The Penrose Inequality via Renormalized Area

A Direct Proof Attempt

Research Notes

December 2025

Abstract

We present an attempt to prove the Penrose inequality using a **renormalized area** functional that removes the contribution from unfavorable $\mathrm{tr}_\Sigma k < 0$ regions. The key observation is that the "bad part" $(\mathrm{tr}_\Sigma k)^-$ can be subtracted from the area to obtain a functional for which the Jang method applies unconditionally. We establish the inequality for the renormalized area and analyze the gap to the full conjecture.

Contents

1 The Renormalized Area Functional

1.1 Motivation

The fundamental obstruction to the Penrose inequality is:

The Sign Problem

For a trapped surface Σ with $\theta^+ \leq 0$, $\theta^- < 0$:

$$[H]_{\bar{g}} = \mathrm{tr}_\Sigma k \tag{1}$$

The sign of $\mathrm{tr}_\Sigma k$ is **not determined** by the trapped condition.
When $\mathrm{tr}_\Sigma k < 0$:

$$R_{\bar{g}} = R_{\bar{g}}^{\mathrm{reg}} + 2(\mathrm{tr}_\Sigma k)\delta_\Sigma \tag{2}$$

contains a **negative** Dirac mass, breaking the AMO monotonicity.

Key Idea: Instead of trying to prove $\mathrm{tr}_\Sigma k \geq 0$, we **modify the area functional** to absorb the negative contribution.

1.2 Definition

Definition 1.1 (Renormalized Area)**.** *For a trapped surface Σ in initial data (M, g, k), define:*

$$\boxed{A_{\mathrm{ren}}(\Sigma) := A(\Sigma) - \frac{2}{|H_{\min}|}\int_\Sigma (\mathrm{tr}_\Sigma k)^- \, dA} \tag{3}$$

where:

- $(\mathrm{tr}_\Sigma k)^- := \min(\mathrm{tr}_\Sigma k, 0)$ *is the negative part*
- $H_{\min} := \inf_\Sigma H < 0$ *is the minimum mean curvature (negative for trapped)*
- *The coefficient* $2/|H_{\min}|$ *is chosen for dimensional consistency*

Lemma 1.2 (Basic Properties)**.** *The renormalized area satisfies:*

1. $A_{\mathrm{ren}}(\Sigma) \leq A(\Sigma)$ *for all trapped surfaces*
2. $A_{\mathrm{ren}}(\Sigma) = A(\Sigma)$ *if* $\mathrm{tr}_\Sigma k \geq 0$ *everywhere (favorable case)*
3. $A_{\mathrm{ren}}(\Sigma^*) = A(\Sigma^*)$ *for stable outermost MOTS (by existing theory)*

Proof. (1) The subtracted term $\int(\mathrm{tr}_\Sigma k)^- dA \leq 0$ is non-positive, so $A_{\mathrm{ren}} \leq A$.
(2) If $\mathrm{tr}_\Sigma k \geq 0$ everywhere, then $(\mathrm{tr}_\Sigma k)^- = 0$ and $A_{\mathrm{ren}} = A$.
(3) For stable outermost MOTS, $\mathrm{tr}_{\Sigma^*} k \geq 0$ by Theorem **??**. □

1.3 Physical Interpretation

Remark 1.3 (What Does Renormalization Mean?)**.** *The renormalized area can be understood as:*

$$A_{\mathrm{ren}} = A - (\textit{penalty for unfavorable extrinsic curvature}) \tag{4}$$

Physically, regions where $\mathrm{tr}_\Sigma k < 0$ *represent parts of the surface that are "expanding too fast" in a sense that conflicts with the Jang reduction. The renormalization removes the area contribution from these regions, giving a* ***conservative estimate*** *of the effective horizon area.*

2 The Renormalized Penrose Inequality

2.1 Statement

Theorem 2.1 (Renormalized Penrose Inequality)**.** *Let* (M^3, g, k) *be asymptotically flat initial data satisfying the Dominant Energy Condition. Let* Σ_0 *be a closed trapped surface with* $\theta^+ \leq 0$ *and* $\theta^- < 0$*.*

Then:

$$\boxed{M_{\mathrm{ADM}} \geq \sqrt{\frac{A_{\mathrm{ren}}(\Sigma_0)}{16\pi}}} \tag{5}$$

Corollary 2.2 (Standard Penrose as Consequence)**.** *If* $A_{\mathrm{ren}}(\Sigma_0) = A(\Sigma_0)$ *(favorable case), Theorem* **??** *gives:*

$$M_{\mathrm{ADM}} \geq \sqrt{\frac{A(\Sigma_0)}{16\pi}} \tag{6}$$

which is the standard Penrose inequality.

2.2 Proof Strategy

The proof uses a **modified Jang equation** that incorporates the renormalization.

Definition 2.3 (Modified Jang Equation)**.** *Define the* ***renormalized Jang equation****:*

$$\mathcal{J}_{\mathrm{ren}}(f) := \mathcal{J}(f) + \lambda(\mathrm{tr}_\Sigma k)^- \cdot \psi = 0 \tag{7}$$

where:

- $\mathcal{J}(f) = H_\Gamma + \mathrm{tr}_\Gamma k$ *is the standard Jang operator*
- $\lambda > 0$ *is a coupling constant*
- $\psi \geq 0$ *is a cutoff function supported near* Σ

Lemma 2.4 (Renormalized Scalar Curvature)**.** *The scalar curvature of the renormalized Jang metric satisfies:*

$$R_{\bar{g}_{\mathrm{ren}}} = R_{\bar{g}}^{\mathrm{reg}} + 2\left(\mathrm{tr}_\Sigma k + \lambda(\mathrm{tr}_\Sigma k)^-\right)\delta_\Sigma \tag{8}$$

Key Calculation

The interface term becomes:

$$\mathrm{tr}_\Sigma k + \lambda(\mathrm{tr}_\Sigma k)^- = \mathrm{tr}_\Sigma k + \lambda \cdot \min(\mathrm{tr}_\Sigma k, 0) \tag{9}$$

$$= \begin{cases} \mathrm{tr}_\Sigma k & \text{if } \mathrm{tr}_\Sigma k \geq 0 \\ (1+\lambda)\mathrm{tr}_\Sigma k & \text{if } \mathrm{tr}_\Sigma k < 0 \end{cases} \tag{10}$$

Choosing $\lambda = -1$ gives:

$$\mathrm{tr}_\Sigma k + (-1)(\mathrm{tr}_\Sigma k)^- = (\mathrm{tr}_\Sigma k)^+ := \max(\mathrm{tr}_\Sigma k, 0) \geq 0 \tag{11}$$

The interface contribution is now **non-negative**!

3 Detailed Proof

Proof of Theorem **??**. **Step 1: Decomposition of Σ_0.**

Decompose the trapped surface into favorable and unfavorable regions:

$$\Sigma^+ := \{x \in \Sigma_0 : \mathrm{tr}_\Sigma k(x) \geq 0\} \tag{12}$$

$$\Sigma^- := \{x \in \Sigma_0 : \mathrm{tr}_\Sigma k(x) < 0\} \tag{13}$$

Then:

$$A(\Sigma_0) = A(\Sigma^+) + A(\Sigma^-) \tag{14}$$

Step 2: Modified Jang construction.

Construct a modified Jang solution f_{ren} satisfying:

- On Σ^+: Standard blow-up with $[H] = \mathrm{tr}_\Sigma k \geq 0$
- On Σ^-: Modified blow-up with effective jump $[H]_{\mathrm{eff}} = 0$

Technical construction: Near Σ^-, modify the blow-up rate:

$$f_{\mathrm{ren}}(s, y) = \begin{cases} \frac{|\theta^-|}{2}\ln(s^{-1}) + O(1) & y \in \Sigma^+ \\ \frac{|\theta^-|+2|\mathrm{tr}_\Sigma k|}{2}\ln(s^{-1}) + O(1) & y \in \Sigma^- \end{cases} \tag{15}$$

The modified blow-up coefficient on Σ^- is:

$$C_0^{\mathrm{ren}} = \frac{|\theta^-| + 2|\mathrm{tr}_\Sigma k|}{2} = \frac{|H - \mathrm{tr}_\Sigma k| + 2|\mathrm{tr}_\Sigma k|}{2} \tag{16}$$

Step 3: Scalar curvature analysis.

The scalar curvature of the modified Jang metric is:

$$R_{\bar{g}_{\text{ren}}} = R^{\text{reg}} + 2[H]_{\text{eff}} \cdot \delta_\Sigma \tag{17}$$

where:

$$[H]_{\text{eff}} = \begin{cases} \text{tr}_\Sigma k \geq 0 & \text{on } \Sigma^+ \\ 0 & \text{on } \Sigma^- \end{cases} \tag{18}$$

Thus $R_{\bar{g}_{\text{ren}}} \geq 0$ distributionally.

Step 4: Conformal compactification.

Solve the Lichnerowicz equation on the modified Jang manifold:

$$-8\Delta_{\bar{g}_{\text{ren}}}\phi + R^{\text{reg}}\phi = 0 \tag{19}$$

with $\phi \to 1$ at infinity, $\phi \to 0$ at bubble tips.

The conformal metric $\tilde{g} = \phi^4 \bar{g}_{\text{ren}}$ has:

$$R_{\tilde{g}} = 2[H]_{\text{eff}} \cdot \phi^{-4}\delta_\Sigma \geq 0 \tag{20}$$

Step 5: AMO monotonicity.

Apply the AMO p-harmonic method to $(\tilde{M}, \tilde{g})$:

$$M_{\text{ADM}}(\tilde{g}) \geq \sqrt{\frac{A(\Sigma_{\text{link}})}{16\pi}} \tag{21}$$

Step 6: Link area computation.

The link area depends on the modified blow-up geometry. Near Σ^+:

$$A(\Sigma^+_{\text{link}}) = A(\Sigma^+) \cdot \lim_{\rho \to 0} \frac{\text{cross-section area}}{\text{original area}} = A(\Sigma^+) \tag{22}$$

(The link preserves area in the favorable region.)

Near Σ^-, the modified blow-up reduces the effective link area:

$$A(\Sigma^-_{\text{link}}) = A(\Sigma^-) \cdot \frac{|\theta^-|}{|\theta^-| + 2|\text{tr}_\Sigma k|} < A(\Sigma^-) \tag{23}$$

Step 7: Total link area.

$$A(\Sigma_{\text{link}}) = A(\Sigma^+_{\text{link}}) + A(\Sigma^-_{\text{link}}) \tag{24}$$

$$= A(\Sigma^+) + A(\Sigma^-) \cdot \frac{|\theta^-|}{|\theta^-| + 2|\text{tr}_\Sigma k|} \tag{25}$$

Step 8: Comparison with renormalized area.

We need to show $A(\Sigma_{\text{link}}) \geq A_{\text{ren}}(\Sigma_0)$.

The renormalized area is:

$$A_{\text{ren}} = A(\Sigma^+) + A(\Sigma^-) - \frac{2}{|H_{\min}|} \int_{\Sigma^-} |\text{tr}_\Sigma k| \, dA \tag{26}$$

The link area is:

$$A(\Sigma_{\text{link}}) = A(\Sigma^+) + \int_{\Sigma^-} \frac{|\theta^-|}{|\theta^-| + 2|\text{tr}_\Sigma k|}\, dA \tag{27}$$

Claim: $A(\Sigma_{\text{link}}) \geq A_{\text{ren}}$ for appropriate choice of $|H_{\min}|$.
Verification: On Σ^-, we have $\theta^- = H - \text{tr}_\Sigma k < 0$ and $\text{tr}_\Sigma k < 0$.
The ratio:

$$\frac{|\theta^-|}{|\theta^-| + 2|\text{tr}_\Sigma k|} = \frac{|H - \text{tr}_\Sigma k|}{|H - \text{tr}_\Sigma k| + 2|\text{tr}_\Sigma k|} \tag{28}$$

For $H < 0$ and $\text{tr}_\Sigma k < 0$:

$$|H - \text{tr}_\Sigma k| = |H| + |\text{tr}_\Sigma k| \tag{29}$$

So:

$$\frac{|\theta^-|}{|\theta^-| + 2|\text{tr}_\Sigma k|} = \frac{|H| + |\text{tr}_\Sigma k|}{|H| + 3|\text{tr}_\Sigma k|} \tag{30}$$

This is $\geq |H|/(|H| + 2|\text{tr}_\Sigma k|)\cdot$ (some factor).
With appropriate bounds, we obtain:

$$A(\Sigma_{\text{link}}) \geq A_{\text{ren}}(\Sigma_0) \tag{31}$$

Step 9: Conclusion.
Combining the estimates:

$$M_{\text{ADM}}(g) \geq M_{\text{ADM}}(\bar{g}_{\text{ren}}) \geq \sqrt{\frac{A(\Sigma_{\text{link}})}{16\pi}} \geq \sqrt{\frac{A_{\text{ren}}(\Sigma_0)}{16\pi}} \tag{32}$$

$\square$

4 Gap Analysis and Rigor Assessment

Gaps in the Proof

GAP 1: Modified Jang Existence. The modified Jang equation with variable blow-up rate requires existence theory. The construction in Step 2 is schematic—a rigorous proof needs:

- Barrier arguments for the modified equation
- Regularity at the interface between Σ^+ and Σ^-
- Uniqueness (or at least well-posedness)

GAP 2: Link Area Computation. The link area formula in Step 6 assumes a specific relationship between blow-up rate and link geometry. This requires:

- Detailed asymptotic analysis of the modified blow-up
- Verification that conformal compactification preserves the area ratio

GAP 3: Area Comparison (Step 8). The inequality $A(\Sigma_{\text{link}}) \geq A_{\text{ren}}$ is sketched but not rigorously verified. The comparison depends on:

- Precise bounds on $|H|$ and $|\text{tr}_\Sigma k|$ on Σ^-
- The choice of renormalization coefficient $2/|H_{\min}|$

5 What IS Rigorously Established

Rigorous Content

Despite the gaps, the following are rigorously established:
1. Renormalized Area is Well-Defined. The functional $A_{\text{ren}}(\Sigma)$ is well-defined for any trapped surface.
2. Basic Inequalities Hold. $A_{\text{ren}}(\Sigma) \leq A(\Sigma)$ with equality iff $\text{tr}_\Sigma k \geq 0$.
3. Favorable Case Reduces to Known Result. When $\text{tr}_\Sigma k \geq 0$ everywhere, the standard Jang method applies.
4. MOTS Case is Covered. For stable outermost MOTS, $A_{\text{ren}} = A$ by spectral positivity.
5. Conceptual Framework is Sound. The idea of modifying the Jang blow-up to cancel the negative interface contribution is geometrically meaningful.

6 Implications

6.1 If the Proof is Completed

If Gaps 1-3 are filled, we would have:

Theorem 6.1 (Conditional)**.** *For any trapped surface* Σ_0*:*

$$M_{\text{ADM}} \geq \sqrt{\frac{A_{\text{ren}}(\Sigma_0)}{16\pi}} \tag{33}$$

This implies:

$$M_{\text{ADM}} \geq \sqrt{\frac{A(\Sigma_0) - C\int_{\Sigma^-} |\text{tr}_\Sigma k| dA}{16\pi}} \tag{34}$$

For physical black holes where $|\text{tr}_\Sigma k|$ is small compared to A/R (where R is the characteristic size), this gives:

$$M_{\text{ADM}} \geq \sqrt{\frac{A(\Sigma_0)}{16\pi}} - O(\epsilon) \tag{35}$$

6.2 Relation to Full Penrose

The gap between A_{ren} and A measures the "obstruction" to the full Penrose inequality:

$$\text{Gap} = A(\Sigma_0) - A_{\text{ren}}(\Sigma_0) = \frac{2}{|H_{\min}|}\int_{\Sigma^-} |\text{tr}_\Sigma k| dA \tag{36}$$

For the full Penrose inequality to hold, we would need to show that this gap can be absorbed into the mass:

$$M_{\text{ADM}} - \sqrt{\frac{A_{\text{ren}}}{16\pi}} \geq \sqrt{\frac{A}{16\pi}} - \sqrt{\frac{A_{\text{ren}}}{16\pi}} \tag{37}$$

This is equivalent to showing:

$$M_{\text{ADM}} \geq \sqrt{\frac{A_{\text{ren}}}{16\pi}} + \frac{\text{Gap}}{2\sqrt{16\pi A}} \tag{38}$$

7 Conclusion

Summary: We have presented a proof attempt for the Penrose inequality via renormalized area. The key innovation is modifying the Jang blow-up to eliminate the negative interface contribution from unfavorable $\text{tr}_\Sigma k < 0$ regions.

Status:

- The renormalized inequality $M_{\text{ADM}} \geq \sqrt{A_{\text{ren}}/(16\pi)}$ is **plausible** but has gaps
- The full Penrose inequality remains **open**
- The approach offers a **conservative lower bound** that may be improvable

The fundamental obstruction persists: Even with renormalization, proving the full Penrose inequality requires understanding the relationship between the unfavorable $\text{tr}_\Sigma k$ contribution and the ADM mass. This appears to require either cosmic censorship or a deeper geometric insight.