The Penrose Inequality via Trapping Product

Exploiting $\theta^+\theta^- > 0$

Research Notes

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Abstract

We develop a novel approach to the Penrose inequality based on the **trapping product** $\mathcal{P} = \theta^+\theta^-$, which is strictly positive for trapped surfaces and vanishes exactly on MOTS. Unlike $\mathrm{tr}_\Sigma k = \frac{1}{2}(\theta^+ - \theta^-)$, this quantity has **definite sign**. We construct a modified Jang equation using this product and analyze its properties.

1 The Trapping Product

Key Observation

For a trapped surface Σ with $\theta^+ \leq 0$ and $\theta^- < 0$:

$$\mathcal{P} := \theta^+\theta^- \geq 0 \quad \text{(always non-negative!)} \tag{1}$$

$$H = \frac{1}{2}(\theta^+ + \theta^-) < 0 \quad \text{(definite sign)} \tag{2}$$

$$\mathrm{tr}_\Sigma k = \frac{1}{2}(\theta^+ - \theta^-) \quad \text{(undetermined sign - THE PROBLEM)} \tag{3}$$

Crucially:

- $\mathcal{P} > 0$ for strictly trapped surfaces ($\theta^+ < 0$)
- $\mathcal{P} = 0$ iff $\theta^+ = 0$ (MOTS case)
- $\mathcal{P}$ is a **symmetric** function of θ^+ and θ^-

1.1 Algebraic Identities

Lemma 1.1 (Product-Sum Relations)**.** *For any surface with null expansions $\theta^\pm$:*

$$\mathcal{P} = \theta^+\theta^- = H^2 - (\mathrm{tr}_\Sigma k)^2 \tag{4}$$

$$\theta^\pm = H \pm \mathrm{tr}_\Sigma k \tag{5}$$

Proof. Direct computation:

$$H^2 - (\mathrm{tr}_\Sigma k)^2 = \frac{1}{4}(\theta^+ + \theta^-)^2 - \frac{1}{4}(\theta^+ - \theta^-)^2 \tag{6}$$

$$= \frac{1}{4}\left[(\theta^+)^2 + 2\theta^+\theta^- + (\theta^-)^2 - (\theta^+)^2 + 2\theta^+\theta^- - (\theta^-)^2\right] \tag{7}$$

$$= \theta^+\theta^- = \mathcal{P} \tag{8}$$

□

Corollary 1.2 (Sign of $\mathrm{tr}_\Sigma k$)**.** *For trapped surfaces:*

$$(\mathrm{tr}_\Sigma k)^2 = H^2 - \mathcal{P} \tag{9}$$

Since $H^2 > 0$ and $\mathcal{P} \geq 0$, we have:

$$|\mathrm{tr}_\Sigma k| \leq |H| \tag{10}$$

with equality iff $\mathcal{P} = 0$ (MOTS).

2 The Product-Modified Jang Equation

2.1 Motivation

The standard Jang equation:

$$\mathcal{J}(f) = H_\Gamma + \mathrm{tr}_\Gamma k = \theta^+_\Gamma = 0 \tag{11}$$

produces scalar curvature:

$$R_{\bar{g}} = R^{\mathrm{reg}} + 2(\mathrm{tr}_\Sigma k)\delta_\Sigma \tag{12}$$

The problem: $\mathrm{tr}_\Sigma k$ can be negative.
Idea: Construct an equation using $\mathcal{P}$ instead.

2.2 The Product-Modified Equation

Definition 2.1 (Product-Jang Equation)**.** *Define the* ***product-Jang equation****:*

$$\boxed{\mathcal{J}_\mathcal{P}(f) := \theta^+_\Gamma \cdot \theta^-_\Gamma = \mathcal{P}_\Gamma = 0} \tag{13}$$

where $\theta^\pm_\Gamma$ are the null expansions of the graph Γ_f.

Lemma 2.2 (Equivalent Formulations)**.** *The product-Jang equation is equivalent to:*

$$\mathcal{P}_\Gamma = H^2_\Gamma - (\mathrm{tr}_\Gamma k)^2 = 0 \tag{14}$$

$$|H_\Gamma| = |\mathrm{tr}_\Gamma k| \tag{15}$$

$$\theta^+_\Gamma = 0 \text{ OR } \theta^-_\Gamma = 0 \tag{16}$$

The Key Insight

The equation $\mathcal{P}_\Gamma = 0$ has **two branches**:

1. $\theta^+_\Gamma = 0$: Standard MOTS (Jang equation)
2. $\theta^-_\Gamma = 0$: Past MOTS (dual Jang equation)

For trapped surfaces, both branches are accessible, but the solution naturally chooses the branch with better regularity at each point.

3 Existence Theory for Product-Jang

3.1 The Variational Formulation

The product-Jang equation can be written as a degenerate elliptic equation:

$$(H_\Gamma - \mathrm{tr}_\Gamma k)(H_\Gamma + \mathrm{tr}_\Gamma k) = 0 \tag{17}$$

This is a **fully nonlinear** equation of the form:

$$F[D^2 f, Df, f] = 0 \tag{18}$$

where:

$$F = \left(\frac{\mathrm{div}(\nabla f / W)}{1 + |\nabla f|^2 / W^2} + \frac{k(\nabla f, \nabla f)}{W} + H - \mathrm{tr}k \right) \times (\text{same with } k \to -k) \tag{19}$$

3.2 Viscosity Solution Approach

Definition 3.1 (Viscosity Solution)**.** *A function $f \in C(\bar{M})$ is a **viscosity solution** of $\mathcal{P}_\Gamma = 0$ if:*

1. ***Subsolution:*** *For any smooth ϕ such that $f - \phi$ has a local max at x_0:*

$$\min(\theta^+_{\Gamma_\phi}(x_0), \theta^-_{\Gamma_\phi}(x_0)) \leq 0 \tag{20}$$

2. ***Supersolution:*** *For any smooth ϕ such that $f - \phi$ has a local min at x_0:*

$$\max(\theta^+_{\Gamma_\phi}(x_0), \theta^-_{\Gamma_\phi}(x_0)) \geq 0 \tag{21}$$

Theorem 3.2 (Existence - Conditional)**.** *Let (M^3, g, k) be asymptotically flat with trapped surface Σ_0. Assume:*

- *Σ_0 is strictly trapped: $\theta^+ < 0$, $\theta^- < 0$*
- *The DEC holds: $\mu \geq |J|$*

Then there exists a viscosity solution f to $\mathcal{P}_\Gamma = 0$ with:

1. *$f \to +\infty$ as we approach Σ_0 from the exterior*
2. *$f \to 0$ at spatial infinity*

GAP: Existence Proof

The existence theorem requires developing viscosity solution theory for the product equation. Key challenges:

- The equation is **not** uniformly elliptic
- Branch switching between $\theta^+ = 0$ and $\theta^- = 0$
- Regularity at branch points

Potential approach: Perron's method using sub/supersolutions from the individual Jang equations.

4 Geometric Analysis of Product-Jang

4.1 Scalar Curvature of the Product-Jang Metric

Theorem 4.1 (Scalar Curvature Formula)**.** *Let f be a smooth solution to $\mathcal{P}_\Gamma = 0$ away from Σ_0. The induced metric $\bar{g}$ on the graph satisfies:*

$$R_{\bar{g}} \geq 2(\mu - |J|) - \frac{(\mathcal{P}_\Gamma)'}{W} \geq 0 \tag{22}$$

where the inequality holds because $\mathcal{P}_\Gamma = 0$ on the solution.

Key Calculation

At any point where $\theta_\Gamma^+ = 0$ (solution on standard branch):

$$R_{\bar{g}} = 2\mu + \mathrm{tr}(k)^2 - |k|^2 - 2\langle X, J\rangle \tag{23}$$

$$+ 2|A - K|^2 + 2(\theta_\Gamma^+)(\theta_\Gamma^-) \tag{24}$$

$$= 2\mu + \mathrm{tr}(k)^2 - |k|^2 - 2\langle X, J\rangle + 2|A - K|^2 \tag{25}$$

The last term $2\theta_\Gamma^+\theta_\Gamma^- = 0$ because we're on a branch.
By DEC and Cauchy-Schwarz:

$$R_{\bar{g}} \geq 2(\mu - |J|) \geq 0 \tag{26}$$

The same holds on the $\theta_\Gamma^- = 0$ branch by symmetry.

4.2 Boundary Behavior

Lemma 4.2 (Blow-up at Trapped Surface)**.** *As we approach Σ_0 from the exterior, the solution $f \to +\infty$ with rate:*

$$f(s, y) \sim C(y) \ln(s^{-1}) \tag{27}$$

where $s = dist(\cdot, \Sigma_0)$ and:

$$C(y) = \begin{cases} \frac{|\theta^-|}{2} & \textit{if solution is on } \theta^+ = 0 \textit{ branch} \\ \frac{|\theta^+|}{2} & \textit{if solution is on } \theta^- = 0 \textit{ branch} \end{cases} \tag{28}$$

The Advantage

The product equation allows the solution to **choose the favorable branch** at each point:

- Where $|\theta^+| < |\theta^-|$: Use $\theta^+ = 0$ branch (faster blow-up)
- Where $|\theta^-| < |\theta^+|$: Use $\theta^- = 0$ branch (faster blow-up)

This **optimizes** the blow-up behavior over the surface.

5 The Interface Contribution

5.1 Mean Curvature Jump

The key question: what is the interface term $[H]$ in the scalar curvature?

Proposition 5.1 (Interface Analysis)**.** *At the blow-up surface Σ_0, the mean curvature jump depends on the branch:*

$$[H]_{\theta^+=0} = \mathrm{tr}_\Sigma k \tag{29}$$
$$[H]_{\theta^-=0} = -\mathrm{tr}_\Sigma k \tag{30}$$

THE FUNDAMENTAL ISSUE

Even with the product equation, we cannot escape the sign of $\mathrm{tr}_\Sigma k$!
If the solution could smoothly transition between branches at each point of Σ_0, choosing:

$$\text{Branch} = \begin{cases} \theta^+ = 0 & \text{where } \mathrm{tr}_\Sigma k \geq 0 \\ \theta^- = 0 & \text{where } \mathrm{tr}_\Sigma k < 0 \end{cases} \tag{31}$$

then:

$$[H]_{\text{optimal}} = |\mathrm{tr}_\Sigma k| \geq 0 \tag{32}$$

But: The solution cannot freely switch branches - it must satisfy a global consistency condition.

6 Attempt at Resolution: The $\sqrt{\mathcal{P}}$ Method

6.1 Using the Square Root

Definition 6.1 (Root-Trapping Equation)**.** *Define:*

$$\sqrt{\mathcal{P}_\Gamma} = \sqrt{\theta_\Gamma^+ \cdot \theta_\Gamma^-} \tag{33}$$

and consider the equation:

$$\sqrt{\mathcal{P}_\Gamma} = \epsilon \to 0 \tag{34}$$

as a regularization that approaches MOTS.

This is well-defined for strictly trapped surfaces where $\mathcal{P} > 0$.

Lemma 6.2 (Gradient of $\sqrt{\mathcal{P}}$)**.** *For the trapping product:*

$$\nabla\sqrt{\mathcal{P}} = \frac{\theta^-\nabla\theta^+ + \theta^+\nabla\theta^-}{2\sqrt{\theta^+\theta^-}} \tag{35}$$

6.2 The $\sqrt{\mathcal{P}}$-Flow

Definition 6.3 (Trapping Flow)**.** *Define the flow:*

$$\frac{\partial\Sigma}{\partial t} = -\sqrt{\mathcal{P}} \cdot \nu \tag{36}$$

where ν is the outward normal.

Proposition 6.4 (Flow Properties)**.** *The trapping flow satisfies:*

1. *Stationary points are MOTS ($\theta^+ = 0$) or past-MOTS ($\theta^- = 0$)*

2. For trapped surfaces, $\sqrt{\mathcal{P}} > 0$, so the flow moves inward

*3. The flow is **well-defined** (no sign ambiguity)*

Theorem 6.5 (Area Evolution - Conditional)**.** *Under the trapping flow:*

$$\frac{dA}{dt} = -\int_\Sigma \sqrt{\mathcal{P}} \cdot H\, dA \tag{37}$$

$$= -\int_\Sigma \sqrt{\theta^+\theta^-} \cdot \frac{\theta^+ + \theta^-}{2}\, dA \tag{38}$$

For trapped surfaces ($\theta^+, \theta^- \leq 0$):

$$\frac{dA}{dt} = -\int_\Sigma \sqrt{|\theta^+||\theta^-|} \cdot \frac{|\theta^+| + |\theta^-|}{2}\, dA \leq 0 \tag{39}$$

Area is decreasing!

7 Connection to Penrose Inequality

7.1 The Strategy

1. Start with trapped surface Σ_0
2. Flow by $\sqrt{\mathcal{P}}$ until reaching a MOTS Σ^*
3. Apply known MOTS Penrose inequality to Σ^*
4. Relate $A(\Sigma^*)$ to $A(\Sigma_0)$

Theorem 7.1 (Area Comparison - Conditional)**.** *If the $\sqrt{\mathcal{P}}$-flow exists globally and converges to a smooth MOTS Σ^*:*

$$A(\Sigma^*) \leq A(\Sigma_0) \tag{40}$$

Proof Sketch. By Theorem **??**, area is monotonically decreasing along the flow. At the limit Σ^*, we have $\sqrt{\mathcal{P}} = 0$, so $\theta^+ = 0$ (MOTS). □

Critical Gaps

GAP A: Flow Existence. Long-time existence of the $\sqrt{\mathcal{P}}$-flow is not established. Key issues:

- Short-time existence (parabolic theory)
- Singularity formation
- Convergence to smooth limit

GAP B: Wrong Direction! The area comparison goes the **wrong way**:

$$A(\Sigma^*) \leq A(\Sigma_0) \tag{41}$$

But for Penrose, we need:

$$M_{\text{ADM}} \geq \sqrt{\frac{A(\Sigma_0)}{16\pi}} \geq \sqrt{\frac{A(\Sigma^*)}{16\pi}} \tag{42}$$

The area decreases, so knowing $M \geq \sqrt{A(\Sigma^*)/(16\pi)}$ does NOT imply $M \geq \sqrt{A(\Sigma_0)/(16\pi)}$.

8 Honest Assessment

THE FUNDAMENTAL OBSTRUCTION

After exploring multiple approaches using the trapping product, we find:
The Sign Problem Cannot Be Avoided.

1. **Product-Jang:** The interface term $[H]$ still involves $\mathrm{tr}_\Sigma k$ with undetermined sign.
2. **Branch Switching:** Even if we could switch branches optimally, this requires a discontinuous solution, violating regularity.
3. **$\sqrt{\mathcal{P}}$-Flow:** Area decreases to MOTS, giving the wrong direction for Penrose.
4. **Symmetric Quantities:** While $\mathcal{P} = \theta^+\theta^- > 0$ has definite sign, the **interface contribution** in any geometric construction still depends on $\mathrm{tr}_\Sigma k$.

Root Cause: The Penrose inequality for general trapped surfaces requires proving:

$$(\text{ADM mass}) \geq (\text{function of area}) \tag{43}$$

Any proof via positive mass theorem requires scalar curvature ≥ 0, which for Jang-type constructions requires:

$$R = R^{\text{reg}} + 2(\text{interface term})\delta_\Sigma \geq 0 \tag{44}$$

The interface term is intrinsically related to $\mathrm{tr}_\Sigma k$, whose sign we cannot control for general trapped surfaces.

9 What Would Be Needed

To prove the Penrose inequality for arbitrary trapped surfaces, one would need:

1. **A geometric construction that bypasses the sign issue:**
   - Use a quantity that depends only on $|\mathrm{tr}_\Sigma k|$, not its sign
   - Relate this to mass via a new positive mass argument
2. **A flow that increases area to MOTS:**
   - Current flows (IMCF, $\sqrt{\mathcal{P}}$) all decrease or fix area
   - Need a flow moving **outward** from trapped to MOTS
3. **Cosmic censorship:**
   - Assume trapped surface evolves to event horizon
   - Event horizon has area $\geq$ trapped surface (area theorem)
   - Apply MOTS inequality at late times
4. **Spacetime approach:**
   - Prove directly using null hypersurface techniques
   - Requires Lorentzian positive mass theorem (not available)

10 Conclusion

The trapping product $\mathcal{P} = \theta^+\theta^- > 0$ provides a symmetric, sign-definite quantity for trapped surfaces. However, exploiting this for the Penrose inequality faces fundamental obstacles:

- The interface contribution in Jang-type constructions unavoidably involves $\mathrm{tr}_\Sigma k$
- Natural flows using $\mathcal{P}$ decrease area, giving the wrong direction
- The obstruction appears to be **structural**, not technical

The 1973 Penrose conjecture for general trapped surfaces remains genuinely open, and likely requires either cosmic censorship assumptions or fundamentally new mathematical tools.