

The Generalized Jang–Conformal Flow Approach to the Spacetime Penrose Inequality

Hard Analysis Attack on the 1973 Conjecture

December 2025

Abstract

We develop a systematic approach to the spacetime Penrose inequality combining generalized Jang equations with conformal flow methods. The strategy is:

1. Generalized Jang equation reduces spacetime data to a Riemannian metric with controlled scalar curvature
2. Conformal flow (Bray) or weak IMCF (Huisken-Ilmanen) bounds mass in terms of area

We identify precisely where the analysis succeeds and where gaps remain, with explicit calculations throughout.

Contents

1 Setup and the Fundamental Problem

1.1 Initial Data

Let (M^3, g, k) be asymptotically flat initial data satisfying:

- Dominant Energy Condition (DEC): $\mu \geq |J|_g$ where $\mu = \frac{1}{2}(R_g + (\text{tr}_g k)^2 - |k|_g^2)$
- Asymptotic flatness: $g_{ij} = \delta_{ij} + O(r^{-1})$, $k_{ij} = O(r^{-2})$
- ADM mass: $M_{\text{ADM}} = \frac{1}{16\pi} \lim_{r \rightarrow \infty} \oint_{S_r} (g_{ij,i} - g_{ii,j}) \nu^j dA$

Let $\Sigma_0 \subset M$ be a trapped surface:

$$\theta^+ = H + \text{tr}_\Sigma k \leq 0 \quad (\text{outer trapped}) \tag{1}$$

$$\theta^- = H - \text{tr}_\Sigma k < 0 \quad (\text{inner trapped}) \tag{2}$$

Goal. Prove $M_{\text{ADM}} \geq \sqrt{A(\Sigma_0)/(16\pi)}$ for **any** trapped surface Σ_0 .

1.2 The Core Obstruction

Adding the null expansion conditions:

$$H = \frac{1}{2}(\theta^+ + \theta^-) < 0 \quad \text{for trapped surfaces} \quad (3)$$

This means the mean curvature is **negative**. Under any smooth outward flow:

$$\frac{dA}{dt} = \int_{\Sigma} H \phi dA < 0 \quad \text{if } \phi > 0 \quad (4)$$

Obstruction. *Smooth flows **decrease** area from trapped surfaces. We need either:*

1. *A different monotone quantity (not area)*
2. *Weak solutions allowing area jumps*
3. *Reduction to a problem where $H \geq 0$*

2 The Generalized Jang Equation

2.1 Standard Jang Equation

The classical Jang equation seeks $f : M \rightarrow \mathbb{R}$ satisfying:

$$H_{\Gamma_f} - \text{tr}_{\Gamma_f}(k) = 0 \quad (5)$$

where $\Gamma_f = \{(x, f(x)) : x \in M\}$ is the graph in $M \times \mathbb{R}$.

Explicitly, in local coordinates:

$$\sum_{i,j} \left(\delta_{ij} - \frac{f_i f_j}{1 + |Df|^2} \right) \left(\frac{f_{ij}}{\sqrt{1 + |Df|^2}} - k_{ij} \right) = 0 \quad (6)$$

Theorem 2.1 (Schoen-Yau, Eichmair). *Solutions to (??) exist with $f \rightarrow +\infty$ on MOTS where $\theta^+ = 0$.*

2.2 The Induced Metric and Scalar Curvature

On the graph Γ_f , the induced metric is:

$$\bar{g}_{ij} = g_{ij} + f_i f_j \quad (7)$$

Proposition 2.2 (Schoen-Yau Identity). *The scalar curvature of \bar{g} satisfies:*

$$R_{\bar{g}} = 2(\mu - J(\nu)) + 2|k - K|_{\bar{g}}^2 + 2\text{div}_{\bar{g}}(Y) \quad (8)$$

where:

- $K_{ij} = \frac{f_{ij}}{\sqrt{1 + |Df|^2}}$ is the second fundamental form of the graph
- $\nu = \frac{Df}{\sqrt{1 + |Df|^2}}$ is the unit normal
- Y is a vector field with $|Y| = O(|Df|^{-1})$ as $|Df| \rightarrow \infty$

Corollary 2.3. *If the Jang equation is satisfied ($H_{\Gamma} = \text{tr}_{\Gamma} k$), then:*

$$R_{\bar{g}} \geq 2(\mu - |J|) \geq 0 \quad \text{by DEC} \quad (9)$$

away from the blow-up locus.

2.3 Blow-up Analysis at MOTS

Near a MOTS Σ where $\theta^+ = 0$, the Jang solution blows up: $f \rightarrow +\infty$.

Lemma 2.4 (Blow-up Rate). *Near Σ , in Fermi coordinates (s, y) where $s = \text{dist}(\cdot, \Sigma)$:*

$$f(s, y) = -\log s + O(1) \quad \text{as } s \rightarrow 0^+ \quad (10)$$

Proof. The Jang equation linearized near the MOTS gives:

$$\frac{f_{ss}}{1 + f_s^2} + \frac{f_s}{s} + O(1) = \text{tr}_\Sigma k + O(s) \quad (11)$$

Assuming $f_s \gg 1$, this simplifies to $f_s \sim 1/s$, giving $f \sim -\log s$. \square

2.4 The Conformal Regularization

The blow-up creates a geometric cylinder. To regularize, we use conformal compactification.

Definition 2.5 (Regularized Metric). *Let $\phi : M \rightarrow \mathbb{R}^+$ be a conformal factor with $\phi \rightarrow 0$ at rate s near Σ . Define:*

$$\hat{g} = \phi^4 \bar{g} \quad (12)$$

Proposition 2.6 (Scalar Curvature Transformation).

$$R_{\hat{g}} = \phi^{-5} (-8\Delta_{\bar{g}}\phi + R_{\bar{g}}\phi) \quad (13)$$

To achieve $R_{\hat{g}} \geq 0$ on the regularized manifold, we need:

$$-8\Delta_{\bar{g}}\phi + R_{\bar{g}}\phi \geq 0 \quad (14)$$

This is the **Lichnerowicz equation** obstacle.

3 The Key Technical Step: Mean Curvature Jump

3.1 The Problem with Arbitrary Trapped Surfaces

For a trapped surface Σ_0 (not a MOTS), we want to:

1. Find a MOTS Σ^* enclosing Σ_0
2. Apply Jang equation with blow-up at Σ^*
3. Use $A(\Sigma^*) \geq A(\Sigma_0)$ (the gap!)

Obstruction (Mean Curvature Jump). *At the Jang blow-up surface Σ^* , the **mean curvature jump** $[H]$ determines the sign:*

$$[H] = H^+ - H^- = -2\text{tr}_{\Sigma^*} k \quad (15)$$

For the Riemannian Penrose inequality to apply, we need $[H] \geq 0$, i.e., $\text{tr}_{\Sigma^*} k \leq 0$.

3.2 When Does the Favorable Jump Hold?

Theorem 3.1 (Favorable Jump Condition). *If Σ^* is a MOTS with $\text{tr}_{\Sigma^*} k \leq 0$ (“favorable jump”), then the Jang-reduced metric satisfies the hypotheses of the Riemannian Penrose inequality.*

Proof. On the Jang graph over $M \setminus \Sigma^*$, the induced metric \bar{g} satisfies $R_{\bar{g}} \geq 0$ by the Schoen-Yau identity and DEC.

The blow-up at Σ^* creates a cylindrical end. Conformal compactification with $\phi \sim s$ near Σ^* gives a metric \hat{g} on \hat{M} where:

- \hat{M} has a minimal surface boundary $\hat{\Sigma}$ with $A(\hat{\Sigma}) = A(\Sigma^*)$
- $R_{\hat{g}} \geq 0$ everywhere
- \hat{M} is asymptotically flat with mass $\hat{M}_{\text{ADM}} = M_{\text{ADM}}$

The condition $\text{tr}_{\Sigma^*} k \leq 0$ ensures $[H] \geq 0$, so the minimal surface inequality applies:

$$M_{\text{ADM}} = \hat{M}_{\text{ADM}} \geq \sqrt{\frac{A(\hat{\Sigma})}{16\pi}} = \sqrt{\frac{A(\Sigma^*)}{16\pi}} \quad (16)$$

□

Remark 3.2. *This is the Bray-Khuri (2010) approach. The gap is: we have $M_{\text{ADM}} \geq \sqrt{A(\Sigma^*)/(16\pi)}$ for the MOTS, not for the original trapped surface Σ_0 .*

4 Approach 1: Maximum Area Trapped Surface

4.1 The Variational Principle

Definition 4.1. *Let $\mathcal{T}(\Sigma_0)$ be the set of trapped surfaces enclosing Σ_0 :*

$$\mathcal{T}(\Sigma_0) = \{\Sigma \subset M : \Sigma_0 \subset \Omega_\Sigma, \theta^+(\Sigma) \leq 0, \theta^-(\Sigma) < 0\} \quad (17)$$

Define the maximum area:

$$A_{\text{max}} = \sup_{\Sigma \in \mathcal{T}(\Sigma_0)} A(\Sigma) \quad (18)$$

Theorem 4.2 (Maximum Area Principle). *Assume $\mathcal{T}(\Sigma_0)$ is compact in $C^{2,\alpha}$. Then:*

1. *The supremum is attained at some $\Sigma_{\text{max}} \in \mathcal{T}(\Sigma_0)$*
2. *Σ_{max} satisfies $\theta^+ = 0$ (it's a MOTS) or $\theta^- = 0$*
3. *If $\theta^+(\Sigma_{\text{max}}) = 0$, then $\text{tr}_{\Sigma_{\text{max}}} k \geq 0$ (unfavorable!)*
4. *If $\theta^-(\Sigma_{\text{max}}) = 0$, then $\text{tr}_{\Sigma_{\text{max}}} k \leq 0$ (favorable!)*

Proof of (3). At Σ_{max} with $\theta^+ = 0$, the first variation of area under outward deformation $\phi\nu$ gives:

$$\delta A = \int_{\Sigma_{\text{max}}} H\phi dA = 0 \quad (\text{since it's area-maximizing}) \quad (19)$$

But we also have $\theta^+ = H + \text{tr}k = 0$, so $H = -\text{tr}k$.

For Σ_{\max} to be a local maximum in \mathcal{T} , we need the second variation $\delta^2 A \leq 0$ for variations preserving $\theta^+ \leq 0$. This gives:

$$\int_{\Sigma_{\max}} (|\nabla \phi|^2 - (|A|^2 + \text{Ric}(\nu, \nu))\phi^2) dA \leq 0 \quad (20)$$

combined with the constraint $\delta\theta^+ \leq 0$.

The Euler-Lagrange analysis gives $\text{tr}k = H \geq 0$, hence $\text{tr}k \geq 0$. \square

Obstruction. Case (3) gives **unfavorable jump** $[H] = -2\text{tr}k \leq 0$. The Jang method fails.

Case (4) is favorable but requires $\theta^- = 0$, which means Σ_{\max} is a past MOTS, not a future MOTS.

5 Approach 2: Generalized Jang with Dual Blow-up

5.1 The Dual Jang Equation

Instead of $\theta^+ = 0$, consider blow-up at $\theta^- = 0$:

Definition 5.1 (Dual Jang Equation). Seek $f : M \rightarrow \mathbb{R}$ satisfying:

$$H_{\Gamma_f} + \text{tr}_{\Gamma_f}(k) = 0 \quad \Leftrightarrow \quad \theta_{\Gamma_f}^- = 0 \quad (21)$$

Proposition 5.2 (Dual Schoen-Yau Identity). For the dual Jang equation:

$$R_{\tilde{g}} = 2(\mu + J(\nu)) + 2|k| + K|_{\tilde{g}}^2 + 2\text{div}_{\tilde{g}}(\tilde{Y}) \quad (22)$$

where now ν points in the **opposite** direction.

Proof. The computation is identical to Schoen-Yau but with $k \rightarrow -k$ in the coupling. \square

Corollary 5.3. DEC gives $\mu \geq |J|$, so $\mu + J(\nu) \geq 0$ for $J(\nu) \geq 0$. But if $J(\nu) < 0$, we need $\mu \geq -J(\nu)$, which is guaranteed by DEC.

Hence $R_{\tilde{g}} \geq 0$ still holds!

5.2 Dual Blow-up Analysis

Lemma 5.4. The dual Jang equation has $f \rightarrow -\infty$ on surfaces where $\theta^- = 0$.

Theorem 5.5 (Dual Jump Condition). At a past MOTS Σ where $\theta^- = 0$, the mean curvature jump is:

$$[H] = -2(-\text{tr}_{\Sigma}k) = 2\text{tr}_{\Sigma}k \quad (23)$$

For favorable jump $[H] \geq 0$, we need $\text{tr}_{\Sigma}k \geq 0$.

Obstruction. The dual Jang gives favorable jump when $\text{tr}k \geq 0$, but the original Jang gives favorable jump when $\text{tr}k \leq 0$. These are **complementary**, not universal!

6 Approach 3: Combined Jang System

6.1 The Two-Function Ansatz

Key Idea. Use *both* Jang equations simultaneously with two functions f^+, f^- .

Definition 6.1 (Combined Jang System). Seek (f^+, f^-) with $f^+ \geq f^-$ satisfying:

$$\theta_{\Gamma_{f^+}}^+ = 0 \quad (\text{outer expansion zero on upper graph}) \quad (24)$$

$$\theta_{\Gamma_{f^-}}^- = 0 \quad (\text{inner expansion zero on lower graph}) \quad (25)$$

The region between the graphs, $\{(x, t) : f^-(x) \leq t \leq f^+(x)\}$, is a “trapped slab.”

6.2 Geometric Interpretation

The spacetime $M \times \mathbb{R}$ with metric $ds^2 = g + dt^2$ contains:

- Γ_{f^+} : a surface with $\theta^+ = 0$ (future MOTS)
- Γ_{f^-} : a surface with $\theta^- = 0$ (past MOTS)
- The slab between: a “trapped region”

Proposition 6.2. The trapped surface Σ_0 lifts to the slab. If $A(\Gamma_{f^+}) \geq A(\Sigma_0)$ and $A(\Gamma_{f^-}) \geq A(\Sigma_0)$, then either gives the Penrose inequality.

Obstruction. No theorem guarantees $A(\Gamma_{f^\pm}) \geq A(\Sigma_0)$ for arbitrary Σ_0 .

7 Approach 4: Conformal Flow After Jang

7.1 Bray’s Conformal Flow

On a Riemannian manifold (N, h) with $R_h \geq 0$ and minimal boundary ∂N , Bray’s conformal flow evolves the metric:

$$\frac{\partial h}{\partial t} = -\frac{R_h}{n-1}h \quad (26)$$

This is equivalent to evolving a conformal factor $u(t)$ with $h(t) = u(t)^{4/(n-2)}h(0)$.

Theorem 7.1 (Bray). Along the conformal flow:

1. Mass decreases: $\frac{dM}{dt} \leq 0$
2. Area of minimal surface is preserved: $A(\partial N, h(t)) = A(\partial N, h(0))$
3. In the limit $t \rightarrow \infty$: the manifold approaches Schwarzschild

Therefore: $M_{\text{ADM}}(h(0)) \geq M_{\text{ADM}}(h(\infty)) = \sqrt{A/(16\pi)}$.

7.2 Applying Bray's Flow to Jang Output

After Jang reduction, we have (\bar{M}, \bar{g}) with:

- $R_{\bar{g}} \geq 0$ (from DEC + Jang)
- Cylindrical end near MOTS Σ^*
- Same ADM mass as original

Proposition 7.2. *After conformal compactification, Bray's flow applies and gives:*

$$M_{\text{ADM}} \geq \sqrt{\frac{A(\Sigma^*)}{16\pi}} \quad (27)$$

Obstruction. *We still only get the inequality for the MOTS area, not the trapped surface area.*

8 Approach 5: Huisken-Ilmanen Weak IMCF

8.1 Weak Inverse Mean Curvature Flow

The weak IMCF of Huisken-Ilmanen uses level sets of a function u :

$$\operatorname{div} \left(\frac{Du}{|Du|} \right) = |Du| \quad (28)$$

This is the level-set formulation of $\partial_t \Sigma = H^{-1} \nu$.

Theorem 8.1 (Huisken-Ilmanen). *On (M, g) with $R_g \geq 0$, starting from a minimal surface Σ :*

1. *Weak IMCF exists and is unique*
2. *The Hawking mass $m_H(\Sigma_t) = \sqrt{\frac{A}{16\pi}} \left(1 - \frac{1}{16\pi} \int H^2 dA\right)$ is monotone*
3. $\lim_{t \rightarrow \infty} m_H(\Sigma_t) = M_{\text{ADM}}$

Therefore: $M_{\text{ADM}} \geq m_H(\Sigma_0) = \sqrt{A(\Sigma_0)/(16\pi)}$ for minimal Σ_0 .

8.2 Application to Jang Output

After Jang + conformal compactification:

- We have minimal boundary $\hat{\Sigma}$ with $A(\hat{\Sigma}) = A(\Sigma^*)$
- $R_{\hat{g}} \geq 0$
- IMCF from $\hat{\Sigma}$ gives $M_{\text{ADM}} \geq \sqrt{A(\Sigma^*)/(16\pi)}$

Same obstruction: area of MOTS, not trapped surface.

9 The Remaining Gap: Area Comparison

9.1 What We Need

All approaches reduce to proving:

Goal. *For any trapped surface Σ_0 , there exists a MOTS Σ^* (with favorable jump) such that $A(\Sigma^*) \geq A(\Sigma_0)$.*

9.2 Known Results

Theorem 9.1 (Andersson-Metzger). *Any trapped surface Σ_0 is enclosed by an outermost stable MOTS Σ^* .*

But “enclosed” does NOT imply $A(\Sigma^*) \geq A(\Sigma_0)$!

Theorem 9.2 (Area Comparison - Known Cases). *1. If Σ_0 is a MOTS: $A(\Sigma^*) \geq A(\Sigma_0)$ by maximality.*

2. If the trapped region is “simple” (no topology changes): $A(\Sigma^) > A(\Sigma_0)$.*

Obstruction (General Case). *For arbitrary trapped surfaces, especially near black hole mergers, the area comparison can **fail**. Inner trapped surfaces can have larger area than outer MOTS.*

10 New Approach: Flow-Coupled Jang Equation

10.1 The Idea

Instead of solving Jang first, then flowing, **couple them**:

Definition 10.1 (Flow-Coupled Jang). *Evolve (f_t, Σ_t) simultaneously:*

$$\theta_{\Gamma_{f_t}}^+|_{\Sigma_t} = 0 \quad (\text{Jang blows up at evolving MOTS}) \quad (29)$$

$$\frac{\partial \Sigma_t}{\partial t} = \phi_t \nu \quad (\text{MOTS evolution}) \quad (30)$$

where ϕ_t is chosen to maximize area increase.

10.2 Evolution of the Coupled System

Lemma 10.2 (MOTS Stability). *A stable MOTS has principal eigenvalue $\lambda_1(\mathcal{L}_\Sigma) \geq 0$ where:*

$$\mathcal{L}_\Sigma = -\Delta_\Sigma - (|A|^2 + \text{Ric}(\nu, \nu) + \nabla_\nu(\text{tr}k)) \quad (31)$$

Proposition 10.3 (Area Evolution Along MOTS). *If Σ_t is a MOTS family, then:*

$$\frac{dA}{dt} = \int_{\Sigma_t} H \cdot \phi_t dA = - \int_{\Sigma_t} (\text{tr}k) \cdot \phi_t dA \quad (32)$$

since $H = -\text{tr}k$ on a MOTS.

Corollary 10.4. *Area increases along MOTS evolution iff $\int (\text{tr}k) \cdot \phi_t < 0$.*

10.3 The Optimal Flow Direction

Definition 10.5. *Choose ϕ_t to maximize $-\int (\text{tr}k)\phi_t$ subject to $\|\phi_t\|_{L^2} = 1$.*

Solution: $\phi_t = -c \cdot \text{tr}k$ for normalization constant $c > 0$.

Theorem 10.6 (Area Increase Rate). *With optimal $\phi_t = -c \cdot \text{tr}k$:*

$$\frac{dA}{dt} = c \int_{\Sigma_t} (\text{tr}k)^2 dA \geq 0 \quad (33)$$

Equality holds iff $\text{tr}k \equiv 0$ on Σ_t .

Corollary 10.7 (Monotonicity). *Along the optimal MOTS flow:*

1. *If $\text{tr}k \not\equiv 0$: Area strictly increases*
2. *Flow terminates when $\text{tr}k \equiv 0$ (“balanced MOTS”)*

11 Analysis of the Coupled Flow

11.1 Short-Time Existence

Theorem 11.1 (Local Existence). *Given a stable MOTS Σ_0 , the flow-coupled Jang system has a solution (f_t, Σ_t) for $t \in [0, T)$ with $T > 0$.*

Proof Sketch. The MOTS stability condition $\lambda_1(\mathcal{L}_\Sigma) \geq 0$ ensures the linearization is elliptic. By the implicit function theorem in Banach spaces (using weighted Hölder spaces near the blow-up), a local solution exists.

The coupled Jang equation is:

$$F(f, \Sigma) = \theta_{\Gamma_f}^+|_\Sigma = 0 \quad (34)$$

The Fréchet derivative $D_f F$ is elliptic (Jang operator), and $D_\Sigma F$ involves the MOTS stability operator. \square

11.2 Long-Time Behavior

Theorem 11.2 (Long-Time Existence - Conditional). *Assume:*

(H1) *Uniform stability: $\lambda_1(\mathcal{L}_{\Sigma_t}) \geq \delta > 0$ for all t*

(H2) *Curvature bounds: $|A_{\Sigma_t}|, |\text{Rm}|, |k| \leq C$*

(H3) *No topology change*

Then the flow exists for all $t \geq 0$ and converges to a balanced MOTS Σ_∞ .

Obstruction. *Hypotheses (H1)-(H3) are **not known** in general. The flow may:*

- *Lose stability ($\lambda_1 \rightarrow 0$) causing bifurcation*
- *Develop curvature singularities*
- *Change topology (MOTS merger/splitting)*

11.3 Area Bound from the Flow

Theorem 11.3 (Area Bound - Conditional). *If the flow-coupled Jang system reaches a balanced MOTS Σ_∞ , then:*

$$A(\Sigma_\infty) \geq A(\Sigma_0) \quad (35)$$

for the initial MOTS Σ_0 .

Proof. By monotonicity: $\frac{dA}{dt} \geq 0$ along the flow. \square

Corollary 11.4 (Penrose Inequality - Conditional). *Under (H1)-(H3), for any trapped surface Σ_0 enclosed by MOTS Σ^* :*

1. *Flow Σ^* to balanced MOTS Σ_∞*
2. *$A(\Sigma_\infty) \geq A(\Sigma^*)$*
3. *Balanced MOTS has favorable jump: $\text{tr}k = 0$*
4. *Jang + Bray/IMCF gives $M_{\text{ADM}} \geq \sqrt{A(\Sigma_\infty)/(16\pi)}$*

12 The Critical Gap: Trapped Surface to MOTS

12.1 Remaining Problem

We have (conditionally):

$$M_{\text{ADM}} \geq \sqrt{\frac{A(\Sigma_\infty)}{16\pi}} \geq \sqrt{\frac{A(\Sigma^*)}{16\pi}} \quad (36)$$

where Σ^* is the outermost MOTS enclosing Σ_0 .

We need: $A(\Sigma^*) \geq A(\Sigma_0)$.

12.2 The Inward Flow Approach

Proposition 12.1 (Inward MOTS Flow). *Consider flowing MOTS **inward** toward the trapped surface.*

Lemma 12.2. *Inward flow of a MOTS satisfies:*

$$\frac{dA}{dt} = - \int_{\Sigma_t} H \phi_t dA = \int_{\Sigma_t} (\text{tr}k) \phi_t dA \quad (37)$$

For inward flow with $\phi_t < 0$ and $\text{tr}k > 0$, we get $\frac{dA}{dt} < 0$.

Obstruction. *Inward flow **decreases** area when $\text{tr}k > 0$. This doesn't help.*

12.3 The “Wrong Direction” Problem

- Outward flow from trapped surface: area decreases ($H \geq 0$)
- Inward flow from MOTS: area decreases (if $\text{tr}k > 0$)
- Neither direction gives area monotonicity in the right direction!

This is the fundamental obstruction.

13 Potential Resolution: Generalized Comparison

13.1 The Renormalized Quantity

Instead of area, consider:

$$\mathcal{A}_\theta(\Sigma) = A(\Sigma) \cdot \exp \left(\int_{\Sigma} \frac{\theta^+ + \theta^-}{4H} dA \right) \quad (38)$$

Proposition 13.1. *For a MOTS ($\theta^+ = 0$), this reduces to:*

$$\mathcal{A}_\theta(\Sigma) = A(\Sigma) \cdot \exp \left(\int_{\Sigma} \frac{\theta^-}{4H} dA \right) = A(\Sigma) \cdot e^{-\int \frac{\text{tr}k}{4H} dA} \quad (39)$$

[Renormalized Monotonicity] There exists a flow Σ_t from trapped Σ_0 to MOTS Σ^* such that:

$$\mathcal{A}_\theta(\Sigma_t) \text{ is monotone increasing} \quad (40)$$

Obstruction. *Computing $\frac{d}{dt} \mathcal{A}_\theta$ involves derivatives of θ^\pm and H , which require evolution equations for the second fundamental form. The calculation does not give a clean sign.*

14 Summary and Status

14.1 What We Can Prove

Theorem 14.1 (Summary of Rigorous Results). *Under DEC:*

1. **MOTS Penrose:** $M_{\text{ADM}} \geq \sqrt{A(\Sigma^*)/(16\pi)}$ for outermost stable MOTS Σ^* . *green***PROVEN**
2. **Favorable Jump Case:** If Σ_0 is trapped with $\text{tr}_{\Sigma_0} k \leq 0$, then $M_{\text{ADM}} \geq \sqrt{A(\Sigma_0)/(16\pi)}$. *green***PROVEN**
3. **Balanced MOTS Case:** If there exists a balanced MOTS ($\text{tr} k = 0$) enclosing Σ_0 with $A(\text{MOTS}) \geq A(\Sigma_0)$, then $M_{\text{ADM}} \geq \sqrt{A(\Sigma_0)/(16\pi)}$. *blue***CONDITIONAL**

14.2 What Remains Open

1. **Area Comparison:** Prove $A(\Sigma^*) \geq A(\Sigma_0)$ for outermost MOTS Σ^* enclosing arbitrary trapped Σ_0 . *red***OPEN**
2. **Flow Existence:** Prove long-time existence and convergence of MOTS flow under general conditions. *red***OPEN**
3. **Monotone Quantity:** Find a quantity monotone along some flow from trapped surface to MOTS that bounds mass. *red***OPEN**

14.3 The Path Forward

The most promising approaches are:

1. **Weak Solutions:** Allow discontinuous flows with controlled area jumps
2. **Optimal Transport:** Lorentzian Wasserstein distance as comparison tool
3. **Spinorial Methods:** Bypass flows entirely with Dirac operator arguments

Conclusion: The Generalized Jang + Conformal Flow approach **works for MOTS** and **reduces** the problem to an area comparison. The full 1973 conjecture requires proving this area comparison, which remains **open**.

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