Conceptual Inventions for the Spacetime Penrose Inequality

New Mathematical Frameworks to Solve 1973

Research Notes

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Abstract

We present ten conceptual inventions—new mathematical frameworks and ideas—that could potentially provide an unconditional proof of Penrose's 1973 conjecture. Each invention addresses the fundamental obstruction: the sign of $\mathrm{tr}_\Sigma k$ is not determined by the trapped condition. We assess feasibility and identify key technical challenges.

Contents

1 The Fundamental Obstruction

The Sign Problem

For a trapped surface Σ_0 with $\theta^+ \leq 0$ and $\theta^- < 0$:

- The Jang method requires $[H] = \mathrm{tr}_\Sigma k \geq 0$ (favorable jump)
- The trapped condition only gives: $\theta^+ + \theta^- = 2H < 0$
- The $\mathrm{tr}_\Sigma k$ terms **cancel** in H, leaving the sign undetermined
- When $\mathrm{tr}_\Sigma k < 0$, the distributional scalar curvature contains a **negative** Dirac mass

All existing approaches reduce to this obstruction.

Remark 1.1 (What We Need)**.** *A successful conceptual invention must either:*

(A) Find a ***different monotonicity formula*** *that doesn't require* $\mathrm{tr}_\Sigma k \geq 0$

(B) Prove that $\mathrm{tr}_\Sigma k \geq 0$ *is* ***automatic*** *for some reason*

(C) Find a ***geometric quantity*** *that compensates for negative* $\mathrm{tr}_\Sigma k$

(D) Construct an ***auxiliary surface*** *with favorable properties*

(E) Use ***spacetime structure*** *to bypass the initial data obstruction*

2 Invention 1: The Trapping Product Functional

Key Observation

The product $\theta^+\theta^-$ is **always positive** for trapped surfaces and **vanishes exactly on MOTS**. This is independent of the sign of $\mathrm{tr}_\Sigma k$.

Conceptual Invention 1 (Trapping Product Monotonicity)**.** *Define the **trapping mass functional**:*

$$\mathcal{M}_T(\Sigma) := \sqrt{\frac{A(\Sigma)}{16\pi}} \cdot \exp\left(-\frac{1}{8\pi}\int_\Sigma \sqrt{\theta^+\theta^-}\, dA\right) \tag{1}$$

Conjecture: *Under appropriate flow, $\mathcal{M}_T$ is monotonically increasing from Σ_0 to Σ^*, and $\mathcal{M}_T(\Sigma^*) = \sqrt{A(\Sigma^*)/(16\pi)}$ since $\theta^+ = 0$ on MOTS.*

Proposition 2.1 (Why This Might Work)**.**

1. *The exponential factor $e^{-\int\sqrt{\theta^+\theta^-}}$ penalizes deeply trapped surfaces*
2. *On approach to MOTS, $\theta^+ \to 0$, so the penalty vanishes*
3. *The product structure is symmetric in $\theta^\pm$, avoiding the sign ambiguity*

Technical Challenge: Need a flow under which this functional is monotone. The natural candidate is $\partial_t X = \frac{\nu}{\sqrt{\theta^+\theta^-}}$ but this is singular at MOTS.

Feasibility: ★★★ (Medium) — Requires developing new flow theory.

3 Invention 2: The Dual Jang Equation

Duality Observation

The standard Jang equation blows up where $\theta^+ = 0$. Consider the **dual** equation that blows up where $\theta^- = 0$ instead.

Conceptual Invention 2 (Dual Jang Reduction)**.** *Define the **dual Jang equation**:*

$$\mathcal{J}^*(f^*) := H_{\Gamma^*} + \mathrm{tr}_{\Gamma^*}(k) = 0 \tag{2}$$

*where $\Gamma^* = \{(x, f^*(x))\}$ is the graph with **past-pointing** normal.*

Key property: *$\mathcal{J}^*(f^*) = \theta^-$ (the ingoing null expansion), so:*

- *Blows up at surfaces where $\theta^- = 0$ (past MOTS)*
- *For trapped surfaces, $\theta^- < 0 \neq 0$, so no blow-up inside trapped region*

Proposition 3.1 (Combined Jang System)**.** *Consider the **pair** (f, f^*) solving:*

$$\mathcal{J}(f) = \theta^+ = 0 \quad \text{(standard Jang)} \tag{3}$$

$$\mathcal{J}^*(f^*) = \theta^- = 0 \quad \text{(dual Jang)} \tag{4}$$

*The **average** $\bar{f} = \frac{1}{2}(f + f^*)$ may have better properties:*

$$\mathcal{J}(\bar{f}) + \mathcal{J}^*(\bar{f}) = \theta^+ + \theta^- = 2H \tag{5}$$

This connects to mean curvature, which has definite sign for trapped surfaces.

Technical Challenge: The dual Jang equation needs existence theory. The boundary conditions and blow-up analysis differ from standard Jang.

Feasibility: ★★★★ (High) — Natural extension of existing theory.

4 Invention 3: The Compensated Scalar Curvature

Compensation Principle

Instead of requiring $R \geq 0$ pointwise, require $R + \mathcal{C} \geq 0$ where $\mathcal{C}$ is a **compensating term** constructed from the geometry.

Conceptual Invention 3 (Scalar Curvature with Trapping Compensation)**.** *Define the* ***compensated scalar curvature****:*

$$R_{\text{comp}} := R + \lambda \cdot |\theta^+\theta^-| \cdot \delta_\Sigma \tag{6}$$

where $\lambda > 0$ is chosen so that the negative contribution from $\text{tr}_\Sigma k < 0$ is exactly compensated by the positive contribution from $|\theta^+\theta^-|$.

Proposition 4.1 (Compensation Formula)**.** *On the Jang manifold near Σ:*

$$R_{\bar{g}} = R_{\bar{g}}^{\text{reg}} + 2\,\text{tr}_\Sigma k \cdot \delta_\Sigma \tag{7}$$

If $\text{tr}_\Sigma k < 0$, choose:

$$\lambda = \frac{2|\text{tr}_\Sigma k|}{|\theta^+\theta^-|} \tag{8}$$

Then $R_{\text{comp}} \geq 0$ distributionally.

Technical Challenge: The compensation introduces a new term that must be accounted for in the mass. Need to show this doesn't increase ADM mass.

Feasibility: ★★ (Low-Medium) — Compensation may introduce other problems.

5 Invention 4: The Causal Isoperimetric Inequality

Causal Structure

The Riemannian isoperimetric inequality bounds volume by area. In Lorentzian geometry, the analogous bound should involve **causal structure**.

Conceptual Invention 4 (Causal Isoperimetric Conjecture)**.** *Let Σ be a trapped surface in asymptotically flat spacetime $(N, \mathbf{g})$ satisfying DEC. Let $\mathcal{D}(\Sigma) = J^+(\Sigma) \cap J^-(I^+)$ be the causal future of Σ intersected with the past of future null infinity.*

Conjecture: *There exists a* ***causal isoperimetric inequality****:*

$$\text{Vol}_4(\mathcal{D}(\Sigma)) \leq C \cdot M_{\text{ADM}}^2 \cdot A(\Sigma) \tag{9}$$

with equality iff the spacetime is Schwarzschild.

Proposition 5.1 (Connection to Penrose Inequality)**.** *If the causal isoperimetric inequality holds, then by taking Σ to be the event horizon cross-section (where $\mathcal{D}(\Sigma)$ is maximized), we get a bound relating $A(\Sigma)$ and M_{ADM}.*

Technical Challenge: Causal structure is highly non-local. The spacetime volume $\text{Vol}_4(\mathcal{D})$ depends on the entire future evolution.

Feasibility: ★★ (Low-Medium) — Requires global spacetime analysis.

6 Invention 5: The Entropic Mass

Information Theory

Black hole entropy $S = A/(4\ell_P^2)$ connects area to information. The Penrose inequality might be an **entropic inequality**.

Conceptual Invention 5 (Entropic Mass Functional). *Define the* ***entropic mass*** *of a surface* Σ*:*

$$M_{\text{ent}}(\Sigma) := \sqrt{\frac{A(\Sigma)}{16\pi}} \cdot \exp\left(\frac{S_{\text{rel}}(\Sigma)}{A(\Sigma)}\right) \tag{10}$$

where $S_{\text{rel}}(\Sigma)$ *is a* ***relative entropy*** *measuring the departure from spherical symmetry:*

$$S_{\text{rel}}(\Sigma) := \int_\Sigma \left(\frac{|\theta^+|^2 + |\theta^-|^2}{H^2} - 1\right) dA \tag{11}$$

Proposition 6.1 (Properties).

1. *$S_{\text{rel}} \geq 0$ by Cauchy-Schwarz (equality iff $\theta^+ = \theta^- = H$, i.e., time-symmetric)*
2. *For MOTS: $\theta^+ = 0$ gives $S_{\text{rel}} = \int |\theta^-|^2/H^2 \, dA \geq 0$*
3. *The exponential factor rewards surfaces closer to equilibrium*

Technical Challenge: Need to connect M_{ent} to ADM mass and establish monotonicity.
Feasibility: ★★★ (Medium) — Interesting connection to quantum information.

7 Invention 6: The Symplectic Area

Phase Space Structure

The space of surfaces in a spacetime has a natural symplectic structure. Area might be related to a **symplectic invariant**.

Conceptual Invention 6 (Symplectic Penrose Inequality). *On the phase space $\mathcal{P}$ of closed surfaces in initial data (M, g, k), define the symplectic form:*

$$\omega(\delta_1\Sigma, \delta_2\Sigma) := \int_\Sigma (\delta_1 H \cdot \delta_2\phi - \delta_2 H \cdot \delta_1\phi) \, dA \tag{12}$$

where $\delta_i\Sigma$ are normal variations with speeds ϕ_i.

Conjecture: *The trapped region $\mathcal{T} \subset \mathcal{P}$ is a Lagrangian submanifold, and:*

$$\text{Vol}_\omega(\mathcal{T}) \leq C \cdot M_{\text{ADM}}^4 \tag{13}$$

Technical Challenge: The symplectic structure on infinite-dimensional spaces is delicate. Need to make this rigorous.

Feasibility: ★ (Low) — Highly speculative, unclear path forward.

8 Invention 7: The Bootstrap Flow

Self-Improving Estimates

In many PDEs, weak solutions satisfy better estimates than initially expected. Could there be a flow that **improves the sign** of $\mathrm{tr}_\Sigma k$?

Conceptual Invention 7 (Sign-Improving Flow)**.** *Consider the flow:*

$$\frac{\partial \Sigma}{\partial t} = (\mathrm{tr}_\Sigma k)^- \cdot \nu \tag{14}$$

where $f^- = \min(f, 0)$ is the negative part. This flow:

- *Moves surfaces outward only where $\mathrm{tr}_\Sigma k < 0$*
- *Remains stationary where $\mathrm{tr}_\Sigma k \geq 0$*
- *Attempts to "cure" the unfavorable sign by deformation*

Proposition 8.1 (Bootstrap Mechanism)**.** *If the flow exists and converges to a limit Σ_∞, then either:*

1. *$\mathrm{tr}_{\Sigma_\infty} k \geq 0$ everywhere (favorable!)*
2. *Σ_∞ is a MOTS (where $\theta^+ = 0$ so $H = \mathrm{tr}_\Sigma k$)*

In either case, the Jang method applies to Σ_∞.

Technical Challenge: Flow may develop singularities or not preserve the trapped condition.
Feasibility: ★★★ (Medium) — Worth investigating existence theory.

9 Invention 8: The Double Bubble Construction

Two-Surface Approach

Instead of one surface, consider a **pair** of surfaces that together satisfy a combined inequality.

Conceptual Invention 8 (Double Trapped Surface)**.** *Given a trapped surface Σ_0 with $\mathrm{tr}_{\Sigma_0} k < 0$, construct a companion surface Σ_0' such that:*

$$\mathrm{tr}_{\Sigma_0} k + \mathrm{tr}_{\Sigma_0'} k \geq 0 \tag{15}$$

and $A(\Sigma_0) + A(\Sigma_0') \leq A(\Sigma_0) + \epsilon$ for small ϵ.

Candidate: *Let Σ_0' be the image of Σ_0 under the time-reflection isometry (if it exists). Then $k \mapsto -k$ so $\mathrm{tr}_{\Sigma'} k = -\mathrm{tr}_\Sigma k$.*

Proposition 9.1 (Combined Inequality)**.** *If (M, g, k) admits a time-reflection symmetry, then for any trapped surface Σ:*

$$M_{\mathrm{ADM}} \geq \sqrt{\frac{A(\Sigma) + A(\Sigma')}{32\pi}} = \sqrt{\frac{A(\Sigma)}{16\pi}} \tag{16}$$

using $A(\Sigma') = A(\Sigma)$ by symmetry.

Technical Challenge: Most spacetimes don't have time-reflection symmetry. Need an approximate version.

Feasibility: ★★★ (Medium) — Works for symmetric cases, generalization unclear.

10 Invention 9: The Renormalized Area

Renormalization

In quantum field theory, divergent quantities are made finite by **renormalization**. The "bad sign" of $\mathrm{tr}_\Sigma k$ might be a "UV divergence" that can be subtracted.

Conceptual Invention 9 (Renormalized Area)**.** *Define the* ***renormalized area****:*

$$A_{\mathrm{ren}}(\Sigma) := A(\Sigma) - \frac{1}{\kappa}\int_\Sigma (\mathrm{tr}_\Sigma k)^- \, dA \tag{17}$$

where $\kappa > 0$ is a "renormalization scale" and $f^- = \min(f, 0)$.

Properties:

- *$A_{\mathrm{ren}} \leq A$ with equality iff $\mathrm{tr}_\Sigma k \geq 0$ everywhere*
- *The subtracted term is exactly the "bad part" causing the sign problem*

Conjecture 10.1 (Renormalized Penrose Inequality)**.** *For appropriate choice of κ (possibly depending on M_{ADM}):*

$$M_{\mathrm{ADM}} \geq \sqrt{\frac{A_{\mathrm{ren}}(\Sigma)}{16\pi}} \tag{18}$$

This would imply the original Penrose inequality since $A_{\mathrm{ren}} \leq A$.

Technical Challenge: Need to determine the correct κ and prove the renormalized inequality.
Feasibility: ★★★★ (High) — Natural modification, may be provable.

11 Invention 10: The Null Brane Action

String Theory Inspiration

In string theory, branes sweep out worldvolumes with actions. A trapped surface is the spatial section of a **null hypersurface** (the horizon). The Penrose inequality might be a **brane energy bound**.

Conceptual Invention 10 (Null Hypersurface Action)**.** *Let $\mathcal{N}$ be the outgoing null hypersurface from Σ. Define the* ***null brane action****:*

$$S_{\mathrm{null}}[\mathcal{N}] := \int_{\mathcal{N}} \left(1 - \frac{\theta^+}{|\theta^-|}\right) \sqrt{|\det h|}\, d^3x \tag{19}$$

where h is the induced degenerate metric on $\mathcal{N}$.

Observation: *For trapped surfaces, $\theta^+ \leq 0$ and $\theta^- < 0$, so:*

$$1 - \frac{\theta^+}{|\theta^-|} = 1 + \frac{\theta^+}{\theta^-} \geq 0 \tag{20}$$

The action is non-negative!

Conjecture 11.1 (Null Brane Bound)**.** *Under the null energy condition:*

$$S_{\text{null}}[\mathcal{N}] \geq c \cdot A(\Sigma) \tag{21}$$

for some universal $c > 0$. Combined with a bound $S_{\text{null}} \leq C \cdot M_{\text{ADM}}^2$, this yields the Penrose inequality.

Technical Challenge: Null hypersurfaces can develop caustics. Need to handle singularities.
Feasibility: ★★★ (Medium) — Interesting physics, needs rigorous formulation.

12 Comparison and Assessment

#	Invention	Approach	Feasibility	Novelty
1	Trapping Product	(A) New monotonicity	★★★	High
2	Dual Jang	(D) Auxiliary surface	★★★★	Medium
3	Compensated R	(C) Compensation	★★	Medium
4	Causal Isoperimetric	(E) Spacetime	★★	High
5	Entropic Mass	(A) New monotonicity	★★★	High
6	Symplectic Area	(A) New monotonicity	★	Very High
7	Bootstrap Flow	(B) Automatic sign	★★★	Medium
8	Double Bubble	(D) Auxiliary surface	★★★	Medium
9	Renormalized Area	(C) Compensation	★★★★	Medium
10	Null Brane Action	(E) Spacetime	★★★	High

12.1 Most Promising Directions

Top 3 Candidates

1. Dual Jang Equation (Invention 2): Natural extension of existing machinery. The dual equation $\mathcal{J}^* = \theta^-$ provides a complementary perspective. The average $\bar{f} = (f + f^*)/2$ connects to mean curvature H, which has definite sign.
2. Renormalized Area (Invention 9): Direct attack on the problem. By subtracting the "bad part" $(\text{tr}_\Sigma k)^-$, we get a modified area that should satisfy a cleaner inequality.
3. Trapping Product Monotonicity (Invention 1): Uses the key observation that $\theta^+\theta^- > 0$ for trapped surfaces. The product structure avoids the sign ambiguity entirely.

13 Research Program

13.1 Phase 1: Dual Jang (3 months)

1. Develop existence theory for $\mathcal{J}^*(f^*) = 0$
2. Analyze blow-up behavior (at $\theta^- = 0$ surfaces)

3. Study combined system (f, f^*) and average $\bar{f}$
4. Derive scalar curvature formula for dual Jang metric

13.2 Phase 2: Renormalized Area (3 months)

1. Determine optimal renormalization scale κ
2. Prove renormalized Penrose inequality for favorable cases
3. Study behavior under geometric flows
4. Connect to ADM mass via divergence identities

13.3 Phase 3: Trapping Product Flow (6 months)

1. Define weak solutions for $\partial_t X = (\theta^+\theta^-)^{-1/2}\nu$
2. Prove short-time existence
3. Analyze approach to MOTS (where $\theta^+ \to 0$)
4. Establish monotonicity of $\mathcal{M}_T$

14 Conclusion

The fundamental obstruction—that $\mathrm{tr}_\Sigma k$ can be negative for trapped surfaces—has blocked progress on the unconditional 1973 conjecture for 50+ years. The conceptual inventions above represent new angles of attack:

- **Dual Jang** uses the complementary null expansion θ^-
- **Renormalized Area** surgically removes the problematic contribution
- **Trapping Product** uses the symmetric quantity $\theta^+\theta^-$

None of these is guaranteed to work, but they represent **genuinely new ideas** rather than variations of existing approaches. The 1973 conjecture likely requires such a conceptual breakthrough.

Remark 14.1 (Final Thought)**.** *Penrose's original argument assumed cosmic censorship. Perhaps the truly unconditional inequality requires incorporating quantum effects (Hawking radiation, entanglement entropy). The Penrose inequality might be a* ***classical limit*** *of a more fundamental quantum bound.*