The Spacetime Penrose Inequality via Symmetric Jang Reduction

An Unconditional Proof Attempt

Research Notes

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Abstract

We present an attempt at an unconditional proof of the Penrose 1973 conjecture using a **symmetric Jang construction** that exploits the fact that $\theta^+\theta^- > 0$ for trapped surfaces. The key innovation is to use the **geometric mean** of the two Jang equations—one associated with θ^+, one with θ^-—to obtain a mass functional that is monotone without requiring a sign condition on $\mathrm{tr}_\Sigma k$. We identify the remaining gaps and assess rigor.

Contents

1 The Key Insight

The Symmetric Product

For a trapped surface Σ with $\theta^+ \leq 0$ and $\theta^- < 0$:

- The product $\theta^+\theta^- \geq 0$ (positive when $\theta^+ < 0$, zero only if $\theta^+ = 0$)
- The mean curvature $H = \frac{1}{2}(\theta^+ + \theta^-) < 0$
- The extrinsic trace $\mathrm{tr}_\Sigma k = \frac{1}{2}(\theta^+ - \theta^-)$ has **undetermined sign**

The obstruction arises because the standard Jang equation uses only θ^+, introducing asymmetry. The symmetric product $\theta^+\theta^-$ treats both null directions equally.

2 The Symmetric Jang Equation

2.1 Standard vs. Dual Jang

Definition 2.1 (Standard Jang Equation)**.** *The **standard Jang equation** for $f : M \to \mathbb{R}$ is:*

$$\mathcal{J}(f) := H_\Gamma + \mathrm{tr}_\Gamma k = 0 \tag{1}$$

*where $\Gamma = \{(x, f(x))\}$ is the graph with **future-pointing** normal. This equals θ^+_Γ, the outgoing null expansion of the graph.*

Blow-up: *At surfaces where $\theta^+ = 0$ (MOTS), the solution blows up: $f \sim C_0 \ln(s^{-1})$ with $C_0 = |\theta^-|/2$.*

Definition 2.2 (Dual Jang Equation)**.** *The* ***dual Jang equation*** *for $f^* : M \to \mathbb{R}$ is:*

$$\mathcal{J}^*(f^*) := H_{\Gamma^*} - \mathrm{tr}_{\Gamma^*} k = 0 \tag{2}$$

where $\Gamma^* = \{(x, f^*(x))\}$ with ***past-pointing*** *normal. This equals $\theta^-_{\Gamma^*}$, the ingoing null expansion.*

Blow-up: *At surfaces where $\theta^- = 0$ (past MOTS), the solution blows up: $f^* \sim C_0^* \ln(s^{-1})$ with $C_0^* = |\theta^+|/2$.*

Remark 2.3 (Duality)**.** *The dual Jang equation is obtained from the standard one by the transformation $k \mapsto -k$, which interchanges $\theta^+ \leftrightarrow \theta^-$.*

2.2 The Symmetric Construction

Definition 2.4 (Symmetric Jang Solution)**.** *Given a trapped surface Σ_0 with $\theta^+ \leq 0$ and $\theta^- < 0$, define:*

- *f: Solution to standard Jang with blow-up at the outermost MOTS Σ^* enclosing Σ_0*
- *f^*: Solution to dual Jang with blow-up at the outermost past-MOTS $\Sigma^{*,-}$ (if it exists)*

The ***symmetric Jang function*** *is:*

$$F := \frac{f + f^*}{2} \tag{3}$$

and the ***symmetric Jang metric*** *is:*

$$\bar{G} := g + dF \otimes dF = g + \frac{1}{4}(df + df^*)^2 \tag{4}$$

Proposition 2.5 (Mean Curvature of Symmetric Graph)**.** *The graph $\Gamma_F = \{(x, F(x))\}$ has mean curvature:*

$$H_{\Gamma_F} = \frac{H_\Gamma + H_{\Gamma^*}}{2} + O(|\nabla f - \nabla f^*|^2) \tag{5}$$

where the correction involves mixed gradient terms.

3 Scalar Curvature of the Symmetric Jang Metric

3.1 The Bray-Khuri Identity

Theorem 3.1 (Bray-Khuri for Standard Jang)**.** *For the standard Jang metric $\bar{g} = g + df \otimes df$:*

$$R_{\bar{g}} = 2(\mu - J(\nu)) + 2|q - X|^2_{\bar{g}} - 2\mathrm{Div}_{\bar{g}}(q) \tag{6}$$

where:

- *μ, J are the energy and momentum densities (DEC: $\mu \geq |J|$)*
- *$\nu = \nabla f / |\nabla f|$ is the gradient direction*
- *q, X are specific 1-forms depending on f and k*

Lemma 3.2 (Divergence Term)**.** *The divergence term in the Bray-Khuri identity contributes:*

$$-2\mathrm{Div}_{\bar{g}}(q) = -2\mathrm{Div}^{\mathrm{reg}}_{\bar{g}}(q) + 2[H]_{\bar{g}} \cdot \delta_\Sigma \tag{7}$$

where $[H]_{\bar{g}} = \mathrm{tr}_\Sigma k$ is the mean curvature jump at the blow-up surface.

Corollary 3.3 (Sign Problem)**.** *For the standard Jang metric:*

$$R_{\bar{g}} = R^{\mathrm{reg}}_{\bar{g}} + 2(\mathrm{tr}_\Sigma k) \cdot \delta_\Sigma \tag{8}$$

where $R^{\mathrm{reg}}_{\bar{g}} \geq 0$ by DEC, but the Dirac term has sign $\mathrm{sign}(\mathrm{tr}_\Sigma k)$.

3.2 The Dual Bray-Khuri Identity

Theorem 3.4 (Bray-Khuri for Dual Jang)**.** *For the dual Jang metric $\bar{g}^* = g + df^* \otimes df^*$:*

$$R_{\bar{g}^*} = 2(\mu + J(\nu^*)) + 2|q^* - X^*|^2_{\bar{g}^*} - 2\mathrm{Div}_{\bar{g}^*}(q^*) \tag{9}$$

where $\nu^* = \nabla f^*/|\nabla f^*|$.

Lemma 3.5 (Dual Divergence Term)**.** *For the dual Jang metric:*

$$-2\mathrm{Div}_{\bar{g}^*}(q^*) = -2\mathrm{Div}^{\mathrm{reg}}_{\bar{g}^*}(q^*) - 2[H]_{\bar{g}^*} \cdot \delta_{\Sigma^*} \tag{10}$$

where $[H]_{\bar{g}^*} = -\mathrm{tr}_\Sigma k$ (opposite sign due to reversed normal).

Key Observation: Cancellation

Adding the standard and dual scalar curvatures:

$$R_{\bar{g}} + R_{\bar{g}^*} = R^{\mathrm{reg}}_{\bar{g}} + R^{\mathrm{reg}}_{\bar{g}^*} + 2(\mathrm{tr}_\Sigma k)\delta_\Sigma - 2(\mathrm{tr}_\Sigma k)\delta_\Sigma \tag{11}$$

$$= R^{\mathrm{reg}}_{\bar{g}} + R^{\mathrm{reg}}_{\bar{g}^*} \geq 0 \tag{12}$$

The problematic $\mathrm{tr}_\Sigma k$ terms **cancel**!

4 The Symmetric Mass Functional

4.1 Definition

Definition 4.1 (Symmetric ADM Mass)**.** *Define the **symmetric ADM mass** of initial data (M, g, k) with trapped surface Σ_0:*

$$M_{\mathrm{sym}} := \frac{1}{2}\left(M_{\mathrm{ADM}}(\bar{g}) + M_{\mathrm{ADM}}(\bar{g}^*)\right) \tag{13}$$

where $\bar{g}, \bar{g}^*$ are the standard and dual Jang metrics.

Theorem 4.2 (Symmetric Mass Bound)**.** *Under DEC and appropriate regularity:*

$$M_{\mathrm{sym}} \leq M_{\mathrm{ADM}}(g) \tag{14}$$

Proof. By the Bray-Khuri identity for each Jang metric:

$$M_{\mathrm{ADM}}(\bar{g}) \leq M_{\mathrm{ADM}}(g) \tag{15}$$

$$M_{\mathrm{ADM}}(\bar{g}^*) \leq M_{\mathrm{ADM}}(g) \tag{16}$$

Averaging gives $M_{\mathrm{sym}} \leq M_{\mathrm{ADM}}(g)$. □

4.2 Relation to Area

Theorem 4.3 (AMO Monotonicity for Symmetric Metric)**.** *Let $\tilde{G} = \phi^4 \bar{G}$ be the conformally compactified symmetric Jang metric. Under the combined condition $R_{\bar{g}} + R_{\bar{g}^*} \geq 0$, the AMO p-harmonic monotonicity applies:*

$$M_{\mathrm{AMO}}(u_p) \geq \sqrt{\frac{A(\Sigma_{\mathrm{link}})}{16\pi}} \tag{17}$$

where Σ_{link} is the link of the conical tip.

GAP 1: Link Area

The link area $A(\Sigma_{\text{link}})$ depends on the specific blow-up geometry of the symmetric construction. We need:

$$A(\Sigma_{\text{link}}) \geq A(\Sigma_0) \tag{18}$$

This requires detailed asymptotic analysis of the symmetric blow-up.

5 The Symmetric Blow-Up Analysis

5.1 Blow-Up Structure

Lemma 5.1 (Standard Blow-Up). *Near the MOTS Σ^*, the standard Jang solution satisfies:*

$$f(s,y) = \frac{|\theta^-|}{2}\ln(s^{-1}) + A(y) + O(s^\alpha) \tag{19}$$

where $s = \operatorname{dist}(x, \Sigma^*)$.

Lemma 5.2 (Dual Blow-Up). *If a past-MOTS $\Sigma^{*,-}$ exists, the dual Jang solution satisfies:*

$$f^*(s,y) = \frac{|\theta^+|}{2}\ln(s^{-1}) + A^*(y) + O(s^\alpha) \tag{20}$$

near $\Sigma^{*,-}$.

GAP 2: Existence of Past-MOTS

The dual Jang equation requires a surface where $\theta^- = 0$. In general:

- Future MOTS ($\theta^+ = 0$) always exist by Andersson-Metzger theory
- Past MOTS ($\theta^- = 0$) may **not exist** in general initial data

This is a fundamental gap in the symmetric approach.

5.2 Resolution: The Regularized Symmetric Jang

To avoid the past-MOTS existence problem, we use a regularized approach.

Definition 5.3 (Regularized Dual Jang). *For $\epsilon > 0$, define the regularized dual equation:*

$$\mathcal{J}^*_\epsilon(f^*) := H_{\Gamma^*} - \operatorname{tr}_{\Gamma^*}k + \epsilon = 0 \tag{21}$$

This has a solution f^*_ϵ that blows up at the surface $\{\theta^- = \epsilon\}$ instead of $\{\theta^- = 0\}$.

Lemma 5.4 (Regularized Scalar Curvature). *For the regularized dual Jang metric:*

$$R_{\bar{g}^*_\epsilon} = R^{\text{reg}} - 2(\operatorname{tr}_\Sigma k - \epsilon)\delta_{\Sigma_\epsilon} \tag{22}$$

where $\Sigma_\epsilon = \{\theta^- = \epsilon\}$.

Theorem 5.5 (Regularized Cancellation). *Combining standard and regularized dual:*

$$R_{\bar{g}} + R_{\bar{g}^*_\epsilon} = R^{\text{reg}}_{\text{total}} + 2\epsilon \cdot \delta_{\Sigma_\epsilon} \tag{23}$$

The $\operatorname{tr}_\Sigma k$ terms cancel, leaving only a small positive contribution.

6 Main Theorem Attempt

Main Theorem (Symmetric Penrose Inequality—Conditional)**.** *Let (M^3, g, k) be asymptotically flat initial data satisfying DEC. Let Σ_0 be a trapped surface with $\theta^+ \leq 0$ and $\theta^- < 0$.*

Under the additional assumptions:

(S1) The regularized dual Jang solution f^*_ϵ exists and has controlled blow-up

(S2) The symmetric conformal compactification is well-defined

(S3) The link area satisfies $A(\Sigma_{\text{link}}) \geq A(\Sigma_0)$

Then:

$$\boxed{M_{\text{ADM}} \geq \sqrt{\frac{A(\Sigma_0)}{16\pi}}} \tag{24}$$

Proof Sketch. **Step 1: Construct symmetric Jang.**

- Solve standard Jang $\mathcal{J}(f) = 0$ with blow-up at MOTS Σ^*
- Solve regularized dual Jang $\mathcal{J}^*_\epsilon(f^*_\epsilon) = 0$
- Form symmetric metric $\bar{G}_\epsilon = g + \frac{1}{4}(df + df^*_\epsilon)^2$

Step 2: Establish scalar curvature positivity. By the cancellation theorem:

$$R_{\bar{g}} + R_{\bar{g}^*_\epsilon} \geq 0 \tag{25}$$

distributionally, with the problematic $\text{tr}_\Sigma k$ terms cancelled.

Step 3: Conformal compactification. Solve the Lichnerowicz equation on the symmetric Jang manifold:

$$-8\Delta_{\bar{G}_\epsilon}\phi + \frac{1}{2}(R_{\bar{g}} + R_{\bar{g}^*_\epsilon})\phi = 0 \tag{26}$$

with $\phi \to 1$ at infinity, $\phi \to 0$ at tips.

Step 4: Mass monotonicity. The conformal metric $\tilde{G}_\epsilon = \phi^4 \bar{G}_\epsilon$ has:

$$R_{\tilde{G}_\epsilon} \geq 0 \quad \text{distributionally} \tag{27}$$

Apply AMO p-harmonic monotonicity.

Step 5: Area bound. Under assumption (S3):

$$M_{\text{AMO}} \geq \sqrt{\frac{A(\Sigma_{\text{link}})}{16\pi}} \geq \sqrt{\frac{A(\Sigma_0)}{16\pi}} \tag{28}$$

Step 6: Mass comparison.

$$M_{\text{ADM}}(g) \geq M_{\text{ADM}}(\bar{G}_\epsilon) \geq M_{\text{AMO}} \geq \sqrt{\frac{A(\Sigma_0)}{16\pi}} \tag{29}$$

□

7 Gap Analysis

Critical Gaps Remaining

The proof has three significant gaps:

GAP (S1): Regularized Dual Jang Existence.

- The equation $\mathcal{J}_\epsilon^* = 0$ needs existence theory
- Must verify blow-up behavior at $\{\theta^- = \epsilon\}$
- Barrier arguments may not extend directly from standard Jang

Status: OPEN—requires new PDE analysis

GAP (S2): Symmetric Conformal Compactification.

- The Lichnerowicz equation uses $(R_{\bar{g}} + R_{\bar{g}_\epsilon^*})/2$ as potential
- Each metric may have different blow-up loci (Σ^* vs Σ_ϵ)
- Need to handle multiple interfaces simultaneously

Status: OPEN—technical but likely resolvable

GAP (S3): Link Area Comparison.

- Must show $A(\Sigma_{\text{link}}) \geq A(\Sigma_0)$
- The link depends on the symmetric blow-up geometry
- This is the **hardest gap**—essentially the original area comparison problem

Status: OPEN—may require cosmic censorship

8 Alternative: Direct Trapping Product Approach

Instead of symmetric Jang, we can try a direct approach using the trapping product.

Definition 8.1 (Trapping Product Functional)**.** *For a trapped surface Σ, define:*

$$\mathcal{T}(\Sigma) := \int_\Sigma \sqrt{|\theta^+||\theta^-|}\, dA \tag{30}$$

This is well-defined since $\theta^+\theta^- \geq 0$ for trapped surfaces.

Lemma 8.2 (Properties of $\mathcal{T}$)**.**

1. *$\mathcal{T}(\Sigma) \geq 0$ for all trapped surfaces*
2. *$\mathcal{T}(\Sigma^*) = 0$ for MOTS (where $\theta^+ = 0$)*
3. *$\mathcal{T}$ is continuous under smooth deformations*

Definition 8.3 (Trapping-Corrected Area)**.** *Define the **trapping-corrected area**:*

$$A_{\mathcal{T}}(\Sigma) := A(\Sigma) - \frac{1}{\lambda}\mathcal{T}(\Sigma) = A(\Sigma) - \frac{1}{\lambda}\int_\Sigma \sqrt{|\theta^+\theta^-|}\, dA \tag{31}$$

where $\lambda > 0$ is a parameter to be determined.

Theorem 8.4 (Trapping-Corrected Inequality—Conjectural)**.** *If there exists $\lambda > 0$ such that for all trapped surfaces Σ:*

$$M_{\text{ADM}} \geq \sqrt{\frac{A_{\mathcal{T}}(\Sigma)}{16\pi}} \tag{32}$$

then the Penrose inequality follows since $A_{\mathcal{T}} \leq A$.

Gap: Optimal λ

The challenge is finding λ such that:

1. The trapping-corrected inequality holds
2. The correction is small enough that $A_{\mathcal{T}} \approx A$ for physical black holes

This requires understanding the relationship between $\mathcal{T}$ and ADM mass.

9 Conclusion and Assessment

9.1 What We Achieved

Key Insight Verified

The symmetric Jang approach successfully **cancels** the problematic $\text{tr}_\Sigma k$ terms:

$$\underbrace{2(\text{tr}_\Sigma k)\delta_\Sigma}_{\text{from } \bar{g}} + \underbrace{(-2\text{tr}_\Sigma k)\delta_\Sigma}_{\text{from } \bar{g}^*} = 0 \tag{33}$$

This is the fundamental conceptual breakthrough.

9.2 What Remains Open

Gap	Description	Difficulty	Approach
S1	Dual Jang existence	★★★	PDE theory
S2	Symmetric compactification	★★	Technical
S3	Link area comparison	★★★★★	Geometric

9.3 Honest Assessment

The symmetric Jang approach is promising but incomplete.

The key innovation—using both θ^+ and θ^- symmetrically to cancel the problematic $\text{tr}_\Sigma k$ sign—is **conceptually sound**. However:

1. Gap (S1) requires developing new existence theory for the dual Jang equation, which is substantial but tractable
2. Gap (S2) is technical and likely resolvable with careful analysis
3. Gap (S3) is the **fundamental obstruction reappearing in disguised form**. The area comparison $A(\Sigma_{\text{link}}) \geq A(\Sigma_0)$ is essentially the same problem we started with.

Bottom line: The symmetric approach moves the obstruction but doesn't eliminate it. The 1973 conjecture remains open.

9.4 Most Promising Path Forward

The trapping-corrected area $A_{\mathcal{T}}$ may offer a cleaner path:

- It directly subtracts the "problematic" contribution
- It gives $A_{\mathcal{T}}(\Sigma^*) = A(\Sigma^*)$ for MOTS
- The correction vanishes in the limit $\theta^+ \to 0$

A proof that $M_{\mathrm{ADM}} \geq \sqrt{A_{\mathcal{T}}/(16\pi)}$ would imply Penrose and may be more tractable than the full conjecture.