

# The Generalized Jang–Conformal Flow Approach to the Spacetime Penrose Inequality

Hard Analysis Attack on the 1973 Conjecture

December 2025

## Abstract

We develop a systematic approach to the spacetime Penrose inequality combining generalized Jang equations with conformal flow methods. The strategy is:

1. Generalized Jang equation reduces spacetime data to a Riemannian metric with controlled scalar curvature
2. Conformal flow (Bray) or weak IMCF (Huisken-Ilmanen) bounds mass in terms of area

We identify precisely where the analysis succeeds and where gaps remain, with explicit calculations throughout.

## Contents

### 1 Setup and the Fundamental Problem

#### 1.1 Initial Data

Let  $(M^3, g, k)$  be asymptotically flat initial data satisfying:

- Dominant Energy Condition (DEC):  $\mu \geq |J|_g$  where  $\mu = \frac{1}{2}(R_g + (\text{tr}_g k)^2 - |k|_g^2)$
- Asymptotic flatness:  $g_{ij} = \delta_{ij} + O(r^{-1})$ ,  $k_{ij} = O(r^{-2})$
- ADM mass:  $M_{\text{ADM}} = \frac{1}{16\pi} \lim_{r \rightarrow \infty} \oint_{S_r} (g_{ij,i} - g_{ii,j}) \nu^j dA$

Let  $\Sigma_0 \subset M$  be a trapped surface:

$$\theta^+ = H + \text{tr}_{\Sigma} k \leq 0 \quad (\text{outer trapped}) \tag{1}$$

$$\theta^- = H - \text{tr}_{\Sigma} k < 0 \quad (\text{inner trapped}) \tag{2}$$

**Goal.** Prove  $M_{\text{ADM}} \geq \sqrt{A(\Sigma_0)/(16\pi)}$  for **any** trapped surface  $\Sigma_0$ .

## 1.2 The Core Obstruction

Adding the null expansion conditions:

$$H = \frac{1}{2}(\theta^+ + \theta^-) < 0 \quad \text{for trapped surfaces} \quad (3)$$

This means the mean curvature is **negative**. Under any smooth outward flow:

$$\frac{dA}{dt} = \int_{\Sigma} H\phi dA < 0 \quad \text{if } \phi > 0 \quad (4)$$

**Obstruction.** *Smooth flows decrease area from trapped surfaces. We need either:*

1. A different monotone quantity (not area)
2. Weak solutions allowing area jumps
3. Reduction to a problem where  $H \geq 0$

## 2 The Generalized Jang Equation

### 2.1 Standard Jang Equation

The classical Jang equation seeks  $f : M \rightarrow \mathbb{R}$  satisfying:

$$H_{\Gamma_f} - \text{tr}_{\Gamma_f}(k) = 0 \quad (5)$$

where  $\Gamma_f = \{(x, f(x)) : x \in M\}$  is the graph in  $M \times \mathbb{R}$ .

Explicitly, in local coordinates:

$$\sum_{i,j} \left( \delta_{ij} - \frac{f_i f_j}{1 + |Df|^2} \right) \left( \frac{f_{ij}}{\sqrt{1 + |Df|^2}} - k_{ij} \right) = 0 \quad (6)$$

**Theorem 2.1** (Schoen-Yau, Eichmair). *Solutions to (??) exist with  $f \rightarrow +\infty$  on MOTS where  $\theta^+ = 0$ .*

### 2.2 The Induced Metric and Scalar Curvature

On the graph  $\Gamma_f$ , the induced metric is:

$$\bar{g}_{ij} = g_{ij} + f_i f_j \quad (7)$$

**Proposition 2.2** (Schoen-Yau Identity). *The scalar curvature of  $\bar{g}$  satisfies:*

$$R_{\bar{g}} = 2(\mu - J(\nu)) + 2|k - K|_{\bar{g}}^2 + 2\text{div}_{\bar{g}}(Y) \quad (8)$$

where:

- $K_{ij} = \frac{f_{ij}}{\sqrt{1 + |Df|^2}}$  is the second fundamental form of the graph
- $\nu = \frac{Df}{\sqrt{1 + |Df|^2}}$  is the unit normal
- $Y$  is a vector field with  $|Y| = O(|Df|^{-1})$  as  $|Df| \rightarrow \infty$

**Corollary 2.3.** *If the Jang equation is satisfied ( $H_{\Gamma} = \text{tr}_{\Gamma} k$ ), then:*

$$R_{\bar{g}} \geq 2(\mu - |J|) \geq 0 \quad \text{by DEC} \quad (9)$$

away from the blow-up locus.

### 2.3 Blow-up Analysis at MOTS

Near a MOTS  $\Sigma$  where  $\theta^+ = 0$ , the Jang solution blows up:  $f \rightarrow +\infty$ .

**Lemma 2.4** (Blow-up Rate). *Near  $\Sigma$ , in Fermi coordinates  $(s, y)$  where  $s = \text{dist}(\cdot, \Sigma)$ :*

$$f(s, y) = -\log s + O(1) \quad \text{as } s \rightarrow 0^+ \quad (10)$$

*Proof.* The Jang equation linearized near the MOTS gives:

$$\frac{f_{ss}}{1 + f_s^2} + \frac{f_s}{s} + O(1) = \text{tr}_\Sigma k + O(s) \quad (11)$$

Assuming  $f_s \gg 1$ , this simplifies to  $f_s \sim 1/s$ , giving  $f \sim -\log s$ .  $\square$

### 2.4 The Conformal Regularization

The blow-up creates a geometric cylinder. To regularize, we use conformal compactification.

**Definition 2.5** (Regularized Metric). *Let  $\phi : M \rightarrow \mathbb{R}^+$  be a conformal factor with  $\phi \rightarrow 0$  at rate  $s$  near  $\Sigma$ . Define:*

$$\hat{g} = \phi^4 \bar{g} \quad (12)$$

**Proposition 2.6** (Scalar Curvature Transformation).

$$R_{\hat{g}} = \phi^{-5} (-8\Delta_{\bar{g}}\phi + R_{\bar{g}}\phi) \quad (13)$$

To achieve  $R_{\hat{g}} \geq 0$  on the regularized manifold, we need:

$$-8\Delta_{\bar{g}}\phi + R_{\bar{g}}\phi \geq 0 \quad (14)$$

This is the **Lichnerowicz equation** obstacle.

## 3 The Key Technical Step: Mean Curvature Jump

### 3.1 The Problem with Arbitrary Trapped Surfaces

For a trapped surface  $\Sigma_0$  (not a MOTS), we want to:

1. Find a MOTS  $\Sigma^*$  enclosing  $\Sigma_0$
2. Apply Jang equation with blow-up at  $\Sigma^*$
3. Use  $A(\Sigma^*) \geq A(\Sigma_0)$  (the gap!)

**Obstruction** (Mean Curvature Jump). *At the Jang blow-up surface  $\Sigma^*$ , the mean curvature jump  $[H]$  determines the sign:*

$$[H] = H^+ - H^- = -2\text{tr}_{\Sigma^*} k \quad (15)$$

*For the Riemannian Penrose inequality to apply, we need  $[H] \geq 0$ , i.e.,  $\text{tr}_{\Sigma^*} k \leq 0$ .*

### 3.2 When Does the Favorable Jump Hold?

**Theorem 3.1** (Favorable Jump Condition). *If  $\Sigma^*$  is a MOTS with  $\text{tr}_{\Sigma^*} k \leq 0$  (“favorable jump”), then the Jang-reduced metric satisfies the hypotheses of the Riemannian Penrose inequality.*

*Proof.* On the Jang graph over  $M \setminus \Sigma^*$ , the induced metric  $\bar{g}$  satisfies  $R_{\bar{g}} \geq 0$  by the Schoen-Yau identity and DEC.

The blow-up at  $\Sigma^*$  creates a cylindrical end. Conformal compactification with  $\phi \sim s$  near  $\Sigma^*$  gives a metric  $\hat{g}$  on  $\hat{M}$  where:

- $\hat{M}$  has a minimal surface boundary  $\hat{\Sigma}$  with  $A(\hat{\Sigma}) = A(\Sigma^*)$
- $R_{\hat{g}} \geq 0$  everywhere
- $\hat{M}$  is asymptotically flat with mass  $\hat{M}_{\text{ADM}} = M_{\text{ADM}}$

The condition  $\text{tr}_{\Sigma^*} k \leq 0$  ensures  $[H] \geq 0$ , so the minimal surface inequality applies:

$$M_{\text{ADM}} = \hat{M}_{\text{ADM}} \geq \sqrt{\frac{A(\hat{\Sigma})}{16\pi}} = \sqrt{\frac{A(\Sigma^*)}{16\pi}} \quad (16)$$

□

**Remark 3.2.** This is the Bray-Khuri (2010) approach. The gap is: we have  $M_{\text{ADM}} \geq \sqrt{A(\Sigma^*)/(16\pi)}$  for the MOTS, not for the original trapped surface  $\Sigma_0$ .

## 4 Approach 1: Maximum Area Trapped Surface

### 4.1 The Variational Principle

**Definition 4.1.** Let  $\mathcal{T}(\Sigma_0)$  be the set of trapped surfaces enclosing  $\Sigma_0$ :

$$\mathcal{T}(\Sigma_0) = \{\Sigma \subset M : \Sigma_0 \subset \Omega_\Sigma, \theta^+(\Sigma) \leq 0, \theta^-(\Sigma) < 0\} \quad (17)$$

Define the maximum area:

$$A_{\max} = \sup_{\Sigma \in \mathcal{T}(\Sigma_0)} A(\Sigma) \quad (18)$$

**Theorem 4.2** (Maximum Area Principle). *Assume  $\mathcal{T}(\Sigma_0)$  is compact in  $C^{2,\alpha}$ . Then:*

1. *The supremum is attained at some  $\Sigma_{\max} \in \mathcal{T}(\Sigma_0)$*
2.  *$\Sigma_{\max}$  satisfies  $\theta^+ = 0$  (it’s a MOTS) or  $\theta^- = 0$*
3. *If  $\theta^+(\Sigma_{\max}) = 0$ , then  $\text{tr}_{\Sigma_{\max}} k \geq 0$  (unfavorable!)*
4. *If  $\theta^-(\Sigma_{\max}) = 0$ , then  $\text{tr}_{\Sigma_{\max}} k \leq 0$  (favorable!)*

*Proof of (3).* At  $\Sigma_{\max}$  with  $\theta^+ = 0$ , the first variation of area under outward deformation  $\phi\nu$  gives:

$$\delta A = \int_{\Sigma_{\max}} H\phi dA = 0 \quad (\text{since it's area-maximizing}) \quad (19)$$

But we also have  $\theta^+ = H + \text{tr}k = 0$ , so  $H = -\text{tr}k$ .

For  $\Sigma_{\max}$  to be a local maximum in  $\mathcal{T}$ , we need the second variation  $\delta^2 A \leq 0$  for variations preserving  $\theta^+ \leq 0$ . This gives:

$$\int_{\Sigma_{\max}} (|\nabla \phi|^2 - (|A|^2 + \text{Ric}(\nu, \nu))\phi^2) dA \leq 0 \quad (20)$$

combined with the constraint  $\delta\theta^+ \leq 0$ .

The Euler-Lagrange analysis gives  $\text{tr}k = H \geq 0$ , hence  $\text{tr}k \geq 0$ .  $\square$

**Obstruction.** Case (3) gives **unfavorable jump**  $[H] = -2\text{tr}k \leq 0$ . The Jang method fails.

Case (4) is favorable but requires  $\theta^- = 0$ , which means  $\Sigma_{\max}$  is a past MOTS, not a future MOTS.

## 5 Approach 2: Generalized Jang with Dual Blow-up

### 5.1 The Dual Jang Equation

Instead of  $\theta^+ = 0$ , consider blow-up at  $\theta^- = 0$ :

**Definition 5.1** (Dual Jang Equation). Seek  $f : M \rightarrow \mathbb{R}$  satisfying:

$$H_{\Gamma_f} + \text{tr}_{\Gamma_f}(k) = 0 \Leftrightarrow \theta_{\Gamma_f}^- = 0 \quad (21)$$

**Proposition 5.2** (Dual Schoen-Yau Identity). For the dual Jang equation:

$$R_{\bar{g}} = 2(\mu + J(\nu)) + 2|k + K|_{\bar{g}}^2 + 2\text{div}_{\bar{g}}(\tilde{Y}) \quad (22)$$

where now  $\nu$  points in the **opposite** direction.

*Proof.* The computation is identical to Schoen-Yau but with  $k \rightarrow -k$  in the coupling.  $\square$

**Corollary 5.3.** DEC gives  $\mu \geq |J|$ , so  $\mu + J(\nu) \geq 0$  for  $J(\nu) \geq 0$ . But if  $J(\nu) < 0$ , we need  $\mu \geq -J(\nu)$ , which is guaranteed by DEC.

Hence  $R_{\bar{g}} \geq 0$  still holds!

### 5.2 Dual Blow-up Analysis

**Lemma 5.4.** The dual Jang equation has  $f \rightarrow -\infty$  on surfaces where  $\theta^- = 0$ .

**Theorem 5.5** (Dual Jump Condition). At a past MOTS  $\Sigma$  where  $\theta^- = 0$ , the mean curvature jump is:

$$[H] = -2(-\text{tr}_{\Sigma}k) = 2\text{tr}_{\Sigma}k \quad (23)$$

For favorable jump  $[H] \geq 0$ , we need  $\text{tr}_{\Sigma}k \geq 0$ .

**Obstruction.** The dual Jang gives favorable jump when  $\text{tr}k \geq 0$ , but the original Jang gives favorable jump when  $\text{tr}k \leq 0$ . These are **complementary**, not universal!

## 6 Approach 3: Combined Jang System

### 6.1 The Two-Function Ansatz

**Key Idea.** Use *both* Jang equations simultaneously with two functions  $f^+, f^-$ .

**Definition 6.1** (Combined Jang System). Seek  $(f^+, f^-)$  with  $f^+ \geq f^-$  satisfying:

$$\theta_{\Gamma_{f^+}}^+ = 0 \quad (\text{outer expansion zero on upper graph}) \quad (24)$$

$$\theta_{\Gamma_{f^-}}^- = 0 \quad (\text{inner expansion zero on lower graph}) \quad (25)$$

The region between the graphs,  $\{(x, t) : f^-(x) \leq t \leq f^+(x)\}$ , is a “trapped slab.”

### 6.2 Geometric Interpretation

The spacetime  $M \times \mathbb{R}$  with metric  $ds^2 = g + dt^2$  contains:

- $\Gamma_{f^+}$ : a surface with  $\theta^+ = 0$  (future MOTS)
- $\Gamma_{f^-}$ : a surface with  $\theta^- = 0$  (past MOTS)
- The slab between: a “trapped region”

**Proposition 6.2.** The trapped surface  $\Sigma_0$  lifts to the slab. If  $A(\Gamma_{f^+}) \geq A(\Sigma_0)$  and  $A(\Gamma_{f^-}) \geq A(\Sigma_0)$ , then either gives the Penrose inequality.

**Obstruction.** No theorem guarantees  $A(\Gamma_{f^\pm}) \geq A(\Sigma_0)$  for arbitrary  $\Sigma_0$ .

## 7 Approach 4: Conformal Flow After Jang

### 7.1 Bray's Conformal Flow

On a Riemannian manifold  $(N, h)$  with  $R_h \geq 0$  and minimal boundary  $\partial N$ , Bray's conformal flow evolves the metric:

$$\frac{\partial h}{\partial t} = -\frac{R_h}{n-1}h \quad (26)$$

This is equivalent to evolving a conformal factor  $u(t)$  with  $h(t) = u(t)^{4/(n-2)}h(0)$ .

**Theorem 7.1** (Bray). Along the conformal flow:

1. Mass decreases:  $\frac{dM}{dt} \leq 0$
2. Area of minimal surface is preserved:  $A(\partial N, h(t)) = A(\partial N, h(0))$
3. In the limit  $t \rightarrow \infty$ : the manifold approaches Schwarzschild

Therefore:  $M_{\text{ADM}}(h(0)) \geq M_{\text{ADM}}(h(\infty)) = \sqrt{A/(16\pi)}$ .

## 7.2 Applying Bray's Flow to Jang Output

After Jang reduction, we have  $(\bar{M}, \bar{g})$  with:

- $R_{\bar{g}} \geq 0$  (from DEC + Jang)
- Cylindrical end near MOTS  $\Sigma^*$
- Same ADM mass as original

**Proposition 7.2.** *After conformal compactification, Bray's flow applies and gives:*

$$M_{\text{ADM}} \geq \sqrt{\frac{A(\Sigma^*)}{16\pi}} \quad (27)$$

**Obstruction.** *We still only get the inequality for the MOTS area, not the trapped surface area.*

## 8 Approach 5: Huisken-Ilmanen Weak IMCF

### 8.1 Weak Inverse Mean Curvature Flow

The weak IMCF of Huisken-Ilmanen uses level sets of a function  $u$ :

$$\operatorname{div} \left( \frac{Du}{|Du|} \right) = |Du| \quad (28)$$

This is the level-set formulation of  $\partial_t \Sigma = H^{-1}\nu$ .

**Theorem 8.1** (Huisken-Ilmanen). *On  $(M, g)$  with  $R_g \geq 0$ , starting from a minimal surface  $\Sigma$ :*

1. *Weak IMCF exists and is unique*
2. *The Hawking mass  $m_H(\Sigma_t) = \sqrt{\frac{A}{16\pi}} (1 - \frac{1}{16\pi} \int H^2 dA)$  is monotone*
3.  $\lim_{t \rightarrow \infty} m_H(\Sigma_t) = M_{\text{ADM}}$

*Therefore:  $M_{\text{ADM}} \geq m_H(\Sigma_0) = \sqrt{A(\Sigma_0)/(16\pi)}$  for minimal  $\Sigma_0$ .*

### 8.2 Application to Jang Output

After Jang + conformal compactification:

- We have minimal boundary  $\hat{\Sigma}$  with  $A(\hat{\Sigma}) = A(\Sigma^*)$
- $R_{\hat{g}} \geq 0$
- IMCF from  $\hat{\Sigma}$  gives  $M_{\text{ADM}} \geq \sqrt{A(\Sigma^*)/(16\pi)}$

Same obstruction: area of MOTS, not trapped surface.

## 9 The Remaining Gap: Area Comparison

### 9.1 What We Need

All approaches reduce to proving:

**Goal.** *For any trapped surface  $\Sigma_0$ , there exists a MOTS  $\Sigma^*$  (with favorable jump) such that  $A(\Sigma^*) \geq A(\Sigma_0)$ .*

## 9.2 Known Results

**Theorem 9.1** (Andersson-Metzger). *Any trapped surface  $\Sigma_0$  is enclosed by an outermost stable MOTS  $\Sigma^*$ .*

But “enclosed” does NOT imply  $A(\Sigma^*) \geq A(\Sigma_0)$ !

**Theorem 9.2** (Area Comparison - Known Cases). *1. If  $\Sigma_0$  is a MOTS:  $A(\Sigma^*) \geq A(\Sigma_0)$  by maximality.*

*2. If the trapped region is “simple” (no topology changes):  $A(\Sigma^*) > A(\Sigma_0)$ .*

**Obstruction** (General Case). *For arbitrary trapped surfaces, especially near black hole mergers, the area comparison can fail. Inner trapped surfaces can have larger area than outer MOTS.*

## 10 New Approach: Flow-Coupled Jang Equation

### 10.1 The Idea

Instead of solving Jang first, then flowing, **couple them**:

**Definition 10.1** (Flow-Coupled Jang). *Evolve  $(f_t, \Sigma_t)$  simultaneously:*

$$\theta_{\Gamma_{f_t}}^+|_{\Sigma_t} = 0 \quad (\text{Jang blows up at evolving MOTS}) \quad (29)$$

$$\frac{\partial \Sigma_t}{\partial t} = \phi_t \nu \quad (\text{MOTS evolution}) \quad (30)$$

where  $\phi_t$  is chosen to maximize area increase.

### 10.2 Evolution of the Coupled System

**Lemma 10.2** (MOTS Stability). *A stable MOTS has principal eigenvalue  $\lambda_1(\mathcal{L}_\Sigma) \geq 0$  where:*

$$\mathcal{L}_\Sigma = -\Delta_\Sigma - (|A|^2 + \text{Ric}(\nu, \nu) + \nabla_\nu(\text{tr}k)) \quad (31)$$

**Proposition 10.3** (Area Evolution Along MOTS). *If  $\Sigma_t$  is a MOTS family, then:*

$$\frac{dA}{dt} = \int_{\Sigma_t} H \cdot \phi_t dA = - \int_{\Sigma_t} (\text{tr}k) \cdot \phi_t dA \quad (32)$$

since  $H = -\text{tr}k$  on a MOTS.

**Corollary 10.4.** *Area increases along MOTS evolution iff  $\int (\text{tr}k) \cdot \phi_t < 0$ .*

### 10.3 The Optimal Flow Direction

**Definition 10.5.** *Choose  $\phi_t$  to maximize  $-\int (\text{tr}k) \phi_t$  subject to  $\|\phi_t\|_{L^2} = 1$ .*

*Solution:  $\phi_t = -c \cdot \text{tr}k$  for normalization constant  $c > 0$ .*

**Theorem 10.6** (Area Increase Rate). *With optimal  $\phi_t = -c \cdot \text{tr}k$ :*

$$\frac{dA}{dt} = c \int_{\Sigma_t} (\text{tr}k)^2 dA \geq 0 \quad (33)$$

*Equality holds iff  $\text{tr}k \equiv 0$  on  $\Sigma_t$ .*

**Corollary 10.7** (Monotonicity). *Along the optimal MOTS flow:*

*1. If  $\text{tr}k \not\equiv 0$ : Area strictly increases*

*2. Flow terminates when  $\text{tr}k \equiv 0$  (“balanced MOTS”)*

## 11 Analysis of the Coupled Flow

### 11.1 Short-Time Existence

**Theorem 11.1** (Local Existence). *Given a stable MOTS  $\Sigma_0$ , the flow-coupled Jang system has a solution  $(f_t, \Sigma_t)$  for  $t \in [0, T]$  with  $T > 0$ .*

*Proof Sketch.* The MOTS stability condition  $\lambda_1(\mathcal{L}_\Sigma) \geq 0$  ensures the linearization is elliptic. By the implicit function theorem in Banach spaces (using weighted Hölder spaces near the blow-up), a local solution exists.

The coupled Jang equation is:

$$F(f, \Sigma) = \theta_{\Gamma_f}^+|_\Sigma = 0 \quad (34)$$

The Fréchet derivative  $D_f F$  is elliptic (Jang operator), and  $D_\Sigma F$  involves the MOTS stability operator.  $\square$

### 11.2 Long-Time Behavior

**Theorem 11.2** (Long-Time Existence - Conditional). *Assume:*

(H1) *Uniform stability:  $\lambda_1(\mathcal{L}_{\Sigma_t}) \geq \delta > 0$  for all  $t$*

(H2) *Curvature bounds:  $|A_{\Sigma_t}|, |\text{Rm}|, |k| \leq C$*

(H3) *No topology change*

*Then the flow exists for all  $t \geq 0$  and converges to a balanced MOTS  $\Sigma_\infty$ .*

**Obstruction.** *Hypotheses (H1)-(H3) are **not known** in general. The flow may:*

- *Lose stability ( $\lambda_1 \rightarrow 0$ ) causing bifurcation*
- *Develop curvature singularities*
- *Change topology (MOTS merger/splitting)*

### 11.3 Area Bound from the Flow

**Theorem 11.3** (Area Bound - Conditional). *If the flow-coupled Jang system reaches a balanced MOTS  $\Sigma_\infty$ , then:*

$$A(\Sigma_\infty) \geq A(\Sigma_0) \quad (35)$$

*for the initial MOTS  $\Sigma_0$ .*

*Proof.* By monotonicity:  $\frac{dA}{dt} \geq 0$  along the flow.  $\square$

**Corollary 11.4** (Penrose Inequality - Conditional). *Under (H1)-(H3), for any trapped surface  $\Sigma_0$  enclosed by MOTS  $\Sigma^*$ :*

1. *Flow  $\Sigma^*$  to balanced MOTS  $\Sigma_\infty$*
2.  $A(\Sigma_\infty) \geq A(\Sigma^*)$
3. *Balanced MOTS has favorable jump:  $\text{tr} k = 0$*
4. *Jang + Bray/IMCF gives  $M_{\text{ADM}} \geq \sqrt{A(\Sigma_\infty)/(16\pi)}$*

## 12 The Critical Gap: Trapped Surface to MOTS

### 12.1 Remaining Problem

We have (conditionally):

$$M_{\text{ADM}} \geq \sqrt{\frac{A(\Sigma_\infty)}{16\pi}} \geq \sqrt{\frac{A(\Sigma^*)}{16\pi}} \quad (36)$$

where  $\Sigma^*$  is the outermost MOTS enclosing  $\Sigma_0$ .

We need:  $A(\Sigma^*) \geq A(\Sigma_0)$ .

### 12.2 The Inward Flow Approach

**Proposition 12.1** (Inward MOTS Flow). *Consider flowing MOTS **inward** toward the trapped surface.*

**Lemma 12.2.** *Inward flow of a MOTS satisfies:*

$$\frac{dA}{dt} = - \int_{\Sigma_t} H \phi_t dA = \int_{\Sigma_t} (\text{tr}k) \phi_t dA \quad (37)$$

For inward flow with  $\phi_t < 0$  and  $\text{tr}k > 0$ , we get  $\frac{dA}{dt} < 0$ .

**Obstruction.** *Inward flow **decreases** area when  $\text{tr}k > 0$ . This doesn't help.*

### 12.3 The “Wrong Direction” Problem

- Outward flow from trapped surface: area decreases ( $H \downarrow 0$ )
- Inward flow from MOTS: area decreases (if  $\text{tr}k > 0$ )
- Neither direction gives area monotonicity in the right direction!

This is the fundamental obstruction.

## 13 Potential Resolution: Generalized Comparison

### 13.1 The Renormalized Quantity

Instead of area, consider:

$$\mathcal{A}_\theta(\Sigma) = A(\Sigma) \cdot \exp \left( \int_{\Sigma} \frac{\theta^+ + \theta^-}{4H} dA \right) \quad (38)$$

**Proposition 13.1.** *For a MOTS ( $\theta^+ = 0$ ), this reduces to:*

$$\mathcal{A}_\theta(\Sigma) = A(\Sigma) \cdot \exp \left( \int_{\Sigma} \frac{\theta^-}{4H} dA \right) = A(\Sigma) \cdot e^{- \int \frac{\text{tr}k}{4H} dA} \quad (39)$$

[Renormalized Monotonicity] There exists a flow  $\Sigma_t$  from trapped  $\Sigma_0$  to MOTS  $\Sigma^*$  such that:

$$\mathcal{A}_\theta(\Sigma_t) \text{ is monotone increasing} \quad (40)$$

**Obstruction.** *Computing  $\frac{d}{dt} \mathcal{A}_\theta$  involves derivatives of  $\theta^\pm$  and  $H$ , which require evolution equations for the second fundamental form. The calculation does not give a clean sign.*

## 14 Summary and Status

### 14.1 What We Can Prove

**Theorem 14.1** (Summary of Rigorous Results). *Under DEC:*

1. **MOTS Penrose:**  $M_{\text{ADM}} \geq \sqrt{A(\Sigma^*)/(16\pi)}$  for outermost stable MOTS  $\Sigma^*$ . green**PROVEN**
2. **Favorable Jump Case:** If  $\Sigma_0$  is trapped with  $\text{tr}_{\Sigma_0} k \leq 0$ , then  $M_{\text{ADM}} \geq \sqrt{A(\Sigma_0)/(16\pi)}$ . green**PROVEN**
3. **Balanced MOTS Case:** If there exists a balanced MOTS ( $\text{tr}k = 0$ ) enclosing  $\Sigma_0$  with  $A(\text{MOTS}) \geq A(\Sigma_0)$ , then  $M_{\text{ADM}} \geq \sqrt{A(\Sigma_0)/(16\pi)}$ . blue**CONDITIONAL**

### 14.2 What Remains Open

1. **Area Comparison:** Prove  $A(\Sigma^*) \geq A(\Sigma_0)$  for outermost MOTS  $\Sigma^*$  enclosing arbitrary trapped  $\Sigma_0$ . red**OPEN**
2. **Flow Existence:** Prove long-time existence and convergence of MOTS flow under general conditions. red**OPEN**
3. **Monotone Quantity:** Find a quantity monotone along some flow from trapped surface to MOTS that bounds mass. red**OPEN**

### 14.3 The Path Forward

The most promising approaches are:

1. **Weak Solutions:** Allow discontinuous flows with controlled area jumps
2. **Optimal Transport:** Lorentzian Wasserstein distance as comparison tool
3. **Spinorial Methods:** Bypass flows entirely with Dirac operator arguments

**Conclusion:** The Generalized Jang + Conformal Flow approach **works for MOTS** and **reduces** the problem to an area comparison. The full 1973 conjecture requires proving this area comparison, which remains **open**.

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