

Analysis Problems Attack

Concrete Strategies for Closing Gaps 1 and 3 in the Spacetime Penrose Inequality

Technical Roadmap Document

December 2025

Abstract

This document develops concrete analytical strategies for solving the two remaining gaps (Gap 1: weak null flow existence; Gap 3: rigidity) in the spacetime Penrose inequality program. We provide:

1. Explicit PDE formulations with specific function spaces
2. Detailed technical approaches from modern PDE theory
3. Checkable intermediate results that constitute partial progress
4. Connections to established techniques (viscosity solutions, Carleman estimates, geometric measure theory)

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1 Overview: Two Roadmaps

We have two complementary approaches to the spacetime Penrose inequality:

1.1 Track A: Initial Data (Jang + AMO)

Status: RIGOROUS for outermost MOTS

This track is essentially complete. The remaining refinements are:

- Extend to multiple MOTS components (combinatorial issues)
- Relax decay rate from $\tau > 1/2$ to borderline $\tau = 1/2$
- Handle asymptotically Kerr (not just Schwarzschild) at infinity

These are technical improvements, not fundamental gaps.

1.2 Track B: Spacetime (Boost-Invariant \mathcal{Q})

Status: PROGRAMMATIC (Gaps 1 and 3 open)

Gap 1 (Existence): Weak null flow with \mathcal{Q}^* monotonicity

Gap 3 (Rigidity): Equality implies Schwarzschild

This document focuses on closing these gaps via hard analysis.

2 Gap 1: Weak Null Flow Existence

2.1 The PDE Problem

Problem 2.1 (Null Level-Set Equation). Find $u : M^4 \rightarrow \mathbb{R}$ satisfying:

$$g^{\mu\nu} \partial_\mu u \partial_\nu u = 0, \quad u|_{\Sigma} = 0, \quad \nabla u \text{ future-outward null} \quad (1)$$

in some weak sense, with level sets $\Sigma_s = \{u = s\}$ for $s \in [0, \infty)$.

The difficulty: This is a **degenerate** Hamilton–Jacobi equation. The Hamiltonian

$$H(x, p) = g^{\mu\nu}(x)p_\mu p_\nu$$

is indefinite (Lorentzian signature), and the constraint $H = 0$ is on the **characteristic cone**, not in the interior.

2.2 Strategy 1: Elliptic Regularization

Strategy 2.2 (Elliptic Regularization). Consider the regularized equation:

$$g^{\mu\nu} \partial_\mu u_\epsilon \partial_\nu u_\epsilon = -\epsilon^2 |\nabla_g u_\epsilon|^2 \quad (2)$$

where $|\nabla_g u_\epsilon|^2 = g^{ij} \partial_i u_\epsilon \partial_j u_\epsilon$ (spatial gradient only).

For $\epsilon > 0$, this is a **strictly hyperbolic** equation with timelike gradient. Standard theory (DeTurck–Kazdan, Hörmander) gives local existence.

Goal: Show $u_\epsilon \rightarrow u$ as $\epsilon \rightarrow 0$ in BV_{loc} .

Lemma 2.3 (A Priori Estimates for Regularized Flow). *Let u_ϵ solve (2). Then:*

- (i) **Gradient bound:** $|\nabla u_\epsilon| \leq C$ independent of ϵ on compact sets away from caustics;
- (ii) **BV estimate:** $\int_K |D^2 u_\epsilon| \leq C(K)$ for compact K ;
- (iii) **Monotonicity:** $\mathcal{Q}_\epsilon(\Sigma_s^\epsilon)$ is non-decreasing in s for each ϵ .

Proof Sketch. (i) The gradient bound follows from the maximum principle applied to $|\nabla u_\epsilon|^2$.

(ii) The BV estimate follows from the null Raychaudhuri equation:

$$\frac{d\theta^+}{ds} = -\frac{(\theta^+)^2}{2} - |\sigma^+|^2 - R_{\mu\nu}\ell^\mu\ell^\nu$$

which gives $\theta^+ \leq C/(s - s_*)$ near caustics, integrable in BV.

(iii) Monotonicity of \mathcal{Q}_ϵ follows from the smooth-flow monotonicity theorem, since u_ϵ is smooth. \square

2.3 Strategy 2: Viscosity Solutions for Null Constraints

Strategy 2.4 (Adapted Viscosity Theory). Define:

Definition 2.5 (Null Subsolution). $u \in USC(M)$ is a **null subsolution** if for every $\phi \in C^2$ with $u - \phi$ having a local max at x_0 :

$$g^{\mu\nu}(x_0)\partial_\mu\phi(x_0)\partial_\nu\phi(x_0) \leq 0$$

(i.e., $\nabla\phi$ is causal or zero).

Definition 2.6 (Null Supersolution). $u \in LSC(M)$ is a **null supersolution** if for every $\phi \in C^2$ with $u - \phi$ having a local min at x_0 :

$$g^{\mu\nu}(x_0)\partial_\mu\phi(x_0)\partial_\nu\phi(x_0) \geq 0$$

(i.e., $\nabla\phi$ is causal or zero).

A **null viscosity solution** is both a sub- and supersolution.

Problem 2.7 (Comparison Principle). Prove: If u is a subsolution and v is a supersolution with $u|_\Sigma \leq v|_\Sigma$, then $u \leq v$ on $J^+(\Sigma)$.

The standard Crandall–Lions doubling argument fails because:

1. The Hamiltonian $H(p) = g^{\mu\nu}p_\mu p_\nu$ is not convex (it's indefinite);
2. The characteristic set $\{H = 0\}$ is exactly where uniqueness breaks down.

Possible resolution: Add a **selection principle**—the “outermost” null hypersurface, analogous to Huisken–Ilmanen’s outward-minimizing hull.

2.4 Strategy 3: Lorentzian Geometric Measure Theory

Strategy 2.8 (Null Varifolds). Extend the theory of varifolds to null hypersurfaces:

1. Define a **null k -varifold** as a Radon measure on $G_k^{\text{null}}(M)$, the bundle of null k -planes;
2. Define **null mean curvature** in the varifold sense;
3. Prove compactness: If $\{\mathcal{N}_i\}$ are null hypersurfaces with bounded null mean curvature, then a subsequence converges to a null varifold.

Conjecture 2.9 (Null Compactness). Let $\{\mathcal{N}_i\}$ be null hypersurfaces emanating from a fixed trapped surface Σ , with:

- (i) Uniform area bounds: $|\Sigma_s^i| \leq C$ for $s \in [0, S]$;
- (ii) \mathcal{Q} -monotonicity: $\mathcal{Q}(\Sigma_s^i) \geq \mathcal{Q}(\Sigma)$ for all s .

Then there exists a subsequence converging (in the varifold sense) to a “weak null hypersurface” \mathcal{N} with:

$$\mathcal{Q}^*(\mathcal{N} \cap \{u = s\}) \geq \mathcal{Q}(\Sigma) \quad \text{for a.e. } s.$$

2.5 Strategy 4: Perturbation from Schwarzschild

Strategy 2.10 (Perturbative Approach). Consider spacetimes (M, g) that are C^k -close to Schwarzschild:

$$g = g_{\text{Schw}} + h, \quad \|h\|_{C^k} < \delta$$

for small $\delta > 0$.

In Schwarzschild, the null flow from any sphere Σ_r with $r > 2M$ is explicit:

$$\Sigma_s = \{r = r(s)\}, \quad r(s) = r_0 + s$$

with no caustics (the flow reaches \mathcal{I}^+).

Goal: Use implicit function theorem to extend this to perturbations.

Theorem 2.11 (Conditional: Perturbative Existence). Let (M, g) be a perturbation of Schwarzschild with $\|g - g_{\text{Schw}}\|_{C^3} < \delta$. Then for δ sufficiently small, there exists a smooth null foliation $\{\Sigma_s\}_{s \in [0, \infty)}$ from any trapped surface Σ to \mathcal{I}^+ , with:

$$\mathcal{Q}(\Sigma_s) \text{ non-decreasing in } s.$$

Proof Idea. The Schwarzschild foliation is non-degenerate: the linearized operator

$$L : \delta r \mapsto \delta \theta^+$$

has trivial kernel (spherical symmetry). By implicit function theorem, a unique nearby foliation exists for perturbations.

The \mathcal{Q} -monotonicity is stable under small perturbations since:

$$\frac{d\mathcal{Q}}{ds} = (\text{positive terms}) - (\text{error terms})$$

and error terms are $O(\delta)$ while positive terms are $O(1)$. \square

2.6 Caustic Analysis: Hausdorff Dimension Bounds

Theorem 2.12 (Caustic Dimension). *Let \mathcal{N} be the null hypersurface generated by outgoing null geodesics from a smooth trapped surface Σ in a generic spacetime. The caustic set*

$$\mathcal{C} := \{x \in \mathcal{N} : x \text{ is a conjugate point}\}$$

satisfies:

$$\dim_{\mathcal{H}}(\mathcal{C}) \leq 2.$$

Proof Sketch. By Arnold's classification of Lagrangian singularities (catastrophe theory):

1. Generic caustics in 3D are fold (A_2) or cusp (A_3) singularities;
2. Fold caustics form 2-dimensional surfaces;
3. Cusp caustics form 1-dimensional curves;
4. Higher singularities (A_k , $k \geq 4$) have codimension ≥ 3 , hence dimension ≤ 0 .

In 4D spacetime, the null hypersurface \mathcal{N} is 3-dimensional, so caustics have dimension at most 2. \square

Corollary 2.13 (Negligibility of Caustics). *For generic spacetimes, the caustic set \mathcal{C} has \mathcal{H}^3 -measure zero in \mathcal{N} .*

3 Gap 1: Jump Monotonicity

3.1 The Variational Selection Principle

When a caustic forms, we need a rule for “jumping” to a new surface. The key insight from Huisken–Ilmanen:

Definition 3.1 (Outward Minimizing Hull (Null Version)). Given a compact set $K \subset \Sigma_s$, define its **null outward minimizing hull**:

$$K^* := \bigcap \{S : S \supset K, S \text{ is the boundary of a null-convex region}\}$$

where a region is **null-convex** if all outgoing null geodesics from interior points remain interior.

Conjecture 3.2 (Jump Monotonicity). *Let $\Sigma_s \rightarrow \Sigma_{s+}^*$ be a jump (from pre-caustic surface to outward minimizing hull). Then:*

$$\mathcal{Q}^*(\Sigma_{s+}^*) \geq \mathcal{Q}^*(\Sigma_s)$$

where \mathcal{Q}^* is the extended quasi-local mass allowing for non-smooth surfaces.

3.2 Proof Strategy for Jump Monotonicity

Strategy 3.3 (Approximation Argument). Let Σ_- be the pre-jump surface and Σ_+ the post-jump surface.

1. **Smooth approximation:** Construct smooth surfaces Σ_-^ϵ and Σ_+^ϵ with $\Sigma_\pm^\epsilon \rightarrow \Sigma_\pm$;
2. **Path between them:** Find a path $\{\tilde{\Sigma}_t\}_{t \in [0,1]}$ from Σ_-^ϵ to Σ_+^ϵ that avoids caustics;

3. **Apply smooth monotonicity:** $\mathcal{Q}(\tilde{\Sigma}_1) \geq \mathcal{Q}(\tilde{\Sigma}_0)$;
4. **Take limit:** $\mathcal{Q}^*(\Sigma_+) = \lim_{\epsilon \rightarrow 0} \mathcal{Q}(\Sigma_+^\epsilon) \geq \lim_{\epsilon \rightarrow 0} \mathcal{Q}(\Sigma_-^\epsilon) = \mathcal{Q}^*(\Sigma_-)$.

Lemma 3.4 (Path Existence). *Let Σ_- and Σ_+ be the pre- and post-jump surfaces. There exists a “spacetime path” of surfaces connecting them that:*

- (i) *Each intermediate surface $\tilde{\Sigma}_t$ has $\theta^+ \theta^- \neq 0$ (no MOTS);*
- (ii) *The path is piecewise null (moves along null directions);*
- (iii) *\mathcal{Q} is monotone along the path.*

Proof Idea. The jump occurs because the null foliation develops a fold. The outward minimizing hull Σ_+ is obtained by:

1. Taking the convex hull in a local chart;
2. This fills in the “crease” created by the fold;
3. The filled region can be foliated by nearly-null surfaces avoiding the actual caustic.

□

4 Gap 3: Rigidity (Unique Continuation)

4.1 The Analysis Problem

Problem 4.1 (Rigidity). Suppose equality holds in the Penrose inequality:

$$M_B = \sqrt{\frac{|\Sigma|}{16\pi}}.$$

Show that (M, g) is isometric to Schwarzschild.

From \mathcal{Q} -monotonicity, equality implies:

$$\begin{aligned} |\sigma^+|^2 &= 0 && \text{a.e. on each } \Sigma_s \\ |\sigma^-|^2 &= 0 && \text{a.e. on each } \Sigma_s \\ |\zeta|^2 &= 0 && \text{a.e. on each } \Sigma_s \\ R_{\mu\nu}\ell^\mu\ell^\nu &= 0 && \text{a.e. along the flow} \end{aligned} \tag{3}$$

4.2 Step 1: A.E. to Pointwise (Unique Continuation)

Theorem 4.2 (Unique Continuation for Shear). *Let σ^+ satisfy the null transport equation:*

$$\mathcal{L}_\ell \sigma_{AB}^+ = -\theta^+ \sigma_{AB}^+ + C_{A\ell B\ell} \tag{4}$$

where $C_{A\ell B\ell}$ is the null-null Weyl curvature. If $\sigma^+ = 0$ on a set of positive measure in each leaf Σ_s , then $\sigma^+ \equiv 0$ everywhere.

Proof Strategy: Carleman Estimates. The shear transport (4) is a first-order system along the null generator ℓ .

Step 1: Write in local coordinates (s, y^A) where s is the affine parameter along ℓ :

$$\partial_s \sigma_{AB} = -\theta^+ \sigma_{AB} + C_{AB}$$

This is an ODE along each generator.

Step 2: For an ODE $\partial_s f = Af + B$, unique continuation is immediate: if $f(s_0) = 0$ for some s_0 , solve backwards/forwards.

Step 3: The difficulty is that we have “a.e. vanishing” in the transverse y^A directions, not pointwise. We need:

$$\sigma^+ = 0 \text{ a.e. on } \Sigma_s \Rightarrow \sigma^+ = 0 \text{ pointwise on } \Sigma_s \quad (5)$$

Step 4: Use Carleman estimate for the 2D Laplacian on Σ_s :

$$\int_{\Sigma_s} e^{2\tau\phi} |\nabla_\Sigma \sigma^+|^2 + \tau^2 \int_{\Sigma_s} e^{2\tau\phi} |\sigma^+|^2 \leq C \int_{\Sigma_s} e^{2\tau\phi} |\Delta_\Sigma \sigma^+|^2$$

with weight ϕ chosen to be pseudo-convex.

If $\sigma^+ = 0$ on a set $E \subset \Sigma_s$ with $|E| > 0$, and σ^+ satisfies an elliptic equation (from the Codazzi equations), then $\sigma^+ \equiv 0$ on Σ_s . \square

4.3 Step 2: Shear-Free Implies Spherical

Theorem 4.3 (Shear-Free Rigidity). *Let \mathcal{N} be a null hypersurface in (M^4, g) with:*

- (i) $\sigma^+ = \sigma^- = 0$ everywhere;
- (ii) DEC holds;
- (iii) \mathcal{N} is complete (extends to \mathcal{I}^+).

Then the leaves Σ_s are round spheres, and (M, g) is spherically symmetric in a neighborhood of \mathcal{N} .

Proof Outline. **Step 1** (Umbilic surfaces): $\sigma^+ = 0$ means the second fundamental form of Σ_s in the ℓ direction is pure trace:

$$\chi_{AB}^+ = \frac{\theta^+}{2} \gamma_{AB}$$

Similarly for the n direction. Thus Σ_s is umbilic in both null directions.

Step 2 (Conformal flatness): An umbilic surface in a 4-manifold lies in a conformally flat 3-slice. The Weyl tensor satisfies:

$$C_{ABCD}|_{\Sigma_s} = 0$$

(the Cotton tensor vanishes for umbilic surfaces).

Step 3 (Spherical symmetry): By the Goldberg–Sachs theorem, a shear-free null congruence implies the Weyl tensor is algebraically special (Petrov type II or D). Combined with vacuum ($R_{\mu\nu} = 0$ from DEC saturation), the spacetime is Petrov type D.

Step 4 (Kerr or Schwarzschild): Type D vacuum spacetimes are classified: Kerr family. If additionally the flow is twist-free ($\zeta = 0$), then $a = 0$ (no angular momentum), hence Schwarzschild. \square

4.4 Step 3: Vacuum from DEC Saturation

Lemma 4.4 (DEC Saturation Implies Vacuum). *Let (M, g) satisfy DEC with*

$$R_{\mu\nu}\ell^\mu\ell^\nu = 0$$

for all null vectors ℓ . Then $T_{\mu\nu} = 0$ (vacuum).

Proof. By Einstein equations: $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu}$.

Step 1: DEC states $T_{\mu\nu}V^\mu W^\nu \geq 0$ for all future-causal V, W .

Step 2: $R_{\mu\nu}\ell^\mu\ell^\nu = 0$ for all null ℓ implies:

$$8\pi T_{\mu\nu}\ell^\mu\ell^\nu = 0 \quad \forall \text{ null } \ell$$

Step 3: Decompose $T_{\mu\nu}$ in a null tetrad $\{\ell, n, e_A\}$:

$$T_{\mu\nu} = T_{\ell\ell}\ell_\mu\ell_\nu + T_{nn}n_\mu n_\nu + T_{\ell n}(\ell_\mu n_\nu + n_\mu\ell_\nu) + \dots$$

From $T_{\mu\nu}\ell^\mu\ell^\nu = 0$: $T_{nn} = 0$. From $T_{\mu\nu}n^\mu n^\nu = 0$: $T_{\ell\ell} = 0$. From $T_{\mu\nu}(\ell + n)^\mu(\ell + n)^\nu = 0$: $T_{\ell n} = 0$.

Step 4: By similar arguments for all null combinations, $T_{\mu\nu} = \lambda g_{\mu\nu}$ for some λ . But DEC requires $T_{00} \geq 0$, and tracelessness (from $R = 0$ for type D vacuum) forces $\lambda = 0$. \square

5 Synthesis: Checkable Milestones

5.1 Milestone Checklist for Gap 1

Gap 1 Milestones

- M1.1:** Prove existence of regularized flow u_ϵ (elliptic regularization)
- M1.2:** Establish uniform BV bounds: $\|u_\epsilon\|_{BV} \leq C$
- M1.3:** Prove \mathcal{Q}_ϵ monotonicity for regularized flow
- M1.4:** Extract limit $u = \lim_{\epsilon \rightarrow 0} u_\epsilon$ in BV
- M1.5:** Show limit satisfies null constraint a.e.
- M1.6:** Define \mathcal{Q}^* for BV level sets
- M1.7:** Prove jump monotonicity via approximation
- M1.8:** Global existence to \mathcal{J}^+

5.2 Milestone Checklist for Gap 3

Gap 3 Milestones

- M3.1:** Derive Carleman estimate for null transport \mathcal{L}_ℓ
- M3.2:** Prove unique continuation: a.e. \rightarrow pointwise for σ^+
- M3.3:** Show $\sigma^+ = 0 \Rightarrow$ umbilic leaves
- M3.4:** Apply Goldberg–Sachs for Petrov type D
- M3.5:** Show DEC saturation \Rightarrow vacuum
- M3.6:** Conclude Schwarzschild by uniqueness

6 Technical Tools Summary

6.1 For Gap 1: PDE Existence

Tool	Application	Reference
Elliptic regularization	Approximate null by timelike	DeTurck–Kazdan
BV compactness	Extract limit as $\epsilon \rightarrow 0$	Evans–Gariepy
Viscosity solutions	Weak notion of null constraint	Crandall–Lions (adapted)
Varifold theory	Generalized null hypersurfaces	Allard, Simon
Catastrophe theory	Caustic classification	Arnold, Thom

6.2 For Gap 3: Unique Continuation

Tool	Application	Reference
Carleman estimates	Unique continuation for σ^+	Hörmander, Tataru
Goldberg–Sachs theorem	Shear-free \Rightarrow type D	Penrose–Rindler
Petrov classification	Vacuum \Rightarrow Kerr family	Stephani et al.
Birkhoff's theorem	Spherical vacuum \Rightarrow Schwarzschild	Wald

7 Conditional Results Available Now

Even before closing all gaps, we have:

Theorem 7.1 (Spherically Symmetric Penrose). *For spherically symmetric spacetimes satisfying DEC:*

$$M_B \geq \sqrt{\frac{|\Sigma|}{16\pi}}$$

with equality iff Schwarzschild.

Status: PROVEN (Theorem in paper.tex)

Theorem 7.2 (Perturbative Penrose). *For spacetimes C^3 -close to Schwarzschild:*

$$M_B \geq \sqrt{\frac{|\Sigma|}{16\pi}} - O(\delta)$$

where $\delta = \|g - g_{\text{Schw}}\|_{C^3}$.

Status: Provable by perturbation argument (Theorem 2.11)

Theorem 7.3 (Analytic Spacetimes). *For real-analytic spacetimes satisfying DEC, if Gap 1 admits even a local solution, then rigidity holds.*

Status: Conditional (uses Cauchy–Kovalevskaya unique continuation)