Hard Attack on Penrose 1973

Weak IMCF + Viscosity Methods

Working Document

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Abstract

We attempt a rigorous proof of Penrose 1973 using weak inverse mean curvature flow with viscosity solution techniques. We identify precisely where each approach succeeds or fails, with explicit calculations.

Contents

1 Setup and Strategy

Goal: Prove $M_{\text{ADM}} \geq \sqrt{A(\Sigma_0)/(16\pi)}$ for any trapped surface Σ_0.

Given:

- (M^3, g, k) asymptotically flat initial data, DEC holds
- Σ_0 closed trapped surface: $\theta^+ = H +_\Sigma k \leq 0$, $\theta^- = H -_\Sigma k < 0$
- Hence $H = \frac{1}{2}(\theta^+ + \theta^-) < 0$ on Σ_0

The Obstruction: Standard IMCF $\partial_t \Sigma = \frac{\nu}{H}$ undefined when $H < 0$.

Strategy: Construct a *weak solution* that jumps over the $H < 0$ region.

2 Attempt 1: Elliptic Regularization

2.1 The Regularized Problem

Following Huisken-Ilmanen, we seek $u : M \setminus \Sigma_0 \to \mathbb{R}$ with level sets $\Sigma_t = \{u = t\}$ satisfying a regularized equation.

Standard IMCF equation: $\div\left(\frac{\nabla u}{|\nabla u|}\right) = |\nabla u|$

This is the level set formulation: if $\Sigma_t = \{u = t\}$, then $H = |\nabla u| \div (\nabla u/|\nabla u|)$ and velocity $= 1/H$, giving the equation.

Problem: When $H < 0$, we need $|\nabla u| < 0$ which is impossible.

2.2 Regularization with Sign Flip

Define the *signed regularization*:

$$\div\left(\frac{\nabla u}{\sqrt{|\nabla u|^2 + \epsilon^2}}\right) = \sqrt{|\nabla u|^2 + \epsilon^2} \tag{1}$$

Lemma 2.1 (Existence for Regularized Problem)**.** *For each $\epsilon > 0$, there exists a unique solution $u_\epsilon \in C^{2,\alpha}(M \setminus \Sigma_0)$ to (**??**) with $u_\epsilon|_{\Sigma_0} = 0$ and $u_\epsilon \to \infty$ at spatial infinity.*

Proof. This is a quasilinear elliptic equation with uniformly elliptic principal part (the ϵ^2 removes degeneracy). Standard theory (Gilbarg-Trudinger, Theorem 13.8) gives existence and regularity.

Ellipticity check: The linearization at u has principal symbol

$$a^{ij}(\nabla u) = \frac{\delta^{ij}}{\sqrt{|\nabla u|^2 + \epsilon^2}} - \frac{\partial_i u \partial_j u}{(|\nabla u|^2 + \epsilon^2)^{3/2}}$$

Eigenvalues: $\lambda_1 = (|\nabla u|^2 + \epsilon^2)^{-3/2}\epsilon^2 > 0$ (in ∇u direction), $\lambda_2 = \lambda_3 = (|\nabla u|^2 + \epsilon^2)^{-1/2}$ (perpendicular). Uniform ellipticity: $\lambda_{\min}/\lambda_{\max} \geq \epsilon^2/(|\nabla u|^2 + \epsilon^2) > 0$. □

2.3 The $\epsilon \to 0$ Limit

Proposition 2.2 (Compactness)**.** *There exists a subsequence $\epsilon_j \to 0$ and $u \in W^{1,1}_{\mathrm{loc}}(M \setminus \Sigma_0)$ such that $u_{\epsilon_j} \to u$ in L^1_{loc}.*

Proof. We need uniform bounds on u_ϵ and ∇u_ϵ.

Step 1: Maximum principle bound on u_ϵ. The maximum principle gives $0 \leq u_\epsilon \leq C \cdot \mathrm{dist}(\cdot, \Sigma_0)$ near Σ_0.

At infinity, the equation becomes approximately $\Delta u \approx |\nabla u|$, giving $u_\epsilon \sim \log r$ growth.

Step 2: Gradient bound. Multiply (**??**) by a test function ϕ^2 and integrate:

$$\int_M \phi^2 (|\nabla u|^2 + \epsilon^2)\, dV = \int_M \phi^2 \frac{\nabla u}{\sqrt{|\nabla u|^2 + \epsilon^2}} \cdot \nabla u\, dV + \text{boundary}$$

This gives $\int |\nabla u_\epsilon|^2 \leq C$ uniformly in ϵ.

Step 3: Compactness. By Rellich-Kondrachov, $W^{1,2} \hookrightarrow L^2$ compactly, giving the result. □

2.4 Critical Analysis: What is the Limit?

GAP (Limit Behavior Near $H < 0$ Region)*.* The limit u satisfies the IMCF equation weakly where $|\nabla u| > 0$. But what happens at points where $H < 0$?

Possibilities:

1. $|\nabla u| = 0$ on a set of positive measure (jump)
2. u is constant on the $H < 0$ region
3. The limit develops a discontinuity

The Problem: The regularized equation (**??**) has RHS $\geq \epsilon > 0$, so level sets always move outward. But the *rate* depends on the geometry.

In the $H < 0$ region, the regularized flow moves slowly (velocity $\sim 1/\epsilon$), and as $\epsilon \to 0$, the time to cross this region $\to \infty$.

Conclusion: Elliptic regularization does NOT naturally produce the jump we need.

3 Attempt 2: Parabolic IMCF with Jump Prescription

3.1 Modified Flow

Consider the flow:

$$\frac{\partial \Sigma_t}{\partial t} = \frac{\nu}{\max(H, \delta)} \tag{2}$$

where $\delta > 0$ is a cutoff.

Idea: When $H < \delta$, the surface moves at rate $1/\delta$. As $\delta \to 0$, surfaces with $H \approx 0$ (MOTS) become stationary while $H < 0$ surfaces move fast.

Lemma 3.1 (Short-time Existence)**.** *For $\delta > 0$ fixed, the flow* (**??**) *exists for short time $t \in [0, T_\delta)$.*

Proof. Standard parabolic theory. The velocity $V = 1/\max(H, \delta) \geq 1/\delta$ is bounded, so the flow is uniformly parabolic. □

3.2 Area Evolution

Lemma 3.2 (Area Formula)**.** *Along the modified flow:*

$$\frac{dA}{dt} = \int_{\Sigma_t} \frac{H}{\max(H, \delta)} \, dA \tag{3}$$

Proof. First variation: $\frac{dA}{dt} = \int H \cdot V \, dA = \int \frac{H}{\max(H,\delta)} \, dA$. □

Analysis of (**??**)**:**

- Where $H \geq \delta$: contribution is $\int_{H \geq \delta} 1 \, dA = A(H \geq \delta) > 0$
- Where $H < \delta$: contribution is $\int_{H<\delta} \frac{H}{\delta} \, dA$
  - If $H > 0$: positive contribution
  - If $H < 0$: **negative contribution** $= \frac{1}{\delta} \int_{H<0} H \, dA < 0$

GAP (Sign of Area Derivative)*.* For a trapped surface with $H < 0$ everywhere:

$$\frac{dA}{dt} = \frac{1}{\delta} \int_{\Sigma_t} H \, dA < 0$$

Area still decreases! The modification doesn't help.

4 Attempt 3: Null Flow Approach

4.1 The Null Mean Curvature Flow

Instead of spacelike IMCF, use *null* evolution. Given Σ_0, flow along the outgoing null direction ℓ^+ (the null normal with θ^+ as expansion).

Definition 4.1 (Null Flow)**.** *The null mean curvature flow evolves Σ_t by:*

$$\frac{\partial \Sigma}{\partial t} = \frac{\ell^+}{\theta^+} \tag{4}$$

when $\theta^+ < 0$.

Key difference from spacelike IMCF: The denominator is $\theta^+ = H +_\Sigma k$, not H alone.

4.2 Area Evolution Under Null Flow

Lemma 4.2 (Null Raychaudhuri). *Along the null flow (**??**):*

$$\frac{dA}{dt} = \int_{\Sigma_t} \frac{\theta^+}{\theta^+} \, dA - \textit{shear terms} = A(\Sigma_t) - \int \frac{|\sigma|^2}{\theta^+} \, dA \tag{5}$$

where σ is the null shear.

Wait, this isn't right. Let me recalculate.

Correct calculation: Under null flow with velocity $\phi = 1/\theta^+$ along ℓ^+:

$$\frac{dA}{dt} = \int_{\Sigma_t} \theta^+ \cdot \phi \, dA = \int_{\Sigma_t} 1 \, dA = A(\Sigma_t)$$

This looks promising! Area grows linearly if we use the null flow.

GAP (Existence of Null Flow). The flow (**??**) is singular when $\theta^+ = 0$ (MOTS). As the surface approaches the MOTS, $\theta^+ \to 0^-$ and velocity $\to -\infty$.

Problem: The flow accelerates to infinite speed as it approaches the MOTS, potentially overshooting or becoming ill-defined.

4.3 Viscosity Solution for Null Flow

Define the null arrival time function $\tau : M \to \mathbb{R}$ by:

$$\tau(p) = \inf\{t \geq 0 : p \in J^+(\Sigma_t)\}$$

where J^+ is the causal future.

Definition 4.3 (Viscosity Solution). *τ is a* viscosity solution *of the null flow if:*

1. *(Subsolution) At any point p where τ has a smooth upper contact ϕ:*

$$g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \leq 0 \quad \textit{(null or timelike)}$$

2. *(Supersolution) At any point where τ has a smooth lower contact: the analogous inequality.*

Theorem 4.4 (Existence of Viscosity Solution). *There exists a unique viscosity solution τ with $\tau|_{\Sigma_0} = 0$.*

Proof Sketch. Use Perron's method. Define:

$$\tau(p) = \sup\{v(p) : v \text{ is a subsolution}, v|_{\Sigma_0} \leq 0\}$$

Step 1: The function $\tau_0(p) =$ (Lorentzian distance from Σ_0 to p) is a subsolution.

Step 2: Comparison principle holds for the eikonal equation $g^{\mu\nu}\partial_\mu\tau\partial_\nu\tau = 0$.

Step 3: The supremum is a viscosity solution. □

GAP (Connection to Area). Even with a viscosity solution τ, we need to prove:

$$A(\{\tau = t\}) \geq A(\{\tau = 0\}) = A(\Sigma_0)$$

For smooth null flows, we showed $dA/dt = A$, giving $A(t) = A(0)e^t$. But for the viscosity solution:

1. Level sets $\{\tau = t\}$ may not be smooth
2. The "area" of a non-smooth set needs careful definition
3. Monotonicity may fail at jump points

5 Attempt 4: Capacity Method

5.1 The Key Insight

The Riemannian Penrose inequality (Bray, Huisken-Ilmanen) uses:

$$M_{\mathrm{ADM}} \geq \frac{C(\Sigma)}{4\pi}$$

where $C(\Sigma)$ is the capacity. For MOTS, $C(\Sigma^*) = \sqrt{4\pi A(\Sigma^*)}$.

Idea: Find a modified capacity $\tilde{C}$ such that:

1. $\tilde{C}(\Sigma_0) \geq \sqrt{4\pi A(\Sigma_0)}$ for trapped Σ_0
2. $M_{\mathrm{ADM}} \geq \tilde{C}(\Sigma_0)/(4\pi)$

5.2 Weighted Capacity

Definition 5.1 (Trapping-Weighted Capacity)**.**

$$\tilde{C}(\Sigma) := \inf_{u \in \mathcal{A}} \int_M w(x)^2 |\nabla u|^2 \, dV \tag{6}$$

where $\mathcal{A} = \{u : u|_\Sigma = 1, u \to 0 \text{ at } \infty\}$ *and* $w(x) = e^{-\psi(x)}$ *for some function* ψ *to be determined.*

Goal: Choose ψ so that $\tilde{C}(\Sigma) \geq \sqrt{4\pi A(\Sigma)}$.

5.3 Euler-Lagrange and Monotonicity

The minimizer u satisfies:

$$\div(w^2 \nabla u) = 0 \quad \text{in } M \setminus \Sigma$$

Lemma 5.2. *If* ψ *satisfies* $\Delta\psi \geq |\nabla\psi|^2 + f$ *for some* $f \geq 0$*, then* $\tilde{C}$ *has good monotonicity properties.*

GAP (Choice of Weight)*.* The natural choice related to trapping would be $\psi \sim \int \theta^+$, but:

1. θ^+ is only defined on surfaces, not in bulk
2. Any extension of θ^+ to bulk is non-canonical
3. The PDE for ψ may not have solutions with required sign

6 Attempt 5: Jang Equation with Different Blow-up

6.1 Review of Jang Approach

The Jang equation $H_{\mathrm{graph}(f)} -_{\mathrm{graph}(f)} k = 0$ has solutions that blow up at MOTS. The induced metric $\hat{g}$ on the graph satisfies:

$$R_{\hat{g}} \geq 2(\mu - J(\nu)) \geq 0 \quad \text{(DEC)}$$

The Problem: At blow-up surface Σ, the jump in mean curvature is:

$$[H_{\hat{g}}] = 2|_\Sigma k|$$

This has the WRONG SIGN when $_\Sigma k < 0$.

6.2 Modified Jang Equation

Consider the *dual Jang equation*:

$$H_{\mathrm{graph}(f)} +_{\mathrm{graph}(f)} k = 0 \tag{7}$$

This blows up where $\theta^- = H - k = 0$ (past MOTS).

Lemma 6.1. *Solutions to* (**??**) *blow up to* $-\infty$ *at surfaces where* $\theta^- = 0$.

Issue: Trapped surfaces have $\theta^- < 0$, so they are NOT blow-up surfaces of the dual Jang equation.

6.3 Interpolated Jang

Try interpolating:

$$H_{\mathrm{graph}(f)} - \lambda_{\mathrm{graph}(f)} k = 0 \tag{8}$$

for $\lambda \in [-1, 1]$.

Blow-up occurs where $H - \lambda k = 0$, i.e., where $\lambda = H/k$.

For a trapped surface: $\theta^+ = H + k \leq 0$ and $\theta^- = H - k < 0$.

If $k > 0$: $H/k < 1$ (from $\theta^- < 0$) and $H/k \leq -1$ (from $\theta^+ \leq 0$), so $\lambda = H/k \leq -1$.

If $k < 0$: $H/k > 1$ (from $\theta^- < 0$), so $\lambda = H/k > 1$.

GAP (No $\lambda \in [-1, 1)$. $Works]For a generic trapped surface, there is no \lambda \in [-1, 1]$ such that the λ-Jang equation blows up exactly at that surface.

7 Attempt 6: Direct Spacetime Argument

7.1 Setup

Embed (M, g, k) into spacetime $(N^4, \bar{g})$. Let $\Sigma_0 \subset M$ be trapped.

Assume: Weak cosmic censorship holds, so there exists event horizon $\mathcal{H}^+$.

Goal: Prove $A(\Sigma_0) \leq A(\mathcal{H}^+ \cap M)$.

7.2 Causal Argument

Lemma 7.1 (Penrose's Original Observation)**.** *If* Σ_0 *is trapped, then* $\Sigma_0 \subset \overline{J^-(\mathcal{H}^+)}$.

Proof. By definition, trapped surfaces cannot communicate with $\mathscr{I}^+$, hence lie in the black hole region $B = N \setminus J^-(\mathscr{I}^+)$. The event horizon $\mathcal{H}^+ = \partial B$, so $\Sigma_0 \subset B \subset \overline{J^-(\mathcal{H}^+)}$. □

7.3 Area Comparison

We want: $A(\Sigma_0) \leq A(\mathcal{H}^+ \cap M)$.

Attempt via null geodesics: Fire null geodesics from Σ_0 toward $\mathcal{H}^+$.

If we use outgoing null geodesics (along ℓ^+): - Initial expansion $\theta^+ \leq 0$ - Raychaudhuri: $\frac{d\theta^+}{d\lambda} = -\frac{1}{2}(\theta^+)^2 - |\sigma|^2 - R_{\mu\nu}\ell^\mu\ell^\nu$ - Under NEC: $\frac{d\theta^+}{d\lambda} \leq -\frac{1}{2}(\theta^+)^2$ - So θ^+ becomes more negative, area decreases.

GAP (Wrong Direction)*.* Outgoing null geodesics from a trapped surface go INTO the black hole, not toward the horizon. They don't reach $\mathcal{H}^+$.

Attempt via ingoing null geodesics (along ℓ^-): - Initial expansion $\theta^- < 0$ - These go toward the horizon, but θ^- is already negative - Area evolution: $\frac{dA}{d\lambda} = \int \theta^- \, dA < 0$ - Area DECREASES along ingoing null geodesics too!

GAP (Both Directions Fail). For a trapped surface, BOTH null expansions are negative, so area decreases in BOTH null directions. There is no direction in which area increases!

8 Attempt 7: Optimal Transport

8.1 Riemannian Optimal Transport Review

In Riemannian geometry, if $\mathrm{Ric} \geq (n-1)K$, then for probability measures μ_0, μ_1 on M:

$$W_2(\mu_0, \mu_1) \leq \text{(diameter bound)}$$

and entropy is convex along geodesics.

8.2 Lorentzian Optimal Transport

For spacetime $(N, \bar{g})$ with timelike Ricci bound, Mondino-Suhr define a Lorentzian Wasserstein distance using the cost $c(x, y) = -\tau(x, y)^2$ where τ is the time separation.

Theorem 8.1 (Mondino-Suhr, 2022). *If $(N, \bar{g})$ satisfies timelike curvature-dimension condition $TCD_p(K, N)$, then certain entropy functionals are convex along timelike geodesics.*

8.3 Application Attempt

Let $\mu_0 =$ uniform measure on Σ_0 (trapped surface). Let $\mu_1 =$ uniform measure on $\mathcal{H}^+ \cap M$ (horizon cross-section).

Idea: Use optimal transport to compare $A(\Sigma_0)$ and $A(\mathcal{H}^+ \cap M)$.

GAP (Technical Issues).
1. TCD requires **timelike** geodesics between supports. But Σ_0 and $\mathcal{H}^+ \cap M$ may not be connected by timelike geodesics (the horizon is null).
2. The curvature-dimension condition TCD requires something like SEC (strong energy condition), not just DEC or NEC.
3. The entropy functional in Lorentzian OT is not directly related to area.

9 Attempt 8: Variational Approach

9.1 The Maximum Area Problem

Consider:

$$A_{\max} := \sup\{A(\Sigma) : \Sigma \text{ trapped}, \Sigma \supset \Sigma_0\}$$

If the supremum is achieved by some $\Sigma_{\max}$, what can we say?

Lemma 9.1 (First Variation). *If $\Sigma_{\max}$ achieves the supremum among trapped surfaces, then:*

$$H = 0 \quad \textit{at any point where } \theta^+ = 0, \theta^- < 0$$

Proof. Vary in the direction $\phi\nu$. First variation of area: $\delta A = \int H\phi \, dA$. For $\Sigma_{\max}$ to be critical among trapped surfaces, we need $\delta A = 0$ for all variations preserving trappedness.

If $\theta^+ = 0$ (boundary of trapped condition) and we vary inward ($\phi < 0$), then θ^+ can increase (become positive), violating trappedness. So we can only vary outward at such points, giving $H \geq 0$. But $H = \frac{1}{2}(\theta^+ + \theta^-) = \frac{1}{2}\theta^- < 0$ at such points. Contradiction.

Thus $\Sigma_{\max}$ cannot have $\theta^+ = 0$ points interior to the trapped region. It must be a MOTS itself. □

GAP (Existence of Maximum). The supremum $A_{\max}$ may not be achieved!

Example: The trapped region may be unbounded, with surfaces of arbitrarily large area. Or the maximizing sequence may "escape to infinity" or develop singularities.

10 Synthesis: The Core Obstruction

After all attempts, the obstruction is clear:

Theorem 10.1 (Fundamental Obstruction)**.** *For a trapped surface Σ_0 with $\theta^+ < 0$ and $\theta^- < 0$ everywhere:*

1. *Any smooth outward evolution decreases area (since $H < 0$)*
2. *Any smooth inward evolution also decreases area (since $\theta^- < 0$)*
3. *Any null evolution decreases area in both directions*
4. *The Jang equation cannot produce favorable jump*

There is NO smooth flow that increases area starting from Σ_0.

10.1 What Would Be Needed

To prove Penrose 1973 unconditionally, one would need:

Option A: Discontinuous Flow A weak solution theory where:

- Area can "jump up" at certain instants
- The jump is controlled by the geometry
- The final area equals $A(\text{MOTS})$

Option B: Different Monotone Quantity A functional $F(\Sigma)$ such that:

- F is monotone along some flow
- $F(\Sigma_0)$ = something depending only on $A(\Sigma_0)$
- $F(\text{MOTS})$ = something giving $M_{\text{ADM}} \geq \sqrt{A/(16\pi)}$

Option C: Direct Argument A proof that bypasses flows entirely, perhaps using:

- Spinorial methods (Witten-style)
- Index theory
- Algebraic/topological arguments

11 A New Attempt: Conformal Flow

11.1 Idea

Instead of moving the surface, conformally change the metric to make $H > 0$.

Let $\tilde{g} = e^{2\phi} g$ for some function ϕ. The mean curvature transforms as:

$$\tilde{H} = e^{-\phi}(H + 2\partial_\nu \phi)$$

To make $\tilde{H} > 0$, we need $\partial_\nu \phi > -H/2 > 0$ (since $H < 0$).

11.2 The Conformal Factor

Solve:

$$\Delta \phi = f, \quad \phi|_\infty = 0, \quad \partial_\nu \phi|_{\Sigma_0} = -H/2 + \epsilon \tag{9}$$

This is a mixed boundary value problem.

Lemma 11.1. *There exists ϕ solving (**??**) for appropriate f.*

11.3 Effect on Mass

Under conformal change $\tilde{g} = e^{2\phi} g$, the ADM mass transforms.

GAP (Mass Change)*.* The ADM mass is NOT conformally invariant.

$$\tilde{M} = M + (\text{terms involving } \phi)$$

If ϕ is significant near infinity, the mass can increase or decrease arbitrarily.

12 Final Assessment

Status after hard analysis:

All eight approaches encounter fundamental obstructions:

Approach	**Obstruction**
Elliptic regularization	Limit doesn't produce jump
Parabolic IMCF	Area still decreases
Null flow	Singular at MOTS, direction issues
Capacity method	No canonical weight function
Modified Jang	No λ produces blow-up at trapped surface
Spacetime causal	Both null directions decrease area
Optimal transport	TCD needs SEC, not applicable
Variational	Existence of maximum unproven
Conformal	Changes mass, not helpful

Honest Conclusion:

The 1973 Penrose conjecture for arbitrary trapped surfaces remains **OPEN**. The fundamental issue is geometric: trapped surfaces have $H < 0$, and no known technique can overcome this without additional assumptions.

The most promising directions for future work:

1. Viscosity solutions for null flows with rigorous existence theory
2. New monotone quantities beyond area
3. Spinorial methods avoiding flows entirely