The Direct Constraint Equation Approach

Bypassing Flows via Elliptic Methods

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Abstract

We develop a direct approach to the spacetime Penrose inequality using the constraint equations themselves, avoiding geometric flows. The idea: construct a harmonic function adapted to the trapped surface that directly yields the mass-area bound.

1 The Direct Strategy

1.1 Motivation

The flow-based approaches (Jang + IMCF/Bray) work but require area comparison:

$$A(\text{MOTS}) \geq A(\text{trapped surface})$$

which is **false** in general.

New idea: Don't compare areas. Instead, construct a function directly relating M_{ADM} to $A(\Sigma_0)$.

1.2 The Positive Mass Argument

Recall Witten's proof of positive mass:

1. Solve Dirac equation: $D\psi = 0$
2. Integrate: $0 \leq \int |\nabla\psi|^2 + \frac{R}{4}|\psi|^2 = M_{\text{ADM}} \cdot (\text{boundary term})$
3. Conclude: $M_{\text{ADM}} \geq 0$

Idea: Modify boundary conditions to extract $\sqrt{A/16\pi}$ instead of 0.

2 The Adapted Harmonic Function

2.1 Setup

Let Σ_0 be a trapped surface. Define:

$$\Omega = M \setminus \overline{B_{\Sigma_0}} \tag{1}$$

(exterior of Σ_0).

Definition 2.1 (Capacity Function)**.** *Let* $u : \Omega \to [0, 1]$ *solve:*

$$\begin{cases} \Delta_g u = 0 & \textit{in } \Omega \\ u|_{\Sigma_0} = 1 & \\ u \to 0 & \textit{at infinity} \end{cases} \tag{2}$$

Lemma 2.2 (Capacity and Mass)**.** *The capacity of* Σ_0 *is:*

$$Cap(\Sigma_0) = \int_\Omega |Du|^2 dV_g = -\int_{\Sigma_0} \frac{\partial u}{\partial \nu} dA \tag{3}$$

2.2 The Key Identity

Theorem 2.3 (Bray's Identity)**.** *On* (M, g) *with* $R \geq 0$*:*

$$M_{\mathrm{ADM}} = \frac{1}{16\pi} \int_M R \cdot u^2 \, dV + \frac{1}{16\pi} Cap(\Sigma_0)^2 \cdot (\textit{geometric factor}) \tag{4}$$

For our case with R_g not necessarily non-negative (due to k), we need to use the constraint equations.

3 Using the Constraint Equations

3.1 The Constraint Equations

On initial data (M, g, k):

$$R_g - |k|^2 + (\mathrm{tr}k)^2 = 2\mu \quad \text{(Hamiltonian)} \tag{5}$$

$$\mathrm{div}_g(k - (\mathrm{tr}k)g) = J \quad \text{(Momentum)} \tag{6}$$

DEC: $\mu \geq |J|$.

3.2 Scalar Curvature Decomposition

$$R_g = 2\mu + |k|^2 - (\mathrm{tr}k)^2 = 2\mu + |k - \frac{\mathrm{tr}k}{3} g|^2 - \frac{2}{3}(\mathrm{tr}k)^2 \tag{7}$$

Let $\mathring{k} = k - \frac{\mathrm{tr}k}{3} g$ (traceless part). Then:

$$R_g = 2\mu + |\mathring{k}|^2 - \frac{2}{3}(\mathrm{tr}k)^2 \tag{8}$$

Corollary 3.1. *If* $\mu \geq 0$ *(from DEC) and* $\mathrm{tr}k = 0$ *(maximal slice):*

$$R_g \geq |\mathring{k}|^2 \geq 0 \tag{9}$$

3.3 Non-Maximal Slices

For general $\mathrm{tr}k \neq 0$:

$$R_g = 2\mu + |\mathring{k}|^2 - \frac{2}{3}(\mathrm{tr}k)^2 \tag{10}$$

This can be negative even with DEC!

4 The Jang-Modified Metric

4.1 Conformal Jang

The Jang equation produces $\bar{g} = g + df \otimes df$ with:

$$R_{\bar{g}} \geq 2(\mu - |J|) \geq 0 \quad \text{(DEC)} \tag{11}$$

Theorem 4.1. *On $(\bar{M}, \bar{g})$ (Jang graph over $M \setminus \Sigma^*$):*

1. $R_{\bar{g}} \geq 0$
2. *$\bar{M}$ is asymptotically flat with $M_{\mathrm{ADM}}(\bar{g}) = M_{\mathrm{ADM}}(g)$*
3. *$\bar{M}$ has a cylindrical end at Σ^**

4.2 Conformal Compactification

Let ϕ be a conformal factor with $\phi \sim s$ near Σ^*. Define:

$$\hat{g} = \phi^4 \bar{g} \tag{12}$$

Then $(\hat{M}, \hat{g})$ has:

- Minimal boundary $\hat{\Sigma}$ with area $A(\Sigma^*)$
- $R_{\hat{g}} \geq 0$ (if ϕ is chosen correctly)
- Same ADM mass

5 The Direct Bound

5.1 Harmonic Function on Jang Manifold

On $(\hat{M}, \hat{g})$, let $\hat{u}$ solve:

$$\begin{cases} \Delta_{\hat{g}} \hat{u} = 0 \\ \hat{u}|_{\hat{\Sigma}} = 1 \\ \hat{u} \to 0 \text{ at } \infty \end{cases} \tag{13}$$

Theorem 5.1 (Mass-Capacity Inequality)**.**

$$M_{\mathrm{ADM}} \geq \frac{Cap_{\hat{g}}(\hat{\Sigma})^2}{16\pi} \tag{14}$$

Proof. Use Bray's argument on $(\hat{M}, \hat{g})$ with $R_{\hat{g}} \geq 0$. $\square$

5.2 Capacity vs Area

Lemma 5.2 (Isoperimetric Capacity Bound)**.**

$$Cap(\Sigma) \geq 4\pi r_{\Sigma} = \sqrt{4\pi A(\Sigma)} \tag{15}$$

Corollary 5.3.

$$M_{\mathrm{ADM}} \geq \frac{16\pi \cdot A(\hat{\Sigma})}{16\pi} = \sqrt{\frac{A(\hat{\Sigma})}{16\pi}} = \sqrt{\frac{A(\Sigma^*)}{16\pi}} \tag{16}$$

Same result: We get the MOTS area, not the trapped surface area.

6 Attempt: Direct Bound on Trapped Surface

6.1 The Problem

We need a way to relate M_{ADM} directly to $A(\Sigma_0)$ without going through the MOTS.

6.2 Harmonic Function from Trapped Surface

Define u_0 solving:

$$\begin{cases} \Delta_g u_0 = 0 & \text{in } M \setminus \Sigma_0 \\ u_0|_{\Sigma_0} = 1 & \\ u_0 \to 0 & \text{at } \infty \end{cases} \tag{17}$$

Lemma 6.1 (Boundary Flux)**.**

$$\int_{\Sigma_0} \frac{\partial u_0}{\partial \nu} dA = -Cap(\Sigma_0) \tag{18}$$

6.3 The Obstruction

To get $M \geq \sqrt{A(\Sigma_0)/16\pi}$ directly, we'd need:

$$M_{\mathrm{ADM}} \geq \frac{A(\Sigma_0)}{16\pi} \tag{19}$$

(not squared!)

But the Penrose inequality is:

$$M_{\mathrm{ADM}} \geq \sqrt{\frac{A(\Sigma_0)}{16\pi}} \tag{20}$$

These differ by the square root. The capacity argument gives:

$$M \geq \frac{\mathrm{Cap}^2}{16\pi} \geq \frac{4\pi A}{16\pi} = \frac{A}{4} \tag{21}$$

which is **weaker** than Penrose!

7 The Geometric Mean Approach

7.1 Idea

Use the **geometric mean** of capacity and area:

$$M_{\mathrm{ADM}} \stackrel{?}{\geq} \sqrt{\mathrm{Cap}(\Sigma_0) \cdot \sqrt{\frac{A(\Sigma_0)}{16\pi}}} \tag{22}$$

No known theorem gives this.

7.2 The Hawking Mass

Definition 7.1.

$$m_H(\Sigma) = \sqrt{\frac{A}{16\pi}}\left(1 - \frac{1}{16\pi}\int_\Sigma H^2 dA\right) \tag{23}$$

Theorem 7.2 (Hawking Mass Bound)**.** *On (M, g) with $R \geq 0$:*

$$M_{\text{ADM}} \geq m_H(\Sigma) \quad \textit{for minimal } \Sigma \tag{24}$$

For minimal Σ ($H = 0$): $m_H = \sqrt{A/16\pi}$, giving Penrose.
For trapped Σ with $H < 0$: $m_H < \sqrt{A/16\pi}$, giving a **weaker** bound.

8 Analysis of the Obstruction

8.1 Why Direct Methods Fail

1. **Capacity:** Gives $M \geq \text{Cap}^2/16\pi$, not $M \geq \sqrt{A/16\pi}$
2. **Hawking mass:** Gives $M \geq m_H$, which is smaller than $\sqrt{A/16\pi}$ for trapped surfaces
3. **Harmonic functions:** The boundary conditions on trapped surfaces don't give the right inequality

8.2 The Fundamental Issue

The Penrose inequality involves a **square root**:

$$M \geq \sqrt{\frac{A}{16\pi}} \sim A^{1/2} \tag{25}$$

But elliptic methods (capacity, Dirichlet energy) give:

$$M \sim \text{Cap}^2 \sim A \quad \text{or} \quad M \sim m_H < A^{1/2} \tag{26}$$

The flow methods (IMCF, Bray) achieve the square root by:

- Starting from minimal surface ($H = 0$)
- Using monotonicity of Hawking mass along the flow
- The square root emerges from the Geroch formula

Key insight: The square root comes from the **minimal surface condition** $H = 0$, which trapped surfaces violate!

9 A Potential New Approach

9.1 The Modified Hawking Mass

Definition 9.1 (Trapping-Corrected Hawking Mass)**.**

$$\tilde{m}_H(\Sigma) = \sqrt{\frac{A}{16\pi}}\left(1 - \frac{1}{16\pi}\int_\Sigma \theta^+\theta^- dA\right) \tag{27}$$

Lemma 9.2. *For trapped surfaces: $\theta^+\theta^- > 0$, so $\tilde{m}_H < \sqrt{A/16\pi}$.*
For MOTS: $\theta^+ = 0$, so $\tilde{m}_H = \sqrt{A/16\pi}$.

Proposition 9.3 (Evolution under θ^+-flow)**.**

$$\frac{d\tilde{m}_H}{dt} = \text{(complicated expression involving } R, k, A, \nabla\theta^{\pm}\text{)} \tag{28}$$

No clear sign, so no monotonicity.

9.2 The θ-Capacity

Definition 9.4.

$$Cap_\theta(\Sigma) = \inf_{u|_\Sigma=1} \int_M w_\theta |Du|^2 dV \tag{29}$$

where $w_\theta = e^{\int \theta^+/H}$ is a weight adapted to trapping.

Problem: $H < 0$ for trapped surfaces, so $\theta^+/H > 0$, making $w_\theta > 1$, which increases capacity, not helps.

10 Honest Conclusion

10.1 Summary of Attempts

Method	Gives	Needed
Capacity	$M \geq \mathrm{Cap}^2/16\pi$	$M \geq A^{1/2}$
Hawking mass	$M \geq m_H < A^{1/2}$	$M \geq A^{1/2}$
Jang + IMCF	$M \geq A(\mathrm{MOTS})^{1/2}$	$M \geq A(\mathrm{trapped})^{1/2}$
Direct elliptic	No clean inequality	—

10.2 The Real Obstruction

The square root in the Penrose inequality:

$$M \geq \sqrt{\frac{A}{16\pi}} \tag{30}$$

arises from:

1. The Geroch monotonicity formula for Hawking mass
2. Which requires $H = 0$ (minimal surface) as the starting point
3. Trapped surfaces have $H < 0$, breaking the argument

To solve 1973: We need either:

- A new monotone quantity that works for $H < 0$
- A way to "correct" the Hawking mass for trapping
- A completely different approach (spinors? optimal transport?)

10.3 Current State

The spacetime Penrose inequality for arbitrary trapped surfaces remains **OPEN**.

All known methods give the inequality for MOTS only:

$$M_{\mathrm{ADM}} \geq \sqrt{\frac{A(\Sigma^*)}{16\pi}} \tag{31}$$

where Σ^* is the outermost MOTS.

The gap $A(\Sigma^*) \overset{?}{\geq} A(\Sigma_0)$ is **not provable** in general.

References

[1] H. Bray, J. Differ. Geom. **59**, 177 (2001).

[2] G. Huisken, T. Ilmanen, J. Differ. Geom. **59**, 353 (2001).