Boost-

Spacetime Penrose Inequality Program

December 2025

Abstract

"" boost- $\mathcal{Q}$ (1) (2) (3) MOTS/caustic/ Penrose

Contents

1 Boost-

1.1 Hawking

Hawking

$$m_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}} \left(1 - \frac{1}{16\pi}\int_\Sigma \theta^+\theta^- \, dA\right) \tag{1}$$

Hawking

null

$$\frac{dm_H}{ds} = \frac{\sqrt{|\Sigma|/16\pi}}{16\pi}\int_\Sigma \left[\mu - |J| \cdot (\text{something}) - 2\sigma^+ : \sigma^- - |\zeta|^2\right] dA \tag{2}$$

- $-2\sigma^+ : \sigma^- \sigma^+ : \sigma^- = \mathrm{tr}(\sigma^+_{ab}\sigma^{-ab})$
- $-|\zeta|^2$

Hawking null Penrose

1.2 #1

σ^+, σ^-

$$\boxed{-\sigma^+ : \sigma^- = -\frac{1}{4}|\sigma^+ + \sigma^-|^2 + \frac{1}{4}|\sigma^+ - \sigma^-|^2} \tag{3}$$

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1.3 #2Boost

null (ℓ, n) boost

$$\ell \mapsto \lambda\ell, \quad n \mapsto \lambda^{-1}n \tag{4}$$

		boost
θ^+	$\lambda\theta^+$	$+1$
θ^-	$\lambda^{-1}\theta^-$	-1
σ^+	$\lambda\sigma^+$	$+1$
σ^-	$\lambda^{-1}\sigma^-$	-1
$\theta^+\theta^-$		0
$\sigma^+ : \sigma^-$		0
σ^+/θ^+		0

Boost

boost $\theta^\pm$ ""

$$\frac{\sigma^+}{\theta^+} - \frac{\sigma^-}{\theta^-} \quad \text{boost} \tag{5}$$

$\theta^+ \to 0$MOTS $\theta^- \to 0$

2 $\mathcal{Q}$

2.1

Definition 1 (Boost-)**.**

$$\mathcal{Q}(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}} \left(1 - \frac{1}{16\pi} \int_\Sigma \left[\theta^+\theta^- + |\zeta|^2 + \frac{1}{4} \left| \frac{\sigma^+}{\theta^+} - \frac{\sigma^-}{\theta^-} \right|^2 \theta^+\theta^- \right] dA \right) \tag{6}$$

2.2 outgoing null

Theorem 2 (Q)**.** *DEC outgoing null*

$$\frac{d\mathcal{Q}}{ds} = \frac{\sqrt{|\Sigma|/16\pi}}{16\pi} \int_\Sigma \Phi \, dA \tag{7}$$

Φ

$$\Phi = \underbrace{(\mu - |J|) \cdot ()}_{DEC \geq 0} \tag{8}$$

$$+ \underbrace{\frac{1}{4}|\sigma^+ - \sigma^-|^2 \cdot ()}_{\geq 0} \tag{9}$$

$$+ \underbrace{(/)}_{caustic} \tag{10}$$

3 --

$-2\sigma^+ : \sigma^-$ ()	$-\sigma^+ : \sigma^- = -\frac{1}{4}\lvert\sigma^+ + \sigma^-\rvert^2 + \frac{1}{4}\lvert\sigma^+ - \sigma^-\rvert^2$	$+\frac{1}{4}\lvert\sigma^+ + \sigma^-\rvert^2$	MOTS θ^+
$-\lvert\zeta\rvert^2$ ()		$+\lvert\zeta\rvert^2$	Causticζ
σ^+/θ^+ (MOTS)		$\lvert\theta^+\rvert < \delta$ outer hull	MOTS$\theta^+ = 0$
$\theta^+ \to -\infty$ (Caustic)		Huisken-Ilmanen outer hull	Caustic
()	$\mathcal{Q} \to M_B$	Bondi +	I^+

4 MOTS- null

4.1

Gap 1+2

- **Gap 1**Caustic$\theta^+ \to -\infty$
- **Gap 2**MOTS $\theta^+ \to 0$

$\mathcal{Q}$ θ^+

outer hull

4.2 MOTS- null

Definition 3 (MOTS-Avoiding Weak Null Flow)**.** Σ ***MOTS- null*** $\{\Sigma_s\}_{s\geq 0}$

(WA1) $\Sigma_0 = \Sigma$ $\theta^+(\Sigma_0) < 0$

(WA2) Σ_s *outgoing null* $|\theta^{\pm}| \geq \delta > 0$

(WA3) ***Caustic*** $\theta^+ \to -\infty$ *outward minimizing hull*

(WA4) ***MOTS-*** $|\theta^+| < \delta$ *outward minimizing hull*

(WA5) $\Sigma_s \to I^+ null$

4.3

[] Σ^- Σ^+ outer hull

$$\mathcal{Q}(\Sigma^+) \geq \mathcal{Q}(\Sigma^-) \tag{11}$$

- Outer hull Lorentzian
- $\mathcal{Q}$ Σ^- MOTS caustic
- ””

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5.1 Raychaudhuri

outgoing null ℓ

$$\frac{d\theta^+}{ds} = -\frac{1}{2}(\theta^+)^2 - |\sigma^+|^2 - R_{\mu\nu}\ell^\mu\ell^\nu \tag{12}$$

$$= -\frac{1}{2}(\theta^+)^2 - |\sigma^+|^2 - 8\pi(\mu - J\cdot\ell) \tag{13}$$

ingoing null n

$$\frac{d\theta^-}{ds} = -\frac{1}{2}(\theta^-)^2 - |\sigma^-|^2 - 8\pi(\mu - J\cdot n) \tag{14}$$

5.2 $\theta^+\theta^-$

$$\frac{d(\theta^+\theta^-)}{ds} = \theta^- \frac{d\theta^+}{ds} + \theta^+ \frac{d\theta^-}{ds} \tag{15}$$

$$= -\frac{1}{2}\theta^-(\theta^+)^2 - \theta^-|\sigma^+|^2 - 8\pi\theta^-(\mu - J\cdot\ell) \tag{16}$$

$$-\frac{1}{2}\theta^+(\theta^-)^2 - \theta^+|\sigma^-|^2 - 8\pi\theta^+(\mu - J\cdot n) \tag{17}$$

5.3

$$-\theta^-|\sigma^+|^2 - \theta^+|\sigma^-|^2 - 2\sigma^+ : \sigma^- \tag{18}$$

$$-\theta^-|\sigma^+|^2 - \theta^+|\sigma^-|^2 - 2\sigma^+ : \sigma^- \tag{19}$$

$$= -\theta^-|\sigma^+|^2 - \theta^+|\sigma^-|^2 + \frac{1}{2}|\sigma^+ + \sigma^-|^2 - \frac{1}{2}|\sigma^+ - \sigma^-|^2 \tag{20}$$

$\Delta\sigma = \sigma^+ - \sigma^-$boost $+1-(-1) = +2$ boost
boost

$$\frac{\sigma^+}{\theta^+} - \frac{\sigma^-}{\theta^-} \quad \text{boost} \tag{21}$$

$$\left|\frac{\sigma^+}{\theta^+} - \frac{\sigma^-}{\theta^-}\right|^2 \theta^+\theta^- = \frac{|\sigma^+\theta^- - \sigma^-\theta^+|^2}{\theta^+\theta^-} \tag{22}$$

$\theta^+\theta^- < 0$untrapped
trapped $\theta^+\theta^- > 0$
trapped surface

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6.1 MOTS ($\theta^+ = 0$)

$\theta^+ \to 0^-$

- $\sigma^+/\theta^+ \to \pm\infty$ $\sigma^+ = 0$
- $\mathcal{Q}$ $\left|\frac{\sigma^+}{\theta^+} - \frac{\sigma^-}{\theta^-}\right|^2 \theta^+\theta^- \to -\infty$ $\theta^- < 0$
- $\mathcal{Q}$

  $|\theta^+| < \delta$

6.2 Caustic ($\theta^+ \to -\infty$)

caustic

- $\theta^+ \to -\infty$
- $\to 0$
- $\mathcal{Q}$

  Huisken-Ilmanen outer hull

6.3 Null Infinity (I^+)

- $|\Sigma_r| \sim 4\pi r^2$
- $\theta^+ \sim 2/r$, $\theta^- \sim -1/r$
- $\sigma^{\pm} \sim O(r^{-2})$news function
- $\zeta \sim O(r^{-2})$

$\mathcal{Q}(\Sigma_r) = M_B + O(r^{-1})$ Bondi

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Main Conditional Theorem

Theorem 4 (Penrose ——). (M^4, g) *DEC Bondi* $M_B\Sigma$ *MOTS- null* $\{\Sigma_s\}_{s\in[0,\infty)}$

(H1) (WA1)–(WA5)

(H2) $\mathcal{Q}(\Sigma^+) \geq \mathcal{Q}(\Sigma^-)$

$$\boxed{M_B \geq \sqrt{\frac{|\Sigma|}{16\pi}}} \tag{23}$$

. 1. $\mathcal{Q}(\Sigma_0) = \sqrt{|\Sigma|/16\pi}$ outermost trapped surface

2. DEC + $d\mathcal{Q}/ds \geq 0$
3. (H2)$\mathcal{Q}$
4. $\lim_{s\to\infty} \mathcal{Q}(\Sigma_s) = M_B$
5. $M_B \geq \mathcal{Q}(\Sigma_0) = \sqrt{|\Sigma|/16\pi}$

□

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1. **null** (WA1)–(WA5)
2. **Outer hull Lorentzian** null "outward minimizing hull"
3. $\mathcal{Q}(\Sigma^+) \geq \mathcal{Q}(\Sigma^-)$
4.
5. Schwarzschild

9 ””

	Kakeya	**Penrose 1973**
	/ Kakeya	$\sigma^+ : \sigma^-$
	+	+ boost
		Caustic, MOTS
		null + outer hull
		DEC +

10

Boost- $\mathcal{Q}$

- $-\sigma^+ : \sigma^- \ + \frac{1}{4}|\sigma^+ - \sigma^-|^2 - \frac{1}{4}|\sigma^+ + \sigma^-|^2$
- DEC $(\mu - |J|) \geq 0$
-

null caustic/MOTS

- $\theta^+ \to -\infty$caustic $\theta^+ \to 0$MOTS
- outer hull
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