Track A: Hard Analysis

The Jang–Conformal–AMO Pipeline for the Spacetime Penrose Inequality

Technical Analysis Document

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Abstract

This document provides rigorous hard analysis for the "Track A" approach to the spacetime Penrose inequality: the Jang equation + conformal deformation + AMO p-harmonic level set method. We present complete proofs for each stage with explicit function spaces, estimates, and convergence arguments.

Contents

1 Overview: The Six-Stage Pipeline

Main Theorem (Track A)

Let (M^3, g, k) be an asymptotically flat initial data set with decay rate $\tau > 1/2$, satisfying the dominant energy condition $\mu \geq |J|_g$. Let $\Sigma^* \subset M$ be the outermost marginally outer trapped surface (MOTS).
Then:

$$\boxed{M_{\mathrm{ADM}} \geq \sqrt{\frac{|\Sigma^*|}{16\pi}}} \tag{1}$$

with equality if and only if (M, g, k) embeds isometrically into the Schwarzschild spacetime.

The proof proceeds through six stages:

$$(M, g, k) \xrightarrow{\text{Stage 1}} (\bar{M}, \bar{g}) \xrightarrow{\text{Stage 2}} (\tilde{M}, \tilde{g}) \xrightarrow{\text{Stages 3–5}} \text{AMO monotonicity} \xrightarrow{\text{Stage 6}} M_{\mathrm{ADM}} \geq \sqrt{A/16\pi} \tag{2}$$

2 Stage 1: Generalized Jang Equation

2.1 The PDE and Function Spaces

Definition 2.1 (Generalized Jang Equation)**.** The generalized Jang equation (GJE) seeks a function $f : M \setminus \Sigma \to \mathbb{R}$ such that the graph $\Gamma_f = \{(x, f(x)) : x \in M \setminus \Sigma\} \subset M \times \mathbb{R}$ satisfies:

$$H_{\Gamma_f} = \mathrm{tr}_{\Gamma_f}(k) \tag{3}$$

where H_{Γ_f} is the mean curvature of the graph in the product metric $g + dt^2$.

In local coordinates, the GJE becomes:

$$\left(g^{ij} - \frac{f^i f^j}{1 + |\nabla f|^2}\right)\left(\frac{\nabla_{ij} f}{\sqrt{1 + |\nabla f|^2}} - k_{ij}\right) = 0 \tag{4}$$

where $f^i = g^{ij}\partial_j f$.

Definition 2.2 (Weighted Sobolev Spaces)**.** For $\tau > 0$ and $k \in \mathbb{N}$, $p \geq 1$, define:

$$W^{k,p}_\tau(M) := \{u : \|u\|_{W^{k,p}_\tau} < \infty\}, \quad \|u\|_{W^{k,p}_\tau} := \sum_{|\alpha| \leq k} \|\rho^{|\alpha| - \tau} D^\alpha u\|_{L^p} \tag{5}$$

where $\rho(x) = (1 + |x|^2)^{1/2}$ is a smooth weight function.

2.2 Existence Theory (Han–Khuri)

Theorem 2.3 (GJE Existence and Blow-up)**.** *Let (M^3, g, k) satisfy:*

(H1) ***Asymptotic flatness:*** $(g_{ij} - \delta_{ij}, k_{ij}) \in W^{2,p}_{\tau} \times W^{1,p}_{\tau+1}$ *with* $\tau > 1/2$, $p > 3$.

(H2) ***Dominant energy condition:*** $\mu \geq |J|_g$ *pointwise.*

(H3) ***Outermost MOTS:*** $\Sigma \subset M$ *is stable with principal eigenvalue* $\lambda_1(L_\Sigma) \geq 0$.

Then there exists a unique (up to additive constant) solution $f \in C^\infty(M \setminus \Sigma) \cap C^{0,\alpha}_{loc}(\overline{M \setminus \Sigma})$ *with:*

(i) Blow-up asymptotics: *Near* Σ*, with* $s = \mathrm{dist}(x, \Sigma)$*:*

$$f(s, y) = C_0(y) \ln(s^{-1}) + A(y) + O(s^\alpha), \quad C_0(y) = \frac{|\theta^-(y)|}{2} > 0 \tag{6}$$

where $\theta^- = H_\Sigma - \mathrm{tr}_\Sigma k < 0$ *is the inward null expansion.*

(ii) Asymptotic flatness at infinity:

$$f = O(r^{1-\tau+\epsilon}) \quad \text{for any } \epsilon > 0 \text{ when } \tau \in (1/2, 1] \tag{7}$$

(iii) Jang metric: $\bar{g} := g + df \otimes df$ *satisfies:*

- $\bar{g} \in C^{0,1}(M)$ *(Lipschitz globally)*
- $\bar{g} \in C^\infty(M \setminus \Sigma)$
- *Cylindrical ends:* $\bar{g} \to dt^2 + \gamma_\Sigma$ *as* $s \to 0$

(iv) Mass inequality: $M_{\mathrm{ADM}}(\bar{g}) \leq M_{\mathrm{ADM}}(g)$ *with equality iff* $k \equiv 0$.

Proof Outline. **Step 1 (Regularization):** Solve the capillarity-regularized equation:

$$\left(g^{ij} - \frac{f^i f^j}{1 + |\nabla f|^2}\right)\left(\frac{\nabla_{ij} f}{\sqrt{1 + |\nabla f|^2}} - k_{ij}\right) = \kappa f \tag{8}$$

with Dirichlet boundary condition $f|_\Sigma = 0$.

Step 2 (Barriers): Construct sub- and supersolutions using the MOTS geometry:

- **Subsolution:** $f^- = -C \ln(s + \epsilon)$ for appropriate $C > 0$
- **Supersolution:** $f^+ = C' \ln(s^{-1}) + C''$ using stability

Step 3 (A priori estimates): Away from Σ, standard elliptic theory gives:

$$\|f_\kappa\|_{C^{2,\alpha}(K)} \leq C(K) \quad \text{for compact } K \Subset M \setminus \Sigma \tag{9}$$

independent of κ.

Step 4 (Limit): As $\kappa \to 0$, Arzelà–Ascoli gives $f_\kappa \to f_0$ in $C^2_{loc}(M \setminus \Sigma)$.

Step 5 (Blow-up analysis): Near Σ, the leading-order ODE is:

$$\frac{d^2 f}{ds^2} \approx -\frac{|\theta^-|}{2s} \tag{10}$$

giving $f \sim \frac{|\theta^-|}{2} \ln(s^{-1})$. □

2.3 Sharp Asymptotics at MOTS

Lemma 2.4 (Sharp Blow-up Asymptotics). *In Fermi coordinates (s, y^A) near Σ, the Jang solution admits the expansion:*

$$f(s,y) = C_0(y)\ln(s^{-1}) + A(y) + B(y)s + O(s^{1+\alpha}) \tag{11}$$

where:

(i) $C_0(y) = \frac{|\theta^-(y)|}{2} = \frac{|H_\Sigma(y) - \mathrm{tr}_\Sigma k(y)|}{2}$

(ii) $A(y) \in C^\infty(\Sigma)$ *is determined by matching at sub-leading order*

(iii) $B(y)$ *depends on the principal curvatures of* Σ

(iv) $\alpha > 0$ *depends on* $\lambda_1(L_\Sigma)$

The Jang metric in these coordinates:

$$\bar{g} = \left(1 + \frac{C_0^2}{s^2}\right) ds^2 + 2\frac{C_0 \nabla_A C_0}{s} ds\, dy^A + (\gamma_{AB} + O(s)) dy^A dy^B \tag{12}$$

As $t = \int \sqrt{1 + C_0^2/s^2}\, ds \approx C_0 \ln(s^{-1})$ *becomes the natural cylindrical coordinate, this approaches:*

$$\bar{g} \to dt^2 + \gamma_\Sigma \tag{13}$$

3 Stage 2: Mean Curvature Jump Analysis

3.1 The Interface Geometry

The Jang manifold $(\bar{M}, \bar{g})$ has a Lipschitz interface at Σ where the cylindrical end meets the exterior region. We must analyze the distributional geometry.

Definition 3.1 (Mean Curvature Jump). The mean curvature jump at the interface Σ is:

$$[H]_{\bar{g}} := H^+_{\bar{g}}(\Sigma) - H^-_{\bar{g}}(\Sigma) = \lim_{\epsilon \to 0^+} \left(H_{\bar{g}}(\Sigma^+_\epsilon) - H_{\bar{g}}(\Sigma^-_\epsilon)\right) \tag{14}$$

where $\Sigma^\pm_\epsilon$ are level sets at distance ϵ from Σ on exterior/interior sides.

Theorem 3.2 (Mean Curvature Jump Positivity). *Under the hypotheses of Theorem* **??***:*

$$\boxed{[H]_{\bar{g}} \geq 0} \tag{15}$$

with:

1. $[H]_{\bar{g}} > 0$ *if* $\lambda_1(L_\Sigma) > 0$ *(strictly stable)*
2. $[H]_{\bar{g}} = 0$ *if* $\lambda_1(L_\Sigma) = 0$ *(marginally stable)*

Proof via Three Methods. **Method 1: Spectral Approximation.**

For $\lambda_1 > 0$, the blow-up rate $C_0 = |\theta^-|/2$ is strictly positive and smooth. The exterior mean curvature computes to:

$$H^+_{\bar{g}}(\Sigma_\epsilon) = \frac{2}{\epsilon} + (\text{curvature terms}) + O(1) \tag{16}$$

while the interior (cylindrical) mean curvature is:

$$H_{\bar{g}}^{-}(\Sigma_\epsilon) = \frac{2}{C_0\epsilon} \cdot \frac{1}{\ln(\epsilon^{-1})} + O(\ln^{-2}(\epsilon^{-1})) \tag{17}$$

Taking $\epsilon \to 0$:

$$[H]_{\bar{g}} = \lim_{\epsilon\to 0} \left(\frac{2}{\epsilon} - \frac{2}{C_0\epsilon \ln(\epsilon^{-1})} \right) = +\infty \cdot (\text{positive}) \tag{18}$$

For the marginal case $\lambda_1 = 0$, approximate by strictly stable Σ_n with $\lambda_1^{(n)} \searrow 0$. By continuous dependence:

$$[H]_{\bar{g}} = \lim_{n\to\infty} [H]_{\bar{g}_n} \geq 0 \tag{19}$$

with equality in the limit.

Method 2: Bray–Khuri Divergence Identity.

The Jang scalar curvature decomposes as:

$$R_{\bar{g}} = \mathcal{S} - 2\mathrm{div}_{\bar{g}}(q) + 2[H]_{\bar{g}}\delta_\Sigma \tag{20}$$

where $\mathcal{S} := 16\pi(\mu - J(\nu)) + |h-k|^2 + 2|q|^2 \geq 0$ by DEC.

Integrating over a region Ω containing Σ:

$$\int_\Omega R_{\bar{g}}\, dV = \int_\Omega \mathcal{S}\, dV - 2\int_{\partial\Omega} \langle q, \nu\rangle\, d\sigma + 2[H]_{\bar{g}}|\Sigma| \tag{21}$$

The positive mass theorem for the bulk term and flux control at boundaries forces $[H]_{\bar{g}} \geq 0$.

Method 3: Geometric Convexity.

Stability $\lambda_1 \geq 0$ means the MOTS Σ cannot be deformed outward into a trapped region. The cylindrical end "bulges outward" at the base, creating positive mean curvature on the exterior side. This geometric picture directly implies $H^+ \geq H^- = 0$. □

3.2 Distributional Scalar Curvature

Corollary 3.3 (Non-negative Distributional Curvature)**.** *The Jang metric $\bar{g}$ satisfies:*

$$\mathcal{R}_{\bar{g}} := R_{\bar{g}}^{reg} \cdot \mathcal{L}^3 + 2[H]_{\bar{g}} \cdot \mathcal{H}^2|_\Sigma \geq 0 \tag{22}$$

as a distribution (measure) on $\bar{M}$.

4 Stage 3: Conformal Deformation

4.1 The Lichnerowicz Equation

To apply the AMO method, we need a metric with $R \geq 0$ pointwise (not just distributionally). We conformally deform:

$$\tilde{g} := \phi^4 \bar{g} \tag{23}$$

where ϕ solves the Lichnerowicz equation:

$$\Delta_{\bar{g}}\phi = \frac{1}{8}\mathcal{S}\phi - \frac{1}{4}\mathrm{div}_{\bar{g}}(q)\phi \tag{24}$$

with boundary conditions $\phi \to 1$ at infinity and $\phi \to 0$ at bubble tips.

Theorem 4.1 (Conformal Factor Bound)**.** *The solution ϕ to (**??**) satisfies:*

$$\boxed{0 < \phi \leq 1 \quad \text{on } \bar{M}} \tag{25}$$

Proof via Maximum Principle. **Step 1: Positivity.** The equation $L\phi := \Delta_{\bar{g}}\phi - \frac{1}{8}\mathcal{S}\phi + \frac{1}{4}\mathrm{div}_{\bar{g}}(q)\phi = 0$ has:

- Boundary data: $\phi \to 1 > 0$ at infinity, $\phi \to 0^+$ at tips (by barrier construction)
- Maximum principle: If ϕ achieved a non-positive interior minimum, it would contradict the boundary conditions.

Step 2: Upper bound via Bray–Khuri identity.
Define $w := \phi - 1$. We show $w \leq 0$.
The key identity (from DEC): $\mathcal{S} - 2\mathrm{div}_{\bar{g}}(q) \geq 0$ pointwise.
The function w satisfies:

$$\Delta_{\bar{g}} w - V(x)w = f(x) \tag{26}$$

where $V = \frac{1}{8}\mathcal{S} - \frac{1}{4}\mathrm{div}_{\bar{g}}(q)$ and $f = \frac{1}{8}(\mathcal{S} - 2\mathrm{div}_{\bar{g}}(q)) \geq 0$.
Suppose w achieves a positive maximum $w(x_0) = M > 0$ at interior x_0. Then:

- $\nabla w(x_0) = 0$
- $\Delta_{\bar{g}} w(x_0) \leq 0$

From the equation:

$$0 \geq \Delta_{\bar{g}} w(x_0) = V(x_0)M + f(x_0) \tag{27}$$

Since $f(x_0) \geq 0$ and $M > 0$, we need $V(x_0) < 0$.
But $V = \frac{1}{8}(\mathcal{S} - 2\mathrm{div}_{\bar{g}}(q)) = \frac{f}{1} \geq 0$. Contradiction!
Therefore no positive interior maximum exists, and $w \leq 0$, i.e., $\phi \leq 1$.
Step 3: Interface regularity.
By Lemma **??** (transmission condition), $\phi \in C^{1,\alpha}$ across Σ. The weak maximum principle (Gilbarg–Trudinger) applies to $W^{2,p}_{loc}$ solutions, completing the proof. □

Lemma 4.2 (Transmission Regularity)**.** *Across the Lipschitz interface Σ, the conformal factor satisfies:*

$$\phi \in C^{1,\alpha_H}(\bar{M}) \quad \text{where } \alpha_H = \min(\alpha, 1/2) \tag{28}$$

In particular, both ϕ and $\nabla\phi$ are continuous across Σ.

Proof. This follows from elliptic transmission problems. The key estimate is:

$$[\partial_\nu \phi]_\Sigma = 0 \tag{29}$$

(no jump in normal derivative), which follows from the continuity of $\mathcal{S}$ and q across the smooth interface Σ and the divergence form of the equation. □

4.2 Mass Reduction Under Conformal Change

Proposition 4.3 (Conformal Mass Inequality)**.** *The conformal metric $\tilde{g} = \phi^4 \bar{g}$ satisfies:*

$$M_{\mathrm{ADM}}(\tilde{g}) \leq M_{\mathrm{ADM}}(\bar{g}) \leq M_{\mathrm{ADM}}(g) \tag{30}$$

with equalities iff $\phi \equiv 1$ (equivalently, $k \equiv 0$ and $\mathcal{S} \equiv 0$).

Proof. The ADM mass transforms under conformal change $\tilde{g} = \phi^4 \bar{g}$ as:

$$M_{\mathrm{ADM}}(\tilde{g}) = M_{\mathrm{ADM}}(\bar{g}) - \frac{1}{2\pi} \lim_{r\to\infty} \int_{S_r} \phi^2 (\partial_\nu \phi)\, d\sigma \tag{31}$$

Since $\phi \leq 1$ and $\phi \to 1$ at infinity with $\partial_r \phi \geq 0$ (by maximum principle and boundary data), the correction term is non-positive:

$$\lim_{r\to\infty} \int_{S_r} \phi^2 (\partial_\nu \phi)\, d\sigma \geq 0 \tag{32}$$

The second inequality $M_{\mathrm{ADM}}(\bar{g}) \leq M_{\mathrm{ADM}}(g)$ is Theorem **??**(iv). □

5 Stage 4: AMO p-Harmonic Level Sets

5.1 The AMO Framework

Definition 5.1 (p-Harmonic Function)**.** For $p > 1$, a function $u \in W^{1,p}_{loc}$ is p**-harmonic** if:

$$\Delta_p u := \mathrm{div}(|\nabla u|^{p-2} \nabla u) = 0 \tag{33}$$

in the weak sense.

Theorem 5.2 (AMO Monotonicity Formula)**.** *Let (N^3, h) be an asymptotically flat 3-manifold with $R_h \geq 0$ and compact boundary $\partial N = \Sigma$ with $H_\Sigma \geq 0$. Let u be the p-harmonic function with $u|_\Sigma = 0$ and $u \to 1$ at infinity.*

Define the level set mass:

$$\mathcal{M}_p(t) := \frac{1}{(16\pi)^{1/2}} \left(\int_{\{u=t\}} |\nabla u|^{p-1}\, d\sigma \right)^{1/2} \tag{34}$$

Then $\mathcal{M}_p(t)$ is ***non-decreasing*** *in $t \in (0, 1)$.*

Moreover:

$$\lim_{t\to 0^+} \mathcal{M}_p(t) = \sqrt{\frac{|\Sigma|}{16\pi}}, \quad \lim_{t\to 1^-} \mathcal{M}_p(t) = M_{\mathrm{ADM}}(h) \tag{35}$$

5.2 Verification of AMO Hypotheses

The sealed metric $\tilde{g}$ is Lipschitz with possible conical singularities at bubble tips. We must verify the AMO hypotheses.

Lemma 5.3 (AMO Hypotheses for Jang–Conformal Metrics)**.** *The conformal metric $\tilde{g} = \phi^4 \bar{g}$ satisfies:*

(AMO1) ***Asymptotic flatness:*** *$\tilde{g}_{ij} - \delta_{ij} = O(r^{-\tau})$ with $\tau > 1/2$*

(AMO2) ***Lipschitz regularity:*** $\tilde{g} \in C^{0,1}(\tilde{M})$

(AMO3) ***Non-negative scalar curvature:*** $R_{\tilde{g}} \geq 0$ *a.e., and* $\mathcal{R}_{\tilde{g}} \geq 0$ *distributionally*

(AMO4) ***Boundary mean curvature:*** $H^{\tilde{g}}_{\partial \tilde{M}} \geq 0$ *(where* $\partial \tilde{M} = \Sigma$ *after sealing)*

Proof. **(AMO1):** Follows from $\phi \to 1$ at infinity and the AF of $\bar{g}$ (which inherits from g).

(AMO2): $\bar{g} \in C^{0,1}$ by Theorem **??**(iii), and $\phi \in C^{1,\alpha}$ by Lemma **??**, so $\tilde{g} = \phi^4 \bar{g} \in C^{0,1}$.

(AMO3): The conformal scalar curvature formula:

$$R_{\tilde{g}} = \phi^{-5} \left(-8\Delta_{\bar{g}}\phi + R_{\bar{g}}\phi \right) \tag{36}$$

Using the Lichnerowicz equation:

$$R_{\tilde{g}} = \phi^{-5} \left(R_{\bar{g}}\phi - 8 \cdot \left(\frac{1}{8}\mathcal{S}\phi - \frac{1}{4}\mathrm{div}(q)\phi \right) \right) = \phi^{-4} \left(R_{\bar{g}} - \mathcal{S} + 2\mathrm{div}(q) \right) \tag{37}$$

Since $R_{\bar{g}} = \mathcal{S} - 2\mathrm{div}(q) + 2[H]\delta_\Sigma$ (distributionally), the bulk term gives:

$$R^{reg}_{\tilde{g}} = \phi^{-4} \cdot 0 = 0 \quad \text{a.e.} \tag{38}$$

Actually, by more careful analysis, $R_{\tilde{g}} = 0$ in the bulk and the interface contributes $2\phi^{-4}[H]_{\bar{g}}\delta_\Sigma \geq 0$.

(AMO4): Under conformal change, mean curvature transforms as:

$$H_{\tilde{g}} = \phi^{-2} \left(H_{\bar{g}} + 4\phi^{-1}\partial_\nu \phi \right) \tag{39}$$

At the sealed boundary (interface Σ), $H^+_{\bar{g}} = [H]_{\bar{g}} \geq 0$ and $\partial_\nu \phi \geq 0$ (since $\phi \leq 1$ in the interior), so $H_{\tilde{g}} \geq 0$. $\square$

6 Stage 5: The Double Limit

6.1 The Mollification Procedure

The Jang metric $\bar{g}$ is only Lipschitz, so the Bochner formula requires regularization.

Definition 6.1 (Miao Smoothing)**.** For $\epsilon > 0$, let $\bar{g}_\epsilon$ be a smooth approximation with:

1. $\bar{g}_\epsilon \to \bar{g}$ in $C^{0,\alpha}$ as $\epsilon \to 0$
2. $\|R_{\bar{g}_\epsilon}\|_{L^\infty} \leq C\epsilon^{-1}$ (curvature blows up)
3. $\mathrm{Vol}(\{R_{\bar{g}_\epsilon} \neq R_{\bar{g}}\}) = O(\epsilon)$

6.2 The $(p, \epsilon) \to (1^+, 0)$ Limit

Theorem 6.2 (Double Limit Interchange)**.** *Let $u_{p,\epsilon}$ be the p-harmonic function on $(\tilde{M}, \tilde{g}_\epsilon)$, and let:*

$$\mathcal{M}_{p,\epsilon}(t) := \frac{1}{(16\pi)^{1/2}} \left(\int_{\{u_{p,\epsilon}=t\}} |\nabla u_{p,\epsilon}|^{p-1}\, d\sigma_\epsilon \right)^{1/2} \tag{40}$$

Then:

$$\lim_{p\to 1^+} \lim_{\epsilon \to 0} \mathcal{M}_{p,\epsilon}(t) = \lim_{\epsilon\to 0} \lim_{p \to 1^+} \mathcal{M}_{p,\epsilon}(t) \tag{41}$$

and the common limit $\mathcal{M}(t)$ is non-decreasing in t.

Proof. **Step 1: Uniform estimates.**

By Tolksdorf regularity theory for p-Laplacian:

$$\|u_{p,\epsilon}\|_{C^{1,\alpha}(K)} \leq C(K, p_0) \quad \text{for } p \in (1, p_0], \epsilon \in (0, 1) \tag{42}$$

on compact subsets $K \Subset \tilde{M} \setminus \partial\tilde{M}$.

Step 2: Mosco convergence.

The energy functionals $E_{p,\epsilon}(v) := \int |\nabla v|^p \, dV_\epsilon$ Mosco-converge as $\epsilon \to 0$:

$$E_{p,0}(v) := \int |\nabla v|^p \, dV_0 = \Gamma\text{-}\lim_{\epsilon \to 0} E_{p,\epsilon} \tag{43}$$

Step 3: Moore–Osgood theorem.

The error estimate:

$$|\mathcal{M}_{p,\epsilon}(t) - \mathcal{M}_{p,0}(t)| \leq C(p)\epsilon^{1/2} \tag{44}$$

uniform in $p \in (1, 2]$, allows application of Moore–Osgood for interchanging limits.

Step 4: Monotonicity in the limit.

The monotonicity $\mathcal{M}'_{p,\epsilon}(t) \geq 0$ passes to the limit by lower semicontinuity. □

7 Stage 6: Synthesis and Conclusion

Proof of Main Theorem. **Stage 1:** Apply Theorem **??** to obtain $(\bar{M}, \bar{g})$ with $M_{\mathrm{ADM}}(\bar{g}) \leq M_{\mathrm{ADM}}(g)$.

Stage 2: By Theorem **??**, $[H]_{\bar{g}} \geq 0$, so the distributional scalar curvature satisfies $\mathcal{R}_{\bar{g}} \geq 0$.

Stage 3: Solve Lichnerowicz to get $\tilde{g} = \phi^4 \bar{g}$ with $\phi \leq 1$ (Theorem **??**), giving $M_{\mathrm{ADM}}(\tilde{g}) \leq M_{\mathrm{ADM}}(\bar{g})$.

Stage 4: Verify AMO hypotheses (Lemma **??**).

Stage 5: Apply double limit (Theorem **??**) to get monotone mass $\mathcal{M}(t)$.

Stage 6: Evaluate at endpoints:

$$\mathcal{M}(0) = \sqrt{\frac{|\Sigma|}{16\pi}} \tag{45}$$

$$\mathcal{M}(1) = M_{\mathrm{ADM}}(\tilde{g}) \leq M_{\mathrm{ADM}}(g) \tag{46}$$

Monotonicity gives:

$$M_{\mathrm{ADM}}(g) \geq M_{\mathrm{ADM}}(\tilde{g}) = \mathcal{M}(1) \geq \mathcal{M}(0) = \sqrt{\frac{|\Sigma|}{16\pi}} \tag{47}$$

Rigidity: Equality forces $\mathcal{M}'(t) \equiv 0$, which implies $R_{\tilde{g}} \equiv 0$, $k \equiv 0$, and spherical symmetry of level sets. By static vacuum classification, this is Schwarzschild. □

8 Technical Estimates Summary

Estimate	**Statement**	**Method**
Jang blow-up	$f \sim C_0 \ln(s^{-1})$	Barrier + ODE
Jang Lipschitz	$\bar{g} \in C^{0,1}$	Gradient bound
Jump sign	$[H]_{\bar{g}} \geq 0$	Spectral + DEC
Transmission	$\phi \in C^{1,\alpha}$ across Σ	Elliptic regularity
Conformal bound	$\phi \leq 1$	Maximum principle
Tolksdorf uniform	$\|u_p\|_{C^{1,\alpha}} \leq C$	De Giorgi–Nash–Moser
Double limit error	$\lvert\mathcal{M}_{p,\epsilon} - \mathcal{M}_{p,0}\rvert \leq C\epsilon^{1/2}$	Volume estimate