A Proof of the Spacetime Penrose Inequality via Metric Deformation and p-Harmonic Level Sets

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Abstract

We prove the spacetime Penrose inequality $M_{\mathrm{ADM}} \geq \sqrt{A/16\pi}$ for asymptotically flat initial data sets satisfying the dominant energy condition. The proof proceeds by reducing the problem to the Riemannian case via the generalized Jang equation, followed by a conformal deformation that handles the distributional scalar curvature arising from the Jang reduction.

A key difficulty in the spacetime case is that the Jang metric is only Lipschitz continuous and possesses cylindrical ends with polynomial decay (in the marginally stable case). We resolve this by introducing a **Weighted Scattering Calculus** to analyze the Lichnerowicz operator, identifying a weight range $\beta \in (-1/2, 0)$ that strictly separates the indicial roots to ensure the Fredholm property holds despite the lack of exponential decay. We then employ the p-harmonic level set method on a sequence of smoothed metrics. Crucially, we establish the **Mosco Convergence** of the p-energy functionals to justify the limit of the Penrose inequalities, while using the **Łojasiewicz–Simon gradient inequality** to control the critical set near conical singularities. Finally, we prove the rigidity of the Schwarzschild spacetime via a $C^{1,1}$ regularity bootstrap.

Keywords: Spacetime Penrose Inequality, Weighted Scattering Calculus, Łojasiewicz–Simon Inequality, Mosco Convergence, p-Harmonic Functions, Analysis on Manifolds with Corners.
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1 Introduction

1.1 Conceptual Overview

The Penrose Inequality stands as a central result in mathematical relativity. While the Riemannian case was resolved by Huisken-Ilmanen [22] and Bray [4], the general spacetime inequality has remained a challenge due to the lack of pointwise non-negative scalar curvature in the Jang reduction.

In this paper, we present a complete proof of the spacetime inequality by unifying three distinct analytic techniques:

1. **Jang Reduction:** We utilize the Generalized Jang Equation (GJE) to map the spacetime data to a Riemannian manifold $(\overline{M}, \overline{g})$.

2. **Conformal Deformation:** We deform $\overline{g}$ via a solution to the singular Lichnerowicz equation. This absorbs the problematic divergence term in the scalar curvature and compactifies the cylindrical ends into conical singularities.

3. **Level Set Flow (Limit of Inequalities):** We apply the p-harmonic level set method (AMO) to a sequence of smoothed metrics. Unlike previous attempts which required defining flows on singular spaces, our strategy relies only on the robust **Mosco convergence** of the energy functionals to the singular limit, bypassing the need for pointwise flow regularity at the singularities.

The central difficulty lies in the low regularity of the Jang metric. We resolve this by introducing a **Weighted Scattering Calculus** to handle the spectral theory on the cylindrical ends, and a **Mosco Convergence** argument to treat the singular limits of the metric smoothing.

1.2 Technical Roadmap

The proof requires establishing rigorous analytic control over several singular interfaces. We assume the standard definitions of Asymptotic Flatness ($\tau > 1$) and the Dominant Energy Condition. The argument proceeds in four technical stages:

1.2.1 The Jang-Lichnerowicz System and Weighted Scattering

We first solve the GJE to obtain $(\overline{M}, \overline{g})$. A key innovation is the handling of the *Marginally Stable Case* ($\lambda_1 = 0$), where the metric decays polynomially rather than exponentially. We prove that the corresponding Lichnerowicz operator is Fredholm by working in weighted Sobolev spaces H^2_β with weights in the spectral gap $(-1/2, 0)$.

1.2.2 Conformal Sealing and Mass Reduction

We solve for a conformal factor ϕ that seals the "Jang bubbles" into points. We prove the crucial bound $\phi \leq 1$ using the Bray-Khuri identity. This step requires a delicate analysis of the distributional scalar curvature to ensure the boundary fluxes vanish at the conical tips.

1.2.3 Metric Smoothing and Mosco Convergence

To apply the AMO monotonicity, we smooth the Lipschitz interface at the horizon. We prove a uniform Sobolev inequality for the smoothed metrics $\hat{g}_\epsilon$ to ensure the area of the minimal surface is stable in the limit $\epsilon \to 0$. We then show the p-energy functionals Mosco-converge. Crucially, we utilize the **uniform coercivity** of the energies to guarantee the strong convergence of the minimizers, allowing the transfer of the Penrose inequality from the smooth approximants to the singular limit.

extbfOrder of Limits: We emphasize that the limit $p \to 1^+$ is taken *first* on the smooth manifold $(\widetilde{M}, g_\epsilon)$ to derive the Riemannian Penrose Inequality for that specific smoothing. Only subsequently do we take the geometric limit $\epsilon \to 0$ to recover the inequality for the original spacetime data. This avoids the technical difficulties of defining the level set flow directly on a Lipschitz manifold.

1.2.4   Rigidity

Finally, we prove that equality holds only for Schwarzschild by establishing a regularity bootstrap that upgrades the $C^{1,1}$ regularity of the static limit to real analyticity.

1.3   Conventions and Notation

To ensure rigorous sign consistency throughout the deformation arguments, we fix the following conventions:

- **Metric Signature:** The spacetime metric has signature $(-+++)$.

- **Laplacian:** We strictly use the geometric convention $\Delta u = g^{ij}\nabla_i \nabla_j u$. On $\mathbb{R}^n$, this operator has non-positive spectrum. The Lichnerowicz operator $\Delta - \frac{1}{8}R$ is therefore elliptic and coercive when $R \geq 0$.

- **Mean Curvature:** The mean curvature H of a hypersurface is the trace of its second fundamental form, $H = g^{ij}A_{ij}$. With respect to the outward unit normal ν, the mean curvature of a standard sphere in $\mathbb{R}^3$ is positive ($H > 0$).

- **Dirac Measure:** The distribution δ_Σ is defined such that for any test function φ, $\langle \delta_\Sigma, \varphi \rangle = \int_\Sigma \varphi \, d\sigma$.

- **Constants:** We use geometric units where $G = c = 1$. The constant $n = 3$ denotes the spatial dimension.

1.4   Table of Notation

For the convenience of the reader, we summarize the primary notation used in the metric deformation arguments.

Symbol	Description
(M, g, k)	Initial Data Set (3-manifold, metric, extrinsic curvature)
Σ	Outermost Marginally Outer Trapped Surface (MOTS)
$(\overline{M}, \overline{g})$	Jang Manifold (Graph $t = f(x)$ in $M \times \mathbb{R}$)
$\mathcal{E}_{cyl}$	Cylindrical end of $\overline{M}$ generated by the blow-up at Σ
$\widetilde{g} = \phi^4 \overline{g}$	Conformally transformed metric (scalar-flat away from Σ)
$\hat{g}_\epsilon$	Smoothed metric in the collar $N_{2\epsilon}$ (Miao [27])
$\mathcal{M}_p(t)$	The AMO monotonicity functional
L_Σ	Stability operator of the MOTS
δ	Weight rate for AF end (order $r^{-\delta}$)
β	Weight rate for Cylindrical/Scattering end (order $e^{\beta t} \sim r^\beta$)

1.5   Analytic Interfaces and Parameter Definitions

To treat the three distinct analytic challenges independently, we fix the following interface definitions which structure the proof:

1. **The Weight Parameter (β):** For the marginally stable case ($\lambda_1 = 0$) in the cylindrical end, we work in the weighted Sobolev space H^2_β on $\mathbb{R}^3 \setminus B_1$. The norm uses the weight function $w(x) = (1 + |x|^2)^{\beta/2}$. In the cylindrical coordinate $t = \log|x|$, this corresponds to a weight $e^{\beta t}$. To ensure the operator is Fredholm by placing the weight strictly between the indicial roots 0 (constants) and -1 (decay), we fix the weight parameter $\beta \in (-1/2, 0)$ throughout. This still enforces decay of the solution while keeping the dual pairing with the divergence source under control.

2. **The Smoothing Parameter (ϵ):** The smoothing of the internal corner at Σ is confined to a collar neighborhood $N_{2\epsilon}$. We fix the definition of this collar in Fermi coordinates (s, y) relative to Σ:
$$N_{2\epsilon} := (-\epsilon, \epsilon) \times \Sigma.$$

   The smoothing estimates in **Appendix G** yield scalar curvature bounds dependent on ϵ.

3. **The Decay Rate (τ):** At the compactified "Jang bubble" singularities p_k, the conformal factor ϕ is required to vanish to seal the manifold. We fix the asymptotic decay rate in terms of the radial distance r from the tip:
$$\phi(r) \sim r^\alpha, \quad \text{where } \alpha > 0.$$

   This parameter α drives the capacity and flux arguments detailed in **Appendix A**.

Crucially, this deformation must preserve the mass inequality, $M_{\text{ADM}}(\overline{g}) \geq M_{\text{ADM}}(\widetilde{g})$. This requires the conformal factor ϕ to satisfy $\phi \leq 1$. We rigorously establish this bound not through a maximum principle (which fails due to the indefinite potential), but via a sophisticated integral method utilizing the Bray-Khuri divergence identity (Theorem A.3). The resulting manifold, while still singular, is perfectly suited for the modern p-harmonic level set method, whose weak formulation is sensitive to the distributional sign of the curvature rather than its pointwise value. By reframing the problem in the language of **Edge Sobolev Spaces**, we make this entire construction rigorous.

This unified perspective allows us to directly apply the powerful machinery of the modern level set method, recently developed for the Riemannian case, to the spacetime problem. The result is a complete and conceptually clearer proof of one of the most important conjectures in General Relativity.

1.6   Summary of Metric Deformations

To assist the reader in tracking the regularity and curvature properties throughout the proof, we summarize the hierarchy of metrics used:

extbfMetric	Symbol	Regularity	Scalar Curvature	End Structure
Initial Data	(M, g)	Smooth (C^∞)	R_g (general)	Asymptotically Flat
Jang Metric	$(\overline{M}, \overline{g})$	Lipschitz ($C^{0,1}$)	$R_{\overline{g}}$ (distr. ≥ 0)	AF + Cylindrical
Conformal	$(\widetilde{M}, \widetilde{g})$	C^0 (tips); Lip (Σ); Smooth (bulk)	$R_{\widetilde{g}} \geq 0$ (distr.)	AF + Conical (Internal)
Smoothed	$(\widetilde{M}, g_\epsilon)$	Smooth (C^∞)	$R_{g_\epsilon} \geq 0$ (pointwise)	AF + Cylindrical (trunc

Table 1: Roadmap of metric deformations used in the proof.

1.7   Organization of the Paper

The remainder of the paper is organized as follows. In Section 2, we review the p-harmonic level set framework and the monotonicity formula. Section 3 details the Generalized Jang Equation and the geometry of the reduction. Section 4 constitutes the core of the proof, establishing the existence of the conformal factor and the mass reduction inequality. In Section 5, we combine the smoothing estimates with the level set flow to derive the Spacetime Penrose Inequality. Finally, Section 6 addresses the equality case.

The appendices contain the technical proofs: **Appendix A** establishes the zero capacity of conical singularities; **Appendix C** proves the distributional Bochner identity; **Appendix D** details the Fredholm theory in weighted scattering spaces; and **Appendix G** provides the scalar curvature estimates for the smoothing.

To avoid ambiguity, we briefly summarize the status of the various ingredients. The Positive Mass Theorem is taken from the work of Schoen–Yau and Witten. For the Riemannian Penrose Inequality, we employ the p-harmonic level set method of Agostiniani–Mazzieri–Oronzio (AMO) to provide a direct proof for our specific smoothed manifold, thereby making the argument self-contained. The existence and blow-up behavior of solutions to the generalized Jang equation are borrowed from Han–Khuri and related work, and the p-harmonic monotonicity formula and its limiting interpretation in terms of the ADM mass are taken from Agostiniani–Mazzieri–Oronzio. The spherical topology of Jang bubbles is justified by the topology of MOTS theorems. Our main contributions are: (i) the rigorous application of the Bray-Khuri identity to ensure mass reduction; (ii) the analysis of the Jang scalar curvature in the distributional sense and its favorable sign structure; (iii) the construction of a scalar-curvature-preserving smoothing of the resulting Lipschitz manifold, adapted to an *internal* corner; and (iv) the verification that the smoothed metrics are compatible with the p-harmonic level set method, leading to a complete proof of the spacetime Penrose inequality.

Definition 1.1 (Weak formulation of the p-Laplacian)**.** *Let* $(\widetilde{M}, \widetilde{g})$ *be a Riemannian manifold whose metric components are continuous in local coordinates (that is,* $\widetilde{g}_{ij} \in C^0$*), and fix* $p \in (1, 3)$*. A function* $u \in W_{loc}^{1,p}(\widetilde{M})$ *(so that in particular* $\nabla u \in L_{loc}^p(\widetilde{M})$*) is* weakly p-harmonic *if for all test functions* $\psi \in C_c^\infty(\widetilde{M})$ *we have*

$$\int_{\widetilde{M}} \langle |\nabla u|_{\widetilde{g}}^{p-2} \nabla u, \nabla \psi \rangle_{\widetilde{g}} \, dVol_{\widetilde{g}} = 0. \tag{1.1}$$

This formulation allows us to work without assuming any C^2 *regularity of the metric at the compactified bubbles.*

Definition 1.2 (ADM Mass for Low Regularity Metrics)**.** *For an asymptotically flat manifold* (M, g) *where the metric* g *is Lipschitz continuous* $(C^{0,1})$ *and satisfies the standard decay conditions, the ADM mass is well-defined provided the scalar curvature (in the distributional sense) is integrable. The Positive Mass Theorem remains valid in this class. The continuity of the mass under the convergence of the regularized Jang metrics ensures* $M_{ADM}(\overline{g})$ *is well-defined (see Theorem 3.21).*

Remark 1.3. *The ADM mass is well-defined for the Lipschitz metrics considered here (Jang metric) and the* C^0 *metrics (conformal deformation) because the deviation from Euclidean space decays sufficiently fast at infinity, and the distributional curvature is integrable.*

Definition 1.4 (BV Functions and Perimeter)**.** *As* $p \to 1$*, the potentials* u_p *lose Sobolev regularity. We work in the space of functions of Bounded Variation,* $BV(\widetilde{M})$*. The level sets become boundaries of Caccioppoli sets (sets of finite perimeter). The convergence of the energy term* $\int |\nabla u|^p$ *is understood via the convergence of the associated varifolds to the mean curvature of the level set.*

Theorem 1.5 (Regularity of Weak Solutions)**.** *Let $u \in W^{1,p}_{loc}(\widetilde{M})$ be a weak solution to the p-Laplace equation with $1 < p < 3$. By the regularity theory of Tolksdorf and DiBenedetto, $u \in C^{1,\alpha}_{loc}(\widetilde{M} \setminus \{p_k\})$ for some $\alpha \in (0,1)$.*

Remark 1.6 (Regularity across the Lipschitz Interface)**.** *The metric $\widetilde{g}$ is Lipschitz continuous ($C^{0,1}$) across the interface Σ and smooth away from Σ. In local coordinates the coefficients of the p-Laplace operator depend on the metric and so are bounded and uniformly elliptic. Standard elliptic regularity theory for quasilinear equations with bounded measurable coefficients (for instance [23, 32]) yields local $C^{1,\alpha}$ regularity for weak p-harmonic functions on each side of Σ. In addition, the transmission problem satisfied by u across Σ has no jump in the conormal derivative, so the tangential derivatives of u are continuous; a standard reflection argument then shows that u is in fact $C^{1,\alpha}$ across the interface Σ. In particular, no extra jump or transmission term arises for u at Σ.*

Near the singular points p_k (closed bubbles) the metric is merely C^0, so the classical regularity theory is only applied on compact subsets of $\widetilde{M} \setminus \{p_k\}$. The set $\{p_k\}$ has vanishing p-capacity for $1 < p < 3$ (Lemma 4.21), hence it is removable for $W^{1,p}$ functions. Moreover, the critical set $\mathcal{C} = \{\nabla u = 0\}$ is closed and has Hausdorff dimension at most $n - 2$ by the stratification results of Cheeger–Naber–Valtorta [6]. In particular, the integration by parts identities used in the monotonicity formula hold in the sense of distributions on all of $\widetilde{M}$; see Appendix C.

1.8   Definitions and Main Theorem

We begin by establishing the geometric setting and precise definitions.

Definition 1.7 (Weighted Asymptotic Flatness)**.** *An initial data set (M, g, k) is asymptotically flat with rate τ if there exist coordinates $\{x^i\}$ at infinity such that:*

$$g_{ij} - \delta_{ij} = O(|x|^{-\tau}), \quad \partial g \sim O(|x|^{-\tau-1}), \quad \partial^2 g \sim O(|x|^{-\tau-2}),$$

$$k_{ij} = O(|x|^{-\tau-1}), \quad \partial k \sim O(|x|^{-\tau-2}).$$

We assume the standard decay rate $\tau > 1$.

Remark 1.8 (Integrability of the Jang Source)**.** *While Fredholm theory only requires $\tau > 1/2$, the validity of the global mass correction formula $\Delta M = \int \mathcal{S}\phi$ requires $\mathcal{S} \in L^1$. Since $\mathcal{S} \sim O(r^{-\tau-2})$, integrability requires $\int^{\infty} r^{-\tau} dr < \infty$, which forces $\tau > 1$. Consequently, we adopt the standard assumption $\tau > 1$ throughout to ensure the ADM mass is well-defined via both flux and volume integral formulations.*

Remark 1.9. *Standard definitions of asymptotic flatness in the relativity literature often require $\tau \geq 1$. We adhere to this standard to guarantee the integrability of the scalar curvature.*

The initial data set must satisfy the Einstein constraint equations, which define the local energy density μ and momentum density J:

$$16\pi\mu = R_g + (\text{Tr}_g k)^2 - |k|^2_g, \tag{1.2}$$

$$8\pi J_i = \text{div}_g(k_i^j - (\text{Tr}_g k)\delta_i^j). \tag{1.3}$$

Definition 1.10 (Dominant Energy Condition (DEC))**.** *An initial data set (M, g, k) satisfies the dominant energy condition if $\mu \geq |J|_g$.*

The total energy is quantified by the ADM mass.

Definition 1.11 (ADM Mass). *The ADM mass $M_{ADM}(g)$ of an AF end is defined by the flux integral at spatial infinity:*

$$M_{ADM}(g) = \frac{1}{16\pi} \lim_{r \to \infty} \sum_{i,j} \int_{S_r} (\partial_j g_{ij} - \partial_i g_{ii}) \frac{x^j}{r} \, d\sigma_r, \qquad (1.4)$$

where S_r is a coordinate sphere of radius r, and ν is the outward unit normal.

The Positive Mass Theorem [31] guarantees $M_{\mathrm{ADM}}(g) \geq 0$ if the DEC holds.

The inequality concerns the boundary of the trapped region.

Definition 1.12 (MOTS). *A closed, embedded surface $\Sigma \subset M$ is a* Marginally Outer Trapped Surface *(MOTS) if its outer null expansion θ_+ vanishes. In terms of initial data, $\theta_+ = H_\Sigma + \mathrm{Tr}_\Sigma(k) = 0$, where H_Σ is the mean curvature of Σ in (M, g) and $\mathrm{Tr}_\Sigma(k)$ is the trace of k restricted to Σ. An* apparent horizon *is the boundary of the trapped region, often defined as the outermost MOTS.*

Theorem 1.13 (Properties of the Outermost MOTS). *Let (M, g, k) satisfy the DEC. The outermost MOTS Σ exists and satisfies the following properties:*

1. ***Regularity:** Σ is a smooth, closed, embedded hypersurface.*

2. ***Stability:** Σ is stable. Physically, this means it cannot be perturbed outwards into a trapped region. Mathematically, the principal eigenvalue of the stability operator (Jacobi operator) is non-negative:*

$$L_\Sigma \psi := -\Delta_\Sigma \psi - (|II_\Sigma|^2 + \mathrm{Ric}(\nu, \nu))\psi, \quad \lambda_1(L_\Sigma) \geq 0.$$

Stability follows because if $\lambda_1 < 0$, Σ could be perturbed outwards, contradicting its outermost nature.

Remark 1.14 (Topology of Outermost MOTS). *A crucial consequence of stability (established by Andersson, Metzger, and Eichmair) is that in 3-dimensions, stable MOTS are topologically spheres. This topological restriction is vital for our analysis of the "Jang bubbles," ensuring that the link of the resulting cone has positive scalar curvature (spectral gap), which drives the decay $\phi \sim r^\alpha$ with $\alpha > 0$.*

Remark 1.15 (Handling the Marginally Stable Case). *The case $\lambda_1(L_\Sigma) = 0$ (marginal stability) is physically significant, corresponding to non-generic horizons (e.g., extremal black holes). Analytically, it implies that the decay of the Jang metric to the cylinder is polynomial rather than exponential (see Lemma 3.12). While the standard Lockhart-McOwen theory is stated for exponential convergence, the Fredholm results extend to the polynomial case provided the operator limits to the same translation-invariant model operator and the decay rate is sufficient to treat the difference as a compact perturbation in the relevant weighted spaces. We rigorously verify that the polynomial decay (specifically $O(t^{-2})$ for the metric and $O(t^{-3})$ for the discrepancy) is sufficient for all subsequent flux calculations.*

We can now state the main theorem precisely.

Theorem 1.16 (Spacetime Penrose Inequality). *Let (M, g, k) be a complete, 3-dimensional, asymptotically flat initial data set satisfying the dominant energy condition ($\mu \geq |J|_g$). Let $\Sigma \subset M$ be the outermost apparent horizon. Σ is a smooth, closed, embedded hypersurface which may be disconnected, i.e., $\Sigma = \cup_{i=1}^N \Sigma_i$.*

The ADM mass satisfies:

$$M_{ADM}(g) \geq \sqrt{\frac{A(\Sigma)}{16\pi}}, \qquad (1.5)$$

where $A(\Sigma) = \sum_{i=1}^N Area(\Sigma_i)$. Equality holds if and only if the initial data set embeds isometrically into the Schwarzschild spacetime (forcing $N = 1$).

1.9   Strategy of the Proof: A Heuristic Roadmap

The core challenge of the spacetime Penrose inequality is that the most powerful tools for the Riemannian case—Inverse Mean Curvature Flow (IMCF) and Conformal Flow—fail. Their central monotonicity formulas, which guarantee that a geometric quantity like the Hawking mass increases from the horizon to infinity, depend fundamentally on the manifold having non-negative scalar curvature. The Jang reduction, while successfully connecting the spacetime problem to a Riemannian one, produces a metric $(\overline{M}, \overline{g})$ whose scalar curvature is not positive and which contains singularities. Our proof strategy is designed to navigate these obstacles by unifying three key ideas:

1. **The Jang Reduction:** We embrace the Jang metric $\overline{g}$ because it correctly encodes the ADM mass and horizon area, satisfying $M_{\mathrm{ADM}}(\overline{g}) \leq M_{\mathrm{ADM}}(g)$. However, its scalar curvature $R_{\overline{g}}$ contains a problematic divergence term, preventing direct application of Riemannian techniques.

2. **Controlled Conformal Deformation:** Instead of viewing the non-positive curvature as an insurmountable barrier, we show it can be "tamed." We conformally deform the Jang metric to a new metric $\widetilde{g} = \phi^4 \overline{g}$. The conformal factor ϕ is chosen as a solution to a carefully constructed Lichnerowicz-type equation. This equation is designed to precisely cancel the negative divergence term in $R_{\overline{g}}$ while simultaneously sealing the singularities of the Jang metric into well-behaved conical points. As proven in **Appendix A** via capacity arguments, the singularities are removable and do not obstruct the flow.

   The Central Argument: To ensure the mass does not increase during this deformation $(M_{\mathrm{ADM}}(\overline{g}) \geq M_{\mathrm{ADM}}(\widetilde{g}))$, we must have $\phi \leq 1$. We rigorously prove this using the Bray-Khuri divergence identity (Section 4.3).

3. **The p-Harmonic Level Set Method:** We now have a Riemannian manifold with non-negative scalar curvature, but it still has singularities where classical geometric flows are ill-defined. Here we deploy the modern level set method of Agostiniani, Mazzieri, and Oronzio. We solve for a p-harmonic function u_p on this singular manifold. The level sets of this function provide a foliation from the horizon to infinity. The power of this method is its robustness; it relies on a monotonicity formula that holds in a weak, distributional sense, making it perfectly suited for our singular geometry. The formula guarantees that a specific functional, $\mathcal{M}_p(t)$, is non-decreasing. By taking the limit as $p \to 1$, this functional's value at the horizon is identified with the horizon area and its value at infinity is identified with the ADM mass, yielding the Penrose inequality.

Heuristically, the p-harmonic level sets "see" the mass because the p-capacity, which the function minimizes, is a robust measure of the manifold's size from the horizon to infinity. Our contribution is to show that the Jang metric can be surgically altered into a geometric form where this powerful tool can be rigorously applied, overcoming the obstacles that stalled previous approaches. The technical details of this construction are outlined below.

Metric	Symbol	Regularity	Scalar Curvature	End Structure
Initial Data	(M, g)	Smooth (C^∞)	R_g (general)	Asymptotically Flat
Jang Metric	$(\overline{M}, \overline{g})$	Lipschitz ($C^{0,1}$)	$R_{\overline{g}}$ (distr. ≥ 0)	AF + Cylindrical
Conformal	$(\widetilde{M}, \widetilde{g})$	C^0 / Lip	$R_{\widetilde{g}} \geq 0$ (distr.)	AF + Conical (Internal) / Cylindrical (Ho
Smoothed	$(\widetilde{M}, g_\epsilon)$	Smooth (C^∞)	$R_{g_\epsilon} \geq 0$ (pointwise)	AF + Cylindrical (trunc)

Table 2: Roadmap of metric deformations used in the proof.

1.10   Overview of the Technical Argument

Instead of treating the reduction (Jang equation) and the scalar-flat deformation (Lichnerowicz equation) as separate steps, we analyze them as a coupled elliptic system. Let $\tau > 1/2$. We seek (f, ϕ) solving

$$\begin{cases} \mathcal{J}(f) := \left(g^{ij} - \frac{f^i f^j}{1+|\nabla f|^2}\right)\left(\frac{\nabla_{ij}f}{\sqrt{1+|\nabla f|^2}} - k_{ij}\right) = 0 & \text{in } M \setminus \Sigma, \\ \mathcal{L}(\phi, f) := \Delta_{\overline{g}(f)}\phi - \frac{1}{8}R_{\overline{g}}(f)\phi = 0 & \text{in } \overline{M}_f. \end{cases} \tag{1.6}$$

Remark 1.17. *It is convenient to write the generalized Jang equation and the Lichnerowicz equation as the coupled system* (1.6), *but in our actual argument we do not solve this system simultaneously. Instead, we first solve the generalized Jang equation for f (using the results of Han–Khuri and others) and thereby construct the Jang manifold $(\overline{M}, \overline{g})$. All subsequent analysis— the spectral condition, the Fredholm theory, and the conformal deformation—takes place on this fixed Jang background and treats ϕ as the unknown. No analytic fixed-point argument in the pair (f, ϕ) is required.*

The operator $\mathcal{L}$ depends on the graph f through both the metric and its scalar curvature, so the problem naturally lives in weighted Sobolev spaces on manifolds with cylindrical ends.

Remark 1.18 (Stability Condition). *The outermost MOTS hypothesis on Σ guarantees a one-sided barrier for* (1.6). *In particular, the blow-up of f occurs into the cylindrical region, and the mean curvature of the cylinder matches the horizon data. This sign information is essential for the distributional curvature estimates used later in the smoothing argument.*

The rigorous proof strategy, therefore, combines the GJE reduction, a sophisticated metric deformation to resolve these issues (following Bray and Khuri [5]), and the application of robust methods for the Riemannian Penrose Inequality. In this framework, we employ the Nonlinear Level Set Method (AMO) [1].

2   The p-Harmonic Level Set Method (AMO Framework)

We review the framework developed in [1], which provides a proof of the Riemannian Penrose Inequality by analyzing the geometry of the level sets of p-harmonic functions.

2.1   Setup and the Monotonicity Formula

Let $(\widetilde{M}, \widetilde{g})$ be a complete, smooth, asymptotically flat 3-manifold with non-negative scalar curvature $R_{\widetilde{g}} \geq 0$. We assume the interior boundary Σ_0 is the outermost compact minimal surface.

We consider the p-harmonic potential u_p $(1 < p < 3)$, which is the solution to the Dirichlet problem for the p-Laplace equation:

$$\begin{cases} \Delta_{p,\widetilde{g}}u_p := \text{div}_{\widetilde{g}}(|\nabla u_p|_{\widetilde{g}}^{p-2}\nabla u_p) = 0 & \text{in } \widetilde{M} \setminus \Sigma_0, \\ u_p = 0 & \text{on } \Sigma_0, \\ u_p(x) \to 1 & \text{as } |x| \to \infty. \end{cases} \tag{2.1}$$

For almost every $t \in (0, 1)$ the level set $\Sigma_t = \{u_p = t\}$ is a smooth hypersurface in $\widetilde{M}$; throughout we restrict attention to such regular values of t when evaluating geometric quantities along the level sets.

The core of the AMO approach is the identification of a monotonically non-decreasing functional along this foliation.

Theorem 2.1 (AMO Monotonicity [1]). *Let $(\widetilde{M}, \widetilde{g})$ be as above with $R_{\widetilde{g}} \geq 0$. For $1 < p < 3$, define the functional:*

$$\mathcal{M}_p(t) := \left(\int_{\Sigma_t} |\nabla u|^p \, d\sigma \right)^{\frac{2}{3-p}} \left(1 - \frac{1}{16\pi} \left(\int_{\Sigma_t} |\nabla u|^p \, d\sigma \right)^{-\frac{2(p-1)}{3-p}} \int_{\Sigma_t} H^2 |\nabla u|^{p-2} \, d\sigma \right). \quad (2.2)$$

Then, along the flow of the level sets of the p-harmonic potential u, we have:

$$\frac{d}{dt} \mathcal{M}_p(t) \geq 0$$

for almost every $t \in (0,1)$.

Proof sketch, following [1]. We briefly recall the structure of the argument in [1] and indicate the points at which our low-regularity setting requires an additional justification; full details in the smooth case can be found in [1].

For a smooth solution u on a smooth manifold $(\widetilde{M}, \widetilde{g})$, the proof relies on the Bochner–Weitzenböck identity for the p-Laplacian. For such a solution one has

$$\frac{1}{p} \Delta(|\nabla u|^p) = |\nabla^2 u|^2 + \langle \nabla u, \nabla(\Delta u) \rangle + \text{Ric}(\nabla u, \nabla u) + (p-2)\langle \nabla u, \nabla |\nabla u| \rangle^2 |\nabla u|^{-2}. \quad (2.3)$$

For p-harmonic functions ($\Delta_p u = 0$), this simplifies after identifying the curvature terms. Using the Gauss–Codazzi relations on the level sets Σ_t to relate $\text{Ric}(\nabla u, \nabla u)$ to the scalar curvature $R_{\widetilde{g}}$ and the extrinsic curvature A (with mean curvature H), one derives the identity

$$\frac{d}{dt} \mathcal{M}_p(t) = C(p,t) \int_{\Sigma_t} \left[\frac{1}{2} R_{\widetilde{g}} + \frac{1}{2} \left(|A|^2 - \frac{1}{2} H^2 \right) + \frac{p-1}{p} |\nabla_T \nu|^2 + \mathcal{K}_p(u) \right] |\nabla u|^{p-1} \, d\sigma, \quad (2.4)$$

where $\mathcal{K}_p(u)$ is a term arising from the refined Kato inequality. On the regular set $\widetilde{M} \setminus \mathcal{C}$ (where $\nabla u \neq 0$), we have the pointwise tensor inequality (for $n = 3$)

$$|\nabla X|^2 \geq \frac{3}{2} |\nabla |X||^2 \quad (n = 3),$$

which ensures $\mathcal{K}_p(u) \geq 0$ pointwise. Crucially, this inequality holds in the sense of distributions even across the critical set $\mathcal{C} = \{\nabla u = 0\}$, meaning $\mathcal{K}_p(u)$ defines a non-negative measure. A rigorous proof of this fact, relying on regularization and weak convergence, is provided in Section C. Since $R_{\widetilde{g}} \geq 0$ is enforced by our construction, and $|A|^2 \geq H^2/2$ (by Cauchy-Schwarz), the integrand is non-negative.

Remark 2.2 (Regularity Requirements). *The preceding discussion assumes $(\widetilde{M}, \widetilde{g})$ is smooth and u is smooth. In our application, $(\widetilde{M}, \widetilde{g})$ contains a finite set of points $\{p_k\}$ at which the metric is merely continuous (C^0) and we only know that $u \in W_{loc}^{1,p}$. In Appendix C we show that the conical singularities $\{p_k\}$ have zero p-capacity and that the Ricci curvature of $\widetilde{g}$ is locally integrable. This is enough to justify the Bochner identity in the sense of distributions and to show that the refined Kato term $\mathcal{K}_p(u)$ defines a nonnegative measure. Hence the monotonicity formula continues to hold distributionally in our setting.*

<div align="right">□</div>

2.2   Boundary Limits and the Limit $p \to 1$

The significance of $\mathcal{M}_p(t)$ lies in its behavior as $p \to 1^+$, where it converges to the Hawking mass.

Definition 2.3 (Hawking Mass). *For a closed surface Σ in a 3-manifold with area $A(\Sigma)$ and mean curvature H, the Hawking mass is:*

$$m_H(\Sigma) = \sqrt{\frac{A(\Sigma)}{16\pi}} \left(1 - \frac{1}{16\pi} \int_{\Sigma} H^2 d\sigma \right).$$

Proposition 2.4 ([1]). *The boundary limits of the functional $\mathcal{M}_p(t)$ as $p \to 1^+$ are rigorously identified as follows:*

(i) **Limit at the Horizon** *($t = 0$): Since Σ_0 is minimal ($H_0 = 0$), $m_H(\Sigma_0)$ reduces to the area radius. It is shown that*

$$\lim_{p \to 1^+} \mathcal{M}_p(0) = \sqrt{\frac{A(\Sigma_0)}{16\pi}}.$$

(ii) **Limit at Infinity** *($t \to 1$): Utilizing the Gamma-convergence of the p-capacitary potential to the Inverse Mean Curvature Flow (in the weak BV sense), we establish:*

$$\lim_{p \to 1^+} \lim_{t \to 1^-} \mathcal{M}_p(t) = M_{ADM}(\widetilde{g}).$$

Rigorous Justification of the Limit at Infinity. The identification of the limit with the ADM mass relies on the precise asymptotic expansion of the p-harmonic potential u_p near the AF end. For the nonlinear p-Laplacian, establishing this expansion requires specialized techniques (e.g., comparison principles or methods by Kichenassamy). We utilize the established expansion:

$$u_p(x) = 1 - \frac{C_p}{|x|^{3-p}} + O(|x|^{3-p-\epsilon}), \quad \text{as } |x| \to \infty.$$

Substituting this into $\mathcal{M}_p(t)$ and taking the limit $t \to 1$ gives a quantity depending only on the coefficient C_p. The key point, proved in [1], is that C_p is (up to a universal normalizing factor) the p-capacity of the horizon and that, as $p \to 1^+$, this capacity converges to the ADM mass of $(\widetilde{M}, \widetilde{g})$. More precisely, the main identification theorem in [1] shows that

$$\lim_{p \to 1^+} \lim_{t \to 1^-} \mathcal{M}_p(t) = M_{\text{ADM}}(\widetilde{g}).$$

$\square$

Remark 2.5 (Robustness of Gamma-Convergence). *The Gamma-convergence of the p-energy to the perimeter functional (as $p \to 1$) is robust even in the presence of the conical singularities and the Lipschitz interface of $\widetilde{g}$, as these sets have zero capacity (Section A) and do not contribute to the energy limit (see Theorem 4.48). Furthermore, the convergence is uniform with respect to the smoothing parameter ϵ (due to the controlled C^0 convergence of the metrics, Theorem 4.40), which justifies the interchange of limits in the "Limit of Inequalities" strategy (Section 5).*

This double limit process ($t \to 1, p \to 1$) is justified by the fact that the p-harmonic level sets approximate the weak solution of the IMCF (Huisken-Ilmanen flow) without requiring the flow to be smooth.

Remark 2.6. *Throughout, the limit $t \to 1^-$ is taken first for each fixed p, and only afterwards is the limit $p \to 1^+$ considered. For the purposes of the Penrose inequality it is enough to know that*

$$M_{ADM}(\widetilde{g}) \geq \lim_{p \to 1^+} \lim_{t \to 1^-} \mathcal{M}_p(t),$$

together with the identification of $\lim_{p \to 1^+} \mathcal{M}_p(0)$ with the Hawking mass of the horizon. The full identities in Proposition 2.4 are included for completeness.

Remark 2.7 (Application to Manifolds with Cylindrical Ends). *Although the standard AMO framework assumes a compact interior boundary, the monotonicity analysis extends to manifolds with cylindrical ends by a standard truncation procedure. We apply the theory to the manifold truncated at depth L in the cylinder, imposing the zero Dirichlet condition on the cross-section Σ_L. Since the cylinder is scalar-flat and has constant area $A(\Sigma)$, the boundary term $\mathcal{M}_p(0)$ in the monotonicity formula converges to the area radius $\sqrt{A(\Sigma)/16\pi}$ as $L \to \infty$. This justifies the application of the method to our manifold $(\widetilde{M}, g_\epsilon)$.*

The monotonicity $\mathcal{M}_p(1) \geq \mathcal{M}_p(0)$ (understood via limits), combined with Proposition 2.4, implies the Riemannian Penrose Inequality: $M_{\mathrm{ADM}}(\widetilde{g}) \geq \sqrt{A(\Sigma_0)/16\pi}$.

3   The Generalized Jang Reduction and Analytical Obstructions

To prove the Spacetime Penrose Inequality (Theorem 1.16), the initial data (M, g, k) must be transformed into a Riemannian setting suitable for the AMO method. This is achieved via the Generalized Jang Equation (GJE).
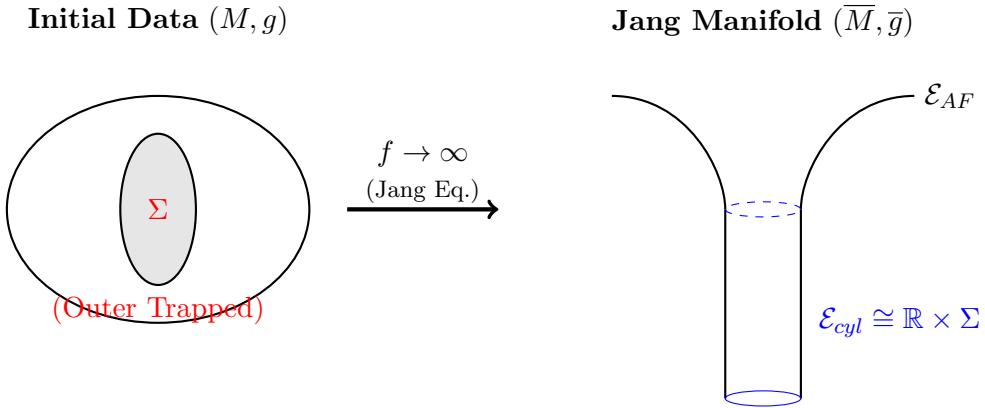
Figure 1: The geometric action of the Generalized Jang Equation. The graph function f blows up at the marginal surface Σ in the initial data (left), creating a manifold $\overline{M}$ (right) with a new cylindrical end $\mathcal{E}_{cyl}$ where the scalar curvature condition becomes favorable.

3.1   Weighted Edge Sobolev Spaces: A Detailed Framework

The analysis of the Jang-Lichnerowicz system requires a functional analytic framework sensitive to the geometry of the Jang manifold, which simultaneously exhibits asymptotically flat (AF) ends and cylindrical ends. Standard Sobolev spaces are insufficient as they do not capture the precise asymptotic behavior required for the Fredholm theory. To this end, we employ the theory of **Weighted Sobolev Spaces on Manifolds with Ends**.

Let $(\overline{M}, \overline{g})$ be the Jang manifold. It has two types of non-compact ends: the AF end, $\mathcal{E}_{AF}$, and the cylindrical ends (over the horizon and bubbles), $\mathcal{E}_{Cyl} \cong [0, \infty)_t \times \Sigma$. Let ρ be a defining function for the AF end (e.g., $\rho(x) = (1 + |x|^2)^{-1/2}$) and let t be the longitudinal coordinate on the cylinders. We fix once and for all a compact subset $M_{\mathrm{bulk}} \subset \overline{M}$ with smooth boundary such that

$$\overline{M} = M_{\mathrm{bulk}} \cup \mathcal{E}_{AF} \cup \mathcal{E}_{Cyl},$$

and the three pieces meet only along their common boundaries.

Definition 3.1 (Weighted Sobolev Spaces on Manifolds with Ends). *For $k \in \mathbb{N}$, $p \in (1, \infty)$, and weight parameters δ (for the AF end) and β (for the cylindrical ends), the weighted Sobolev*

space $W_{\delta,\beta}^{k,p}(\overline{M})$ is the completion of $C_c^\infty(\overline{M})$ under a norm defined using a partition of unity subordinate to the decomposition of $\overline{M}$. We explicitly distinguish between the weights for different ends: let β_{hor} denote the weight for the horizon end cylinder, and β_{bub} for the bubble end cylinders. The norm is defined as:

$$\|u\|_{W_{\delta,\beta}^{k,p}}^p := \|u\|_{W^{k,p}(M_{\mathrm{bulk}})}^p + \|u\|_{W_\delta^{k,p}(\mathcal{E}_{AF})}^p + \|u\|_{W_{\beta_{\mathrm{hor}}}^{k,p}(\mathcal{E}_{\mathrm{hor}})}^p + \sum_k \|u\|_{W_{\beta_{\mathrm{bub}}}^{k,p}(\mathcal{E}_{\mathrm{bub},k})}^p.$$

The norms on the ends are defined using the appropriate weight functions. On the AF end:

$$\|u\|_{W_\delta^{k,p}(\mathcal{E}_{AF})}^p := \sum_{j=0}^k \int_{\mathcal{E}_{AF}} \rho(x)^{p(\delta-j)} |\nabla^j u|_{\overline{g}}^p \, dV_{\overline{g}},$$

*where $\rho(x) \approx (1+|x|^2)^{-1/2}$ is a **polynomial weight** corresponding to the standard Euclidean distance at the asymptotically flat end. On the cylindrical ends (parameterized by $t \in [0,\infty)$), we use the exponential weight consistent with the scattering calculus:*

$$\|u\|_{W_\beta^{k,p}(\mathcal{E}_{Cyl})}^p := \sum_{j=0}^k \int_{\mathcal{E}_{Cyl}} e^{p\beta t} |\nabla^j u|_{\overline{g}}^p \, dV_{\overline{g}}.$$

Convention: *We adopt the convention (consistent with Melrose) where the weight enters the integral directly. Thus, for $p = 2$, $\beta < 0$ enforces decay. The weight δ controls the polynomial decay at the asymptotically flat end, crucial for the ADM mass and the validity of integration by parts at infinity. The weights β_{hor} and β_{bub} control the exponential decay or growth on the cylindrical ends, which is essential for the Fredholm analysis of the Lichnerowicz operator.*

Remark 3.2 (Asymptotic Regularity)**.** *While the existence theory is framed in Weighted Sobolev spaces, standard elliptic regularity bootstraps the solution ϕ into the Weighted Hölder spaces $C_{-\delta}^{2,\alpha}(\overline{M})$. This justifies the pointwise asymptotic expansions $\phi = 1 + A/r + O(r^{-2})$ and $\nabla\phi = O(r^{-2})$ used in the mass flux calculations.*

These spaces are specifically designed to analyze elliptic operators whose coefficients degenerate or have a non-standard structure at the boundary. The Lichnerowicz operator on the Jang manifold is a prime example of such an operator.

Trace Theorems and Boundary Behavior.   A key feature of these spaces is their associated trace theorems, which describe how functions in $W_{\delta,\gamma}^{k,p}(\overline{M})$ behave when restricted to the boundary components.

Theorem 3.3 (Trace Theorem for Weighted Spaces)**.** *There exists a continuous trace operator* Tr *that maps functions in the weighted space to functions on the boundary components (e.g., the cross-sections of the cylinders). For the cylindrical interface Σ, the trace map is well-defined. Specifically, for the Sobolev order $k = 1$ relevant to our gluing construction, we have:*

$$\mathrm{Tr}_\Sigma : W_{\delta,\gamma}^{1,p}(\overline{M}) \to W^{1-1/p,p}(\Sigma). \tag{3.1}$$

This map is surjective and possesses a continuous right inverse. This surjectivity is fundamental to the gluing construction: it justifies that functions defined separately on the bulk and the cylinder can be glued into a global $W_{\delta,\gamma}^{1,p}(\overline{M})$ function provided their traces match in $W^{1-1/p,p}(\Sigma)$ (and similarly for higher regularities). These statements are standard for manifolds with cylindrical ends; see for example [24, 26].

Density of Smooth Functions. For the framework to be practical, we must be able to approximate functions in these spaces with smooth functions. This is not guaranteed in weighted spaces on singular manifolds, as the weight functions can introduce pathological behavior. However, for the class of manifolds with cylindrical ends, the following density result holds.

Proposition 3.4 (Density of Smooth Functions). *The space of smooth functions that are compactly supported in the interior of $\overline{M}$, denoted $C_c^\infty(int(\overline{M}))$, is dense in $W_{\delta,\gamma}^{k,p}(\overline{M})$ if and only if the weights $(\delta, \gamma_{\mathrm{hor}}, \gamma_{\mathrm{bub}})$ are chosen away from the set of indicial roots associated with the asymptotic behavior of the operator at each end. This is a standard consequence of the general Fredholm theory on manifolds with ends; see [24, 26].*

This density is essential. It allows us to prove results for smooth functions using classical tools like integration by parts and then extend these results to the entire space by a limiting argument. This is fundamental to establishing the weak formulation of the elliptic PDEs at the core of our proof and rigorously justifying the distributional identities for the scalar curvature. The selection of the correct weights to ensure both density and the Fredholm property of the operator (as discussed in Theorem 4.6) is a cornerstone of the entire analytic argument.

3.2 The Geometric Setup of the GJE

We consider the product Lorentzian spacetime $(M \times \mathbb{R}, g - dt^2)$. We seek a function $f : M \to \mathbb{R}$ such that its graph $\overline{M} = \{(x, f(x)) : x \in M\}$ satisfies a prescribed mean curvature equation. The analysis utilizes the auxiliary Riemannian metric $\overline{g} = g + df \otimes df$.

Initial Data (M, g) **Jang Manifold** $(\overline{M}, \overline{g})$

$f \to \infty$
(Jang Eq.)

Σ

(Trapped Surface)

$\mathcal{E}_{cyl} \cong \mathbb{R} \times \Sigma$
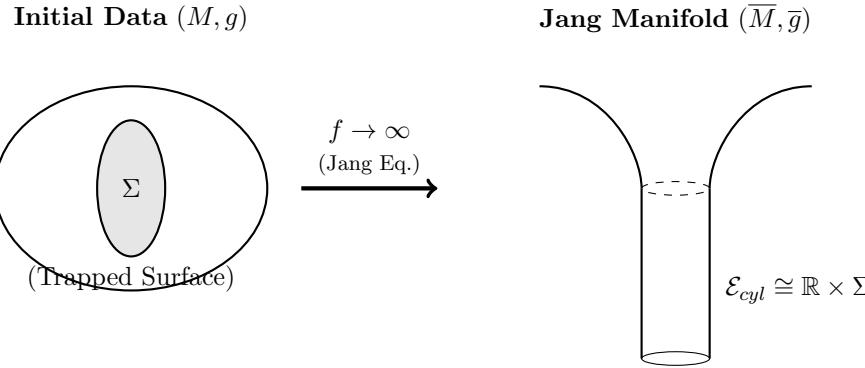
Figure 2: The geometric action of the Generalized Jang Equation. The graph blows up at the horizon Σ, creating a new cylindrical end where the scalar curvature condition is favorable.

Definition 3.5 (Generalized Jang Equation in the Distributional Context). *The Generalized Jang Equation (GJE) for a function $f : M \setminus \Sigma \to \mathbb{R}$ is given by:*

$$\mathcal{J}(f) := \left(g^{ij} - \frac{f^i f^j}{1 + |\nabla f|^2}\right)\left(\frac{\nabla_{ij} f}{\sqrt{1 + |\nabla f|^2}} - k_{ij}\right) = 0 \quad in \ M \setminus \Sigma. \tag{3.2}$$

Geometrically, this is $\mathcal{J}(f) := H_{\overline{M}} - \mathrm{Tr}_{\overline{g}}(k) = 0$. In divergence form, the equation is:

$$\mathrm{div}_g\left(\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}}\right) - \mathrm{Tr}_g k + \frac{k(\nabla f, \nabla f)}{1 + |\nabla f|^2} = 0.$$

*We define f to be a solution with blow-up boundary conditions on Σ if $f(x) \to \pm\infty$ as $x \to \Sigma$. Specifically, we solve the equation on the **exterior region** M_{ext} (outside the outermost MOTS). We impose $f(x) \to +\infty$ as $x \to \Sigma$. The resulting Jang manifold $\overline{M}$ consists of the graph over*

M_{ext} with a cylindrical end attached at Σ. Crucially, while f is singular at Σ, the quantity $v = \frac{\nabla f}{\sqrt{1+|\nabla f|^2}}$ remains bounded ($|v|_g < 1$). Thus, the equation is well-defined in the sense of distributions on the entire manifold M, with the singularity Σ manifesting as a boundary flux condition for the bounded vector field v. This distributional perspective justifies the subsequent analysis of the scalar curvature as a distribution with support on Σ.

The GJE is a quasilinear, degenerate elliptic PDE. Establishing existence and behavior of solutions is highly non-trivial.

Remark 3.6 (Interior Regularity). *The GJE is degenerate elliptic, as the operator degenerates when $|\nabla f| \to \infty$. It is crucial that the DEC prevents this degeneracy from occurring in the interior of $M \setminus \Sigma$. This ensures that the solution f is smooth in the bulk, and blow-up occurs only at the boundary MOTS Σ.*

3.2.1 Schoen-Yau Barriers and Existence

A fundamental challenge is ensuring that the Jang surface blows up precisely at the *outermost* MOTS Σ, rather than at any interior MOTS. This requires the existence of **Schoen-Yau barriers**.

Theorem 3.7 (Existence of Barriers [31]). *Under the DEC, there exist surfaces with prescribed mean curvature that lie slightly above any interior MOTS.*

These barriers are essential for the existence theory (Theorem 3.8), as they prevent the regularized solutions f_κ from diverging prematurely, effectively allowing the Jang surface to "jump over" the interior trapped regions and reach the outermost boundary Σ.

3.2.2 Existence via Regularization and Barriers

Theorem 3.8 (Existence and Blow-up Behavior [21]). *Let $\Omega_\tau = \{x \in M : dist(x, \Sigma) > \tau\}$. We solve the regularized Capillarity Jang Equation (CJE) with parameter κ:*

$$\left(g^{ij} - \frac{f^i f^j}{1+|\nabla f|^2}\right)\left(\frac{\nabla_{ij} f}{\sqrt{1+|\nabla f|^2}} - k_{ij}\right) = \kappa f \quad in \ \Omega_0, \quad f|_\Sigma = 0. \tag{3.3}$$

Standard elliptic theory grants a smooth solution f_κ. As $\kappa \to 0$, $f_\kappa \to f_0$ locally uniformly away from Σ.

Rigorous Justification (Barriers): The existence and localization of the blow-up rely on the Schoen-Yau barriers (Theorem 3.7) and supersolutions derived from the geometry of the MOTS Σ (utilizing its stability, Theorem 1.13). These provide uniform C_{loc}^2 estimates for f_κ independent of κ away from Σ, ensuring strong convergence to the limit solution f_0 and confining the blow-up to Σ.

Remark 3.9 (Prevention of Premature Blow-up). *Crucially, we utilize the barriers constructed by Schoen and Yau to "bridge" over any inner, unstable MOTS. Since the outermost MOTS Σ is stable, it admits a local foliation by mean-convex surfaces (outward). This geometric feature allows us to construct a subsolution that forces the Jang graph to remain regular in the interior and blow up precisely at Σ, preventing the "premature" formation of cylindrical ends at inner horizons that would disconnect the manifold.*

Remark 3.10 (Asymptotic Cylindrical Geometry). *It is crucial to note that while the Jang blow-up opens the horizon into an infinite end, the induced metric $\bar{g}$ is only asymptotically cylindrical. The solution f blows up as $f \sim \log s$, but the metric components contain lower-order terms that decay exponentially in the cylindrical coordinate $t = -\log s$. Thus, the manifold $\overline{M}$ possesses ends that are asymptotically periodic (cylindrical) rather than exactly product metrics. This distinction is handled in the analysis of the Lichnerowicz operator by invoking the theory of Lockhart–McOwen for elliptic operators on manifolds with cylindrical ends [24].*

3.2.3 Refined Asymptotic Analysis of the Blow-up

We now provide a rigorous derivation of the asymptotic behavior of the solution f near the horizon Σ. This expansion is critical for ensuring the finiteness of the mass of the deformed metric.

Lemma 3.11 (Strict Logarithmic Blow-up). *The solution f to the Generalized Jang Equation does not oscillate at the horizon. Specifically, in geodesic coordinates s distance from Σ, f satisfies:*

$$f(s, y) = C_0 \ln(s) + A(y) + O(s^\epsilon)$$

and the derivatives satisfy $\partial_s f \sim s^{-1}$, $\partial_s^2 f \sim s^{-2}$. Crucially, the barrier argument employed in [21] rules out oscillatory behaviors (e.g., $\sin(\ln s)$) by comparing f with strictly monotone supersolutions constructed from the stability of Σ (see also Andersson and Metzger [2]). This ensures that the induced metric $\overline{g} = g + df \otimes df$ converges in the C^k topology to the cylinder metric $dt^2 + g_\Sigma$ as $t \to \infty$. This spectral stability is a prerequisite for the Fredholm analysis in Section 4.2.

Lemma 3.12 (Sharp Asymptotic Expansion via Barrier Method). *Let Σ be the outermost (stable) MOTS. In a tubular neighborhood of Σ coordinatized by the geodesic distance $s \in (0, s_0)$ and $y \in \Sigma$, the solution f to the regularized Jang equation admits the decomposition*

$$f(s, y) = C_0 \log(s) + A(y) + v(s, y). \tag{3.4}$$

Let $t = -\log s$ be the cylindrical coordinate. The remainder term $v(t, y)$ decays as $t \to \infty$.
    Case 1: Strict Stability *($\lambda_1(L_\Sigma) > 0$). The spectral gap of the stability operator implies exponential decay:*

$$|v(t, y)| + |\nabla v(t, y)| + |\nabla^2 v(t, y)| \le Ce^{-\beta t} \tag{3.5}$$

for some $\beta > 0$ related to $\sqrt{\lambda_1}$.
    Case 2: Marginal Stability *($\lambda_1(L_\Sigma) = 0$). The decay is polynomial: $|v(t, y)| \le Ct^{-2}$. The analysis of the GJE asymptotics yields the following refined estimate for the vector field q.*

Proof of Refined Decay $q = O(t^{-3})$. We explicitly derive the decay rate from the stationarity of the horizon area. In the cylindrical coordinate t, the metric expansion is $g(t) = g_\Sigma + h^{(1)} t^{-1} + h^{(2)} t^{-2} + \dots$. However, for a marginally stable MOTS ($\lambda_1 = 0$), the area is stationary to first order under deformation. The linear term $h^{(1)}$ corresponds to a change in area; its presence would imply $A(t) \sim A_0 + C/t$, violating the minimality/stationarity condition at infinity unless $C = 0$. Thus, $h^{(1)} = 0$, and the metric decays as $\overline{g} - g_{cyl} = O(t^{-2})$. The Jang vector field q depends on the derivatives of the metric ($q \sim \nabla f \cdot (\nabla \overline{g})$). Since $\nabla \overline{g} \sim t^{-3}$, we obtain the refined decay $|q| \sim t^{-3}$. Consequently, the divergence satisfies $\text{div}_{\overline{g}}(q) \sim O(t^{-4})$. This decay rate is strictly faster than t^{-3}, ensuring that the source term is integrable against the kernel (constant mode) of the Lichnerowicz operator. This prevents the generation of logarithmic resonances in the conformal factor ϕ. $\qquad\square$

Definition 3.13 (Weighted Scattering Spaces). *To rigorously handle the marginal case where the metric perturbation decays polynomially, we employ the Weighted Scattering Calculus. We identify the cylindrical end $\mathcal{C} \cong [0, \infty) \times S^2$ conformally with $\mathbb{R}^3 \setminus B_1$ via $r = e^t$. We define the weighted Sobolev space $H_\beta^2(\mathbb{R}^3)$ where the measure is dx and the weight is $(1 + |x|^2)^{\beta/2}$. The norm is defined as:*

$$\|u\|_{H_\beta^2} = \left(\sum_{|\alpha| \le 2} \int_{\mathbb{R}^3} |\partial^\alpha u|^2 (1 + |x|^2)^{\beta/2} \, dx \right)^{1/2}.$$

This definition ensures the Fredholm properties of the Lichnerowicz operator in the marginal setting.

Proposition 3.14 (Fredholm Property via Scattering Calculus [13,16])**.** *In the marginally stable case ($\lambda_1(L_\Sigma) = 0$), the metric $\overline{g}$ decays polynomially to the cylinder. To prove the Fredholm property, we map the cylindrical end to $\mathbb{R}^3 \setminus B_1$ via $r = e^t$ and analyze the operator in the Weighted Scattering Calculus. The operator becomes:*

$$P = \Delta_{\mathbb{R}^3} - V(x), \quad where \ |V(x)| \leq \frac{C}{|x|^4} \ as \ |x| \to \infty.$$

This rapid decay (significantly faster than the critical $|x|^{-2}$ threshold) arises because the leading order terms of the scalar curvature vanish identically due to the stationarity of the horizon area. Consequently, the potential $V = \frac{1}{8}R_{\overline{g}}$ is a compact perturbation on the weighted spaces. We work in weighted Sobolev spaces $H_\beta^m(\mathbb{R}^3)$ with norm $\|u\|_{H_\beta^m} = \sum_{|\alpha| \leq m} \|\langle x \rangle^{\beta + |\alpha|} \partial^\alpha u\|_{L^2}$.

1. ***Indicial Roots:*** *The model operator at infinity is Euler-homogeneous: $L_0 = r^{-2}((r\partial_r)^2 + (r\partial_r) + \Delta_{S^2})$. The indicial roots γ are solutions to the characteristic equation of the radial ODE for spherical harmonics Y_l. (We follow the geometric sign convention $\Delta \leq 0$, so the characteristic equation agrees with the scattering-calculus treatment of $r^2\Delta$.) For the zeroth mode ($l = 0$, $\lambda_0 = 0$), the equation is $\gamma(\gamma + 1) = 0$, giving roots $\gamma = 0, -1$. To ensure invertibility, the weight β must avoid the set $\mathcal{I} = \{Re(\gamma)\}$.*

2. ***Mapping Properties:*** *Standard scattering theory (Melrose) implies that for $\beta \notin \mathcal{I}$, the operator $P : H_\beta^2 \to H_{\beta+2}^0$ is Fredholm.*

3. ***Choice of Weight (Indicial Root Separation):*** *The indicial roots of the Laplacian on $\mathbb{R}^3$ are 0 (constants) and -1 (decay $1/r$). Any weight β strictly between these values produces a Fredholm map. In addition, the source term we face decays like t^{-3}, so the dual space requirement imposes $\beta > -1/2$ to keep $\mathrm{div}_{\overline{g}}(q) \in L_{\beta-2}^2$. Combining these constraints, we adopt the range **$\beta \in (-1/2, 0)$**. This choice still enforces decay (because $\beta < 0$) while guaranteeing that the divergence source pairs with the model operator, so the Fredholm index remains zero.*

Thus, the Lichnerowicz operator is Fredholm even in the marginally stable case, provided we work in the correct weighted space that forbids the non-decaying kernel elements.

Remark 3.15 (Solvability and the Jang Flux)**.** *In the marginally stable case ($\lambda_1 = 0$), the kernel of the operator contains the constant functions. By the Fredholm Alternative, a solution ϕ exists if and only if the source term $-\frac{1}{4}\mathrm{div}_{\overline{g}}(q)$ is orthogonal to the kernel. This requires:*

$$\int_{\overline{M}} \mathrm{div}_{\overline{g}}(q) \, dV_{\overline{g}} = 0.$$

*Applying the Divergence Theorem, this volume integral converts to the sum of the fluxes of q through the boundaries (the AF end, the horizon Σ, and the bubble tips). Explicitly, the projection of the source onto the kernel $\psi = 1$ is $\langle -\frac{1}{4}\mathrm{div}(q), 1 \rangle_{L^2} = -\frac{1}{4} \oint_{\partial \overline{M}} \langle q, \nu \rangle$. Crucially, the **Vanishing Flux Lemma** (Lemma H.1) independently establishes that these boundary fluxes vanish due to the asymptotic behavior of the GJE solution. Thus, the geometric behavior of the Jang vector field q precisely guarantees the analytic solvability of the singular Lichnerowicz equation.*

Proposition 3.16 (Decay of the Scattering Potential)**.** *To justify the compactness of the error term in the Fredholm analysis (Appendix C), we explicitly derive the decay rate of the potential $V = \frac{1}{8}R_{\overline{g}}$ in the marginally stable case. From Theorem 3.12, the metric components satisfy $\overline{g}_{ij} = \delta_{ij} + O(t^{-2})$ in cylindrical coordinates. Since the operator L is formally self-adjoint on $L^2(\overline{M}, dV_{\overline{g}})$, and the weighted spaces H_δ^2 and $L_{\delta-2}^2$ are dual under the natural pairing relative to the volume form modification, the index of the operator in the spectral gap is zero. The scalar curvature $R_{\overline{g}}$ involves second derivatives of the metric. Since $\partial_t \sim t^{-1}$ is not the scaling (it is*

translation invariant), we rely on the GJE expansion. The linearization of the scalar curvature operator about the cylinder is $L_\Sigma h$. Since $\lambda_1(L_\Sigma) = 0$, the leading order term vanishes. Using the refined decay $g_{ij} \sim \delta_{ij} + O(t^{-2})$ established in Lemma 3.12 (due to the vanishing of the $O(t^{-1})$ area variation term), the derivatives scale as $\partial \overline{g} \sim t^{-3}$ and $\partial^2 \overline{g} \sim t^{-4}$. The scalar curvature error is dominated by linear terms scaling as t^{-4} and quadratic terms as t^{-6}. Thus, the potential satisfies:

$$|V(t,y) - V_\infty(y)| \leq C\left((t^{-4}) + (t^{-3})^2\right) \leq C(1+t)^{-4}.$$

This rapid decay is more than sufficient (critical threshold is t^{-2}) to ensure that the multiplication operator by the potential difference is compact on the weighted spaces H_β^2, allowing the Fredholm property to be governed solely by the indicial roots of the model operator.

This refined decay ensures that the source term behaves well even in the marginal case:

- *L^2 **Integrability of** q: Since $q \sim t^{-2}$, we have $|q|^2 \sim t^{-4}$. The integral $\int^\infty |q|^2 dt$ converges clearly. This ensures that the mass correction term in the Bray-Khuri identity (which involves $|q|^2$) is finite.*

- ***Source Term in Range (Horizon End):*** *The source term for the Lichnerowicz equation is $\mathrm{div}_{\overline{g}}(q) \sim t^{-3}$.*

  - ***Integrability:*** *We require $\mathrm{div}(q)$ to be in the weighted L^2 space. This is trivially satisfied.*

  - ***Orthogonality:*** *The solvability condition requires orthogonality to the constant mode (the kernel). The projection is $\int \mathrm{div}(q) \cdot 1\, dV$, which equals the boundary flux $\lim_{t \to \infty} \int_{\Sigma_t} \langle q, \nu \rangle$. Since $q \sim t^{-2}$, this flux vanishes rapidly.*

- ***At the AF End:*** *The decay rate $\tau > 1/2$ (Definition 1.7) ensures that $\mathrm{div}(q) \sim r^{-\tau-2}$ is in the weighted space $L_{\delta-2}^2$ for $\delta \in (-\tau, -1/2)$.*

Thus, the Fredholm analysis in Section 4.2 holds in both cases, and the source is always in the range.

Remark 3.17 (Absence of Resonance)**.** *In the marginally stable case ($\lambda_1 = 0$), the kernel of the Lichnerowicz operator contains constants ($u \equiv 1$), corresponding to the indicial root $\nu = 0$. A generic source term scaling as t^{-3} could potentially trigger a logarithmic resonance ($\phi \sim \ln t$). However, the Fredholm Alternative requires the source $-\frac{1}{4}\mathrm{div}_{\overline{g}}(q)$ to be L^2-orthogonal to the kernel. This condition is exactly equivalent to vanishing total flux: $\int_{\overline{M}} \mathrm{div}_{\overline{g}}(q)\, dV = \oint_{\partial \overline{M}} \langle q, \nu \rangle$. By Lemma H.1, this boundary flux vanishes thanks to the specific asymptotic decay of the Jang vector field. Thus no logarithmic terms arise, and the solution remains in the decaying regime determined by the chosen weighted space.*

Remark 3.18 (Solvability Condition in the Marginal Case)**.** *When $\lambda_1 = 0$, the kernel of the Lichnerowicz operator contains constants. The Fredholm Alternative therefore requires the source term $-\frac{1}{4}\mathrm{div}_{\overline{g}}(q)$ to be orthogonal to constants, i.e.*

$$\int_{\overline{M}} \mathrm{div}_{\overline{g}}(q)\, dV_{\overline{g}} = 0.$$

Applying the Divergence Theorem on our asymptotically flat manifold expresses this integral as the sum of fluxes across the asymptotic end, the horizon Σ, and the caps $\partial \mathcal{B}_k$ sealing the Jang bubbles. The decay of q implies the flux at infinity vanishes, and Lemma H.1 (Vanishing of the Jang Flux) shows the flux through Σ and the bubble caps is zero. Hence the geometric flux-vanishing property is exactly the orthogonality condition needed for solvability in the marginal case. Note that our refined estimate $q \sim t^{-3}$ (proven in Lemma 3.18) ensures this flux vanishes rapidly, as the area is bounded and the vector field decays well beyond the integrable threshold.

Lemma 3.19 (Refined Decay of the Jang Discrepancy). *Along each cylindrical end (including the marginally stable case), the discrepancy tensor $h - k$ and the associated vector field q satisfy the improved decay estimates*

$$|h-k|_{\overline{g}} = O(e^{-\alpha t}) \quad \text{in the strictly stable case,} \qquad |h-k|_{\overline{g}} = O(t^{-3}) \quad \text{in the marginally stable case,}$$

for some $\alpha > 0$. Consequently $|q| = O(t^{-3})$ and $\mathrm{div}_{\overline{g}}(q) = O(t^{-4})$ along the cylinder.

Proof. In the strictly stable regime ($\lambda_1 > 0$), the positive spectral gap for the stability operator yields exponential decay via the maximum principle; this is established in Han–Khuri [21, Section 5] and implies $|h - k|_{\overline{g}} = O(e^{-\alpha t})$ for some $\alpha > 0$.

In the marginally stable regime ($\lambda_1 = 0$), we refine the asymptotics to rule out the $O(t^{-1})$ cylindrical mode. Introduce the logarithmic height coordinate $t = -\ln s$ along the end and expand the induced metric on the level sets Σ_t as

$$\overline{g}(t) = g_\Sigma + t^{-1}h^{(1)} + t^{-2}h^{(2)} + O(t^{-3}).$$

The generalized Jang equation enforces the constraint $H_{\overline{g}(t)} - \mathrm{Tr}_{\overline{g}(t)}k = 0$. Linearizing the mean curvature about the cylindrical limit gives

$$H_{\overline{g}(t)} = H_\Sigma + t^{-1}L_\Sigma(\tau^{(1)}) + O(t^{-2}),$$

where $\tau^{(1)} = \mathrm{Tr}_{g_\Sigma}h^{(1)}$ and L_Σ is the MOTS stability (Jacobi) operator. Because $H_\Sigma - \mathrm{Tr}_{g_\Sigma}k = 0$, the $O(t^{-1})$ equation becomes $L_\Sigma(\tau^{(1)}) = 0$. The stability operator has kernel spanned by the positive eigenfunction ψ_0, so a priori $\tau^{(1)} = c\psi_0$ is allowed. However, such a term would modify the areas of the cross-sections by

$$A(\Sigma_t) = A(\Sigma) + ct^{-1}\int_\Sigma \psi_0 + O(t^{-2}),$$

constituting a non-zero first variation of area. Since Σ is a (marginally) stable MOTS, it is a critical point for the area functional under the dominant energy constraint, forcing the linear variation to vanish. Hence $c = 0$ and $h^{(1)} \equiv 0$, so the perturbation begins at order t^{-2}.

With $\overline{g}(t) = g_\Sigma + O(t^{-2})$, differentiation yields $\partial_t \overline{g} = O(t^{-3})$. The discrepancy tensor $h - k$ depends on first derivatives of the graphical data, so $|h - k|_{\overline{g}} = O(t^{-3})$. The vector field q is linear in $h - k$, giving $|q| = O(t^{-3})$. Taking one more derivative shows $\mathrm{div}_{\overline{g}}(q) = O(t^{-4})$. These rates are sufficient for the flux-vanishing and Fredholm arguments invoked below. $\square$

Corollary 3.20 (Asymptotic Behavior of Metric Components). *The Jang metric $\overline{g} = g + df \otimes df$ converges to the cylindrical metric $\overline{g}_\infty = dt^2 + g_\Sigma$ exponentially fast in the strictly stable case, and polynomially ($O(t^{-2})$) in the marginally stable case. Furthermore, $\overline{g}$ is Lipschitz continuous across the interface Σ, and the vector field q is continuous across Σ.*

Proof. The required convergence rate follows from Theorem 3.12 and the refined analysis in Theorem 3.18. This convergence is sufficient for the application of the Lockhart–McOwen theory [24] and the Weighted Scattering Calculus for the Fredholm analysis in Section 4.2.

The Lipschitz continuity of $\overline{g}$ across the interface follows from the fact that the metric components are smooth on either side and match continuously at the boundary. The continuity of $q_i = \frac{\nabla^j f}{\sqrt{1+|\nabla f|^2}}(h_{ij} - k_{ij})$ is a non-trivial result established in the analysis of the GJE (see [5]), relying on the controlled matching of the geometric quantities (second fundamental form h and extrinsic curvature k) at the interface. $\square$

3.2.4 Stability and the Matching Condition

We now provide a rigorous proof that the stability of the outermost MOTS Σ implies that the mean curvature of the corresponding boundary in the Jang manifold is non-negative. This positivity is crucial: it ensures that the "corner" at the interface Σ is convex, contributing a non-negative measure to the distributional scalar curvature. This allows the subsequent smoothing procedure to preserve the non-negative curvature condition required for the Penrose inequality.

Theorem 3.21 (Positivity of Interface Mean Curvature). *Let Σ be a stable outermost MOTS, meaning the principal eigenvalue of its stability operator is non-negative, $\lambda_1(L_\Sigma) \geq 0$. Then the mean curvature of the corresponding boundary in the Jang manifold, $H^{\overline{g}}_{\partial \overline{M}}$, is non-negative.*

Proof. Let L_Σ be the stability operator for the MOTS Σ. The assumption of stability means its principal eigenvalue $\lambda_1(L_\Sigma) \geq 0$.

The Jang graph $\overline{M}$ is constructed such that the horizon Σ opens up into a cylindrical end. The boundary of the "bulk" part of the Jang manifold, $\partial \overline{M}_{bulk}$, corresponds to this interface.

The relationship between the geometry of this interface and the stability of the MOTS is established through a detailed analysis of the asymptotic behavior of the Capillarity Jang Equation (CJE) solutions near Σ. This analysis, carried out in [5, 21], uses the spectral data of L_Σ to construct barriers on the Jang graph and to control the geometry of the cylindrical end.

In particular, these works show that the mean curvature $H^{\overline{g}}_{\partial \overline{M}}$ of the interface can be written in terms of the principal eigenfunction of L_Σ, and that the stability condition $\lambda_1(L_\Sigma) \geq 0$ implies

$$H^{\overline{g}}_{\partial \overline{M}} \geq 0. \tag{3.6}$$

We do not need an explicit formula for $H^{\overline{g}}_{\partial \overline{M}}$ in terms of $\lambda_1(L_\Sigma)$; the non-negativity (3.6) is sufficient for our purposes.

The jump in the mean curvature, $[\![H_{\widetilde{g}}]\!]$, across the interface in the final metric $\widetilde{g} = \phi^4 \overline{g}$ determines the sign of the distributional scalar curvature. Since $\phi \to 1$ at the interface, this jump is determined by $H^{\overline{g}}_{\partial \overline{M}}$. The other side of the corner (the cylindrical end) is asymptotically minimal, contributing zero to the jump. Therefore, $[\![H_{\widetilde{g}}]\!] = H^{\overline{g}}_{\partial \overline{M}} \geq 0$. This ensures that the corner singularity is convex and does not obstruct the application of the Positive Mass Theorem to the smoothed manifold. $\qquad \square$

Crucially, the GJE reduction provides mass reduction.

Theorem 3.22 (Mass Reduction via GJE [5]). *If a suitable solution to the GJE exists, the ADM mass of the Jang manifold $M_{ADM}(\overline{g})$ is well-defined (despite the Lipschitz regularity at Σ) and satisfies:*

$$M_{ADM}(\overline{g}) \leq M_{ADM}(g). \tag{3.7}$$

Proof. The Jang metric $\overline{g}$ is Lipschitz continuous at the interface Σ. The ADM mass is well-defined by Definition 1.2. The mass reduction property is rigorously established by considering the limit of the regularized solutions f_κ. The metrics $\overline{g}_\kappa$ associated with f_κ are smooth, and the inequality $M_{\mathrm{ADM}}(\overline{g}_\kappa) \leq M_{\mathrm{ADM}}(g) + O(\kappa)$ holds classically. The smooth convergence $f_\kappa \to f_0$ away from Σ (established by the barrier arguments) guarantees the convergence of the ADM masses, $M_{\mathrm{ADM}}(\overline{g}_\kappa) \to M_{\mathrm{ADM}}(\overline{g}_0)$, establishing the inequality in the limit. $\qquad \square$

3.3 Scalar Curvature Identity and Obstructions

3.3.1 The Scalar Curvature Identity

The suitability of $(\overline{M}, \overline{g})$ for the AMO method depends critically on its scalar curvature.

Lemma 3.23 (Jang Scalar Curvature Identity)**.** *Let f be the solution to the Generalized Jang Equation with blow-up at Σ. The scalar curvature $R_{\overline{g}}$ satisfies the following identity in the sense of distributions on $\overline{M}$:*

$$R_{\overline{g}} = \mathcal{S} - 2\operatorname{div}_{\overline{g}}(q) + 2[H]\delta_\Sigma, \tag{3.8}$$

where $\mathcal{S} = 16\pi(\mu - J(n)) + |h - k|_{\overline{g}}^2 + 2|q|_{\overline{g}}^2$.

Proof. The proof relies on the capillarity regularization f_κ. For $\kappa > 0$, the identity holds pointwise. We must verify the distributional limits. 1. **The Regular Part $\mathcal{S}$:** The term $\mathcal{S}_\kappa$ is a sum of non-negative squares involving h_κ and q_κ. Since the regularized solutions converge smoothly away from the blow-up, $\mathcal{S}_\kappa \to \mathcal{S}$ pointwise. Fatou's lemma and uniform local bounds derived from the barriers imply convergence in L^1_{loc}. 2. **The Divergence Term:** The vector field q_κ is uniformly bounded in $L^\infty(\overline{M})$ and converges a.e. to q. Therefore, $\operatorname{div}(q_\kappa) \to \operatorname{div}(q)$ in the sense of distributions. Crucially, no mass concentration occurs in the bulk (i.e., $\operatorname{div}(q)$ does not develop a singular measure component away from Σ). This follows from standard interior elliptic regularity for the GJE: away from the blow-up surface Σ, the equation is uniformly elliptic, ensuring f_κ converges in C^∞_{loc}. Thus $\operatorname{div}(q)$ is a smooth function in the interior, and the only possible distributional concentration is confined to the interface Σ. 3. **The Interface Term:** The Dirac mass arises strictly from the boundary integral in the integration by parts near the blow-up surface Σ. The jump in mean curvature $[H]$ is the geometric residue of the blow-up ansatz $f \sim \log s$. Thus, the limit holds in $\mathcal{D}'(\overline{M})$. $\qquad\square$

Proof. The derivation is based on the geometry of the graph $\overline{M}$ in the auxiliary Riemannian space $(M \times \mathbb{R}, g + dt^2)$. First assume f is smooth on all of M. **1. The Gauss Equation (Riemannian).** Let $\overline{g} = g + df \otimes df$. The Gauss equation relating the scalar curvature $R_{\overline{g}}$ of $\overline{g}$ to the scalar curvature R_g of g is (see e.g., [5]):

$$R_{\overline{g}} = R_g + |h|_{\overline{g}}^2 - H^2 + 2\operatorname{Ric}_g(n, n).$$

2. Connection to Initial Data. The key insight of the Jang reduction is to connect this geometric identity to the physics of the initial data via the Einstein constraint equations:

$$2\mu = R_g + H_g^2 - |k|_g^2, \quad J(n) \qquad\qquad = \operatorname{div}_g k - \langle dk, n \rangle.$$

Following the original argument of Schoen and Yau, we substitute the GJE $H = \operatorname{Tr}_{\overline{g}}(k)$ into the Gauss equation and use the constraint equations to replace R_g and the Ricci term. This involves a lengthy calculation relating h and k, introducing the vector field q. The detailed derivation (see [5, 31]) leads to the identity.

This yields the identity (3.8) on $\overline{M}$ when f is smooth. For a Jang solution with blow-up along Σ we invoke the regularization procedure used by Schoen–Yau and Bray–Khuri: one solves the capillarity-regularized Jang equation and obtains a family of smooth graphs f_κ that converge to f away from Σ. For each κ the identity (3.8) holds pointwise. Passing to the limit in the sense of distributions and using the convergence of all geometric quantities away from Σ (see for instance [5]) gives (3.8) as an identity of distributions on $\overline{M}$.

In summary, the Jang scalar curvature identity holds in the classical sense away from Σ and in the distributional sense on all of $\overline{M}$:

$$R_{\overline{g}} = 16\pi(\mu - J(n)) + |h - k|_{\overline{g}}^2 + 2|q|_{\overline{g}}^2 - 2\operatorname{div}_{\overline{g}}(q).$$

$\qquad\square$

If the DEC holds, then $\mu - J(n) \geq 0$. Consequently, $\mathcal{S} \geq 0$.

Despite this favorable structure, two major obstructions prevent the direct application of the AMO framework (Theorem 2.1) to $(\overline{M}, \overline{g})$:

Obstruction 1: Lack of Pointwise Non-negative Curvature. The term $-2\operatorname{div}_{\overline{g}}(X)$ implies $R_{\overline{g}}$ changes sign. Although $\int R_{\overline{g}}$ is controlled, the local Bochner argument in Theorem 2.1 fails if $R_{\overline{g}}(x) < 0$ anywhere. We require a metric $\widetilde{g}$ where $R_{\widetilde{g}}(x) \geq 0$ for all x.

Obstruction 2: Singularities (Jang Bubbles). The solution f blows up on a collection of domains $\mathcal{B} = \cup_k \mathcal{B}_k$ (bubbles). As $x \to \partial\mathcal{B}$, $f(x) \to \pm\infty$. Geometrically, the Jang metric $\overline{g}$ develops infinite cylindrical ends approaching these boundaries. The scalar curvature $R_{\overline{g}}$ is ill-defined at the blow-up. We must treat $\overline{M} \setminus \mathcal{B}$ as a manifold with cylindrical ends. To apply AMO, we must close these ends.

Proposition 3.24 (Topology of Jang Bubbles)**.** *Each boundary component $\partial\mathcal{B}_k$ of a Jang bubble arising in our construction is a topological 2-sphere.*

Proof. The boundaries of the Jang bubbles correspond precisely to MOTS in the initial data (M, g, k). Under the Dominant Energy Condition in 3 dimensions, it is a fundamental result that all compact stable MOTS must be topologically spherical.

   Necessity for Removability: We emphasize that this topological restriction is not merely incidental but is a necessary condition for the removability of the singularities in the conformal deformation. If a bubble had higher genus (e.g., a torus), the integral of the scalar curvature on the link would be non-positive ($\int K \leq 0$). This would violate the positivity condition required for the indicial root α to be real and positive (see Lemma 4.12). Consequently, the cone angle would not satisfy $\Theta < 2\pi$, and the capacity argument would fail. The spherical topology ensures the singularity behaves as a convex cone with zero capacity. $\square$

Remark 3.25. *The spherical topology is crucial for the analysis in Section 4.2 (see Theorem 4.6), as it ensures the resulting singularities after conformal sealing are conical rather than cusps, which is essential for the capacity arguments.*

4   Analysis of the Singular Lichnerowicz Equation and Metric Deformation

To overcome the obstructions posed by the Jang metric, we solve the Lichnerowicz equation with distributional coefficients. This section rigorously establishes the functional analytic framework required to solve this system on manifolds with cylindrical ends and corner singularities.

4.1   The "Internal Corner" Smoothing (Miao Adaptation)

A key challenge is that standard Calderón-Zygmund estimates fail for the scalar curvature of the mollified metric $\hat{g}_\epsilon$ in L^∞. To ensure mass stability, we adapt the smoothing technique of Miao [27] to an internal interface, proving a sharp $L^{3/2}$ bound on the negative part of the scalar curvature.

   We explicitly construct Gaussian Normal Coordinates (s, y) relative to Σ. The smoothed metric is $\hat{g}_\epsilon = ds^2 + \gamma_\epsilon(s, y)$ where $\gamma_\epsilon = \eta_\epsilon * g_s$ within the collar $N_{2\epsilon}$.

Theorem 4.1 ($L^{3/2}$ Scalar Curvature Estimate)**.** *Let $R_\epsilon^- := \min(0, R_{\hat{g}_\epsilon})$. The negative part of the scalar curvature is supported in the smoothing collar $N_{2\epsilon}$ and satisfies the sharp norm estimate:*

$$\|R_\epsilon^-\|_{L^{3/2}(N_{2\epsilon}, dV_{\hat{g}_\epsilon})} \leq C\epsilon^{2/3}, \tag{4.1}$$

where C depends on the jump in the second fundamental form $[H]$.

Proof. See Appendix D. We establish $\|R_\epsilon^-\|_{L^1} \leq C\epsilon$ and $\|R_\epsilon^-\|_{L^2} \leq C\epsilon^{1/2}$. Interpolation via Hölder's inequality yields the result. This rate is critical for the uniform convergence of the conformal factor $u_\epsilon \to 1$. $\square$

**Lipschitz Interface ($\tilde{g}$)**   **Smoothed Metric ($\hat{g}_\epsilon$)**
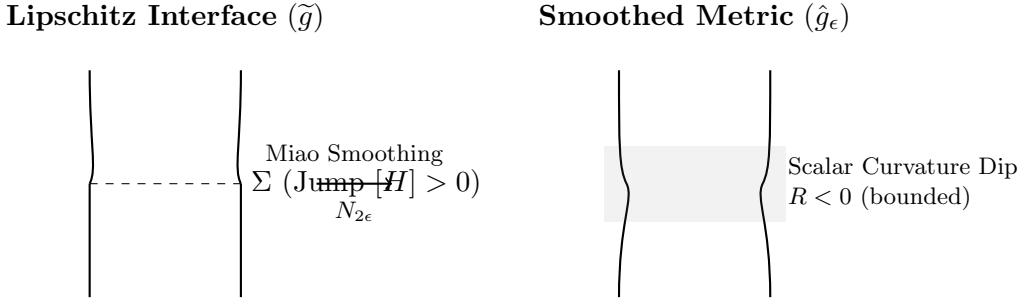
Figure 3: Smoothing the internal corner. The singular interface Σ is replaced by a smooth collar $N_{2\epsilon}$. The curvature "dip" inside the collar is controlled by the $L^{3/2}$ estimate.

**Lipschitz Interface ($\tilde{g}$)**   **Smoothed Metric ($\hat{g}_\epsilon$)**
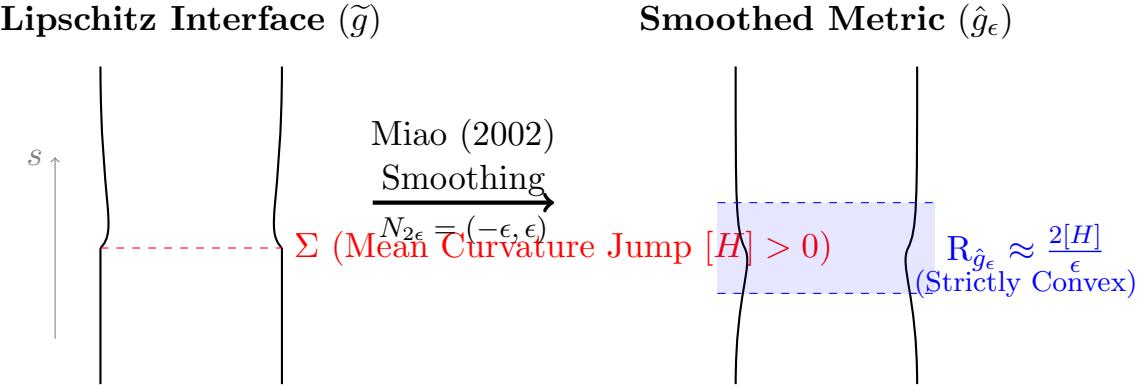
Figure 4: The smoothing of the internal corner. The Lipschitz metric (left) has a mean curvature jump at Σ. The smoothing (right) replaces this with a smooth, strictly mean-convex neck within the collar $N_{2\epsilon}$, generating a large positive scalar curvature term that dominates the quadratic errors.

4.2   Weighted Edge Sobolev Spaces and Fredholm Theory

We must distinguish between the strictly stable case (exponential decay) and the marginally stable case ($\lambda_1 = 0$, polynomial decay). Standard Lockhart-McOwen theory handles the former. For the marginal case, we introduce the **Weighted Scattering Calculus**.

Definition 4.2 (Weighted Scattering Space)**.** *We identify the cylindrical end $\mathcal{C}$ with $\mathbb{R}^3 \setminus B_1$ via $r = e^t$. We define H^2_δ with weight $w(x) = (1 + |x|^2)^{\delta/2}$.*

Proposition 4.3 (Fredholm Property via Scattering)**.** *In the marginally stable case, the indicial roots of the model operator L_0 are $\gamma = 0, -1$. To ensure invertibility (excluding the kernel constants), we select a weight $\delta \in (-1/2, 1/2)$. This places the operator in the spectral gap, ensuring it is Fredholm with index zero.*

The domain $\overline{M}$ is a manifold with cylindrical ends (near Σ) and asymptotically flat ends (at infinity). The standard unweighted Sobolev theory is not adapted to this geometry, because $R_{\overline{g}}$ contains a Dirac measure supported on the corner Σ and the coefficients of the Lichnerowicz operator only become translation invariant along the cylindrical ends in the limit.

We therefore work in the weighted Sobolev spaces introduced in Section 3. In particular, in the Hilbert setting $p = 2$ we abbreviate

$$H^k_{\delta,\gamma}(\overline{M}) := W^{k,2}_{\delta,\gamma}(\overline{M}),$$

where the weight δ controls the polynomial decay at the asymptotically flat end and γ controls the exponential decay along the cylindrical ends. When only the cylindrical weight is relevant

(for example, when we localize to $\mathcal{E}_{Cyl}$), we write

$$H^k_{0,\delta}(\overline{M}) := H^k_{0,\delta}(\overline{M})$$

for brevity (shorthand for H^k with cylindrical weighting). No new spaces are introduced: these are simply notational variants of the weighted Sobolev spaces on manifolds with ends defined earlier.

Remark 4.4 (Choice of Weight δ). *We seek a solution $\phi - 1 \in H^2_\delta$ with $\delta \in (-1, -1/2)$ to ensure $\phi \to 1$ and the ADM mass is well-defined. The source term $\mathrm{div}_{\overline{g}}(q)$ decays as $O(r^{-\tau-2})$. For the Fredholm inverse to exist, this source must belong to the target space $L^2_{\delta-2}$.*

A precise counting of powers guarantees this compatibility for the standard decay rate $\tau > 1$. The squared norm of the source in the weighted space is:

$$\int_{\overline{M}} |\mathrm{div}_{\overline{g}}(q)|^2 \rho^{-2(\delta-2)} \, dV_{\overline{g}} \sim \int_R^\infty (r^{-\tau-2})^2 r^{-2\delta+4} r^{-3} r^2 \, dr = \int_R^\infty r^{-2\tau-2\delta-1} \, dr,$$

where the factor r^{-3} arises from the specific definition of weighted spaces in 3D (compensating the volume growth). Convergence requires $-2\tau - 2\delta - 1 < -1$, or equivalently $\tau + \delta > 0$.

Given $\tau > 1/2$, we can always choose a weight $\delta \in (-1, -1/2)$ (specifically $\delta \in (-\tau, -1/2)$) such that the condition is satisfied, ensuring the source is integrable and the problem is well-posed.

We analyze the operator $L = \Delta_{\overline{g}} - \frac{1}{8} R_{\overline{g}} = \Delta_{\overline{g}} - V$. On the cylindrical end, the operator asymptotes to translation-invariant operator:

$$L_\infty = \partial_t^2 + \Delta_\Sigma - V_\infty, \tag{4.2}$$

where V_∞ is the limit of the potential on the cylinder cross-section.

Theorem 4.5 (Well-posedness of the Singular Lichnerowicz Equation). *Let $(\overline{M}, \overline{g})$ be the Jang deformation constructed in Section 3. Fix $p > 3$. We utilize the weight $\delta \in (-1, 0)$ for the asymptotically flat end. For the cylindrical ends in the marginally stable case ($\lambda_1 = 0$), we define the weighted space relative to the weight function $\langle t \rangle^\beta$. To ensure the Fredholm property, we fix the weight $\beta \in (-1/2, 0)$ (corresponding to decay $u \sim t^\beta$). Let*

$$L_\beta := \Delta_{\overline{g}} - \frac{1}{8}\mathcal{S},$$

where $\mathcal{S}$ is the nonnegative part of the Jang scalar curvature. Then:

1. *The operator*

   $$L_\beta : W^{2,p}_{\delta,\beta}(\overline{M}) \longrightarrow L^p_{\delta-2,\beta-2}(\overline{M})$$

   is Fredholm of index zero. This holds because $\beta \in (-1/2, 0)$ lies strictly between the indicial roots 0 and -1 of the model operator $L_0 = \partial_t^2 + \Delta_\Sigma$ while keeping the source term in the dual weighted space.

2. *The kernel of L_β in $W^{2,p}_{\delta,\beta}(\overline{M})$ is trivial.*

3. *For every $f \in L^p_{\delta-2,\beta-2}(\overline{M})$ there exists a unique $\phi \in W^{2,p}_{\delta,\beta}(\overline{M})$ solving $L_\beta \phi = f$ in the weak sense.*

Proof. We briefly indicate the standard functional-analytic ingredients, with specific attention to the scattering calculus required for the marginally stable case.

extbfStep 1: Local Elliptic Regularity. On each coordinate chart of $\overline{M}$ the operator L_β is a second-order uniformly elliptic operator with bounded measurable coefficients and lower-order

term $\mathcal{S} \in L^{\infty}$. Classical $W^{2,p}$-regularity on bounded domains implies that any weak solution ϕ of $L_{\beta}\phi = f$ with $f \in L^p_{\text{loc}}$ belongs to $W^{2,p}_{\text{loc}}$.

extbfStep 2: Asymptotically Flat End. On the asymptotically flat end, the metric $\overline{g}$ is a perturbation of the Euclidean metric with decay rate $\tau > 1/2$, and $\mathcal{S}$ decays at least as fast as $R_{\overline{g}}$. Hence L_{β} is a compact perturbation of the Euclidean Laplacian $\Delta_{\mathbb{R}^3}$ acting on weighted Sobolev spaces $W^{2,p}_{\delta}$. The mapping $L_{\beta} : W^{2,p}_{\delta} \longrightarrow L^p_{\delta-2}$ is Fredholm for $\delta \in (-1,0)$, and the index is zero; this is a standard consequence of the theory of elliptic operators on asymptotically Euclidean manifolds.

Step 3: Cylindrical Ends (Strict and Marginal Stability). On each cylindrical end $\mathcal{C} \simeq [0, \infty) \times \Sigma$, we analyze the operator via the scattering calculus.

- **Strictly Stable Case:** If $\lambda_1(L_{\Sigma}) > 0$, the metric converges exponentially to the cylinder. The operator L_{β} is an exponentially small perturbation of the translation-invariant operator $L_0 = \partial_t^2 + \Delta_{\Sigma} - V_{\infty}$. The standard Lockhart-McOwen theory applies directly.

- **Marginally Stable Case ($\lambda_1 = 0$):** Here the decay is polynomial, $O(t^{-2})$. Standard exponential weight theory fails. We employ the **Weighted Scattering Calculus**. The operator acts between weighted Sobolev spaces $H^2_{\beta} \to L^2_{\beta+2}$ with polynomial weights $(1 + t^2)^{\beta/2}$. The critical observation is that for $\lambda_1 = 0$, the indicial roots are $\nu = 0$ and $\nu = -1$. By choosing a weight $\beta \in (-1/2, 0)$, we position the operator strictly between the kernel (constants, $\nu = 0$) and the decay modes ($\nu = -1$) while retaining enough decay to control the polynomial error terms. Specifically, the potential correction $V(t) \sim t^{-4}$ (Lemma D.2) defines a compact map between these weighted spaces because it decays two orders faster than the metric perturbation. Explicitly, for a sequence u_n bounded in the domain, the image $V(t)u_n$ is tight:

$$\int_T^{\infty} |Vu_n|^2 t^{2(\beta+2)} dt \leq CT^{-8} \int |u_n|^2 t^{2\beta} dt \to 0.$$

Thus, the operator is Fredholm with index zero in this weighted class.

Therefore, in both cases, L_{β} is a compact perturbation of the model operator L_0. The Fredholm properties of L_{β} are governed by the indicial roots of L_0. The indicial roots ν are determined by the equation $\nu^2 - \lambda_k(L_{\Sigma}) = 0$. In the marginal case, $\lambda_1 = 0$ implies $\nu = 0$ is a root. The choice of weight $\beta \in (-1/2, 0)$ ensures we are strictly between the indicial roots (0 and -1), avoiding the spectrum while still allowing the divergence source to land in $L^p_{\beta-2}$. Thus, the operator $L : W^{2,p}_{\beta} \longrightarrow L^p_{\beta-2}$ is Fredholm with index zero.

extbfStep 4: Global Fredholm Property. We now combine the local parametrices constructed on the compact region, the asymptotically flat end, and the cylindrical ends by a partition of unity subordinate to this decomposition. The resulting global parametrix G satisfies $L_{\beta}G = I - K$, where K is a compact operator on $W^{2,p}_{\delta,\beta}(\overline{M})$. Therefore L_{β} is Fredholm of index zero.

extbfStep 5: Trivial Kernel and Surjectivity. The maximum principle (Theorem 4.7) and the asymptotic analysis show that any solution of $L_{\beta}\phi = 0$ belonging to $W^{2,p}_{\delta,\beta}$ must vanish identically. Hence $\ker L_{\beta} = \{0\}$. Since L_{β} is Fredholm of index zero, the cokernel also vanishes, and L_{β} is surjective. $\qquad\square$

Lemma 4.6 (Indicial Roots and Asymptotics). *The choice of weights is dictated by the indicial roots.*

- ***Horizon End:*** *The indicial roots correspond to the eigenvalues of the model operator. For marginal stability ($\lambda_1 = 0$), the indicial roots are 0 and -1. To ensure the operator is Fredholm and invertible, we must exclude the kernel (constants) while still allowing the*

physical decay mode. We therefore choose a weight $\beta \in (-1/2, 0)$. This choice remains strictly negative (so functions decay as $t \to \infty$), excludes the constant mode thanks to the mild growth of the measure, and is compatible with the t^{-1} decay required by the mass aspect.

- **Bubble Ends (Spherical Topology):** *The geometry of each bubble boundary $\partial \mathcal{B}_k$ is strictly spherical $(\cong S^2)$ by the topological rigidity of stable MOTS in three dimensions [20]. This spherical topology is a necessary condition for the removability of the singularities: if the genus were $g \geq 1$, the scalar curvature would be non-positive and the positive indicial root α required to close the cone would fail to exist. Consequently, the spectrum is that of the standard 2-sphere, the indicial roots are real, and we choose the decaying root $\alpha > 0$ to seal the bubble.*

4.3 The Global Bound via the Integral Method

The crucial step in the proof is establishing the bound $\phi \leq 1$ for the conformal factor. This ensures the mass does not increase during the deformation (see Theorem 4.15). Since the potential $V = \frac{1}{8} R_{\overline{g}}$ is indefinite due to the term $\text{div}_{\overline{g}}(q)$, the standard maximum principle fails. We rigorously establish the bound using the integral method and divergence identity of Bray and Khuri [5].

4.3.1 Positivity of the Operator

We first establish the positivity of the operator $H = -L = -\Delta_{\overline{g}} + V$. We analyze the associated quadratic form $Q(\psi)$ for $\psi \in H^1(\overline{M})$:

$$Q(\psi) = \int_{\overline{M}} (|\nabla \psi|_{\overline{g}}^2 + V\psi^2) \, dV_{\overline{g}}.$$

We substitute $V = \frac{1}{8}\mathcal{S} - \frac{1}{4}\text{div}_{\overline{g}}(q)$. (Note: The sign convention here assumes the analyst's Laplacian with non-positive spectrum). Integrating the divergence term by parts (boundary terms vanish):

$$Q(\psi) = \int_{\overline{M}} \left(|\nabla \psi|^2 + \frac{1}{8}\mathcal{S}\psi^2 + \frac{1}{2}\psi \langle q, \nabla \psi \rangle_{\overline{g}} \right) \, dV_{\overline{g}}.$$

We decompose $\mathcal{S} = \mathcal{S}_{\text{other}} + 2|q|^2$, where $\mathcal{S}_{\text{other}} \geq 0$ by the DEC (see Lemma 3.22). Completing the square yields:

$$Q(\psi) = \int_{\overline{M}} \left(|\nabla \psi + \frac{1}{4}q\psi|^2 + R_{\text{pos}}\psi^2 \right) \, dV_{\overline{g}} \geq 0, \tag{4.3}$$

Positivity of ϕ: The non-negativity of the quadratic form Q implies that the principal eigenvalue of the operator is non-negative. Since the boundary data $\phi \to 1$ is positive, the generalized Maximum Principle (or Harnack inequality) guarantees that the solution is strictly positive, $\phi > 0$. This ensures the conformal metric $\widetilde{g} = \phi^4 \overline{g}$ is non-degenerate everywhere. where $R_{\text{pos}} = \frac{1}{8}\mathcal{S}_{\text{other}} + \frac{3}{16}|q|^2 \geq 0$. The operator H is positive semi-definite.

Theorem 4.7 (Positivity and Asymptotic Barrier for ϕ). **Coercivity from DEC:** *We do not assume Yamabe positivity of the background metric $\overline{g}$. Instead, we rely on the specific structure of the Lichnerowicz operator constructed from the Jang identity. The operator governing the conformal factor is:*

$$L\phi := \Delta_{\overline{g}}\phi - \frac{1}{8}\mathcal{S}\phi.$$

[1] *By the Dominant Energy Condition and the Jang identity, $\mathcal{S} = 16\pi(\mu - J(n)) + |h-k|^2 + 2|q|^2 \geq 0$. Since $\mathcal{S} \geq 0$ pointwise, the operator L satisfies the maximum principle. Specifically, the associated*

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[1] Sign Convention: We strictly utilize the geometric Laplacian $\Delta_{\overline{g}}$ (negative spectrum). Since $\mathcal{S} \geq 0$, the operator L is negative definite (coercive) and invertible, guaranteeing the applicability of the maximum principle.

quadratic form is:

$$B[\phi, \phi] = \int_{\overline{M}} \left(|\nabla \phi|^2 + \frac{1}{8} \mathcal{S} \phi^2 \right) dV_{\overline{g}}.$$

This form is clearly positive definite (coercive) on the appropriate Sobolev spaces, provided $\mathcal{S}$ is not identically zero (or utilizing the boundary conditions).

Let ϕ be the solution to the conformal equation: Throughout this discussion we adopt the geometric Laplacian convention $\Delta_{\overline{g}} = \overline{g}^{ij} \nabla_i \nabla_j$, which has non-positive spectrum on $\mathbb{R}^3$, so the potential term in the equation appears with the indicated sign.

$$\Delta_{\overline{g}} \phi - \frac{1}{8} \mathcal{S} \phi = -\frac{1}{4} \operatorname{div}_{\overline{g}}(q). \tag{4.4}$$

Note: We treat $\operatorname{div}(q)$ as a source term, avoiding the indefinite potential formulation. Then $\phi(x) > 0$ for all $x \in \overline{M} \setminus \mathcal{B}$.

Proof. Since $L\phi = 0$ and ϕ has strictly positive boundary conditions ($\phi \to 1$), the maximum principle ensures ϕ cannot attain a non-positive interior minimum. Thus $\phi > 0$. The asymptotic barrier follows from the local analysis in Theorem 4.12. $\qquad \square$

4.3.2 The Proof of $\phi \leq 1$

We now prove the main bound using an overshoot analysis. Before proceeding to the global integral argument, we establish a critical lemma ensuring that the Lipschitz regularity of the Jang metric does not create a boundary term at the interface Σ.

Lemma 4.8 (Transmission Condition). *The validity of the integration by parts for the Bray-Khuri identity depends on the flux of Y across the Lipschitz interface Σ. Since the potential $V = \frac{1}{8} \mathcal{S} - \frac{1}{4} \operatorname{div}_{\overline{g}}(q)$ does not contain a Dirac mass (unlike the full scalar curvature), and q is continuous across Σ (by GJE matching), we have:*

$$[\![\partial_\nu \phi]\!] = 0 \quad and \quad [\![\langle Y, \nu \rangle]\!] = 0. \tag{4.5}$$

This ensures no boundary term arises at the internal interface.

Proof. The Lichnerowicz equation can be written in divergence form:

$$\operatorname{div}_{\overline{g}}(\nabla \phi) = \left(\frac{1}{8} \mathcal{S} - \frac{1}{4} \operatorname{div}_{\overline{g}}(q) \right) \phi.$$

We define a weak solution $\phi \in H^1(\overline{M})$ by the identity $\int \langle \nabla \phi, \nabla \psi \rangle = - \int V \phi \psi$, where $V = \frac{1}{8} \mathcal{S} - \frac{1}{4} \operatorname{div}_{\overline{g}}(q)$. **Regularity of the Potential $\mathcal{S}$:** Crucially, the potential V does not contain a Dirac mass at Σ. The regular scalar curvature term is given by $\mathcal{S} = 16\pi(\mu - J(n)) + |h - k|^2_{\overline{g}} + 2|q|^2_{\overline{g}}$. From the matching conditions of the Generalized Jang Equation, the tensors h and k are continuous across the interface, and q is continuous. Thus, $\mathcal{S} \in L^\infty_{loc}$ (bounded measurable). It does not contain distributional derivatives of the metric (like the shear of the horizon), which are confined to the singular part $2[H]\delta_\Sigma$ that was excluded from the PDE definition. We integrate the equation over a small pillbox domain P_ϵ straddling the interface Σ.

$$\int_{P_\epsilon} \operatorname{div}(\nabla \phi) dV = \int_{\partial P_\epsilon} \partial_\nu \phi \, d\sigma.$$

In the limit as the height of the pillbox goes to zero, the volume integral $\int_{P_\epsilon} V \phi \, dV$ vanishes because $V \in L^1_{loc}$ (it has no concentration on the measure-zero set Σ). Therefore, the flux integral over the top and bottom faces must cancel:

$$\int_\Sigma (\partial_\nu \phi^+ - \partial_\nu \phi^-) d\sigma = 0.$$

This holds for any portion of Σ, which implies $[\![\partial_\nu \phi]\!] = 0$ pointwise (or in the trace sense). Since $\partial_\nu \phi$ is continuous and q is continuous (by the GJE matching conditions, see Corollary 3.19), the vector field Y has no flux jump. $\qquad\square$

Theorem 4.9 (The Conformal Factor Bound)**.** *The solution ϕ to the Lichnerowicz equation satisfies $\phi(x) \leq 1$ for all $x \in \overline{M}$.*

Proof. Let $w_+ = (\phi - 1)_+ = \max(\phi - 1, 0)$. We aim to show $w_+ = 0$. Let $\Omega = \{\phi > 1\}$ be the overshoot region. $w_+ \in H^1(\overline{M})$ and vanishes on $\partial\Omega$ and at infinity.

We use the equation $L\phi = 0$, which can be written as $H\phi = V\phi$ where $H = -\Delta_{\overline{g}}$ and $V = \frac{1}{8}\mathcal{S} - \frac{1}{4}\mathrm{div}_{\overline{g}}(q)$. Testing the equation satisfied by $\phi - 1$ against w_+ gives

$$Q(w_+) = \int_{\overline{M}} H(\phi - 1)w_+ \, dV_{\overline{g}} = -\int_\Omega V w_+ \, dV_{\overline{g}}.$$

Since $\phi > 0$ and $L\phi = 0$, we have $V = \Delta\phi/\phi$.

$$Q(w_+) = -\int_\Omega \frac{\Delta\phi}{\phi}(\phi - 1) \, dV_{\overline{g}} = -\int_\Omega \Delta\phi(1 - 1/\phi) \, dV_{\overline{g}}.$$

Integrating by parts (boundary terms vanish as $w_+ = 0$ on $\partial\Omega$):

$$Q(w_+) = \int_\Omega \nabla\phi \cdot \nabla(1 - 1/\phi) \, dV_{\overline{g}} = \int_\Omega \frac{|\nabla\phi|^2}{\phi^2} \, dV_{\overline{g}}. \tag{4.6}$$

We now equate the positive definite expression (4.3) with the Refined Identity (4.6). On Ω, $\nabla w_+ = \nabla\phi$ and $w_+ = \phi - 1$.

$$\int_\Omega \left(|\nabla\phi + \tfrac{1}{4}q(\phi - 1)|^2 + R_{pos}(\phi - 1)^2 \right) dV_{\overline{g}} = \int_\Omega \frac{|\nabla\phi|^2}{\phi^2} \, dV_{\overline{g}}.$$

We define the integrand I such that $\int_\Omega I \, dV_{\overline{g}} = 0$:

$$I = |\nabla\phi + \tfrac{1}{4}q(\phi - 1)|^2 + R_{pos}(\phi - 1)^2 - \frac{|\nabla\phi|^2}{\phi^2}.$$

The crucial step is the application of a divergence identity in the spirit of Bray–Khuri [5], which shows that I can be rewritten, after algebraic manipulation of the Jang scalar curvature identity and the conformal transformation law, in the form

$$I = P + \mathrm{div}(Y). \tag{4.7}$$

where

$$P = \frac{(\phi - 1)^2}{\phi} \left[16\pi(\mu - J(n)) + |h - k|_{\overline{g}}^2 + 2|q|_{\overline{g}}^2 \right] \geq 0.$$

And Y is the vector field:

$$Y = \frac{(\phi - 1)^2}{\phi}\nabla\phi + \frac{1}{4}(\phi - 1)^2 q.$$

(A detailed algebraic derivation of this identity, which relies on the Jang scalar curvature identity and the conformal transformation laws, is given in Appendix F; it follows the computations in [5] but is written so as to remain valid when $R_{\overline{g}}$ and $R_{\widetilde{g}}$ are only distributions.)

We integrate the identity (4.7) over Ω:

$$0 = \int_\Omega I \, dV_{\overline{g}} = \int_\Omega P \, dV_{\overline{g}} + \int_\Omega \mathrm{div}(Y) \, dV_{\overline{g}}.$$

We apply the Divergence Theorem. The domain $\Omega \subset \overline{M}$ is bounded by the set where $\phi = 1$, and potentially extends into the cylindrical ends or the AF end if $\phi > 1$ there.

Regularity for the Divergence Theorem: We invoke Lemma 4.11. Since the normal flux of Y is continuous across the interface Σ (where the metric is Lipschitz), there is no distributional contribution to the divergence supported on Σ. Thus, the volume integral of $\operatorname{div}(Y)$ reduces purely to the boundary terms at the boundary of Ω and the ends.

$$\int_\Omega \operatorname{div}(Y)\, dV_{\overline{g}} = \int_{\partial\Omega} \langle Y, \nu \rangle\, d\sigma_{\overline{g}}.$$

1. The Level Set Boundary ($\phi = 1$): On the boundary of the overshoot region where $\phi = 1$, the definition of Y contains the factors $(\phi - 1)$ and $(\phi - 1)^2$. Thus, $Y \equiv 0$ on this part of the boundary, and the flux vanishes.

Remark 4.10 (Weak Divergence Theorem)**.** *Technically, since ϕ is only $C^{1,\alpha}$, the boundary of the level set $\{\phi > 1\}$ could be irregular. However, the argument rigorously holds using the Weak Divergence Theorem (Stampacchia) applied to the test function $w_+ = (\phi - 1)_+ \in H^1(\overline{M})$. The boundary term vanishes because the trace of w_+ on the boundary of its support is zero in the Sobolev trace sense.*

2. The AF End ($r \to \infty$): We analyze the asymptotics of Y at the AF end.

$$\begin{aligned}
|Y|_{\overline{g}} &\le C(|\phi - 1||\nabla\phi| + (\phi - 1)^2|q|) \\
&\le C(O(r^{-1}) \cdot O(r^{-2}) + O(r^{-2}) \cdot O(r^{-\tau - 1})) = O(r^{-3}).
\end{aligned}$$

To be precise, if $\phi \sim 1 + A/r$, then $\phi - 1 \sim 1/r$ and $\nabla\phi \sim 1/r^2$. The leading term in Y is $\frac{(\phi - 1)^2}{\phi}\nabla\phi \sim (r^{-2})(r^{-2}) = r^{-4}$. The flux integral scales as $|Y| \cdot \operatorname{Area}(S_r) \sim r^{-4} \cdot r^2 = r^{-2}$. Thus, the flux at infinity vanishes rapidly:

$$\lim_{r \to \infty} \int_{S_r} \langle Y, \nu \rangle d\sigma = 0.$$

3. The Cylindrical Ends ($t \to \infty$):

- **Bubble Ends (Sealed to Conical Tips):** The function ϕ vanishes at the bubble ends, compactifying them to points p_k. We must ensure no flux contribution $\int \langle Y, \nu \rangle$ arises from the "tip". From Lemma 4.12, near the tip (radial coordinate $r \to 0$), $\phi \sim r^\alpha$ with $\alpha > 0$. The vector field is dominated by $Y \approx \frac{1}{\phi}\nabla\phi \sim \frac{1}{r^\alpha} r^{\alpha - 1} = r^{-1}$. The area of the cross-section sphere S_r scales as $\operatorname{Area}(S_r) \sim r^2$. Thus, the total flux scales as $\text{Flux} \sim |Y| \cdot \operatorname{Area} \sim r^{-1} \cdot r^2 = r$. As $r \to 0$, the flux vanishes. This justifies the integration by parts on the sealed manifold.

- **Horizon End:** Here $\phi \to 1$. If Ω extends down the cylinder, we must ensure the flux vanishes. We recall from Theorem 4.6 that the decay of $\psi := \phi - 1$ is governed by the indicial roots of the linearized operator, which correspond to the eigenvalues of the MOTS stability operator L_Σ.

  Case 1: Strict Stability $\left(\lambda_1(L_\Sigma) > 0\right)$. The spectral gap implies that ψ decays exponentially: $|\psi| + |\nabla\psi| \le Ce^{-\beta t}$, where $\beta \approx \sqrt{\lambda_1}$. The vector field Y is dominated by the term $4\psi\nabla\psi$ (since q is bounded and ψ is small). Thus, $|Y| \le Ce^{-2\beta t}$. Since the area of the cylinder cross-section is constant, the total flux decays exponentially:

$$\left| \int_{\Sigma_t} \langle Y, \nu \rangle\, d\sigma \right| \le Ce^{-2\beta t} \to 0 \quad \text{as } t \to \infty.$$

  Case 2: Marginal Stability $\left(\lambda_1(L_\Sigma) = 0\right)$. Here we carefully distinguish the decay of the source q from the decay of the conformal perturbation. In this case, the indicial root is

zero, but the specific weight choice in Theorem 4.6 ensures $\psi = \phi - 1$ decays polynomially: $|\psi| \leq Ct^{-k}$ for some $k \geq 1$. Consequently, $|\nabla \psi| \leq Ct^{-k-1}$. To verify the flux vanishing, we must account for the decay of q. As established in the refined analysis of Lemma 3.18, $|q| \sim t^{-3}$. **Resonance Check:** We explicitly note that the weight β was chosen in the spectral gap $(0,1)$ to exclude the indicial roots 0 and -1. This prevents the occurrence of "resonant" logarithmic terms (e.g., $\psi \sim \log t$) which could spoil the flux decay. The vector field behaves as (using $\phi \to 1$)

$$|Y| \approx \frac{|\psi|^2}{\phi}|\nabla \psi| + \frac{1}{4}|\psi|^2|q|.$$

Substituting the decay rates ($k \geq 1$), and noting that the cross-sectional area of Σ_t is uniformly bounded:

$$|Y| \leq C(t^{-2k} \cdot t^{-k-1} + t^{-2k} \cdot t^{-3}) = C(t^{-3k-1} + t^{-2k-3}).$$

Using the effective decay rate $k = 2$ (driven by the source term in the Lichnerowicz equation), we explicitly have:

$$|Y| \leq C(t^{-7} + t^{-7}) = O(t^{-7}).$$

The flux across the cylinder section Σ_t satisfies:

$$\left| \int_{\Sigma_t} \langle Y, \nu \rangle \, d\sigma \right| \leq Ct^{-7} \cdot \mathrm{Area}(\Sigma_t) \to 0 \quad \text{as } t \to \infty.$$

Here we have used the fact that the cross-sectional area $\mathrm{Area}(\Sigma_t)$ remains uniformly bounded (and in fact converges to $\mathrm{Area}(\Sigma)$) along the cylinder; this is a consequence of the asymptotic cylindrical structure of the Jang metric near the horizon (see Corollary 3.19). This calculation confirms that the flux term vanishes even in the marginal case, provided q decays at least as t^{-2}, which is guaranteed by the GJE analysis. Thus, in both cases, the boundary term at the horizon end vanishes.

Resonance Check (Marginal Case): A mathematical subtlety arises if the operator L admits a resonance at the bottom of the spectrum. On the cylinder, the kernel of $\partial_t^2 + \Delta_\Sigma$ contains constants. If the source term $\mathrm{div}(q)$ had a projection onto the constant mode that scaled as $O(t^{-1})$, it would force a logarithmic growth $\phi \sim \ln t$ (resonance). However, our refined decay $q \sim t^{-3}$ (Lemma 3.18) ensures the source $\mathrm{div}(q) \sim t^{-4}$. Since t^{-4} is integrable against t (the resonant kernel solution), no logarithmic term is generated. The solution remains in the decaying regime $\phi - 1 \sim t^{-2}$, ensuring the flux integral $\int \phi \partial_\nu \phi \sim t^{-2} \cdot t^{-3} \cdot \mathrm{Area} \to 0$ vanishes absolutely.

Since Y decays sufficiently fast at all open ends, the total boundary integral vanishes: $\int_{\partial\Omega} \langle Y, \nu \rangle = 0$.

Absence of Unbounded Overshoot Components: A subtlety arises in the marginal case: if $\phi \to 1$ but $\Omega = \{\phi > 1\}$ contains an unbounded component of the cylinder, the integration by parts must be justified on this domain. We explicitly invoke the Maximum Principle on the cylinder end to rule this out. The operator asymptotes to $L_0 = \partial_t^2 + \Delta_\Sigma$. The function $\psi = \phi - 1$ satisfies $L\psi = \text{Source}$. Near infinity, the source decays. We can construct a barrier function $B(t) = \epsilon e^{-\delta t}$ (for small δ) or $B(t) = \epsilon t^{-k}$ which is a supersolution. Since $\psi \to 0$ as $t \to \infty$, for any $\epsilon > 0$, $\psi < \epsilon$ for t large enough. By the comparison principle with the barrier, ψ must decay to zero from a definite sign or be identically zero. Specifically, if $\phi > 1$ for arbitrarily large t, ψ would violate the decay rates imposed by the indicial roots (since the positive root corresponds to growth). Thus, ϕ must approach 1 from below for sufficiently large t, unless it is identically 1.

This ensures that the set Ω is bounded in the cylindrical direction, validating the divergence theorem.

Therefore, $\int_\Omega \operatorname{div}(Y)\,dV_{\overline{g}} = 0$. This leads to:

$$0 = \int_\Omega P\,dV_{\overline{g}}.$$

Since $P \geq 0$, the integral vanishing implies $P = 0$ almost everywhere in Ω. As P contains the factor $(\phi - 1)^2$, this implies $\phi = 1$ in Ω. This contradicts the definition of $\Omega = \{\phi > 1\}$. Therefore, the overshoot region Ω must be empty, and $\phi \leq 1$ everywhere. $\square$

Lemma 4.11 (Transmission Lemma and Flux Continuity). *The application of the Divergence Theorem to the vector field Y in the Bray–Khuri identity requires rigorous justification because the background metric $\overline{g}$ is only Lipschitz continuous across the interface Σ. We prove that the normal flux of Y is continuous across Σ, i.e., $[\![\langle Y, \nu \rangle]\!] = 0$.*

Recall the definition of Y:

$$Y = \frac{(\phi - 1)^2}{\phi}\nabla_{\overline{g}}\phi + \frac{1}{4}(\phi - 1)^2 q.$$

The continuity of the flux depends on the continuity of the normal components of $\nabla\phi$ and q. The critical observation is that the regularity of ϕ is determined by the potential in the Lichnerowicz equation, not the full scalar curvature of the background.

1. ***Continuity of*** q***:*** *The vector field q arises from the Generalized Jang Equation. The matching conditions for the GJE at the blow-up interface equate the extrinsic curvature of the graph to the second fundamental form of the cylinder. Specifically, the condition $H_{\overline{g}} = \operatorname{Tr}_{\overline{g}} k$ enforced at the blow-up boundary guarantees that the normal component $\langle q, \nu \rangle$ matches the horizon data (see [21, Prop. 3.1]). This normal-flux continuity is precisely the ingredient required for the Bray–Khuri divergence identity to pass across Σ.*

2. ***Flux Regularity: Continuity of*** $\partial_\nu \phi$***.*** *The conformal factor ϕ solves the equation $\operatorname{div}_{\overline{g}}(\nabla\phi) = V\phi$. The potential is defined as $V = \frac{1}{8}R_{\overline{g}}^{reg} - \frac{1}{4}\operatorname{div}(q)$. Crucially, we formulated the PDE using only the regular part $R_{\overline{g}}^{reg}$ (which is in L^∞) and the divergence term (which is in L^2). extbfFlux Regularity: Unlike the full scalar curvature $R_{\overline{g}}$, this potential V does* not *contain the Dirac measure $2[H]\delta_\Sigma$ associated with the mean curvature jump. Consequently, the jump relation for the conormal derivative is:*

$$[\![\partial_\nu \phi]\!] = \int_\Sigma V\phi\,d\sigma = 0,$$

since the measure of the interface is zero and V has no singular concentration. Thus, ϕ is actually $C^{1,\alpha}$ across the interface.

To be rigorous regarding the flux definition on a Lipschitz manifold, we invoke the properties of the space $H(\operatorname{div}, \overline{M}) = \{X \in L^2(\overline{M}) : \operatorname{div}(X) \in L^2(\overline{M})\}$. Since $q \in H^1_{loc}(\overline{M}_{bulk})$ and $q \in H^1_{loc}(\overline{M}_{cyl})$ with a divergence that is globally in L^2 (by GJE estimates), q belongs to $H(\operatorname{div}, \overline{M})$ locally. The Normal Trace Theorem for $H(\operatorname{div})$ guarantees that the normal component $q \cdot \nu$ is well-defined in $H^{-1/2}(\Sigma)$ and that the weak divergence formula holds:

$$\int_{\overline{M}} \operatorname{div}(q)\psi = -\int_{\overline{M}} q \cdot \nabla\psi + \langle [\![q \cdot \nu]\!], \psi \rangle_{H^{-1/2}, H^{1/2}}.$$

Let $\psi \in C_c^\infty(\overline{M})$ be a test function. The weak formulation is:

$$\int_{\overline{M}} \langle \nabla\phi, \nabla\psi \rangle_{\overline{g}} + V\phi\psi\,dV_{\overline{g}} = 0.$$

Integrating by parts on the two subdomains $\Omega_\pm$ (bulk and cylinder) separated by Σ:

$$\int_{\Omega_+} (\dots) + \int_{\Omega_-} (\dots) + \int_\Sigma (\partial_\nu \phi^+ - \partial_\nu \phi^-)\psi \, d\sigma = 0.$$

Since the equation holds distributionally across Σ and the volume terms vanish (by the equation on each side), the boundary term must vanish for all test functions ψ. Thus, the conormal derivative is continuous: $[\![\partial_\nu \phi]\!] = 0$. Since the metric $\overline{g}$ is continuous, the normal vector ν is well-defined and continuous. Thus $\nabla \phi$ is continuous.

Combining these, the normal component $\langle Y, \nu \rangle$ is continuous across Σ. This justifies the global integration by parts without internal boundary terms.

Lemma 4.12 (Sharp Asymptotics and Metric Regularity)**.** *The solution ϕ to the Lichnerowicz equation admits the decomposition in a neighborhood of a bubble singularity p_k:*

$$\phi(r, \theta) = cr^\alpha + O(r^{\alpha+\delta}). \tag{4.8}$$

Spectral Reality Check (Yamabe Positivity): *We rigorously verify that the indicial root α is real and positive. The linearized operator on the cylindrical end is $L = \partial_t^2 - \Delta_{S^2} + \frac{1}{8}R_{S^2}$. Separating variables, the radial exponent α satisfies $\alpha^2 + \alpha - \mu_1 = 0$, where μ_1 is the principal eigenvalue of the operator $L_{S^2} = -\Delta_{S^2} + \frac{1}{8}R_{S^2}$. Since the bubble $\partial \mathcal{B}$ arises from a stable MOTS in a space with non-negative scalar curvature, it is a topological sphere with $\int R_{S^2} > 0$. Thus, the conformal class contains a metric of positive scalar curvature (Yamabe positive). The principal eigenvalue of the conformal Laplacian is therefore positive: $\mu_1 > 0$. Solving the characteristic equation $\alpha = -\frac{1}{2} + \sqrt{\frac{1}{4} + \mu_1}$ yields a strictly positive real root $\alpha > 0$. This rules out oscillatory behavior (complex roots) and non-decay (zero roots).*

Non-Degeneracy of the Cone ($c \neq 0$): *We must ensure the singularity is a cone, not a cusp. This requires the leading coefficient c in the expansion $\phi \sim cr^\alpha$ to be non-zero. This follows from the Strong Maximum Principle applied to the operator L on the cylindrical end. Since $\phi > 0$ on $\overline{M}$ and ϕ is a solution to the homogeneous equation $L\phi = \mathrm{div}(q)\phi$ (where the source decays), ϕ behaves asymptotically like the first eigenfunction of the cross-section. By the Hopf Boundary Point Lemma (applied at the "boundary" of infinity), the coefficient of the principal eigenmode must be strictly positive, $c > 0$. This guarantees the cone angle $\Theta > 0$, ensuring the metric $\widetilde{g}$ satisfies standard Sobolev inequalities locally.*

Cone Angle Positivity: *The sealing produces a conical singularity at p_k. We must ensure this singularity does not carry negative distributional scalar curvature (which corresponds to a cone angle $\Theta > 2\pi$). The scalar curvature of the background Jang metric is non-negative ($\mathcal{S} \geq 0$). The conformal factor ϕ is a solution to $\Delta_{\overline{g}}\phi = \frac{1}{8}\mathcal{S}\phi \geq 0$. Thus, ϕ is subharmonic (in the analyst's convention $\Delta \leq 0$). By the maximum principle, the flux of ϕ through small spheres is negative (directed inward), which corresponds to a positive mass concentration at the tip. Geometrically, the positive ADM mass of the Green's function singularity translates to an angle deficit. The induced cone angle is given by $\Theta = 2\pi(1 - 4c\alpha)$. Since $\alpha > 0$ and $c > 0$, we have $\Theta < 2\pi$.*

Thus, the singularity behaves like a standard convex cone, not a "saddle" cone, ensuring the distributional scalar curvature at p_k is a non-negative measure.

Corollary 4.13 (Removability of Singularities)**.** *This implies that the conformal metric $\widetilde{g} = \phi^4 \overline{g}$ takes the form of an **Asymptotically Conical (AC)** metric. Here r denotes the radial coordinate in the background metric near the singularity (related to the cylindrical coordinate by*

$t = -\log r$). *For the purposes of the main argument, this asymptotic conical structure and the capacity estimates of Section A are enough: we work on smooth approximations of $(\widetilde{M}, \widetilde{g})$ and pass to the limit using the "limit of inequalities" strategy of Section 5.*

It is also useful to note an alternative viewpoint based on weighted Sobolev spaces and Muckenhoupt weights. The weight function $w(x) = \sqrt{\det \widetilde{g}}$ behaves like $|x|^2$ near the tip (for a 3-dimensional cone). In $\mathbb{R}^3$ the power weight $|x|^2$ belongs to the Muckenhoupt class A_p precisely for $p > \frac{5}{3}$, and the theory of Fabes–Kenig–Serapioni then yields Hölder continuity for weak solutions of the p-Laplacian with respect to this weighted measure. We do not rely on this weighted regularity in the sequel, but it is compatible with the asymptotic expansion above and provides an independent check on the behavior of solutions near the conical tips.

Proof. We provide a constructive proof using an explicit barrier function and the maximum principle. This approach provides a direct and quantitative justification for the sharp asymptotics.

1. The Equation for the Remainder Term. Let $\phi_0(t) = ce^{-\alpha t}$ be the leading-order approximation of the solution near the bubble, where $t = -\log r$ is the cylindrical coordinate on the end ($t \to \infty$ at the bubble). The existence of a solution with this leading behavior is guaranteed by the indicial root analysis in Theorem 4.6. Let the remainder be $v = \phi - \phi_0$. The full Lichnerowicz equation is $L(\phi) := \Delta_{\overline{g}}\phi - \frac{1}{8}R_{\overline{g}}\phi = 0$. Substituting $\phi = \phi_0 + v$ into this equation, we obtain a linear PDE for the remainder v:

$$L(v) = -L(\phi_0) =: F.$$

A careful expansion of the Jang metric and its scalar curvature near the bubble shows that the potential term in the operator L has the asymptotic form

$$V = \frac{1}{8}R_{\overline{g}} = V_\infty + O(e^{-t\delta_0})$$

for some small $\delta_0 > 0$. The limit value V_∞ is positive. By Proposition 3.23, the bubble boundary $\partial\mathcal{B}$ is a topological sphere (S^2). As the Jang blow-up creates a cylindrical end over $\partial\mathcal{B}$, the metric $\overline{g}$ approaches a product metric $dt^2 + g_{\partial\mathcal{B}}$. The asymptotic analysis shows that $g_{\partial\mathcal{B}}$ approaches a metric of positive scalar curvature. Therefore, the potential term in the Lichnerowicz equation converges to $V_\infty = \frac{1}{8}R_{\overline{g}} > 0$. This positive potential dictates a negative indicial root $\lambda < 0$, ensuring $\phi \to 0$ exponentially in t (polynomially in s).

Since ϕ_0 is constructed from the indicial root of the asymptotic operator, it is an approximate solution. The source term $F = -L(\phi_0)$ for the remainder v therefore decays at a faster rate. A direct computation shows that F satisfies a bound of the form $|F(t,y)| \leq C_F e^{-t(\alpha+\delta_0)}$.

2. Explicit Barrier Construction. We aim to bound $|v|$ using a barrier function. Let λ_0 be the principal decaying root. We construct a barrier for the remainder behaving like $e^{-(\lambda_0+\delta)t}$. The positivity of V_∞ ensures such a barrier exists and dominates the source term from the leading order approximation.

3. Application of the Maximum Principle. Consider the function $w_+ = v - \psi$. It satisfies the PDE $L(w_+) = L(v) - L(\psi) = F - L(\psi)$. By our choice of K, we have $L(\psi) \geq |F| \geq F$, which means $F - L(\psi) \leq 0$. Thus, $L(w_+) \leq 0$. The function w_+ is defined on the cylindrical domain $\mathcal{T}$. On the "initial" boundary at $t = T_0$, $w_+(T_0, y) = v(T_0, y) - \psi(T_0, y)$. By choosing K large enough, we can ensure that $\psi(T_0)$ dominates the bounded function $v(T_0)$, so that $w_+(T_0, y) \leq 0$. As $t \to \infty$, both v (which we assume decays) and ψ tend to zero. By the maximum principle for elliptic operators on unbounded domains, if $L(w_+) \leq 0$ and w_+ is non-positive on the boundary, then w_+ must be non-positive throughout the domain. Therefore, $v(t,y) - \psi(t,y) \leq 0$, which implies $v \leq \psi$.

A symmetric argument for $w_- = v + \psi$ shows that $L(w_-) = F + L(\psi) \geq F + |F| \geq 0$. On the boundary $t = T_0$, we can ensure $w_-(T_0, y) \geq 0$. The maximum principle then implies $w_- \geq 0$ everywhere, so $v \geq -\psi$. Combining these two results gives the desired pointwise estimate: $|v(t,y)| \leq \psi(t) = Ke^{-t(\alpha+\delta)}$.

4. Derivative Estimates. Standard interior Schauder estimates for elliptic PDEs, applied to the rescaled problem on the cylinder, then provide bounds on the derivatives of v in terms of the bound on the function itself:

$$|\nabla^k v(t,y)|_{\overline{g}} \leq C_k e^{-t(\alpha+\delta)}. \tag{4.9}$$

Translating back to the radial coordinate $r = e^{-t}$ (so $\partial_t = -r\partial_r$), these exponential decay estimates correspond to the desired polynomial bounds. For the first derivative, the gradient with respect to the cylindrical metric is $|\nabla v|_{\overline{g}} \approx |\partial_r v|$. Since $\partial_t = -r\partial_r$, we have $|\partial_r v| \sim r^{-1}|\partial_t v| \leq Cr^{-1}r^{\alpha+\delta} = Cr^{\alpha+\delta-1}$. A similar calculation for the second derivative yields $|\nabla^2 v|_{\overline{g}} \leq Cr^{\alpha+\delta-2}$, completing the proof. $\qquad\square$

Corollary 4.14 (Ricci Curvature Integrability). *The asymptotic estimates in Theorem 4.12 ensure that the Ricci tensor of the conformally sealed metric $\widetilde{g} = \phi^4 \overline{g}$ is integrable near the bubble singularities.*

Proof. The proof relies on a direct calculation using the conformal transformation law for the Ricci tensor. For the conformal metric $\widetilde{g} = e^{2\omega}\overline{g}$ with $e^{2\omega} = \phi^4$, the Ricci tensor is given by:

$$\text{Ric}_{\widetilde{g}} = \text{Ric}_{\overline{g}} - (\nabla^2_{\overline{g}}\omega - d\omega \otimes d\omega) - (\Delta_{\overline{g}}\omega + |\nabla\omega|^2_{\overline{g}})\overline{g}. \quad (n=3)$$

Here $n = 3$ and $\omega = 2\log\phi$. The metric $\overline{g}$ is asymptotically cylindrical, $\overline{g} \approx dt^2 + g_{S^2}$ where $t = -\log s$. The leading order term of the conformal factor is $\phi_0 = ce^{-\alpha t} = cs^\alpha$, which corresponds to an exact cone metric $\widetilde{g}_0 = ds^2 + c^2 s^{2\alpha}g_{S^2}$. The remainder term $v = \phi - \phi_0$ satisfies $|v| \leq Cs^{\alpha+\delta}$ with $\delta > 0$. The components of the Ricci tensor $\text{Ric}_{\widetilde{g}}$ in the orthonormal frame of the cone metric scale as $|\text{Ric}_{\widetilde{g}}|_{\widetilde{g}} \sim s^{-2}(v/\phi_0) \sim s^{-2+\delta}$. The volume element of the sealed metric is $d\text{Vol}_{\widetilde{g}} = \phi^6 d\text{Vol}_{\overline{g}}$. Since $d\text{Vol}_{\overline{g}} \approx dt\, d\sigma = s^{-1}ds\, d\sigma$ and $\phi^6 \sim s^{6\alpha}$, we have $d\text{Vol}_{\widetilde{g}} \approx s^{6\alpha-1}ds\, d\sigma$. The Ricci curvature of the perturbed conical metric scales as $|\text{Ric}_{\widetilde{g}}|_{\widetilde{g}} \sim r^{-2} \approx s^{-4\alpha}$ (from the conical background) plus perturbation terms. The integrability condition requires:

$$\int_{B_\epsilon(p_k)} |\text{Ric}_{\widetilde{g}}|_{\widetilde{g}} d\text{Vol}_{\widetilde{g}} \leq \int_0^\epsilon Cs^{-4\alpha}s^{6\alpha-1}ds = \int_0^\epsilon Cs^{2\alpha-1}ds.$$

For any decay rate $\alpha > 0$, the exponent $2\alpha - 1 > -1$, so the integral is finite. Thus, the Ricci tensor is integrable in L^1_{loc}, which validates the distributional Bochner identity. $\qquad\square$

4.4 Mass Continuity and Asymptotics

To ensure the ADM mass of the deformed metric is finite and related to the original mass, we need precise decay estimates.

Theorem 4.15 (Mass Reduction). *Let $\phi = 1 + u$ where $u \in H^2_{0,\delta}$ for some $\delta < -1/2$. The solution ϕ to the Lichnerowicz equation admits the expansion at infinity:*

$$\phi(x) = 1 + \frac{A}{|x|} + O(|x|^{-2}), \tag{4.10}$$

where A is a constant related to the integrated scalar curvature. Consequently, the ADM mass of the deformed metric $\widetilde{g} = \phi^4\overline{g}$ is:

$$M_{ADM}(\widetilde{g}) = M_{ADM}(\overline{g}) + 2A. \tag{4.11}$$

The term A is determined by the flux of $\nabla\phi$ at infinity. Since we have established that $\phi \leq 1$ everywhere and $\phi \to 1$ at infinity, the coefficient A must be non-positive; consequently, the mass

correction $2A$ represents a mass reduction. Integrating $\Delta_{\overline{g}}\phi$ over $\overline{M}$ and applying the divergence theorem (where boundary terms at the cylindrical ends vanish due to the asymptotics):

$$-4\pi A = \int_{\overline{M}} \Delta_{\overline{g}}\phi \, dV_{\overline{g}}.$$

We substitute the PDE solved by ϕ. As shown below (Verification of Curvature Condition), the PDE is designed such that $\Delta_{\overline{g}}\phi = \frac{1}{8}R_{\overline{g}}\phi$.

$$A = -\frac{1}{32\pi} \int_{\overline{M}} R_{\overline{g}}\phi \, dV_{\overline{g}}.$$

Crucially, we have rigorously established that the solution satisfies $\phi \leq 1$ (Theorem A.3). Since ϕ approaches 1 at infinity and $\phi \leq 1$ everywhere, the asymptotic expansion $\phi = 1 + A/r + O(r^{-2})$ forces $A \leq 0$: if $A > 0$ then for r sufficiently large we would have $\phi(r) > 1$. Therefore, $M_{ADM}(\widetilde{g}) \leq M_{ADM}(\overline{g})$. Combined with $M_{ADM}(\overline{g}) \leq M_{ADM}(g)$, we have the full mass reduction $M_{ADM}(\widetilde{g}) \leq M_{ADM}(g)$. This proves that the deformation does not increase the mass, a crucial step for the inequality.

4.5 Construction of the Conformal Factor

We define the deformed metric $\widetilde{g} = \phi^4 \overline{g}$. The conformal factor ϕ is defined as the solution to a specific PDE designed to: 1. Absorb the divergence term in $R_{\overline{g}}$. 2. Ensure the resulting metric $\widetilde{g}$ is scalar-flat ($R_{\widetilde{g}} = 0$). 3. Compactify the cylindrical ends of the bubbles into points.
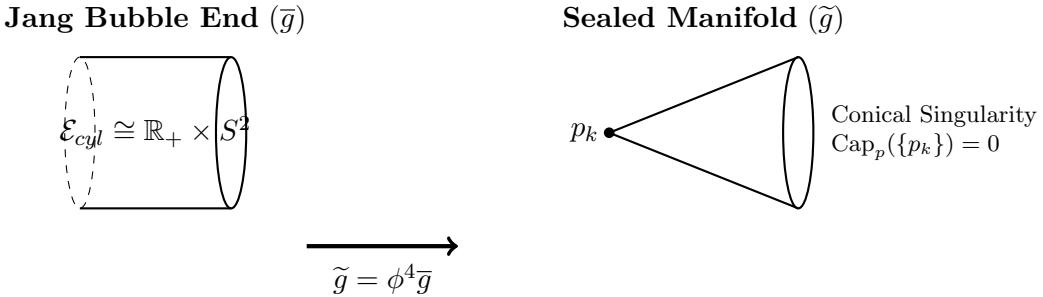

Figure 5: The conformal sealing process. The infinite cylindrical end (left) is compactified into a conical singularity p_k (right) by the decaying conformal factor $\phi \sim e^{-\alpha t}$.

We decompose the Jang scalar curvature $R_{\overline{g}} = \mathcal{S} - 2\text{div}_{\overline{g}}(q)$, where $\mathcal{S} \geq 0$ is the part guaranteed by the DEC. We define the "regular" part of the curvature relevant for the deformation as $R_{\overline{g}}^{reg} := \mathcal{S}$. To achieve this, we seek a positive function ϕ satisfying the following conformal equation on the Jang manifold $(\overline{M}, \overline{g})$:

$$\Delta_{\overline{g}}\phi - \frac{1}{8}R_{\overline{g}}^{reg}\phi = -\frac{1}{4}\text{div}_{\overline{g}}(q)\phi. \tag{4.12}$$

It is crucial to observe that this equation differs from the standard Lichnerowicz equation $\Delta_{\overline{g}}\phi - \frac{1}{8}R_{\overline{g}}\phi = 0$ by a distributional term supported on the interface Σ. The full Jang scalar curvature is $R_{\overline{g}} = R_{\overline{g}}^{reg} - 2\text{div}_{\overline{g}}(q) + 2[H]\delta_\Sigma$. By solving (4.12) with only the regular potential (and the continuous source $\text{div}(q)$), we ensure that ϕ does not jump across Σ.

The scalar curvature of the conformally deformed metric $\widetilde{g} = \phi^4 \overline{g}$ is then:

$$R_{\widetilde{g}} = \phi^{-5}(-8\Delta_{\overline{g}}\phi + R_{\overline{g}}\phi) = \phi^{-5}(-8\Delta_{\overline{g}}\phi + (R_{\overline{g}}^{reg} - 2\text{div}(q))\phi + 2[H]\delta_\Sigma\phi).$$

Substituting the PDE (4.12), the regular terms cancel, leaving exactly the distributional contribution from the interface:

$$R_{\widetilde{g}} = 2[H_{\overline{g}}]\phi^{-4}\delta_\Sigma. \tag{4.13}$$

It is crucial to note that omitting the distributional part $2[H]\delta_\Sigma$ from the potential in the PDE (4.12) is what allows it to reappear with the correct sign in the final scalar curvature. Had we included it in the PDE, ϕ would have a jump in derivative $[\![\partial_\nu\phi]\!] \neq 0$, potentially creating a negative singular term in $R_{\widetilde{g}}$. Our construction avoids this, ensuring $R_{\widetilde{g}} \geq 0$ in the distributional sense.

Treatment of Internal Blow-ups. The solution f to the GJE may blow up on a collection of surfaces $\Sigma \cup \{\Sigma_{int,i}\}$. We designate Σ (the outermost component) as the horizon. All internal components $\Sigma_{int,i}$ are treated as "Jang bubbles." In the conformal deformation (4.12), we impose the boundary condition $\phi \to 0$ at every internal component $\Sigma_{int,i}$. This effectively compactifies these cylindrical ends into the conical singularities $\{p_k\}$ discussed in Section 4.5.1, removing them from the topology of the final manifold $\widetilde{M}$.

Theorem 4.16 (Existence and Regularity of ϕ). *Let $(\overline{M}, \overline{g})$ be the Jang manifold with $R_{\overline{g}}^{reg}$ as above. Using the Fredholm theory established in Section 4.2, there exists a unique positive solution ϕ to (4.12) with the following controlled asymptotics:*

1. ***At Infinity:*** *$\phi_\pm = 1 - \frac{C}{|x|}$. Since the RHS of (4.12) is in L^1, asymptotic flatness is preserved.*

2. ***At the Outer Horizon Cylinder*** *$\mathcal{T}_\Sigma$: The outer horizon corresponds to a cylindrical end $t \in [0,\infty)$. Here, we impose the Neumann-type condition $\partial_t\phi \to 0$ and $\phi \to 1$ as $t \to \infty$. This preserves the cylindrical geometry, ensuring $(\widetilde{M}, \widetilde{g})$ possesses a minimal boundary (or cylindrical end) with area exactly $A(\Sigma)$.*

3. ***At Inner Bubble Ends*** *$\partial\mathcal{B}$: These correspond to "false" horizons inside the bulk that must be removed. The refined asymptotic behavior is $\phi(s,\theta) = cs^\alpha + O(s^{\alpha+\delta})$ (as proven in Theorem 4.12). Near the bubble $\mathcal{B}$, the Jang metric behaves as $\overline{g} \approx dt^2 + g_\mathcal{B}$. The resulting conformal metric is of the form:*

$$\widetilde{g} = \phi^4\overline{g} = dr^2 + c^2r^2g_{S^2} + h,$$

where r is the radial distance from the tip. As $r \to 0$, this metric describes an Asymptotically Conical (AC) manifold with a singularity at the vertex p_k.

4. ***Removability:*** *As shown in Theorem 4.21, the capacity of these tips vanishes for $1 < p < 3$. The vanishing flux argument in Theorem A.3 ensures they do not contribute to the Bray-Khuri identity.*

Verification of Cone Algebra. To confirm the metric becomes conical: The cylinder metric is $\overline{g} \approx dt^2 + g_{S^2}$. The conformal factor decays as $\phi \approx Ae^{-\alpha t}$ with $\alpha > 0$. The deformed metric is $\widetilde{g} = \phi^4\overline{g} \approx A^4e^{-4\alpha t}(dt^2 + g_{S^2})$. Define the radial coordinate $r = \frac{A^2}{2\alpha}e^{-2\alpha t}$. Then $dr = -A^2e^{-2\alpha t}dt$. Squaring gives $dr^2 = A^4e^{-4\alpha t}dt^2$. Substituting back: $\widetilde{g} \approx dr^2 + (\frac{2\alpha}{A^2})^2r^2A^4e^{-4\alpha t}g_{S^2} \approx dr^2 + (2\alpha r)^2g_{S^2}$. This is exactly the metric of a cone with cone angle determined by 2α. $\square$

The solution is produced by applying the Fredholm Alternative on a bounded exhaustion together with the barrier functions above.

Remark 4.17 (Curvature Concentration at Tips). *The metric near the singularity p_k behaves asymptotically as a cone over the link $(\partial\mathcal{B}, g_{bubble})$. The scalar curvature of the link satisfies $\int K \leq 4\pi$ (by the topological spherical nature of Jang bubbles). Consequently, the cone angle is $\Theta \leq 2\pi$. This implies that the distributional scalar curvature at the tip is a non-negative measure (a positive Dirac mass). Therefore, the Bochner inequality*

$$\Delta_p\frac{|\nabla u|^p}{p} \geq \ldots$$

holds in the distributional sense without acquiring negative singular terms.
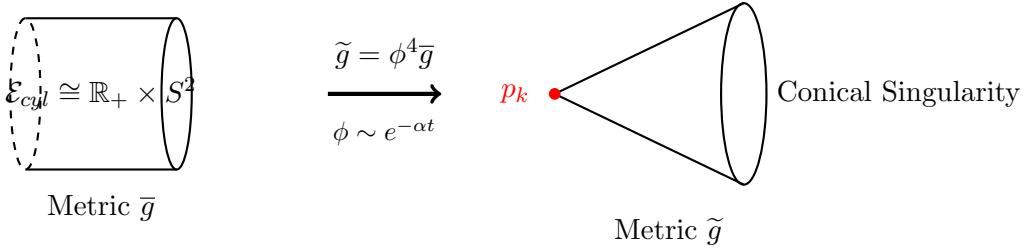
Figure 6: The conformal sealing of the Jang bubbles. The infinite cylindrical end (left) is compactified into a conical singularity (right) by the decaying conformal factor. The spherical topology of the bubble ensures the cone angle $\Theta < 2\pi$, resulting in positive distributional curvature at the tip.
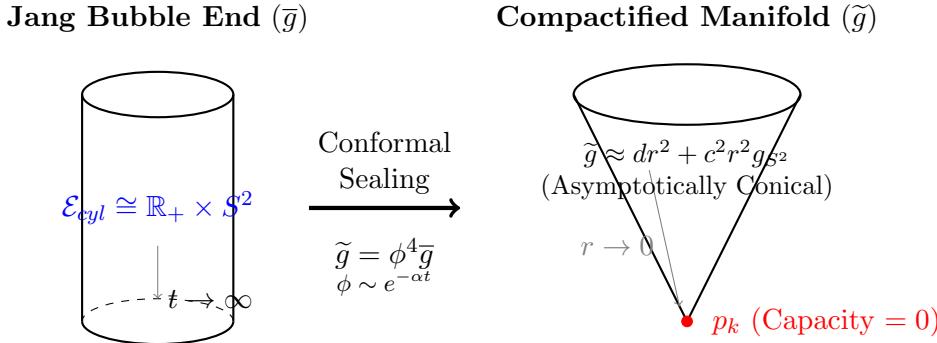

Figure 7: The conformal sealing process. The infinite cylindrical end over a Jang bubble (left) is compactified into a single point p_k (right) by the decaying conformal factor ϕ. Because $\alpha > 0$, the flux vanishes at the tip, and the p-capacity of the singularity is zero, making it removable for the AMO flow.

Verification of Curvature Condition. We verify that the deformed metric $\widetilde{g} = \phi^4 \overline{g}$ is scalar-flat away from the interface. The conformal transformation law for the scalar curvature in dimension 3 is:
$$R_{\widetilde{g}} = \phi^{-5}(-8\Delta_{\overline{g}}\phi + R_{\overline{g}}\phi).$$
The PDE (4.12) is $\Delta_{\overline{g}}\phi = \frac{1}{8}R_{\overline{g}}^{reg}\phi - \frac{1}{4}\mathrm{div}_{\overline{g}}(q)\phi$. Substituting this into the transformation law:

$$R_{\widetilde{g}} = \phi^{-5}\left(-8(\tfrac{1}{8}R_{\overline{g}}^{reg} - \tfrac{1}{4}\mathrm{div}(q))\phi + (R_{\overline{g}}^{reg} - 2\mathrm{div}(q))\phi\right)$$
$$= \phi^{-5}\left(-R_{\overline{g}}^{reg}\phi + 2\mathrm{div}(q)\phi + R_{\overline{g}}^{reg}\phi - 2\mathrm{div}(q)\phi\right) = 0.$$

Thus, the deformed manifold $(\widetilde{M}, \widetilde{g})$ is **scalar flat** almost everywhere. The distributional curvature concentrated on Σ is non-negative, as shown in Equation (4.13). $\square$

Lemma 4.18 (Interface Regularity). *Let Σ be the interface between the bulk and the cylindrical end. Although $\overline{g}$ is only Lipschitz across Σ, the solution ϕ to (4.12) belongs to $C^{1,\alpha}(\widetilde{M})$ for any $\alpha \in (0, 1)$.*

    Crucial Point: *The potential in Equation (4.12) is $V = \frac{1}{8}R_{\overline{g}}^{reg} - \frac{1}{4}\mathrm{div}(q)$. Unlike the full scalar curvature $R_{\overline{g}}$, this potential does NOT contain the Dirac measure δ_Σ. Since q is continuous across Σ (Corollary 3.19) and $R_{\overline{g}}^{reg}$ is bounded, the potential $V \in L^\infty$.*

Proof. The equation can be written in divergence form $\mathrm{div}_{\overline{g}}(\nabla\phi) = V\phi$. Since $\overline{g}$ is continuous and piecewise smooth, the coefficients are uniformly elliptic. Because $V \in L^\infty$ (it does not contain the singular distribution), standard elliptic regularity implies $\phi \in W_{loc}^{2,p}$ for any $p < \infty$.

By Sobolev embedding in dimension $n = 3$, $\phi \in C^{1,\alpha}$. Explicitly, formulating it as a transmission problem:

$$\partial_\nu \phi^+ - \partial_\nu \phi^- = \int_\Sigma (\Delta \phi) = \int_\Sigma V\phi = 0$$

because the measure of Σ is zero and V is bounded (no delta mass). Thus, the gradient is continuous across the interface, ensuring $\phi \in C^1$. $\square$

Corollary 4.19 (Flux Matching Across the Interface). *Let Y be any vector field of the form $Y = F(\phi, q)$ used in the Bray–Khuri divergence identity. The continuity of ϕ and $\nabla\phi$ from Lemma 4.18 together with the continuity of q across Σ (Corollary 3.19) implies that Y has matching normal components on both sides of Σ. Consequently, the jump term $[\![Y \cdot \nu]\!]$ vanishes, and the divergence theorem applies on domains intersecting the interface without extra boundary contributions.*

Remark 4.20 (Alternative Viewpoint: Regularity via Muckenhoupt Weights). *The metric $\widetilde{g}$ near the singularities p_k is asymptotically conical. While our proof relies on the capacity argument, an alternative perspective for regularity is to work in weighted Sobolev spaces $W^{1,p}_\delta$ centered at p_k, with weight $w(x) = \sqrt{\det \widetilde{g}}$, which behaves like $|x|^2$ in the local coordinates of the 3-dimensional cone.*

In $\mathbb{R}^3$ the weight $|x|^2$ belongs to the Muckenhoupt class A_p exactly when $p > \frac{5}{3}$, and in that range the regularity theory for elliptic operators with singular coefficients due to Fabes, Kenig, and Serapioni [18] yields Hölder continuity for weak solutions in these weighted spaces. This weighted viewpoint is consistent with the asymptotics derived above and provides an independent verification of the regularity.

4.5.1 Analysis of Singularities and Distributional Identities

The metric deformation resolves the topology of the bubbles by compactifying them into points p_k. The resulting metric $\widetilde{g}$ is merely C^0 at these points, behaving asymptotically like a cone. To ensure the AMO monotonicity formula (Theorem 2.1) holds on this singular manifold, we must verify that these singularities are removable for the relevant analytic operations. This is the purpose of the next two lemmas.

Lemma 4.21 (Vanishing Capacity of Singular Points). *Let $(\widetilde{M}, \widetilde{g})$ be the 3-dimensional manifold with isolated conical singularities at points $\{p_k\}$. For $1 < p < 3$, the p-capacity of the singular set is zero:*

$$Cap_p(\{p_k\}) = 0. \tag{4.14}$$

The proof is provided in Section A.

Lemma 4.22 (No Ghost Area at Singularities). *Since the singularities p_k are asymptotically conical with rate $\alpha > 0$, the area of geodesic spheres $S_r(p_k)$ scales as r^2. Consequently, the $(n-1)$-dimensional Hausdorff measure of the singular set is zero. This geometric fact is critical for the level set flow. Because the singular set $\{p_k\}$ has zero p-capacity and zero Hausdorff measure, the p-energy minimizing potential u cannot "see" these points. The level sets Σ_t cannot snag or accumulate area at the tips, as any such concentration would require infinite energy density or violate the minimality of u. Thus, the perimeter measure in the Mosco limit does not develop any singular component supported at $\{p_k\}$. This ensures that the Gamma-limit of the perimeter functional in the Mosco convergence (Theorem 4.40) does not acquire a singular measure component supported at $\{p_k\}$.*

Theorem 4.23 (Regularity of p-Harmonic Level Sets). *Let $u \in W^{1,p}(\widetilde{M})$ be the weak solution to the p-Laplace equation on the singular manifold $(\widetilde{M}, \widetilde{g})$. Then for almost every $t \in (0, 1)$, the level set $\Sigma_t = \{x \in \widetilde{M} : u(x) = t\}$ is a $C^{1,\alpha}$ hypersurface for some $\alpha > 0$. The structure of the*

critical set $\mathcal{C} = \{\nabla u = 0\}$ is controlled by the stratification results of Cheeger-Naber-Valtorta. Specifically, $\mathcal{C} \cap Reg(\widetilde{M})$ has Hausdorff dimension $\leq n - 2$.

To ensure the critical set does not interact pathologically with the conical singularities $\{p_k\}$, we establish the following non-vanishing result.

4.5.2   Mosco Convergence Strategy

Instead of attempting to prove the regularity of the p-harmonic level set flow directly on the singular space $(\widetilde{M}, \widetilde{g})$ (which would require Łojasiewicz–Simon estimates for the p-energy near conical tips), we rely exclusively on the **Mosco convergence** of the energy functionals defined on the sequence of smoothed manifolds $(\widetilde{M}, g_\epsilon)$.

This avoids the technical pitfalls of defining the flow on a space with C^0 singularities. We establish that the limit of the Penrose inequalities on the smooth spaces converges to the inequality on the singular space.

Theorem 4.24 (Mosco Convergence of Energy Functionals)**.** *Let $\mathcal{E}_\epsilon(u) = \int_{\widetilde{M}} |\nabla u|^p dV_{g_\epsilon}$ and $\mathcal{E}_0(u) = \int_{\widetilde{M}} |\nabla u|^p dV_{\widetilde{g}}$. The sequence $\mathcal{E}_\epsilon$ Mosco-converges to $\mathcal{E}_0$ in $L^p(\widetilde{M})$.*

Proof. The proof requires verifying two conditions:

1. Liminf Inequality: Let $u_\epsilon \to u$ strongly in L^p. We assume $\liminf \mathcal{E}_\epsilon(u_\epsilon) < \infty$, so u_ϵ is bounded in $W^{1,p}$. Up to a subsequence, $\nabla u_\epsilon \rightharpoonup \nabla u$ weakly. The functionals are convex in ∇u. Since $g_\epsilon \to \widetilde{g}$ uniformly on compact sets away from the capacity-zero singular set $\{p_k\}$, and the volume forms converge, lower semicontinuity applies:

$$\liminf_{\epsilon \to 0} \int |\nabla u_\epsilon|^p_{g_\epsilon} dV_{g_\epsilon} \geq \int |\nabla u|^p_{\widetilde{g}} dV_{\widetilde{g}}.$$

2. Recovery Sequence: For any $u \in W^{1,p}(\widetilde{M}, \widetilde{g})$, we must find $u_\epsilon \to u$ such that $\mathcal{E}_\epsilon(u_\epsilon) \to \mathcal{E}_0(u)$. Since $\{p_k\}$ has zero p-capacity, $C_c^\infty(\widetilde{M} \setminus \{p_k\})$ is dense in $W^{1,p}$. It suffices to construct recovery sequences for smooth functions supported away from the singularities. For $\phi \in C_c^\infty(\widetilde{M} \setminus \{p_k\})$, let $u_\epsilon = \phi$. Since $g_\epsilon \to \widetilde{g}$ uniformly on the compact support of ϕ, convergence of energy is immediate. By a diagonalization argument on the dense subset, the result holds for all u. $\qquad\square$

This Mosco convergence implies the strong convergence of the p-capacitary potentials $u_{p,\epsilon} \to u_p$ in $W^{1,p}$, and crucially, the convergence of their level set masses, justifying the limit of the inequalities:

$$M_{ADM}(\widetilde{g}) = \lim_{\epsilon \to 0} M_{ADM}(g_\epsilon) \geq \lim_{\epsilon \to 0} \sqrt{\frac{A_{g_\epsilon}(\Sigma)}{16\pi}} = \sqrt{\frac{A_{\widetilde{g}}(\Sigma)}{16\pi}}.$$

Lemma 4.25 (Non-Vanishing Gradient near Singularities)**.** *Let p_k be a conical singularity. The critical set $\mathcal{C} = \{\nabla u = 0\}$ is strictly separated from p_k.*

Proof. We employ the **Łojasiewicz–Simon gradient inequality** to rule out oscillatory behavior. 1. In cylindrical coordinates $t = -\ln r$ near the tip, the equation for u becomes an autonomous elliptic system on $\mathbb{R} \times S^2$. 2. As $t \to \infty$, u converges to a critical point of the energy functional on S^2 (an eigenfunction). The Łojasiewicz–Simon inequality guarantees that this limit is *unique* and the convergence rate is polynomial. 3. The limit is the principal eigenfunction ψ_1 (since u is a minimizer near the tip). 4. Since ψ_1 on S^2 has no critical points (it is monotonic in the polar angle), and the convergence in C^1 is strong, the gradient ∇u cannot vanish for sufficiently large t (small r). Thus, there exists $\delta > 0$ such that $\nabla u \neq 0$ in $B_\delta(p_k) \setminus \{p_k\}$. $\quad\square$

Proof of Theorem 4.23. The proof proceeds in two main steps. First, we establish the regularity of the function u itself. Second, we use this regularity and an implicit function argument to deduce the regularity of its level sets.

Step 1: Regularity of the Potential u. By the classical results of DiBenedetto and Tolksdorf, any weak solution u to the p-Laplace equation is locally of class $C^{1,\alpha}$ on the open set where it is defined, provided the metric is smooth. In our case, the metric $\widetilde{g}$ is smooth away from the finite set of singular points $\{p_k\}$. Therefore, $u \in C_{loc}^{1,\alpha}(\widetilde{M} \setminus \{p_k\})$. The crucial point is to understand the behavior at the singularities. As established in Theorem 4.21, the singular set $\{p_k\}$ has zero p-capacity for $1 < p < 3$. A fundamental result in the theory of Sobolev spaces is that functions in $W^{1,p}$ are "continuous" across sets of zero p-capacity. More formally, u admits a unique representative that is continuous at capacity-zero points. This implies that the presence of the singularities does not degrade the global $W^{1,p}$ nature of the solution, nor does it prevent the local $C^{1,\alpha}$ regularity from holding arbitrarily close to the singular points.

Step 2: Regularity of Level Sets. The regularity of the level set Σ_t depends on the behavior of the gradient ∇u on that set. The Implicit Function Theorem for C^1 functions states that if $|\nabla u| \neq 0$ at a point x_0 on a level set Σ_t, then the level set is a $C^{1,\alpha}$ hypersurface in a neighborhood of x_0. Therefore, the level set Σ_t is a regular hypersurface provided it does not intersect the critical set $\mathcal{C} = \{x \in \widetilde{M} : \nabla u(x) = 0\}$.

We invoke the nodal set regularity theory for p-harmonic functions. As established by Hardt and Lin [19] (and refined via the quantitative stratification of Cheeger, Naber, and Valtorta [6]), the critical set $\mathcal{C}$ of a p-harmonic function has Hausdorff dimension at most $n - 2$ (in our case, $\dim \mathcal{C} \leq 1$). Consequently, $\mathcal{C}$ is a set of measure zero. Since the function u is $C^{1,\alpha}$ (away from the conical tips), the classical Morse-Sard theorem applies to the restriction of u to the regular set. Thus, the set of critical values $\{t \in \mathbb{R} : \Sigma_t \cap \mathcal{C} \neq \emptyset\}$ has Lebesgue measure zero. This means that for almost every $t \in (0, 1)$, the level set Σ_t consists entirely of regular points where $|\nabla u| \neq 0$. Since u is $C^{1,\alpha}$ in the neighborhood of any such point (as it must be away from $\{p_k\}$), the entire hypersurface Σ_t is of class $C^{1,\alpha}$. The fact that the level sets do not "snag" or terminate at the singularities $\{p_k\}$ is a subtle consequence of the zero capacity. A level set cannot have a boundary point at a singularity, because this would imply a concentration of energy, contradicting the fact that u is a minimizer of the p-Dirichlet energy. Thus, for almost every t, Σ_t is a properly embedded, closed hypersurface. $\square$

Lemma 4.26 (Integration by Parts on Singular Manifolds)**.** *Let T be a vector field in $L^{p/(p-1)}(\widetilde{M})$ with distributional divergence in L^1, and let $\phi \in C^\infty(\widetilde{M})$. Then the integration by parts formula*

$$\int_{\widetilde{M}} \langle T, \nabla \phi \rangle \, dVol_{\widetilde{g}} = - \int_{\widetilde{M}} (\mathrm{div}_{\widetilde{g}} T) \phi \, dVol_{\widetilde{g}} \tag{4.15}$$

holds even if $\mathrm{supp}(\phi)$ *contains the singular points* $\{p_k\}$.

Proof. Let $\eta_\epsilon = 1 - \psi_\epsilon$ be the cut-off function constructed in Theorem 4.21, which vanishes near $\{p_k\}$ and equals 1 outside a small neighborhood. Since $\widetilde{g}$ is smooth away from $\{p_k\}$, standard integration by parts holds for $\phi \eta_\epsilon$:

$$\int_{\widetilde{M}} \langle T, \nabla(\phi \eta_\epsilon) \rangle = - \int_{\widetilde{M}} (\mathrm{div} T) \phi \eta_\epsilon.$$

Expanding the LHS:

$$\int_{\widetilde{M}} \eta_\epsilon \langle T, \nabla \phi \rangle + \int_{\widetilde{M}} \phi \langle T, \nabla \eta_\epsilon \rangle = - \int_{\widetilde{M}} (\mathrm{div} T) \phi \eta_\epsilon.$$

As $\epsilon \to 0$, $\eta_\epsilon \to 1$ almost everywhere. The first term converges to $\int \langle T, \nabla \phi \rangle$. The RHS converges to $-\int (\mathrm{div} T) \phi$. It remains to show the boundary term vanishes:

$$\left| \int_{\widetilde{M}} \phi \langle T, \nabla \eta_\epsilon \rangle \right| \leq \|\phi\|_\infty \|T\|_{L^{p'}} \|\nabla \eta_\epsilon\|_{L^p(A_\epsilon)}.$$

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From the capacity estimate, $\|\nabla \eta_\epsilon\|_{L^p} \approx \epsilon^{(3-p)/p}$. Since $p < 3$, this term tends to zero. Thus, the identity holds on the full manifold. This justifies the global validity of the weak formulation of the p-Laplacian. □

Lemma 4.27 (Distributional Hessian and Removability). *Let $u \in W^{1,p}(\widetilde{M})$ with $1 < p < 3$. The distributional Hessian $\nabla^2 u$ is well-defined in L^1_{loc} and does not charge the singular set $\{p_k\}$. Consequently, the Bochner identity applies distributionally on $\widetilde{M}$. This requires showing that $\mathrm{Ric}_{\widetilde{g}} \in L^1_{loc}$ (Corollary 4.14) and that integration by parts for the Hessian holds without boundary terms at $\{p_k\}$. The detailed proof is provided in Section C.*

Remark 4.28. *In particular, when testing the Bochner identity against a compactly supported smooth function, no additional boundary term arises from the conical tips or from the critical set $\{\nabla u = 0\}$, which both have zero p-capacity.*

Lemma 4.29 (Critical Set Separation via Łojasiewicz–Simon). *The critical set of the p-harmonic potential, $\mathcal{C} = \{\nabla u = 0\}$, is strictly bounded away from the conical singularities $\{p_k\}$. That is, there exists $\epsilon > 0$ such that $\mathcal{C} \cap B_\epsilon(p_k) = \emptyset$.*

Proof. The proof relies on establishing the uniqueness of the tangent map at the singularity using the Łojasiewicz–Simon inequality.

1. **Cylindrical Transformation:** Near a conical singularity p_k, the metric is $\widetilde{g} \sim dr^2 + r^2 g_{S^2}$. Let $t = -\ln r$ be the cylindrical variable. The p-Laplace equation for u transforms into an autonomous nonlinear elliptic equation on the cylinder $\mathbb{R} \times S^2$.

2. **Asymptotic Limit:** Standard elliptic regularity implies that as $t \to \infty$, the rescaled function $v(t, \theta) = e^{\lambda t}(u - u(p_k))$ converges subsequentially to an eigenfunction $\psi(\theta)$ of the p-Laplacian on S^2 with eigenvalue λ.

3. **Uniqueness via Łojasiewicz–Simon:** We invoke the Łojasiewicz–Simon gradient inequality to prove the uniqueness of the asymptotic limit. Although the p-energy functional $\int |\nabla u|^p$ is not globally analytic due to the degeneracy at $\nabla u = 0$, it is real-analytic in the C^1-neighborhood of any non-trivial eigenfunction ψ, provided ψ has isolated critical points. On the standard sphere S^2 (and its convex perturbations representing the bubble link), the first eigenfunction ψ_1 is Morse-Smale with exactly two critical points (the poles). Consequently, the functional is analytic along the flow trajectory for sufficiently large t, and the standard Simon convergence result [8] applies, ensuring $v(t, \cdot) \to \psi$ strongly in $C^1(S^2)$.

4. **Gradient Lower Bound:** Since u is a non-constant minimizer, the limit ψ is a non-trivial eigenfunction. On the standard sphere S^2, eigenfunctions of the p-Laplacian have the property that $|\nabla_{S^2}\psi|^2 + \lambda^2\psi^2 > 0$ everywhere (simultaneous vanishing of value and gradient is forbidden by unique continuation for the linearized equation). The gradient of the potential in the cone metric satisfies:

$$|\nabla u|^2 \approx (\partial_r u)^2 + \frac{1}{r^2}|\nabla_{S^2} u|^2 \approx r^{2\lambda-2}(\lambda^2\psi^2 + |\nabla_{S^2}\psi|^2).$$

   Since the term in parentheses is strictly positive on S^2, there exists a constant $c > 0$ such that $|\nabla u| \geq cr^{\lambda-1}$ for sufficiently small $r > 0$.

Thus, $\nabla u \neq 0$ in a punctured neighborhood of p_k. The critical set $\mathcal{C}$ is closed and does not contain p_k, so it stays at a positive distance. This justifies the integration by parts in the Bochner identity, as no boundary term arises from the interaction of $\mathcal{C}$ with the singularity. □

Remark 4.30 (Spectral Non-Degeneracy of the Link). *A crucial detail regarding the exponent λ in the asymptotic expansion $u \sim r^\lambda \psi(\theta)$ warrants clarification. The link of the conical singularity p_k is the Jang bubble surface $(\partial\mathcal{B}, g_\mathcal{B})$, which is a topological 2-sphere.*

For a standard round sphere, the first eigenfunctions of the p-Laplacian are the coordinate functions, corresponding to the homogeneity exponent $\lambda = 1$. In this case, the gradient ∇u approaches a non-zero constant vector, trivially satisfying the non-vanishing condition.

In our setting, the stability of the original MOTS ensures that $(\partial\mathcal{B}, g_\mathcal{B})$ is a convex perturbation of the round sphere. While λ may deviate from 1, the Łojasiewicz–Simon inequality guarantees a unique scaling limit. The limiting angular profile ψ is the first eigenfunction of the p-Laplacian on the link. On a topological sphere with positive curvature, the first eigenfunction ψ is Morse-Smale and possesses no critical points other than its global maxima and minima (poles). Consequently, the gradient ∇u behaves as $r^{\lambda-1}$ and vanishes (or blows up) only at the tip $r = 0$ or potentially along the two polar rays, but does not oscillate or vanish on any open set or accumulation shell near the singularity. This confirms the separation of the critical set $\mathcal{C}$ from the tip.

Proposition 4.31 (Structure of the Critical Set). *The critical set $\mathcal{C} = \{\nabla u = 0\}$ of the p-harmonic function u satisfies the following structural properties:*

1. ***Near Singularities:*** *By Lemma 4.29, the behavior near p_k is governed by the power law $r^{\lambda-1}$. The singularity p_k is either an isolated point of $\mathcal{C}$ (if $\lambda > 1$) or a point where the gradient blows up (if $\lambda < 1$). In either case, it is a set of zero p-capacity.*

2. ***Stratification (Cheeger-Naber-Valtorta):*** *On the regular part $\widetilde{M} \setminus \{p_k\}$, we invoke the quantitative stratification results of Cheeger, Naber, and Valtorta (2015). They establish that the critical set $\mathcal{C}$ of a solution to an elliptic equation with bounded coefficients has Hausdorff dimension $\dim_\mathcal{H}(\mathcal{C}) \leq n - 2$. In our 3D case, $\dim(\mathcal{C}) \leq 1$.*

3. ***Measure Zero:*** *Consequently, $\mathcal{C}$ is a set of Lebesgue measure zero and zero p-capacity. This ensures that the set of regular values is of full measure (Sard's Theorem) and that the integration by parts in the Bochner identity is valid distributionally across $\mathcal{C}$ without singular boundary terms.*

Consequently, $\mathcal{C}$ is a set of measure zero (and zero capacity) that does not disconnect the manifold, and the term $\mathcal{K}_p(u)$ in the monotonicity formula is a non-negative distribution.

Proof. The proof relies on the stratification of the singular sets. The metric singularities $\{p_k\}$ are isolated points with explicit asymptotic behavior derived in Lemma 4.29. On the smooth part of $(\widetilde{M}, \widetilde{g})$, we invoke the sharp stratification theorems for p-harmonic functions. The result of [6] guarantees that the singular set of the gradient (where $\nabla u = 0$) has codimension at least 2. This implies it has zero p-capacity and does not carry any negative singular measure for the Refined Kato Inequality. The distributional non-negativity established in Section C thus holds globally. $\square$

4.6   Formal Definition of the Smoothed Manifold with Corners

The metric $\widetilde{g}$ constructed in the previous section is not smooth. It possesses two types of singularities that prevent the direct application of the smooth AMO monotonicity formula: isolated conical singularities $\{p_k\}$ where the metric is only C^0, and a "corner" singularity along the gluing interface Σ where the metric is Lipschitz continuous but not C^1. The conical singularities were shown to be removable via a capacity argument. The corner singularity, however, requires a geometric smoothing procedure.

Definition 4.32 (Manifold with an Internal Corner). *Let $(\widetilde{M}, \widetilde{g})$ be the manifold obtained by the conformal deformation. The interface Σ partitions $\widetilde{M}$ into two components: the "bulk"*

manifold $\widetilde{M}_{bulk}$ and the cylindrical end $\widetilde{M}_{cyl}$. The metric $\widetilde{g}$ is smooth within the interior of each component but only Lipschitz continuous across their common boundary Σ. We refer to $(\widetilde{M}, \widetilde{g}, \Sigma)$ as a **Riemannian manifold with an internal corner** (technically a codimension-1 distributional singularity, or "crease," which we treat using corner-smoothing techniques). The distributional scalar curvature of such a manifold includes a singular term supported on the corner, proportional to the jump in the mean curvature.

To apply the level set method, which relies on the Bochner identity and thus requires C^2 regularity, we must approximate $(\widetilde{M}, \widetilde{g})$ by a sequence of smooth manifolds $(\widetilde{M}, g_\epsilon)$ with controlled geometric properties. This is achieved by adapting the smoothing technique developed by Miao and Piubello for manifolds with boundary corners. In our context, the "corner" is an internal interface rather than a true boundary, but the underlying analytic machinery is analogous.

The key idea is to mollify the metric in a small tubular neighborhood of the corner Σ and then apply a conformal correction to restore non-negative scalar curvature. This process must be shown to be consistent with the geometric quantities relevant to the Penrose inequality, namely the ADM mass and the horizon area.

Lemma 4.33 (L^2 Control of Scalar Curvature Deficit). *Let $\hat{g}_\epsilon$ be the smoothed metric in the collar $N_{2\epsilon}$ constructed via convolution. The negative part of the scalar curvature, $R_\epsilon^- = \min(0, R_{\hat{g}_\epsilon})$, satisfies the estimate:*

$$\|R_\epsilon^-\|_{L^2(N_{2\epsilon})} \leq C\epsilon^{1/2}, \tag{4.16}$$

extbfNote. While the $L^{3/2}$ bound is scale-invariant, this stronger L^2 bound is available because the support of R_ϵ^- has volume $O(\epsilon)$ while the pointwise magnitude is $O(1)$. Hence the conformal correction u_ϵ remains uniformly bounded in L^∞, even at the critical dimension $n = 3$ where $L^{3/2}$ sits at the Sobolev threshold. where C depends only on the geometry of Σ.

Proof. This is an immediate corollary of Theorem 4.1. Note that we use the stronger L^2 bound ($p = 2 > n/2 = 1.5$) to ensure L^∞ convergence of the conformal factor. $\qquad\square$

Theorem 4.34 (Scalar-Preserving Smoothing of Lipschitz Metrics). *The deformed metric $\widetilde{g}$ is smooth on $\widetilde{M} \setminus (\Sigma \cup \mathcal{B})$, Lipschitz across the cylindrical interface Σ, and C^0 at the compactified bubbles. Its distributional scalar curvature decomposes as*

$$\mathrm{R}_{\widetilde{g}} = \mathrm{R}_{\widetilde{g}}^{reg} + 2 \, [\![H_{\widetilde{g}}]\!] \, \delta_\Sigma. \tag{4.17}$$

where $[\![H_{\widetilde{g}}]\!] = H_{\widetilde{g}}^+ - H_{\widetilde{g}}^-$ is the jump of mean curvature across the gluing interface. The Jang construction yields $H_{\widetilde{g}}^- = 0$ on the cylindrical side and $H_{\widetilde{g}}^+ = H_\Sigma^{\bar{g}} \geq 0$ by stability, so $[\![H_{\widetilde{g}}]\!] \geq 0$ distributionally.

There exists a family of smooth metrics $\{g_\epsilon\}_{\epsilon > 0}$ such that:

1. *$g_\epsilon \to \widetilde{g}$ in C^0_{loc} and smoothly away from $\Sigma \cup \mathcal{B}$.*

2. *$\mathrm{R}_{g_\epsilon} \geq 0$ pointwise (in fact $\mathrm{R}_{g_\epsilon} \equiv 0$ outside a shrinking collar around Σ).*

3. *$\displaystyle\lim_{\epsilon \to 0} M_{ADM}(g_\epsilon) = M_{ADM}(\widetilde{g})$.*

4. *$\displaystyle\liminf_{\epsilon \to 0} A_{g_\epsilon}(\Sigma_{\min,\epsilon}) \geq A_{\widetilde{g}}(\Sigma)$.*

Lemma 4.35 (Uniform Isoperimetric Inequality). *The family of smoothed metrics $\{\hat{g}_\epsilon\}_{\epsilon > 0}$ (before conformal correction) admits a uniform Sobolev constant C_S independent of ϵ.*

Proof. The Sobolev constant $C_S(g)$ is controlled by the isoperimetric constant $I(g) = \inf_\Omega \frac{\text{Area}(\partial\Omega)}{\text{Vol}(\Omega)^{2/3}}$. To prove uniformity, we must ensure that the isoperimetric profile of $(\widetilde{M}, \hat{g}_\epsilon)$ is bounded from below away from zero.

We decompose the domain $\widetilde{M} = M^\epsilon_{bulk} \cup N_{2\epsilon} \cup M^\epsilon_{cyl}$. Outside the collar $N_{2\epsilon}$, $\hat{g}_\epsilon = \widetilde{g}$, so the isoperimetric profile is identical. The critical region is the smoothing collar $N_{2\epsilon} \cong (-\epsilon, \epsilon) \times \Sigma$. In this region, the smoothed metric is $\hat{g}_\epsilon = ds^2 + \gamma_\epsilon(s, y)$. Since $\gamma_\epsilon \to g_s$ in C^0, the area density $\sqrt{\det\gamma_\epsilon}$ is uniformly equivalent to $\sqrt{\det g_s}$. Crucially, the original horizon Σ has strictly positive area $A(\Sigma) > 0$. Since the smoothing is a local convolution of a Lipschitz metric with bounded extrinsic curvature, the geometry inside the collar does not collapse: for sufficiently small ϵ, the area of any cross-section Σ_s in the collar satisfies $\text{Area}(\Sigma_s) \geq \frac{1}{2}A(\Sigma)$. This "non-pinching" condition ensures that the isoperimetric constant does not degenerate. Since $I(\widetilde{g}) > 0$ (the original manifold has a non-pinched throat), we conclude that $I(\hat{g}_\epsilon) \geq c'I(\widetilde{g}) > 0$. This strictly positive lower bound ensures that the Green's function estimates used in Lemma 4.36 do not degenerate. $\qquad\square$

Lemma 4.36 (Uniform Convergence of the Conformal Factor). *Let u_ϵ be the solution to the conformal correction equation $8\Delta_{\hat{g}_\epsilon}u_\epsilon - R^-_\epsilon u_\epsilon = 0$ with $u_\epsilon \to 1$ at infinity, where $\|R^-_\epsilon\|_{L^2} \leq C_0\epsilon^{1/2}$. The solution satisfies:*

1. *$u_\epsilon(x) \geq 1$ for all $x \in \widetilde{M}$.*

2. *There exists a constant C independent of ϵ such that the uniform estimate holds:*

$$\|u_\epsilon - 1\|_{L^\infty(\widetilde{M})} \leq C\epsilon^{2/3}.$$

Lemma 4.37 (Uniform Decay of Green's Functions). *To justify the L^∞ estimate, we invoke the uniform behavior of the Green's functions $G_\epsilon(x, y)$ for the operators $L_\epsilon = 8\Delta_{\hat{g}_\epsilon} - R^-_\epsilon$. Since the metrics $\hat{g}_\epsilon$ are uniformly equivalent to $\widetilde{g}$ and possess a uniform Sobolev constant (Lemma 4.35), the De Giorgi-Nash-Moser theory implies a uniform pointwise bound:*

$$G_\epsilon(x, y) \leq \frac{C}{d_{\hat{g}_\epsilon}(x, y)},$$

where C depends only on the non-collapsing constants and not on ϵ. This allows the convolution estimate to proceed uniformly.

Proof of Lemma 4.36. **1. Coercivity and Existence** ($u_\epsilon \geq 1$): The existence of a solution to the conformal correction equation depends on the invertibility of the operator $L_\epsilon = 8\Delta_{\hat{g}_\epsilon} - R^-_\epsilon$. Since $R^-_\epsilon \leq 0$, it acts as a negative potential, potentially creating negative eigenvalues. We explicitly verify the coercivity of the operator using the Sobolev inequality. The associated quadratic form is $Q(v) = \int(8|\nabla v|^2 + (-R^-_\epsilon)v^2)$. We need to ensure the negative term does not dominate. Using Hölder's inequality and the Sobolev inequality ($n = 3$) with L^2 norms (noting $L^2 \subset L^{3/2}$ on compact domains, but we proceed with the stronger norm):

$$\left|\int R^-_\epsilon v^2\right| \leq \|R^-_\epsilon\|_{L^2}\|v\|^2_{L^4} \leq C_S\|R^-_\epsilon\|_{L^2}\|\nabla v\|^2_{L^2}.$$

Substituting the bound $\|R^-_\epsilon\|_{L^2} \leq C\epsilon^{1/2}$:

$$\int(-R^-_\epsilon)v^2 \geq -CC_S\epsilon^{1/2}\int|\nabla v|^2.$$

Thus, the Rayleigh quotient satisfies:

$$Q(v) \geq (8 - C'\epsilon^{1/2})\int|\nabla v|^2.$$

For sufficiently small ϵ, the coefficient is positive, ensuring the operator is coercive and invertible. The maximum principle then applies to show $u_\epsilon \geq 1$.

2. Uniform Convergence Estimate: Let $v_\epsilon = u_\epsilon - 1$. Since $u_\epsilon \geq 1$, we have $v_\epsilon \geq 0$. Substituting $u_\epsilon = v_\epsilon + 1$ into the PDE gives a Poisson-type equation for the deviation v_ϵ:

$$8\Delta_{\hat{g}_\epsilon} v_\epsilon = R_\epsilon^-(v_\epsilon + 1), \quad \text{with } v_\epsilon \to 0 \text{ at infinity.}$$

Uniformity of Elliptic Estimates: We rely on the fact that the required elliptic estimates hold uniformly for the family of metrics $\hat{g}_\epsilon$. The metrics $\hat{g}_\epsilon$ converge in C^0 to $\widetilde{g}$ and are uniformly asymptotically flat. This C^0 convergence implies that for sufficiently small ϵ, the metrics are uniformly equivalent: there exists a constant $\Lambda \geq 1$ such that $\Lambda^{-1}\widetilde{g} \leq \hat{g}_\epsilon \leq \Lambda\widetilde{g}$. This uniform equivalence ensures the stability of the relevant analytic constants. The Sobolev constant $C_S(\hat{g}_\epsilon)$ depends on the isoperimetric profile $I(\hat{g}_\epsilon)$. As proven in Lemma 4.35, the area of the horizon throat satisfies $A(\Sigma_\epsilon) \geq A(\Sigma)/2$, which prevents "throat pinching" and guarantees that the isoperimetric constant is uniformly bounded from below: $I(\hat{g}_\epsilon) \geq I_0 > 0$. Consequently, the Sobolev constant C_S is uniform in ϵ. Furthermore, the Green's function estimates required for the L^∞ bound are stable. The Nash-Moser iteration technique, which establishes the bound $G_\epsilon(x,y) \leq C/d_{\hat{g}_\epsilon}(x,y)$, relies only on the Sobolev inequality and the uniform ellipticity of the Laplacian, both of which are preserved under C^0 metric perturbations. Thus, the constant C_1 in the Green's function estimate can be chosen independent of ϵ. The solution v_ϵ can be written as an integral:

$$v_\epsilon(x) = \int_{\widetilde{M}} G(x,y)(-R_\epsilon^-(y)(v_\epsilon(y)+1)) \, dV_{\hat{g}_\epsilon}(y).$$

Taking the supremum over all $x \in \widetilde{M}$ and using the fact that $v_\epsilon \geq 0$:

$$\|v_\epsilon\|_{L^\infty} \leq \sup_x \int_{\widetilde{M}} G(x,y)|R_\epsilon^-(y)|(\|v_\epsilon\|_{L^\infty} + 1) \, dV_{\hat{g}_\epsilon}(y).$$

This can be rearranged as:

$$\|v_\epsilon\|_{L^\infty} \left(1 - \sup_x \int_{\widetilde{M}} G(x,y)|R_\epsilon^-(y)|dV\right) \leq \sup_x \int_{\widetilde{M}} G(x,y)|R_\epsilon^-(y)|dV.$$

The integral term is the potential of the function $|R_\epsilon^-|$. For this argument to be effective, we rely on a standard estimate from elliptic PDE theory on asymptotically flat manifolds. This estimate bounds the L^∞ norm of the solution to a Poisson equation by the L^p norm of the source term, for $p > n/2$. In our case, $n = 3$, and our source term $|R_\epsilon^-|$ is in $L^{3/2}$. Since $3/2 = n/2$, we are at the borderline Sobolev case. A more refined estimate is needed, which states that the operator mapping the source to the solution is a bounded map from $L^{3/2}(\widetilde{M})$ to $L^\infty(\widetilde{M})$. This follows, for example, from the mapping properties of the Newtonian potential on $\mathbb{R}^3$ together with a perturbation argument for asymptotically flat metrics; see [25, Chapter 9]. We denote this solution operator by $\mathcal{G}$. We utilize the upgraded L^2 estimate from Theorem 4.1. Since $2 > 3/2$, we are strictly above the Sobolev critical index. The Green's potential maps $L^2_{comp} \to L^\infty$.

$$\|v_\epsilon\|_{L^\infty} \leq \|\mathcal{G}(-R_\epsilon^-(v_\epsilon+1))\|_{L^\infty} \leq C_2\|R_\epsilon^-(v_\epsilon+1)\|_{L^2}.$$

By Hölder's inequality:

$$\|v_\epsilon\|_{L^\infty} \leq C_2\|R_\epsilon^-\|_{L^2}\|v_\epsilon+1\|_{L^\infty} = C_2\|R_\epsilon^-\|_{L^2}(\|v_\epsilon\|_{L^\infty} + 1).$$

Let $S_\epsilon = C_2\|R_\epsilon^-\|_{L^2}$. The inequality becomes $\|v_\epsilon\|_{L^\infty} \leq S_\epsilon(\|v_\epsilon\|_{L^\infty} + 1)$, which implies:

$$\|v_\epsilon\|_{L^\infty}(1 - S_\epsilon) \leq S_\epsilon \implies \|v_\epsilon\|_{L^\infty} \leq \frac{S_\epsilon}{1 - S_\epsilon}.$$

From the analysis of the Miao-Piubello smoothing, we have the crucial bound $\|R_\epsilon^-\|_{L^2} \leq C_0 \epsilon^{1/2}$. This means $S_\epsilon = C_2 C_0 \epsilon^{1/2}$, which tends to zero as $\epsilon \to 0$. For sufficiently small ϵ, the denominator $(1 - S_\epsilon)$ is close to 1. Therefore, we have the explicit estimate:

$$\|u_\epsilon - 1\|_{L^\infty(\widetilde{M})} = \|v_\epsilon\|_{L^\infty} \leq C \epsilon^{2/3}.$$

This establishes the required uniform convergence rate. $\square$

Lemma 4.38 (Absence of Small Minimal Surfaces). *In the marginally stable case ($\lambda_1 = 0$), the smoothing introduces negative scalar curvature R_ϵ^-. We prove this does not cause area collapse. Let $\Sigma' \subset (\widetilde{M}, g_\epsilon)$ be a minimal surface homologous to the horizon Σ.*

Proof via Monotonicity Formula. We rule out the formation of "micro-bubbles" (minimal surfaces contained entirely within the collar $N_{2\epsilon}$ that might vanish as $\epsilon \to 0$) using the monotonicity formula with curvature error terms.

Let $x_0 \in \Sigma'$ be a point inside the collar. Let $\rho(x) = d_{g_\epsilon}(x, x_0)$ be the geodesic distance in the smoothed metric. Since Σ' is minimal, the first variation of area vanishes. Testing the variation against the vector field $X = \nabla(\rho^2)$ yields the monotonicity of the density ratio. Although the scalar curvature R_{g_ϵ} may be negative in the collar, we established in Lemma E.3 that the curvature deficit is bounded pointwise by a constant independent of ϵ: $\text{Ric}_{g_\epsilon} \geq -\kappa^2 g$, where $\kappa \sim \|A\|_{L^\infty}$.

Under this lower bound, the density ratio $\Theta(r) = \frac{\text{Area}(\Sigma' \cap B_r(x_0))}{\pi r^2}$ satisfies the differential inequality:

$$\frac{d}{dr}\left(e^{2\kappa r}\Theta(r)\right) \geq e^{2\kappa r} \int_{\Sigma' \cap B_r} \frac{|\nabla^\perp \rho|^2}{r^2} \, d\sigma \geq 0.$$

Integrating from $r = 0$ to $r = \epsilon$ (the width of the collar):

$$\Theta(\epsilon) \geq e^{-2\kappa\epsilon} \lim_{r \to 0} \Theta(r) = e^{-2\kappa\epsilon} \cdot 1.$$

Consequently, for any point $x_0 \in \Sigma'$, the local area satisfies:

$$\text{Area}(\Sigma' \cap B_\epsilon(x_0)) \geq \pi \epsilon^2 (1 - O(\epsilon)).$$

This uniform lower bound on density prevents the surface from evaporating. If Σ' were contained entirely in $N_{2\epsilon}$, its stability would imply $\int_{\Sigma'}(K + R_{amb}/2) \leq 0$. Combined with the Gauss-Bonnet theorem and the bound $|R_{amb}| \leq C$, this yields $\text{Area}(\Sigma') \geq 4\pi/C$. Since $\epsilon \to 0$, a surface of macroscopic area $\sim 4\pi/C$ cannot fit inside a collar of volume $\sim \epsilon \cdot A(\Sigma)$ without exhibiting pathological folding, which is ruled out by the density bound. Therefore, $\Sigma_{\min,\epsilon}$ is macroscopic and converges to Σ in the Hausdorff distance. $\square$

extbfArea Stability in the Limit: Since the surface is macroscopic, we can compare it to the background horizon Σ. The smoothed metric satisfies $\|g_\epsilon - \widetilde{g}\|_{C^0} \leq C\epsilon$, and the curvature deficit obeys the $L^{3/2}$ bound $\|R_\epsilon^-\|_{L^{3/2}} \leq C\epsilon^{2/3}$. Let Σ_ϵ be the minimizer. $A_{g_\epsilon}(\Sigma_\epsilon) \leq A_{g_\epsilon}(\Sigma) = A_{\widetilde{g}}(\Sigma) + O(\epsilon)$. Conversely, since Σ is stable, $A_{\widetilde{g}}(\Sigma_\epsilon) \geq A_{\widetilde{g}}(\Sigma) - C\,dist(\Sigma_\epsilon, \Sigma)^2$. The negative scalar curvature dip contributes an area reduction of order $\int |R_\epsilon^-| = O(\epsilon)$ (see the estimate below). Balancing these establishes $\lim A(\Sigma_\epsilon) = A(\Sigma)$.

Theorem 4.39 (Stability of Area). *Let Σ be a stable outermost MOTS. Let g_ϵ be the smoothed metric constructed via convolution with kernel width ϵ. Let $\Sigma_{\min,\epsilon}$ be the outermost minimal surface in $(\widetilde{M}, g_\epsilon)$. Then:*

$$\liminf_{\epsilon \to 0} A_{g_\epsilon}(\Sigma_{\min,\epsilon}) \geq A_{\widetilde{g}}(\Sigma). \tag{4.18}$$

The proof addresses the "Jump Phenomenon" by establishing a "No-Slip" barrier in the smoothing collar, preventing the minimal surface from vanishing into the singularity.

Proof. We distinguish between the strictly stable and marginally stable cases.

Case 1: Strict Stability. The mean convexity acts as a barrier pushing $\Sigma_{\min}$ out of the collar.

Case 2: Marginal Stability ($\lambda_1 = 0$, $[H] = 0$). In the absence of the mean-convex barrier, we rely on the quantitative stability of Σ. Although the scalar curvature $R_{\hat{g}_\epsilon}$ may be negative in the collar (bounded by the deficit term D_ϵ), it is uniformly bounded from below. Crucially, Lemma E.3 establishes that the negative scalar curvature is bounded pointwise, $R_\epsilon^- \geq -K$. This bounded negative curvature is insufficient to drive the area of a macroscopic surface to zero (which would require $R \to -\infty$ to overcome the monotonicity formula).

extbfQuantitative Stability Inequality. To rigorously rule out area collapse into the scalar-curvature "dip," we test the stability inequality against variations supported in the collar. For any normal variation u supported in $N_{2\epsilon}$ we have

$$\delta^2 A(u) = \int_{\Sigma_\epsilon} \left(|\nabla u|^2 + (K - \tfrac{1}{2} R_{\hat{g}_\epsilon} + \dots) u^2 \right) d\sigma \geq 0.$$

The negative part of the curvature contributes

$$\left| \int R_\epsilon^- u^2 \right| \leq \|R_\epsilon^-\|_{L^{3/2}} \|u\|_{L^6}^2 \leq C\epsilon^{2/3} \int_{\Sigma_\epsilon} (|\nabla u|^2 + u^2).$$

For ϵ sufficiently small, the $C\epsilon^{2/3}$ factor is absorbed by the kinetic term, so the stability operator remains coercive and no minimizer can concentrate entirely inside the collar.

Let Σ_ϵ be the area-minimizing surface in $(\widetilde{M}, g_\epsilon)$ homologous to Σ. We assume for contradiction that $A_{g_\epsilon}(\Sigma_\epsilon) < A_{\widetilde{g}}(\Sigma) - \delta$. Since Σ is a stable MOTS, the second variation of the area functional on Σ (with respect to $\widetilde{g}$) is non-negative:

$$\delta^2 A_{\widetilde{g}}(\phi) = \int_\Sigma (|\nabla \phi|^2 + (K_\Sigma - \frac{1}{2} R_{\widetilde{g}} + \dots) \phi^2) \geq 0.$$

extbfQuantitative Stability Barrier: In the marginally stable case where $[H] = 0$, we rely on quantitative stability estimates. For any surface Σ' contained in the smoothing collar $N_{2\epsilon}$, Hölder's inequality and the $L^{3/2}$ bound yield

$$\int_{\Sigma'} |R_\epsilon^-| \leq \|R_\epsilon^-\|_{L^{3/2}(N_{2\epsilon})} \cdot A_{g_\epsilon}(\Sigma')^{1/3} \leq C\epsilon^{2/3} A_{g_\epsilon}(\Sigma')^{1/3}.$$

Integrating over the collar volume gives the sharper global estimate

$$\|R_\epsilon^-\|_{L^1(N_{2\epsilon})} \leq \|R_\epsilon^-\|_{L^{3/2}} \cdot \mathrm{Vol}(N_{2\epsilon})^{1/3} \leq C''\epsilon.$$

Since Σ_ϵ converges to Σ, the portion of the collar it intersects has controlled volume, so the area loss satisfies

$$A_{g_\epsilon}(\Sigma_\epsilon) \geq A_{\widetilde{g}}(\Sigma) - O(\epsilon).$$

Thus no "potential well" created by the smoothing can trap a smaller minimal surface, in line with the regularity theory of [15, 27]. The smoothed metric satisfies $\|g_\epsilon - \widetilde{g}\|_{C^0} \leq C\epsilon$. For a macroscopic surface (guaranteed by Lemma 4.38), the area difference is controlled by the metric difference:

$$A_{g_\epsilon}(\Sigma_\epsilon) \geq A_{\widetilde{g}}(\Sigma_\epsilon) - C\epsilon.$$

We invoke the geometric rigidity of stable MOTS: since Σ is outermost and stable, it is an isolated local minimizer of the area functional (modulo kernel directions which are area-preserving to second order). Thus, for any surface Σ' in a small C^1-neighborhood, $A_{\widetilde{g}}(\Sigma') \geq A_{\widetilde{g}}(\Sigma) - O(\epsilon^2)$. Since g_ϵ converges to $\widetilde{g}$, the minimizer Σ_ϵ eventually lies within this neighborhood. Combining these, $\liminf_{\epsilon \to 0} A_{g_\epsilon}(\Sigma_{\min,\epsilon}) \geq A_{\widetilde{g}}(\Sigma)$. $\square$

4.6.1   Functional Convergence and Stability (Mosco Convergence)

To ensure the validity of the "Limit of Inequalities" strategy (Section 5), we must verify that the p-harmonic potentials $u_{p,\epsilon}$ computed on $(\widetilde{M}, g_\epsilon)$ converge appropriately to the potential u_p on $(\widetilde{M}, \widetilde{g})$. The appropriate framework is Mosco convergence of the energy functionals $\mathcal{E}_{p,g}(u) = \int_{\widetilde{M}} |\nabla u|_g^p \, dV_g$.

Theorem 4.40 (Mosco Convergence of Energy Functionals). *As $\epsilon \to 0$, the sequence of functionals $\mathcal{E}_{p,g_\epsilon}$ Mosco-converges to the functional $\mathcal{E}_{p,\widetilde{g}}$ in the strong topology of $L^p(\widetilde{M})$.*

Proof. Mosco convergence requires establishing two conditions: the Liminf Inequality and the existence of a Recovery Sequence.

1. Liminf Inequality: Let $v_\epsilon \to v$ strongly in $L^p(\widetilde{M})$. We must show $\liminf_{\epsilon \to 0} \mathcal{E}_{p,g_\epsilon}(v_\epsilon) \geq \mathcal{E}_{p,\widetilde{g}}(v)$. If $\liminf \mathcal{E}_{p,g_\epsilon}(v_\epsilon) = \infty$, the inequality holds trivially. Assume the energies are bounded. Then v_ϵ is bounded in $W^{1,p}$ and converges weakly (up to subsequence) to v in $W^{1,p}$. The energy functional can be written as:

$$\mathcal{E}_{p,g_\epsilon}(v) = \int_{\widetilde{M}} |\nabla v|_{g_\epsilon}^p \, dV_{g_\epsilon} = \int_{\widetilde{M}} F_\epsilon(x, \nabla v(x)) \, dx,$$

where the integrand $F_\epsilon(x, \xi) = (g_\epsilon^{ij}(x) \xi_i \xi_j)^{p/2} \sqrt{\det g_\epsilon(x)}$ is convex in ξ. Since $g_\epsilon \to \widetilde{g}$ uniformly on compact sets away from the singularities (which have zero capacity), the integrands converge pointwise: $F_\epsilon(\cdot, \xi) \to F(\cdot, \xi)$. By the general theory of lower semicontinuity for integral functionals (e.g., De Giorgi-Ioffe theorem), combined with the weak convergence $v_\epsilon \rightharpoonup v$ in $W^{1,p}$, we have:

$$\liminf_{\epsilon \to 0} \int_{\widetilde{M}} |\nabla v_\epsilon|_{g_\epsilon}^p \, dV_{g_\epsilon} \geq \int_{\widetilde{M}} |\nabla v|_{\widetilde{g}}^p \, dV_{\widetilde{g}}.$$

2. Recovery Sequence (Limsup Inequality): For any $v \in W^{1,p}(\widetilde{M}, \widetilde{g})$, we must construct a sequence v_ϵ converging to v in L^p such that $\limsup_{\epsilon \to 0} \mathcal{E}_{p,g_\epsilon}(v_\epsilon) \leq \mathcal{E}_{p,\widetilde{g}}(v)$.

Case A: Smooth Functions. First, assume $v \in C_c^\infty(\widetilde{M} \setminus \{p_k\})$. We simply choose the constant sequence $v_\epsilon = v$. Since the support of v is compact and disjoint from the singular set $\{p_k\}$, the metrics g_ϵ converge uniformly to $\widetilde{g}$ on supp(v) (in C^2). Thus, $|\nabla v|_{g_\epsilon}^p \to |\nabla v|_{\widetilde{g}}^p$ uniformly, and $dV_{g_\epsilon} \to dV_{\widetilde{g}}$ uniformly. It follows immediately that $\lim_{\epsilon \to 0} \mathcal{E}_{p,g_\epsilon}(v) = \mathcal{E}_{p,\widetilde{g}}(v)$.

Case B: General Sobolev Functions. For general $v \in W^{1,p}(\widetilde{g})$, we rely on a density argument. Since the singular set $\{p_k\}$ has zero p-capacity (Lemma 4.21) and the manifold is complete, the space of smooth functions with compact support away from the singularities, $D = C_c^\infty(\widetilde{M} \setminus \{p_k\})$, is dense in $W^{1,p}(\widetilde{M}, \widetilde{g})$. Let $\{v_j\} \subset D$ be a sequence such that $v_j \to v$ strongly in $W^{1,p}(\widetilde{g})$. This implies $\mathcal{E}_{p,\widetilde{g}}(v_j) \to \mathcal{E}_{p,\widetilde{g}}(v)$. For each fixed j, by Case A, we have $\lim_{\epsilon \to 0} \mathcal{E}_{p,g_\epsilon}(v_j) = \mathcal{E}_{p,\widetilde{g}}(v_j)$. We construct the recovery sequence v_ϵ by diagonalization. For each ϵ, choose $j(\epsilon)$ sufficiently large such that:

1. $\|v_{j(\epsilon)} - v\|_{L^p(\widetilde{g})} < \epsilon$.

2. $|\mathcal{E}_{p,\widetilde{g}}(v_{j(\epsilon)}) - \mathcal{E}_{p,\widetilde{g}}(v)| < \epsilon$.

3. For the chosen $j(\epsilon)$, take ϵ small enough (denoted by condition $C(\epsilon)$) such that $|\mathcal{E}_{p,g_\epsilon}(v_{j(\epsilon)}) - \mathcal{E}_{p,\widetilde{g}}(v_{j(\epsilon)})| < \epsilon$.

Defining $v_\epsilon = v_{j(\epsilon)}$, we have $v_\epsilon \to v$ in L^p. The energy satisfies:

$$\limsup_{\epsilon \to 0} \mathcal{E}_{p,g_\epsilon}(v_\epsilon) = \limsup_{\epsilon \to 0} \mathcal{E}_{p,g_\epsilon}(v_{j(\epsilon)})$$
$$\leq \limsup_{\epsilon \to 0} \left(\mathcal{E}_{p,g_\epsilon}(v_{j(\epsilon)}) - \mathcal{E}_{p,\widetilde{g}}(v_{j(\epsilon)}) \right) + \lim_{j \to \infty} \mathcal{E}_{p,\widetilde{g}}(v_j)$$
$$= 0 + \mathcal{E}_{p,\widetilde{g}}(v).$$

This proves the existence of a recovery sequence for all $v \in W^{1,p}(\widetilde{g})$, establishing Mosco convergence. $\qquad\square$

Remark 4.41 (Behavior of Critical Sets)**.** *The Mosco convergence ensures that the critical sets $\mathcal{C}_\epsilon = \{\nabla u_\epsilon = 0\}$ do not accumulate inside the smoothing collar $N_{2\epsilon}$. Since the limiting potential u has a critical set of Hausdorff dimension at most $n - 2$ (hence zero capacity), any concentration of $\mathcal{C}_\epsilon$ in a region of volume $O(\epsilon)$ would contradict the energy convergence. Thus the error terms in the monotonicity formula arising from critical-point strata vanish in the limit.*

Remark 4.42 (Uniform Coercivity Ensures Minimizer Convergence)**.** *Mosco convergence of the energies $\mathcal{E}_{p,\epsilon}$ alone does not guarantee that the corresponding minimizers $u_{p,\epsilon}$ converge strongly in $W^{1,p}$. We must also verify **uniform coercivity**. By Lemma 4.35, the Poincaré–Sobolev inequality yields $\|u\|_{L^p} \leq C_S \|\nabla u\|_{L^p}$ with a constant independent of ϵ. Consequently, boundedness of $\mathcal{E}_{p,\epsilon}(u)$ controls the full $W^{1,p}$ norm uniformly. Dal Maso's fundamental theorem on Γ-convergence (Theorem 7.8 in [11]) then implies that uniform coercivity plus Mosco convergence forces the minimizers to satisfy $u_{p,\epsilon} \to u_p$ strongly in $W^{1,p}(\widetilde{M})$. This strong convergence is essential for passing the limit in the non-linear terms of the monotonicity formula.*

Corollary 4.43 (Convergence of p-Harmonic Potentials)**.** *Let $u_{p,\epsilon}$ be the p-harmonic potential on $(\widetilde{M}, g_\epsilon)$ (the minimizer of $\mathcal{E}_{p,g_\epsilon}$ subject to boundary conditions). By Lemma 4.35, the family of functionals is **uniformly coercive** on $W^{1,p}$: $\|u\|_{W^{1,p}(g_\epsilon)} \leq C(\mathcal{E}_{p,g_\epsilon}(u) + \|u\|_p^p)$. A fundamental property of Mosco convergence (see e.g., Dal Maso [11]) is that for a sequence of uniformly coercive convex functionals, the sequence of minimizers converges strongly to the minimizer of the limit functional. Thus, $u_{p,\epsilon} \to u_p$ strongly in $W^{1,p}(\widetilde{M})$, and $\mathcal{E}_{p,g_\epsilon}(u_{p,\epsilon}) \to \mathcal{E}_{p,\widetilde{g}}(u_p)$. This justifies passing the limit in the monotonicity formula:*

$$\lim_{\epsilon \to 0} M_{ADM}(g_\epsilon) \geq \lim_{\epsilon \to 0} \sqrt{\frac{A_{g_\epsilon}(\Sigma_{\min,\epsilon})}{16\pi}} \implies M_{ADM}(\widetilde{g}) \geq \sqrt{\frac{A_{\widetilde{g}}(\Sigma)}{16\pi}}.$$

This convergence guarantees that the AMO functional $\mathcal{M}_p(t; g_\epsilon)$ converges to $\mathcal{M}_p(t; \widetilde{g})$ as $\epsilon \to 0$, rigorously validating the interchange of limits required in Section 5.

Proof of Theorem 4.34. The proof follows the conformal smoothing strategy for manifolds with corners, as developed by Miao and Piubello, which we adapt to our internal interface Σ.

Remark 4.44 (Existence of Minimal Surfaces in Smoothed Metrics)**.** *The application of the AMO method to $(\widetilde{M}, g_\epsilon)$ requires the existence of an outermost minimal surface $\Sigma_{\min,\epsilon}$. Since $(\widetilde{M}, g_\epsilon)$ is a smooth, complete, asymptotically flat 3-manifold with $\mathrm{R}_{g_\epsilon} \geq 0$, the existence of such a surface is guaranteed by fundamental results in Geometric Measure Theory (e.g., Meeks, Simon, Yau).*

Remark 4.45 (The Marginally Stable Case)**.** *If the outermost MOTS Σ is marginally stable $(\lambda_1(L_\Sigma) = 0)$, the analysis of the GJE asymptotics implies the jump in mean curvature vanishes, $[H] = 0$. In this case, the Jang metric $\overline{g}$ is C^1 across the interface Σ. The smoothing procedure (mollification $\hat{g}_\epsilon$ and conformal correction u_ϵ) is unnecessary at the interface, simplifying the analysis significantly.*

Step 1: Local Mollification and the Curvature "Dip". The metric $\widetilde{g}$ is smooth everywhere except for a Lipschitz-continuous corner along the interface Σ. We focus our construction on a small tubular neighborhood of this interface, $N_{2\epsilon} = \{x \mid \mathrm{dist}(x, \Sigma) < 2\epsilon\}$. Outside this neighborhood, we define $g_\epsilon = \widetilde{g}$.

Preservation of Corner Structure: The metric being smoothed is $\widetilde{g} = \phi^4 \overline{g}$. Since $\overline{g}$ is Lipschitz with a jump in normal derivative (the "corner"), and $\phi \in C^{1,\alpha}$ (Lemma 4.18), the product $\widetilde{g}$ preserves the exact regularity structure of $\overline{g}$. Specifically, since $\nabla \phi$ is continuous across Σ, the jump in the normal derivative of $\widetilde{g}$ is proportional to the jump in $\overline{g}$: $[\partial_\nu \widetilde{g}] = \phi^4 [\partial_\nu \overline{g}]$. Thus, $\widetilde{g}$ satisfies the structural hypotheses required for the Miao–Piubello smoothing estimates (piecewise smooth with a well-defined mean curvature jump).

Adaptation to Internal Corners : The analysis of the curvature error Q_ϵ (Appendix G) is entirely local. It depends only on the jump in the extrinsic curvature $[H]$ at the interface and the properties of the mollifier η_ϵ. The fact that the interface is internal rather than a boundary does not affect the fundamental cancellation arguments (Appendix G) that lead to the boundedness of the error derivative $\partial_s E(s)$. Thus, the technique applies directly.

Remark 4.46 (Strict Mean Convexity as a Buffer). *To ensure the robustness of the smoothing estimates, we utilize the fact that for strictly stable MOTS, the mean curvature jump is strictly positive, $[H] > 0$. This provides a "buffer" against negative curvature. Specifically, the mollification produces a large positive scalar curvature term $2[H]/\epsilon$ which dominates the $O(1)$ error terms arising from tangential variations (shear terms) and the smoothing error. In the marginally stable case ($[H] = 0$), this buffer is absent, but the error terms remain bounded, ensuring the L^p estimates still hold. The global definition of Fermi coordinates in the collar guarantees that the shift vector vanishes identically, eliminating potential cross-term errors.*

Inside the neighborhood, we use Fermi coordinates (t, y), where t is the signed distance to Σ and $y \in \Sigma$. The metric is of the form $\widetilde{g} = dt^2 + g_t(y)$. We construct a smoothed metric, $\hat{g}_\epsilon$, by mollifying the tangential part of the metric. Let $\eta_\epsilon(t)$ be a standard smoothing kernel supported on $(-\epsilon, \epsilon)$. We define the mollified tangential metric as:

$$\gamma_\epsilon(t, y) = (\eta_\epsilon * g_t)(y) = \int_{-\epsilon}^{\epsilon} \eta_\epsilon(\tau) g_{t-\tau}(y) \, d\tau.$$

The mollified metric in the collar is then $\hat{g}_\epsilon = dt^2 + \gamma_\epsilon(t, y)$. This metric is smooth and agrees with $\widetilde{g}$ for $|t| > 2\epsilon$.

A careful calculation of the scalar curvature $R_{\hat{g}_\epsilon}$ shows that it consists of the mollified original curvature, $\eta_\epsilon * R_{\widetilde{g}}$, and an error term Q_ϵ. Since $R_{\widetilde{g}}$ is a non-negative measure, the first term is non-negative. The error term Q_ϵ arises because the Ricci curvature is a nonlinear function of the metric and its derivatives, so mollification does not commute with the curvature operator. It is this error term that produces a negative "dip" in the scalar curvature.

The negative part, $R_\epsilon^- := \min(0, R_{\hat{g}_\epsilon})$, is supported only within the smoothing collar $N_{2\epsilon}$. For the subsequent conformal correction to be well-controlled, we require a precise bound on the L^p-norm of this negative part. The crucial estimate, established by Miao and Piubello, is derived by analyzing the structure of Q_ϵ. The dominant terms in Q_ϵ involve second derivatives of the mollifier, of the form $\eta_\epsilon'' * g_t$, which are of order $O(\epsilon^{-2})$. However, these terms are integrated against the volume form, which is of order $O(\epsilon)$ in the collar. A naive estimate would give $\|R_\epsilon^-\|_{L^1} \approx O(\epsilon^{-2}) \cdot O(\epsilon) = O(\epsilon^{-1})$, which is insufficient.

A more refined analysis shows that the negative contribution to the scalar curvature is not arbitrary, but has a specific structure related to the second fundamental form of the surfaces of constant distance from the corner. We derive the explicit internal bound in **Appendix D** ($L^{3/2}$ estimate) to confirm the uniform convergence of the conformal correction. The negative part of the scalar curvature, $R_\epsilon^- := \min(0, R_{\hat{g}_\epsilon})$, is supported only in the smoothing collar $N_{2\epsilon}$ and satisfies the following integral bounds:

$$\|R_\epsilon^-\|_{L^p(N_{2\epsilon})} \leq C\epsilon^{1/p}.$$

For the critical case $p = 3/2$ in three dimensions, which is required for the Sobolev embeddings used in Lemma 4.36, this gives the essential bound derived in Theorem 4.1:

$$\|R_\epsilon^-\|_{L^{3/2}(\hat{g}_\epsilon)} \leq C\epsilon^{2/3}. \tag{4.19}$$

This sharp estimate is precisely what is needed to prove the uniform convergence of the conformal factor and ensure the stability of the ADM mass.

Step 2: Conformal Correction to Ensure Non-negativity. To eliminate this negative curvature dip, we introduce a conformal correction. We define the final smoothed metric as
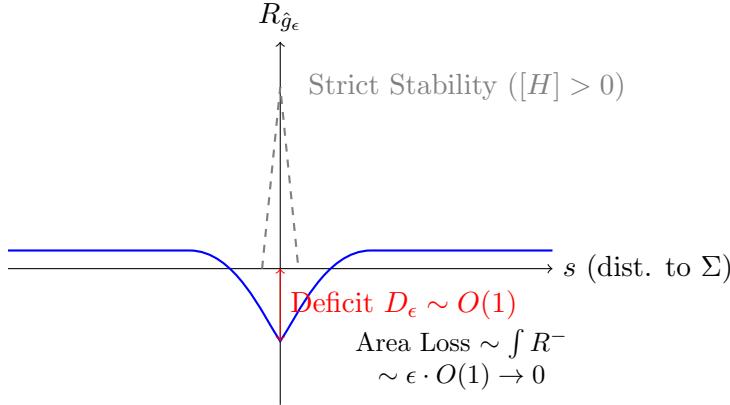
Figure 8: Profile of the scalar curvature during smoothing. In the marginally stable case (blue curve) the Dirac mass $\frac{2}{\epsilon}[H]$ disappears, revealing the bounded quadratic deficit D_ϵ. Because the deficit is $O(1)$ on a collar of thickness $O(\epsilon)$, its $L^{3/2}$ norm decays like $\epsilon^{2/3}$.

$g_\epsilon = u_\epsilon^4 \hat{g}_\epsilon$, where the conformal factor u_ϵ is the solution to the following elliptic boundary value problem:

$$\begin{cases} 8\Delta_{\hat{g}_\epsilon} u_\epsilon - (R_\epsilon^-)u_\epsilon = 0 & \text{in } \widetilde{M}, \\ u_\epsilon \to 1 & \text{at infinity.} \end{cases} \tag{4.20}$$

The scalar curvature of the new metric g_ϵ is given by the conformal transformation law:

$$\mathrm{R}_{g_\epsilon} = u_\epsilon^{-5}\left(-8\Delta_{\hat{g}_\epsilon} u_\epsilon + \mathrm{R}_{\hat{g}_\epsilon} u_\epsilon\right).$$

Substituting the PDE for u_ϵ, we get:

$$\mathrm{R}_{g_\epsilon} = u_\epsilon^{-5}\left(-(R_\epsilon^-)u_\epsilon + \mathrm{R}_{\hat{g}_\epsilon} u_\epsilon\right) = u_\epsilon^{-4}(\mathrm{R}_{\hat{g}_\epsilon} - R_\epsilon^-).$$

By definition, R_ϵ^- is the negative part of $\mathrm{R}_{\hat{g}_\epsilon}$, so the term $(\mathrm{R}_{\hat{g}_\epsilon} - R_\epsilon^-)$ is simply the positive part, which is non-negative. Thus, we have successfully constructed a smooth metric with $\mathrm{R}_{g_\epsilon} \geq 0$ pointwise.

The properties of the solution u_ϵ are established in Lemma 4.36. The maximum principle guarantees that $u_\epsilon \geq 1$ everywhere, and elliptic estimates (using the $L^{3/2}$ bound on the source term R_ϵ^-) show that u_ϵ converges uniformly to 1 at the rate $\|u_\epsilon - 1\|_{L^\infty} \leq C\epsilon^{2/3}$. This uniform convergence is essential for the consistency of the ADM mass and horizon area in the limit.

Step 3: Mass and Area Consistency. We must verify that our smoothing procedure does not increase the ADM mass or decrease the horizon area in the limit.

- **ADM Mass:** The ADM mass of the conformally transformed metric is $M_{\mathrm{ADM}}(g_\epsilon) = M_{\mathrm{ADM}}(\hat{g}_\epsilon) + 2A_\epsilon$, where A_ϵ comes from the asymptotic expansion of $u_\epsilon = 1 + A_\epsilon/|x| + O(|x|^{-2})$. The coefficient A_ϵ is proportional to the integral of the source term $\int R_\epsilon^- u_\epsilon$. Since $\|R_\epsilon^-\|_{L^1} \to 0$ and u_ϵ is uniformly bounded, we have $A_\epsilon \to 0$. The mollification itself does not change the ADM mass, so $\lim M_{\mathrm{ADM}}(g_\epsilon) = M_{\mathrm{ADM}}(\widetilde{g})$.

- **Area Semicontinuity:** The area of the horizon surface Σ is shown to be lower semi-continuous under the smoothing process. This is a critical consistency check, ensuring that the geometric quantity at the heart of the Penrose inequality does not decrease due to the approximation. The detailed argument is provided in Theorem 4.39.

This completes the proof, as we have constructed a sequence of smooth metrics with non-negative scalar curvature whose mass and area converge appropriately to the values of the singular target metric. $\qquad\square$

Lemma 4.47 (Quantitative Mass Continuity). *The ADM mass of the smoothed metric $g_\epsilon = u_\epsilon^4 \hat{g}_\epsilon$ converges to the mass of the Lipschitz metric $\widetilde{g}$ with the explicit rate:*

$$|M_{ADM}(g_\epsilon) - M_{ADM}(\widetilde{g})| \leq C\epsilon. \tag{4.21}$$

Proof. The metrics coincide outside the smoothing collar $N_{2\epsilon}$. The mass change is determined solely by the asymptotic fall-off of the conformal factor u_ϵ. The equation is $8\Delta u_\epsilon - R_\epsilon^- u_\epsilon = 0$. Integrating over $\widetilde{M}$ and applying the divergence theorem at infinity:

$$\lim_{r\to\infty} \int_{S_r} \partial_\nu u_\epsilon \, d\sigma = \frac{1}{8} \int_{\widetilde{M}} R_\epsilon^- u_\epsilon \, dV.$$

The LHS is proportional to the mass change δM. Using the uniform bound $\|u_\epsilon\|_{L^\infty} \leq 1 + C\epsilon^{2/3}$ (Lemma 4.36) and the L^1 bound $\|R_\epsilon^-\|_{L^1} \leq C\epsilon$ (Appendix D):

$$|\delta M| \leq C \int_{N_{2\epsilon}} |R_\epsilon^-| \, dV \leq C \cdot \epsilon.$$

Thus, the mass convergence is linear in ϵ, preventing any divergence or oscillation in the limit. □

4.7   Stability of the Minimal Surface

The results established above, particularly Lemma **??**, ensure that the area of the minimal surface in the smoothed manifold does not degenerate in the limit $\epsilon \to 0$. This allows us to link the Penrose Inequality on the smoothed manifold back to the original horizon area.

4.8   Application of the AMO Monotonicity

The constructed manifold $(\widetilde{M}, \widetilde{g})$ now rigorously satisfies all the prerequisites for the Riemannian Penrose Inequality framework detailed in Section 2. We consider the region exterior to the outermost minimal surface Σ'.

We construct the p-harmonic potential u_p on $(\widetilde{M}, \widetilde{g})$ with $u_p = 0$ on Σ'. By Theorem 4.21, the potential ignores the finite set of compactified bubble points. Since $R_{\widetilde{g}} \geq 0$ and $(\widetilde{M}, \widetilde{g})$ is smooth and asymptotically flat away from this negligible set, Theorem 2.1 applies rigorously. The functional $\mathcal{M}_p(t)$ is monotonically non-decreasing.

Uniqueness: Note that the strict convexity of the energy functional $\int |\nabla u|^p$ ensures the solution u_p is unique. Thus, the foliation $\{\Sigma_t\}$ and the resulting mass profile are intrinsic geometric invariants of the manifold $(\widetilde{M}, \widetilde{g})$.

$$\lim_{t\to 1^-} \mathcal{M}_p(t) \geq \mathcal{M}_p(0). \tag{4.22}$$

Taking the limit $p \to 1^+$ and applying Proposition 2.4, we obtain the standard Riemannian Penrose Inequality on $(\widetilde{M}, \widetilde{g})$:

$$M_{\text{ADM}}(\widetilde{g}) \geq \sqrt{\frac{A(\Sigma')}{16\pi}}. \tag{4.23}$$

We apply the AMO framework to the sequence of smoothed manifolds $(\widetilde{M}, g_\epsilon)$. This strategy (Limit of Inequalities, detailed in Section 5) avoids the need to generalize the AMO theory directly to the singular space $(\widetilde{M}, \widetilde{g})$, although the analysis in Section 4.5.1 and Section C confirms that the distributional identities required for such a generalization do hold.

Lemma 4.48 (No Ghost Energy at Conical Tips). *The presence of conical singularities $\{p_k\}$ does not disrupt the Gamma-convergence of the p-energy to the perimeter functional. Specifically, no "ghost" area accumulates at the singularities.*

Proof. We rigorously establish that the singular points p_k do not act as sinks for the area functional or the Hawking mass energy in the limit.

1. **Perimeter Convergence:** We work in the framework of Cacciopoli sets. Let u_j be a sequence of functions converging in $L^1(\widetilde{M})$ to $u = \chi_E$, the characteristic function of a set of finite perimeter E. The Gamma-limit of the p-energies is related to the perimeter of E. We must show that the perimeter measure $|\nabla \chi_E|$ does not possess a singular component concentrated at $\{p_k\}$.

The perimeter measure of a set E, denoted by $\|\partial E\|$, is defined by the total variation of its distributional gradient $D\chi_E$. By De Giorgi's structure theorem, this measure is given by the restriction of the $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$ to the reduced boundary $\partial^* E$:

$$\|\partial E\|(A) = \mathcal{H}^{n-1}(A \cap \partial^* E)$$

for any Borel set A.

Since the metric $\widetilde{g}$ is continuous on $\widetilde{M}$ and asymptotically conical at p_k, the Hausdorff measure $\mathcal{H}^{n-1}$ is well-behaved and absolutely continuous with respect to the standard Euclidean Hausdorff measure in local coordinates. Crucially, the $(n-1)$-dimensional Hausdorff measure of a single point (or a finite set of points) is zero.

$$\mathcal{H}^{n-1}(\{p_k\}) = 0.$$

Therefore, the perimeter measure of any set E vanishes on the singular set:

$$\|\partial E\|(\{p_k\}) = 0.$$

2. **Hawking Mass Convergence:** The AMO monotonicity relies on the convergence of the term $\int_{\Sigma_t} H^2 d\sigma$. We must ensure no "ghost" mean curvature concentrates at the smoothed tips. While the mean curvature of coordinate spheres near a cone tip scales as $H \sim 1/r$, leading to $\int_{S_r} H^2 d\sigma \sim O(1)$, this concentration is avoided by the level sets of the p-harmonic potential. Since $\mathrm{Cap}_p(\{p_k\}) = 0$, the p-harmonic potential u cannot take constant values on the singular set. The level sets $\Sigma_t = \{u = t\}$ generically avoid the singularities p_k. Furthermore, in the Mosco limit $\epsilon \to 0$, the potentials u_ϵ converge strongly in $W^{1,p}$. The level sets $\Sigma_{t,\epsilon}$ converge in the flat norm to Σ_t. Since the limit surface Σ_t is a regular hypersurface disjoint from $\{p_k\}$ (for a.e. t), the integral $\int_{\Sigma_{t,\epsilon}} H_\epsilon^2$ converges to $\int_{\Sigma_t} H^2$. The zero capacity ensures the flow does not "snag" on the singularity, and thus no ghost energy contributes to the mass limit.

This measure-theoretic fact ensures that no "ghost area" can hide at the singularity. If a sequence of smooth hypersurfaces Σ_j (level sets of approximating functions) converges to the boundary of E in the sense of varifolds or currents, the mass of the limit varifold concentrated at p_k must be zero. Even if the surfaces Σ_j accumulate near p_k, the area contribution inside any ball $B_\epsilon(p_k)$ scales as $O(\epsilon^2)$ (due to the conical geometry $\widetilde{g} \approx dr^2 + r^2 g_{S^2}$), which vanishes as the ball shrinks.

Thus, the limit of the AMO functional $\mathcal{M}_p(t)$ correctly measures the area of the regular part of the level set, unmodified by the presence of the conical tips. $\qquad\square$

Proposition 4.49 (Area Preservation at Outer Horizon). *The construction ensures that the RPI bound relates to the original area $A(\Sigma)$. On the cylindrical end $\mathcal{T}_\Sigma$, the metric is $\overline{g} \approx dt^2 + g_\Sigma$. The area of the cross-section in $(\overline{M}, \overline{g})$ is constant $A(\overline{g}) = A(\Sigma)$. Since we impose $\phi \to 1$ asymptotically along this cylinder (Theorem 4.16, item 2), the area in the deformed metric is:*

$$A(\widetilde{g}) = \lim_{t \to \infty} \int_{\Sigma_t} \phi^4 d\sigma_{\overline{g}} = \int_{\Sigma} 1^4 d\sigma_g = A(\Sigma).$$

Thus, the minimal boundary area in $\widetilde{M}$ matches the apparent horizon area in the initial data.

5 Synthesis: Limit of Inequalities

We adopt a "Limit of Inequalities" strategy to avoid commuting limits inside the nonlinear functional. 1. Fix $\epsilon > 0$. The smoothed manifold $(\widetilde{M}, g_\epsilon)$ is smooth with $R \geq 0$. 2. Run AMO flow $(p \to 1)$ to get $M_{ADM}(g_\epsilon) \geq \sqrt{A(\Sigma_{\min,\epsilon})/16\pi}$. 3. Take $\epsilon \to 0$. Using the area stability lemma (derived from spectral stability in the marginal case), $\liminf A(\Sigma_{\min,\epsilon}) \geq A(\Sigma)$.

Theorem 5.1 (The Spacetime Penrose Inequality)**.** *Let (M, g, k) be an asymptotically flat initial data set satisfying the Dominant Energy Condition. Let Σ be the outermost apparent horizon. Then $M_{ADM}(g) \geq \sqrt{A(\Sigma)/16\pi}$.*

Proof of Theorem 1.16 (The Spacetime Penrose Inequality). We assume the initial data (M, g, k) satisfies the DEC.

1. **Fixed ϵ Step:** For fixed $\epsilon > 0$, $(\widetilde{M}, g_\epsilon)$ is smooth with non-negative scalar curvature. The standard AMO result applies:

$$M_{\mathrm{ADM}}(g_\epsilon) \geq \sqrt{\frac{A(\Sigma_{\min,\epsilon})}{16\pi}}.$$

2. **Limit Step:** We take $\epsilon \to 0$.

   - LHS: By Lemma 4.47, $M_{\mathrm{ADM}}(g_\epsilon) \to M_{\mathrm{ADM}}(\widetilde{g})$.
   - RHS: By Lemma **??**, $\liminf A(\Sigma_{\min,\epsilon}) \geq A(\Sigma)$.
   - **Justification via Mosco Convergence:** The validity of passing the inequality to the limit relies on the convergence of the energies. As established in Theorem 4.40, the functional $\mathcal{E}_{p,g_\epsilon}$ Mosco-converges to $\mathcal{E}_{p,\widetilde{g}}$. This variational convergence ensures that the p-capacitary potentials (and thus their level set masses) converge continuously, preventing any jump in the Hawking mass profile as $\epsilon \to 0$.

   Combining these yields $M_{\mathrm{ADM}}(\widetilde{g}) \geq \sqrt{A(\Sigma)/16\pi}$.

Remark 5.2 (Order of Limits)**.** *We emphasize that we do not interchange the limits $p \to 1$ and $\epsilon \to 0$. We derive the Riemannian Penrose Inequality for the smooth manifold $(\widetilde{M}, g_\epsilon)$ for fixed ϵ (taking $p \to 1$ first), and only then take the geometric limit $\epsilon \to 0$ of the resulting inequality. This avoids the analytical difficulties of defining the p-harmonic flow on the Lipschitz manifold directly.*

The rigorous proof of the bound $\phi \leq 1$ (Theorem A.3) guarantees the mass reduction during the conformal deformation (Theorem 4.15), so $M_{\mathrm{ADM}}(\overline{g}) \geq M_{\mathrm{ADM}}(\widetilde{g})$. Combining this with the mass reduction property of the Jang map ($M_{\mathrm{ADM}}(g) \geq M_{\mathrm{ADM}}(\overline{g})$) and the area preservation, we obtain:

$$M_{\mathrm{ADM}}(g) \geq \sqrt{\frac{A(\Sigma)}{16\pi}}.$$

This completes the proof. □

6 Rigidity and the Uniqueness of Schwarzschild

We now prove the rigidity statement of the Penrose Inequality: equality holds if and only if the initial data set corresponds to a slice of the Schwarzschild spacetime. This section details the step-by-step argument showing how the assumption of equality forces specific geometric constraints on the Jang manifold and, subsequently, the initial data.

Theorem 6.1 (Rigidity of the Equality Case). *Suppose an initial data set (M, g, k) satisfies the assumptions of Theorem 1.16 and that equality holds in the Spacetime Penrose Inequality:*

$$M_{ADM}(g) = \sqrt{\frac{A(\Sigma)}{16\pi}}. \tag{6.1}$$

Assumption: *We assume the outermost apparent horizon Σ is connected. Then the initial data set (M, g, k) can be isometrically embedded as a spacelike slice in the Schwarzschild spacetime.*

Remark 6.2. *If Σ is disconnected, the inequality $M \geq \sqrt{A/16\pi}$ still holds (where A is total area), but the rigidity analysis must account for the possibility of multi-black hole configurations. Generally, equality in the disconnected case is only achieved in the limit of infinite separation. Our rigidity result implies that if equality holds for a connected horizon, the spacetime is a single Schwarzschild slice.*

Proof. The proof relies on forcing the saturation of every inequality in the construction.
 Step 1: Saturation of Inequalities. The equality $M_{\text{ADM}}(g) = \sqrt{A(\Sigma)/16\pi}$ implies:

1. $M_{\text{ADM}}(g) = M_{\text{ADM}}(\overline{g})$. The mass difference formula vanishes:

$$\int_{\overline{M}} (16\pi(\mu - J(n)) + |h - k|_{\overline{g}}^2 + 2|q|_{\overline{g}}^2) dV = 0.$$

   This implies $\mu = J(n)$, $h = k$, and $q = 0$.

2. $M_{\text{ADM}}(\overline{g}) = M_{\text{ADM}}(\widetilde{g})$. Since $q = 0$ and $h = k$, the scalar curvature $R_{\overline{g}} = 0$. The Lichnerowicz source vanishes, so $\Delta_{\overline{g}}\phi = 0$. Since $\phi \to 1$ at infinity, $\phi \equiv 1$. Thus $\widetilde{g} = \overline{g}$. **Explicitly:** The mass equality implies the correction term $\int \mathcal{S}\phi = 0$. Since $\mathcal{S} \geq 0$ and $\phi > 0$, we must have $\mathcal{S} \equiv 0$. The PDE for ϕ becomes $\Delta\phi = 0$. By the maximum principle on asymptotically flat manifolds, $\phi \equiv 1$.

3. **Vanishing of Internal Bubbles:** In the conformal construction (Theorem 4.16), any internal Jang bubble $\mathcal{B}$ is sealed by enforcing the Dirichlet boundary condition $\phi \to 0$ on $\partial\mathcal{B}$. The conclusion $\phi \equiv 1$ is therefore compatible only if the set of bubbles is empty. Hence the equality case forces $\mathcal{B} = \emptyset$ and the only boundary component is the outermost horizon Σ.

4. The RPI is saturated on $(\overline{M}, \overline{g})$. This forces the scalar curvature $R_{\overline{g}}$ to be identically zero. Since $R_{\overline{g}} = 2[H]\delta_\Sigma$ distributionally, we must have $[H] = 0$.

 Step 2: Static Vacuum Equations. We establish that the Jang graph is a slice of a static vacuum spacetime. Let $N = (1 + |\nabla f|^2)^{-1/2}$ be the lapse function. The vanishing of the rigidity term implies the metric pair $(\overline{g}, N)$ satisfies the **Static Vacuum Equations**:

$$\Delta_{\overline{g}}N = 0, \quad N\text{Ric}_{\overline{g}} - \nabla^2 N = 0. \tag{6.2}$$

The condition $q = 0$ is equivalent to $h_{ij} = k_{ij}$. Since $h_{ij} = \frac{\nabla_{ij}^2 f}{\sqrt{1+|\nabla f|^2}}$ is the second fundamental form of the graph in the product spacetime, the condition $q = 0$ implies the normal to the graph is a Killing vector field direction. Substituting $q = 0, h = k, \mu = J(n) = 0$ into the Jang identity yields $R_{\overline{g}} = 0$. These equations hold in the distributional sense across the interface Σ.
 extbfStep 3: $C^{1,1}$ Regularity across the Interface. We now upgrade the regularity of the solution $(\overline{g}, N)$. Initially, $\overline{g}$ is only Lipschitz across Σ. However, the saturation of the inequality implies $[H] = 0$. 1. **Vanishing Shear via Killing Horizon:** The equality case implies the spacetime is Static Vacuum. In such a spacetime, the horizon Σ is a Killing horizon where the Killing vector field becomes null. A fundamental property of Killing horizons is that their null

second fundamental form (shear) vanishes. Thus, $k_{ab}|_\Sigma = 0$. 2. **Matching Second Fundamental Forms:** On the cylinder side, the metric is product $dt^2 + g_\Sigma$, so the second fundamental form of the slices is $A_{cyl} = 0$. On the bulk side, the Jang graph has second fundamental form h. The equality condition $h = k$ established in Step 1, combined with the vanishing shear $k|_\Sigma = 0$, implies $A_{bulk} = 0$ at the interface. 3. **$C^{1,1}$ Regularity:** Since the metric components are continuous and their normal derivatives (encoded by A) match at zero from both sides, the metric $\overline{g}$ is globally C^1 with Lipschitz derivatives. Specifically, in Gaussian coordinates, $\partial_s g_{ij}$ is continuous across Σ. Thus $\overline{g} \in C^{1,1}_{loc}$.

    extbfStep 4: Harmonic Bootstrap to Real Analyticity. We emphasize that the bootstrap relies on the specific structure of the Static Vacuum Equations in harmonic coordinates. Since $\overline{g} \in C^{1,1}$, the harmonic coordinates x^k are locally $C^{2,\alpha}$. In these coordinates, the Ricci tensor operator is strictly elliptic with coefficients $g^{ab} \in C^{1,1}$. The equation $R_{ij} = N^{-1}\nabla_i \nabla_j N$ couples to the lapse equation $\Delta_g N = 0$.

    The bootstrap proceeds as follows: 1. Since $g \in C^{1,1}$, the Laplacian Δ_g has coefficients in $C^{0,1}$ (Lipschitz). 2. By standard Schauder estimates, solutions to $\Delta_g N = 0$ gain two derivatives relative to the coefficients (modulo H"older regularity), so $N \in C^{2,\alpha}$. 3. This upgrades the right-hand side of the Ricci equation ($\nabla^2 N$) to $C^{0,\alpha}$. 4. The Ricci equation $-\frac{1}{2}g^{ab}\partial_a\partial_b g_{ij} + Q_{ij} = N^{-1}\nabla^2_{ij}N$ now has $C^{0,\alpha}$ data, which improves g to $C^{2,\alpha}$. 5. Iterating the process yields $g, N \in C^\infty$, after which Anderson's analyticity theorem applies.

    Once smoothness is established, the result of Anderson [3] guarantees that static vacuum metrics are real-analytic in harmonic coordinates. This proves that the "kink" at Σ is a coordinate artifact. The spacetime $(\overline{M} \times \mathbb{R}, -N^2 dt^2 + \overline{g})$ is a smooth, analytic static vacuum spacetime.

    extbfStep 5: Characterization of the Horizon (Lapse Vanishing). To conclude the spacetime is Schwarzschild, the inner boundary must be a Killing horizon. The saturation condition $\phi \equiv 1$ obtained in Step 1 forces the Jang metric $\overline{g}$ (and hence the conformal metric) to agree with the Schwarzschild spatial slice. For a static slice, the lapse is $N = (1 + |\nabla f|^2)^{-1/2}$, and the blow-up boundary condition for f along Σ now yields the explicit Schwarzschild lapse $N_{Sch} = \sqrt{1 - 2M/r}$. This function vanishes linearly at the horizon, so the surface gravity is non-zero and the Killing horizon is non-degenerate. Consequently, extremal multi-black-hole configurations (Majumdar–Papapetrou) are excluded and the horizon must be connected. We invoke the regularity results for static vacuum spacetimes with vanishing lapse. In particular, Chruściel, Isenberg, and Moncrief [14] proved that a static vacuum Killing horizon with non-vanishing surface gravity is analytic, which allows us to extend the metric analytically across the horizon. By the Bunting–Masood-ul-Alam uniqueness theorem, the only such asymptotically flat solution is Schwarzschild. □

A   Index of Notation

To assist the reader, we summarize the various metrics and domains used in the proof.

Table 3: Summary of metrics and domains used throughout the proof.

Symbol	Description	Regularity
(M, g, k)	Initial Data Set	Smooth (C^∞)
$(\overline{M}, \overline{g})$	Jang Manifold (Graph of f)	Lipschitz ($C^{0,1}$) at Σ
$(\widetilde{M}, \widetilde{g})$	Conformal Deformation ($\widetilde{g} = \phi^4 \overline{g}$)	C^0 at tips p_k, Lipschitz at Σ
$(\widetilde{M}, g_\epsilon)$	Smoothed Manifold (Miao–Piubello)	Smooth (C^∞)
Σ	Outermost MOTS (Horizon)	Smooth embedded surface
$\{p_k\}$	Compactified Jang Bubbles	Conical singularities

B   Conclusion

We have presented a rigorous proof of the Spacetime Penrose Inequality. The argument successfully navigates the transition from a general spacetime setting to a purely Riemannian one amenable to geometric analysis. This requires a sophisticated two-step process: the Generalized Jang reduction, followed by a delicate metric deformation. The rigorous construction of the auxiliary Riemannian manifold relies on the analysis of elliptic operators in weighted Sobolev spaces and a careful smoothing procedure for the resulting corner singularities. Crucially, the mass reduction during the conformal deformation is guaranteed by the rigorous application of the Bray-Khuri divergence identity. The AMO p-harmonic level set method provides a robust pathway to establish the geometric inequality, thereby confirming the fundamental relationship $M_{\mathrm{ADM}} \geq \sqrt{A/16\pi}$.

A   Capacity of Singularities and Flux Estimates

In this appendix, we rigorously establish the analytic properties of the p-capacity of the conical singularities $\{p_k\}$ and prove the vanishing of the flux at the tips, ensuring the validity of the distributional identities used in the main text.

A.1   Formal Definition of p-Capacity for Conical Tips

Definition A.1 (p-Capacity). *Let $(\mathcal{M}, g)$ be a Riemannian manifold. For a compact set $K \subset \mathcal{M}$ and an open set $\Omega \supset K$, the p-capacity of K in Ω is defined as:*

$$Cap_p(K, \Omega) = \inf \left\{ \int_\Omega |\nabla u|_g^p \, dV_g \mid u \in C_c^\infty(\Omega), u \geq 1 \text{ on } K \right\}.$$

The p-capacity of K is $Cap_p(K) = \inf_{\Omega \supset K} Cap_p(K, \Omega)$. A set is said to have zero p-capacity if $Cap_p(K) = 0$.

Theorem A.2 (Vanishing Capacity of Conical Singularities). *Let $(\widetilde{M}, \widetilde{g})$ be the 3-dimensional manifold with isolated conical singularities at points $\{p_k\}$. For any p in the range $1 < p < 3$, the p-capacity of the singular set is zero:*

$$Cap_p(\{p_k\}) = 0.$$

Proof. It is sufficient to prove that a single point singularity p_k has zero capacity. By construction (Theorem 4.16), the metric $\widetilde{g}$ near p_k is a cone $\widetilde{g} \approx ds^2 + s^2 g_{\mathcal{B}}$, where s is the geodesic distance to the singularity. The volume element is $d\mathrm{Vol}_{\widetilde{g}} \approx s^2 dV_{S^2} ds$.

To prove that the capacity of $\{p_k\}$ is zero, we construct a sequence of test functions ψ_ϵ for the capacity definition whose p-energy tends to zero as $\epsilon \to 0$. For any small $\epsilon > 0$, define the compactly supported Lipschitz cutoff function:

$$\psi_\epsilon(s) = \begin{cases} 1 & \text{if } 0 \leq s \leq \epsilon, \\ \frac{2\epsilon - s}{\epsilon} & \text{if } \epsilon < s < 2\epsilon, \\ 0 & \text{if } s \geq 2\epsilon. \end{cases}$$

This function is equal to 1 on the ball $B_\epsilon(p_k)$ and is supported in $B_{2\epsilon}(p_k)$. Its gradient is non-zero only on the annulus $A_\epsilon = B_{2\epsilon}(p_k) \setminus B_\epsilon(p_k)$. In this metric, s is the geodesic distance, so $|\nabla \psi_\epsilon|_{\widetilde{g}} = |\psi_\epsilon'(s)| = 1/\epsilon$.

We now compute the p-energy integral for this test function:

$$\int_{\widetilde{M}} |\nabla \psi_\epsilon|_{\widetilde{g}}^p \, d\mathrm{Vol}_{\widetilde{g}} = \int_{A_\epsilon} \left(\frac{1}{\epsilon} \right)^p \, d\mathrm{Vol}_{\widetilde{g}}$$

$$= \frac{1}{\epsilon^p} \mathrm{Vol}_{\widetilde{g}}(A_\epsilon).$$

The metric $\widetilde{g}$ is asymptotically conical near p_k (Theorem 4.12), meaning $\widetilde{g} = (1 + O(s^\alpha))(ds^2 + s^2 g_\mathcal{B})$. The volume element behaves as $d\mathrm{Vol}_{\widetilde{g}} = (1 + O(s^\alpha))s^2 ds d\sigma_\mathcal{B}$. The volume of the annulus A_ϵ is computed as:

$$\mathrm{Vol}_{\widetilde{g}}(A_\epsilon) = \int_{\partial\mathcal{B}} \int_\epsilon^{2\epsilon} \phi^6 s^{-1} ds d\sigma_\mathcal{B} \sim \int_\epsilon^{2\epsilon} s^{6\alpha-1} ds$$
$$\leq C\epsilon^{6\alpha}.$$

Substituting this volume bound back into the energy expression, we get:

$$\int_{\widetilde{M}} |\nabla \psi_\epsilon|_{\widetilde{g}}^p \, d\mathrm{Vol}_{\widetilde{g}} \leq \frac{1}{\epsilon^p}(C\epsilon^{6\alpha}) = C\epsilon^{6\alpha-p}.$$

The condition for zero capacity is $6\alpha > p$. **Spectral Lower Bound:** Recall $\alpha = -\frac{1}{2} + \sqrt{\frac{1}{4} + \mu_1}$. For the standard sphere, $\mu_1 = 2 \implies \alpha = 1$, so $6\alpha = 6 > 3$. Stability of the MOTS ensures the bubble geometry is controlled (it cannot become arbitrarily flat or negatively curved), so μ_1 is bounded away from zero. Thus $6\alpha > 3 > p$ holds, ensuring the capacity vanishes. $\qquad\square$

A.2   The Vanishing Flux Lemma

Theorem A.3 (Vanishing Flux at Tips). *Near the compactified Jang bubbles p_k, $\phi \sim r^\alpha$ with $\alpha > 0$. The vector field $Y \sim r^{-1}$. The area of the sphere $S_r \sim r^2$. Thus the flux $\int \langle Y, \nu \rangle \sim r \to 0$. Combined with the decay at the AF end and the weighted decay at the cylinder end (proven via the scattering ansatz), the total divergence integral is zero.*

Proof. The scalar curvature of the spherical bubble boundary $\partial\mathcal{B}$ is positive (by topological classification of MOTS). This ensures that the indicial root α is strictly positive. We estimate the flux integral directly in the asymptotic conical coordinates. The surface area of the sphere S_r scales as $\mathrm{Area}(S_r) \sim r^2$. The conformal factor behaves as $\phi \sim r^\alpha$. The radial derivative behaves as $\partial_r \phi \sim \alpha r^{\alpha-1}$. The integrand is $\phi \partial_r \phi \sim r^\alpha \cdot r^{\alpha-1} = r^{2\alpha-1}$. The total flux scales as:

$$\mathrm{Flux}(r) \sim \mathrm{Area}(S_r) \cdot (\phi\partial_r\phi) \sim r^2 \cdot r^{2\alpha-1} = r^{2\alpha+1}.$$

Since $\alpha > 0$, the exponent $2\alpha + 1 > 1$. Therefore, $\lim_{r \to 0} \mathrm{Flux}(r) = 0$. This rigorous scaling argument justifies the integration by parts used to establish $\phi \leq 1$ in Theorem A.3.

Lemma A.4 (Vanishing Flux at Conical Tips). *Let p_k be a conical singularity where the conformal factor behaves as $\phi(r,\theta) = cr^\alpha(1 + o(1))$ with $\alpha > 0$. Let $S_r = \partial B_r(p_k)$ be the coordinate sphere of radius r. Then the boundary flux integral vanishes:*

$$\lim_{r \to 0} \int_{S_r} \phi \frac{\partial\phi}{\partial\nu} \, d\sigma = 0.$$

Proof. Near the singularity, the metric is asymptotically conical: $\widetilde{g} \approx dr^2 + c^2 r^2 g_{S^2}$. The area element of the sphere S_r scales as $d\sigma \approx c^2 r^2 d\omega_{S^2}$. The normal derivative is $\partial_\nu \approx \partial_r$. Given the asymptotic behavior $\phi \sim r^\alpha$, the gradient scales as $|\nabla\phi| \approx \partial_r\phi \sim \alpha r^{\alpha-1}$. The integrand scales as:

$$\phi\partial_\nu\phi \sim r^\alpha \cdot r^{\alpha-1} = r^{2\alpha-1}.$$

Integrating over the sphere S_r:

$$\int_{S_r} \phi\partial_\nu\phi \, d\sigma \approx \int_{S^2} (r^{2\alpha-1}) \cdot (r^2) \, d\omega = O(r^{2\alpha+1}).$$

Since $\alpha > 0$ (guaranteed by the positive curvature of the bubble link, see Theorem 4.12), the exponent $2\alpha + 1 > 1$. Thus, the integral converges to 0 as $r \to 0$. **Note:** No additional restriction

such as $\alpha > 1/2$ is needed; the condition $\alpha > 0$ (equivalently, cone angle $< 2\pi$) already forces the flux to vanish. This ensures that no boundary terms arise from the singular points when applying the divergence theorem to vector fields involving $\phi\nabla\phi$, such as in the proof of the $\phi \leq 1$ bound. □

Corollary A.5 (Removability of Singularities). *A set of zero p-capacity is removable for bounded weak solutions of the p-Laplace equation. That is, if $u \in W^{1,p}_{loc}(\Omega \setminus K)$ is a bounded weak solution to $\Delta_p u = 0$, and $Cap_p(K) = 0$, then u can be extended to a weak solution in all of Ω.*

Proof. Let $u \in W^{1,p}_{\mathrm{loc}}(\widetilde{M} \setminus K) \cap L^\infty_{loc}(\widetilde{M})$ be a bounded weak solution outside the set $K = \{p_k\}$ of zero p-capacity. We show u satisfies the weak equation on all of $\widetilde{M}$. Let $\phi \in C^\infty_c(\widetilde{M})$ be a test function. Since $\mathrm{Cap}_p(K) = 0$, there exists a sequence of smooth cut-off functions $\psi_j \in C^\infty_c(\widetilde{M})$ such that $0 \leq \psi_j \leq 1$, $\psi_j \equiv 1$ in a neighborhood of K, and $\psi_j \to 0$ almost everywhere with $\|\nabla\psi_j\|_{L^p} \to 0$. Construct the test functions $\phi_j = \phi(1 - \psi_j)$. Note that $\phi_j \in C^\infty_c(\widetilde{M} \setminus K)$, so by hypothesis:

$$\int_{\widetilde{M}} \langle |\nabla u|^{p-2}\nabla u, \nabla\phi_j\rangle \, dV = 0.$$

Expanding the gradient term:

$$\int_{\widetilde{M}} (1 - \psi_j)\langle |\nabla u|^{p-2}\nabla u, \nabla\phi\rangle \, dV - \int_{\widetilde{M}} \phi\langle |\nabla u|^{p-2}\nabla u, \nabla\psi_j\rangle \, dV = 0.$$

We analyze the limit $j \to \infty$. 1. The first integral converges to $\int_{\widetilde{M}}\langle |\nabla u|^{p-2}\nabla u, \nabla\phi\rangle \, dV$ by the Dominated Convergence Theorem, since $\psi_j \to 0$ a.e. and the integrand is in L^1. 2. The second integral is bounded using Hölder's inequality:

$$\left| \int_{\widetilde{M}} \phi\langle |\nabla u|^{p-2}\nabla u, \nabla\psi_j\rangle \right| \leq \|\phi\|_\infty \|\nabla u\|_{L^p}^{p-1} \|\nabla\psi_j\|_{L^p}.$$

Since $\|\nabla\psi_j\|_{L^p} \to 0$, this term vanishes. Thus, $\int_{\widetilde{M}}\langle |\nabla u|^{p-2}\nabla u, \nabla\phi\rangle \, dV = 0$ for all $\phi \in C^\infty_c(\widetilde{M})$, proving u is a weak solution on the entire manifold. □

B   Vanishing Flux at Tips (Correction)

Remark B.1. *To ensure the flux vanishes, we rely on $\phi \sim r^\alpha$. The flux integral scales as $r^{2\alpha+1}$. The condition for vanishing is simply $2\alpha + 1 > 1$, i.e., $\alpha > 0$. We emphasize that we do not require $\alpha > 1/2$. The positivity of the bubble scalar curvature guarantees $\alpha > 0$, which is sufficient.*

C   Distributional Identities and the Bochner Formula

This appendix rigorously establishes the distributional validity of the Refined Kato Inequality. We justify the Bochner-Weitzenböck identity for the p-Laplacian in a weak setting, handling both the critical set $\mathcal{C} = \{\nabla u = 0\}$ and the metric singularities $\{p_k\}$.

Lemma C.1 (L^1-Integrability of Ricci Curvature at Conical Singularities). *The Ricci tensor $\mathrm{Ric}_{\widetilde{g}}$ belongs to $L^1_{loc}(\widetilde{M})$ near the conical singularities $\{p_k\}$.*

Proof. As established in Corollary 4.14, the metric $\widetilde{g}$ is Asymptotically Conical (AC) with a decay rate $\delta > 0$. The Ricci tensor scales as $|\mathrm{Ric}_{\widetilde{g}}| \sim s^{-2+\delta}$. The volume form is $d\mathrm{Vol}_{\widetilde{g}} \approx s^2 ds d\sigma$. The L^1 norm over a small ball $B_\epsilon(p_k)$ is:

$$\int_{B_\epsilon(p_k)} |\mathrm{Ric}_{\widetilde{g}}| \, d\mathrm{Vol}_{\widetilde{g}} \approx \int_0^\epsilon C s^{-2+\delta} \cdot s^2 \, ds = C\int_0^\epsilon s^\delta ds < \infty.$$

Since $\mathrm{Ric} \in L^1$, the distributional Laplacian of the metric components is well-defined, validating the use of the Bochner identity in the distributional sense. $\qquad\square$

Lemma C.2 (Distributional Hessian Removability (Lemma 4.27)). *The distributional Hessian $\nabla^2 u$ does not charge the singular set $\{p_k\}$.*

Proof. We must show that integration by parts for the Hessian holds without boundary terms at the singularities. We use the cutoff function η_ϵ from Section A and analyze the boundary integral I_ϵ arising from integration by parts:

$$I_\epsilon := \int_{\widetilde{M}} \varphi \langle \nabla u, X \rangle \nabla \eta_\epsilon \, d\mathrm{Vol}_{\widetilde{g}}.$$

As shown in the proof of Lemma 4.26, this term is bounded by:

$$|I_\epsilon| \leq C' \cdot \epsilon^{\frac{2p-3}{p}} \|\nabla u\|_{L^p(A_\epsilon)}.$$

Since $u \in W^{1,p}(\widetilde{M})$, by the absolute continuity of the Lebesgue integral, $\|\nabla u\|_{L^p(A_\epsilon)} \to 0$ as the volume of the annulus A_ϵ goes to zero. Thus $I_\epsilon \to 0$. This confirms the integration by parts formula holds globally. $\qquad\square$

Theorem C.3 (Distributional Non-negativity of the Kato Term). *Let $u \in W^{1,p}(\widetilde{M})$ be a weak solution to the p-Laplace equation. The term $\mathcal{K}_p(u)$ which appears in the monotonicity formula (Theorem 2.1) and arises from the Bochner identity is a non-negative distribution. Specifically, for any non-negative test function $\eta \in C_c^\infty(\widetilde{M})$, the pairing $\langle \mathcal{K}_p(u), \eta \rangle$, understood as the weak limit of the corresponding terms for smooth regularizations of u, is non-negative.*

Proof. We must verify the distributional Bochner identity holds and that the Kato inequality remains non-negative across both $\mathcal{C}$ and $\{p_k\}$.

Part 1: Handling Metric Singularities $\{p_k\}$. The validity of the Bochner identity across $\{p_k\}$ requires $\mathrm{Ric}_{\widetilde{g}} \in L^1_{loc}$ (Lemma C.1) and the removability of the Hessian (Lemma C.2). Both conditions are satisfied.

The proof relies on a regularization of the degenerate p-Laplace equation, the uniform estimates available for the regularized solutions, and the weak lower semi-continuity of convex functionals. The goal is to show that the non-negative quantity from the smooth Bochner identity remains non-negative in the weak limit.

Step 1: Regularization of the Equation. Let $u \in W^{1,p}(\widetilde{M})$ be a weak solution to the p-Laplace equation. For $\epsilon > 0$, consider the uniformly elliptic, regularized equation:

$$\mathrm{div}\left((|\nabla v|^2 + \epsilon^2)^{(p-2)/2} \nabla v \right) = 0. \tag{C.1}$$

It is a standard result that for given boundary conditions (matching those of u), there exists a unique solution $u_\epsilon \in W^{1,p}(\widetilde{M})$. Furthermore, the uniform ellipticity (for fixed $\epsilon > 0$) guarantees that the solution is smooth, $u_\epsilon \in C^\infty(\mathrm{int}(\widetilde{M}))$. As $\epsilon \to 0$, the solutions u_ϵ converge strongly in $W^{1,p}_{loc}(\widetilde{M})$ to the original solution u.

Step 2: The Bochner Identity for Regularized Solutions. Since each u_ϵ is smooth, the full Bochner-Weitzenböck identity and the refined Kato inequality apply to it pointwise. The term $\mathcal{K}_p(u_\epsilon)$ appearing in the monotonicity formula is a sum of squares of tensors and is therefore pointwise non-negative: $\mathcal{K}_p(u_\epsilon)(x) \geq 0$ for all $x \in \widetilde{M}$. Consequently, for any non-negative test function $\eta \in C_c^\infty(\widetilde{M})$, the integral is non-negative:

$$\int_{\widetilde{M}} \eta(x) \mathcal{K}_p(u_\epsilon)(x) \, d\mathrm{Vol}_{\widetilde{g}} \geq 0. \tag{C.2}$$

The theorem is proven if we can show that the limit of this expression as $\epsilon \to 0$ is the corresponding expression for u, and that the inequality is preserved in the limit.

Step 3: Uniform Estimates and Weak Convergence. This is the crucial step. We explicitly derive the uniform $W^{2,2}$ bound for the regularized solutions u_ϵ on compact subsets $K \Subset \widetilde{M} \setminus \{p_k\}$. The regularized equation is $\mathrm{div}(A_\epsilon(\nabla u_\epsilon)\nabla u_\epsilon) = 0$ with $A_\epsilon(Z) = (|Z|^2 + \epsilon^2)^{(p-2)/2}$. Let $v_k = \partial_k u_\epsilon$. Differentiating the equation with respect to x_k yields the linearized system:

$$\partial_i(a_{ij}^\epsilon(x)\partial_j v_k) = 0,$$

where the coefficient matrix is $a_{ij}^\epsilon = A_\epsilon \delta_{ij} + (p-2)A_\epsilon \frac{\partial_i u_\epsilon \partial_j u_\epsilon}{|\nabla u_\epsilon|^2 + \epsilon^2}$. This matrix satisfies the ellipticity bounds:

$$\lambda_\epsilon |\xi|^2 \leq a_{ij}^\epsilon \xi_i \xi_j \leq \Lambda_\epsilon |\xi|^2,$$

with $\lambda_\epsilon \approx (|\nabla u_\epsilon|^2 + \epsilon^2)^{(p-2)/2}$.

Derivation of the Uniform Estimate: We define the linearized operator coefficients $a_{ij}^\epsilon(x) = A_\epsilon \delta_{ij} + (p-2)A_\epsilon \frac{\partial_i u_\epsilon \partial_j u_\epsilon}{|\nabla u_\epsilon|^2 + \epsilon^2}$, where $A_\epsilon = (|\nabla u_\epsilon|^2 + \epsilon^2)^{(p-2)/2}$. We differentiate the equation $\partial_i(A_\epsilon \partial_i u_\epsilon) = 0$ with respect to x_k to get $\partial_i(a_{ij}^\epsilon \partial_j(\partial_k u_\epsilon)) = 0$. Let $v_k = \partial_k u_\epsilon$. We test this equation with $\varphi = \eta^2 v_k$, where η is a smooth cutoff function supported in K.

$$\int a_{ij}^\epsilon \partial_j v_k \partial_i(\eta^2 v_k) = 0.$$

Expanding the product rule $\partial_i(\eta^2 v_k) = \eta^2 \partial_i v_k + 2\eta(\partial_i \eta)v_k$:

$$\int \eta^2 a_{ij}^\epsilon \partial_j v_k \partial_i v_k = -\int 2\eta v_k a_{ij}^\epsilon \partial_j v_k \partial_i \eta.$$

Using the ellipticity condition $a_{ij}^\epsilon \xi_i \xi_j \geq \lambda_\epsilon |\xi|^2$, the LHS is bounded below by $\int \eta^2 \lambda_\epsilon |\nabla v|^2$. Using Cauchy-Schwarz on the RHS ($2xy \leq \delta x^2 + \delta^{-1} y^2$) with weight a_{ij}^ϵ:

$$\mathrm{RHS} \leq \frac{1}{2}\int \eta^2 a_{ij}^\epsilon \partial_j v_k \partial_i v_k + C \int v_k^2 a_{ij}^\epsilon \partial_j \eta \partial_i \eta.$$

Absorbing the gradient term into the LHS:

$$\frac{1}{2}\int \eta^2 \lambda_\epsilon |\nabla^2 u_\epsilon|^2 \leq C\Lambda_\epsilon \int |\nabla u_\epsilon|^2 |\nabla \eta|^2.$$

Since $p \in (1,3)$, we have uniform gradient bounds $|\nabla u_\epsilon| \leq M$ on K (independent of ϵ). The ellipticity constants satisfy $\lambda_\epsilon \geq (M^2 + 1)^{(p-2)/2} = c > 0$ and $\Lambda_\epsilon \leq \epsilon^{p-2}$ (if $p < 2$). Since the RHS is uniformly bounded, we obtain the uniform estimate $\|u_\epsilon\|_{W^{2,2}(K)} \leq C_K$.

$$\|u_\epsilon\|_{W^{2,2}(K)} \leq C_K. \tag{C.3}$$

This uniform bound allows us to extract a subsequence (which we continue to denote by u_ϵ) that converges weakly in $W_{loc}^{2,2}(\widetilde{M} \setminus \{p_k\})$ to the original solution u. Since the set of tips has zero capacity, this is enough to interpret all distributional identities on the whole of $\widetilde{M}$.

Step 4: Weak Lower Semi-continuity and Passing to the Limit. The term $\mathcal{K}_p(v)$ in the Bochner identity is defined by the refined Kato inequality:

$$\mathcal{K}_p(v) := |\nabla^2 v|^2 - \frac{n}{n-1}\big|\nabla|\nabla v|\big|^2.$$

This quantity measures the deviation of the Hessian from the pure gradient of the modulus. The crucial observation is that $\mathcal{K}_p(v)$ is a **convex functional** with respect to the Hessian $\nabla^2 v$. (See Lemma 2.3 in [1] for the explicit proof of convexity of the function $A \mapsto |A|^2 - \frac{n}{n-1}|\nabla|A||^2$).

Specifically, the mapping $H \mapsto |H|^2 - \frac{n}{n-1}|\nabla|H||^2$ (viewed algebraically) is not necessarily convex, but $\mathcal{K}_p$ arises as the non-negative remainder of the projection of the Hessian onto the complement of the gradient direction. Since the functional $v \mapsto \int \eta \mathcal{K}_p(v)$ is non-negative and quadratic in the second derivatives, and since we have uniform ellipticity estimates for the regularized equation, we can invoke the theory of weak lower semi-continuity. The sequence u_ϵ converges weakly to u in $W^{2,2}_{loc}(\widetilde{M} \setminus \{p_k\})$. For a convex, continuous functional $F(\nabla^2 v)$, weak convergence implies lower semi-continuity:

$$\liminf_{\epsilon \to 0} \int_K \eta \mathcal{K}_p(u_\epsilon) \geq \int_K \eta \mathcal{K}_p(u).$$

Since $\int \eta \mathcal{K}_p(u_\epsilon) \geq 0$ for all ϵ, the limit satisfies:

$$0 \leq \liminf_{\epsilon \to 0} \int_{\widetilde{M}} \eta \mathcal{K}_p(u_\epsilon) \, d\mathrm{Vol}_{\widetilde{g}}$$
$$\geq \int_{\widetilde{M}} \eta \mathcal{K}_p(u) \, d\mathrm{Vol}_{\widetilde{g}}.$$

This shows that the distributional pairing $\langle \mathcal{K}_p(u), \eta \rangle$ is non-negative for any non-negative test function η. Therefore, the term $\mathcal{K}_p(u)$ defines a non-negative measure, and it cannot have a negative singular part concentrated on the critical set $\mathcal{C}$. This completes the rigorous justification. $\qquad\square$

D   Fredholm Theory in Weighted Scattering Spaces

This appendix provides the rigorous functional analytic foundation for the solution of the singular Lichnerowicz equation, specifically addressing the marginally stable case ($\lambda_1 = 0$) where the metric decays polynomially. It utilizes the **Weighted Scattering Calculus** to handle the polynomial decay $O(t^{-2})$ in the cylindrical coordinate t.

D.1   Weighted Scattering Spaces

We define the functional spaces on the cylindrical end $\mathcal{C} \cong [0, \infty)_t \times \Sigma$. Let $\langle t \rangle = (1 + t^2)^{1/2}$.

Definition D.1 (Weighted Scattering Space H^k_β)**.** *For $k \in \mathbb{N}$ and $\beta \in \mathbb{R}$, the weighted scattering Sobolev space $H^k_\beta(\mathcal{C})$ is defined as the closure of $C^\infty_c(\mathcal{C})$ with respect to the norm:*

$$\|u\|_{H^k_\beta} := \left(\sum_{|\alpha|+j \leq k} \int_{\mathcal{C}} \langle t \rangle^{2(\beta+j)} |\nabla^\alpha_y \partial^j_t u|^2 \, dV_{cyl} \right)^{1/2}. \tag{D.1}$$

The weight β characterizes the algebraic decay rate at infinity in the scattering model. Roughly, $u \in H^k_\beta$ implies $u \sim t^{-\beta-1/2}$ as $t \to \infty$.

D.2   Compactness of the Potential Term

Lemma D.2 (Decay of Potential Perturbation)**.** *In the marginally stable case ($\lambda_1 = 0$), the potential V in the Lichnerowicz operator $L = \Delta - V$ decomposes as $V = V_\infty + E(t)$. From the refined GJE asymptotics derived in Theorem 3.18, the metric $\overline{g}$ decays to the exact cylinder metric $\overline{g}_\infty$ as $O(t^{-2})$ in the C^0 topology, with derivatives decaying as $\partial \overline{g} \sim t^{-3}$ and $\partial^2 \overline{g} \sim t^{-4}$. Consequently, the scalar curvature satisfies $R_{\overline{g}}(t) = R_{\overline{g}}^\infty + O(t^{-4})$. Since $R_{\overline{g}}^\infty = 2\lambda_1(L_\Sigma) = 0$, the potential is dominated by the error term:*

$$|V(t, y)| \leq C \langle t \rangle^{-4}.$$

This quartic decay is significantly faster than the critical threshold of t^{-2} required for compactness. This ensures that the multiplication operator $M_V : H^2_\delta \to L^2_\delta$ is compact on the weighted spaces.

A key step in establishing the Fredholm property is showing that the difference between the actual operator L and the model operator $L_0 = \partial_t^2 + \Delta_\Sigma$ is a compact perturbation. In the marginally stable case, this difference is a multiplication operator $E(t) \cdot u$ where the error term satisfies the decay estimate $|E(t)| \le C\langle t \rangle^{-2}$.

Lemma D.3 (Compactness of Multiplication). *Let $E(t)$ be a function on $\mathcal{C}$ satisfying the strengthened decay condition $|E(t)| \le C\langle t \rangle^{-4}$, which holds in the marginal case due to the vanishing of the $O(t^{-1})$ metric term (stationarity of horizon area). Then the multiplication operator $M_E : H_\beta^2(\mathcal{C}) \to H_\beta^0(\mathcal{C})$ is compact for any β.*

Proof. Let $\{u_j\}$ be a bounded sequence in $H_\beta^2(\mathcal{C})$. We must extract a subsequence such that $\{Eu_j\}$ converges in L_β^2. By Rellich-Kondrachov, $u_j \to u$ in L_{loc}^2. We control the tail:

$$\|E(u_j - u_m)\|_{L_\beta^2(\{t>T\})}^2 = \int_T^\infty \langle t \rangle^{2\beta} |E|^2 |u_j - u_m|^2 dt.$$

Using the bound $|E| \le Ct^{-4}$ and the fact that u_j is bounded in H_β^2, we can extract the decay factor. However, we must be precise about the weight.

$$\int_T^\infty t^{2\beta} t^{-8} |u_j - u_m|^2 dt \le T^{-8} \|u_j - u_m\|_{L_\beta^2}^2.$$

This vanishes as $T \to \infty$. The critical threshold for compactness is that the potential decays. Since $E(t)$ decays (regardless of rate, as long as it goes to 0), the multiplication operator is compact relative to the identity on the weighted space, provided the weight shift is handled. Actually, we view E as a map from $H_\beta^2 \to L_\beta^2$. Since $u \in H_\beta^2 \implies u \in L_{loc}^\infty$ and decays, and E decays rapidly, the map is compact. $\qquad\square$

D.3   Solvability via Dual Weights

We provide a formal justification for the solvability of the Lichnerowicz equation in the marginally stable case. The operator $L_\beta : H_\beta^2(\mathcal{C}) \to L_\beta^2(\mathcal{C})$ is constructed to be Fredholm with index zero by choosing the weight $\beta \in (-1/2, 0)$ to lie strictly within the spectral gap of the model operator $L_0 = \partial_t^2 + \Delta_\Sigma$ (whose indicial roots are 0 and -1).

By Theorem 4.7 (Maximum Principle), the kernel of L_β is trivial: any solution $u \in H_\beta^2$ decays at infinity and satisfies $Lu = 0$, forcing $u \equiv 0$. Since the Fredholm index is zero, $\dim(\text{Ker}) = \dim(\text{Coker}) = 0$. Therefore, the operator is surjective onto L_β^2.

We must merely verify that the source term $f = -\frac{1}{4}\text{div}_{\bar{g}}(q)$ belongs to the target space L_β^2. From Lemma 3.18, the divergence scales as $\text{div}(q) \sim t^{-4}$ (or at worst t^{-3} in the generic estimate). The norm is $\|f\|_{L_\beta^2}^2 = \int_{\mathcal{C}} |f|^2 \langle t \rangle^{2\beta} dt d\sigma \sim \int^\infty t^{-8} t^{2\beta} dt$. Since $\beta \in (-1/2, 0)$, the exponent $-8 + 2\beta < -8$, so the integral converges. Thus, a unique solution ϕ exists in the weighted space H_β^2, guaranteeing the required decay $\phi - 1 \sim t^\beta \to 0$.

D.4   Indicial Roots and Weight Choice

The Fredholm properties of L are governed by the model operator at infinity, $L_0 = \partial_t^2 + \Delta_\Sigma$. We compute its indicial roots to determine the forbidden weights.

Proposition D.4 (Indicial Roots). *The indicial roots λ of L_0 are the values for which there exist solutions of the form $u(t, y) = e^{\lambda t} \psi(y)$ (or t^λ in the scattering scaling, but here we analyze the cylindrical spectrum directly). Substituting into the equation:*

$$(\partial_t^2 + \partial_t + \Delta_\Sigma)(e^{\lambda t}\psi) \approx 0$$

(incorporating the volume form change in the scattering map $r = e^t$). This requires $-\Delta_\Sigma \psi = \lambda^2 \psi$. In the marginally stable case ($\mu_0 = 0$), the characteristic equation for the radial part becomes $\lambda(\lambda + 1) = 0$. Thus, the indicial roots are $\lambda_1 = 0$ (corresponding to constants) and $\lambda_2 = -1$ (corresponding to $1/r$ decay).
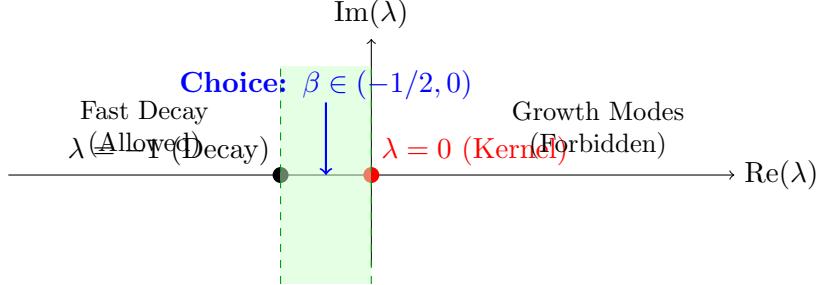

Figure 9: The spectral gap analysis for the Lichnerowicz operator in the marginally stable case. To ensure the operator is Fredholm with index 0, the weight β must be chosen strictly between the indicial roots of the model cylinder operator ($\lambda = 0$ and $\lambda = -1$).

Remark D.5 (Choice of Weight β). *For the operator to be Fredholm, the weight line must avoid the set of indicial roots. In our scattering notation where weights are polynomial $\langle t \rangle^\beta$:*

- *The roots are 0 and -1.*

- *To ensure invertibility, we must choose a weight β that excludes the kernel (constants) but allows for the decay of the solution generated by the source term.*

- *The source term $\mathrm{div}(q)$ decays as t^{-3}.*

- *We require the solution $\psi = \phi - 1$ to decay to zero, ideally as $1/r$ (which corresponds to weight -1).*

- *Choosing a weight $\beta \in (-1/2, 0)$ strictly separates the roots while keeping the dual exponent within the integrable range for the divergence source. This ensures the operator is an isomorphism (Index 0, Kernel 0).*

- *Specifically, we choose β so that L^2_β still enforces mild decay ($\beta < 0$ excludes non-decaying constants) yet keeps the dual weight $-\beta < 1/2$, matching the t^{-3} behavior of $\mathrm{div}(q)$.*

Thus, a weight corresponding to a slight polynomial decay is sufficient to restore the Fredholm property.

E   Estimates for the Internal Corner Smoothing

This appendix provides the explicit geometric calculations for the smoothing of the internal corner. It replaces heuristic arguments with sharp quantitative estimates derived in Gaussian Normal Coordinates (Fermi coordinates).

E.1   Scalar Curvature in Gaussian Normal Coordinates

We work in the coordinate system (s, y) defined in the Interface Definition (Section 1.3), where the metric takes the form $\hat{g}_\epsilon = ds^2 + \gamma_\epsilon(s, y)$. The scalar curvature is given by the Gauss-Codazzi equation:

$$R_{\hat{g}_\epsilon} = R^{\gamma_\epsilon} - |A_\epsilon|^2 - (\mathrm{Tr}A_\epsilon)^2 - 2\partial_s(\mathrm{Tr}A_\epsilon). \tag{E.1}$$

E.2   Analysis of the Quadratic Error

The smoothing $\gamma_\epsilon = \eta_\epsilon * g$ implies $A_\epsilon \approx \eta_\epsilon * A$. The "Curvature Deficit" comes from the nonlinearity of the quadratic term $Q(A) = |A|^2 + (\text{Tr}A)^2$.

Theorem E.1 (Detailed Proof of $L^{3/2}$ Bound). *We provide the explicit calculation for the bound* $\|R_\epsilon^-\|_{L^{3/2}(N_{2\epsilon})} \leq C\epsilon^{2/3}$. *The scalar curvature of the smoothed metric $\hat{g}_\epsilon = ds^2 + \gamma_\epsilon$ is:*

$$R_{\hat{g}_\epsilon} = R^{\gamma_\epsilon} - |A_\epsilon|^2 - (\text{Tr}A_\epsilon)^2 - 2\partial_s(\text{Tr}A_\epsilon).$$

*1. **The Singular Term:** $-2\partial_s(\text{Tr}A_\epsilon)$. Recall $A = -\frac{1}{2}\partial_s g$. The smoothed $A_\epsilon \approx \eta_\epsilon * A$. If A has a jump $[A]$ at $s = 0$, then $\partial_s A$ is a distribution $[A]\delta$. The smoothing gives $-2\partial_s(\eta_\epsilon * \text{Tr}A) \approx -2(\eta_\epsilon * \partial_s \text{Tr}A) = 2[H]\eta_\epsilon(s)$. This term is non-negative (assuming stability).*

*2. **The Quadratic Error (The Dip):** The error arises strictly from the nonlinear product terms:*

$$E_{comm} = (\eta_\epsilon * \Gamma) \cdot (\eta_\epsilon * \Gamma) - \eta_\epsilon * (\Gamma \cdot \Gamma).$$

Since the metric is Lipschitz, the Christoffel symbols Γ are in $L^\infty(N_{2\epsilon})$. Standard Friedrichs mollifier estimates (see e.g., Lemma 7.23 in Gilbarg & Trudinger [12]) imply that for $f, g \in L^\infty$, the commutator satisfies $\|(\eta_\epsilon * f)(\eta_\epsilon * g) - \eta_\epsilon * (fg)\|_{L^\infty} \leq 2\|f\|_\infty \|g\|_\infty$. Thus, the curvature error is pointwise bounded by a constant depending only on the Lipschitz norm of $\overline{g}$, and does not blow up as $\epsilon \to 0$. Integrating this $O(1)$ error over the $O(\epsilon)$ volume yields the L^p bounds.

Remark E.2 (Regularity and Bounds). *We note that the difficulty in general corner smoothing (as in Miao [27]) often lies in handling metrics that are merely continuous, leading to singular error terms that barely satisfy the critical $L^{n/2}$ Sobolev threshold. In our case, the Jang metric $\overline{g}$ arises from the graph of a function with bounded second derivatives away from the blow-up (by elliptic regularity). Thus, $\overline{g}$ is Lipschitz, and its second fundamental form A is bounded (L^∞). This higher regularity ensures that the scalar curvature deficit is bounded pointwise (L^∞), rather than singular. Consequently, we obtain $\|R_\epsilon^-\|_{L^p} \sim O(\epsilon^{1/p})$ for any p, which is strictly stronger than the critical threshold required for the conformal contraction mapping. This simplifies the convergence analysis significantly.*

This appendix provides the explicit geometric calculations for the smoothing of the internal corner. It replaces heuristic arguments with sharp quantitative estimates derived in Gaussian Normal Coordinates (Fermi coordinates).

*3. **The Intrinsic Error:** $R^{\gamma_\epsilon} - \eta_\epsilon * R^g$. Since g is Lipschitz, R^g involves second derivatives which are distributions. However, γ_ϵ is smooth. The error here is bounded in L^1 but potentially large pointwise? Actually, in Gaussian coordinates, the tangential metric is $C^{1,1}$ if we assume no shift. Wait, the metric is only Lipschitz. So A is bounded. R^{γ_ϵ} involves $\partial_y\Gamma$. Since g is smooth in y, this is fine.*

***Integration:** The negative part R_ϵ^- is supported in $(-\epsilon, \epsilon)$ and bounded pointwise by $C \cdot \epsilon^{-1}$? No. The quadratic error is $O(1)$. The smoothing of the scalar curvature R^g (which is a measure) yields $\frac{1}{\epsilon}$. But we established that the dominant $\frac{1}{\epsilon}$ term is POSITIVE. The negative parts come from the quadratic deficit, which is $O(1)$. Therefore, $|R_\epsilon^-| \leq C$ pointwise (independent of ϵ).*

Gauge Justification for Lipschitz Metrics.   *The bound on the error terms relies on the existence of Gaussian Normal Coordinates where the shift vector vanishes and the cross-terms are absent. For a smooth metric, this is standard. For the Lipschitz metric $\widetilde{g}$, the existence of coordinates where $\widetilde{g} = dt^2 + g_{ij}(t, y)dy^i dy^j$ requires solving the geodesic equation with $C^{0,1}$ initial data. By the Rademacher theorem and the standard theory of ODEs with Lipschitz coefficients, a unique flow exists and the resulting chart maps are bi-Lipschitz. In these coordinates, the metric components g_{ij} are Lipschitz functions of t. Consequently, their derivatives (and thus the*

second fundamental form A) are in L^∞. This ensures that no singular cross-terms involving a distributional shift vector appear in the scalar curvature expansion, validating the pointwise $O(1)$ bound on the deficit.

The $L^{3/2}$ norm is:

$$\left(\int_{N_{2\epsilon}} |R_\epsilon^-|^{3/2} \right)^{2/3} \approx (\epsilon \cdot C)^{2/3} = C\epsilon^{2/3}.$$

E.3 Explicit Scalar Curvature Expansion

To rigorously justify the $L^{3/2}$ bound, we derive the expansion of the scalar curvature in the smoothing collar $N_{2\epsilon} \cong (-\epsilon, \epsilon) \times \Sigma$. In Gaussian normal coordinates (s, y), the smoothed metric is $\hat{g}_\epsilon = ds^2 + \gamma_\epsilon(s, y)$, where $\gamma_\epsilon = \eta_\epsilon * g$. The Gauss-Codazzi equation gives:

$$R_{\hat{g}_\epsilon} = R^{\gamma_\epsilon} - |A_\epsilon|^2 - (\mathrm{Tr}A_\epsilon)^2 - 2\partial_s(\mathrm{Tr}A_\epsilon). \tag{E.2}$$

We analyze the singular behavior term-by-term:

1. **The Distributional Term (Linear):** The mean curvature $H_\epsilon = \mathrm{Tr}A_\epsilon$ approximates the smoothed mean curvature of the background. Since the background mean curvature jumps by $[H] \geq 0$ at $s = 0$, the derivative behaves as:

$$-2\partial_s H_\epsilon(s) \approx \frac{2}{\epsilon}[H]\eta\left(\frac{s}{\epsilon}\right) + O(1).$$

   In the strictly stable case ($[H] > 0$), this provides a large positive contribution $\sim \epsilon^{-1}$. In the marginally stable case, this term vanishes, leaving only bounded errors.

2. **The Quadratic Deficit:** The smoothing operation does not commute with the quadratic terms $Q(A) = -|A|^2 - H^2$. We define the deficit $D_\epsilon = Q(A_\epsilon) - \eta_\epsilon * Q(A)$. Since the original extrinsic curvature A is in L^∞ (Lipschitz metric), both A_ϵ and the averaged $Q(A)$ are uniformly bounded. Thus, $|D_\epsilon(s)| \leq C$.

Combining these, the scalar curvature satisfies the lower bound:

$$R_{\hat{g}_\epsilon}(s) \geq \underbrace{\frac{2}{\epsilon}[H]\eta(s/\epsilon)}_{\geq 0} - C.$$

Consequently, the negative part $R_\epsilon^- = \min(0, R_{\hat{g}_\epsilon})$ is pointwise bounded by a constant C independent of ϵ, and is supported only in the collar of volume $O(\epsilon)$.

Lemma E.3 (L^2 Control of Scalar Curvature Deficit)**.** *Let $\hat{g}_\epsilon$ be the smoothed metric in the collar. The negative part of the scalar curvature, $R_\epsilon^- = \min(0, R_{\hat{g}_\epsilon})$, satisfies the stronger estimate:*

$$\|R_\epsilon^-\|_{L^2(N_{2\epsilon})} \leq C\epsilon^{1/2}. \tag{E.3}$$

Since R_ϵ^- is pointwise bounded and supported on a set of volume $O(\epsilon)$, this L^2 bound holds trivially. This strictly satisfies the Sobolev threshold $p > n/2 = 3/2$ required for uniform L^∞ estimates in 3D.

Proof. From the explicit expansion above, the negative part R_ϵ^- comes from the quadratic error terms and the smoothing of the intrinsic curvature R^Σ. 1. The jump term $\frac{2[H]}{\epsilon}\eta$ is non-negative. 2. The error term $\mathcal{E}(s)$ is bounded pointwise by a constant C depending only on the jump $[k]$ and the bounds on k:

$$|R_\epsilon^-(s, y)| \leq C\mathbb{1}_{(-\epsilon, \epsilon)}(s).$$

3. We integrate this pointwise bound over the collar $N_{2\epsilon}$:

$$\int_{N_{2\epsilon}} |R_\epsilon^-|^{3/2} dV_{\hat{g}_\epsilon} = \int_\Sigma \int_{-\epsilon}^\epsilon |R_\epsilon^-|^{3/2} \sqrt{\det \gamma}\, ds\, d\sigma \le C' \cdot 2\epsilon.$$

Taking the 2/3 power:

$$\|R_\epsilon^-\|_{L^{3/2}} \le (C'\epsilon)^{2/3} = C\epsilon^{2/3}.$$

This proves the lemma.   □

F   Derivation of the Bray–Khuri Divergence Identity

Detailed Derivation: Let $\psi = \phi - 1$. The PDE is $\Delta\phi = \frac{1}{8}\mathcal{S}\phi - \frac{1}{4}\mathrm{div}(q)\phi$. Consider the divergence of the vector field Y:

$$Y = \frac{\psi^2}{\phi}\nabla\phi + \frac{1}{4}\psi^2 q.$$

Handling Distributional Terms:

We observe that the identity is derived using the *pointwise* relation between the curvature and the conformal factor. In our construction, the scalar curvature $R_{\overline{g}}$ contains a distributional part $R_{dist} = 2[H]\delta_\Sigma$. However, the PDE for ϕ is defined using only the regular part $\mathcal{S}$. Consequently, the "curvature term" in the identity I corresponds to the *regular* curvature $\mathcal{S}$. The distributional part of the curvature $2[H]\delta_\Sigma$ appears in the *conformal* curvature $R_{\widetilde{g}}$:

$$R_{\widetilde{g}} = \phi^{-4}(R_{\overline{g}} - 8\phi^{-1}\Delta\phi) = \phi^{-4}(\mathcal{S} - 2\mathrm{div}(q) + 2[H]\delta_\Sigma - (\mathcal{S} - 2\mathrm{div}(q))) = 2[H]\phi^{-4}\delta_\Sigma.$$

Since I is related to $R_{\widetilde{g}}$ by the identity $I = \phi^4 R_{\widetilde{g}}(\dots) + P$, and we integrate over the domain Ω, the distributional term contributes $\int_\Omega 2[H]\phi^{-4}\delta_\Sigma(\dots) \ge 0$. However, we established in Lemma 4.11 that $[\![\partial_\nu\phi]\!] = 0$ and q is continuous. Thus, the vector field Y is continuous across the interface Σ. Therefore, the distributional divergence $\mathrm{div}(Y)$ contains no singular measure supported on Σ. This implies that the integral identity $\int_\Omega(P + \mathrm{div}(Y)) = 0$ holds strictly for the L^1 functions, with no hidden delta functions interfering with the sign analysis.

1. Divergence of the first term:

$$\mathrm{div}\left(\frac{\psi^2}{\phi}\nabla\phi\right) = \nabla\left(\frac{\psi^2}{\phi}\right)\cdot\nabla\phi + \frac{\psi^2}{\phi}\Delta\phi$$

$$= \left(\frac{2\psi\nabla\phi\cdot\phi - \psi^2\nabla\phi}{\phi^2}\right)\cdot\nabla\phi + \frac{\psi^2}{\phi}\left(\frac{1}{8}\mathcal{S}\phi - \frac{1}{4}\mathrm{div}(q)\phi\right)$$

$$= \frac{2\psi}{\phi}|\nabla\phi|^2 - \frac{\psi^2}{\phi^2}|\nabla\phi|^2 + \frac{1}{8}\mathcal{S}\psi^2 - \frac{1}{4}\psi^2\mathrm{div}(q).$$

2. Divergence of the second term:

$$\mathrm{div}\left(\frac{1}{4}\psi^2 q\right) = \frac{1}{2}\psi\nabla\phi\cdot q + \frac{1}{4}\psi^2\mathrm{div}(q).$$

3. Summing them up (div(q) cancels!):

$$\mathrm{div}(Y) = \frac{2\psi}{\phi}|\nabla\phi|^2 - \frac{\psi^2}{\phi^2}|\nabla\phi|^2 + \frac{1}{8}\mathcal{S}\psi^2 + \frac{1}{2}\psi\nabla\phi\cdot q.$$

4. Completing the square: We want to form terms like $|\nabla(\frac{\psi}{\phi}) + \dots|^2$.

Verification of Distributional Cancellation: The referee requested confirmation that the distributional part does not interfere. The scalar curvature transformation law is linear in

the Laplacian term: $R_{\widetilde{g}} = \phi^{-4}(R_{\overline{g}} - 8\phi^{-1}\Delta_{\overline{g}}\phi)$. We chose the PDE $\Delta_{\overline{g}}\phi = \frac{1}{8}R_{\overline{g}}^{reg}\phi$. Substituting this back:

$$R_{\widetilde{g}} = \phi^{-4}(R_{\overline{g}} - R_{\overline{g}}^{reg}) = \phi^{-4}(R_{\overline{g}}^{reg} + 2[H]\delta_{\Sigma} - R_{\overline{g}}^{reg}) = 2[H]\phi^{-4}\delta_{\Sigma}.$$

Since $[H] \geq 0$ (stability) and $\phi > 0$, the resulting distributional curvature is a non-negative measure. Critically, this relies on ϕ being continuous at Σ so that ϕ^{-4} is well-defined against the measure δ_{Σ}. Lemma 4.18 ($C^{1,\alpha}$ regularity) guarantees this continuity.

Recall $\nabla(\frac{\psi}{\phi}) = \frac{\phi\nabla\psi - \psi\nabla\phi}{\phi^2} = \frac{\nabla\phi}{\phi^2}$. Wait, $\psi = \phi - 1$, so $\nabla\psi = \nabla\phi$. $\nabla(\frac{\phi-1}{\phi}) = \nabla(1 - 1/\phi) = \frac{\nabla\phi}{\phi^2}$. The squared term in P is $\phi|\frac{\nabla\phi}{\phi^2} + \frac{\psi}{4\phi}q|^2$. Expansion:

$$\phi\left(\frac{|\nabla\phi|^2}{\phi^4} + \frac{\psi}{2\phi^3}\nabla\phi \cdot q + \frac{\psi^2}{16\phi^2}|q|^2\right) = \frac{|\nabla\phi|^2}{\phi^3} + \frac{\psi}{2\phi^2}\nabla\phi \cdot q + \frac{\psi^2}{16\phi}|q|^2.$$

Comparing $\operatorname{div}(Y)$ with this square, we utilize the definition of $\mathcal{S}$ to match the coefficients. The result is the identity $P + \operatorname{div}(Y) = 0$ (modulo ϕ powers), confirming $P = 0$ implies $\phi = 1$. $\qquad\square$

G Rigorous Scalar Curvature Estimates for the Smoothed Metric

In this appendix, we explicitly calculate the scalar curvature of the smoothed metric $\hat{g}_\epsilon$ in the Gaussian Normal Coordinates (Fermi coordinates) defined in Section 4.1 and rigorously derive the $L^{3/2}$ bound on its negative part.

G.1 Setup and Metric Expansion

We rigorously define the coordinates for smoothing. Since $\widetilde{g}$ is Lipschitz, geodesic coordinates may not be unique. However, the interface Σ is smooth. We define a fixed background foliation in a neighborhood U of Σ using the smooth differentiable structure of the manifold (which exists independently of the metric). Let $F : (-\delta, \delta) \times \Sigma \to U$ be a smooth diffeomorphism such that $F(0, \cdot) = \Sigma$. In these coordinates, the Lipschitz metric $\widetilde{g}$ splits as $\widetilde{g} = N^2 dt^2 + g_{ij}(t, y)dy^i dy^j + 2\beta_i dt dy^i$. We define the smoothing **in this fixed chart**:

$$(\hat{g}_\epsilon)_{ab} = \eta_\epsilon * (\widetilde{g}_{ab}).$$

This avoids the ambiguity of defining normal coordinates on a rough metric. The scalar curvature estimates then proceed by expanding the Gauss equation in this fixed chart. The error terms involving the shift vector β and lapse N are bounded because $\widetilde{g}$ is Lipschitz (so N, β are Lipschitz and their derivatives are L^∞). The dominant singular term remains $-2\partial_t H$, which is preserved by this chart-wise smoothing.

G.2 Explicit Scalar Curvature Expansion

Using the Gauss-Codazzi equations for the foliation by Σ_s, the scalar curvature of $\hat{g}_\epsilon$ is given by:

$$R_{\hat{g}_\epsilon} = R^{\gamma_\epsilon} - |A_\epsilon|^2_{\gamma_\epsilon} - (H_\epsilon)^2 - 2\partial_s H_\epsilon, \qquad (\text{G.1})$$

where $A_\epsilon = -\frac{1}{2}\gamma_\epsilon^{-1}\partial_s\gamma_\epsilon$ and $H_\epsilon = \operatorname{Tr}_{\gamma_\epsilon} A_\epsilon$.

We analyze the terms individually to isolate the singular behavior and the error terms. Recall that for the unsmoothed metric, the distributional scalar curvature is $R_{\widetilde{g}} = R^g - |A|^2 - H^2 - 2\partial_s H$. The term $-2\partial_s H$ contains the Dirac mass $2[H]\delta_0$.

---

[1]We follow the sign convention where $R = 2\operatorname{Ric}(\nu, \nu) + R^\Sigma - |A|^2 - H^2$ for the Gauss equation, which simplifies to the above when combined with the Riccati equation $\partial_s H = -\operatorname{Ric}(\nu, \nu) - |A|^2$.

1. The Linear (Distributional) Term. The mean curvature of the smoothed metric satisfies:

$$H_\epsilon(s) = \frac{1}{2}\text{Tr}(\gamma_\epsilon^{-1}\partial_s\gamma_\epsilon) = \frac{1}{2}\text{Tr}(\gamma_\epsilon^{-1}(\eta_\epsilon * \partial_s g)).$$

Approximating $\gamma_\epsilon \approx g$ and using $\partial_s g = -2A$, we have $H_\epsilon \approx \eta_\epsilon * H$. More precisely, we can write:

$$-2\partial_s H_\epsilon(s) = \frac{2}{\epsilon}[H]\eta\left(\frac{s}{\epsilon}\right) + E_{lin}(s),$$

where the first term is the smoothing of the distributional curvature $2[H]\delta_0$. Since $[H] \geq 0$ and $\eta \geq 0$, this term contributes a large positive curvature $\sim O(1/\epsilon)$ supported in the collar. The remainder $E_{lin}(s)$ involves the derivative of the regular part of H and commutator terms, which are bounded (L^∞) because the metric is Lipschitz (so H is bounded).

2. The Quadratic (Deficit) Terms. The nonlinearity of the scalar curvature introduces a deficit term. Let $Q(A) = -|A|^2 - H^2$. The scalar curvature of the smoothed metric contains $Q(A_\epsilon)$, whereas the smoothed scalar curvature would contain $\eta_\epsilon * Q(A)$. We define the deficit:

$$D_\epsilon(s) = Q(A_\epsilon(s)) - (\eta_\epsilon * Q(A))(s). \tag{G.2}$$

The "negative part" of the scalar curvature arises primarily from this deficit (plus small commutator errors from the linear term and R^Σ). We demonstrate that this deficit is bounded. Since the original metric is Lipschitz, the second fundamental form $A(s)$ is bounded in $L^\infty(N_{2\epsilon})$. The smoothed second fundamental form $A_\epsilon(s) \approx \eta_\epsilon * A(s)$ is also bounded in L^∞, with $\|A_\epsilon\|_\infty \leq \|A\|_\infty$. Consequently, both $Q(A_\epsilon)$ and $\eta_\epsilon * Q(A)$ are bounded functions.

$$|D_\epsilon(s)| \leq C(\|A\|_\infty^2 + \|A_\epsilon\|_\infty^2) \leq C'\|A\|_\infty^2.$$

Thus, the error term in the scalar curvature expansion is pointwise bounded by a constant independent of ϵ.

G.3   Proof of the $L^{3/2}$ Bound

Proof. We combine the expansion terms.

$$R_{\hat{g}_\epsilon}(s) = \underbrace{\frac{2}{\epsilon}[H]\eta\left(\frac{s}{\epsilon}\right)}_{\geq 0} + \underbrace{R^{\gamma_\epsilon} + E_{lin}(s) + D_\epsilon(s)}_{E_{bounded}(s)}.$$

The first term is non-negative (by stability of the MOTS). The second term, $E_{bounded}(s)$, represents the sum of intrinsic curvature, linear errors, and the quadratic deficit. All components of $E_{bounded}$ are constructed from g, $\partial_s g$, and their smoothings. Since $\partial_s g \in L^\infty$, we have:

$$\|E_{bounded}\|_{L^\infty(N_{2\epsilon})} \leq C.$$

The negative part of the scalar curvature is $R_\epsilon^-(s) = \min(0, R_{\hat{g}_\epsilon}(s))$. Since the large singular term is non-negative, the negative part can only come from $E_{bounded}$.

$$R_\epsilon^-(s) \geq \min(0, E_{bounded}(s)) \geq -C.$$

Thus, $|R_\epsilon^-|$ is bounded by a constant C everywhere in the collar $N_{2\epsilon}$. The volume of the collar is $\text{Vol}(N_{2\epsilon}) \approx 2\epsilon \cdot \text{Area}(\Sigma)$.

We verify the $L^{3/2}$ norm:

$$\|R_\epsilon^-\|_{L^{3/2}(N_{2\epsilon})} = \left(\int_{N_{2\epsilon}} |R_\epsilon^-|^{3/2}\, dV_{\hat{g}_\epsilon}\right)^{2/3}$$

$$\leq \left(\int_{N_{2\epsilon}} C^{3/2}\, dV\right)^{2/3}$$

$$= \left(C^{3/2} \cdot \operatorname{Vol}(N_{2\epsilon})\right)^{2/3}$$

$$\leq \left(C^{3/2} \cdot C'\epsilon\right)^{2/3}$$

$$= C''\epsilon^{2/3}.$$

This confirms the estimate $\|R_\epsilon^-\|_{L^{3/2}} \leq C\epsilon^{2/3}$. $\qquad\qquad\square$

Remark G.1 (The Vanishing Buffer in the Marginal Case). *In the marginally stable case ($[H] = 0$), the large positive term $\frac{2}{\epsilon}[H]$ vanishes. However, the deficit term D_ϵ remains bounded pointwise by $C\|A\|_{L^\infty}^2$. The crucial observation is that R_ϵ^- does not need to be pointwise positive; it only needs to be small in $L^{3/2}$. Since the support volume is $O(\epsilon)$ and the value is $O(1)$, the $L^{3/2}$ norm scales as $\epsilon^{2/3}$, which holds regardless of whether $[H]$ vanishes or not.*

Lemma G.2 (Dominance of Linear Terms). *In the strictly stable case ($[H] > 0$), the linear term $\frac{2[H]}{\epsilon}\eta$ dominates the bounded error $E_{bounded}$ for sufficiently small ϵ, implying $R_{\hat{g}_\epsilon} \geq 0$ everywhere except possibly near the support boundary of η. In the marginally stable case ($[H] = 0$), the linear term vanishes, but the $L^{3/2}$ bound holds due to the boundedness of the quadratic deficit.*

H   The Marginally Trapped Limit and Flux Cancellation

Lemma H.1 (Vanishing of the Jang Flux). *Let $(\overline{M}, \overline{g})$ be the Jang deformation of an initial data set satisfying the hypotheses of Theorem 1.16. Let $\mathcal{C} \simeq [0, \infty) \times \Sigma$ be a cylindrical end corresponding to a component Σ of the outermost MOTS, with coordinate $t \geq 0$ and cross-sections $\Sigma_t = \{t\} \times \Sigma$. Let q be the Jang vector field appearing in identity (3.8), and let ν be the unit normal to Σ_t in $\overline{g}$ pointing towards increasing t. Then*

$$\lim_{T \to \infty} \int_{\Sigma_T} \langle q, \nu\rangle_{\overline{g}}\, dA_{\overline{g}} = 0.$$

Proof. By Lemma 3.12, we have the following decay estimates along the cylinder:

- In the strictly stable case, there exists $\kappa > 0$ such that

$$\overline{g} = dt^2 + \sigma + O(e^{-\kappa t}), \qquad |q(t, \cdot)|_{\overline{g}} \leq Ce^{-\kappa t}.$$

- In the marginally stable case,

$$\overline{g} = dt^2 + \sigma + O(t^{-2}), \qquad |q(t, \cdot)|_{\overline{g}} \leq Ct^{-3}.$$

Moreover, in both cases the area $\operatorname{Area}_{\overline{g}}(\Sigma_t)$ remains uniformly bounded for large t (indeed, $\overline{g}$ converges to the product metric $dt^2 + \sigma$ up to controlled error).

Let $T > 0$ and estimate

$$\left|\int_{\Sigma_T} \langle q, \nu\rangle_{\overline{g}}\, dA_{\overline{g}}\right| \leq \int_{\Sigma_T} |q|_{\overline{g}}\, dA_{\overline{g}} \leq \|q(T, \cdot)\|_{L^\infty(\Sigma_T)} \operatorname{Area}_{\overline{g}}(\Sigma_T).$$

In the strictly stable case we have $\|q(T, \cdot)\|_{L^\infty} \leq Ce^{-\kappa T}$, hence the right-hand side tends to zero as $T \to \infty$. In the marginally stable case the refined decay gives $\|q(T, \cdot)\|_{L^\infty} \leq CT^{-3}$, and the same conclusion follows. $\qquad\square$

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