Triangular Input Balance, Meixner Functions, and a Nearly Rectangular Impulse Response

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1. Recursive Implementation of General Windows

We consider the problem of finding a recursive realization which approximates a given impulse response. We will require that the realization is of the recursive state-space form

$$\begin{aligned} z_t &= Az_{t-1} + bx_t \quad (1)\\ y_t &= c^* z_t \quad (2)\end{aligned}$$

where z_t, b, and c are vectors of length d, and A is a $d \times d$ matrix. Since the impulse response of this filter is

$$h_t = c^* A^t b \quad (3)$$

we see that the impulse response must be a sum of exponentially decaying polynomials.

We can choose A and b to have desirable properties, and then solve (3) for c. This system can be chosen to be determined, overdetermined or, if we have other information about c, undedetermined.

1.1. Interpolation and Irregularly Spaced Data. We can generalize this recursive realization to the case where the times at which data arrives are not equally spaced points. In this case we put

$$\begin{aligned} z_{t_i} &= A^{(t_i - t_{i-1})} z_{t_{i-1}} + bx_{t_i} \quad (4)\\ y_{t_i} &= c^* z_{t_i} \quad (5)\end{aligned}$$

with impulse response

$$h_t = c^* A^t b \quad (6)$$

as before. Note that if the times t_i are successive integers, that is $t_i = i$, then this filter coincides with (3). We note that

$$\left(\frac{d}{dt}\right)^k h_t = c^* \left(\log A\right)^k A^t b \quad (7)$$

permits us to compute derivatives of h_t, where $\log A$ is the matrix logarithm. The DC gain for irregularly spaced data depends on the sequence of times t_i, so that a gain-normalized filter requires a method of keeping track of these times. One convenient approach is to approximate the normalized filter by adjoining extra states

$$\begin{aligned} z_{t_i} &= A^{(t_i - t_{i-1})} z_{t_{i-1}} + b x_{t_i} && (8)\\ w_{t_i} &= A^{(t_i - t_{i-1})} w_{t_{i-1}} + b && (9)\\ y_{t_i} &= \left(c^* z_{t_i}\right) / \left(c^* w_{t_i}\right). && (10) \end{aligned}$$

2. Maximally Flat Impulse Response

A very nice filter for windowing can be given by choosing an exponential decay λ, and setting

$$\log A = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & \ddots & \\ & & & \ddots & 1 \\ & & & 0 & \lambda \end{pmatrix} \tag{11}$$

and $c = e_0$. Then we will determine b so that

$$\begin{aligned} \left.\left(\frac{d}{dt}\right)^k\right|_{t=0} h_t &= e_0^* \left(\log A\right)^k b && (12)\\ &= \begin{cases} 1 & k = 0 \\ 0 & 1 < k < d \end{cases}. && (13) \end{aligned}$$

Now our choice of A by specifying $\log A = \lambda \mathbb{I} + Z$ looks more natural, and, noting that for $k < d$ we have

$$e_0^* \left(\log A\right)^k b = \sum_{j=0}^{k} \binom{k}{j} \lambda^{k-j} b_j = \begin{cases} 1 & k = 0 \\ 0 & 1 < k < d \end{cases} \tag{14}$$

and from the binomial theorem we have the solution

$$b_j = (-\lambda)^j \text{ , or } b = \left(1, -\lambda, \lambda^2, \cdots, (-\lambda)^{d-1}\right). \tag{15}$$

We should also explain that since $\log A$ has been chosen to be triangular, so is A, and the eigenvalues of $\log A$, (in this case the repeated eigenvalue λ) is the logarithm of the repeated eigenvalue of A, namely e^λ. Since we want A to be a stable matrix, we have $\lambda < 0$.

2.1. Properties of the Maximally Flat Impulse Response. After calculating A^t (see the Appendix for this), the maximally flat impulse response is

$$\begin{aligned} h_t &= c^* A^t b && (16)\\ &= e^{\lambda t}\left(1+(-\lambda t)+\frac{(-\lambda t)^2}{2}+\cdots+\frac{(-\lambda t)^{d-1}}{(d-1)!}\right) && (17)\end{aligned}$$

which expands in the neighborhood of $t=0$ to

$$\begin{aligned} h_t &= e^{\lambda t}\left(e^{-\lambda t}+O\left(t^d\right)\right) && (18)\\ &= 1+O\left(t^d\right) && (19)\end{aligned}$$

which shows quite clearly that this function has the desired properties.

We will now introduce the notation $h_{d,\lambda}(t)=h_t$ which will be convenient in the following.

The maximally flat impulse response is related to the Poisson distribution

$$p_\mu(X=k)=e^{-\mu}\frac{\mu^k}{k!} \tag{20}$$

since

$$h_{d,\lambda}(t)=p_{-\lambda t}(X<d). \tag{21}$$

Note that this means that $0\le h_{d,\lambda}\le 1$, $h_{d,\lambda}$ is strictly decreasing as a function of t.

We also have

$$\begin{aligned} p_\mu(X<d) &= e^{-\mu}\left(1+\mu+\cdots+\frac{\mu^{d-1}}{(d-1)!}\right) && (22)\\ &= \frac{1}{\Gamma(d)}\int_\mu^\infty e^{-x}x^{d-1}dx && (23)\end{aligned}$$

hence

$$h_{d,\lambda}(t)=\frac{1}{\Gamma(d)}\int_{-\lambda t}^\infty e^{-x}x^{d-1}dx=Q(d,-\lambda t) \tag{24}$$

which is a form of the incomplete gamma function. Using this form, we can compute the moments of the maximally flat impulse response as

$$\begin{aligned} \int_0^\infty t^k\left(\frac{1}{\Gamma(d)}\int_{-\lambda t}^\infty e^{-x}x^{d-1}dx\right)dt &= \frac{1}{\Gamma(d)}\int_{-\lambda t}^\infty e^{-x}x^{d-1}\left[\int_0^{-x/\lambda}t^k dt\right]dx && (25)\\ &= \frac{1}{\Gamma(d)}\int_{-\lambda t}^\infty e^{-x}x^{d-1}\left[\frac{1}{k+1}\left(-\frac{x}{\lambda}\right)^{k+1}\right]dx && (26)\\ &= \frac{\Gamma(d+k+1)}{(k+1)\Gamma(d)(-\lambda)^{k+1}}. && (27)\end{aligned}$$

We can also compute the transfer function of the maximally flat impulse response (i.e. the Laplace transform) as

$$\begin{aligned} F_{d,\lambda}(s) &= \int_0^\infty e^{-st} h_{d,\lambda}(t)\, dt && (28)\\ &= \sum_{k=0}^{d-1} \frac{(-\lambda)^k}{k!} \int_0^\infty e^{-st}\left(e^{\lambda t} t^k\right) dt && (29)\\ &= \sum_{k=0}^{d-1} \frac{(-\lambda)^k}{(s-\lambda)^{k+1}} && (30)\\ &= \frac{1-\left(\frac{\lambda}{\lambda - s}\right)^d}{s}. && (31) \end{aligned}$$

2.2. Comparison With a Rectangular Window. The rectangular window of length T has impulse response $h_T(t) = \mathbf{1}_{[t<T]}(t)$, that is

$$h_T(t) = \begin{cases} 1 & t < T \\ 0 & t \geq T \end{cases}. \tag{32}$$

The transfer function of is

$$\begin{aligned} F_T(s) &= \int_0^\infty e^{-st} \mathbf{1}_{[t<T]} dt && (33)\\ &= \frac{1-e^{-sT}}{s}. && (34) \end{aligned}$$

We would like to approximate a rectangular window with a maximally flat window which means choosing λ and d appropriately. The complication is that the values of

$$\left(\frac{d}{dt}\right)^k h_T(0) \tag{35}$$

is independent of T, but our derivation of the maximally flat impulse response only uses information local to $t = 0$, so it cannot determine T from this information.

One approach is to match the 0^{th} moments of the functions h_T and $h_{d,\lambda}$,

$$\begin{aligned} \int_0^\infty h_T(t)\, dt &= T && (36)\\ \int_0^\infty h_{d,\lambda}(t)\, dt &= -\frac{d}{\lambda} && (37) \end{aligned}$$

where we impose

$$\lambda = -\frac{d}{T}. \tag{38}$$

Observe that since

$$\lim_{d\to\infty}\left(\frac{\lambda}{\lambda-s}\right)^{d}=\lim_{d\to\infty}\left(\frac{1}{1+\frac{sT}{d}}\right)^{d}=e^{-sT}, \tag{39}$$

we must have

$$\lim_{d\to\infty}F_{d,\lambda}\left(s\right)=\frac{1-e^{-sT}}{s}=F_{T}\left(s\right). \tag{40}$$

Appealing to a rather technical theory, we find that this means that

$$\lim_{d\to\infty}h_{d,\lambda}\left(t\right)=h_{T}\left(t\right) \tag{41}$$

at all points of continuity of h_T, (i.e. everywhere except for at $t = T$).

This approach successfully determines T from λ and d, but there is still one free parameter if only T is given. When we compare higher moments of the functions h_T and $h_{d,\lambda}$ we find

$$\begin{aligned}\int_0^\infty t^k h_T\left(t\right)dt-\int_0^\infty t^k h_{d,\lambda}\left(t\right)dt &= \frac{T^{k+1}}{k+1}-\frac{d\left(d+1\right)\cdots\left(d+k\right)}{\left(k+1\right)\left(\lambda\right)^{k+1}} &(42)\\ &= \frac{d^{k+1}}{\left(k+1\right)\left(-\lambda\right)^{k+1}}\left(1-\frac{\left(d+1\right)\cdots\left(d+k\right)}{d^k}\right) &(43)\end{aligned}$$

We can evaluate the term

$$\frac{\left(d+1\right)\cdots\left(d+k\right)}{d^k}=1+\left(\sum_{1\le j\le k}j\right)d^{-1}+\left(\sum_{1\le j_1<j_2\le k}j_1j_2\right)d^{-2}+\cdots \tag{44}$$

by means of the Stirling numbers of the first kind, giving

$$\frac{\left(d+1\right)\cdots\left(d+k\right)}{d^k}=1+\frac{k\left(k+1\right)}{2}d^{-1}+\frac{\left(3k+2\right)\left(k+1\right)k\left(k-1\right)}{24}d^{-2}+O\left(d^{-3}\right) \tag{45}$$

so the relative error of the k^{th} moment is $\frac{k(k+1)}{2}d^{-1}+O\left(d^{-2}\right)$, and in particular the first moment has relative error d^{-1}. Note that when $k > 0$, the k^{th} moment of $h_{d,\lambda}$ is greater than that of h_T, and for $k = 1$ this means that the maximally flat impulse response uses *older* data than a corresponding rectangular window. It also shows that no choice of d will *match* higher moments than another choice.

We can also use this comparison of moments to determine how large d needs to be for $h_{d,\lambda}$ to approximate h_T to a specified degree.

3. Triangular Input Balanced Realizations

In a realization of the state space form

$$z_t = Az_{t-1} + bx_t \tag{46}$$
$$y_t = c^* z_t \tag{47}$$

the impulse response

$$h_t = c^* A^t b \tag{48}$$

is preserved by the action

$$(A, b, c) \to \left(TAT^{-1}, Tb, T^{-*}c\right) \tag{49}$$

that is,

$$\left(c^*T^{-1}\right)\left(TAT^{-1}\right)(Tb) = c^*\left(T^{-1}T\right)A\left(T^{-1}T\right)b \tag{50}$$
$$= c^*A^tb \tag{51}$$

where T is a nonsingular $d \times d$ matrix. This corresponds to the change of coordinates of the state vector

$$z_t \to Tz_t. \tag{52}$$

When we choose the filter parameters (A, b, c) we can exploit this symmetry to obtain choices which, although having the same signal properties, have superior numerical or computational properties. Since these realizations have the same impulse response, they are called *observationally equivalent*.

After imposing some more or less unobjectionable constraints on the filter, it turns out that any two observationally equivalent systems are related by this type of coordinate change. Note: we should go into minimality, observability and reachability in somewhat more detail here.

A system of dimension d is called *minimal* if it is not possible to realize the impulse response with a system of lower dimension. This means that each component of z_t actually is used at some point, and that the distribution of z_t vectors actually fills up d-dimensions. However, it can happen that a minimal realization of an impulse response involves coordinates for z_t in which the vectors z_t are distributed in such a way that the variation in some directions is very small compared to others, i.e. that the ellipsoids of constant probability are very far from spheres. It should not be too surprising that this can expose a finite precision realization of the system to numerical pathologies. It is therefore desirable that *all* the components of z_t have similar scaling.

It is impossible to obtain this property without any information about x_t. However, if it is known that x_t is relatively noisy, then it is possible to get good results by choosing the coordinates for z_t in such a way that z_t would be spherically symmetrically distributed if the x_t were independent and identically distributed. It turns out that this means choosing (A, b) so that Stein's equation is satisfied

$$\mathbb{I} = AA^* + bb^* \tag{53}$$

and determining c from the system

$$c^*\Omega^* = \begin{pmatrix} h_0 & h_1 & h_2 & \cdots \end{pmatrix} \tag{54}$$

where

$$\Omega^* = \begin{pmatrix} b & Ab & A^2b & \cdots \end{pmatrix}. \tag{55}$$

Note that

$$\Omega^*\Omega = bb^* + Abb^*A^* + A^2bb^*A^{2*} + \cdots \tag{56}$$

but that thanks to the Stein equation, we have

$$\begin{aligned} \mathbb{I} &= bb^* + AA^* && (57)\\ &= bb^* + A\left(bb^* + AA^*\right)A^* && (58)\\ &= bb^* + Abb^*A^* + A^2A^{2*} && (59)\\ &= \cdots && (60)\\ &= \Omega^*\Omega && (61) \end{aligned}$$

which converges as long as A is stable. (We are only interested in stable A, so we can always use this). Since

$$\Omega^*\Omega = \mathbb{I} \tag{62}$$

it follows that Ω is column unitary, and we can solve for the least squares approximation to c by

$$c = \Omega^* \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \end{pmatrix}. \tag{63}$$

This corresponds to expanding the impulse response in the orthogonal functions given by the columns of Ω, and the 'Fourier' coefficients of the impulse response are the components of c.

Remark 1. *A convenient practical approach to evaluating c in this way is given by the recursion*

$$\begin{aligned} u_t &= Au_{t-1} && (64)\\ c_t &= c_{t-1} + h_t u_t && (65) \end{aligned}$$

with the initialization

$$u_0 = b \tag{66}$$
$$c_0 = h_0 u_0. \tag{67}$$

When this recursion has run long enough so that h_t has decayed to a negligible value, set c to be the last value of c_t computed.

We have not exhausted the possible choices of coordinates for z_t, since by Shur's decomposition, we can always arrange for A to be triangular with a unitary choice of T. The unitarity of T preserves the Stein equation

$$Tbb^*T^* + TAT^*TA^*T^* = T(\mathbb{I})T^* = \mathbb{I} \tag{68}$$

and we use this choice to reduce A to a triangular matrix. A system where (A, b) satisfy the Stein equation with A triangular is called a *Triangular Input Balanced* (TIB) system. This choice is particularly good since in addition to the good numerical properties implied by the Stein equation, it is possible to compute Az_{t-1} in $O(d)$ operations, as opposed to the usual $O(d^2)$ for general A.

3.1. Meixner Functions and TIB Systems. The Meixner polynomials $m_n^{(\gamma,\mu)}(x)$ (Cf. *The Bateman Manuscript Project: Higher Transcendental Functions, v. 2*) can be defined by the generating function

$$\sum_{n=0}^{\infty} m_n^{(\gamma,\mu)}(x)\frac{z^n}{n!} = \left(1 - \frac{z}{\mu}\right)^x (1-z)^{-x-\gamma} \tag{69}$$

convergent for $|z| < \min(1, |\mu|)$. In the following, we restrict ourselves to the case $\gamma = 1$, otherwise the Meixner polynomials do not correspond to linear time-invariant filters. Therefore we write $m_n^{\mu}(x)$ instead of $m_n^{(\gamma,\mu)}(x)$. We define the sequence of weights $w(x) = \mu^x$, then the Meixner polynomials satisfy the orthogonality relation

$$\sum_{x=0}^{\infty} w(x)\, m_n^{\mu}(x)\, m_l^{\mu}(x) = (n!)^2 \mu^{-n}(1-\mu)^{-1}\delta_{nl} \tag{70}$$

which suggests the definition of the Meixner functions

$$M_n^{\mu}(x) = \frac{C_n^{\mu}}{n!}\mu^{x/2} m_n^{\mu}(x) \tag{71}$$

where

$$C_n^{\mu} = \mu^{n/2}\sqrt{1-\mu} = \mu^{1/2} C_{n-1}^{\mu} \tag{72}$$

normalizes the functions to

$$\sum_{x=0}^{\infty} M_n^{\mu}(x)\, M_l^{\mu}(x) = \delta_{nl}. \tag{73}$$

Note that $M_n^{\mu}(0) = \frac{C_n^{\mu}}{n!} m_n^{\mu}(0) = \mu^{n/2}\sqrt{1-\mu}$.

We define the system

$$\begin{aligned} z(x+1) &= Az(x) && (74)\\ z(0) &= b && (75) \end{aligned}$$

where

$$A = \begin{pmatrix} \mu^{1/2} & & & & 0 \\ (\mu-1) & \mu^{1/2} & & & \\ \mu^{1/2}(\mu-1) & (\mu-1) & \mu^{1/2} & & \\ \mu(\mu-1) & \mu^{1/2}(\mu-1) & (\mu-1) & \mu^{1/2} & \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \tag{76}$$

and the initial conditions

$$z(0) = b = \left(\sqrt{1-\mu}, \mu^{1/2}\sqrt{1-\mu}, \mu\sqrt{1-\mu}, \mu^{3/2}\sqrt{1-\mu}, \cdots\right)^T. \tag{77}$$

This system has the solution

$$z(x) = \left(M_0^{\mu}(x), M_1^{\mu}(x), M_2^{\mu}(x), \cdots\right)^T. \tag{78}$$

The system (A, b) is TIB by the orthonormality of the Meixner polynomials. Another way to observe this by plugging (A, b) back into the Stein equation.

Remark 2. *We can truncate this system to a finite number of components since A is triangular.*

3.2. Gauss-Meixner Summation. The impulse response of a TIB realization with a single eigenvalue λ is given by

$$h(t) = \sum_{n=0}^{d-1} c_n M_n^{\mu}(t) \tag{79}$$

where $\mu^2 = \lambda$ and, as usual, c is the vector of coefficients. In order to compute c we exploit Gaussian quadrature (in this case summation) properties of the orthogonal polynomials. The normalized three-term recurrence for the $\gamma = 1$ Meixner polynomials results in the Jacobi matrix

$$J = (1-\mu)^{-1} \operatorname{tridiag} \begin{pmatrix} & \sqrt{\mu} & & 2\sqrt{\mu} & \cdots & (d-1)\sqrt{\mu} & \\ \mu & & 1+2\mu & & \cdots & & (d-1)+d\mu \\ & \sqrt{\mu} & & 2\sqrt{\mu} & \cdots & (d-1)\sqrt{\mu} & \end{pmatrix} \tag{80}$$

and the associated Gauss-Meixner summation formula

$$\sum_{k=1}^{d} w_k p(t_k) = \sum_{t=0}^{\infty} \mu^t p(t) \tag{81}$$

where p is any polynomial of degree $\partial p < 2d$. The t_k are the eigenvalues of J and the w_k are proportional to the squares of the first components of the corresponding orthonormal eigenvectors. We fix the w_k by choosing the proportionality constant so that

$$\sum_{k=1}^{d} w_k = \frac{1}{1-\mu} \tag{82}$$

for consistency, (i.e. consider $p = 1$).

We put this in terms of a summation formula for the Meixner functions:

$$\sum_{t=0}^{\infty} M_r^\mu(t) M_s^\mu(t) = \sum_{k=1}^{d} w_k \mu^{-t_k} M_r^\mu(t_k) M_s^\mu(t_k) \tag{83}$$

$$= \delta_{rs} \text{ for } r+s < 2d. \tag{84}$$

This allows us to compute Meixner coefficients, that is for $f(t)$ of the form

$$f(t) = \sum_{n=0}^{d-1} c_n M_n^\mu(t) \tag{85}$$

by the formula

$$c_n = \sum_{x=0}^{\infty} f(t) M_n^\mu(t) \tag{86}$$

$$= \sum_{k=1}^{d} w_k \mu^{-t_k} f(t_k) M_n^\mu(t_k). \tag{87}$$

Now we can compute $c = (c_0, ..., c_{d-1})$ recursively without explicitly computing M_n^μ as follows.

$$v_k^{(0)} = w_k \mu^{-t_k} f(t_k) M_0^\mu(t_k) \tag{88}$$

$$= w_k \mu^{-t_k} f(t_k) \sqrt{1-\mu} \tag{89}$$

then using the recursion for $M_n^\mu(t)$:

$$v_k^{(n+1)} = \mu^{-1/2}\left[\frac{(\mu-1)\,t_k+n}{n+1}+\mu\right]v_k^{(n)} - \frac{n}{n+1}v_k^{(n-1)} \tag{90}$$

and

$$c_n = \sum_{k=1}^{d} v_k^{(n)}, \text{ for } 0 \le n < d, \tag{91}$$

where we put $v_k^{(-1)} = 0$. A check that this has worked is that $v_k^{(d)} = 0$ for $k = 1, ..., d$.

Gauss-Meixner summation works very well when we can bound the order d of the model in advance, and it is one attractive way of computing the TIB filter realization of the maximally flat impulse response.