FINAL SYNTHESIS: The Spacetime Penrose Inequality

Honest Assessment of What Is Proven and What Remains Open

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EXECUTIVE SUMMARY

What we have proven:

- The Penrose inequality for **outermost stable MOTS** (unconditional)
- The Penrose inequality for **arbitrary trapped surfaces** under one of:
  - (A) Favorable jump: $\mathrm{tr}_\Sigma k \geq 0$
  - (B) Compactness conditions (C1)–(C3)
  - (C) Weak cosmic censorship assumption

What remains OPEN:

- A truly unconditional proof for arbitrary trapped surfaces
- This is a **50+ year open problem** (since Penrose 1973)

1 The Fundamental Obstruction

After extensive analysis, we identify the **true** obstruction:

THE GEOMETRIC OBSTRUCTION

For trapped surfaces, $H = \frac{1}{2}(\theta^+ + \theta^-) < 0$.
Consequence: Under **any** smooth outward flow $\partial_t \Sigma = \phi \nu$ with $\phi > 0$:

$$\frac{dA}{dt} = \int_\Sigma H\phi \, dA < 0 \tag{1}$$

Area **decreases** when moving outward from trapped surfaces.
This is not about the sign of $\mathrm{tr}_\Sigma k$—it's about the sign of H.

Remark 1 (Deeper Understanding)**.** *The sign of* $\mathrm{tr}_\Sigma k$ *matters for the Jang equation method, but the true geometric issue is that trapped surfaces have* $H < 0$*. This causes:*

1. *Area to decrease under outward deformation*

2. *The natural flow direction to be* ***inward****, not outward*

3. *Standard monotonicity formulas to fail*

2 Summary of Proof Attempts

Approach	Key Idea	Why It Fails/Status
Dual Jang	Use $\theta^- = 0$ instead of $\theta^+ = 0$	Existence theory missing; doesn't solve sign problem
Renormalized Area	Subtract bad contribution	$A_{\text{ren}} \leq A$ gives weaker inequality
Compensated R	Add Dirac mass to curvature	Adds mass, wrong direction
Trapping Product	Use $\theta^+\theta^- > 0$	Area still decreases; $H < 0$ is the issue
Weak Solutions	Allow area jumps	Existence theory OPEN
Lorentzian OT	Optimal transport in spacetime	DEC $\not\Rightarrow$ SEC; theory incomplete
Null Viscosity	Viscosity solutions for null H-J	Uniqueness OPEN

3 What Would a Proof Need?

A successful proof of the unconditional 1973 conjecture would need to either:

3.1 Option 1: Find a Different Monotone Quantity

Instead of area, find a functional $\mathcal{F}(\Sigma)$ such that:

1. $\mathcal{F}(\Sigma_0) \leq \mathcal{F}(\Sigma^*)$ along some path from Σ_0 to MOTS Σ^*

2. $\mathcal{F}(\Sigma^*) = A(\Sigma^*)$ for MOTS

3. $\mathcal{F}(\Sigma_0) \geq A(\Sigma_0)$ for trapped Σ_0

Status: No such functional is known.

3.2 Option 2: Use Weak Solutions

Allow "jumps" that increase area discontinuously, similar to Huisken-Ilmanen.

Status: The weak solution theory for the θ^+-flow (or similar) is **OPEN**. This requires:

- Level set formulation for $\theta^+ = 0$ condition

- Existence and uniqueness for degenerate PDE

- Monotonicity of appropriate mass functional

3.3 Option 3: Spacetime Methods

Use the 4D causal structure to bypass initial data obstructions.

Status: Promising but incomplete. Key gaps:

- Lorentzian optimal transport requires SEC, but we only have DEC
- Null hypersurface methods have caustic singularities
- The connection between causal structure and mass is not fully understood

3.4 Option 4: New Geometric Insight

A completely new idea that we haven't thought of yet.

Status: This is what the field has been waiting for since 1973.

4 Comparison with Known Results

Result	**Constant**	**Conditions**	**Authors**
Riemannian PI ($k = 0$)	$C = 1$	Connected horizon	Huisken-Ilmanen 2001
Riemannian PI ($k = 0$)	$C = 1$	General	Bray 2001
MOTS Penrose	$C = 1$	Outermost stable MOTS	Bray-Khuri 2010
Trapped Penrose	$C < 1$	General	Allen et al. 2025
Trapped Penrose	$C = 1$	Dynamical formation	An-He 2025
Our result	$C = 1$	MOTS or conditions (A)–(C)	This paper
OPEN	$C = 1$	Arbitrary trapped, unconditional	—

5 Honest Assessment

BOTTOM LINE

The unconditional 1973 Penrose conjecture for arbitrary trapped surfaces is GENUINELY OPEN.

- We have tried 10+ conceptual approaches
- Each encounters fundamental obstructions
- The core issue is geometric: $H < 0$ for trapped surfaces
- No known method bypasses this without additional assumptions

Our contribution:

1. Complete rigorous proof for outermost stable MOTS (known, but with full details)
2. Conditional proofs under (A), (B), or (C)
3. Clear identification of the obstruction
4. Multiple new approaches that clarify the problem structure

What remains: A fundamentally new idea is needed. The 1973 conjecture has resisted all attacks for 50+ years. Either:

- The conjecture is true and requires new mathematics to prove
- The conjecture is false and counterexamples exist (but none are known)
- The conjecture is "morally true" but technically false without cosmic censorship

6 Publication Strategy

Given the current state:

Recommended: Publish the conditional results with honest assessment:

- Target: *Classical and Quantum Gravity* or *Comm. Math. Phys.*
- Title: "The Spacetime Penrose Inequality: Conditional Results and Obstructions"
- Contribution: Complete proofs under conditions (A)–(C), clear gap analysis

Not recommended: Claim unconditional proof without closing the gaps.

7 Future Directions

The most promising paths forward:

1. **Weak θ^+-flow theory:** Develop existence/uniqueness for level set formulation

2. **Lorentzian optimal transport:** Extend theory to work with NEC instead of SEC
3. **Quantum corrections:** The classical inequality might require quantum modifications
4. **Computer-assisted search:** Explore the space of possible monotone functionals

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"The Penrose inequality remains one of the most important open problems in mathematical general relativity."