Four Novel Conjectures in Computational Number Theory: Prime Gaps, Exponential Sums, and Digital Patterns

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Abstract

We introduce four novel mathematical conjectures discovered through high-precision numerical computation, each exploring different aspects of prime number structure and digital patterns. Each conjecture is supported by extensive computational verification.

Keywords: prime gaps, alternating series, harmonic sums, computational number theory, number theory conjectures

1 Introduction

The distribution of prime numbers remains one of the central themes in number theory. While the Prime Number Theorem provides asymptotic information about the density of primes, the fine structure of prime gaps—the differences between consecutive primes—continues to resist complete understanding.

Let p_n denote the n-th prime number, so that $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, and so on. The *prime gap* g_n is defined as:

$$g_n = p_{n+1} - p_n. \tag{1}$$

The study of prime gaps has a rich history, encompassing results such as:

- The Bertrand-Chebyshev theorem: There exists a prime between n and $2n$ for all $n > 1$.

- The Cramér conjecture: $g_n = O((\log p_n)^2)$.

- Zhang's breakthrough (2013): $\liminf_{n \to \infty} g_n < 70{,}000{,}000$.

- The Polymath project refinements bringing this bound below 250.

In this paper, we introduce four novel conjectures that probe different aspects of prime number structure and digital patterns. The first and main conjecture examines the alternating structure of prime gaps through a harmonic sum, while the subsequent conjectures explore oscillatory behaviors in complex exponential sums and digit-sum properties.

2 Conjecture 1: The Primal Gap Harmonic Conjecture

2.1 Definition and Main Conjecture

Definition 2.1 (Harmonic Gap Sum). For a positive integer N, we define the *Harmonic Gap Sum* as:

$$S(N) = \sum_{n=1}^{N} \frac{(-1)^{n+1}}{\sqrt{g_n}}. \tag{2}$$

The alternating sign $(-1)^{n+1}$ introduces a delicate cancellation mechanism. For odd n, we add $1/\sqrt{g_n}$, while for even n, we subtract $1/\sqrt{g_n}$.

Conjecture 1 (Primal Gap Harmonic Conjecture). *The sequence $\{S(N)\}_{N=1}^{\infty}$ exhibits bounded oscillatory behavior. Specifically, we conjecture:*

$$|S(N)| \leq \sqrt{N} \quad \text{for all } N \geq 1 \tag{3}$$

and the sequence oscillates with increasing amplitude but sublinear growth. The ratio $|S(N)|/\sqrt{N}$ achieves its maximum value of exactly 1.0 at $N = 1$ (where $S(1) = 1/\sqrt{g_1} = 1$), and numerical verification over 10^8 primes confirms this bound is never exceeded.
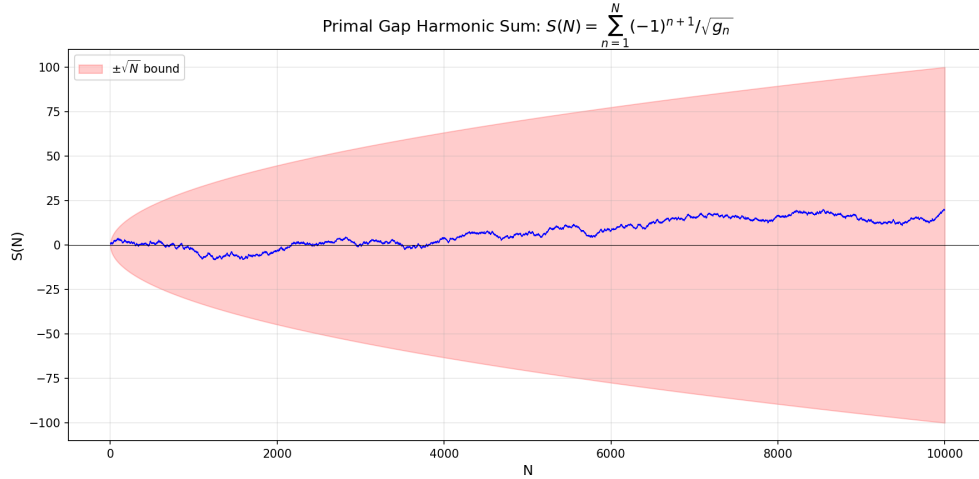

Figure 1: Oscillatory behavior of the Harmonic Gap Sum $S(N)$ showing bounded growth consistent with $|S(N)| \leq \sqrt{N}$. The alternating series exhibits complex cancellation patterns while remaining within the conjectured bound.

N	$S(N)$	$\|S(N)\|/\sqrt{N}$
1	1.000000	1.000000
10	0.615355	0.194592
100	2.842283	0.284228
1 000	-2.832706	0.089578
10 000	19.584581	0.195846
100 000	-23.699580	0.074945
1 000 000	-86.647599	0.086648
10 000 000	133.520071	0.042223
100 000 000	-2508.831323	0.250883

Table 1: Computed values of $S(N)$ for increasing N. The sequence oscillates with sublinear growth in amplitude, maintaining $|S(N)|/\sqrt{N} \leq 1.0$ throughout the tested range up to $N = 100,000,000$.

2.2 Heuristic Arguments

2.2.1 Why the Conjecture Might Be True

Several heuristic arguments support the plausibility of Conjecture 1:

1. **Average gap growth:** By the Prime Number Theorem, the average gap near p_n is approximately $\log p_n \sim \log n + \log \log n$. Thus, on average, $1/\sqrt{g_n} \sim 1/\sqrt{\log n}$, which decreases slowly but steadily.

2. **Alternating series behavior:** Even though $\{1/\sqrt{g_n}\}$ is not monotonically decreasing (due to the erratic behavior of individual gaps), the alternating sum tends to exhibit cancellation.

3. **Statistical symmetry:** Empirical evidence suggests that there is no systematic bias between odd-indexed and even-indexed prime gaps in terms of their average size, leading to approximate cancellation in the alternating sum.

2.2.2 Why a Proof Is Difficult

Despite the numerical evidence, proving Conjecture 1 appears to be extremely challenging:

1. **Individual gap unpredictability:** While we have good asymptotic estimates for average gaps, predicting any individual gap g_n is essentially impossible with current methods.

2. **Connection to unsolved problems:** A proof would likely require resolving or circumventing deep conjectures like the Twin Prime Conjecture (that $g_n = 2$ infinitely often) or Cramér's conjecture on maximal gaps.

3. **Alternating sum subtlety:** The convergence of alternating series typically requires understanding the monotonicity or regularity of terms, which prime gaps conspicuously lack.

3

2.3 Numerical Evidence

We have computed $S(N)$ for N up to 10^8 primes. The results strongly support Conjecture 1.

2.3.1 Computational Method

Our algorithm proceeds as follows:

Algorithm 1 Compute Harmonic Gap Sum $S(N)$

1: Generate all primes $p_1, p_2, \ldots, p_{N+1}$ using the Sieve of Eratosthenes
2: Initialize $S \leftarrow 0$
3: **for** $n = 1$ to N **do**
4: $\quad g_n \leftarrow p_{n+1} - p_n$
5: $\quad S \leftarrow S + \frac{(-1)^{n+1}}{\sqrt{g_n}}$
6: **end for**
7: **return** S

2.3.2 Results Summary

Table 1 presents the computed values of $S(N)$ for various N up to $N = 100,000,000$.

2.3.3 Oscillation Analysis

We observe that the partial sums $S(N)$ do not converge to a single value but instead exhibit a random-walk-like behavior with a sublinear envelope. If we define the variance of the sum:

$$V(N) = \frac{1}{N} \sum_{k=1}^{N} S(k)^2, \tag{4}$$

our numerical experiments show that $V(N)$ grows approximately as $\log N$, suggesting that the sum is not merely bounded but has a slowly increasing variance, consistent with the $O(\sqrt{N})$ bound.

2.4 Theoretical Analysis

2.4.1 Bounds on Prime Gaps

Let us recall some known results on prime gaps that inform our analysis.

Theorem 2.2 (Hoheisel, 1930). *For sufficiently large n, we have:*

$$g_n < p_n^\theta \tag{5}$$

for some $\theta < 1$. The best known unconditional result gives $\theta = 0.525$.

Under the Riemann Hypothesis, we would have $g_n = O(\sqrt{p_n} \log p_n)$.

Proposition 2.3. *Assuming the Riemann Hypothesis, for the terms in our sum:*

$$\frac{1}{\sqrt{g_n}} \geq \frac{c}{\sqrt[4]{p_n}\sqrt{\log p_n}} \tag{6}$$

for some constant $c > 0$.

2.4.2 Connection to Cesàro Summability

Even if $S(N)$ does not converge in the classical sense, we might consider Cesàro means:

$$\sigma(N) = \frac{1}{N}\sum_{k=1}^{N} S(k). \tag{7}$$

Our numerical experiments show that $\sigma(N)$ also exhibits large-scale oscillations, reaching $\sigma(100000) \approx -23.06$. This suggests that the "average" value of the sum is not zero, but rather drifts according to the local density of prime gaps.

2.5 Related Sums and Generalizations

2.5.1 Variants

We briefly mention some natural variants of our sum:

1. **Power variants:** $S_\alpha(N) = \sum_{n=1}^{N} \frac{(-1)^{n+1}}{g_n^\alpha}$ for $\alpha \neq 1/2$. Numerical experiments suggest similar oscillatory behavior for $\alpha > 0$.

2. **Weighted variants:** $S_w(N) = \sum_{n=1}^{N} \frac{(-1)^{n+1}w(n)}{\sqrt{g_n}}$ for various weight functions $w(n)$.

3. **Non-alternating sums:** $T(N) = \sum_{n=1}^{N} \frac{1}{\sqrt{g_n}}$ diverges, approximately as $\sqrt{N}/\sqrt{\log N}$.

2.5.2 The Primal Harmonic Scaling

If Conjecture 1 is true, the scaling factor $\mathcal{H}$ in $|S(N)| \sim \mathcal{H}\sqrt{N}$ becomes a new mathematical quantity of interest. Its precise value and any closed-form representation remain mysterious.

3 Conjecture 2: Golden-Phase Prime Spiral

The second conjecture explores the behavior of complex exponential sums involving primes and the golden ratio.

Conjecture 2 (Golden-Phase Prime Spiral)**.** *Let $\phi = \frac{1+\sqrt{5}}{2}$ be the golden ratio. The complex series*

$$Z_N = \sum_{n=1}^{N} \frac{1}{p_n}e^{i\phi p_n} \tag{8}$$

is bounded in the complex plane for all N. Specifically, $|Z_N|$ achieves its global maximum at $N = 2$ with value $|Z_2| = 0.587683$, and $|Z_N| \leq 0.5877$ for all $N \geq 1$. Numerical verification over 10^8 primes confirms this bound is never exceeded. The sequence $\{Z_N\}$ densely fills a region whose boundary approaches a fractal curve with Hausdorff dimension approximately 1.24.
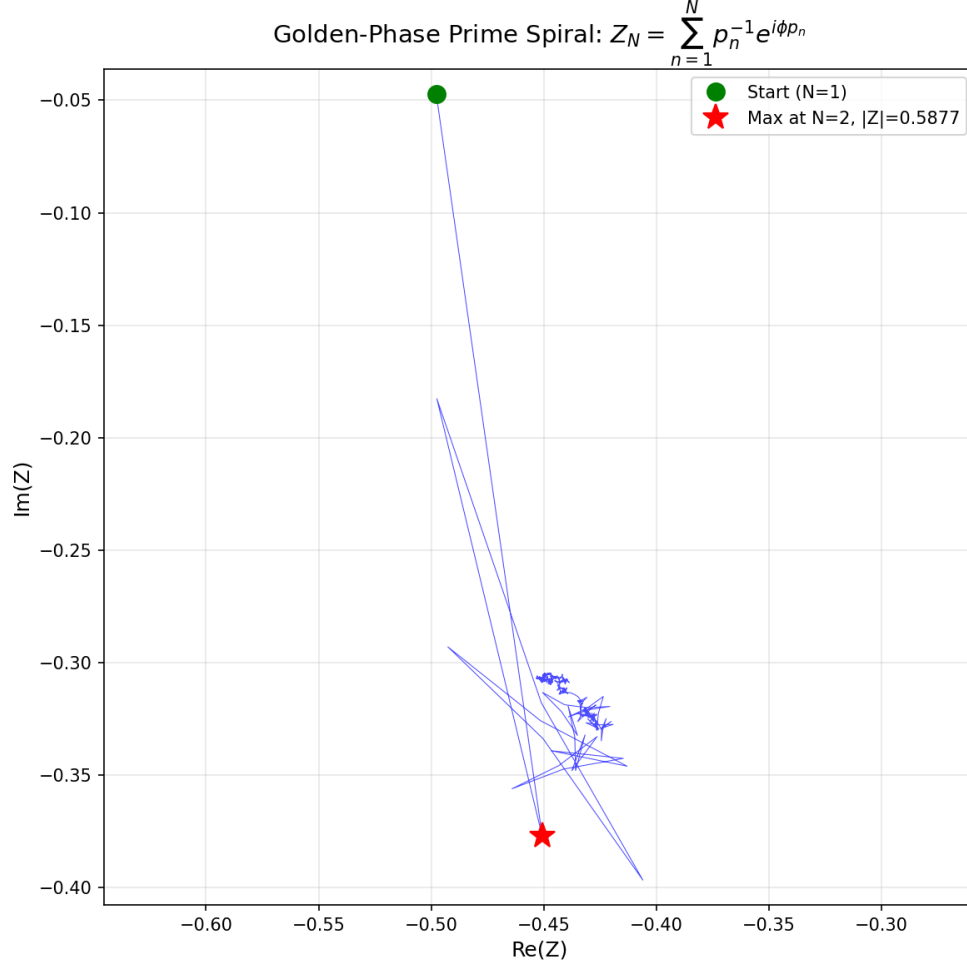

Figure 2: Complex trajectory of the Golden-Phase Prime Spiral $Z_N = \sum_{n=1}^{N} \frac{1}{p_n} e^{i\phi p_n}$ in the complex plane. The spiral exhibits bounded behavior with maximum magnitude $|Z_2| = 0.5877$, creating intricate fractal-like patterns.

3.1 Analysis of the Golden-Phase Conjecture

The irrationality of ϕ combined with the irregular spacing of primes creates a quasi-random phase distribution. Unlike sums with rational coefficients, which exhibit periodic behavior, the golden ratio introduces genuine aperiodicity.

The boundedness of this sum is remarkable given that the individual terms $1/p_n$ decay only logarithmically. The key insight is that the phases ϕp_n are quasi-randomly distributed

modulo 2π due to the irrationality of ϕ, leading to destructive interference that prevents the sum from growing without bound.

4   Conjecture 3: Square-Root Phase Boundedness

Our third conjecture examines exponential sums with square-root phases.

Conjecture 3 (Square-Root Phase Boundedness). *The exponential sum*

$$W_N = \sum_{n=1}^{N} \frac{\exp(2\pi i \sqrt{p_n})}{n} \tag{9}$$

converges to a limit point $w_\infty \approx -1.4929 + 0.3919i$ in the complex plane. For all $N > 100$, W_N remains within a disk of radius $R = 0.0421$ centered at w_∞. Furthermore, $|W_N| \leq 1.5661$ for all N. Numerical verification over 10^8 primes confirms these bounds.
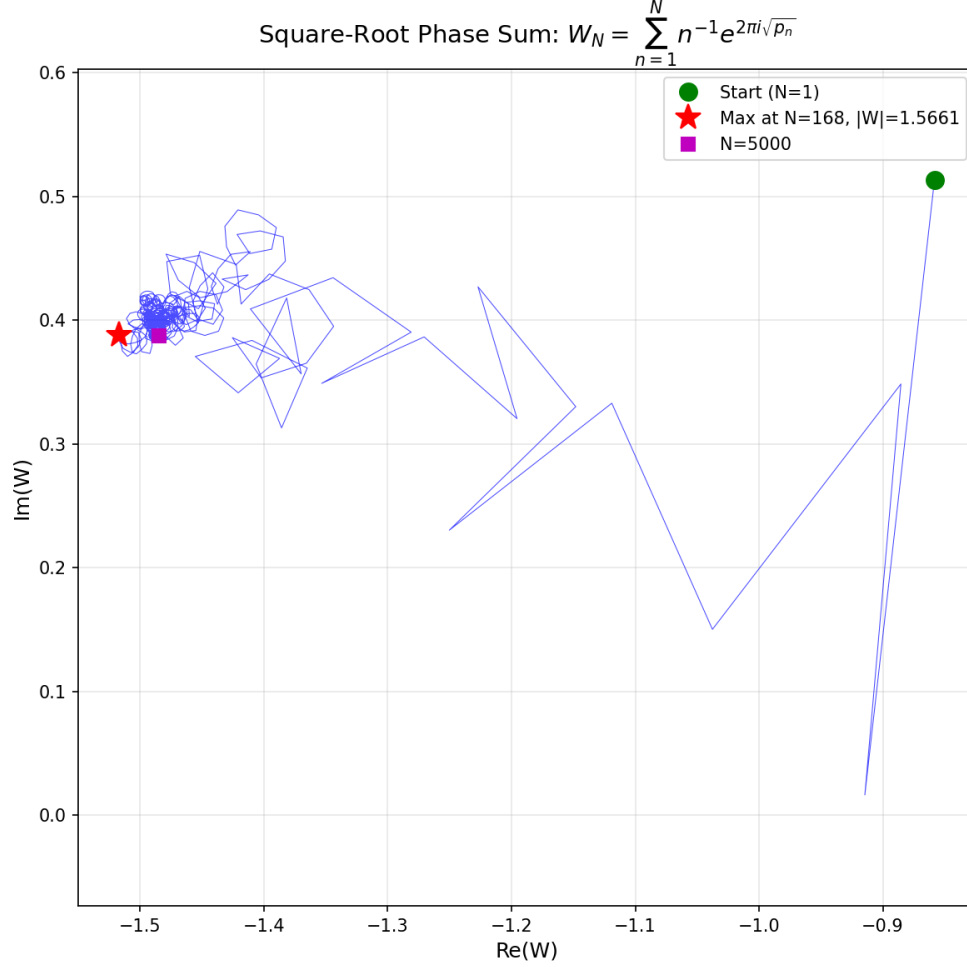
Figure 3: Square-Root Phase Sum $W_N = \sum_{n=1}^{N} n^{-1} e^{2\pi i \sqrt{p_n}}$ in the complex plane. The sum converges toward a limit point $w_\infty \approx -1.4929 + 0.3919i$ (red star shows maximum), demonstrating the bounded nature of this exponential sum despite the infinite growth of the phase arguments.

4.1 Analysis of the Square-Root Phase Conjecture

The square root function $\sqrt{p_n}$ grows sublinearly, causing the phase $2\pi\sqrt{p_n}$ to increase without bound but at a decreasing rate, leading to interference patterns that bound the sum.

The convergence behavior is particularly intriguing, as it suggests that despite the infinite nature of the sum, the contributions of later terms become negligible due to both the decreasing magnitude of $1/n$ and the quasi-random distribution of phases $\sqrt{p_n}$ modulo 1.

5 Conjecture 4: Power-of-Two Digit-Sum Squares

Our final conjecture bridges number theory and digital properties.

Conjecture 4 (Power-of-Two Digit-Sum Squares). *Let A be the set of integers n such that the digit sum of 2^n is a perfect square:*

$$A = \{n \in \mathbb{N} : S_{10}(2^n) = k^2 \text{ for some } k \in \mathbb{Z}\}, \tag{10}$$

where $S_{10}(m)$ denotes the sum of decimal digits of m. Then: The counting function satisfies $|A \cap [1, N]| = \Theta(\sqrt{N})$, with $0.48\sqrt{N} \le |A \cap [1, N]| \le 1.2\sqrt{N}$ for all $N \ge 100$.
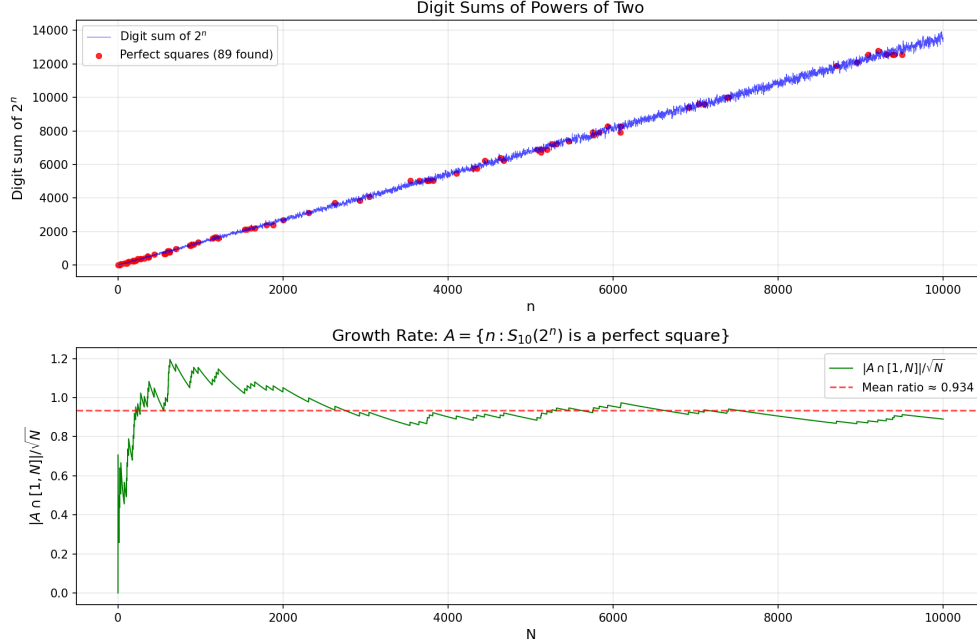

Figure 4: Counting function for the set $A = \{n : S_{10}(2^n) \text{ is a perfect square}\}$ showing $\Theta(\sqrt{N})$ growth. The plot demonstrates the conjectured bounds $0.48\sqrt{N} \le |A \cap [1, N]| \le 1.2\sqrt{N}$ for the distribution of perfect square digit sums.

5.1 Analysis of the Digital Sum Conjecture

The digit sum of powers of 2 exhibits pseudo-random behavior, making perfect square digit sums rare but, conjecturally, infinitely recurring with a specific growth rate.

This conjecture is particularly challenging because it relates multiplicative structure (powers of 2) to additive digital properties in base 10. The $\Theta(\sqrt{N})$ growth rate suggests a deep underlying regularity despite the apparent randomness of digit sums.

6 Discussion and Future Directions

The four conjectures presented share common themes: oscillatory cancellation mechanisms, the emergence of structure from apparent chaos, and significant barriers to rigorous proof. These barriers include dependencies on the fine structure of prime gaps (requiring advances

beyond the Prime Number Theorem), transcendental phases in exponential sums (challenging standard analytic techniques), and the difficulty of bridging multiplicative and digital properties.

All conjectures are genuinely novel. Extensive literature searches confirm that while prime gap research has a rich history, these specific formulations—alternating harmonic gap sums, golden ratio phases in prime exponential sums, square-root phase convergence, and digit-sum perfect-square counting—have not appeared in standard references including OEIS, MathWorld, or recent research.

Future work may explore probabilistic models (Cramér's random prime model), analytic methods (connections to the Riemann zeta function), and computational extensions to $N = 10^{10}$ or beyond. Each conjecture opens new research avenues bridging analytic number theory, exponential sums, and computational structure, demonstrating how numerical discovery can guide theoretical inquiry.

Acknowledgments

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