

2. Implicit function theorems and applications

2.1 Implicit functions

The implicit function theorem is one of the most useful single tools you'll meet this year. After a while, it will be second nature to think of this theorem when you want to figure out how a change in variable x affects variable y .

First a reminder of what an implicit function looks like. A *non*-implicit function (an explicit function?) is the kind of thing you're accustomed to, which looks like:

$$y = f(x)$$

and you call y the dependent variable and x the independent variable. But you could also express this same relationship as a zero of the implicit function:

$$g(x, y) = y - f(x) = 0$$

The pairs of x and y which satisfy the first relationship will also satisfy the second relationship. The reason for calling it an “implicit” function is that it doesn't say outright that y depends on x —but it *is* a function: as you vary x , you have to vary y as well, in order to maintain the “equals zero” relationship.

To give it a definition, an **implicit function** of x and y is simply any relationship that takes the form:

$$g(x, y) = 0$$

Any explicit function can be changed into an implicit function using the trick above, just setting $g(x, y) = y - f(x) = 0$. In theory, any implicit function could be converted into an explicit function by solving for y in terms of x . In practice, this may be rather challenging, though. Consider:

$$\ln(x + y) + xy - 12 = 0$$

Try as hard as you like, you'll never be able to isolate either of the variables. This is one motivation for working with implicit functions.

Another motivation is that we often work with general functions, rather than a particular functional form. For instance, once can generally state the FOC from a typical utility maximization problem in this form:

$$u(\bar{x})/x_i - p_i = 0$$

It would be desirable to talk about properties of demands without assuming that the utility function is Cobb-Douglas, CES, or anything in particular—a general form would encompass all these cases.

The final reason to learn how to work with implicit functions is that implicit function naturally arise in economics. Every time we do a constrained optimization problem, we end up with some condition set equal to zero. This is already an implicit function. Why try to solve it for one variable in terms of the others, when we don't need to? These are the sorts of things that we will be asking from the implicit function theorem:

Example 2.1.1: *The first order condition from a utility maximization problem is $u(w_1 - s) + (1 + r)u(w_2 + (1 + r)s) = 0$. Find ds/dr .*

Example 2.1.2: *The first order conditions from a utility maximization problem are:*

$$\begin{aligned} x_1^{-1} x_2^{-1} - p_1 &= 0 \\ (1 - \alpha) x_1^{-1} x_2^{-1} - p_2 &= 0 \end{aligned}$$

Find all the partial derivatives of the demand functions, x_i/p_j .

2.2 The implicit function theorem (two variable case)

When we have an implicit function of the form $g(x, y) = 0$, $x, y \in \mathbb{R}^1$, the **implicit function theorem** says that we can figure out dy/dx quite easily.¹ Provided that we have some continuity and a non-zero denominator, this derivative is:

$$g(x, y) = 0 \quad \frac{dy}{dx} = -\frac{g'_x}{g'_y}$$

Remember that for dy/dx , the *denominator* of the right-hand side is g'_y . In other words, whatever variable is on top on the left is the derivative which is on bottom on the right. One trick to remember this is to pretend that you can cancel out the -whatever terms, so that you get:

$$\frac{g'_x}{g'_y} = \frac{g'_x}{g'_y} = \frac{1}{1} \frac{x}{y} = \frac{y}{x}$$

This is *not* how the implicit function theorem works, though. It's just a useful trick for checking what goes where.

¹ At least, you can figure it out quite easily provided that you know which derivative you're trying to find—these variables aren't going to be called x and y in the problems you encounter, they'll be s and r and p and h and w . Half the time, the question won't ask, "Find ds/dr " as in Example 2.1.1—instead it'll be worded, "Find how much savings changes when the interest rate changes." It'll be up to you to figure out what that means.

So how does the implicit function theorem work? Again, an implicit function is like a normal function in that y must change as x changes, in order to maintain the relationship. You can think of $y(x)$. The function is really just a function of one independent variable, like this:

$$g(x, y(x)) = 0$$

There's a y in there, but not really—the value of y is dictated by x . By totally differentiating this function with respect to x , we get:

$$\begin{aligned}\frac{d}{dx}[g(x, y(x))] &= \frac{d}{dx}[0] \\ \frac{g}{x} + \frac{g}{y} y'(x) &= 0 \\ \frac{dy}{dx} = y'(x) &= -\frac{g/x}{g/y}\end{aligned}$$

And thus, we get the implicit function theorem by doing nothing more than treating y as a function of x and totally differentiating. An alternative way of deriving this result is to just take the function $g(x, y)$, and write its change in differential form. Since we know that 0 is unchanging, this form is:

$$\begin{aligned}\frac{g}{y} dy + \frac{g}{x} dx &= 0 \\ \frac{g}{y} dy &= -\frac{g}{x} dx \\ \frac{dy}{dx} &= -\frac{g/x}{g/y}\end{aligned}$$

The derivation of the implicit function theorem is quick and simple, and it might be worth remembering in case you can't remember the theorem itself.

2.3 Multivariate versions of the implicit function theorem

When y is an implicit function of many variables. Now we're talking about implicit functions that look like:

$$g(x_1, x_2, \dots, x_n, y) = 0$$

It turns out that nothing really changes for these functions. If we write out the differential form of this function, we get:

$$\frac{g}{x_1} dx_1 + \frac{g}{x_2} dx_2 + \dots + \frac{g}{x_n} dx_n + \frac{g}{y} dy = 0$$

By setting all the $dx_i = 0$ except for one dx_ℓ , we get the partial derivative of y with respect to x_ℓ . This expression turns out to be much the same as for the single- x case:

$$\frac{y}{x_\ell} = - \frac{g / x_\ell}{g / y}$$

When there are many y variables and many x variables. Let's say that we have m variables that we call y_1, y_2, \dots, y_m , and that these variables are implicitly a function of some n variables, called x_1, x_2, \dots, x_n . It will take m equations to describe the entire system of y variables, like:

$$\begin{aligned} g_1(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m) &= 0 \\ g_2(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m) &= 0 \\ &\vdots \\ g_m(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m) &= 0 \end{aligned}$$

There are several ways to express the implicit function theorem in this form. One is to imagine that those m implicit functions are a single vector-valued function $g: \mathbb{R}^{n+m} \rightarrow \mathbb{R}^m$, such that:

$$g(\bar{x}, \bar{y}) = 0$$

If we write this in differential form, we get that:

$$\underbrace{D_{\bar{x}}(\bar{x}, \bar{y})}_{m \times n} \underbrace{d\bar{x}}_{n \times 1} + \underbrace{D_{\bar{y}}(\bar{x}, \bar{y})}_{m \times m} \underbrace{d\bar{y}}_{m \times 1} = \underbrace{0}_{m \times 1}$$

(You can verify that the dimensions make sense, right?) You can sort of solve this equation to get one multivariate formulation of the theorem:

$$d\bar{y} = -D_{\bar{y}}g(\bar{x}, \bar{y})^{-1} D_{\bar{x}}g(\bar{x}, \bar{y}) d\bar{x}$$

Alternatively, you could think about only the partial derivatives of y with respect to one of the x variables. To find this out, you do:

$$\frac{y_1}{x_i}, \frac{y_2}{x_i}, \dots, \frac{y_m}{x_i}^T = -D_{\bar{y}}g(\bar{x}, \bar{y})^{-1} D_{x_i}g(\bar{x}, \bar{y})$$

Finally, we can use a version of **Cramer's rule** to solve systems of implicit functions. Suppose that we're given:

$$\begin{aligned} g_1(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m) &= 0 \\ g_2(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m) &= 0 \\ &\vdots \\ g_m(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m) &= 0 \end{aligned}$$

and we want to know one partial derivative, like y_k / x_ℓ . The differential form of this function looks like:

$$\begin{aligned} \frac{g_1}{y_1} dy_1 + \dots + \frac{g_1}{y_m} dy_m + \frac{g_1}{x_1} dx_1 + \dots + \frac{g_1}{x_n} dx_n &= 0 \\ \vdots & \\ \frac{g_m}{y_1} dy_1 + \dots + \frac{g_m}{y_m} dy_m + \frac{g_m}{x_1} dx_1 + \dots + \frac{g_m}{x_n} dx_n &= 0 \end{aligned}$$

Now, the idea of a partial derivative is that only x_ℓ changes and none of the other x variables; but this can still mean that lots of y variables change around, not just y_k . So we set $dx_i = 0$ for all $i \neq \ell$ (which means that we're really talking about a *partial* derivative at this point—holding everything constant except dx_ℓ . I'll change the notation to partial derivatives). This results in a system of m equations in m unknowns—these unknowns are the y_j / x_ℓ :

$$\begin{aligned} \frac{g_1}{y_1} \frac{y_1}{x_\ell} + \dots + \frac{g_1}{y_m} \frac{y_m}{x_\ell} &= -\frac{g_1}{x_\ell} \\ \vdots & \\ \frac{g_m}{y_1} \frac{y_1}{x_\ell} + \dots + \frac{g_m}{y_m} \frac{y_m}{x_\ell} &= -\frac{g_m}{x_\ell} \end{aligned}$$

(See how it's a system of linear equations?) There are several ways that we can solve this system of equations—one of them being Cramer's rule. Recall that if we want to solve for the unknown variable y_k / x_ℓ on the left-hand side, we establish the $m \times m$ matrix of coefficients on the unknown y_j / x_ℓ variables—this turns out to be the same as the matrix of first derivatives, $D_y g(\bar{x}, \bar{y})$. Cramer's rule says that y_k / x_ℓ is equal to a fraction, the denominator of which is $\det(D_y g(\bar{x}, \bar{y}))$, the numerator of which is the determinant of the same matrix with the k -th column replaced by the vector on the right-hand side, or $-g / x_\ell$. More clearly:

$$\frac{y_k}{x_\ell} = - \frac{\det \begin{pmatrix} \frac{g_1}{y_1} & \dots & \frac{g_1}{x_\ell} & \dots & \frac{g_1}{y_m} \\ \vdots & & \vdots & & \vdots \\ \frac{g_m}{y_1} & \dots & \frac{g_m}{x_\ell} & \dots & \frac{g_m}{y_m} \end{pmatrix}}{\det \begin{pmatrix} \frac{g_1}{y_1} & \dots & \frac{g_1}{y_1} & \dots & \frac{g_1}{y_m} \\ \vdots & & \vdots & & \vdots \\ \frac{g_m}{y_1} & \dots & \frac{g_m}{y_m} & \dots & \frac{g_m}{y_m} \end{pmatrix}}$$

That's the formula to remember for the multivariate version of the implicit function theorem.

Note that the minus sign in front of the $-g / x_\ell$ in the system of equations above has turned into a minus sign out in front of the fraction. This comes because we've changed the signs of every element in a row of the matrix in the numerator; there's a rule which says that multiplying every element of one row or column of a matrix by some scalar means that the determinant of the matrix also changes by the same. In this case, -1 .

2.4 Applications and exercises

Some applications of the implicit function theorem show up in micro, but most of them are in macro. Here are some problems that you will face:

Exercise 2.4.1: For the utility function $u(x_1, x_2)$, a particular vector (x_1, x_2) will put you on some indifference curve \bar{u} . Find the slope of this curve at (x_1, x_2) .

Step 1: Figure out what derivative you want.

Step 2: Define an implicit function of these variables.

Step 3: Apply the implicit function theorem.

Exercise 2.4.2: The individual lives for two periods. He has a utility function $u(c_1) + \beta u(c_2)$. His budget constraint requires that his period 1 consumption be his period 1 endowment minus any savings, $c_1 = w_1 - s$. In the second period, his consumption will be $c_2 = w_2 + (1 + r)s + T$. The government has taxed savings at the rate $1 - \tau$, and uses it to finance a lump-sum transfer of T . The individual takes w_1 and w_2 as given, but for the government the government's budget to balance, we must have $T = (1 - \tau)(1 + r)s$. Find $ds/d\tau$.

Step 1: Set up the utility maximization problem.

Step 2: Find FOCs. These should define an implicit function.

Step 3: **Insert the GBC, after taking the FOC.**

Step 4: Apply the implicit function theorem.

Exercise 2.4.3: People live for two periods (in overlapping generations), and the utility function is the same as before. The government has implemented a pay-as-you-go social security system. The way this works is that every young person pays a lump-sum tax of τ , and every old person collects a pension of τ . This means that $c_1 = w_1 - \tau$ and $c_2 = w_2 + (1 + r)s + \tau$. The size of each cohort (or generation) increases at rate n , which means that $N_{t+1} = (1 + n)N_t$, where N_t represents the number of people born at time t . Find $ds/d\tau$.

Step 1: Set up the utility maximization problem.

Step 2: Figure out what the GBC is supposed to be.

Step 3: Find FOCs.

Step 4: Insert the GBC.

Step 5: Apply the implicit function theorem.

Exercise 2.4.4: The individual lives for two periods. His utility function depends on consumption at time 1 and consumption at time 2, $u(c_1, c_2)$. He is endowed with w_1 when young and w_2 when old. He can choose savings s at $t = 1$, which get a gross rate of return $1 + r$. For each of the following utility functions, find s/r .

- a. $u(c_1, c_2) = \ln c_1 + \ln c_2$.
- b. $u(c_1, c_2) = \ln c_1 + c_2$.
- c. $u(c_1, c_2) = (1 - \alpha)^{-1} \{ c_1^{1-\alpha} + c_2^{1-\alpha} \}$, and $w_2 = 0$.
- d. $u(c_1, c_2) = \exp(-c_1) + \exp(-c_2)$, and $w_2 = 0$.

Step 1: Set up the utility maximization problem.

Step 2: Find FOCs.

Step 3: **Evaluate:** is it easier to solve for s , or to use the IFT?

Step 4: Do it.

If you have some extra time, you might want to try solving those problems using both methods—solving for s directly, as well as using the implicit function theorem. Which way is easier? What's different about the way the answers look? Can you reconcile them?

Exercise 2.4.5: The individual has a utility function represented by $u(x_1, x_2) = x_1 x_2^{1-\alpha}$. Using the first order conditions, find x_i/p_i . Now solve the demand functions, and take the same derivatives. Do they match up? Why or why not?

Exercise 2.4.6: Same thing, except that the utility function is now $u(x_1, x_2) = (x_1 + x_2)^{1/\alpha}$.