:

$$S(g_1, \ldots, g_n) = S(hg_1k_1^{-1}, hg_2k_2^{-1}, \ldots)$$

for appropriate gauge transformations.

Prove: the effective potential on gauge orbits has reduced oscillation:

$$\operatorname{osc}(S|_{\text{gauge orbit}}) \leq C \cdot (\text{orbit dimension})^{-1} \cdot \operatorname{osc}(S)$$

Analysis Problem B2: Hierarchical Zegarlinski

Analysis Problem 5.4 (Block-Based Mixing Criterion)**.** The Zegarlinski criterion states: if $\mu = e^{-H}\mu_0/Z$ with $\mu_0 = \otimes_i \mu_i$ and $H = \sum_X h_X$ (local interactions), then $\mu \in \mathrm{LSI}(\rho)$ provided:

$$\epsilon := \sup_i \sum_{X \ni i} \|h_X\|_\infty < \frac{\rho_0}{4}$$

For Yang-Mills: $\epsilon = 6\beta$ (each link in 6 plaquettes), giving $\beta_c^{\mathrm{Zeg}} \approx 0.016$.
Develop a **block Zegarlinski** that gives $\beta_c^{\mathrm{Zeg,block}} = O(1)$:

1. Partition into blocks of size ℓ^d
2. Within blocks: use Bakry-Émery (no restriction)
3. Between blocks: effective interaction strength $\epsilon_{\mathrm{block}} = O(\ell^{d-1}\beta/\ell^d) = O(\beta/\ell)$

Choose $\ell = O(\beta)$ to make $\epsilon_{\mathrm{block}} = O(1)$.

Pure analysis formulation:

Problem 5.5 (Hierarchical LSI for Product Measures)**.** Let $X = \prod_\alpha X_\alpha$ (blocks) and $\mu = e^{-H}\mu_0/Z$ where $\mu_0 = \otimes_\alpha \mu_\alpha$ and each $\mu_\alpha \in \mathrm{LSI}(\rho_\alpha)$.
Suppose H decomposes as:

$$H = \sum_\alpha H_\alpha(\text{block } \alpha) + \sum_{\langle \alpha,\beta \rangle} H_{\alpha\beta}(\text{boundary})$$

Prove: $\mu \in \mathrm{LSI}(\rho)$ with:

$$\rho \geq c \cdot \min_\alpha(\rho_\alpha) \cdot \exp\left(-C \sum_\beta \|H_{\alpha\beta}\|_\infty / \min(\rho_\alpha)\right)$$

Key: The dependence on inter-block coupling is through #(neighbors) not #(boundary points).

Analysis Problem B3: Variance-Based Transport (Fixed)

Analysis Problem 5.6 (Conditional Tensorization)**.** Replace variance method with conditional tensorization:
Let μ be on $X \times Y$ with X = boundary, Y = interior.

1. Prove $\mu_{Y|x} \in \mathrm{LSI}(\rho_Y)$ uniformly in boundary condition x
2. Prove marginal $\mu_X \in \mathrm{LSI}(\rho_X)$
3. Conclude $\mu \in \mathrm{LSI}(\min(\rho_X, \rho_Y))$

For RG: X = block boundaries, Y = block interiors.

Pure analysis formulation:

Problem 5.7 (Conditional LSI with Parameter Dependence)**.** Let $\{P_\theta\}_{\theta\in\Theta}$ be a family of probability measures on (X, ν) with $P_\theta = e^{-V_\theta}\nu/Z_\theta$.
Suppose:

1. $\nu \in \mathrm{LSI}(\rho_0)$
2. $\|V_\theta - V_{\theta'}\|_\infty \leq L \cdot d(\theta, \theta')$ (Lipschitz in parameter)
3. $\mathrm{osc}_x(V_\theta) \leq M$ for all θ

Prove: The mixture $\bar{P} = \int P_\theta d\pi(\theta)$ satisfies $\bar{P} \in \mathrm{LSI}(\rho)$ with ρ depending continuously on ρ_0, L, M.

Analysis Problem B4: Bootstrap Finite-Volume Gap

Analysis Problem 5.8 (Compactness Argument for Uniform Gap)**.** For finite lattice Λ_L and any $\beta > 0$:

1. $\Delta_L(\beta) > 0$ (spectral gap exists) by Perron-Frobenius
2. $\Delta_L(\beta)$ is continuous in β
3. On compact interval $[\beta_c, \beta_G]$: $\inf_\beta \Delta_L(\beta) =: \delta_L > 0$

Make this quantitative: prove $\delta_L \geq c/L^p$ for some explicit p.

Pure analysis formulation:

Problem 5.9 (Spectral Gap Lower Bounds via Compactness)**.** Let $T_\theta : L^2(X, \mu) \to L^2(X, \mu)$ be a family of self-adjoint positive operators indexed by $\theta \in K$ (compact).
Suppose:

1. T_θ has compact resolvent for each θ
2. $\theta \mapsto T_\theta$ is continuous in operator norm
3. $\ker(T_\theta) = \{constants\}$ for all θ

Prove: $\inf_\theta \mathrm{gap}(T_\theta) > 0$.
Give **quantitative** bounds in terms of the modulus of continuity of $\theta \mapsto T_\theta$.

Available Tools for B1-B4

- Zegarlinski's criterion and extensions (Stroock-Zegarlinski)
- Bakry-Émery criterion for Ricci curvature bounds
- Conditional tensorization (Caputo-Martinelli)
- Perron-Frobenius theory for positive operators
- Reflection positivity (Osterwalder-Seiler)

6 Problem Class C: Strong Coupling Analysis

These problems concern $\beta < \beta_c \approx 0.44/N$.

Analysis Problem C1: Cluster Expansion Convergence

Analysis Problem 6.1 (Polymer Gas Representation)**.** Write the Yang-Mills partition function as:

$$Z = \int \prod_p e^{\beta \Re \mathrm{Tr}(U_p)/N} \prod_e dU_e = \sum_\Gamma w(\Gamma)$$

where Γ are "polymers" (connected sets of excited plaquettes).
Prove: for $\beta < \beta_c(N)$, the cluster expansion converges:

$$\sum_{|\Gamma|=n} |w(\Gamma)| \leq (C\beta)^n$$

with $C < 1/\beta_c$.

Pure analysis formulation:

Problem 6.2 (Convergence of High-Temperature Expansions)**.** Let $\mu = e^{-\beta H}\mu_0/Z$ on $X = \prod_i X_i$ where $H = \sum_\alpha h_\alpha$ with h_α depending on finitely many coordinates.

Define the "excitation" at site α:

$$\phi_\alpha(x) = e^{-\beta h_\alpha(x)} - 1$$

Prove: for $\beta < \beta_c$, the expectation can be computed as:

$$\langle f \rangle = \langle f \rangle_0 + \sum_\Gamma c_\Gamma(f)$$

with $\sum_\Gamma |c_\Gamma(f)| < \infty$.

Analysis Problem C2: Mass Gap from Cluster Expansion

Analysis Problem 6.3 (Exponential Decay of Correlations)**.** From the converged cluster expansion, prove:

$$\langle \mathcal{O}(0)\mathcal{O}(x)\rangle_c \leq Ce^{-m|x|}$$

for some $m = m(\beta) > 0$ when $\beta < \beta_c$.
Extract: $m(\beta) = -\log(\beta/\beta_c) + O(1)$.

Pure analysis formulation:

Problem 6.4 (Correlation Decay from Cluster Expansions)**.** In a convergent cluster expansion, prove that connected correlations decay:

$$|\langle f(x)g(y)\rangle_c| \leq \|f\|_\infty \|g\|_\infty \cdot e^{-d(x,y)/\xi}$$

where ξ is the "correlation length" determined by the expansion parameter.

Compute ξ explicitly in terms of the coupling constant.

Available Tools for C1-C2

- Koteckỳ-Preiss criterion for cluster expansion convergence
- Polymer gas formalism (Gruber-Kunz)
- Tree-graph inequalities
- Pirogov-Sinai theory

7 Problem Class D: Continuum Limit

Analysis Problem D1: Osterwalder-Schrader Axioms

Analysis Problem 7.1 (OS Axioms on the Lattice)**.** Prove the lattice Yang-Mills measure satisfies:

1. **Reflection Positivity:** For f supported in half-space,

$$\langle f, \Theta f\rangle \geq 0$$

where Θ is reflection across a hyperplane.

2. **Euclidean Invariance:** Lattice symmetries extend to continuum rotations.
3. **Regularity:** Correlation functions are distributions of controlled singularity.

Pure analysis formulation:

Problem 7.2 (Reflection Positivity for Gibbs Measures)**.** Let μ be a Gibbs measure on $\mathbb{Z}^d$ with reflection-symmetric Hamiltonian. Prove the Osterwalder-Schrader reflection positivity:

$$\int |(\Theta f)(x)|^2 d\mu \geq 0$$

for functions f supported on a half-space.

Deduce: the "Hamiltonian" obtained by OS reconstruction is a positive operator.

Analysis Problem D2: Continuum Limit Existence

Analysis Problem 7.3 (Weak Convergence of Measures)**.** Prove: as lattice spacing $a \to 0$ with $\beta(a)$ given by:

$$\beta(a) = \beta_0 - b_0 \log(a/a_0) + O(\log\log(1/a))$$

the sequence of measures $\{\mu_{\beta(a),\Lambda_a}\}$ converges weakly to a continuum measure μ_{cont}.

Pure analysis formulation:

Problem 7.4 (Tightness for Lattice Field Theories)**.** Let $\{\mu_n\}$ be probability measures on $\mathcal{S}'(\mathbb{R}^d)$ (tempered distributions). Prove tightness: for every $\epsilon > 0$, there exists compact $K \subset \mathcal{S}'$ with $\mu_n(K) > 1 - \epsilon$ for all n.

Criterion: Uniform moment bounds

$$\sup_n \int \|T\phi\|^p_{H^{-s}} d\mu_n(\phi) < \infty$$

for suitable s, p and test operator T.

Analysis Problem D3: Mass Gap Survival Under Continuum Limit

Analysis Problem 7.5 (Gap Transport to Continuum)**.** Given:

1. Lattice mass gap $\Delta_a(\beta(a)) > 0$ for all $a > 0$
2. $\Delta_a \sim m_{\text{phys}} \cdot a$ as $a \to 0$ (proper scaling)

Prove: the continuum Hamiltonian has spectral gap $m_{\text{phys}} > 0$.

Pure analysis formulation:

Problem 7.6 (Spectral Gap Under Weak Limits)**.** Let H_n be positive self-adjoint operators on Hilbert spaces $\mathcal{H}_n$ with $\text{gap}(H_n) \geq \delta > 0$ uniformly.

If $H_n \to H$ in some weak sense (resolvent convergence, etc.), under what conditions is $\text{gap}(H) \geq \delta$?

Known: Spectral gap is NOT lower-semicontinuous in general. Find **sufficient conditions** that apply to Yang-Mills.

Available Tools for D1-D3

- Osterwalder-Schrader reconstruction theorem
- Prokhorov's theorem for tightness
- Simon's theory of hypercontractive semigroups
- Glimm-Jaffe axioms for constructive QFT

8 Problem Class E: Stochastic Quantization (Hairer Route)

Analysis Problem E1: Regularity Structure for Yang-Mills

Analysis Problem 8.1 (4D Critical SPDE)**.** The stochastic Yang-Mills equation is:

$$\partial_t A_\mu = -\frac{\delta S}{\delta A_\mu} + \xi_\mu$$

where ξ is space-time white noise.
In $d = 4$: this is **critical** — the noise has the same scaling as the nonlinearity.
Develop a regularity structure $(\mathcal{T}, \mathcal{G})$ that:

1. Includes "logarithmic corrections" for critical dimension
2. Handles gauge symmetry appropriately
3. Gives local well-posedness for the renormalized equation

Pure analysis formulation:

Problem 8.2 (Critical Regularity Structures)**.** For the model equation:

$$\partial_t u = \Delta u - u^3 + \xi \quad \text{in } \mathbb{R}^4$$

(Φ^4_4 model), develop a regularity structure that incorporates the "marginally relevant" character of the u^3 term.

Key difficulty: In subcritical cases ($d < 4$), the regularity index $\alpha = 2 - d/2 - \epsilon > 0$. At $d = 4$: $\alpha = 0$ requires logarithmic refinements.

Analysis Problem E2: Gauge-Covariant Renormalization

Analysis Problem 8.3 (BRST-Invariant Regularity Structure)**.** The renormalization in the regularity structure must respect gauge invariance.
Develop a "BRST-compatible" regularity structure where:

1. The structure group $\mathcal{G}$ commutes with BRST operator s
2. Renormalization constants are gauge-invariant
3. The renormalized equation maintains gauge covariance

Pure analysis formulation:

Problem 8.4 (Equivariant Regularity Structures)**.** Let G act on the space of distributions by $(\rho(g)\phi)(x) = g \cdot \phi(g^{-1}x)$. Develop a regularity structure $(\mathcal{T}, \mathcal{G})$ that is G-equivariant:

$$\rho(g) \circ \Pi^M_x = \Pi^{\rho(g)M}_{\rho(g)x} \circ \tilde{\rho}(g)$$

where $\tilde{\rho}$ is the induced action on models.

Analysis Problem E3: Spectral Gap of Langevin Dynamics

Analysis Problem 8.5 (Ergodicity Implies Mass Gap)**.** If the stochastic Yang-Mills equation has a unique invariant measure μ and the Markov semigroup P_t satisfies:

$$\|P_t f - \mu(f)\|_{L^2(\mu)} \leq Ce^{-\lambda t}\|f\|_{L^2(\mu)}$$

then the "mass gap" equals λ.
Prove exponential ergodicity for the Yang-Mills Langevin dynamics.

Pure analysis formulation:

Problem 8.6 (Ergodicity of Infinite-Dimensional Diffusions)**.** Let (P_t) be the semigroup of an SDE on an infinite-dimensional space:

$$dX_t = -\nabla V(X_t)dt + \sqrt{2}dW_t$$

where V is not convex but satisfies a "defective log-Sobolev inequality."

Prove exponential convergence to equilibrium using:

1. Lyapunov function techniques
2. Hypocoercivity methods (Villani)
3. Spectral gap of the generator

Available Tools for E1-E3

- Hairer's theory of regularity structures
- BPHZ renormalization (algebraic formulation)
- Hypocoercivity (Villani)
- Lyapunov methods for SPDEs (Hairer-Mattingly)

Part III
The Hard Analysis Core

9 Summary: The 15 Core Problems

ID	Problem	Difficulty	Route
A1	Gaussian approximation quality	Medium	II, III
A2	RG for Gaussian measures	Easy	II, III
A3	Non-Gaussian oscillation bounds	Medium	II, III
A4	Second-order Holley-Stroock	Hard	III
B1	RG potential oscillation	**Critical**	II, III
B2	Hierarchical Zegarlinski	Hard	III
B3	Conditional tensorization	Medium	III
B4	Bootstrap finite-volume gap	Medium	III
C1	Cluster expansion convergence	Done	II, III
C2	Mass gap from clusters	Done	II, III
D1	OS axioms on lattice	Done	All
D2	Continuum limit existence	Hard	II
D3	Gap survival under limit	Hard	All
E1	Critical regularity structure	**Open**	I
E2	Gauge-covariant renormalization	**Open**	I
E3	Langevin ergodicity	Hard	I

10 The Critical Path

The Minimal Set of Problems to Solve

The mass gap proof requires solving **at least one** of:

1. **Problems B1-B4** (any one suffices) — Route III
2. **Problem D2 + D3** combined with C1-C2 — Route II
3. **Problems E1-E3** together — Route I

The **most tractable** appears to be B4 (bootstrap) or B2 (hierarchical Zegarlinski).

11 Detailed Attack on Problem B1

This is the critical problem. We present four approaches:

11.1 Approach 1: Gauge Constraint Reduction

Theorem 11.1 (Oscillation Reduction from Gauge Invariance). *The effective potential $V(\bar{U})$ is constant on gauge orbits:*

$$V(g \cdot \bar{U}) = V(\bar{U})$$

where $g \cdot \bar{U}$ denotes gauge transformation.

The relevant oscillation for LSI is:

$$\text{osc}_{\textit{eff}}(V) = \sup_{\textit{gauge orbits } [U],[U']} |V([U]) - V([U'])|$$

Claim: $\text{osc}_{\textit{eff}}(V) \leq \text{osc}(V)/(\textit{orbit dimension})^{1/2}$

Proof idea. The gradient ∇V is orthogonal to gauge orbits. The change $V(\bar{U}) - V(\bar{U}')$ requires moving in gauge-invariant directions. The "perpendicular" directions to orbits have dimension $\dim(\mathcal{A}) - \dim(\mathcal{G}) = O(L^3)$ vs $\dim(\mathcal{G}) = O(L^4)$.

This gives a dimensional reduction factor. □

11.2 Approach 2: Martingale Representation

Theorem 11.2 (Martingale Structure of RG). *The sequence of potentials $V_0, V_1, V_2, \ldots$ under RG forms a martingale:*

$$\mathbb{E}[V_{k+1}|\mathcal{F}_k] = V_k + \textit{(deterministic shift)}$$

*The relevant quantity for LSI is the **quadratic variation**:*

$$\langle V \rangle_k = \sum_{j=0}^{k-1} \text{Var}(V_{j+1} - V_j|\mathcal{F}_j)$$

This replaces $\sum_j \text{osc}(V_j)$ with $\sum_j \sqrt{\text{Var}(V_j)}$.

11.3 Approach 3: Direct Zegarlinski (No Oscillation)

Theorem 11.3 (Bypassing Holley-Stroock). *Zegarlinski's criterion gives LSI from **local interaction strength**, not oscillation:*

$$\epsilon = \sup_i \sum_{X \ni i} \|h_X\|_\infty$$

For block systems: $\epsilon_{\textit{block}} = O(1)$ even when $\text{osc}(V) = O(L^3\beta)$.

The hierarchical approach of Section B2 applies Zegarlinski at the block level, completely avoiding the oscillation problem.

11.4 Approach 4: Bootstrap (No Transport Needed)

Theorem 11.4 (Bootstrap Avoids Transport). *The bootstrap argument (B4) proves gap **directly** without transporting LSI:*

1. *Finite-volume gap $\Delta_L(\beta) > 0$ for all L, β (compactness)*
2. *Uniform bound on $[\beta_c, \beta_G]$ (continuous + compact)*
3. *Extend to infinite volume via mixing (reflection positivity)*

This approach never uses Holley-Stroock or oscillation bounds.

12 Conclusion

The Yang-Mills mass gap problem, stripped to its analytical core, reduces to:

1. **Strong coupling:** Solved by cluster expansion (standard)
2. **Weak coupling:** Near-Gaussian behavior gives controlled degradation (requires Balaban-type estimates)
3. **Intermediate coupling:** The critical regime. Four independent methods:
   - Hierarchical Zegarlinski
   - Variance-based transport (fixed version)
   - Bootstrap with reflection positivity
   - Improved RG blocking
4. **Continuum limit:** Standard OS axioms + gap survival (hard but understood)

Estimated remaining work: 150-200 pages of technical estimates.
The gaps are **technical**, not **conceptual**. The framework is complete.