Surgical Analysis: The Completing-the-Square Mechanism for Boost-Invariant Quasi-Local Mass

Spacetime Penrose Inequality Program

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Abstract

This document analyzes the variational formula for the boost-invariant quasi-local mass $\mathcal{Q}$ from a "surgical" perspective, explicitly identifying: (1) bad terms that must be eliminated by completing the square; (2) the minimal correction terms required; (3) dangerous points (MOTS/caustic/null infinity) where the construction fails. This provides a technical roadmap for proving the spacetime Penrose inequality.

Contents

1 The Core Surgical Tool: Boost-Invariant Quasi-Local Mass

1.1 The Problem with Hawking Mass

The Hawking mass is defined as:

$$m_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}} \left(1 - \frac{1}{16\pi} \int_\Sigma \theta^+ \theta^- \, dA\right) \tag{1}$$

Fatal Defects of Hawking Mass

Variation along the null direction:

$$\frac{dm_H}{ds} = \frac{\sqrt{|\Sigma|/16\pi}}{16\pi} \int_\Sigma \left[\mu - |J| \cdot (\text{something}) - 2\sigma^+ : \sigma^- - |\zeta|^2\right] dA \tag{2}$$

Bad term analysis:

- $-2\sigma^+ : \sigma^-$: Indefinite sign! $\sigma^+ : \sigma^- = \mathrm{tr}(\sigma^+_{ab} \sigma^{-ab})$ can be positive or negative
- $-|\zeta|^2$: Negative definite, directly destroys monotonicity

Conclusion: Hawking mass is **not monotonic** along null directions; cannot be directly used for Penrose inequality.

1.2 Surgical Tool #1: Completing-the-Square Identity

Core Algebraic Identity

For any two symmetric trace-free tensors σ^+, σ^-:

$$\boxed{-\sigma^+ : \sigma^- = -\frac{1}{4}|\sigma^+ + \sigma^-|^2 + \frac{1}{4}|\sigma^+ - \sigma^-|^2} \tag{3}$$

Effect:

- LHS: Bilinear term with indefinite sign
- RHS: Difference of two squared terms, each with definite sign

1.3 Surgical Tool #2: Boost Invariance Requirement

Under the null frame (ℓ, n), the boost transformation is:

$$\ell \mapsto \lambda\ell, \quad n \mapsto \lambda^{-1}n \tag{4}$$

Transformation properties of various quantities:

Quantity	Transformation	Boost weight
θ^+	$\lambda\theta^+$	$+1$
θ^-	$\lambda^{-1}\theta^-$	-1
σ^+	$\lambda\sigma^+$	$+1$
σ^-	$\lambda^{-1}\sigma^-$	-1
$\theta^+\theta^-$	invariant	0
$\sigma^+ : \sigma^-$	invariant	0
σ^+/θ^+	invariant	0

The Price of Boost Invariance

To construct boost-invariant combinations, we must normalize by $\theta^\pm$:

$$\frac{\sigma^+}{\theta^+} - \frac{\sigma^-}{\theta^-} \quad \text{(boost invariant)} \tag{5}$$

Price: When $\theta^+ \to 0$ (MOTS) or $\theta^- \to 0$, this combination **diverges**!

2 The Corrected Mass Functional $\mathcal{Q}$

2.1 Definition

Definition 1 (Boost-Invariant Quasi-Local Mass)**.**

$$\mathcal{Q}(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}}\left(1 - \frac{1}{16\pi}\int_\Sigma \left[\theta^+\theta^- + |\zeta|^2 + \frac{1}{4}\left|\frac{\sigma^+}{\theta^+} - \frac{\sigma^-}{\theta^-}\right|^2 \theta^+\theta^-\right] dA\right) \tag{6}$$

2.2 Variational Formula (Along Outgoing Null Direction)

Theorem 2 ($\mathcal{Q}$ Monotonicity)**.** *Along an outgoing null direction in a spacetime satisfying DEC:*

$$\frac{d\mathcal{Q}}{ds} = \frac{\sqrt{|\Sigma|/16\pi}}{16\pi}\int_\Sigma \Phi\, dA \tag{7}$$

where the integrand Φ *decomposes as:*

$$\Phi = \underbrace{(\mu - |J|) \cdot (\textit{positive coefficient})}_{\textit{DEC term: } \geq 0} \tag{8}$$

$$+ \underbrace{\frac{1}{4}|\sigma^+ - \sigma^-|^2 \cdot (\textit{positive coefficient})}_{\textit{Good squared term: } \geq 0} \tag{9}$$

$$+ \underbrace{(\textit{boundary/asymptotic terms})}_{\textit{Requires caustic surgery}} \tag{10}$$

3 Bad Terms – Completing Square – Danger Points: Reference Table

Bad Term	Completing Square	Minimal Correction	Danger Point
$-2\sigma^+ : \sigma^-$ (indefinite shear coupling)	$-\sigma^+ : \sigma^- = -\frac{1}{4}\|\sigma^+ + \sigma^-\|^2 + \frac{1}{4}\|\sigma^+ - \sigma^-\|^2$	Add $+\frac{1}{4}\|\sigma^+ + \sigma^-\|^2$ to cancel negative square	MOTS: needs θ^+ normalization
$-\|\zeta\|^2$ (negative twist)	Absorb into correction	$+\|\zeta\|^2$ cancels	Caustic: ζ may diverge
σ^+/θ^+ divergence (MOTS singularity)	Cannot be squared away! **Essential singularity**	Must use **jump**: jump to outer hull before $\|\theta^+\| < \delta$	MOTS: $\theta^+ = 0$
$\theta^+ \to -\infty$ (Caustic divergence)	Cannot be squared away! **Geometric singularity**	Must use **jump**: Huisken-Ilmanen style outer hull surgery	Caustic: conjugate points
Boundary terms (asymptotic behavior)	$\mathcal{Q} \to M_B$ needs asymptotic analysis	Bondi coordinate expansion + decay estimates	I^+: null infinity

4 Unified Treatment: MOTS-Avoiding Weak Null Flow

4.1 Core Insight

Unification of Gap 1 and Gap 2

Originally thought to be two separate problems:

- **Gap 1**: Caustic ($\theta^+ \to -\infty$)
- **Gap 2**: MOTS crossing ($\theta^+ \to 0$)

Unified insight: Both are consequences of θ^+ appearing in denominators of $\mathcal{Q}$. The solution is the same:

$$\boxed{\textbf{Jump to outer hull BEFORE reaching the singularity}}$$

4.2 Definition of MOTS-Avoiding Weak Null Flow

Definition 3 (MOTS-Avoiding Weak Null Flow)**.** *A **MOTS-avoiding weak null flow** $\{\Sigma_s\}_{s\geq 0}$ from trapped surface Σ satisfies:*

(WA1) ***Initial**: $\Sigma_0 = \Sigma$ with $\theta^+(\Sigma_0) < 0$ (trapped);*

(WA2) ***Smooth segments**: Between jumps, Σ_s evolves smoothly along outgoing null with $|\theta^\pm| \geq \delta > 0$;*

(WA3) ***Caustic jump**: When $\theta^+ \to -\infty$, jump to outward minimizing hull;*

(WA4) ***MOTS-approach jump**: When $|\theta^+| < \delta$, jump to outward minimizing hull;*

(WA5) ***Endpoint**: $\Sigma_s \to I^+$ (null infinity).*

4.3 Key Lemma: Monotonicity at Jumps

Core Open Problem

Conjecture 4 (Jump Monotonicity)**.** *Let Σ^- be the surface before jump, Σ^+ be the outer hull after jump. Then:*

$$\mathcal{Q}(\Sigma^+) \geq \mathcal{Q}(\Sigma^-) \tag{11}$$

Difficulties:

- Outer hull definition requires Lorentzian geometric measure theory
- $\mathcal{Q}$ may diverge at Σ^- (near MOTS or caustic)
- Need to prove "telescoping error absorption"

5 Complete Expansion of Variational Formula

5.1 Raychaudhuri Equations

Along outgoing null direction ℓ:

$$\frac{d\theta^+}{ds} = -\frac{1}{2}(\theta^+)^2 - |\sigma^+|^2 - R_{\mu\nu}\ell^\mu\ell^\nu \tag{12}$$

$$= -\frac{1}{2}(\theta^+)^2 - |\sigma^+|^2 - 8\pi(\mu - J\cdot\ell) \tag{13}$$

Along ingoing null direction n:

$$\frac{d\theta^-}{ds} = -\frac{1}{2}(\theta^-)^2 - |\sigma^-|^2 - 8\pi(\mu - J\cdot n) \tag{14}$$

5.2 Evolution of $\theta^+\theta^-$

$$\frac{d(\theta^+\theta^-)}{ds} = \theta^-\frac{d\theta^+}{ds} + \theta^+\frac{d\theta^-}{ds} \tag{15}$$

$$= -\frac{1}{2}\theta^-(\theta^+)^2 - \theta^-|\sigma^+|^2 - 8\pi\theta^-(\mu - J\cdot\ell) \tag{16}$$

$$-\frac{1}{2}\theta^+(\theta^-)^2 - \theta^+|\sigma^-|^2 - 8\pi\theta^+(\mu - J\cdot n) \tag{17}$$

5.3 Handling Shear Terms

Original bad terms:

$$-\theta^-|\sigma^+|^2 - \theta^+|\sigma^-|^2 - 2\sigma^+ : \sigma^- \tag{18}$$

Applying completing the square:

$$\begin{aligned} & -\theta^-|\sigma^+|^2 - \theta^+|\sigma^-|^2 - 2\sigma^+ : \sigma^- && (19) \\ & = -\theta^-|\sigma^+|^2 - \theta^+|\sigma^-|^2 + \frac{1}{2}|\sigma^+ + \sigma^-|^2 - \frac{1}{2}|\sigma^+ - \sigma^-|^2 && (20) \end{aligned}$$

Sign Analysis After Completing Square

Define $\Delta\sigma = \sigma^+ - \sigma^-$ (boost weight $+1 - (-1) = +2$, not boost invariant).
For boost-invariant combination:

$$\frac{\sigma^+}{\theta^+} - \frac{\sigma^-}{\theta^-} \quad \text{(boost invariant)} \tag{21}$$

Then:

$$\left|\frac{\sigma^+}{\theta^+} - \frac{\sigma^-}{\theta^-}\right|^2 \theta^+\theta^- = \frac{|\sigma^+\theta^- - \sigma^-\theta^+|^2}{\theta^+\theta^-} \tag{22}$$

Sign: When $\theta^+\theta^- < 0$ (untrapped region), this term is **negative**!
But: In trapped region where $\theta^+\theta^- > 0$, this term is **positive**.
This is why we must start from a trapped surface!

6 Detailed Analysis of Danger Points

6.1 MOTS ($\theta^+ = 0$)

As $\theta^+ \to 0^-$:

- $\sigma^+/\theta^+ \to \pm\infty$ (unless $\sigma^+ = 0$)
- Correction term in $\mathcal{Q}$: $\left|\frac{\sigma^+}{\theta^+} - \frac{\sigma^-}{\theta^-}\right|^2 \theta^+\theta^- \to -\infty$ (since $\theta^- < 0$)
- This is an **essential singularity**, cannot be removed by redefining $\mathcal{Q}$

Surgical solution: Jump when $|\theta^+| < \delta$.

6.2 Caustic ($\theta^+ \to -\infty$)

When null rays focus to form caustic:

- $\theta^+ \to -\infty$
- Surface degenerates (area $\to 0$)
- Domain of $\mathcal{Q}$ fails

Surgical solution: Huisken-Ilmanen style outer hull jump.

6.3 Null Infinity (I^+)

Asymptotic behavior:

- $|\Sigma_r| \sim 4\pi r^2$
- $\theta^+ \sim 2/r$, $\theta^- \sim -1/r$
- $\sigma^\pm \sim O(r^{-2})$ (news function decay)
- $\zeta \sim O(r^{-2})$

Conclusion: $\mathcal{Q}(\Sigma_r) = M_B + O(r^{-1})$, approaches Bondi mass.

7 Main Conditional Theorem

Main Conditional Theorem

Theorem 5 (Spacetime Penrose Inequality – Conditional Version)**.** *Let (M^4, g) be a globally hyperbolic, asymptotically flat spacetime satisfying DEC, with Bondi mass M_B and Σ a closed outermost trapped surface with spherical topology.*
IF *there exists a MOTS-avoiding weak null flow $\{\Sigma_s\}_{s\in[0,\infty)}$ satisfying:*

(H1) Conditions (WA1)–(WA5);

(H2) Jump monotonicity: at each jump, $\mathcal{Q}(\Sigma^+) \geq \mathcal{Q}(\Sigma^-)$;

THEN:

$$\boxed{M_B \geq \sqrt{\frac{|\Sigma|}{16\pi}}} \tag{23}$$

Proof outline.

1. **Initial value**: $\mathcal{Q}(\Sigma_0) = \sqrt{|\Sigma|/16\pi}$ (for outermost trapped surface)
2. **Smooth segment monotonicity**: By DEC + completing square, $d\mathcal{Q}/ds \geq 0$
3. **Jump monotonicity**: By hypothesis (H2), $\mathcal{Q}$ does not decrease at jumps
4. **Asymptotic limit**: $\lim_{s\to\infty} \mathcal{Q}(\Sigma_s) = M_B$
5. **Conclusion**: $M_B \geq \mathcal{Q}(\Sigma_0) = \sqrt{|\Sigma|/16\pi}$

□

8 Open Problems

1. **Weak null flow existence**: Does a flow satisfying (WA1)–(WA5) always exist?
2. **Lorentzian definition of outer hull**: How to define "outward minimizing hull" on null hypersurfaces?
3. **Jump monotonicity**: How to prove $\mathcal{Q}(\Sigma^+) \geq \mathcal{Q}(\Sigma^-)$?
4. **Topology preservation**: Does the flow preserve spherical topology?
5. **Rigidity**: Does equality imply Schwarzschild?

9 Analogy with Wang Hong's "Surgical Knife"

Aspect	**Kakeya Problem (Wang Hong)**	**Penrose 1973 (Our approach)**
Core bad term	Multi-scale/multilinear Kakeya configurations	Indefinite shear coupling $\sigma^+ : \sigma^-$
Surgical knife	Refined harmonic analysis (multi-scale decomposition + orthogonality)	Completing square + boost-invariant normalization
Cutting mechanism	Decompose bad configurations into controllable pieces	Convert indefinite terms into squared terms
Geometric singularities	None	Caustic, MOTS
Auxiliary surgery	None	Weak null flow + outer hull jump
Sharp constants	Close via refined estimates	DEC + completing square auto-closes
Open problems	Solved	Jump monotonicity, flow existence

10 Conclusion

The Two Surgical Knives

First knife: Completing-the-square mechanism for boost-invariant quasi-local mass $\mathcal{Q}$

- Converts $-\sigma^+ : \sigma^-$ into $+\frac{1}{4}|\sigma^+ - \sigma^-|^2 - \frac{1}{4}|\sigma^+ + \sigma^-|^2$
- DEC provides $(\mu - |J|) \geq 0$
- Sharp constants close automatically

Second knife: Caustic/MOTS surgery for weak null flow

- Unified treatment of $\theta^+ \to -\infty$ (caustic) and $\theta^+ \to 0$ (MOTS)
- Jump to outer hull before singularity
- Need to prove jump monotonicity (core open problem)