MODELING, LEARNING AND REASONING ABOUT PREFERENCE TREES OVER COMBINATORIAL DOMAINS

DISSERTATION

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the College of Engineering at the University of Kentucky

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ABSTRACT OF DISSERTATION

MODELING, LEARNING AND REASONING ABOUT PREFERENCE TREES OVER COMBINATORIAL DOMAINS

For my PhD dissertation I study problems in preference representation, reasoning and learning, when preferences concern outcomes ranging over combinatorial domains. Preferences is a major research component in artificial intelligence (AI) and decision theory, and is closely related to the social choice theory considered by economists and political scientists. In my research I will exploit emerging connections between preference reasoning in AI and social choice. Most of my research is on qualitative preference representations that extend and combine existing formalisms such as conditional preference nets, lexicographic preference trees and answer-set optimization programs; on learning problems that aim at discovering qualitative preference models and predictive preference information from practical data; and on preference reasoning problems centered around qualitative preference optimization and aggregation methods.

KEYWORDS: preferences, decision theory, social choice theory, knowledge representation and reasoning, computational complexity, artificial intelligence

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To my family, especially to my son Adam (Shude) Liu.

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Time flies. I cannot believe that my graduate school career is coming to the end. It almost feels yesterday when I started my Ph.D. program in computer science at the University of Kentucky (UK) in Fall 2010. Looking back and connecting the dots in the past six years, I owe much gratitude to the wonderful individuals riding along with me from start to end of this journey.

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Chapter 1 Introduction

Preferences are ubiquitous. They arise when we select ice-cream flavors, vote for candidates for an office, and buy cars. Preferences have spurred research in areas such as artificial intelligence, psychology, economics, and operations research. Specifically, there has been growing interest in research problems on preferences in contexts such as knowledge representation and reasoning, constraint satisfaction, decision making, and social choice theory. My research focuses on problems in preference modeling, reasoning and learning.

Preferences can be represented in a *quantitative* and *qualitative* manner. For the former, agents express preferences in a numerical form of a value function that precisely assesses the degree of satisfaction of objects (often called *outcomes* or *alternatives*). Specifying preferences as value functions on alternatives is feasible for humans in some situations, e.g., when the number of alternatives is limited. In other circumstances, particularly when the number of alternatives is large, people often cannot express their preferences directly and accurately as value functions [28].

Assume an agent is given three flavors of ice-cream: *strawberry*, *chocolate* and *vanilla*, and is asked to describe her preference among them. The agent could think of a value function that assigns quantities (*utilities*) to each outcome based on a scale from 1 to 10, with 10 representing the most satisfaction. For instance, the agent could give the following value function:

$$\textit{strawberry} \mapsto 6,\ \textit{chocolate} \mapsto 9 \text{ and } \textit{vanilla} \mapsto 3.$$

This function shows that the favorite alternative to the agent is *chocolate* (it has the highest utility), and *strawberry* is preferred over *vanilla*.

Instead of rating alternatives quantitatively, it is often easier and more intuitive to give preferential information in a qualitative way, so that to specify a binary preference relation. Thus, the same agent could rank the flavors as the following preference order:

$$chocolate \succ strawberry \succ vanilla.$$

Note that one can obtain the qualitative preferences from the value function, but not vice versa. My research deals with qualitative preference relations.

Preferences play an essential role in areas that involve making decision, such as decision theory and social choice theory. Once we have preferences of a user or users, we can reason about the preferences to support decision making. In general, preference reasoning problems can be classified based on the number n of agents from which the preferences are gathered:

1. $n = 1$: individual decision making,
2. $n > 1$: collaborative decision making.

In case $n = 1$, we focus on optimization of the agent's preferences and help her make a better decision by, for example, computing an optimal alternative or comparing two given alternatives. For the case where $n > 1$, it is important to calculate a consensus (e.g., a winning outcome or ranking) of the group of agents.

One of the problems in preference reasoning is to aggregate preferences of a group of agents. The problem is central to collective decision making and has been studied extensively in social choice theory. Let us consider a scenario, where we are given a set of alternatives $X = \{a, b, c, d, e\}$ and a set P_X of 10 preferences (called *votes*) as follows.

$$5 : a > c > b > e > d,$$
$$3 : b > a > e > c > d,$$
$$2 : c > d > b > a > e.$$

Note that the integer associated with every preference order is the number of agents sharing that same preference. We are asked to compute the winning alternative according to some aggregation rule. Plurality, veto and Borda are examples of commonly used voting rules. For instance, Borda rule assigns score $m - i$ to the ith ranked alternative, where m is the number of alternatives. Thus, the winner is the alternative with the highest score. We compute that the Borda winner for P_X is a, since its score 31 is the highest, followed by candidates b and c with the second highest score 26.

While in the cases when the number of alternatives is small the preference-aggregating problems, such as dominance testing and winner determination, have received wide attention in the literature, the problems concerning preferences over *combinatorial domains*, which typically contain large numbers of outcomes, have not been investigated as much.

To illustrate the setting of combinatorial domains, let us consider a taxi company plans to purchase a fleet of cars. The features (or, as we will say, attributes) that will be taken into account are *BodyType*, *Capacity*, *Make*, *Price*, and *Safety*. Each attribute has a domain of values that it can take, e.g., *BodyType* may have four values *minivan*, *sedan*, *sport*, and *suv*. There could be hundreds or thousands of cars, described by different combinations of values on these five attributes, even for a relatively small number of attributes, and the decision makers will soon find it impossible to enumerate all of them from the most preferred to the least.

Consequently, an expressive yet concise representation is needed to specify preferences over combinatorial alternatives. Such preference formalisms are often categorized into *logical models* and *graphical models*. Logical models include penalty logic (*Pen-logic*) [48], possibilistic logic (*Poss-logic*) [29], qualitative choice logic (*Qual-logic*) [20], conditional preference theories (*CP-theories*) [79], and answer set optimization (*ASO*) [22], whereas graphical models found in the literature include generalized additive independence networks (GAI-nets) [7, 46], lexicographic preference trees (*LP-trees*) [11, 59], conditional preference networks (*CP-nets*) [50], conditional preference networks with trade-offs (*TCP-nets*) [14], and conditional importance networks (*CI-nets*) [50].

Once we fix a preference formalism, say $\mathcal{F}$, in which preferences of agents are specified, eliciting and learning preference expressions in $\mathcal{F}$ from agents becomes a fundamental problem. Different techniques have been proposed to preference learning in $\mathcal{F}$ such as *active learning* (or *preference elicitation*) and *passive learning* [38]. In the process of active learning, the algorithm iteratively asks the user for a pairwise preference between two given outcomes and constructs an instance of $\mathcal{F}$ as more preferences are elicited. For passive learning, the learning algorithm assumes that a set of pairwise preferences are obtained over a period of time and builds an instance of $\mathcal{F}$ with no more information from the user.

My research has centered around the language of LP-trees. Extending LP-trees, I have proposed two new tree-like preference formalisms: partial lexicographic preference trees (*PLP-trees*) [62] and preference trees (*P-trees*) [35, 61, 63]. The language of P-trees exploits a natural way humans apply to express preference information in the setting of combinatorial domains. Often a human agent would first consider the most desired *criterion*, possibly represented by a propositional formula φ. Outcomes that agree with it are preferred to those that do not. Then, the same mechanism is applied recursively to further discriminate among the outcomes that satisfy φ and among those that falsify φ. This process ends up with a structured preference system that always induces a total preorder.

My research on tree-like preference formalisms can be categorized into three main directions: preference modeling, preference learning and reasoning about preferences.

Preference Modeling My research formally proposed PLP-trees [62] and P-trees [61, 63]. In particular, I studied the relationship between P-trees and other existing preference languages, and showed that P-trees extend LP-trees, possibilistic logic, and ASO rules. Moreover, my work established computational complexity results of commonly considered decision problems in the setting of P-trees, such as *dominance testing optimality testing*, and *optimality testing w.r.t a property*.

Preference Learning Given a set of pairwise preferences between alternatives, called *examples*, acquired from the user, it is important to learn (i) a PLP-tree, preferably of a small size, consistent with a dataset of examples, and (ii) a PLP-tree correctly ordering as many of the examples as possible in case of inconsistency. In my work, I studied both these problems [62]. I established complexity results for them and, in each case where the problem is in the class P, proposed a polynomial time algorithm. On the experimentation side, I have designed and implemented algorithms, using both Answer-Set Programming (ASP) and approximation methods, to learn PLP-trees and forests of these trees in the passive learning setting. To facilitate experimentation, I developed several datasets based on classification datasets such as *Library for Preferences*, *Preference Learning Site*, and *UCI Machine Learning Repository*. To evaluate the effectiveness and feasibility of our own models, I compared them with machine learning models, such as decision trees and random forests.

Preference Aggregation In this area, I investigated two preference-aggregation problems, the *winner* problem and the *evaluation* problem, based on *positional scoring rules* (such as k-approval and Borda) when votes in elections are given as LP-trees [58, 59]. My work brought new computational complexity results of these problems, and provided computational methods to model and solve the problems using *answer set programming* (ASP) and *weighted partial maximum satisfiability* (WPM).

The outline of the remainder of this dissertation is the following. In Chapter 2, I present necessary technical preliminaries including binary relations, order theory, and computational complexity theory. In Chapter 3, I go through related work that proposed approaches to preference modeling and reasoning in artificial intelligence and social choice theory. In Chapters 4 to 7, I discuss results of my work on modeling, learning and reasoning about preferences over combinatorial domains. I conclude with a brief note in Chapter 8 on my ongoing research, as well as on possible directions of future work.

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Chapter 2 Technical Preliminaries

In this section I will give an overview of mathematical and computational concepts that I will use throughout the rest of this document. First, since I consider preference relations that are modeled as binary relations, I recall the definitions of binary relations and their key properties. I then define several types of preference relations in terms of these properties. Second, I review combinatorial domains as I am interested in preferences over combinatorial domains. Third, I introduce propositional logic to show how propositional formulas are used to compactly represent outcomes. Finally, I review concepts in computational complexity theory, as they are useful in describing the hardness of problems involving reasoning about preferences.

2.1 Relations and Orders

Definition 1. Let A and B be two sets of elements. A *binary relation* R between A and B is a subset of the Cartesian product of A and B, that is,

$$R \subseteq A \times B.$$

The following properties of binary relations are particularly relevant for modeling preferences.

Definition 2. Let R be a binary relation over a set O of objects ($R \subseteq O \times O$). We say that R is

1. reflexive if for every $o \in O$, $(o, o) \in R$.
2. irreflexive if for every $o \in O$, $(o, o) \notin R$.
3. total if for every $o_1, o_2 \in O$, $(o_1, o_2) \in R$ or $(o_2, o_1) \in R$.
4. transitive if for every $o_1, o_2, o_3 \in O$, if $(o_1, o_2) \in R$ and $(o_2, o_3) \in R$, then $(o_1, o_3) \in R$.

5. symmetric if for every $o_1, o_2 \in O$, if $(o_1, o_2) \in R$, then $(o_2, o_1) \in R$.
6. antisymmetric if for every $o_1, o_2 \in O$, if $(o_1, o_2) \in R$ and $(o_2, o_1) \in R$, then $o_1 = o_2$.

For instance, assuming that $\mathbb{N} = \{1, 2, \ldots\}$ is the set of positive integers, the *less-than-or-equal-to* relation $\leq$ over $\mathbb{N}$ is reflexive, total, transitive and antisymmetric, while the *less-than* relation $<$ over $\mathbb{N}$ is irreflexive, transitive and antisymmetric.

Definition 3. A binary relation over O is a *partial preorder* if it is reflexive and transitive, a *total preorder* if it is a partial preorder that is total, a *partial order* if it is a partial preorder that is antisymmetric, and a *total order* if it is a partial order that is total.

We use preorders to model preference relations. Thus, when we describe a preference order, we have in mind a relation that is a partial preorder. Given two objects o and o', we sometimes need to say that o' is at least as good as o or, that o' is strictly preferred over o. In some situations, due to lack of information about the two objects at hand, we cannot determine which object is preferred over the other, and speak about the objects being incomparable. Formally, we have the following definitions.

Definition 4. Let O be a set of objects, and o and o' two objects in O. Let $\succeq$ be a preference relation that is a partial preorder over O. We say that o' is weakly preferred to o if $o' \succeq o$. Object o' is strictly preferred to o, $o' \succ o$, if $o' \succeq o$ and $o \not\succeq o'$. Object o' is equivalent with o, $o' \approx o$, if $o' \succeq o$ and $o \succeq o'$. Object o' is incomparable with o, $o' \bowtie o$, if $o' \not\succeq o$ and $o \not\succeq o'$.

We illustrate these notions with several examples of preorders (preference orders) in Figure 2.1. We assume that a directed edge is from a less preferred object to a more preferred one. It is clear that the relation in Figure 2.1a is a partial preorder, Figure 2.1b a total preorder, Figure 2.1c a partial order, and Figure 2.1d a total order. Note that in these figures, each node represents a set of distinct but equivalent outcomes.

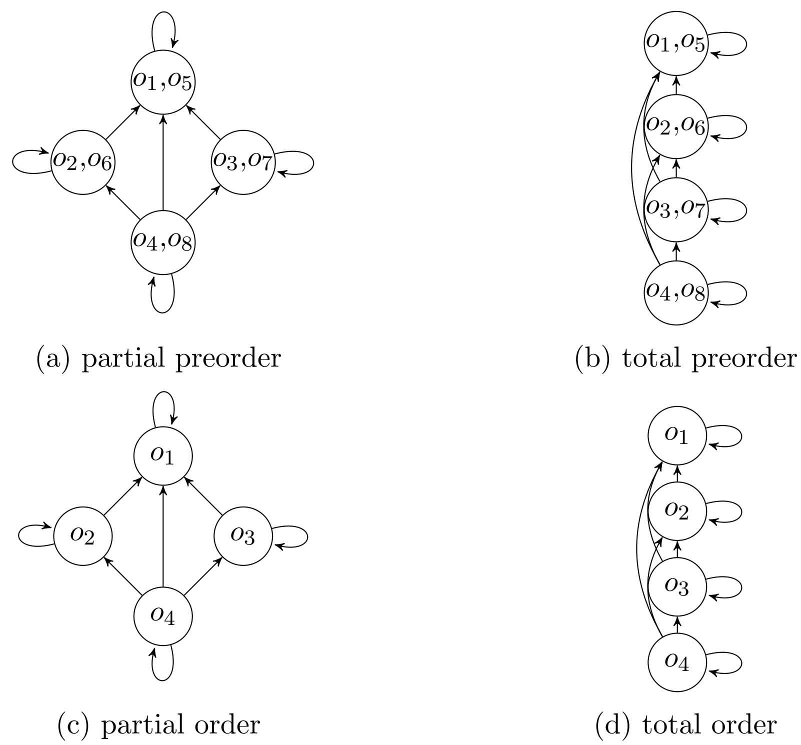

Figure 2.1: Binary relations

Definition 5. Let R and R' be two binary relations, R' *extends* R if $R \subseteq R'$.

As an example of relation extensions, we consider the partial order $\succeq$ in Figure 2.1c. Since $o_2 \bowtie o_3$, we have in total two extensions:

$$o_1 \succeq o_2 \succeq o_3 \succeq o_4,$$

$$o_1 \succeq o_3 \succeq o_2 \succeq o_4.$$

Definition 6. Let $\succeq$ be a preference relation over O, $o \in O$ is *optimal* if there does not exist $o' \in O$ such that $o' \succ o$.

For instance, object o_1 is optimal in the partial preorder shown in Figure 2.1a.

2.2 Combinatorial Domains

One scenario when decision problems involving preferences are difficult is when outcomes are described as combinations of attribute values from finite domains. Take the domain of cars as an example, where the attributes we care about are *Price*, *Safety*, and *Capacity*. Every attribute has a binary domain of values: *Capacity* with domain {*low,high*}, *Price* with domain {*low,high*}, and *Safety* with domain {*low,high*}. Note that, although we mostly use binary attributes in the dissertation, the results and algorithms we have obtained can easily be adjusted to general non-binary cases. Since these are the only aspects of a car we care about, cars can be described as vectors of values from these domains. For instance, vector ⟨*high,low,high*⟩ represents a car that has high capacity, low price and high safety. We clearly see that the number of cars grows exponentially as there are more attributes.

Definition 7. Let $\mathcal{I}$ be a set of attributes $\{X_1, \ldots, X_p\}$, each attribute X_i associated with a finite domain $Dom(X_i)$. A *combinatorial domain* $CD(\mathcal{I})$ is a set of combinations of values from $Dom(X_i)$:

$$CD(\mathcal{I}) = \prod_{X_i \in \mathbf{V}} Dom(X_i).$$

We call the elements of $CD(\mathcal{I})$ outcomes. Clearly, the size of $CD(\mathcal{I})$ is exponential in p, the number of attributes. The exponential growth of $|CD(\mathcal{I})|$ makes it hard, if not impossible, for agents to directly assess their preferences, even when each domain is binary and there are as few as 6-7 attributes. In many practical cases, hard constraints that can be modeled, for instance, by propositional formulas, are identified and imposed to eliminate the infeasible outcomes.

2.3 Propositional Logic

In this work, *propositional logic* plays an important role in compactly representing preferences over combinatorial domains. Propositional logic [49], or propositional calculus, is a logic language concerning propositions (e.g., statements that are true or false) that are built upon atomic propositions by means of logical connectives. We first define the syntax of the language, that is, how formulas are constructed. Then, we show its semantics, i.e., what it means for a formula to be true or false.

Propositions are represented as *well-formed formulas*, or simply *formulas* when no ambiguity. Formulas are built from an alphabet of truth symbols ($\top$ and $\bot$), variables (uppercase letters), connectives ($\neg$, $\wedge$, $\vee$, and $\rightarrow$), and parentheses. A *formula* is either a truth symbol, a variable, or, if φ and ψ are formulas, $(\neg\varphi)$, $(\varphi \vee \psi)$, $(\varphi \wedge \psi)$, and $(\varphi{\rightarrow}\psi)$. (An outside pair of parentheses is often left out. In addition, conventions based on the binding strength of connectives are used to eliminate some other pairs of parentheses.) For example, if X and Y are formulas, we have that $X \wedge (X{\rightarrow}\neg Y)$ is a formula.

A *truth assignment* is a mapping v from variables to logical values *True* and *False*. We now define what it means for a truth assignment to *satisfy* and *falsify* a formula. Let v be a truth assignment, X a variable, and φ and ψ propositional formulas. First, we define that v satisfies X, denoted by $v \models X$, if $v(X) = \mathit{True}$; and that v falsifies X, denoted by $v \not\models X$, if $v(X) = \mathit{False}$. Then, we have $v \models \neg\varphi$ if $v \not\models \varphi$ holds, $v \models \varphi \wedge \psi$ if $v \models \varphi$ and $v \models \psi$ hold, $v \models \varphi \vee \psi$ if $v \models \varphi$ or $v \models \psi$ holds, $v \models \varphi{\rightarrow}\psi$ if $v \not\models \varphi$ holds or $v \models \psi$ holds. We always have $v \models \top$ and $v \not\models \bot$.

A truth assignment can then be viewed as an outcome in a combinatorial domain. We consider two types of combinatorial domains: those of *binary* attributes and those of *non-binary* attributes.

For a combinatorial domain of binary attributes, such attributes correspond to variables in a language of propositional logic, and the outcomes from such a combinatorial domain correspond to truth assignments for that language. To establish the correspondence, it suffices to select one value in the domain of each attribute as *True* (or simply, 1) and the other one as *False* (or simply, 0). Thus, propositional formulas provide a convenient way of representing sets of, possibly exponentially many, outcomes from the corresponding combinatorial domain. We consider the domain of cars as discussed in Section 2.2. We write variables X_1, X_2 and X_3 in propositional logic, corresponding to attributes *Capacity*, *Price*, and *Safety*. We then set that the variables X_1, X_2 and X_3 being *True* (1_1, 1_2 and 1_3, respectively) represent *high* in the attributes *Capacity*, *Price* and *Safety*, respectively. Immediately, the variables being *False* (0_1, 0_2 and 0_3, respectively) means *low* in the attributes. Hence, we now can represent a car with low capacity, high price and low safety by a truth assignment $0_1 1_2 0_3$. As a consequence, formula $X_1 \wedge \neg X_2$ is a shorthand for the set of cars that have high capacities and low prices.

When the attributes in the combinatorial domain become in general non-binary, we now view the *values* in the attribute domains, not the attributes, as variables in propositional logic. A variable assigned *True* (*False*) in M means corresponding attribute value is in o (is not in o, respectively). Then, a formula φ represents the set of outcomes whose counterpart truth assignments satisfying φ. We look at the domain of cars of non-binary attributes: *Capacity* with domain {*2,5,7m*}, *Price* with domain {*low,med,high,vhigh*}, and *Safety* with domain {*low,med,high*}. This domain corresponds to a language of propositional logic of 10 variables, because there are 10 attribute values. We denote these variables by T_C, F_C, S_C, L_P, M_P, H_P, V_P, L_S, M_S and H_S, in order of the values in attributes *Capacity*, *Price*, and *Safety*. A truth assignment, that sets *True* on variables S_C, M_P, and H_S, and *False* on the others, models a car that has small capacity, medium price and high safety. We see that not all truth assignments are legal. Therefore, we need a constraint that, for every attribute domain, exactly one variable is true. Such a constraint can be expressed as a propositional formula Φ. Now it is clear that formula $\Phi \wedge ((H_P \wedge S_C) \vee (M_P \wedge M_S))$ precisely and concisely represents the set of cars that have high price and capacity of 7 or more, and cars that have medium price and medium security. Thus, all results we have obtained for combinatorial domains over binary attributes apply to the general case, too.

2.4 Computational Complexity Theory

Computer scientists looking for algorithms to solve computational problems seek ways to classify problems according to their computational hardness in terms of time (the number of instructions needed to solve the problem) or space (the size of memory needed to solve the problem). In this section, we define classes of computational complexity used for such classification. We assume familiarity with the concept of the *Turing machine* (TM). The definition of this notion and other definitions discussed below can be found in complexity books by Garey and Johnson [39]; Lewis and Papadimitriou [57]; and Arora and Barak [5].

Decision Problems

Let Σ be a finite set of elements. A *string* over alphabet Σ is an ordered tuple of finite elements from Σ. In complexity theory, Σ is typically binary, that is, $\Sigma = \{0, 1\}$. We denote by Σ^* the set of all strings of elements in Σ. A *decision problem* (or a *language*) is a set L of strings such that $L \subseteq \Sigma^*$. For instance, the SAT problem is the set of all finite propositional formulas that have a satisfying truth assignment (assuming some natural representation of propositional formulas as strings over a finite alphabet).

Studying decision problems on preferences involves designing reasoning algorithms and proving complexity results. Hence, it is important to review complexity classes that are related to later discussions of computational complexity results. These classes include **P**, **NP**, **coNP**, classes in the polynomial hierarchy, and **PSPACE**.

P, NP and coNP

What differentiates the two classes **P** and **NP** is whether the decision problem can be solved by a deterministic or a non-deterministic TM. [5]

Let $f(n)$ be the computation time to solve a problem of input size n. We denote by $\mathbf{DTIME}(f(n))$ ($\mathbf{NTIME}(f(n))$) a set of decision problems for which there exists a deterministic (non-deterministic, respectively) TM that solves any instance of the problem in time $f(n)$. We now define the two classes as follows.

Definition 8 (Garey and Johnson, 1979)**.** The class **P** (**NP**) consists of the decision problems that can be solved using a deterministic (non-deterministic, respectively) TM in time polynomial in the size of the input. Formally, we have

$$\mathbf{P} = \bigcup_{d \in \mathbb{N}} \mathbf{DTIME}(n^d),$$
$$\mathbf{NP} = \bigcup_{d \in \mathbb{N}} \mathbf{NTIME}(n^d),$$

where n is the size of the input.

Researchers in the field of complexity theory have studied the relation between these two classes. Clearly, the relation $\mathbf{P} \subseteq \mathbf{NP}$ holds. Whether $\mathbf{NP} \subseteq \mathbf{P}$ holds or not remains an open question. However, it is strongly believed that $\mathbf{P} \neq \mathbf{NP}$ [41].

One of the many complexity classes related to **P** and **NP** [41] is the class **coNP**, which contains problems that are complements of the problems in **NP**. Let $L \subseteq \{0,1\}^*$ be a decision problem, we denote by $\overline{L}$ the complement of L, that is, $\overline{L} = \{0,1\}^* - L$. We have the following definition of the class **coNP**.

Definition 9. $\mathbf{coNP} = \{L : \overline{L} \in \mathbf{NP}\}$.

To characterize the most difficult problems in class C (**NP**, **coNP**, etc), it is helpful to introduce the definition of polynomial-time reducibility [41] and the idea of C-hardness.

Definition 10. A decision problem $L \subseteq \{0,1\}^*$ is *polynomial-time reducible* to a decision problem $L' \subseteq \{0,1\}^*$, $L \leq_p L'$, if there is a polynomial-time computable function $g : \{0,1\}^* \rightarrow \{0,1\}^*$ such that for every instance $x \in L$ *iff* $g(x) \in L'$. If C is a class of decision problems, we say that L' is *C-hard* if $L \leq_p L'$ for every L in class C.

Definition 11. Let C be a complexity class (**NP**, **coNP**, etc). A decision problem L' is *C-complete* if L' is in the class C and L' is C-hard.

It is clear that, in order to prove C-completeness, one needs to show that $L' \in C$ (membership of class C), and prove C-hardness.

TM with Oracles and Polynomial Hierarchy

A *TM with an oracle* for a decision problem L is a TM that makes calls to an oracle that decides L. The *polynomial hierarchy*, denoted by **PH**, is a hierarchy of these complexity classes (i.e., Δ_i^P, Σ_i^P, and Π_i^P) that generalize the classes **P**, **NP** and **coNP** to oracles.

Definition 12. The **PH** is defined iteratively. We first define that $\Delta_0^P = \Sigma_0^P = \Pi_0^P =$ **P**. Then for $i \geq 0$, we define Δ_{i+1}^P (Σ_{i+1}^P) to consist of decision problems solvable by a polynomial-time deterministic (non-deterministic, respectively) TM with an oracle for some Σ_i^P-complete problem. We denote by Π_{i+1}^P the set of decision problems that are complements of problems in Σ_{i+1}^P.

For example, Σ_2^P is the class of decision problems solvable by a non-deterministic TM in polynomial time with an oracle for some **NP**-complete problem.

One may notice that the classes Σ_i^P and Π_i^P consist of problems that are complements to each other. Moreover, we have the inclusion between these classes as shown in Figure 2.2.

$$
\begin{tikzcd}
 & \vdots & \\
\Sigma_3^{\mathrm{P}} & & \Pi_3^{\mathrm{P}} \\
 & \Delta_3^{\mathrm{P}} \arrow[ul] \arrow[ur] & \\
\Sigma_2^{\mathrm{P}} \arrow[uu] \arrow[ur] & & \Pi_2^{\mathrm{P}} \arrow[uu] \arrow[ul] \\
 & \mathrm{P}^{\mathrm{NP}} = \Delta_2^{\mathrm{P}} \arrow[ul] \arrow[ur] & \\
\mathrm{NP} = \Sigma_1^{\mathrm{P}} \arrow[uu] \arrow[ur] & & \Pi_1^{\mathrm{P}} = \mathrm{coNP} \arrow[uu] \arrow[ul] \\
 & \Delta_0^{\mathrm{P}} = \Sigma_0^{\mathrm{P}} = \mathrm{P} = \Pi_0^{\mathrm{P}} = \Delta_1^{\mathrm{P}} \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Figure 2.2: Polynomial hierarchy diagram

PSPACE

In this work, we consider yet another complexity class called **PSPACE** that concerns the complexity of space. It consists of problems that can be decided in polynomial space.

Definition 13. The class **PSPACE** is the class of decision problems solvable by a TM in space polynomial in the size of the input.

It is not hard to see the following relation hold.

$$\mathbf{PH} \subseteq \mathbf{PSPACE}.$$

We illustrate the relationship among the complexity classes in Figure 2.3. Many classes that are not in our focus are omitted from our diagram.

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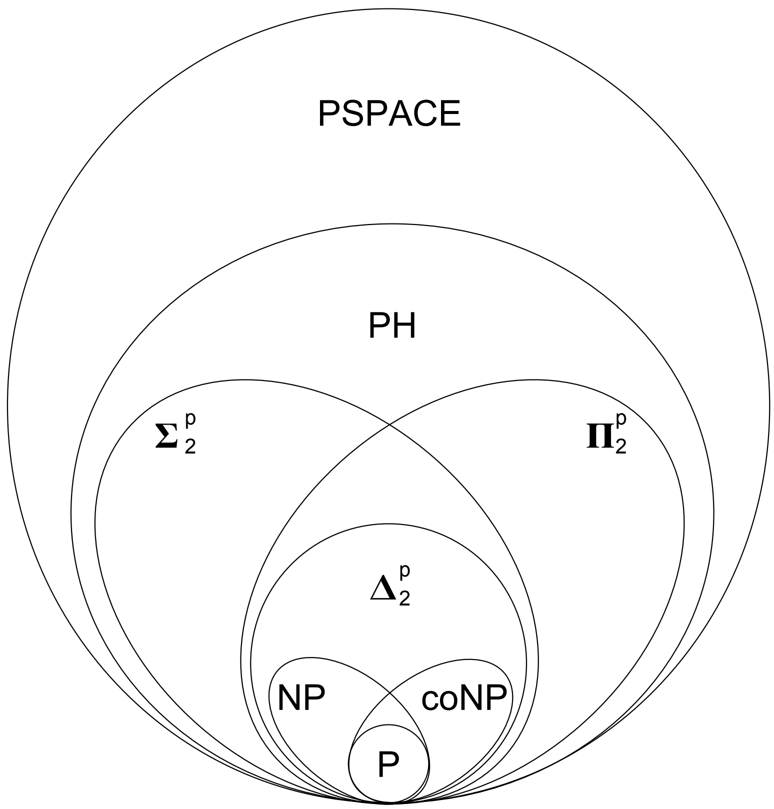


Figure 2.3: Computational complexity diagram

Chapter 3 Related Work

In this chapter I will present research work in the literature that is related to my research for the dissertation. I will first review some of the preference systems that were introduced before, designed to represent qualitative preferences over combinatorial domains. I will then introduce concepts from social choice theory underlying methods to combine individual preferences to reach a common decision.

3.1 Preference Modeling and Reasoning

Researchers have proposed several languages to model preferences. I will now discuss those of them that are closely related to my work. These languages include graphical formalisms: *Conditional Preference Networks* (CP-nets) and *Lexicographic Preference Trees* (LP-trees); and logical formalisms: *Possibilistic Logic* and *Answer Set Optimization* (ASO). They are developed to provide concise and intuitive presentations of preferential information for objects from combinatorial domains.

For all systems, I will focus on two aspects: the language in which preferences are specified, and complexity of and algorithms for problems about the model. The most fundamental of these problems are introduced in the following definitions.

Definition 14. $\mathcal{L}$-CONSISTENCE: given an instance $\mathcal{C}$ of a preference formalism $\mathcal{L}$, decide whether $\mathcal{C}$ is consistent, that is, whether there exists a total order of outcomes that agrees with every preference statement in $\mathcal{C}$.

Definition 15. $\mathcal{L}$-DOMINANCE: given an instance $\mathcal{C}$ of a preference formalism $\mathcal{L}$ and its two distinct outcomes o_1 and o_2, decide whether $o_1 \succ_{\mathcal{C}} o_2$, that is, whether o_1 is strictly preferred to o_2 in the preference order defined by $\mathcal{C}$.

Definition 16. $\mathcal{L}$-OPTIMALITY-I: given an instance $\mathcal{C}$ of a preference formalism $\mathcal{L}$, decide whether $\mathcal{C}$ has an optimal outcome.

Definition 17. $\mathcal{L}$-OPTIMALITY-II: given an instance $\mathcal{C}$ of a preference formalism $\mathcal{L}$ and an outcome o of $\mathcal{C}$, decide whether o is an optimal outcome.

Definition 18. $\mathcal{L}$-OPTIMALITY-III: given an instance $\mathcal{C}$ of a preference formalism $\mathcal{L}$ and some property Φ, decide whether there is an optimal outcome o that satisfies Φ.

Conditional Preference Networks

The Language. Conditional Preference Networks (CP-nets) define preferential relations between outcomes based on the *ceteris paribus* semantics [12]. *Ceteris paribus* is Latin for "everything else being equal."

Let $\mathbf{V}$ be a set of binary attributes.[1] We denote by $Asst(\mathbf{V})$ the set of all truth assignments to the attributes in $\mathbf{V}$. For each attribute $X_i \in \mathbf{V}$, $Pa(X_i)$ denotes the *parent* attributes of X_i, such that preferences over the domain of X_i depend upon how $Pa(X_i)$ are evaluated.

Definition 19. Let $\mathbf{V}$ be a set of binary attributes $\mathbf{V} = \{X_1, \ldots, X_n\}$. A CP-net over $\mathbf{V}$ is a tuple (G,T), where

1. $G = (V, E)$ is a directed graph, also called a *dependency graph*, specifying dependencies among attributes; an arrow in the dependency graph points to a child attribute from a parent attribute; for every $X_i \in V$, we have $Pa(X_i) = \{X_j : (X_j, X_i) \in E\}$; and
2. T is a collection of conditional preference tables (CPTs) for all attributes. A $CPT(X_i)$ consists of preference statements of the form

[1] Attributes in CP-nets can be multi-valued. However, as my research mostly deals with preference models over binary attributes, it suffices to discuss CP-nets in the binary setting.

$$\mathbf{u} :\succ^i_{\mathbf{u}},$$

where $\mathbf{u} \in Asst(Pa(X_i))$ and $\succ^i_{\mathbf{u}}$ is a total order describing preferences over $Dom(X_i)$ for a given assignment u to $Pa(X_i)$.

We say that a CP-net $N = (G, T)$ is *acyclic* if G is acyclic; otherwise, it is *cyclic*. To illustrate, let us consider the domain of cars. For simplicity, we take three binary attributes *Capacity*, *Price*, and *Safety*. Attribute *Capacity* (X_1) has two values *high* (1_1) and *low* (0_1). Attribute *Price* (X_2) has two values *high* (1_2) and *low* (0_2). Attribute *Safety* (X_3) has two values *high* (1_3) and *low* (0_3). An example CP-net $N = (G, T)$ over binary attributes $\mathbf{V} = \{X_1, X_2, X_3\}$ is shown in Figure 3.1a. We see that the preferences on *Price* (*Safety*) depend upon the assignment made to *Capacity* (*Price*, respectively).

To decide if outcome o_1 is preferred to outcome o_2 in a CP-net N, one needs to show that o_2 can be successively improved, in a "ceteris paribus" way, according to the preference statements in N to reach o_1.

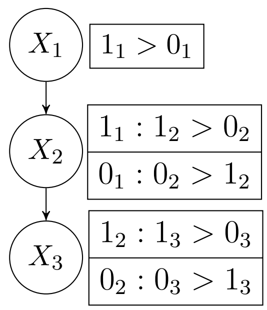
Definition 20. Let N be a CP-net over $\mathbf{V}$, $X_i \in \mathbf{V}$, $\mathbf{U} = Pa(X_i)$, and $\mathbf{Y} = \mathbf{V} - (\mathbf{U} \cup \{X_i\})$. Let $\mathbf{u}x_i\mathbf{y}$ be an outcome, where $x_i \in Dom(X_i)$, $\mathbf{u} \in Asst(\mathbf{U})$, and $\mathbf{y} \in Asst(\mathbf{Y})$. An *improving flip* of $\mathbf{u}x_i\mathbf{y}$ wrt X_i is an outcome $\mathbf{u}x'_i\mathbf{y}$ such that $x'_i \succ^i_{\mathbf{u}} x_i$. A *sequence of improving flips* wrt N is a sequence of outcomes $o_1, \ldots, o_j$ such that, for every $k < j$, o_{k+1} is an improving flip of o_k wrt some attribute in $\mathbf{V}$. We say that outcome o_1 is preferred to outcome o_2 in N, denoted by $o_1 \succ_N o_2$, if there exists a sequence of improving flips from o_2 to o_1.

In a CP-net N, we say that outcome o is *optimal* if there does not exist another outcome o' such that $o' \succ_N o$.

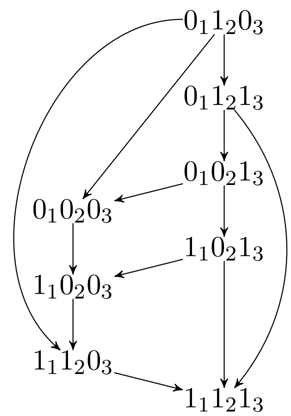
Consider the CP-net N in Figure 3.1. It induces a partial order shown in Figure 3.1b, where each arrow represents an improving flip between two outcomes. We see that $1_1 0_2 0_3 \succ_N 0_1 1_2 1_3$ because of the improving flipping sequence: $0_1 1_2 1_3$, $0_1 0_2 1_3$,

$0_1 0_2 0_3$, and $1_1 0_2 0_3$. Outcome $1_1 1_2 1_3$ is optimal because no other outcome is better. We note that there is no flipping sequence between $1_1 0_2 1_3$ and $0_1 0_2 0_3$. In such case, we say that the two outcomes are *incomparable*. This CP-net is consistent because there exists a total order of outcomes that agrees with the preference graph Figure 3.1b. There are in fact two such total orders:

$$1_1 1_2 1_3 \succ 1_1 1_2 0_3 \succ 1_1 0_2 0_3 \succ \mathbf{1_1 0_2 1_3} \succ \mathbf{0_1 0_2 0_3} \succ 0_1 0_2 1_3 \succ 0_1 1_2 1_3 \succ 0_1 1_2 0_3,$$
$$1_1 1_2 1_3 \succ 1_1 1_2 0_3 \succ 1_1 0_2 0_3 \succ \mathbf{0_1 0_2 0_3} \succ \mathbf{1_1 0_2 1_3} \succ 0_1 0_2 1_3 \succ 0_1 1_2 1_3 \succ 0_1 1_2 0_3.$$



(a) Dependency graph and CPT's



(b) Preference graph

Figure 3.1: Acyclic CP-net

Problems and Complexity. Boutilier, Brafman, Domshlak, Hoos and Poole [12] have proved that every acyclic CP-net is consistent, whereas Goldsmith et al. [45] have shown that the CPN-CONSISTENCE problem is PSPACE-complete in general.

For the CPN-DOMINANCE problem, its complexity depends on the structure of the dependency graph. The CPN-DOMINANCE problem can be solved by a polynomial time algorithm for binary-valued tree-structured CP-nets, and the problem is NP-complete for binary-valued CP-nets with specially structured dependency graphs (e.g., max-δ-connected dependency graphs) [12]. However, it is NP-hard for general binary-valued acyclic CP-nets [12]. Furthermore, in the most general case when the dependency graph could be cyclic, this problem is PSPACE-complete even if the CP-nets are consistent [45].

For acyclic CP-nets, the optimality problems (i.e., CPN-OPTIMALITY-I, CPN-OPTIMALITY-II, and CPN-OPTIMALITY-III) are easy, that is, they are in the class P [12].

Lexicographic Preference Trees

The language of lexicographic preference trees [11] uses trees to model preferences. It is motivated by lexicographic orderings [50] and lexicographic preferences [33]. This formalism and its variants are the primary focus on my research.

The Language. *A lexicographic preference tree* (*LP-tree*) T over a set $\mathcal{I}$ of p binary attributes $X_1, \ldots, X_p$ is a labeled *binary tree*. Each node t in T is labeled by an attribute from $\mathcal{I}$, denoted by $Iss(t)$, and with *preference information* of the form $a > b$ or $b > a$ indicating which of the two values a and b comprising the domain of $Iss(t)$ is preferred (in general the preference may depend on the values of attributes labeling the ancestor nodes). We require that each attribute appears exactly once on each path from the root to a leaf.

Intuitively, the attribute labeling the root of an LP-tree is of highest importance. Alternatives with the preferred value of that attribute are preferred over outcomes with the non-preferred one. The two subtrees refine that ordering. The left subtree determines the ranking of the preferred "upper half" and the right subtree determines the ranking of the non-preferred "lower half." In each case, the same principle is used, with the root attribute being the most important one. We note that the attributes labeling the roots of the subtrees need not be the same (the relative importance of attributes may depend on values for the attributes labeling the nodes on the path to the root).

The precise semantics of an LP-tree T captures this intuition. Given an outcome

$x_1x_2 \dots x_p$, we find its preference ranking in T by traversing the tree from the root to a leaf. When at node t labeled with the attribute X_i, we follow down to the left subtree if x_i is preferred according to the preference information at node t. Otherwise, we follow down to the right subtree.

It is convenient to imagine the existence of yet another level of nodes in the tree, not represented explicitly, with each node in the lowest level "splitting" into two of these implicit nodes, each representing an outcome. Descending the tree given an outcome in the way described above takes us to an (implicit) node that represents precisely that outcome's rank. The more to the left the node representing the outcome, the more preferred it is, with the one in the leftmost (implicit) node being the most desirable one as left links always correspond to preferred values.

To illustrate these notions, let us consider an example LP-tree over the car domain, given by the three binary attributes *Capacity*, *Price*, and *Safety*, described earlier. Our agent prefers cars with high capacity to cars with low capacity, and this preference on *Capacity* is the most important one. Then, for high-capacity cars, the next most important attribute is *Safety* and she prefers cars with high security level, and the least important attribute is *Price*. She prefers low-price cars if security is low, and high-price, otherwise. For low-capacity cars, the importance of *Safety* and *Price* changes with *Price* being more important. The agent prefers low-price cars among the low-capacity. Finally, high-security cars are preferred over low-security cars. These preferences are captured by the LP-tree T in Figure 3.2. The tree shows that the most preferred car for our agent has high capacity, security, and price, and the next in order of preference has high capacity and security but low price.

Sometimes LP-trees can be represented in a more concise way. For instance, if for some node t, its two subtrees are identical (that is, the corresponding nodes are assigned the same attribute), they can be collapsed to a single subtree, with the same assignment of attributes to nodes. To retain preference information, at each node

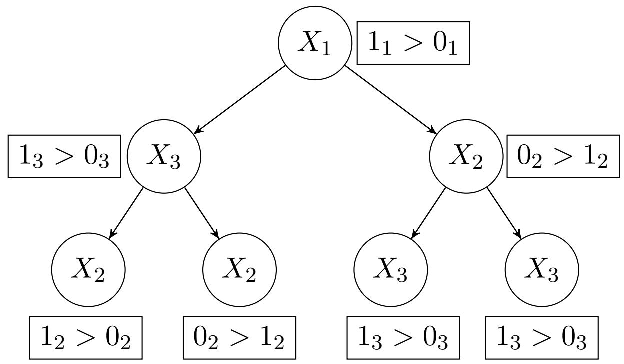


Figure 3.2: An LP-tree T

t' of the subtree we place a *conditional preference table*, and each preference in it specifies the preferred value for the attribute labeling that node given the value of the attribute labeling t. In the extreme case when for every node its two subtrees are identical, the tree can be collapsed to a path.

Formally, given an LP-tree (possibly with some subtrees collapsed), for a node t, let $NonInst(t)$ be the set of ancestor nodes of t whose subtrees were collapsed into one, and let $Inst(t)$ represent the remaining ancestor nodes. A *parent* function $\mathcal{P}$ assigns to each node t in T a set $\mathcal{P}(t) \subseteq NonInst(t)$ of *parents* of t, that is, the nodes whose attributes may have influence on the local preference at $Iss(t)$. Clearly, the conditional preference table at t requires only $2^{|\mathcal{P}(t)|}$ rows, possibly many fewer than in the worst case. In the extreme case, when an LP-tree is a path and each node has a bounded (independent of p) number of parents, the tree can be represented in $O(p)$ space.

If for every node t in an LP-tree, $\mathcal{P}(t) = \emptyset$, all (local) preferences are unconditional and conditional preference tables consist of a single entry. Such trees are called *unconditional preference* LP-trees (UP trees, for short). Similarly, LP-trees with all non-leaf nodes having their subtrees collapsed are called an *unconditional importance* LP-trees (UI trees, for short). This leads to a a natural classification of LP-trees into four classes: unconditional importance and unconditional preference LP-trees (UI-UP

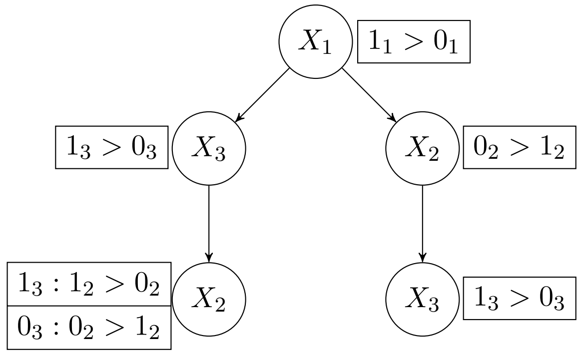


Figure 3.3: A CI-CP LP-tree T

trees), unconditional importance and conditional preference trees (UI-CP trees), etc. The class of CI-CP trees comprises all LP-trees, the class of UI-UP trees is the most narrow one.

The LP-tree T in Figure 3.2 can be represented more concisely as a (collapsed) CI-CP tree v in Figure 3.3. Nodes at depth one have their subtrees collapsed. In the tree in Figure 3.2, the subtrees of the node at depth 1 labeled P are not only identical but also have the same preference information at every node. Thus, collapsing them does not incur growth in the size of the conditional preference table.

An LP-tree consisting of p binary attributes corresponds to a total order over 2^p outcomes. For the example in Figure 3.3, the total order induced by T is

$$1_1 1_2 1_3 \succ 1_1 0_2 1_3 \succ 1_1 0_2 0_3 \succ 1_1 1_2 0_3 \succ 0_1 0_2 1_3 \succ 0_1 0_2 0_3 \succ 0_1 1_2 1_3 \succ 0_1 1_2 0_3.$$

Problems and Complexity. As any LP-tree induces a total order, the *LP*-CONSISTENCE problem is trivial. Moreover, an optimal outcome always exists and the *LP*-OPTIMALITY-I problem is trivial, too. Similarly, the *LP*-OPTIMALITY-II and *LP*-OPTIMALITY-III problems are easy to solve. Deciding whether outcome o_1 dominates outcome o_2 is done by traversing the tree until an attribute X is reached such that $o_1(X) \neq o_2(X)$. Alternatives o_1 and o_2 are then ordered based on the preference information on X [11]. This method works in polynomial time and so, we know that the *LP*-DOMINANCE problem is in **P**.

In addition to problems of reasoning about a single LP-tree, recently researchers have initiated studies of the problem of aggregating LP-trees expressing preferences of multiple agents. The goal is to facilitate collaborative decision making when all agents express their preferences by means of LP-trees. LP-trees are aggregated according to some social choice scheme, such as issue-by-issue voting [31], sequential majority voting rule [80], positional scoring rules (e.g. Borda, k-Approval) [55, 60]. Basics of social choice are discussed later in this chapter. In Chapter 7, I will provide detailed definitions of aggregating problems and results I obtained on their complexity according to positional scoring rules, as well as experimental analysis of computational methods I proposed for aggregating votes given as LP-trees. These methods are based on answer-set programming (ASP) [21] and weighted partial maximum satisfiability (WPM) [73].

Preference Trees

The model of preference trees, proposed by Fraser [34, 35], is a more general formalism than LP-trees. Using formulas as labels of the nodes, preference trees can represent total preorders.

The Language and the Model. A preference tree PT over $\mathcal{A}$ is a binary tree with each node labeled by some preference statement P, which is represented by a propositional formula over $\mathcal{A}$.

By associating the root of PT with $CD(\mathcal{A})$, each node in PT partially orders a subset of $CD(\mathcal{A})$. Particularly, if each node t labeled by φ is associated with the set $Q_t \subseteq CD(\mathcal{A})$, then for the two subtrees of t, the left subtree L_t is associated with outcomes satisfying φ, and the right subtree R_t with outcomes falsifying φ.

Consider the domain of cars over three binary attributes: *Capacity*, *Price* and *Safety*, with values *high* (1_1) and *low* (0_1), *high* (1_2) and *low* (0_2), and *high* (1_3) and *low* (0_3), respectively.

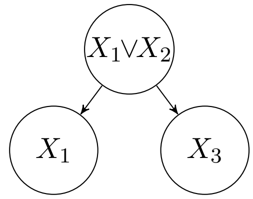


Figure 3.4: A preference tree

We look at the preference tree in Figure 3.4. The most preferred cars according to this preference tree are those with high capacity, followed by the cars with high price. The least preferred cars are those with low capacity and price, but with high safety.

Problems and Complexity. In the setting of preference trees, the PT-CONSISTENCE and the PT-OPTIMALITY-I problems are trivial as any preference tree induces a total preorder. Later in the dissertation I will show, as our new results [63], that the PT-DOMINANCE problem is in P, PT-OPTIMALITY-II is coNP-complete, and PT-OPTIMALITY-III is Δ_2^P-complete.

Possibilistic Logic

Possibilistic logic [29] describes atomic preferences as weighted propositional formulas and uses collections of weighted formulas to specify preference relations.

The Language and the Model. A possibilistic logic theory Π over a vocabulary $\mathcal{I}$ is a set of *preference pairs*

$$\{(\phi_1, a_1), \ldots, (\phi_m, a_m)\},$$

where every ϕ_i is a propositional formula over $\mathcal{I}$, and every a_i is a real number such that $1 \geq a_1 > \ldots > a_m \geq 0$ (if two formulas have the same weight, they can be replaced by their conjunction). Intuitively, a_i represents the importance of ϕ_i, with larger values indicating higher importance.

The *tolerance degree* of an outcome o with regard to a preference pair (ϕ, a), $TD_{(\phi,a)}(o)$, is defined by

$$TD_{(\phi,a)}(o) = \begin{cases} 1, & o \models \phi \\ 1 - a, & o \not\models \phi \end{cases}$$

Based on that, the tolerance degree of an outcome o with regard to a theory Π of preference pairs, $TD_\Pi(o)$, is defined by

$$TD_\Pi(o) = min\{TD_{(\phi_i,a_i)}(o) : 1 \leq i \leq m\}.$$

The larger $TD_\Pi(o)$, the more preferred o is; that is, given two outcomes o_1 and o_2, we have

$$o_1 \succ_\Pi o_2 \textit{ iff } TD_\Pi(o_1) > TD_\Pi(o_2),$$
$$o_1 \approx_\Pi o_2 \textit{ iff } TD_\Pi(o_1) = TD_\Pi(o_2).$$

Intuitively, the most preferred outcomes are those satisfying all the formulas in Π. The next preferred ones are those that falsify some formulas $\phi_i \in \Pi$, the smallest a_m of which is maximal.

Let us look at the combinatorial domain of cars over three binary attributes *Capacity*, *Price*, and *Safety*, with values *high* (1_1) and *low* (0_1), *high* (1_2) and *low* (0_2), and *high* (1_3) and *low* (0_3), respectively. Consider that an agent presents the following possibilistic theory P with two preference pairs.

$$P = \{(X_1 \wedge X_3, 0.8), (X_3 \Leftrightarrow X_2, 0.6)\}.$$

Intuitively, the agent expresses the following preferences. She likes the most the cars that falsifies neither $X_1 \wedge X_3$ nor $X_3 \Leftrightarrow X_2$. Her next preferred cars are those falsifying $X_3 \Leftrightarrow X_2$, but satisfying $X_1 \wedge X_3$. The least preferred cars for her are the ones falsifying $X_1 \wedge X_3$. We now compute the tolerance degrees of outcomes with

Table 3.1: Tolerance degrees with respect to P

Outcomes	$TD_{(\phi_1,a_1)}(o)$	$TD_{(\phi_2,a_2)}(o)$	$TD_P(o)$
$1_1 1_2 1_3$	1	1	1
$1_1 1_2 0_3$	0.2	0.4	0.2
$1_1 0_2 1_3$	1	0.4	0.4
$1_1 0_2 0_3$	0.2	1	0.2
$0_1 1_2 1_3$	0.2	1	0.2
$0_1 1_2 0_3$	0.2	0.4	0.2
$0_1 0_2 1_3$	0.2	0.4	0.2
$0_1 0_2 0_3$	0.2	1	0.2

regard to P and show the results in Table 3.1. The theory P, thus, induces a total preorder:

$$1_1 1_2 1_3 \succ 1_1 0_2 1_3 \succ 1_1 1_2 0_3 \approx 1_1 0_2 0_3 \approx 0_1 1_2 1_3 \approx 0_1 1_2 0_3 \approx 0_1 0_2 1_3 \approx 0_1 0_2 0_3.$$

Answer Set Optimization

The formalism of Answer Set Optimization (ASO) was introduced by Brewka, Niemelä and Truszczynski [23] and later enhanced by Brewka [19].

In this work, we focus on the original framework [23] where the Pareto method is used to order outcomes.

The Language.

Definition 21. Let A be a finite set of atoms. An ASO theory over A is a tuple (P_{gen}, P_{pref}), where

1. P_{gen}, the generating program, is a logic program, built of atoms in A, used to generate answer sets called feasible outcomes,
2. P_{pref}, the selecting program, is a preference program consisting of preference rules of the form

$$C_1 > \ldots > C_m \leftarrow B,$$

where each C_i is a propositional formula over A and B is a conjunction of literals of atoms in A.

A single ASO preference rule specifies a total preorder over outcomes. Applying the Pareto method, a general ASO program with multiple ASO rules determines a partial preorder over the space of outcomes represented by answer sets. We will now provide the details.

We say that outcome o is *irrelevant* to preference rule r if $o \models \neg B \vee (\neg C_1 \wedge \ldots \wedge \neg C_m)$, that is, if o does not satisfy B or o does not satisfy any of the propositional formulas C_i. As mentioned in the work by Brewka et al [23], outcomes irrelevant to r are considered as good as the best outcomes. This default treatment of irrelevance can be overwritten by including formula $\neg B \vee (\neg C_1 \wedge \ldots \wedge \neg C_m)$ in any place of the preference rule r. Formally we define satisfaction degree of an answer set with respect to a preference rule.

Definition 22. Let o be an outcome generated by P_{gen}, r an ASO preference rule. The satisfaction degree of o on r, denoted $d_r(o)$, is defined as follows: $d_r(o) = 1$ if o is irrelevant to r; $d_r(o) = \min\{i : o \models C_i\}$, otherwise.

Definition 23. Let (P_{gen}, P_{pref}) be an ASO theory, o and o' two outcomes. Outcome o' is weakly Pareto-preferred to o, $o' \succeq o$, if, for every rule r in $P_{pref}, d_r(o') \leq d_r(o)$. outcome o' is strictly Pareto-preferred to o, $o' \succ o$, if $o' \succeq o$ and $d_r(o') < d_r(o)$ for some $r \in P_{pref}$. Outcome o is optimal if there exists no outcome o'' such that $o'' \succ o$.

Consider an ASO theory $P = (P_{gen}, P_{pref})$ over the domain of cars, where

$$P_{gen} = \{\ 1\{X_1, \neg X_1\}1.\ \ 1\{X_1, \neg X_1\}1.\ \ 1\{X_1, \neg X_1\}1.\}, \text{ and}$$
$$P_{pref} = \{r.\ \ r'.\},$$

where rule r is $X_3 > \neg X_3 \leftarrow X_1 \wedge X_2$, and rule r' is $\neg X_2 \wedge X_3 > X_2 \wedge X_3 > \neg X_3$. Rule r expresses that, among high-capacity and high-price cars, cars of high safety are

Table 3.2: Satisfaction degrees with respect to P

Outcomes	$d_r(o)$	$d_{r'}(o)$
$1_1 1_2 1_3$	1	2
$1_1 1_2 0_3$	2	3
$1_1 0_2 1_3$	1	1
$1_1 0_2 0_3$	1	3
$0_1 1_2 1_3$	1	2
$0_1 1_2 0_3$	1	3
$0_1 0_2 1_3$	1	1
$0_1 0_2 0_3$	1	3

preferred to cars of low safety. Rule r' describes the preference statement that high-safety and low-price cars are the most preferred, followed by high-safety and high-price cars, which are better than low-safety cars. We now compute the satisfaction degrees of outcomes with regard to P and show the results in Table 3.2. The theory P, thus, induces a total preorder:

$$1_1 0_2 1_3 \approx 0_1 0_2 1_3 \succ 1_1 1_2 1_3 \approx 0_1 1_2 1_3 \succ 1_1 0_2 0_3 \approx 0_1 1_2 0_3 \approx 0_1 0_2 0_3 \succ 1_1 1_2 0_3.$$

In this case, we have two optimal outcomes: $1_1 0_2 1_3$ and $0_1 0_2 1_3$.

Problems and Complexity. Brewka et al [23] proved that the ASO-DOMINANCE problem is in P, ASO-OPTIMALITY-I is NP-complete, ASO-OPTIMALITY-II is coNP-complete, and ASO-OPTIMALITY-III is Σ_2^P-complete. More recently, Zhu and Truszczynski [82] presented complexity results concerning the existence of optimal outcomes similar and dissimilar to a given interpretation.

Brewka et al [23] also introduced a ranked version of ASO. Ranked ASO programs are ASO programs where rules in P_{pref} are given numeric values that represent different levels of importance of preference rules. Let us assume $P_{pref} = \{P_1, \ldots, P_g\}$ is a collection of ranked ASO preferences divided into g sets P_i, with each set P_i consisting of ASO-rules of rank d_i so that $d_1 < d_2 < \ldots d_g$. We assume that a lower rank of a preference rule indicates its higher importance. We define $o' \succeq^{rk} o$ w.r.t P if for every i, $1 \leq i \leq g$, $o' \approx_{P_i} o$, or if there exists a rank i such that $o' \approx_{P_j} o$ for every j, $j < i$, and $o' \succ_{P_i} o$.

Complexity results discussed above stay unchanged, when we move from unranked to ranked ASO programs.

3.2 Social Choice

The study of preference aggregation can be traced back to social choice theory, which dates back to Condorcet's paradox of voting, noted by the Marquis de Condorcet in the 18th century, in which the winning ranking of outcomes could be cyclic even given acyclic individual votes [78]. Kenneth Arrow's work, *Social Choice and Individual Values*, is recognized as the basis of modern social choice [1]. In the book, Arrow states that any preference aggregation method for at least three outcomes cannot meet some fairly desirable axioms, a result known as the Arrow's impossibility theorem. Further extending this result, Gibbard and Satterthwaite showed that any social choice function, again meeting some fair properties, is subject to manipulation [44, 75]. Extending the Gibbard-Satterthwaite theorem, the Duggan-Schwartz theorem deals with voting rules that elect a nonempty set of co-winners rather than a single winner [30].

All these results inform us that it is impossible to design a fair preference aggregation system that is manipulation-proof. However, Bartholdi, Tovey and Trick proposed the idea of protecting social choice schemes from manipulation via computational complexity [9, 8, 10]. The idea is that, if manipulation is computationally hard to achieve, manipulation is unlikely.

That started the field of computational social choice by adding an algorithmic perspective from computer science to the formal approach of social choice theory [16].

Preference Aggregation and Voting Rules

One of the most fundamental problems in social choice theory is how to aggregate individual preferences over outcomes so that a collaborative preference relation is reached. In other settings, people are interested in some optimal outcomes rather than a collective preference relation over all outcomes.

Social Welfare and Social Choice Functions.

Definition 24. Let $A = \{a_1, \ldots, a_m\}$ be a finite set of outcomes, $N = \{1, \ldots, n\}$ a finite set of agents (or voters). A preference relation (or a vote) v_i given by agent i is a total order $\succ_i$, that is, a total, transitive and antisymmetric. A preference profile P is a finite set of preference relations $\{\succ_1, \ldots, \succ_n\}$.

We denote by $\mathcal{L}(A)$ the set of all preference relations over the space of outcomes A, and $\mathcal{L}(A)^n$, the set of all preference profiles.

Definition 25. A social welfare function (SWF) is a function f:

$$\mathcal{L}(A)^n \rightarrow \mathcal{L}(A).$$

We call the resulting relation $\succ \in \mathcal{L}(A)$ the social preference relation.

If there are two outcomes a_1 and a_2, May's theorem [69] suggests that a_1 should be preferred to a_2 in the social preference relation if and only if more agents prefer a_1 to a_2 than a_2 to a_1. This idea is called the majority voting. However, when there are more than two outcomes, the majority voting rule can lead to cycles of outcomes, which is known as the Condorcet's paradox. For instance, we have three voters with the following preference relations:

$$a_1 \succ_1 a_2 \succ_1 a_3$$

$$a_2 \succ_2 a_3 \succ_2 a_1$$

$$a_3 \succ_3 a_1 \succ_3 a_2$$

Based on the pairwise majority rule, we have the following cycle

$$a_1 \succ a_2,\ a_2 \succ a_3,\ a_3 \succ a_1.$$

Definition 26. A social choice function (*SCF*) is a function f:

$$\mathcal{L}(A)^n \to 2^A - \{\emptyset\}.$$

We call the resulting outcome (outcomes) a winner (co-winners, respectively).

Voting Rules

The problem of aggregating individual preferences (or votes) into a single collective preference relation or a single group preferred winner is one of the key problems in social choice theory. Several voting rules and schemas have been proposed over the years. While, when there are three or more candidates, none of these methods is free of some unexpected properties, some of them have gained broad acceptance. I will now introduce some of these commonly used voting rules.

Definition 27. A voting rule r is a specific *SCF* proposed for practical use.

Positional Scoring Rules. For profiles over a set A of outcomes, a *scoring vector* is a sequence $w = (w_1, \ldots, w_m)$ of integers such that $w_1 \geq w_2 \geq \ldots \geq w_m$ and $w_1 > w_m$. Given a vote v with the outcome a in position i ($1 \leq i \leq m$), the score of a in v is given by $s_w(v, a) = w_i$. Given a profile P of votes and an outcome a, the score of a in P is given by $s_w(P, a) = \sum_{v \in P} s_w(v, a)$. These scores determine the ranking generated from P by the scoring vector w (assuming, as is common, some independent tie breaking rule). Common positional scoring rules include the plurality rule, the veto rule, the k-approval rule and Borda's rule.

1. plurality: $(1, 0, \ldots, 0)$
2. veto: $(1, \ldots, 1, 0)$

3. k-approval: $(1, \ldots 1, 0, \ldots 0)$ with k the number of 1's
4. Borda: $(m-1, m-2, \ldots, 1, 0)$

We propose yet another positional scoring rule, called (k,l)-approval [60], with the scoring vector $(a, \ldots, a, b, \ldots, b, 0, \ldots, 0)$, where both a and b are constants ($a \geq b$) and the numbers of a's and b's equal to k and l, respectively. Note that (k,l)-approval allows agents to specify two levels of approval, compared to only one level in k-approval, and thus (k,l)-approval generalizes k-approval.

A voting method, that is closely related to positional scoring rules, is the approval voting [15]. Under approval voting, each voter approves any number of outcomes and the winner, or co-winners, are those with the highest score.

Condorcet Consistent Rules. A *Condorcet winner* is an outcome that wins every pairwise comparisons against each of the other outcomes. Clearly, a Condorcet winner is unique whenever it exists. If a voting rule r always selects the Condorcet winner, if it exists, then r is said to be Condorcet consistent.

Positional scoring rules are not Condorcet consistent [32]. Voting rules that are Condorcet consistent include the following, only to list a few [16].

1. Copeland's rule: An outcome scores 1 for each pairwise comparison it wins, and some number between 0 and 1 for each pairwise comparison it ties. Alternatives with the highest score are the co-winners.
2. Maximin: The Maximin score of an outcome a is the minimum number of votes for a among all pairwise comparisons. Alternatives with the highest Maximin score wins.
3. Kemeny's rule: It selects linear rankings that maximize the number of agreements with pairwise preferences of outcomes in the profile of votes, and the top-ranked outcomes in these rankings are the co-winners.
4. Dodgson's rule: A winner is an outcome that can be made a Condorcet winner by a minimal number of swaps of adjacent outcomes in the votes.

If it is required that only a single winner is eventually elected, we apply some tie-breaking method in case of co-winners. Such a tie-breaking method could be that we break ties in favor of the lexicographically smallest or largest outcome, or in favor of a randomly picked outcome among co-winners.

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Chapter 4 Reasoning with Preference Trees

Preference trees, or *P-trees* for short, offer an intuitive and often concise way of representing preferences over combinatorial domains. In this chapter, we propose an alternative definition of P-trees, and formally define their compact representation that exploits occurrences of identical subtrees. We show that P-trees generalize lexicographic preference trees and are strictly more expressive. We relate P-trees to *answer-set optimization* programs and *possibilistic logic* theories. Finally, we study reasoning with P-trees and establish computational complexity results for key reasoning tasks of comparing outcomes with respect to orders defined by P-trees, and of finding optimal outcomes.

4.1 Introduction

Let us consider preferences on the domain of cars. We will assume that cars are described by four binary variables:

1. *Capacity* (X_1) with values *high* (1_1) and *low* (0_1),
2. *Price* (X_2) with values *high* (1_2) and *low* (0_2),
3. *Safety* (X_3) with values *high* (1_3) and *low* (0_3), and
4. *Transmission* (X_4) could be *automatic* (1_4) and *manual* (0_4).

A truth assignment of these four variables $0_1 0_2 1_3 1_4$ represents the car with low capacity, low price, high safety, and automatic transmission.

Explicitly specifying strict preference orders on $CD(\mathcal{I})$ becomes impractical even for combinatorial domains with as few as 7 or 8 attributes. However, the setting introduced above allows us to specify total preorders on outcomes in terms of desirable properties outcomes should have. For instance, a formula φ might be interpreted as a definition of a total preorder in which outcomes satisfying φ are preferred to those that do not satisfy φ (and outcomes within each of these two groups are equivalent). More generally, we could see an expression (a sequence of formulas)

$$\varphi_1 > \varphi_2 > \ldots > \varphi_k$$

as a definition of a total preorder in which outcomes satisfying φ_1 are preferred to all others, among which outcomes satisfying φ_2 are preferred to all others, etc., and where outcomes not satisfying any of the formulas φ_i are least preferred. This way of specifying preferences is used (with minor modifications) in possibilistic logic [29] and ASO programs [22]. In our example, the expression

$$X_3 \wedge X_4 > \neg X_2 \wedge \neg X_4$$

states that we prefer automatic (1_4) cars with high safety (1_3) to manual (0_4) cars with low price (0_2), with all other cars being the least preferred.

This linear specification of preferred formulas is sometimes too restrictive. An agent might prefer outcomes that satisfy a property φ to those that do not. Within the first group that agent might prefer outcomes satisfying a property ψ_1 and within the other a property ψ_2. Such *conditional* preference can be naturally captured by a form of a decision tree presented in Figure 4.1. Leaves, shown as boxes, represent sets of outcomes satisfying the corresponding conjunctions of formulas ($\varphi \wedge \psi_1$, $\varphi \wedge \neg\psi_1$, etc.).

Trees such as the one in Figure 4.1 are called *preference trees*, or *P-trees*. They were introduced by Fraser [34, 35], who saw them as a convenient way to represent conditional preferences. Despite their intuitive nature they have not attracted much interest in the preference research in AI. In particular, they were not studied for their

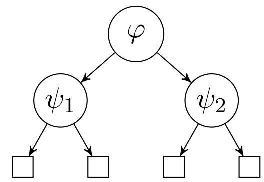


Figure 4.1: A preference tree

relationship to other preference formalisms. The attribute of compact representations received only an informal treatment by Fraser (P-trees in their full representation are often impractically large), and the algorithmic attributes of reasoning with P-trees were also only touched upon.

We propose an alternative definition of preference trees, and formally define their compact representation that exploits occurrences of identical subtrees. P-trees are reminiscent of LP-trees [11]. We discuss the relation between the two concepts and show that P-trees offer a much more general, flexible and expressive way of representing preferences. We also discuss the relationship between preference trees and ASO preferences and possibilistic logic theories. We study the complexity of problems of comparing outcomes with respect to orders defined by preference trees, and of problems of finding optimal outcomes.

This chapter is organized as follows. In the next section, we formally define P-trees and a compact way to represent them. In the following section we present results comparing the language of P-trees with other preference formalisms. We then move on to study the complexity of key reasoning tasks for preferences captured by P-trees and, finally, conclude by outlining some future research directions.

4.2 Preference Trees

In this section, we define preference trees and discuss their representation. Let $\mathcal{I}$ be a set of binary attributes[1]. A *preference tree* (*P-tree*, for short) over $\mathcal{I}$ is a binary tree

[1] In case of multi-value attributes, $\mathcal{I}$ is then a set of binary variables representing attribute values.

with all nodes other than leaves labeled with propositional formulas over $\mathcal{I}$. Each P-tree T defines a natural strict order $\succeq_T$ on the set of its leaves, the order of their enumeration from left to right.

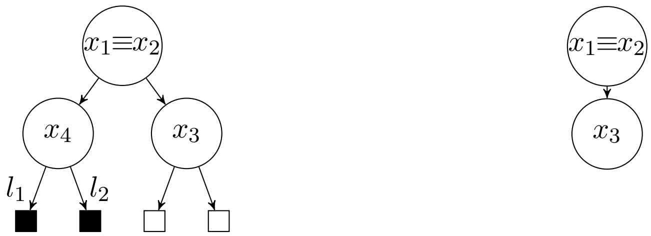
Given an outcome $o \in CD(\mathcal{I})$, we define the *leaf of o in T* as the leaf reached by starting at the root of T and proceeding downwards. When at a node t labeled with φ, if $o \models \varphi$, we descend to the left child of t; otherwise, we descend to the right node of t. We denote the leaf of o in T by $l_T(o)$.

We use the concept of the leaf of an outcome o in a P-tree T to define a total preorder on $CD(\mathcal{I})$. Namely, for outcomes $o_1, o_2 \in CD(\mathcal{I})$, we set $o_1 \succeq_T o_2$, o_1 is *preferred* to o_2, if $l_T(o_1) \succeq_T l_T(o_2)$, and $o_1 \succ_T o_2$, o_1 is *strictly preferred* to o_2, if $l_T(o_1) \succ_T l_T(o_2)$. (We overload the relations $\succeq_T$ and $\succ_T$ by using it both for the order on the leaves of T and the corresponding preorder on the outcomes from $CD(\mathcal{I})$). We say that o_1 is *equivalent* to o_2, $o_1 \approx_T o_2$, if $l_T(o_1) = l_T(o_2)$. Finally, o is *optimal* if there exists no o' such that $o' \succ_T o$.

Let us come back to the car example and assume that an agent prefers small cars with low price or big cars with high price over the other options. This preference is described by the formula $(X_1 \wedge X_2) \vee (\neg X_1 \wedge \neg X_2)$ or, more concisely, as an equivalence $X_1 \equiv X_2$. Within each of the two groups of cars (satisfying the formula and not satisfying the formula), high safety (1_3) is preferred. These preferences can be captured by the P-tree in Figure 4.2a. We note that in this example, the preferences at the second level are *unconditional*, that is, they do not depend on preferences at the top level.

To compare two outcomes, $o_1 = 0_1 0_2 0_3 1_4$ and $o_2 = 1_1 1_2 1_3 0_4$, we walk down the tree and find that $l_T(o_1) = l_1$ and $l_T(o_2) = l_2$. Thus, we have $o_1 \succ_T o_2$ since l_1 precedes l_2.

The key property of P-trees is that they can represent any total preorder on $CD(\mathcal{I})$.



(a) Full (b) Compact

Figure 4.2: P-trees on cars

Proposition 1. *For every set $\mathcal{I}$ of binary attributes, for every set $D \subseteq CD(\mathcal{I})$ of outcomes over $\mathcal{I}$, and for every total preorder $\succeq$ on D into no more than 2^n clusters of equivalent outcomes, there is a P-tree T of depth at most n such that the preorder determined by T on $CD(\mathcal{I})$ when restricted to D coincides with $\succeq$ (that is, $\succeq_{T|D} = \succeq$).*

Proof. Let $\succeq$ be a total preorder on a subset $D \subseteq CD(\mathcal{I})$ of outcomes over $\mathcal{I}$, and let $D_1 \succ D_2 \succ \ldots \succ D_m$ be the corresponding strict ordering of clusters of equivalent outcomes, with $m \leq 2^n$. If $m = 1$, a single-leaf tree (no decision nodes, just a box node) represents this preorder. This tree has depth 0 and so, the assertion holds. Let us assume then that $m > 1$, and let us define $D' = D_1 \cup \ldots \cup D_{\lceil m/2 \rceil}$ and $D'' = D \setminus D'$. Let $\varphi_{D'}$ be a formula such that models of D' are precisely the outcomes in D' (such a formula can be constructed as a disjunction of conjunctions of literals, each conjunction representing a single outcome in D'). If we place $\varphi_{D'}$ in the root of a P-tree, that tree represents the preorder with two clusters, D' and D'', with D' preceding D''. Since each of D' and D'' has no more than 2^{n-1} clusters, by induction, the preorders $D_1 \succ \ldots \succ D_{\lceil m/2 \rceil}$ and $D_{\lceil m/2 \rceil + 1} \succ \ldots \succ D_m$ can each be represented as a P-tree with depth at most $n - 1$. Placing these trees as the left and the right subtrees of $\varphi_{D'}$ respectively results in a P-tree of depth at most n that represents $\succeq$. □

Compact Representation of P-Trees. Proposition 2 shows high expressivity of P-trees. However, the construction described in the proof has little practical use.

First, the P-tree it produces may have a large size due to the large sizes of labeling formulas that are generated. Second, to apply it, one would need to have an explicit enumeration of the preorder to be modeled, and that explicit representation in practical settings is unavailable.

However, preferences over combinatorial domains that arise in practice typically have structure that can be elicited from a user and exploited when constructing a P-tree representation of the preferences. First, decisions at each level are often based on considerations involving only very few attributes, often just one or two and very rarely more than that. Moreover, the subtrees of a node that order the "left" and the"right" outcomes are often identical or similar.

Exploiting these features often leads to much smaller representations. A *compact P-tree over $\mathcal{I}$* is a tree such that

1. every node is labeled with a Boolean formula over $\mathcal{I}$, and
2. every non-leaf node t labeled with φ has either two outgoing edges, with the left one meant to be taken by outcomes that satisfy φ and the right one by those that make φ false (Figure 4.3a), or one outgoing edge pointing
   - straight-down (Figure 4.3b), which indicates that the two subtrees of t are *identical* and the formulas labeling every pair of corresponding nodes in the two subtrees are the *same*,
   - left (Figure 4.3c), which indicates that right subtree of t is empty, or
   - right (Figure 4.3d), which indicates that left subtree of t is empty.

The P-tree in Figure 4.2a can be collapsed as both subtrees of the root are the same (including the labeling formulas). This leads to a tree in Figure 4.2b with a straight-down edge. We note that we drop box-labeled leaves in compact representations of P-trees, as they no longer have an interpretation as distinct clusters.

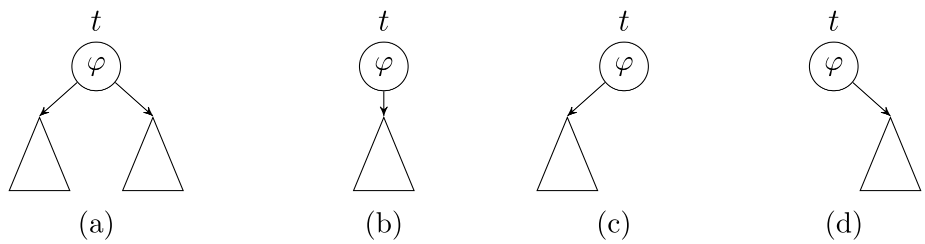


Figure 4.3: Compact P-trees

Empty Leaves in P-Trees. Given a P-tree T one can prune it so that all sets of outcomes corresponding to its leaves are non-empty. However, keeping empty clusters may lead to compact representations of much smaller (in general, even exponentially smaller) size.

A full P-tree T in Figure 4.4a uses labels $\varphi_1 = \neg X_1 \vee X_3$, $\varphi_2 = X_2 \vee \neg X_4$, and $\varphi_3 = X_2 \wedge X_3$. We check that leaves l_1, l_2 and l_3 are empty, that is, the conjunctions $\varphi_1 \wedge \neg\varphi_2 \wedge \varphi_3$, $\neg\varphi_1 \wedge \varphi_2 \wedge \varphi_3$ and $\neg\varphi_1 \wedge \neg\varphi_2 \wedge \varphi_3$ are unsatisfiable. Pruning T one obtains a compact tree T' (Figure 4.4b) that is smaller compared to T, but larger than T'' (Figure 4.4c), another compact representation of T, should we allow empty leaves and exploit the structure of T.

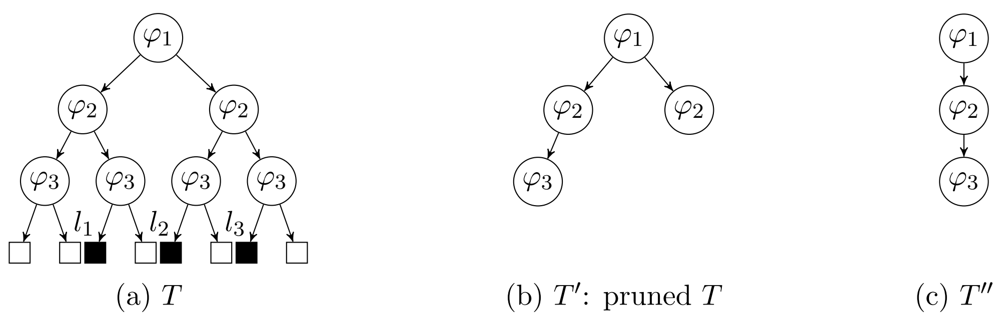


Figure 4.4: P-trees with empty leaves

That example generalizes and leads to the question of finding small sized representations of P-trees. (We conjecture that the problem in its decision version asking about the existence of a compact representation of size at most k is NP-complete). From now on, we assume that P-trees are given in their compact representation.

4.3 P-Trees and Other Formalisms

In this section we compare the preference representation language of P-trees with other preference languages.

P-Trees Generalize LP-Trees. As stated earlier, P-trees are reminiscent of LP-trees, a preference language that has received significant attention recently [11, 56, 60]. In fact, LP-trees over a set $\mathcal{I} = \{X_1, \ldots, X_n\}$ of attributes are simply special P-trees over $\mathcal{I}$. Namely, an LP-tree over $\mathcal{I}$ can be defined as a P-tree over $\mathcal{I}$, in which all formulas labeling nodes are atoms x_i or their negations $\neg x_i$, depending on whether x_i or $\neg x_i$ is the preferred over the other, and every path from the root to a leaf has all atoms x_i appear in it as labels exactly once. Clearly, LP-trees are full binary trees of depth n (assuming the depth of the root is 1) and determine strict *total orders* on outcomes in $CD(\mathcal{I})$ (no indifference between different outcomes). An example of an LP-tree over $\{X_1, X_2, X_3, X_4\}$ for our car example is given in Figure 4.5.

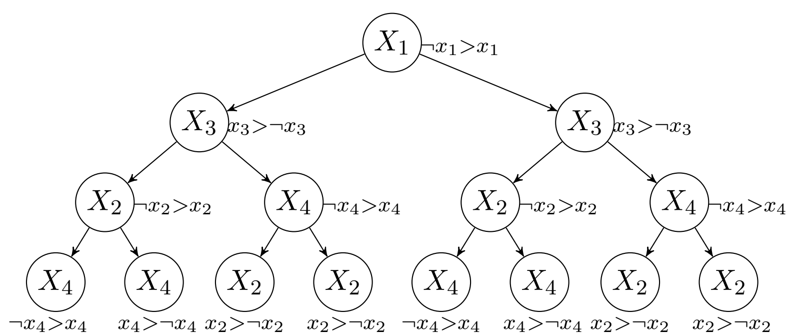
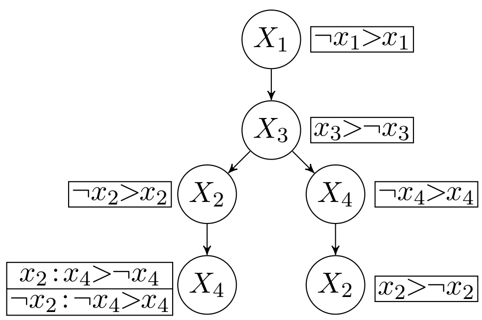
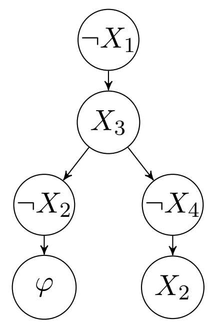


Figure 4.5: A full LP-tree on cars

In general representing preferences by LP-trees is impractical. The size of the representation is of the same order as that of an explicit enumeration of the preference order. However, in many cases preferences on outcomes have structure that leads to LP-trees with similar subtrees. That structure can be exploited, as in P-trees, to represent LP-trees compactly. Figure 4.6a shows a compact representation of the LP-tree in Figure 4.5. We note the presence of conditional preference tables that make up for the lost full binary tree structure. Together with the simplicity of the language, compact representations are behind the practical usefulness of LP-trees. The compact representations of LP-trees translate into compact representations of P-trees, in the sense defined above. This matter is not central to our discussion and we simply illustrate it with an example. The compactly represented P-tree in Figure 4.6b is the counterpart to the compact LP-tree in Figure 4.6a, where $\varphi = (X_2 \wedge X_4) \vee (\neg X_2 \wedge \neg X_4)$.



(a) A compact LP-tree



(b) The corresponding P-tree

Figure 4.6: A compact LP-tree as a compact P-tree

The major drawback of LP-trees is that they can capture only a very small fraction of preference orders. Let us first compute the number, say $G(n)$, of LP-trees over n attributes. We have

$$G(n) = \begin{cases} 1, & n = 0; \\ 2n \cdot G^2(n-1), & n > 0. \end{cases}$$

From this recursive definition of $G(n)$, we calculate that

$$G(n) = \prod_{k=0}^{n-1} (n-k)^{2^k} \cdot 2^{2^k}$$

It is asymptotically much smaller than $L(n) = (2^n)!$, the number of all preference orders of the corresponding domain of outcomes. In particular, we show in Theorem 1 that LP-trees only encode an exponentially small portion of all linear orders.

Theorem 1. *Let $L(n) = 2^n!$ be the number of linear orders of outcomes over n binary attributes, r be the ratio of $G(n)$ to $L(n)$. We have*

$$r = \frac{G(n)}{L(n)} < \frac{1}{2^{(2^n \cdot (n - \log n - 2))}}. \tag{4.1}$$

Proof.

$$\begin{aligned}
r2^n! &= T(n); \qquad (4.2)\\
\log r + \log 2^n! &= \log(\prod_{k=0}^{n-1}(n-k)^{2^k} \cdot 2^{2^k})\\
&= \sum_{k=0}^{n-1}(\log((n-k)^{2^k}) + 2^k)\\
&= \sum_{k=0}^{n-1}(2^k \cdot (\log(n-k) + 1))\\
&< \sum_{k=0}^{n-1}(2^k \cdot (\log n + 1))\\
&= (\log n + 1) \cdot \sum_{k=0}^{n-1} 2^k\\
&= (\log n + 1) \cdot (2^n - 1). \qquad (4.3)
\end{aligned}$$

Let N be such that $N = (\log n + 1) \cdot (2^n - 1)$. By the Stirling's approximation $n! \geq \sqrt{2\pi n} \cdot (\frac{n}{e})^n$, we have the following.

$$
\begin{aligned}
\log r &< N - \log 2^n! \\
&\leq N - \log(\sqrt{2\pi 2^n} \cdot (\frac{2^n}{e})^{2^n}) \\
&< N - \log(\sqrt{2^n} \cdot (\frac{2^n}{e})^{2^n}) \\
&= N - (\frac{n}{2} + \log \frac{2^{n \cdot 2^n}}{e^{2^n}}) \\
&= N - (\frac{n}{2} + n \cdot 2^n - \log e^{2^n}) \\
&< N - (\frac{n}{2} + n \cdot 2^n - \log 2^{2^n}) \\
&= N - (\frac{n}{2} + n \cdot 2^n - 2^n) \\
&= 2^n \cdot (\log n - n + 2) - \log n - 1 - \frac{n}{2} \\
&< 2^n \cdot (\log n - n + 2).
\end{aligned} \tag{4.4}
$$

Therefore, we have

$$r < \frac{1}{2^{(2^n \cdot (n - \log n - 2))}}. \tag{4.5}$$

□

This is in stark contrast with Proposition 2, according to which every total preorder can be represented by a P-tree.

Even very natural orderings, which have simple (and compact) representations by P-trees often cannot be represented as LP-trees. For instance, there is no LP-tree on $\{x_1, x_2\}$ representing the order $00 \succ 11 \succ 01 \succ 10\}$. However, the P-trees (both full and compact) in Figure 4.2 do specify it.

P-Trees Extend ASO-Rules. The formalism of ASO-rules [22] provides an intuitive way to express preferences over outcomes as total preorders. An ASO-rule partitions outcomes into ordered clusters according to the semantics of the formal-

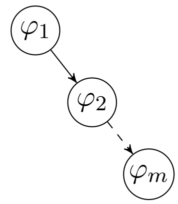


Figure 4.7: A P-tree T_r (T_P)

ism. Formally, an ASO-rule r over $\mathcal{I}$ is a preference rule of the form

$$C_1 > \ldots > C_m \leftarrow B, \tag{4.6}$$

where all C_i's and B are propositional formulas over $\mathcal{I}$.

Let us consider the domain of cars. An agent may prefer a car with high safety and capacity to one with low safety and capacity, if she is going for an expensive car. Such preference can be described as an ASO-rule:

$$X_1 \wedge X_3 > \neg X_1 \wedge \neg X_3 \leftarrow X_2.$$

Under the semantics of ASO, this preference rule specifies that the most desirable cars are all the inexpensive cars, the expensive cars with high capacity and safety, the expensive cars with high capacity but low safety, and the expensive cars with high safety but low capacity.

Given an ASO-rule r of form (4.6), we show how r is encoded in a P-tree. From the ASO-rule r, we build a P-tree T_r in Figure 4.7, where $\varphi_1 = \neg B \vee C_1 \vee (\bigwedge_{2 \leq i \leq m} \neg C_i)$, $\varphi_i = C_i$ ($2 \leq i \leq m$), and the dashed edge represents nodes labeled by the formulas $\varphi_3, \ldots, \varphi_{m-1}$ and every formula φ_i, $3 \leq i \leq m-1$, is constructed such that the parent of φ_i is φ_{i-1}, the left child of φ_i is empty, and the right child of φ_i is φ_{i+1}.

Theorem 2. *Given an ASO-rule r, the P-tree T_r has size linear in the size of r, and for every two outcomes M and M'*

$$M \succeq_r^{ASO} M' \text{ iff } M \succeq_{T_r} M'$$

Proof. The P-tree T_r induces a total preorder $\succeq_{T_r}$ where outcomes satisfying φ_1 are preferred to outcomes satisfying $\neg\varphi_1 \wedge \varphi_2$, which are then preferred to outcomes satisfying $\neg\varphi_1 \wedge \neg\varphi_2 \wedge \varphi_3$, and so on. The least preferred are the ones satisfying $\bigwedge_{1 \leq i \leq m} \neg\varphi_i$. Clearly, this order $\succeq_{T_r}$ is precisely the order $\succeq_r^{ASO}$ given by the ASO rule r. $\square$

There are other ways of translating ASO-rules to P-trees. For instance, it might be beneficial if the translation produced a more balanced tree. We could proceed as in the proof of Proposition 2.

For example, if $m = 6$, we build the P-tree T_r^b in Figure 4.8, where $\psi_1 = \varphi_1 \vee \varphi_2 \vee \varphi_3$, $\psi_2 = \varphi_1$, $\psi_3 = \varphi_2$, $\psi_4 = \varphi_4$, and $\psi_5 = \varphi_5$. The indices i's of the formulas ψ_i's indicate the order in which the corresponding formulas are built recursively.

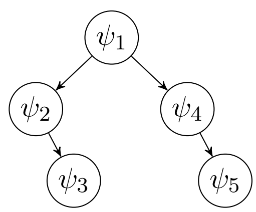


Figure 4.8: T_r^b when $m = 6$

This P-tree representation of a preference r of the form (4.6) is balanced with height $\lceil \log_2 m \rceil$. Moreover, the property in Theorem 2 also holds for the balanced T_r^b of size polynomial in the size of r. In fact, the size of T_r^b is in $O(s_r \log s_r)$, where s_r is the size of rule r. It is clear that, though tree T_r^b is larger than T_r in size, comparing outcomes could be done faster due to a smaller depth of T_r^b.

Representing P-Trees as RASO-Theories. Preferences represented by compact P-trees cannot in general be captured by ASO preferences without a significant (in some cases, exponential) growth in the size of the representation. However, any P-tree can be represented as a set of *ranked* ASO-rules, or an RASO-theory [22], aggregated by the Pareto method.

Given a P-tree T, we construct an RASO-theory Φ_T as follows. We start with $\Phi_T = \emptyset$. For every node t_i in a P-tree T, we update $\Phi_T = \Phi_T \cup \{\varphi_i \overset{d_i}{\leftarrow} \mathit{conditions}\}$, where φ_i is the formula labeling node t_i, d_i, rank of the ASO-rule, is the depth of node t_i, and *conditions* is the conjunction of formulas φ_j or $\neg\varphi_j$ labeling all nodes t_j that are ancestor nodes of t_i in T with two outgoing edges. Whether φ_j or $\neg\varphi_j$ is used depends on how the path from the root to t_i determines whether descending left (φ_j) or right ($\neg\varphi_j$) at t_j.

For instance, the P-tree T in Figure 4.6b gives rise to the following RASO-theory:

$\neg X_1 \overset{1}{\leftarrow}.$

$X_3 \overset{2}{\leftarrow}.$

$\neg X_2 \overset{3}{\leftarrow} X_3. \qquad \neg X_4 \overset{3}{\leftarrow} \neg X_3.$

$(X_2 \wedge X_4) \vee (\neg X_2 \wedge \neg X_4) \overset{4}{\leftarrow} X_3. \qquad X_2 \overset{4}{\leftarrow} \neg X_3.$

Theorem 3. *Given a P-tree T, there exists an RASO-theory Φ_T of size polynomial in the size of T such that for every two outcomes M and M'*

$$M \succeq^{RASO}_{\Phi_T} M' \text{ iff } M \succeq_T M'$$

Proof. ($\Leftarrow$) Let us assume $M \succeq_T M'$. Denote by $(\varphi_{i_1}, \ldots, \varphi_{i_j})$ the order of formulas labeling the path determined by M from the root to a leaf. Let φ_{i_k}, $1 \leq k \leq j$, be the first formula that M and M' evaluate differently, in fact, $M \models \varphi_{i_k}$ and $M' \not\models \varphi_{i_k}$. Denote by d the depth of φ_{i_k} in T. Based on the construction of Φ_T, for every RASO-rule r of rank less than d, we have $M \approx^{ASO}_r M'$. For every RASO-rule r of rank d, we have $M \succ^{ASO}_r M'$ if r comes from φ_{i_k}; $M \approx^{ASO}_r M'$ for other rules of rank d. According to RASO ordering, $M \approx^{RASO}_{\Phi_T} M'$ holds if φ_{i_k} does not exist; $M \succ^{RASO}_{\Phi_T} M'$ holds, otherwise. Therefore, $M \succeq^{RASO}_{\Phi_T} M'$ holds.

($\Rightarrow$) Prove by contradiction. We assume that $M \succeq^{RASO}_{\Phi_T} M'$ and $M' \succ_T M$ hold. We again denote by $(\varphi_{i_1}, \ldots, \varphi_{i_j})$ the order of formulas labeling the path determined by M from the root to a leaf. There must exist some formula φ_{i_k}, $1 \leq k \leq j$, such that $M' \models \varphi_{i_k}$, $M \not\models \varphi_{i_k}$, and all formulas φ_ℓ, $1 \leq \ell \leq k-1$, are evaluated in the same way by M and M'. Based on RASO ordering, we have $M' \succ^{RASO}_{\Phi_T} M$, contradiction. $\square$

Hence, the relationship between P-trees and ASO preferences can be summarized as follows. Every ASO preference rule can be translated into a P-tree, and every P-tree into a theory of ranked ASO preference rules. In both cases, the translations have size polynomial in the size of the input. Examining the reverse direction, the size of the ASO rule translated from a P-tree could be exponential, and the orders represented by ranked ASO theories *strictly include* the orders induced by P-trees as RASO-theories describe *partial* preorders in general.

P-Trees Extend Possibilistic Logic. Similarly as for ASO-rules, we can apply different methods to encode a possibilistic logic theories in P-trees. Here we discuss one of them. We define T_Π to be an unbalanced P-tree shown in Figure 4.7 with with labels φ_i defined as follows: $\varphi_1 = \bigwedge_{1\leq i\leq m} \phi_i$, $\varphi_2 = \bigwedge_{1\leq i\leq m-1} \phi_i \wedge \neg\phi_m$, $\varphi_3 = \bigwedge_{1\leq i\leq m-2} \phi_i \wedge \neg\phi_{m-1}$, and $\varphi_m = \phi_1 \wedge \neg\phi_2$.

Theorem 4. *Given a possibilistic theory* Π*, there exists a P-tree* T_Π *of size polynomial in the size of* Π *such that for every two outcomes* M *and* M'

$$M \succeq^{Poss}_{\Pi} M' \text{ iff } M \succeq_{T_\Pi} M'$$

Proof. It is clear that the size of P-tree T_Π is polynomial in the size of Π. Let $mi(M,\Pi)$ denote the maximal index j such that M satisfies all $\phi_1,\ldots,\phi_j$ in Π. (If M falsifies all formulas in Π, we have $mi(M,\Pi)=0$.) We have that $M \succeq^{Poss}_{\Pi} M'$ if and only if $mi(M,\Pi) \geq mi(M',\Pi)$, and $mi(M,\Pi) \geq mi(M',\Pi)$ if and only if $M \succeq_{T_\Pi} M'$. Therefore, the theorem follows. $\square$

4.4 Reasoning Problems and Complexity

In this section, we study decision problems on reasoning about preferences described as P-trees, and provide computational complexity results for the three reasoning problems defined below.

Definition 28. Dominance-testing (DomTest): given a P-tree T and two distinct outcomes M and M', decide whether $M \succeq_T M'$.

Definition 29. Optimality-testing (OptTest): given a P-tree T and an outcome M of T, decide whether M is optimal.

Definition 30. Optimality-with-property (OptProp): given a P-tree T and some property α expressed as a Boolean formula over the vocabulary of T, decide whether there is an optimal outcome M that satisfies α.

Our first result shows that P-trees support efficient dominance testing.

Theorem 5. *The* DomTest *problem can be solved in time linear in the height of the P-tree T.*

Proof. The DomTest problem can be solved by walking down the tree. The preference between M and M' is determined at the first non-leaf node n where M and M' evaluate φ_n differently. If such node does not exist before arriving at a leaf, $M \approx_T M'$. □

An interesting reasoning problem not mentioned above is to decide whether there exists an optimal outcome with respect to the order given by a P-tree. However, this problem is trivial as the answer simply depends on whether there is any outcome at all. However, optimality *testing* is a different matter. Namely, we have the following result.

Theorem 6. *The* OptTest *problem is coNP-complete.*

Ψ

Figure 4.9: The P-tree T_Φ

Proof. We show that the complementary problem, testing non-optimality of an outcome M, is NP-complete. Membership is obvious. A witness of non-optimality of M is any outcome M' such that $M' \succ_T M$, a property that can be verified in linear time (cf. Theorem 5). NP-hardness follows from a polynomial time reduction from SAT [40]. Given a CNF formula $\Phi = c_1 \wedge \ldots \wedge c_n$ over a set of variables $V = \{X_1, \ldots, X_m\}$, we construct a P-tree T and an outcome M as follows.

1. We choose $X_1, \ldots, X_m, unsat$ as attributes, where *unsat* is a new variable;

2. we define the P-tree T_Φ (cf. Figure 4.9) to consist of a single node labeled by $\Psi = \Phi \wedge \neg unsat$;

3. we set $M = \{unsat\}$.

We show that $M = \{unsat\}$ is not an optimal outcome if and only if $\Phi = \{c_1, \ldots, c_n\}$ is satisfiable.

($\Rightarrow$) Assume that $M = \{unsat\}$ is not an optimal outcome. Since $M \not\models \Psi$, M belongs to the right leaf and there must exist an outcome M' such that $M' \succ M$. This means that $M' \models \Phi \wedge \neg unsat$. Thus, Φ is satisfiable.

($\Leftarrow$) Let M' be a satisfying assignment to Φ over $\{X_1, \ldots, X_m\}$. Since no $c_i \in \Phi$ mentions *unsat*, we can assume $unsat \notin M'$. So $M' \models \Psi$ and M' is optimal. Thus, $M = \{unsat\}$ is not optimal. □

Theorem 7. *The* OptProp *problem is Δ_2^P-complete.*

Proof. (Membership) The problem is in the class Δ_2^P. Let T be a given preference tree. To check whether there is an optimal outcome that satisfies a property α, we start at the root of T and move down. As we do so, we maintain the information about the path we took by updating a formula ψ, which initially is set to $\top$ (a generic tautology). Each time we move down to the left from a node t, we update ψ to $\psi \wedge \varphi_t$, and when we move down to the right, to $\psi \wedge \neg\varphi_t$. To decide whether to move down left or right form a node t, we check if $\varphi_t \wedge \psi$ is satisfiable by making a call to an NP oracle for deciding satisfiability. If $\varphi_t \wedge \psi$ is satisfiable, we proceed to the left subtree and, otherwise, to the right one. We then update t to be the node we moved to and repeat. When we reach a leaf of the tree (which represents a cluster of outcomes), this cluster is non-empty, consists of all outcomes satisfying ψ and all these outcomes are optimal. Thus, returning YES, if $\psi \wedge \alpha$ is satisfiable and NO, otherwise, correctly decides the problem. Since the number of oracle calls is polynomial in the size of the tree T, the problem is in the class Δ_2^P.

(Hardness) The maximum satisfying assignment (MSA) problem[2] [53] is Δ_2^P-complete. We first show that MSA remains Δ_2^P-hard if we restrict the input to Boolean formulas that are satisfiable and have models other than the all-false model (i.e., $0_1 \dots 0_n$).

Lemma 1. *The MSA problem is Δ_2^P-complete when Φ is satisfiable and has models other than the all-false model.*

Proof. Given a Boolean formula Φ over $\{X_1, \dots, X_n\}$, we define $\Psi = \Phi \vee (X_0 \wedge \neg X_1 \wedge \dots \wedge \neg X_n)$ over $\{X_0, X_1, \dots, X_n\}$. It is clear that Ψ is satisfiable, and has at least one model other than the all-false one. Let M be a lexicographically maximum assignment satisfying Φ and M has $X_n = 1$. Extending M by $X_0 = 1$ yields a lexicographically maximum assignment satisfying Ψ and this assignment satisfies $X_n = 1$. Conversely, if M is a lexicographically maximum assignment satisfying Ψ and $X_n = 1$ holds in M, it follows that $M \models \Phi$. Thus, restricted M to $\{X_1, \dots, X_n\}$, the assignment is lexicographically maximal satisfying Φ. $\square$

[2]Given a Boolean formula Φ over $\{X_1, \dots, X_n\}$, the maximum satisfying assignment (MSA) problem is to decide whether $x_n = 1$ in the lexicographically maximum satisfying assignment for Φ. (If Φ is unsatisfiable, the answer is *no*.)

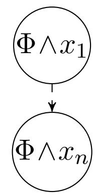


Figure 4.10: The P-tree T_Φ

We now show the hardness of the OptProp problem by a reduction from this restricted version of the MSA problem. Let Φ be a satisfiable propositional formula over variables $X_1, \ldots, X_n$ that has at least one model other than the all-false one. We construct an instance of the OptProp problem as follows. We define the P-tree T_Φ as shown in Figure 4.10, where every node is labeled by formula $\Phi \wedge X_i$, and we set $\alpha = X_n$.

Our P-tree T_Φ induces a total preorder consisting of a sequence of singleton clusters, each containing an outcome satisfying Φ, followed by a single cluster comprising all outcomes that falsify Φ and the all-false model. By our assumption on Φ, the total preorder has at least two non-empty clusters. Moreover, all singleton clusters preceding the last one are ordered lexicographically. Thus, the optimal outcome of T_Φ satisfies α if and only if the lexicographical maximum satisfying outcome of Φ satisfies x_n. $\square$

4.5 Conclusions

We investigated the qualitative preference representation language of *preference trees*, or *P-trees*. This language was introduced in early 1990s (cf. [34, 35]), but have not received a substantial attention as a formalism for preference representation in AI. We studied formally the attribute of compact representations of P-trees, established its relationship to other preference languages such as lexicographic preference trees, possibilistic logic and answer-set optimization. For several preference reasoning problems on P-trees we derived their their computational complexity.

P-trees are quite closely related to possibilistic logic theories or preference expressions in answer-set optimization. However, they allow for much more structure among formulas appearing in these latter two formalisms (arbitrary trees as opposed to the linear structure of preference formulas in the other two formalisms). This structure allows for representations of conditional preferences. P-trees are also more expressive than lexicographic preference trees. This is the case even for P-trees in which every node is labeled with a formula involving just two attributes, as we illustrated with the $00 \succ 11 \succ 01 \succ 01$ example. Such P-trees are still simple enough to correspond well to the way humans formulate hierarchical models of preferences, with all their decision conditions typically restricted to one or two attributes.

Our work shows that P-trees form a rich preference formalism that deserves further studies. Among the open problems of interest are those of learning P-trees and their compact representations, aggregating P-trees coming from different sources (agents), and computing optimal consensus outcomes. These problems will be considered in the future work.

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Chapter 5 Learning Partial Lexicographic Preference Trees

We introduce *partial lexicographic preference trees* (PLP-trees) as a formalism for compact representations of preferences over combinatorial domains. Our main results concern the problem of passive learning of PLP-trees. Specifically, for several classes of PLP-trees, we study how to learn (i) a PLP-tree consistent with a dataset of examples, possibly subject to requirements on the size of the tree, and (ii) a PLP-tree correctly ordering as many of the examples as possible in case the dataset of examples is inconsistent. We establish complexity of these problems and, in all cases where the problem is in the class P, propose polynomial time algorithms.

5.1 Introduction

Recently, there has been a rising interest in representing preferences over combinatorial domains by exploiting the notion of the lexicographic ordering. For instance, assuming attributes are over the binary domain $\{0, 1\}$, with the preferred value for each attribute being 1, a sequence of attributes naturally determines an order on outcomes. This idea gave rise to the language of *lexicographic preference models* or *lexicographic strategies*, which has been extensively studied in the literature [76, 27, 81]. The formalism of complete *lexicographic preference trees* (LP-trees) [11] generalizes the language of lexicographic strategies by arranging attributes into decision trees that assign preference ranks to outcomes. An important aspect of LP-trees is that they allow us to model *conditional* preferences on attributes and *conditional* ordering of attributes. Another formalism, the language of *conditional lexicographic preference trees* (or CLP-trees) [17], extends LP-trees by allowing subsets of attributes as labels of nodes.

A central problem in preference representation concerns learning implicit models of preferences (such as lexicographic strategies, LP-trees or CLP-trees), of possibly small sizes, that are consistent with all (or at least possibly many) given examples, each correctly ordering a pair of outcomes. The problem was extensively studied. Booth et al. [11] considered learning of LP-trees, and Bräuning and Eyke [17] of CLP-trees.

In this work, we introduce *partial lexicographic preference trees* (or *PLP-trees*) as means to represent *total preorders* over combinatorial domains. PLP-trees are closely related to LP-trees requiring that every path in the tree contains all attributes used to describe outcomes. Consequently, LP-trees describe total orders over the outcomes. PLP-trees relax this requirement and allow paths on which some attributes may be missing. Hence, PLP-trees describe total preorders. This seemingly small difference has a significant impact on some of the learning problems. It allows us to seek PLP-trees that minimize the set of attributes on their paths, which may lead to more robust models by disregarding attributes that have no or little influence on the true preference (pre)order.

The rest of the chapter is organized as follows. In the next section, we introduce the language of PLP-trees and describe a classification of PLP-trees according to their complexity. We also define three types of passive learning problems for the setting of PLP-trees. In the following sections, we present algorithms learning PLP-trees of particular types and computational complexity results on the existence of PLP-trees of different types, given size or accuracy. We close with conclusions and a brief account of future work.

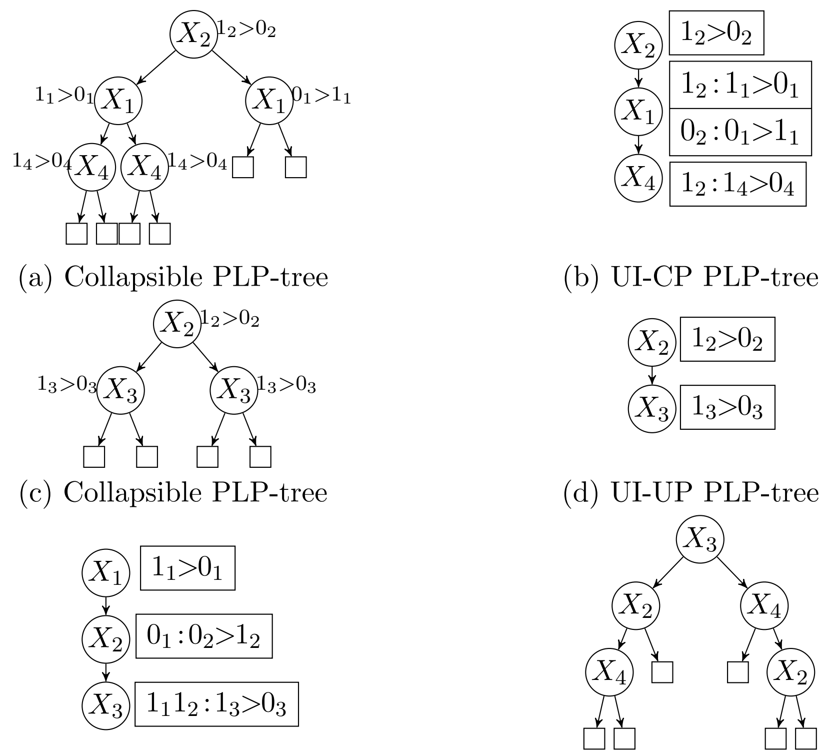
5.2 Partial Lexicographic Preference Trees

Let $\mathcal{I} = \{X_1, \ldots, X_p\}$ be a set of binary attributes, with each X_i having its domain $D_i = \{0_i, 1_i\}$. The corresponding *combinatorial domain* is the set $\mathcal{X} = D_1 \times \ldots \times D_p$. Elements of $\mathcal{X}$ are called *outcomes*.

A PLP-tree over $\mathcal{X}$ is binary tree whose every non-leaf node is labeled by an attribute from $\mathcal{I}$ and by a preference entry $1 > 0$ or $0 > 1$, and whose every leaf node is denoted by a box $\Box$. Moreover, we require that on every path from the root to a leaf each attribute appears *at most* once.

To specify the total preorder on outcomes defined by a PLP-tree T, let us enumerate leaves of T from left to right, assigning them integers $1, 2$, etc. For every outcome α we find its leaf in T by starting at the root of T and proceeding downward. When at a node labeled with an attribute X, we descend to the left or to the right child of that node based on the value $\alpha(X)$ of the attribute X in α and on the preference assigned to that node. If $\alpha(X)$ is the preferred value, we descend to the left child. We descend to the right child, otherwise. The integer assigned to the leaf that we eventually get to is the *rank* of α in T, written $r_T(\alpha)$. The preorder $\succeq_T$ on distinct outcomes determined by T is defined as follows: $\alpha \succeq_T \beta$ if $r_T(\alpha) \leq r_T(\beta)$ (smaller ranks are "better"). We also define derived relations $\succ_T$ (strict order) and $\approx_T$ (equivalence or indifference): $\alpha \succ_T \beta$ if $\alpha \succeq_T \beta$ and $\beta \not\succeq_T \alpha$, and $\alpha_T \approx_T \beta$ if $\alpha \succeq_T \beta$ and $\beta \succeq_T \alpha$. Clearly, $\succeq_T$ is a total preorder on outcomes partitioning them into strictly ordered clusters of equivalent outcomes.

To illustrate the notions just introduced, we consider preference orderings of car options over four binary attributes. The *capacity* (X_1) can be either *low* (0_1) or *high* (1_1). The *price* (X_2) is either *low* (0_2) or *high* (1_2). The *safety* (X_3) can be *low* (0_3) or *high* (1_3). Finally, the *transmission* (X_4) of a car can be *manual* (0_4) or *automatic* (1_4). An agent could specify her preferences over cars as a PLP-tree in Figure 5.1a. *Price* is the most important attribute to the agent and she prefers high to low. Her next most important attribute is *capacity* (independently of her selection for price). She prefers high over low on capacity for expensive cars, and low over high for inexpensive cars. Among the expensive cars, no matter what capacity she considers, her next consideration is the *transmission*, and she prefers automatic to manual. In this example, the attribute *safety* does not figure into preferences at all. The most preferred cars are automatic cars with high price and high capacity, with all possible combinations of choices for safety (and so, the cluster of most preferred cars has two elements).



(a) Collapsible PLP-tree (b) UI-CP PLP-tree

(c) Collapsible PLP-tree (d) UI-UP PLP-tree

(e) Invalid UI-CP PLP-tree (f) CI-FP PLP-tree

Figure 5.1: PLP-trees over the car domain

Classification of PLP-Trees

In the worst case, the size of a PLP-tree is exponential in the number of attributes in $\mathcal{I}$. However, some PLP-trees have a special structure that allows us to "collapse" them and obtain more compact representations. This yields a natural classification of PLP-trees, which we describe below.

Let $R \subseteq \mathcal{I}$ be the set of attributes that appear in a PLP-tree T. We say that T is *collapsible* if there is a permutation $\hat{R}$ of elements in R such that for every path in T

from the root to a leaf, attributes that label nodes on that path appear in the same order in which they appear in $\hat{R}$.

If a PLP-tree T is collapsible, we can represent T by a single path of nodes labeled with attributes according to the order in which they occur in $\hat{R}$, where a node labeled with an attribute X_i is also assigned a *partial conditional preference table* (PCPT) that specifies preferences on X_i, conditioned on values of ancestor attributes in the path. These tables make up for the lost structure of T as different ways in which ancestor attributes evaluate correspond to different locations in the original tree T. Moreover, missing entries in PCPT of X_i imply equivalence (or indifference) between values of X_i under conditions that do not appear in the PCPT. Clearly, the PLP-tree in Figure 5.1a is collapsible, and can be represented compactly as a single-path tree with nodes labeled by attributes in the permutation and PCPTs (cf. Figure 5.1b). Such a collapsed path labeled by attributes is sometimes denoted as a sequence of attributes in $\hat{R}$ connected by $\triangleright$, e.g., $X_2 \triangleright X_3 \triangleright X_1$ for the path in Figure 5.1b.

Collapsible PLP-trees represented by a single path of nodes will be referred to as *unconditional importance* trees or *UI* trees, for short. The name reflects the fact that the order in which we consider attributes when seeking the rank of an outcome is always the same (not conditioned on the values of ancestor attributes of higher importance).

Let L be a collapsible PLP-tree. If for every path in L the order of attributes labeling the path is exactly $\hat{R}$, and L has the same preference $1 > 0$ on *every* node, then every PCPT in the collapsed tree contains the same preference $1 > 0$, no matter the evaluation of the ancestor attributes. Thus, every PCPT in the collapsed form can be simplified to a single *fixed* preference $1 > 0$, a shorthand for its full-sized counterpart. We call the resulting collapsed tree a *UI* tree with *fixed preferences*, or a *UI-FP* PLP-tree.

A similar simplification is possible if every path in L has the same ordering of attributes which again is exactly $\hat{R}$, and for every attribute X_i all nodes in L labeled with X_i have the same preference on values of X_i (either $1_i > 0_i$ or $0_i > 1_i$). Such collapsed trees are called *UI-UP* PLP-trees, with *UP* standing for *unconditional preference*. As an example, the *UI-UP* tree in Figure 5.1d is the collapsed representation of the collapsible tree in Figure 5.1c.

In all other cases, we refer to collapsed PLP-trees as *UI-CP* PLP-trees, with *CP* standing for *conditional preference*. If preferences on an attribute in such a tree depend in an essential way on all preceding attributes, there is no real saving in the size of representation (instead of an exponential PLP-tree we have a small tree but with preference tables that are of exponential size). However, if the preference on an attribute depends only on a few higher importance attributes say, never more than one or two (or, more generally, never more than some fixed bound b), the collapsed representation is significantly smaller.

As an aside, we note that not every path of nodes labeled with attributes and PCPTs is a *UI* tree. An example is given in Figure 5.1e. Indeed, one can see that there is no PLP-tree that would collapse to it. There is a simple condition characterizing paths with nodes labeled with attributes and PCPTs that are valid *UI* trees. This matter is not essential to our discussion later on and we will not discuss it further here.

When a PLP-tree is not collapsible, the importance of an attribute depends on where it is located in the tree. We will refer to such PLP-trees as *conditional importance* trees or *CI* trees.

Let T be a *CI* PLP-tree. We call T a *CI-FP* tree if every non-leaf node in T is labeled by an attribute with preference $1 > 0$. An example of a *CI-FP* PLP-tree is shown in Figure 5.1f, where preferences on each non-leaf node are $1 > 0$ and hence omitted. If, for every attribute X_i, all nodes in T labeled with X_i have the same preference ($1_i > 0_i$ or $0_i > 1_i$) on X_i, we say T is a *CI-UP* PLP-tree. All other non-collapsible PLP-trees are called *CI-CP* PLP-trees.

5.3 Passive Learning

An *example* is a tuple (α, β, v), where α and β are two *distinct* outcomes from combinatorial domain $\mathcal{X}$ over a set $\mathcal{I} = \{X_1, \ldots, X_p\}$ of binary attributes, and $v \in \{0, 1\}$. An example $(\alpha, \beta, 1)$ states that α is strictly preferred to β ($\alpha \succ \beta$). Similarly, an example $(\alpha, \beta, 0)$ states that α and β are equivalent ($\alpha \approx \beta$). Let $\mathcal{E} = \{e_1, \ldots, e_m\}$ be a set of examples over $\mathcal{I}$, with $e_i = (\alpha_i, \beta_i, v_i)$. We set $\mathcal{E}^{\approx} = \{e_i \in \mathcal{E} : v_i = 0\}$, and $\mathcal{E}^{\succ} = \{e_i \in \mathcal{E} : v_i = 1\}$. In the following, we denote by p and m the number of attributes and the number of examples, respectively.

For a PLP-tree T in full representation we denote by $|T|$ the size of T, that is, the number of nodes in T. If T stands for a *UI* tree, we write $|T|$ for the size of T measured by the total size of preference tables associated with attributes in T. The size of a preference table is the total size of preferences in it, each preference measured as the number of values in the condition plus 1 for the preferred value in the domain of the attribute. In particular, the sizes of *UI-FP* and *UI-UP* trees are given by the number of nodes on the path.

A PLP-tree T *satisfies* an example e if T orders the two outcomes of e in the same way as they are ordered in e. Otherwise, T *falsifies* e. Formally, T *satisfies* $e = (\alpha, \beta, 1)$ if $\alpha \succ_T \beta$, and T *satisfies* $e = (\alpha, \beta, 0)$ if $\alpha \approx_T \beta$. We say T is *consistent* with a set $\mathcal{E}$ of examples if T satisfies every example in $\mathcal{E}$.

In this work, we study the following passive learning problems for PLP-trees of all types we introduced.

Definition 31. Consistent-learning (ConsLearn): given an example set $\mathcal{E}$, decide whether there exists a PLP-tree T (of a particular type) such that T is consistent with $\mathcal{E}$.

Definition 32. Small-learning (SMALLLEARN): given an example set $\mathcal{E}$ and a positive integer l ($l \leq |\mathcal{E}|$), decide whether there exists a PLP-tree T (of a particular type) such that T is consistent with $\mathcal{E}$ and $|T| \leq l$.

Definition 33. Maximal-learning (MAXLEARN): given an example set $\mathcal{E}$ and a positive integer k ($k \leq m$), decide whether there exists a PLP-tree T (of a particular type) such that T satisfies at least k examples in $\mathcal{E}$.

5.4 Learning UI PLP-trees

In this section, we study the passive learning problems for collapsible PLP-trees in their collapsed representations as *UI-FP*, *UI-UP* and *UI-CP* trees.

The ConsLearn Problem

The CONSLEARN problem is in the class P for *UI-FP* and *UI-UP* trees. To show it, we present a general template of an algorithm that learns a *UI* tree. Next, for each of the classes *UI-FP* and *UI-UP*, we specialize the template to a polynomial-time algorithm.

The template algorithm is shown as Algorithm 1. The input consists of a set $\mathcal{E}$ of examples and a set $\mathcal{I}$ of attributes from which node labels can be selected. Throughout the execution, the algorithm maintains a set S of unused attributes, initialized to $\mathcal{I}$, and a set of examples that are not yet ordered by the tree constructed so far.

If the set of strict examples is empty, the algorithm returns an empty tree. Otherwise, the algorithm identifies the set $AI(\mathcal{E}, S)$ of attributes in S that are *available* for selection as the label for the next node. If that set is empty, the algorithm terminates with failure. If not, an attribute, say X_l, is selected from $AI(\mathcal{E}, S)$, and a PCPT for that attribute is constructed. Then the sets of examples not ordered yet and of attributes not used yet are updated, and the steps repeat.

Algorithm 1: Procedure *learnUI* that learns a UI tree

Input: $\mathcal{E}$ and $S = \mathcal{I}$

Output: A sequence T of attributes from $\mathcal{I}$ and PCPTs that define a *UI* tree consistent with $\mathcal{E}$, or FAILURE if such a tree does not exist

1 $T \leftarrow$ empty sequence;
2 **while** $\mathcal{E}^{\succ} \neq \emptyset$ **do**
3   Construct $AI(\mathcal{E}, S)$;
4   **if** $AI(\mathcal{E}, S) = \emptyset$ **then**
5   **return** FAILURE;
6   **end**
7     $X_l \leftarrow$ an element from $AI(\mathcal{E}, S)$;
8   Construct $PCPT(X_l)$;
9     $T \leftarrow T, (X_l, PCPT(X_l))$;
10     $\mathcal{E} \leftarrow \mathcal{E} \backslash \{e \in \mathcal{E}^{\succ} : e \text{ is decided on } X_l\}$;
11     $S \leftarrow S \backslash \{X_l\}$;
12 **end**
13 **return** T;

To obtain a learning algorithm for a particular class of *UI* trees (*UI-FP* or *UI-UP*) we need to specify the notion of an available attribute (needed for line 3) and describe how to construct a partial conditional preference table (needed for line 8).

To this end, let us define $NEQ(\mathcal{E}, S)$ to be the set of all attributes in S (where $S \subseteq \mathcal{I}$) that incorrectly handle at least one equivalent example in $\mathcal{E}^{\approx}$. That is, for an attribute $X \in S$ we have $X \in NEQ(\mathcal{E}, S)$ precisely when for some example $(\alpha, \beta, 0)$ in $\mathcal{E}$, $\alpha(X) \neq \beta(X)$. Similarly, let us define $EQ(\mathcal{E}, S)$ to be the set of attributes in S that do not order any of the strict examples in $\mathcal{E}$. That is, for an attribute $X \in S$ we have $X \in EQ(\mathcal{E}, S)$ precisely when for every example $(\alpha, \beta, 1)$ in $\mathcal{E}$, $\alpha(X) = \beta(X)$.

Fixed Preferences. For the problem of learning *UI-FP* trees, we define $AI(\mathcal{E}, S)$ to contain every attribute $X \notin NEQ(\mathcal{E}, S)$ such that

(1) for every $(\alpha, \beta, 1) \in \mathcal{E}^{\succ}$, $\alpha(X) \geq \beta(X)$.

Proposition 2. *If there is a UI-FP tree consistent with all examples in $\mathcal{E}$ and using only attributes from S as labels, then an attribute $X \in S$ is a top node of some such tree if and only if $X \in AI(\mathcal{E}, S)$.*

Proof. Let T be a *UI* tree consistent with $\mathcal{E}$ and having only attributes from S as labels. Let X be the attribute labeling the top node of T. Clearly, $X \notin NEQ(\mathcal{E}, S)$, as otherwise, T would strictly order two outcomes α and β such that $(\alpha, \beta, 0) \in \mathcal{E}^{\approx}$. To prove condition (1), let us consider any example $(\alpha, \beta, 1) \in \mathcal{E}^{\succ}$. Since T is consistent with $(\alpha, \beta, 1)$, $\alpha(X) \geq \beta(X)$. Consequently, $X \in AI(\mathcal{E}, S)$.

Conversely, let $X \in AI(\mathcal{E}, S)$ and let T be a *UI-FP* tree consistent with all examples in $\mathcal{E}$ and using only attributes from S as labels (such a tree exists by assumption). If X labels the top node in T, we are done. Otherwise, let T' be a tree obtained from T by adding at the top of T another node, labeling it with X and removing from T the node labeled by X, if such a node exists. By the definition of $AI(\mathcal{E}, S)$ we have that $X \notin NEQ(\mathcal{E}, S)$ and that condition (1) holds for X. Using these properties, we see that T' is also a *UI-FP* tree consistent with all examples in $\mathcal{E}$. Since the top node of T' is labeled by X, the assertion follows. □

We now specialize Algorithm 1 by using in line 3 the definition of $AI(\mathcal{E}, S)$ given above and by setting each $PCPT(X_l)$ to the fixed unconditional preference $1_l > 0_l$. Proposition 2 directly implies the correctness of this version of Algorithm 1.

Theorem 8. *Let $\mathcal{E}$ be a set of examples over a set $\mathcal{I}$ of binary attributes. Algorithm 1 adjusted as described above terminates and outputs a sequence T representing a UI-FP tree consistent with $\mathcal{E}$ if and only if such a tree exists.*

We note that attributes in $NEQ(\mathcal{E}, S)$ are never used when constructing $AI(\mathcal{E}, S)$. Thus, in the case of *UI-FP* trees, S could be initialized to $\mathcal{I} \setminus NEQ(\mathcal{E}, \mathcal{I})$. In addition, if an attribute selected for the label of the top node belongs to $EQ(\mathcal{E}^{\succ}, S)$, it does not in fact decide any of the strict examples in $\mathcal{E}$ and can be dropped. The resulting tree is also consistent with all the examples. Thus, the definition of $AI(\mathcal{E}, S)$ can be refined by requiring one more condition: $X \notin EQ(\mathcal{E}^{\succ}, S)$. That change does not affect the correctness of the algorithm but eliminates a possibility of generating trees with "redundant" levels.

Unconditional Preferences. The case of learning *UI-UP* trees is very similar to the previous one. Specifically, we define $AI(\mathcal{E}, S)$ to contain an attribute $X \in S$ precisely when $X \notin NEQ(\mathcal{E}, S)$ and

(2) for every $(\alpha, \beta, 1) \in \mathcal{E}^{\succ}$, $\alpha(X) \geq \beta(X)$, or for every $(\alpha, \beta, 1) \in \mathcal{E}^{\succ}$, $\alpha(X) \leq \beta(X)$.

We obtain an algorithm learning *UI-UP* trees by using in line 3 the present definition of $AI(\mathcal{E}, S)$. In line 8, we take for $PCPT(X_l)$ either $1_l > 0_l$ or $0_l > 1_l$ (depending on which of the two cases in (2) holds for X_l).

The correctness of this algorithm follows from a property similar to that in Proposition 2.

As in the previous case, here too S could be initialized to $\mathcal{I} \setminus NEQ$, and the condition $X \notin EQ(\mathcal{E}^{\succ}, S)$ could be added to the definition of $AI(\mathcal{E}, S)$.

Conditional Preferences. The problem is in NP because, if a *UI-CP* tree consistent with $\mathcal{E}$ exists (*a priori*, it does not have to have size polynomial in the size of $\mathcal{E}$), then another such tree of size polynomial in the size of $\mathcal{E}$ exists, as well. We conjecture that the general problem of learning *UI-CP* trees is, in fact, NP-complete. As we have only partial results for this case, the study of the *UI-CP* tree learning will be the subject of future work.

The SmallLearn Problem

Algorithm 1 produces a *UI* PLP-tree consistent with $\mathcal{E}$, if one exists. In many cases, it is desirable to compute a small, sometimes even the smallest, representation consistent with $\mathcal{E}$. We show that these problems for *UI* trees are NP-hard.

Theorem 9. *The* SmallLearn *problem is NP-complete for each class of* $\{UI\} \times \{FP, UP, CP\}$.

Proof. We present the proof only in the case of *UI-FP*. The argument in other cases (*UI-UP* and *UI-CP*) is similar.

(Membership) One can guess a *UI-FP* PLP-tree T in linear time, and verify in polynomial time that T has at most l attributes and satisfies every example in $\mathcal{E}$.

(Hardness) We present a polynomial-time reduction from the *hitting set problem* (HSP), which is NP-complete [40]. To recall, in HSP we are given a finite set $U = \{a_1, \ldots, a_n\}$, a collection $C = \{S_1, \ldots, S_d\}$ of subsets of U with $\bigcup_{S_i \in C} S_i = U$, and a positive integer $k \leq n$, and the problem is to decide whether U has a hitting set U' such that $|U'| \leq k$ ($U' \subseteq U$ is a *hitting* set for C if $U' \cap S_i \neq \emptyset$ for all $S_i \in C$). Given an instance of HSP, we construct an instance of our problem as follows.

1. $\mathcal{I} = \{X_i : a_i \in U\}$ (thus, $p = n$).
2. $\mathcal{E} = \{(\mathbf{s}_i, \mathbf{0}, 1) : S_i \in C\}$, where $\mathbf{s}_i$ is a p-bit vector such that $\mathbf{s}_i[j] = 1 \Leftrightarrow a_j \in S_i$ and $\mathbf{s}_i[j] = 0 \Leftrightarrow a_j \notin S_i$ ($1 \leq j \leq p$), and $\mathbf{0}$ is a p-bit vector of all 0's (thus, $m = d$).
3. We set $l = k$.

We need to show that U has a hitting set of size at most k if and only if there exists a *UI-FP* PLP-tree of size at most l consistent with $\mathcal{E}$.

($\Rightarrow$) Assume U has a hitting set U' of size k. Let U' be $\{a_{j_1}, \ldots, a_{j_k}\}$. Define a *UI-FP* PLP-tree $L = X_{j_1} \triangleright \ldots \triangleright X_{j_k}$. We show that L is consistent with $\mathcal{E}$. Let $e = (\alpha_e, \beta_e, 1)$ be an arbitrary example in $\mathcal{E}$, where $\alpha_e = \mathbf{s}_i$ and $\beta_e = \mathbf{0}$. Since U' is a hitting set, there exists r, $1 \leq r \leq k$, such that $a_{j_r} \in S_i$. Thus, there exists r, $1 \leq r \leq k$, such that $\alpha_e(X_{j_r}) = 1$. Let r be the smallest with this property. It is clear that e is decided at X_{j_r}; thus, we have $\alpha_e \succ_L \beta_e$.

($\Leftarrow$) Assume there is a *UI-FP* PLP-tree L of l attributes in $\mathcal{I}$ such that L is consistent with $\mathcal{E}$. Moreover, we assume $L = X_{j_1} \triangleright \ldots \triangleright X_{j_l}$. Let $U' = \{a_{j_1}, \ldots, a_{j_l}\}$. We show by means of contradiction. Assume that U' is not a hitting set. That is, there exists a set $S_i \in C$ such that $U' \cap S_i = \emptyset$. Then, there exists an example $e = (\alpha_e, \beta_e, 1)$, where $\alpha_e = \mathbf{s}_i$ and $\beta_e = \mathbf{0}$, such that $\alpha_e \approx_L \beta_e$ because none of the attributes

$\{X_i : \alpha_e(X_i) = 1\}$ show up in L. This is a contradiction! Thus, U' is a hitting set. □

Corollary 10. *Given a set $\mathcal{E}$ of examples $\{e_1, \ldots, e_m\}$ over $\mathcal{I} = \{X_1, \ldots, X_p\}$, finding the smallest PLP-tree in each class of $\{UI\} \times \{FP, UP, CP\}$ consistent with $\mathcal{E}$ is NP-hard.*

Consequently, it is important to study fast heuristics that aim at approximating trees of optimal size. Here, we propose a greedy heuristic for Algorithm 1. In every iteration the heuristic selects the attribute $X_l \in AI(\mathcal{E}, S)$ that decides the most examples in $\mathcal{E}^{\succ}$. However, for some dataset the resulting greedy algorithm does not perform well: the ratio of the size of the tree computed by our algorithm to the size of the optimal sequence may be as large as $\Omega(p)$. To see this, we consider the following input.

$(1_1 0_2 0_3 0_4, 0_1 0_2 0_3 0_4, 1)$

$(1_1 1_2 0_3 0_4, 0_1 0_2 0_3 0_4, 1)$

$(1_1 0_2 1_3 0_4, 0_1 0_2 0_3 0_4, 1)$

$(0_1 0_2 0_3 1_4, 1_1 0_2 0_3 0_4, 1)$

For each class of $\{UI\} \times \{FP, UP\}$, Algorithm 1 in the worst case computes $X_2 \triangleright X_3 \triangleright X_4 \triangleright X_1$, whereas the optimal tree is $X_4 \triangleright X_1$ (with the PCPTs omitted as they contain only one preference and so, they do not change the asymptotic size of the tree). This example generalizes to the arbitrary number p of attributes. Thus, the greedy algorithm to learn small *UI* trees is no better than any other algorithm in the worst case.

Approximating HSP has been extensively studied over the last decades. It has been shown [65] that, unless $NP \subset DTIME(n^{poly \log n})$, HSP cannot be approximated in polynomial time within factor of $c \log n$, where $0 < c < \frac{1}{4}$ and n is the number of elements in the input. The reduction we used above shows that this result carries over to our problem.

Theorem 11. *Unless NP $\subset$ DTIME($n^{poly \log n}$), the problem of finding the smallest PLP-tree in each class of $\{UI\} \times \{FP, UP, CP\}$ consistent with $\mathcal{E}$ cannot be approximated in polynomial time within factor of $c \log p$, where $0 < c < \frac{1}{4}$.*

It is an open problem whether this result can be strengthened to a factor linear in p (cf. the example for the worst-case behavior of our simple greedy heuristic).

The MaxLearn Problem

When there is no *UI* PLP-tree consistent with the set of all examples, it may be useful to learn a *UI* PLP-tree satisfying as many examples as possible. We show this problem is in fact NP-hard for all three classes of *UI* trees.

Theorem 12. *The* MaxLearn *problem is NP-complete for each class of $\{UI\} \times \{FP, UP, CP\}$.*

Proof. The problem is in NP. This is evident for the case of *UI-FP* and *UI-UP* trees. If $\mathcal{E}$ is a given set of examples, and k a required lower bound on the number of examples that are to be correctly ordered, then witness trees in these classes (trees that correctly order at least k examples in $\mathcal{E}$) have size polynomial in the size of $\mathcal{E}$. Thus, verification can be performed in polynomial time. For the case of *UI-CP* trees, if there is a *UI-CP* tree correctly ordering at least k examples in $\mathcal{E}$, then there exists such tree of size polynomial in $|\mathcal{E}|$.

The hardness part follows from the proof in the setting of learning lexicographic strategies [76], adapted to the case of *UI* PLP-trees. □

Corollary 13. *Given a set $\mathcal{E}$ of examples $\{e_1, \ldots, e_m\}$ over $\mathcal{I} = \{X_1, \ldots, X_p\}$, finding a PLP-tree in each class of $\{UI\} \times \{FP, UP, CP\}$ satisfying the maximum number of examples in $\mathcal{E}$ is NP-hard.*

5.5 Learning CI PLP-trees

Finally, we present results on the passive learning problems for PLP-trees in classes $\{CI\} \times \{FP, UP, CP\}$. We recall that these trees assume full (non-collapsed) representation.

The ConsLearn Problem

We first show that the ConsLearn problem for class *CI-UP* is NP-complete. We then propose polynomial-time algorithms to solve the ConsLearn problem for the classes *CI-FP* and *CI-CP*.

Theorem 14. *The* ConsLearn *problem is NP-complete for class CI-UP.*

Proof. The problem is in NP because the size of a witness, a *CI-UP* PLP-tree consistent with $\mathcal{E}$, is bounded by $|\mathcal{E}|$ (if a *CI-UP* tree consistent with $\mathcal{E}$ exists, then it can be modified to a tree of size no larger than $O(|\mathcal{E}|)$). Hardness follows from the proof by Booth et al. [11] showing ConsLearn is NP-hard in the setting of LP-trees. □

For the two other classes of trees, the problem is in P. This is demonstrated by polynomial-time Algorithm 2 adjusted for both classes.

Fixed Preference. For class *CI-FP*, we define $AI(\mathcal{E}, S)$ to contain attribute $X \notin NEQ(\mathcal{E}, S)$ if

(3) for every $(\alpha, \beta, 1) \in \mathcal{E}^{\succ}$, $\alpha(X) \geq \beta(X)$.

Proposition 3. *If there is a CI-FP tree consistent with all examples in $\mathcal{E}$ and using only attributes from S as labels, then an attribute $X \in S$ is a top node of some such tree if and only if $X \in AI(\mathcal{E}, S)$.*

Proof. It is clear that if there exists a *CI-FP* PLP-tree consistent with $\mathcal{E}$ and only using attributes from S as labels, then the fact that $X \in S$ labels the root of some such tree implies $X \in AI(\mathcal{E}, S)$.

Algorithm 2: The recursive procedure *learnCI* that learns a CI PLP-tree

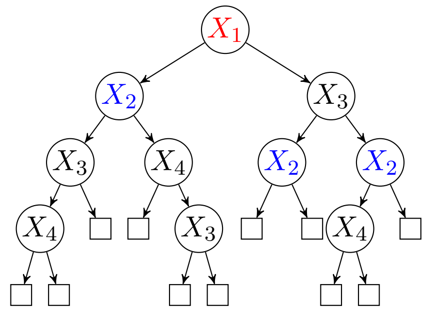
Input: $\mathcal{E}$, $S = \mathcal{I}$, and t: an unlabeled node
Output: A CI PLP-tree over S consistent with $\mathcal{E}$, or FAILURE

if $\mathcal{E}^{\succ} = \emptyset$ **then**
  Label t as a leaf and **return**;
end
Construct $AI(\mathcal{E}, S)$;
if $AI(\mathcal{E}, S) = \emptyset$ **then**
  return FAILURE and terminate;
end
Label t with tuple (X_l, x_l) where X_l is from $AI(\mathcal{E}, S)$, and x_l is the preferred value on X_l;
$\mathcal{E} \leftarrow \mathcal{E} \backslash \{e \in \mathcal{E}^{\succ} : e \text{ is decided on } X_l\}$;
$S \leftarrow S \backslash \{X_l\}$;
Create two edges u_l, u_r and two unlabeled nodes t_l, t_r such that $u_l = \langle t, t_l \rangle$ and $u_r = \langle t, t_r \rangle$;
$\mathcal{E}_l \leftarrow \{e \in \mathcal{E} : \alpha_e(X_j) = \beta_e(X_j) = x_l\}$;
$\mathcal{E}_r \leftarrow \{e \in \mathcal{E} : \alpha_e(X_j) = \beta_e(X_j) = \overline{x_l}\}$;
$learnCI(\mathcal{E}_l, S, t_l)$;
$learnCI(\mathcal{E}_r, S, t_r)$;

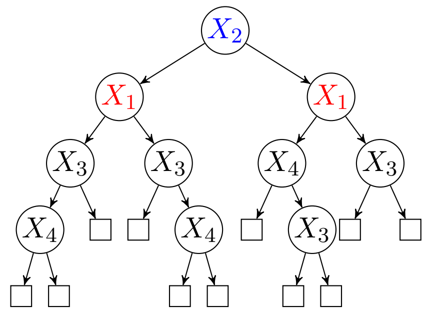
Now we show the other direction. Let T be the *CI-FP* tree over a subset of S consistent with $\mathcal{E}$, X be an attribute such that $X \in AI(\mathcal{E}, S)$. If X is the root attribute in T, we are done. Otherwise, we construct a *CI-FP* tree T' by creating a root, labeling it with X, and make one copy of T the left subtree of T' (T'_l) and another, the right subtree of T' (T'_r). For a node t and a subtree B in T, we write t'_l and B'_l, respectively, for the corresponding node and subtree in T'_l. We define t'_r and B'_r similarly. If X does not appear in T, we are done constructing T'; otherwise, we update T' as follows.

1). For every node $t \in T$ labeled by X such that t has two leaf children, we replace the subtrees rooted at t'_l and t'_r in T'_l and T'_r with leaves.

2). For every node $t \in T$ labeled by X such that t has one leaf child and a non-leaf subtree B, we replace the subtree rooted at t'_l in T'_l with B'_l, and the subtree rooted at t'_r in T'_r with a leaf, if $t \in T$ has a right leaf child; otherwise, we replace the subtree rooted at t'_l in T'_l with a leaf, and the subtree rooted at t'_r in T'_r with B'_r.



(a) T



(b) T'

Figure 5.2: $X_2 \in AI(\mathcal{E}, S)$ is picked at the root

3). Every other node $t \in T$ labeled by X has two non-leaf subtrees: left non-leaf subtree BL and right BR. For every such node $t \in T$, we replace the subtree rooted at t'_l in T'_l with BL'_l, and the subtree rooted at t'_r in T'_r with BR'_r.

As an example, this construction of T' from T is demonstrated in Figure 5.2. We see that this construction results in a *CI-CP* tree consistent with $\mathcal{E}$ and, clearly, it has its root labeled with X. Thus, the assertion follows. □

Proposition 3 clearly implies the correctness of Algorithm 2 with $AI(\mathcal{E}, S)$ defined as above for class *CI-FP* and each $x_l \in (X_l, x_l)$ set to 1.

Theorem 15. *Let $\mathcal{E}$ be a set of examples over a set $\mathcal{I}$ of binary attributes. Algorithm 2 adjusted as described above terminates and outputs a CI-FP tree T consistent with $\mathcal{E}$ if and only if such a tree exists.*

Conditional Preference. For class *CI-CP*, we define that $AI(\mathcal{E}, S)$ contains attribute $X \notin NEQ(\mathcal{E})$ if

(4) for every $(\alpha, \beta, 1) \in \mathcal{E}^{\succ}$, $\alpha(X) \geq \beta(X)$, or for every $(\alpha, \beta, 1) \in \mathcal{E}^{\succ}$, $\alpha(X) \leq \beta(X)$.

We obtain an algorithm learning *CI-CP* trees by using in line 4 the present definition of $AI(\mathcal{E}, S)$. In line 8, we take for x_l either 1 or 0 (depending on which of the two cases in (4) holds for X_l). The correctness of this algorithm follows from a property similar to that in Proposition 3.

The SmallLearn and MaxLearn Problems

We outline the results we have for this case. Both problems for the three *CI* classes are NP-complete. They are in NP since if a witness PLP-tree exists, one can modify it so that its size does not exceed the size of the input. Hardness of the SmallLearn problem for *CI* classes follows from the proof of Theorem 9, whereas the hardness of the MaxLearn problem for *CI* cases follows from the proof by Schmitt and Martignon [76].

5.6 Conclusions

We proposed a preference language, *partial lexicographic preference trees*, *PLP-trees*, as a way to represent preferences over combinatorial domains. For several natural classes of PLP-trees, we studied passive learning problems: ConsLearn, SmallLearn and MaxLearn. All complexity results we obtained are summarized in tables in Table 5.1. The ConsLearn problem for *UI-CP* trees is as of now unsettled. While we are aware of subclasses of *UI-CP* trees for which polynomial-time algorithms are possible, we conjecture that in general, the problem is NP-complete.

Table 5.1: Complexity results for passive learning problems

	FP	UP	CP
UI	P	P	*NP*
CI	P	NPC	P

(a) ConsLearn

	FP	UP	CP
UI	NPC	NPC	NPC
CI	NPC	NPC	NPC

(b) SmallLearn & MaxLearn

For the future research, we will develop good heuristics for our learning algorithms. We will implement these algorithms handling attributes of, in general, finite domains of values, and evaluate them on both synthetic and real-world preferential datasets. With PLP-trees of various classes learned, we will compare our models with the ones learned through other learning approaches on predicting new preferences.

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Chapter 6 Empirical Evaluation of Algorithms to Learn PLP-Trees and PLP-Forests

Partial lexicographic preference trees, or *PLP-trees*, form an intuitive formalism for compact representation of qualitative preferences over combinatorial domains. In this chapter, we show that PLP-trees can be used to accurately model preferences arising in practical situations, and that high-accuracy PLP-trees can be effectively computed. We also propose and study a variant of the model based on the concept of a *PLP-forest*, a *collection* of PLP-trees, where the preference order specified by a PLP-forest is obtained by aggregating the orders of its constituent PLP-trees. The motivation is that learning many PLP-trees, each from a small set of examples, often is faster than learning a single tree from a large example set yet, thanks to aggregation, yields an accurate and robust representation of the preference order being modeled. We propose and implement several algorithms to learn PLP-trees and PLP-forests. To support experimentation, we use datasets that we adapted to the preference learning setting from existing classification datasets. Our results demonstrate the potential of both approaches, with learning PLP-forests showing particularly promising behavior.

6.1 Introduction

Learning preference models, that is, expressions concisely representing a preference order has been central to this research. Much of the attention was focused on learning utility functions that represent preference orders quantitatively [37]. Recently, researchers proposed several *qualitative* models of preference orders arguing that they are more directly aligned with conventions humans use when expressing their preferences. They include *conditional preference networks* (CP-nets) [13], and models ordering outcomes lexicographically such as lexicographic strategies [76], conditional lexicographic preference trees [17], lexicographic preference trees (LP-trees) [11], partial lexicographic preference trees (PLP-trees) [62], and preference trees[36, 64]. As with quantitative models, learning qualitative models is important. Indeed, eliciting them directly from users is often impractical. However, while learning CP-nets has received a fair amount of attention [54, 26, 52, 47], study of learning lexicographic models is still in the early stages. The results obtained so far concern mostly learning LP-trees [11] and conditional lexicographic preference trees [17]. Other models received less attention. In particular, no algorithms for learning PLP-trees have yet been proposed even though PLP-trees retain the simplicity of LP-trees but also offer flexibility that makes them less sensitive to overfitting.

In this chapter, we address the problem of practicality of PLP-trees as a preference representation formalism. To this end, we introduce several best-agreement and approximate algorithms to learn PLP-trees of the four classes: UIUP, UICP, CIUP, and CICP (as discussed in Chapter 5). We show experimentally that they are effective on various domains and datasets and generate trees that accurately approximate the preference order being modeled. To support our experiments, following Bräuning and Eyke [17], we generated a library of datasets of preference examples deriving them from datasets of examples developed by the machine learning community to support research on the classification problem.

PLP-trees are in some aspects similar to decision trees. When learning a decision tree, a problem that may arise is that of overfitting. To reduce its effect, Breiman [18] proposed learning a *random forest*, that is, a set of uncorrelated decision trees learned from randomly selected sets of examples. The random forest learning algorithm [18] has two key steps. First, it generates several decision trees, randomizing the attributes used in their construction. Then, to classify an instance, the algorithm *aggregates* the predictions made by individual trees in the forest by the majority rule.

We adapted both the notion of a decision forest and the idea to aggregate their elements to the setting of PLP-trees. A *PLP-forest* is a collection of PLP-trees. PLP-forests consisting exclusively of UIUP, UICP, CIUP or CICP trees are called UIUP, UICP, CIUP or CICP PLP-forests, respectively. PLP-trees in a PLP-forest are learned using randomly selected *small* fragments of a training set. To predict if one outcome is preferred over another, we apply the *pairwise majority rule* (PMR), a simple and effective voting rule studied in social choice. We adjust algorithms learning PLP-trees to the setting of PLP-forests and study their effectiveness both in terms of time and accuracy.

The key findings supported by our results are: (1) PLP-trees and PLP-forests are expressive preference models. Experiments with the datasets we constructed from commonly used machine learning classification domains showed that the accuracy of learned models typically exceeded 85%, often exceeded 90%, and in some cases was as high as 95%. (2) PLP-forests aggregated by PRM provide in general higher accuracy than PLP-trees. (3) PLP-trees and PLP-forests learned by a greedy approximation method have accuracy comparable to *best-agreement* PLP-trees and PLP-forests learned by maximizing the number of correctly handled examples in the training set. Moreover, because of overfitting arising in "best-agreement" trees and forests, in some cases, heuristic approaches offer an even better accuracy. (4) Approximation learning methods are fast and can work with large datasets; methods based on learning best-agreement trees can also be effective in practice, especially when we learn PLP-forests, where we bound the number of examples each tree in the forest is learned from.

This chapter is organized as follows. In Section 6.2, we begin with reviewing PLP-trees and their classification, extending them to domains with arbitrary multi-valued (that is, not necessarily binary) attributes. We also recall the complexity of the problem to learn PLP-trees [62]. We also discuss the preference learning library that we use in our experiments. Next, we discuss algorithms to learn PLP-trees. We consider two types of algorithms — finding best-agreement PLP-trees and finding PLP-trees based on the greedy heuristics. We then present and discuss empirical results on the performance of our PLP-tree learning algorithms. In Section 6.3, we introduce PLP-forests and specify the pairwise majority rule to aggregate trees in a PLP-forest. This is followed by an analysis of experimental results using the same datasets as before. We conclude the chapter with a brief summary and a look into possible directions for future work.

6.2 Partial Lexicographic Preference Trees

We study the four classes of PLP-trees: UIUP, UICP, CIUP, and CICP. Among UICP trees, of practical interest are those where the number of parents are bounded by some fixed integer k independent of p, and the CPTs are complete. We call this type of trees UICP-k PLP-trees. In this case, the sizes of the CPTs and, consequently, the sizes of the trees are polynomial in the number of attributes. In practice, when deciding a preference order at an attribute, humans rarely condition them on more than two attributes of higher importance. Consider the domain of cars over three binary attributes: *Capacity*, *Price* and *Safety*, with values *high* (1_1) and *low* (0_1), *high* (1_2) and *low* (0_2), and *high* (1_3) and *low* (0_3), respectively. An example of UICP-1 PLP-trees over cars is shown in Figure 6.1.

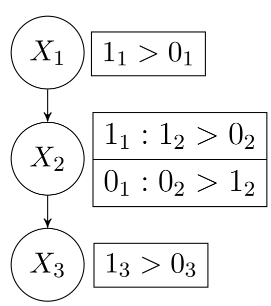


Figure 6.1: UICP-1 PLP-tree

We now discuss the complexity of the problem to learn PLP-trees. The problem assumes that we are given a collection of examples, that is, expressions $(\alpha, \beta, \succ)$ and $(\alpha, \beta, \approx)$, where α and β are outcomes. Examples of the first type are *strict* examples and of the second type *equivalence* examples. A PLP-tree T satisfies a strict example $(\alpha, \beta, \succ)$ if $\alpha \succ_T \beta$. Similarly, T satisfies an equivalence example $(\alpha, \beta, \approx)$ if $\alpha \approx_T \beta$. The objective of the problem is to compute a PLP-tree (of a specified type) that satisfies the maximum number of examples from the input set. We refer to this problem as MaxLearn.

The MaxLearn problem is NP-hard for each of the four classes of PLP-trees we discussed (when applicable, assuming that we learn collapsed representations). This is an easy consequence of the fact that the corresponding decision versions of the problem (asking for the existence of a PLP-tree of a given type satisfying at least k examples from the input set, where k is another input parameter) are NP-complete [62].

Preference Learning Library

We now describe the datasets we used in our study of learning algorithms we present later. These datasets were generated from publicly available classification datasets developed by the machine learning community. When constructing the datasets, we limited the number of attributes in outcomes to ten and the sizes of attribute domains to four.

Classification datasets associate with each outcome α a label $l(\alpha)$. If there is a total (pre)order relation on the labels, say $\succeq$, we can use this relation to produce preference examples out of classification examples. Namely, for each pair of outcomes α and β from the classification dataset, if $l(\alpha) \succ l(\beta)$, we take $(\alpha, \beta, \succ)$ as a strict example, and if $l(\alpha) = l(\beta)$, we take $(\alpha, \beta, \approx)$ as an equivalence example.[1] Through-

[1]Clearly, our preference datasets do not contain incomparability examples. This is not a limitation in our work as the preference models we learn represent total preorders.

Table 6.1: Classification datasets in UCI Machine Learning Repository used to generate preference datasets

Preference Datasets	Original Datasets in UCI MLR
BreastCancerWisconsin	Breast Cancer Wisconsin
CarEvaluation	Car Evaluation
CreditApproval	Credit Approval
GermanCredit	Statlog (German Credit Data)
Ionosphere	Ionosphere
MammographicMass	Mammographic Mass
Mushroom	Mushroom
Nursery	Nursery
SPECTHeart	SPECT Heart
TicTacToe	Tic-Tac-Toe Endgame
Vehicle	Statlog (Vehicle Silhouettes)
Wine	Wine

out the chapter, we write p for the number of attributes in a dataset, $\mathcal{X}$ for the set of outcomes, $\mathcal{E}$ for the set of examples, and $\mathcal{E}^{\succ}$ and $\mathcal{E}^{\approx}$ for the sets of strict and equivalence examples, respectively.

At present, our preference library consists of twelve datasets obtained from the classification datasets listed in Table 6.1. In ten of them there is a natural order on the labels. For the other two of them namely, Vehicle and Wine, there is no domain-specific natural order on the labels. In these two cases, to generate examples we fixed a preference order on the labels arbitrarily (see below). We discuss three preference datasets (CarEvaluation, Vehicle and Wine) in detail and provide a summary description of the remaining ones in Table 6.2, where we use $|\cdot|$ to denote the size of a set.

CarEvaluation The CarEvaluation dataset has 1728 outcomes over 6 attributes. To generate equivalent and strict examples for the dataset, we assume that outcomes labeled by “vgood” are better than those by “good,” which are better than those by “acc,” which are preferred to those by “unacc.”

Vehicle The Vehicle dataset has 455 outcomes over 10 attributes. To generate equivalent and strict examples for the dataset, we assume that outcomes labeled by

Table 6.2: Description of datasets in the library

Dataset	p	$\lvert\mathcal{X}\rvert$	$\lvert\mathcal{E}^{\succ}\rvert$	$\lvert\mathcal{E}^{\approx}\rvert$
BreastCancerWisconsin	9	270	9,009	27,306
CarEvaluation	6	1,728	682,721	809,407
CreditApproval	10	520	66,079	68,861
GermanCredit	10	914	172,368	244,873
Ionosphere	10	118	3,472	3,431
MammographicMass	5	62	792	1,099
Mushroom	10	184	8,448	8,388
Nursery	8	1,266	548,064	252,681
SPECTHeart	10	115	3,196	3,359
TicTacToe	9	958	207,832	250,571
Vehicle	10	455	76,713	26,572
Wine	10	177	10,322	5,254

"bus" are better than those by "opel," which are better than those by "saab," which are preferred to those by "van."

Wine The Wine dataset has 177 outcomes over 10 attributes. To generate equivalent and strict examples for the dataset, we assume that outcomes labeled by "1" are better than those by "2," which are better than those by "3."

Algorithms

We propose and evaluate both best-agreement and greedy algorithms for the MaxLearn problem. For these algorithms and experiments, we focus on solving the MaxLearn problem where the given the set of examples contains only strict examples. The learning algorithms are essentially to learn PLP-trees that approximate the original total preorders of at most five equivalent clusters of outcomes. A small PLP-tree with three or more nodes will already specify a preorder of more clusters. Our algorithms typically learn bigger trees. Thus, learning these tie-breaking trees provides finer-grained approximations of the original orderings and better understanding of the distribution of the outcomes according to the agents' preferences.

To find the best-agreement model, that is, to compute a PLP-tree (of a specified type) that maximizes the number of satisfied examples, we used answer-set programming (ASP) [66, 70] and its *gringo/clasp* grounder-solver tool [43]. This approach consists of two logical programming modules: the data module describing the dataset (i.e., attributes, domains, outcomes and examples), and the rule module applying an optimization statement to search for a PLP-tree that correctly decides as many examples as possible. Given an instance of the MaxLearn problem expressed as the two modules, the ASP tool *gringo/clasp* computes an answer set encoding the PLP-tree that is a solution to the input instance.

Our method to solve the MaxLearn problem approximately, that is, to compute a PLP-tree that satisfies many (but perhaps not the maximum possible) number of examples is based on a greedy approach.

Algorithm 3 provides a detailed description of the method. When the Boolean parameter Δ is set to **true**, the algorithm learns *UI* trees, otherwise, it learns *CI* trees with conditional importance of attributes. The algorithm starts with a non-empty container (e.g., a stack or a queue) C of one item $(\mathcal{E}^{\succ}, \mathcal{A}, n, \Delta)$ and an unlabeled node T set to n. We now describe the remainder of Algorithm 3 for each value of Δ (learning UI and CI trees, respectively).

UI The algorithm pops an item $(\mathcal{E}^{\succ}, \mathcal{A}, n, \Delta)$ from C, and picks the root attribute X_l and $CPT(X_l)$ that correctly handles the most examples in $\mathcal{E}$. Next, it updates the set $\mathcal{A}$ of available attributes and the set $\mathcal{E}$ of remaining examples to be decided, and creates the next node n'. Then, the algorithm creates and pushes the item $(\mathcal{E}^{\succ}, \mathcal{A}, n', \Delta)$ onto C. The algorithm repeats until all strict examples in $\mathcal{E}^{\succ}$ are decided, either correctly or not. For UIUP trees, the $CPT(X_l)$ has only a single local preference. For UICP-1 trees, the table could contain only one local preference as in the UIUP case, or it could be a $CPT(X_l)$ of preferences on D_l dependent on one parent attribute (cf. Figure 5.1b).

CI As in the case of UI trees, the algorithm pops an item from C, and picks

$(X_l, CPT(X_l))$ for the root, where $CPT(X_l)$ contains only one local preference. Having updated $\mathcal{A}$ and $\mathcal{E}$, for each value of X_l, the algorithm constructs a node and partitions $\mathcal{E}$. Next, when the algorithm pushes all items $(\mathcal{E}_i^{\succ}, \mathcal{A}, n_i, \Delta)$ onto to container C, the choice of C could make a difference on the CIUP tree to be learned. This is because the local preference learned for an attribute is fixed for that attribute which could appear elsewhere in the tree. To this end, we implemented C using stack and queue in our experiments to have the learning algorithm for CIUP trees work either in a breadth-first or a depth-first manner. We hereby denote by CIUPB (CIUPD) the class of CIUP trees learned by the breadth-first (depth-first, respectively) implementation of the greedy algorithm. However, for the most general type of CI trees, the CICP PLP-trees, the choice does not influence the quality of the learned models.

Our greedy method is similar to the greedy method proposed by Schmitt and Martegnon [76] to learn the so called *UIFP* trees.[2] Schmitt and Martegnon provided a worst-case performance bound for their method. Given a set $\mathcal{E}$ of preference examples, let $OPT(\mathcal{E})$ be the minimum number of examples falsified by a *UIFP* tree, and $GREEDY(\mathcal{E})$ be the number of examples falsified by the *UIFP* tree computed by the greedy approach. Schmitt and Martegnon proved that for every set $\mathcal{E}$ of preference examples over p-attribute outcomes,

$$GREEDY(\mathcal{E}) \leq p \cdot OPT(\mathcal{E}).$$

This result does not give a tight bound on the performance of the greedy method. Schmitt and Martegnon [76] proved that no polynomial-time algorithm learning *UIUP* trees that would be accurate to within a constant factor c is possible unless P=NP. We conjecture that the same holds for more general classes of trees, although it may be that p in the upper bound can be replaced by a slower growing function of p. We

[2] They are *UIUP* trees in which the order on the values of the domain of every attribute is fixed *a priori* and must be used in the tree.

Algorithm 3: The *greedy* algorithm that learns a PLP-tree

Input: C: a container of items $(\mathcal{E}^{\succ}, \mathcal{A}, n, \Delta)$, where $\mathcal{E}^{\succ}$ is the set of strict example to be decided, $\mathcal{A}$ the set of available attributes, n an unlabeled node to consider next, and Δ a Boolean value indicating the type of PLP-trees (UI or CI) to be learned, and $T = n$: an unlabeled node for which a PLP-tree is to be learned.

Output: A PLP-tree T over $\mathcal{A}$.

$(\mathcal{E}^{\succ}, \mathcal{A}, n, \Delta) \leftarrow$ Pop an item from C;
if $\mathcal{E}^{\succ} = \emptyset$ **then**
    Label n as a leaf;
    if C *is empty* **then**
        return;
    end
else
    $(X_l, CPT(X_l)) \leftarrow$ Pick $X_l \in \mathcal{A}$ and $CPT(X_l)$ that correctly decides the maximum number of examples in $\mathcal{E}^{\succ}$;
    Label n with tuple $(X_l, CPT(X_l))$;
    $\mathcal{E}^{\succ} \leftarrow \mathcal{E}^{\succ} \backslash \{e \in \mathcal{E}^{\succ} : \alpha_e(X_l) \neq \beta_e(X_l)\}$;
    $\mathcal{A} \leftarrow \mathcal{A} \backslash \{X_l\}$;
    if $\Delta = \boldsymbol{true}$ **then**
        Create an edge u and an unlabeled node n' such that $u = \langle n, n' \rangle$;
        Push item $(\mathcal{E}^{\succ}, \mathcal{A}, n', \Delta)$ onto C;
    else
        for $i \leftarrow 1$ *to* $|D_l|$ **do**
            Create an edge u_i and an unlabeled node n_i such that $u_i = \langle n, n_i \rangle$;
            $\mathcal{E}_i^{\succ} \leftarrow \{e \in \mathcal{E}^{\succ} : \alpha_e(X_l) = \beta_e(X_l) = x_{l,i}\}$;
            Push item $(\mathcal{E}_i^{\succ}, \mathcal{A}, n_i, \Delta)$ onto C;
        end
    end
end
greedy(C, T);

leave these questions for future work and focus here on experimental evaluation of the accuracy of our learning algorithms.

Experiments

First, we consider learning UIUP PLP-trees using the best-agreement and greedy methods. The goal is to compare the accuracy of both methods. This is important as the best-agreement method, because of its complexity, can only be used on relatively small example sets.

For a dataset D (where D is one of the twelve datasets we studied), we fix the size of the training set to t, where $1 \leq t \leq 250$. Then, we randomly pick $TR_D \subset \mathcal{E}^{\succ}$, where $|TR_D| = t$, as the set of *training* examples, and $TE_D = \mathcal{E}^{\succ} \setminus TR_D$ as the set of *testing* examples. Then, from TR_D, we train a UIUP PLP-tree T_{BA} using the best-agreement method (that is, T_{BA} decides the maximum possible number of examples in TR_D), and a UIUP PLP-tree T_G using our greedy heuristics. Finally, we test the models T_{BA} and T_G, on the testing examples in TE_D and compute the *accuracy* of each method, the percentage of strict examples in TE_D decided correctly by the corresponding tree. For each t, $1 \leq t \leq 250$, we repeat this process 20 times and compute the average accuracies. We do this for all 12 datasets. Figure 6.2 shows the *learning curves* (the accuracies as the function of the size of the training set) for the the best-agreement method (BA-UIUP) and the greedy algorithm (G-UIUP) for the datasets CarEvaluation, Ionosphere, Mushroom and Wine. We show the accuracies for the two methods on all datasets when $t = |TR_D| = 250$ in Table 6.3.

This experiment shows that, when the number of training examples is small, the greedy approach achieves accuracy comparable with that of the best-agreement method. The results summarized in Table 6.3 show that (1) the greedy algorithm already achieves accuracy exceeding 85% on six datasets (notably, accuracy of 95.5% on Wine); and (2) the greedy algorithm performs very close to the best-agreement method, with the difference within 2 percentage points on *all but two datasets*, Ionosphere and Mushroom. Examining the learning curves in Figure 6.2, we observe that, on all datasets but Ionosphere and Mushroom, the greedy algorithm works well compared with the best-agreement method across the range of the training set sizes. The learning curves for the two datasets on which the greedy method lags behind the best-agreement one are shown in Figure 6.2e and Figure 6.2g.

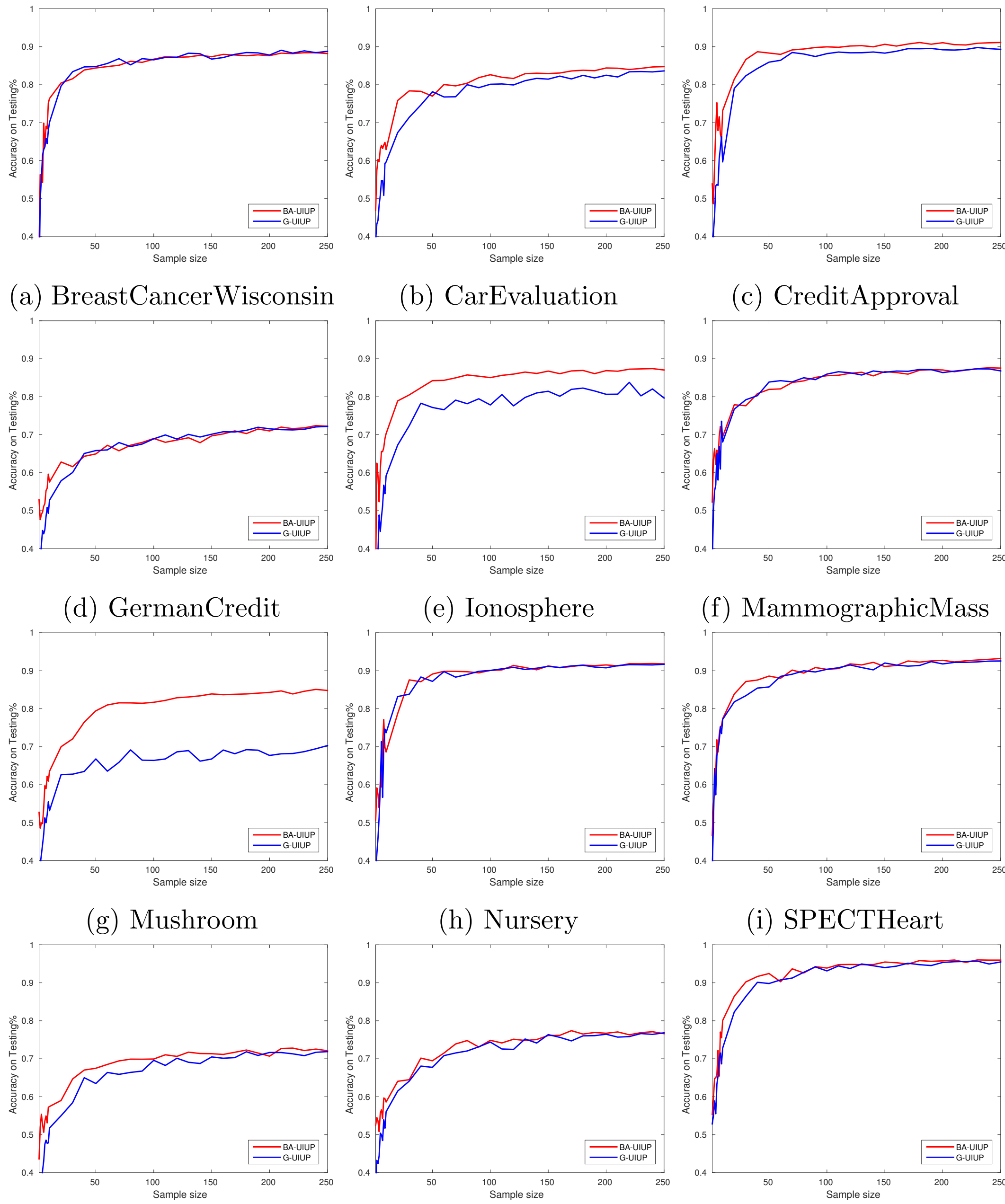
Since the best-agreement method quickly fails as the training sample size grows,

Table 6.3: Accuracy (percentage of correctly handled testing examples) for UIUP PLP-trees learned using the best-agreement and the greedy methods on the learning data (250 of $\mathcal{E}^{\succ}$)

Dataset	BA-UIUP	G-UIUP
BreastCancerWisconsin	88.4	88.2
CarEvaluation	84.8	83.6
CreditApproval	91.1	89.3
GermanCredit	72.2	72.2
Ionosphere	87.0	79.6
MammographicMass	87.5	86.8
Mushroom	84.8	70.3
Nursery	91.8	91.7
SPECTHeart	93.2	92.6
TicTacToe	72.1	71.9
Vehicle	76.8	76.6
Wine	96.0	95.5

in experiments with large learning sets we only used the greedy heuristics to learn PLP-trees from the classes UIUP, UICP-1, CIUP (including CIUPB and CIUPD), and CICP. As demonstrated above, the greedy heuristic is a good alternative to the best-agreement method. For a dataset D, we generate $TR_D \subset \mathcal{E}^{\succ}$ as the training set, and use $TE_D = \mathcal{E}^{\succ} \setminus TR_D$ as the testing set. We learn UIUP, UICP-1, CIUPB, CIUPD and CICP trees based on TR_D using the greedy heuristics, and then we test the the trees learned on the testing set TE_D, computing their accuracy. In Table 6.4, we present results of accuracy on testing using 70% of $\mathcal{E}^{\succ}$ in the training phase. (As in the previous experiment, we computed the learning curves by varying the size of the training set up to 70% of the size of $\mathcal{E}^{\succ}$. The curves show similar behavior to those presented earlier — the accuracy increases with the size of the training set, but gets close to the maximum accuracy already for relatively small training sets.)

From Table 6.4 we note that, for the greedy algorithm, (1) for all datasets, there is a clear gain in the accuracies for the UIUP models using larger training sets; (2) for all but one dataset (Ionosphere), the UICP-1 models, which allow for simple conditional preference statements, are more accurate than the UIUP models; (3) both



(a) BreastCancerWisconsin (b) CarEvaluation (c) CreditApproval

(d) GermanCredit (e) Ionosphere (f) MammographicMass

(g) Mushroom (h) Nursery (i) SPECTHeart

(j) TicTacToe (k) Vehicle (l) Wine

Figure 6.2: Learning UIUP PLP-trees

the CIUPB and CIUPD models are more accurate than the UIUP models for all but one dataset (MammographicMass); and (4) the most general class CICP achieves the best accuracies among all four classes of PLP-trees across all datasets.

The size of a PLP-tree is measured by the total number of preferences in the CPTs in the tree. Clearly, for UIUP, CIUP and CICP trees, it is also the number

Table 6.4: Accuracy percents on the testing data (30% of $\mathcal{E}^{\succ}$) for all four classes of PLP-trees, using models learned by the greedy algorithm from the learning data (the other 70% of $\mathcal{E}^{\succ}$)

Dataset	UIUP	UICP-1	CIUPB	CIUPD	CICP
BreastCancerWisconsin	90.7	91.4	91.0	90.7	91.4
CarEvaluation	85.8	86.0	85.8	85.9	86.0
CreditApproval	91.4	91.7	91.6	92.0	92.2
GermanCredit	74.3	74.6	74.3	74.5	75.7
Ionosphere	87.1	86.9	87.2	88.5	90.4
MammographicMass	88.2	89.5	87.3	86.9	90.0
Mushroom	71.6	74.2	77.1	75.6	76.6
Nursery	92.9	93.0	93.0	93.0	93.0
SPECTHeart	93.4	94.9	95.4	94.8	95.7
TicTacToe	73.9	74.5	74.4	75.4	76.2
Vehicle	79.2	80.4	80.3	80.0	81.2
Wine	95.5	97.8	97.8	97.5	97.8

of non-leaf nodes in the tree. For UICP trees it is the total number of rows in all conditional preference tables in the tree. It is desirable to learn trees that are accurate but small. Trees of a small size provide insights into the structure and properties of the preference order of a user.

The size of a PLP-tree learned by the greedy algorithm is bounded by the number of training examples. On the other hand, it never exceeds the size of the largest possible tree for a domain it models. These maxima are shown for each dataset in Table 6.5. The maximum for CI trees is the common maximum for UICPB, UICPD and CICP trees. The last column in the table shows the size of the training example set used (70% of all examples).

Table 6.6 shows average size of trees learned by our greedy algorithm (for each dataset and for each class of trees considered). The results indicate that the learned trees have indeed relatively small sizes when compared to the upper bounds implied by Table 6.5. The difference is drastic for CIUPB, CIUPD and CICP trees, where trees we learn have sizes that are small fractions of the maximum possible size they potentially might have. For UIUP trees and UICP-1 trees, the difference is smaller

Table 6.5: Maximum sizes of trees for all the classes and the training sample sizes for all datasets

Dataset	UIUP	UICP-1	CI	$\lvert\mathcal{E}^{\succ}_{train}\rvert$
BreastCancerWisconsin	9	33	87,381	6,306
CarEvaluation	6	21	853	477,904
CreditApproval	10	37	91,477	46,255
GermanCredit	10	37	349,525	120,657
Ionosphere	10	19	1,023	2,430
MammographicMass	5	17	341	554
Mushroom	10	37	91,477	5,913
Nursery	8	29	7,765	383,644
SPECTHeart	10	19	1,023	2,237
TicTacToe	9	25	9,841	145,482
Vehicle	10	37	349,525	53,699
Wine	10	37	349,525	7,225

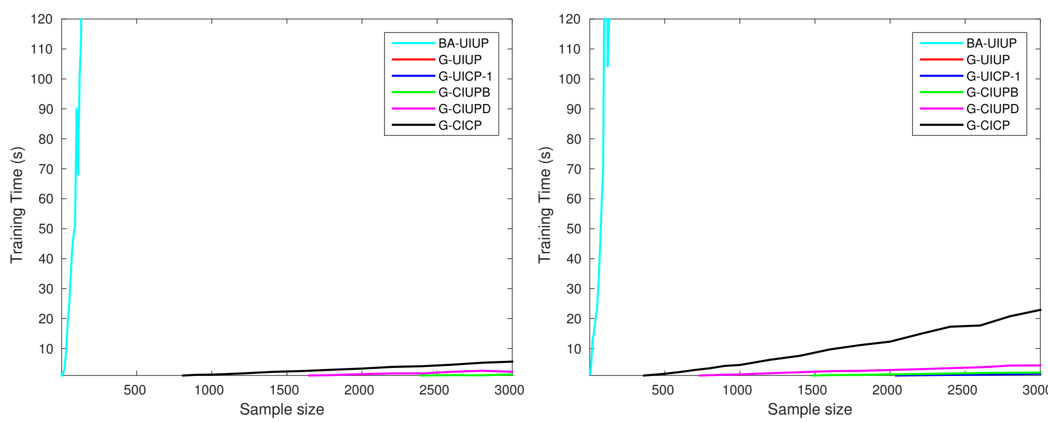
(these trees because of their structure are very small to start with), yet even there is some cases the learned trees have sizes below 80% of the maximum size and occasionally are much smaller (for instance for the Wine dataset). These small-size trees can provide explicit insights into the importance the user assigns to attributes when deciding between outcomes, and into how her preferences of attributes depend on preferences on the more important ones.

We also observe that the sizes of learned CIUPB trees are always smaller than the sizes of the learned CI trees of the other two types. In some cases (datasets GermanCredit, Nursery, TicTacToe, Vehicle), they are significantly smaller. Given that the accuracies of learned CIUPB and CIUPD trees are very close to each other, and the accuracies of the learned CIUPB and CICP trees differ by more than 2 percentage points in only one case (GermanCredit), the results suggests that CIUPB trees provide a a particularly attractive preference model. The results are well aligned with the intuition that when using CIUP trees, agents build them level by level in a breadth-first fashion.

Another important observation concerning our greedy algorithms is that they work fast even on large training sets. This is demonstrated in Figure 6.3, where

Table 6.6: Sizes of trees learned by the greedy algorithm from the training data (70% of $\mathcal{E}^{\succ}$)

Dataset	UIUP	UICP-1	CIUPB	CIUPD	CICP
BreastCancerWisconsin	6.7	21.8	19.8	28.0	25.7
CarEvaluation	6.0	17.0	73.2	108.9	109.5
CreditApproval	9.0	24.7	31.3	78.6	81.1
GermanCredit	9.7	36.0	49.8	210.3	190.0
Ionosphere	9.6	17.2	19.8	31.5	30.6
MammographicMass	4.5	14.7	8.3	10.8	10.0
Mushroom	7.6	20.7	15.7	22.7	16.3
Nursery	8.0	25.7	56.2	121.0	116.9
SPECTHeart	8.4	13.7	13.0	18.4	19.0
TicTacToe	8.0	21.8	36.8	126.8	115.2
Vehicle	9.0	32.7	33.9	101.3	105.4
Wine	5.1	13.3	14.2	16.9	14.6



(a) BreastCancerWisconsin

(b) Vehicle

Figure 6.3: Training time comparison: best-agreement vs. greedy

we show the effectiveness of the greedy method for datasets BreastCancerWisconsin and Vehicle, and contrast it with highly limited range of applicability of the best-agreement method. For other datasets, we observe a similar behavior.

Closing this section, we provide a brief comparison between PLP-trees and decision trees, a commonly-used classification model in machine learning. Decision trees can be used as classifiers that, given two outcomes, can tell if an outcome is better or worse than another. Our experimental results show that decision trees are generally better than PLP-trees, although the difference is mostly within 3 percentage points, on predicting preferences between outcomes in the testing phase. However, PLP-trees offer not only a quick way to determine dominance (the order between two outcomes) but also insights into the structure of the reasoning process of the decision maker. They point to importance of attributes and conditional dependencies between them, and explicitly identify optimal outcomes. This information is hard to glean out of the decision-tree model for the dominance relation.

6.3 Partial Lexicographic Preference Forests

As we see from Table 6.4, our approximation method achieves high accuracy (above 85%) on most of the datasets for all four types of PLP-trees. However, on some datasets such as Mushroom, PLP-trees that we learn have accuracy below 80% across all classes of trees. In an effort to improve on this, we introduce the notion of a *PLP-forest*, that is, a *collection* of PLP-trees. Let $F = \{T_1, \ldots, T_n\}$ be a PLP-forest. We say that F is a $\mathcal{C}$ PLP-forest, where $\mathcal{C}$ is one of the four classes UIUP, UICP-1, CIUP and CICP, if F consists exclusively of $\mathcal{C}$ PLP-trees.

Aggregating PLP-Trees in a PLP-Forest

We use the *pairwise majority rule* (PMR) to aggregate orders defined by trees in a forest. The choice of PMR as the aggregation rule is motivated by three considerations. First, plurality was used in the related work on random forest learning that motivated and influenced our ideas behind PLP forests and PLP forest learning. Second, the task we have at hand is to determine the preferences between outcomes, so PMR is well aligned with this task (the outcome that "wins" on more orders "wins" overall). Finally, the PMR is intuitive and easy to implement.

Let us denote by $N_F(o_1, o_2) = |\{T \in F : o_1 \succ_T o_2\}|$ the number of trees in the forests where the outcome o_1 is preferred to the outcome o_2. Given a forest F, and two outcomes o_1 and o_2, we say that $o_1 \succ_F^{PMR} o_2$ iff $N_F(o_1, o_2) > N_F(o_2, o_1)$, and that $o_1 \approx_F^{PMR} o_2$ iff $N_F(o_1, o_2) = N_F(o_2, o_1)$.

In some cases, PMR may lead to the so-called *Condorcet's Paradox*, where the strict $\succ_F^{PMR}$ relation contains a cycle. Earlier empirical studies, however, conclude that there is little evidence for occurrences of Condorcet's Paradox. Among these studies, one recent work by Mattei et al. on the Netflix dataset showed that the Condorcet's Paradox has a low occurrence percentage of less than 0.11%[68]; that is, on average, out of one thousand elections they ran there was about one election where Condorcet's Paradox accrued. Aligned with this empirical conclusion, our datasets are created in a way that Condorcet's Paradox is prevented from happening. Other possible aggregators are positional scoring rules (adjusted for total *preorders*), Copeland's method, among others. We will leave this and discuss it later in the chapter as part of the future work.

Experimentation

To further boost up performances, we now show empirical results of learning *PLP-forests*.

First, we show results for UIUP PLP-forests using the best-agreement learning and the greedy heuristics. In each experiment, we randomly partition a dataset into training set (70%) and testing set (30%), learn a forest of 5000 trees, where each tree is learned from 50 randomly selected examples from the training set, and then test the forest against the testing set. We repeat it 20 times and report the average accuracy. We present these results in Table 6.7 (we write BA and G to indicate the method used).

We see that G+Forest outperforms G+Tree on all but one dataset (i.e., Ionosphere). This indicates the gain of using a forest of diverse trees against a single tree for UIUP. Similarly, we observe that BA+Forest outperforms G+Forest on all

Table 6.7: Accuracy percents on the testing data (30% of $\mathcal{E}^{\succ}$) for UIUP trees and forests of 5000 UIUP trees, using the greedy and the best-agreement algorithms from the learning data (the other 70% of $\mathcal{E}^{\succ}$)

Dataset	G+Tree	G+Forest	BA+Forest
BreastCancerWisconsin	90.7	93.4	95.1
CarEvaluation	85.8	91.9	89.2
CreditApproval	91.4	91.5	93.1
GermanCredit	74.3	75.4	77.9
Ionosphere	87.1	83.0	92.5
MammographicMass	88.2	89.1	90.8
Mushroom	71.6	78.8	90.2
Nursery	92.9	93.2	94.0
SPECTHeart	93.4	93.7	94.9
TicTacToe	73.9	75.1	77.2
Vehicle	79.2	82.7	81.9
Wine	95.5	95.8	96.9

Table 6.8: Accuracy percents on the testing data (30% of $\mathcal{E}^{\succ}$) for all four classes of PLP-forests of 5000 trees, using the greedy algorithm from the learning data (the other 70% of $\mathcal{E}^{\succ}$)

Dataset	UIUP	UICP-1	CIUPB	CIUPD	CICP
BreastCancerWisconsin	93.4	94.1	93.7	94.1	94.0
CarEvaluation	91.9	88.3	91.4	89.7	91.4
CreditApproval	91.5	91.6	92.8	92.9	93.0
GermanCredit	75.4	73.8	76.1	76.1	76.2
Ionosphere	83.0	87.9	89.3	89.4	89.5
MammographicMass	89.1	90.1	90.0	90.1	90.2
Mushroom	78.8	87.2	92.2	92.2	91.8
Nursery	93.2	89.9	93.3	93.4	93.4
SPECTHeart	93.7	93.5	93.6	93.6	93.7
TicTacToe	75.1	75.2	76.6	76.5	76.9
Vehicle	82.7	81.8	83.2	83.2	83.4
Wine	95.8	95.4	97.5	97.8	97.8

datasets but one (CarEvaluation). This points to another advantage of PLP forest learning: they achieve good accuracy even when individual trees are learned from small example sets and so, the best-agreement learning becomes practical.

Second, we show results for the greedy heuristics and the five types of PLP-forests (under the same setting as before).

The results are shown in Table 6.8. Comparing with Table 6.4, we see that UICP-

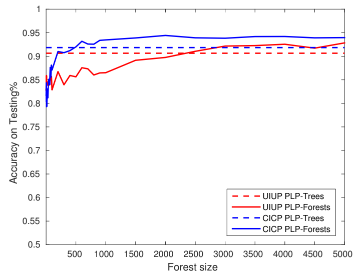
1 trees do not lend themselves well to the use in forests, the accuracies for individual UICP-1 trees are higher than for forests of UICP-1 trees for five out of 12 datasets. However, for all other types of trees, the idea of learning forests of such trees is very effective. We get improvements in the accuracy on all datasets but one for UIUP and CIUPD trees, and in all but two datasets for CIUPB and CICP trees. In the case of the dataset Mushroom, the improvements provided by forest learning are particularly significant.

We also studied how the accuracy of PLP-forests changes with the number of their PLP-trees. In Figure 6.4, we show the results for UIUP and CICP PLP-forests for all twelve datasets.

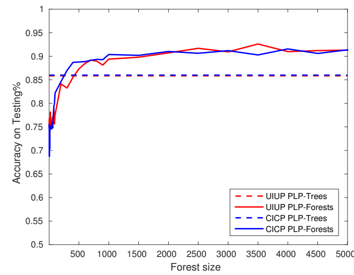
Examining Figure 6.4, we note that with even smaller forests, consisting of 2000 forests, the accuracies are already very close to those we observe for forests consisting of 5000 trees. That suggests that much larger forests would not offer any additional boost in the accuracy. The figure also shows that the number of trees needed in a forest in order to offer a better accuracy than that of an individual tree varies (for only one case with dataset Ionosphere and class UIUP, we do not see forests of trees surpass individual trees in accuracy).

6.4 Conclusions

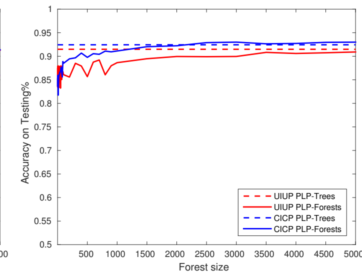
In this chapter, we presented results considering problems concerning learning *partial lexicographic preference trees*, or *PLP-trees*. We showed that PLP-trees are expressive preference models that can be used to accurately model preferences arising in practical situations, and that high-accuracy PLP-trees can be effectively computed. We also proposed and studied a variant of the model based on the concept of a *PLP-forest*, a *collection* of PLP-trees, where the preference order specified by a PLP-forest is obtained by aggregating the orders of its PLP-trees. We proposed and implemented the best-agreement and greedy algorithms to learn PLP-trees and PLP-forests. To



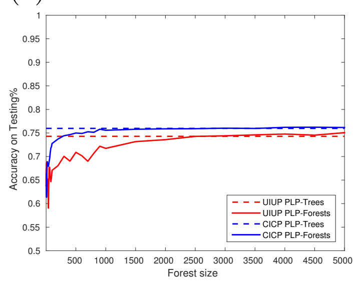
(a) BreastCancerWisconsin



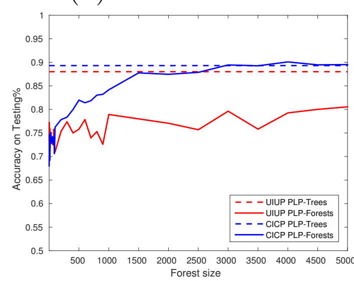
(b) CarEvaluation



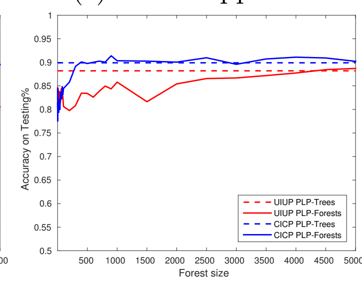
(c) CreditApproval



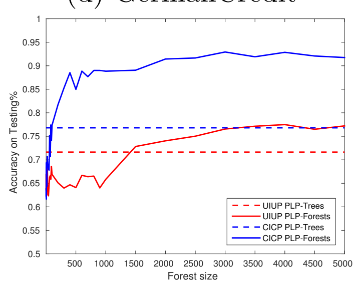
(d) GermanCredit



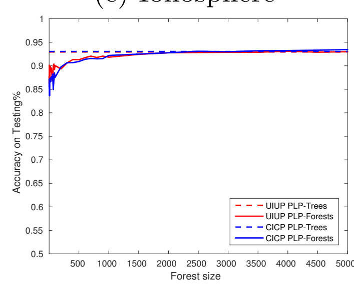
(e) Ionosphere



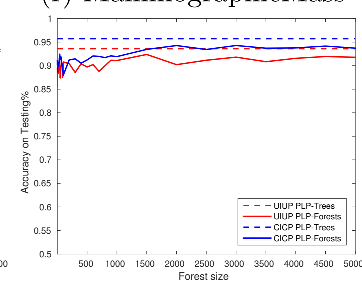
(f) MammographicMass



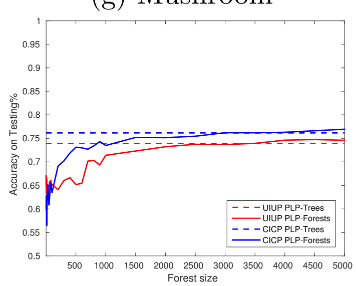
(g) Mushroom



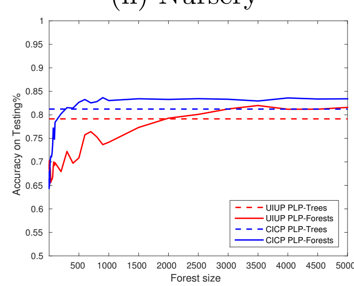
(h) Nursery



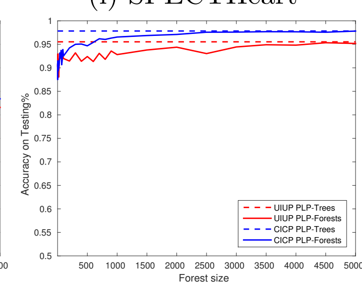
(i) SPECTHeart



(j) TicTacToe



(k) Vehicle



(l) Wine

Figure 6.4: Forests of UIUP trees vs. forests of CICP trees

support experimentation, we used datasets that we adapted to the preference learning setting from existing classification datasets.

Our results demonstrated the potential of both approaches. For learning single trees, our results show the effectiveness of the greedy heuristics and identify learning CIUPB trees as leading to both high accuracy and small tree sizes. Learning PLP-forests improves accuracy and yields effective preference models even when individual trees are learned from small example sets. That allows us to use the best-agreement method for learning PLP forests, the method inapplicable when example sets are large.

Looking into the future, we are interested in expanding our preference learning library by creating real-world datasets through conducting experiments involving human subjects. We also plan to extend the theoretical results on the worst-case bound for the greedy method to more general classes of PLP-trees. Finally, we intend to implement and experiment with other aggregators for PLP-forests, and compare with our results using the desirable and intuitive majority rule.

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Chapter 7 Aggregating Lexicographic Preference Trees

Aggregating *votes* — preference orders over *candidates* or *alternatives* — is a fundamental problem of decision theory and social choice. We study this problem in the setting when alternatives are described as tuples of values of attributes. Such spaces of alternatives are called *combinatorial*. They are characterized by large sizes that make explicit enumerations of alternatives from the most to the least preferred infeasible. Instead, typically votes are specified implicitly in terms of some compact and intuitive preference representation mechanism. In our work, we assume that votes are given as *lexicographic preference trees* and consider two preference-aggregation problems, the *winner* problem and the *evaluation* problem. We study them under the assumption that *positional scoring rules* (such as k-approval and Borda) are used for aggregation. We develop computational complexity results for these two problems. We also propose computational methods to solve them. They are based on encodings of the problems in *Answer-Set Programming* and as instances of the *Weighted Partial Maximum Satisfiability* problem, and exploit off-the-shelf solvers available for these two formalisms. Finally, we present results of an experimental study of the effectiveness of these methods.

7.1 Introduction

Preferences are an essential component of decision making, social choice, knowledge representation, and constraint satisfaction. Fundamental problems of preference reasoning are to *aggregate* individual preference orders of a group of agents (the *votes* of agents in the group) into a consensus best candidate (the *winner*), and to identify candidates with strong consensus support from the group ("good" alternatives). These problems have been studied extensively in social choice [6]. Aggregation methods known as *positional scoring rules*, which include such well-known rules as plurality, k-approval and Borda, are among the best understood and the most widely used ones.

When the number of alternatives is small, the simplest and most effective way to describe a preference order (a vote) is to enumerate the alternatives from the most to the least preferred. Moreover, given a collection of such votes, for many aggregation rules, including all positional scoring rules, computing winners and "good" candidates is easy — it can be done in polynomial time. The situation changes when alternatives are characterized in terms of *attributes* (or issues), and are specified by tuples of attribute values. Spaces of such alternatives, often called *combinatorial domains*, are large. Indeed, the number of alternatives grows exponentially with the number of attributes. This large size of combinatorial domains brings up two problems. First, it is no longer feasible to describe votes by enumerating alternatives in the order of preference. Thus, formalisms offering compact and intuitive representations of votes are needed. Several such *preference formalisms* have been developed over the years including penalty logic [25], possibilistic logic [29], conditional preference networks (CP nets) [12], preference trees [36, 64], and lexicographic preference trees [11].[1] Second, when votes are given as expressions in some preference formalism, computing the winner or a "good" candidate is no longer easy. In fact, it is known that for many preference formalisms these problems are NP-hard even when positional scoring rules are used to aggregate votes. *Issue-by-attribute* aggregation addresses the computational hardness problem but often leads to results different from those obtained by applying common voting rules [31].

In this chapter, we assume that votes are represented as *lexicographic preference trees*, or *LP-trees*, for short [11], and that they are aggregated by some simple positional scoring rules such as Borda, k-approval and a refinement of the latter, (k, l)-approval. Given this setting, we study computing the best alternative, and the

[1] Kaci [50] offers a comprehensive discussion of preference formalisms.

related problem to decide whether an alternative with the score exceeding a given threshold (a "good" alternative) exists. We refer to the former problem as the *winner* problem and to the latter one as the *evaluation* problem. In our setting, these problems are often computationally hard. For Borda, the *winner* problem is NP-hard and the *evaluation* problem is NP-complete [56]. For k-approval, for some specific values of k, both problems are in P but, for some other, they are NP-hard and NP-complete, respectively [56]. Further, when (k, l)-approval is used, for several values of k and l, the problems are similarly hard.

Nevertheless, because the *winner* and the *evaluation* problems arise in practice and the positional scoring rules are common, computational tools for the two problems are needed. To develop such tools, we encode the problems in answer-set programming (ASP) [67, 71] and weighted partial maximum satisfiability (WPM-SAT) [4, 3], and apply to the encodings the ASP solvers *clingo* [42] and *clingcon* [72], and a WPM-SAT solver *toulbar*[2]. We chose the two ASP solvers as they represent substantially different approaches to computing answer sets. The *clingo* solver is a native ASP solver developed along the lines of satisfiability solvers. The *clingcon* solvers enhances *clingo* with specialized treatment of some common classes of numeric constraints by delegating some reasoning tasks to a CP solver *Gecode* [77]. As problems we are considering involve numeric constraints, a comparison of the two solvers is of interest. We study all the resulting methods experimentally. To support the experimentation we propose and implement a method to randomly generate LP-trees of some restricted form.

The main contributions of our work are complexity results and algorithms for the *winner* and the *evaluation* problems when votes are specified as LP-trees. Specifically, we present new complexity results for the two problems for several positional scoring rules: k-approval (for specific values of k), variants of Borda, and (k, l)-approval (for specific combinations of values of k and l). Next, we propose algorithms for the two problems based on their ASP and WPM-SAT encodings and using ASP and WPM-SAT solvers. Finally, we provide an experimental evidence of the effectiveness of the proposed computational methods.

7.2 Computing Ranks

We now show how the *rank* of an outcome in an LP-tree is computed. As we consider positional scoring rules, the scores of an outcome in an LP-tree or an LP-profile under these rules follow directly from its rank in the tree.

Given an LP-tree T and an outcome $o \in CD(\mathcal{I})$, the computation of the rank $r(T, o)$ of o in T is given in Algorithm 4, where $T'(x_j)$ is the left (more-preferred) subtree of T', and $T'(\overline{x_j})$ is the right (less-preferred) subtree of T'. Note that in each case we need to update the CPT's in the subtrees accordingly. Clearly, Algorithm 4 takes $O(p)$. Conversely, it is also easy to compute the outcome at a given rank in a tree.

Algorithm 4: Compute the rank of an outcome in an LP-tree

Input: LP-tree T and outcome o
Output: the rank r of o in T

$r \leftarrow 0$;
$T' \leftarrow T$;
for $i \leftarrow 1$ **to** p **do**
    Let X_j be the root attribute of T' with preference $x_j > \overline{x_j}$;
    if $o(X_j) = x_j$ **then**
        $T' \leftarrow T'(x_j)$;
    else
        $r \leftarrow r + 2^{p-i}$;
        $T' \leftarrow T'(\overline{x_j})$;
    end
end
return r

Now computing the scores of an outcome for the rules k-approval, (k, l)-approval and Borda is straightforward. We have the following.

1. k-approval: $s_{kApp}(T, o) = 1$, if $r(T, o) < k$; 0, otherwise.

2. (k, l)-approval: $s_{klApp}(T, o) = a$, if $r(T, o) < k$; b, if $k \leq r(T, o) < k + l$; 0, otherwise.

3. Borda: $s_{Borda}(T, o) = m - r(T, o) - 1$.

7.3 The Problems and Their Complexity

We consider only *effective implicit* positional scoring rules, that is, rules defined by an algorithm that given m (the number of alternatives and, at the same time, the size of the scoring vector) and a rank r, $0 \leq r \leq m - 1$, (1) returns the value w_r of the scoring vector, and (2) works in time polynomial in the sizes of r and m. The rules k-approval, (k, l)-approval and Borda are examples of effective implicit positional scoring rules:

1. k-approval: $w_{kApp}(r, m) = 1$, if $r < k$; 0, otherwise.

2. (k, l)-approval: $w_{klApp}(r, m) = a$, if $r < k$; b, if $k \leq r < k + l$; 0, otherwise.

3. Borda: $w_{Borda}(r, m) = m - r - 1$.

Let us fix an effective implicit positional scoring rule $\mathcal{D}$ with the scoring vector w. Given an LP profile $\mathcal{V}$, the *winner* problem for $\mathcal{D}$ consists of computing an alternative $o \in \mathcal{X}$ with the maximum score $s_w(\mathcal{V}, o)$. Similarly, given a profile $\mathcal{V}$ and a positive integer R, the *evaluation* problem for $\mathcal{D}$ asks if there exists an alternative $o \in \mathcal{X}$ such that $s_w(\mathcal{V}, o) \geq R$. In each case, w is the scoring vector of $\mathcal{D}$ for m alternatives; we recall that it is given implicitly in term of an algorithm that efficiently computes its entries.

We apply the voting rules listed above to profiles consisting of LP-trees or *LP profiles*, for short. We distinguish four classes of profiles, UI-UP, UI-CP, CI-UP and CI-CP depending on the type of LP-trees they consist of.

Remark The restriction to effective implicit positional scoring rules is essential in the context of combinatorial domains. It is because an explicit specification of the scoring vector has size equal to the number of alternatives and is exponential in the number of attributes. If it were to be given explicitly, it would have to be a part of input. The sheer size of the scoring vector would then make both the winner and the evaluation problems trivially solvable in polynomial time. However, most interesting positional scoring rules are effective implicit, which means that they can be described concisely as an algorithm (implicit) and at the same time provide a fast access to any weight in the scoring vector (effective). In this setting, the complexity of the winner and the evaluation problems is no longer obvious, and it is precisely this setting that models practical situations, where scoring vectors are based on *regular* patterns.

k-Approval

If $k = 2^{p-1}$ the evaluation problem is in P for all four classes of profiles of LP-trees [56]. However, if k equals 2^{p-2} or 2^{p-3}, the problem is NP-complete, again for all four types of profiles [56] (in fact, the result holds for a larger set of values k, we refer for details to the paper by Lang et al. [56]). Clearly, in each case where the evaluation problem is NP-complete, the winner problem is NP-hard.

We first show that the two problems are in P even when the deviation of k from 2^{p-1} is given by a polynomial in p. In other words, if $k = 2^{p-1} + f(p)$ or $k = 2^{p-1} - f(p)$, where $f(p)$ is a polynomial in p such that $f(p) \geq 0$ for $p \geq 1$, both the winner and the evaluation problems for k-approval can be solved by polynomial time algorithms. The next two results address the two cases for k, respectively.

Theorem 16. *Let f be a polynomial such that $f(p) \geq 0$ for $p \geq 1$, and let $k = 2^{p-1} + f(p)$. Given a profile of n LP-trees over p binary attributes $X_1, \ldots, X_p$, the winner under k-approval can be computed in time polynomial in the size of the profile.*

Proof. Let P be a profile of n LP-trees. The score $s_k(o)$ of an alternative o under k-approval in P is given by

$$s_k(o) = s'(o) + s''(o),$$

where $s'(o)$ is the score of o under the 2^{p-1}-approval (the number of votes that place o in the upper half of the order), and $s''(o)$ is the number of votes that place o as one of top $f(p)$ votes in the lower half (we omit references to the profile to simplify the notation).

To find the highest possible score $s'(o)$, we define $x_i = 0$, if the number of votes with the root labeled with X_i and with 0 preferred to 1 is *strictly* larger than the number of votes with the root labeled with X_i with 1 preferred to 0. We define $x_i = 1$ similarly. If x_i does not get set to 0 or 1, it is set to u (undefined). We call the resulting p-tuple a *partial alternative* and denote it by PA. Since it is the root that decides whether an LP-tree contributes 1 to the score of an alternative, it is clear that any alternative consistent with PA achieves the highest possible score under 2^{p-1}-approval, that is, the highest possible s'-score. Finding this score, say W', can then be accomplished by (1) finding an alternative o consistent with PA, and (2) finding its score $s'(o)$. Clearly, both (1) and (2) together can be done in time bounded by a polynomial in the size of the profile.

Next, we consider s''. Let us denote by A the set of all alternatives o with $s''(o) > 0$. To this end, it is enough to find in each tree T in P alternatives with ranks $2^{p-1} + 1, \ldots, 2^{p-1} + f(p)$. Since finding an alternative of a given rank in an LP-tree can be accomplished in time polynomial in p, the set A can indeed be computed in time polynomial in the size of the profile.

We now compute $s_k(o)$ for all alternatives in A. Given the size of A, the task can be computed in time bounded by a polynomial in the size of the profile. Let W be the maximum of these scores achieved, say, by an alternative o. If $W \geq W'$, then o is a winning alternative (has the best score among those in A and the score of any other alternative does not exceed W'). Otherwise, any alternative consistent with PA can be taken for the winner (indeed, in such case, the highest possible score to achieve under k-approval is W'). $\square$

Theorem 17. *Let f be a polynomial such that $f(p) \geq 0$ for $p \geq 1$, and let $k = 2^{p-1} - f(p)$. Given a profile of n LP-trees over p binary attributes $X_1, \ldots, X_p$, the winner under k-approval can be computed in time polynomial in the size of the profile.*

Proof. Let P be a profile of n LP-trees. Similarly as in the proof of the previous result, the score $s_k(o)$ of an alternative o under k-approval is given by

$$s_k(o) = s'(o) - s''(o),$$

where $s'(o)$ is the score of o under the 2^{p-1}-approval (the number of votes that place o in the upper half of the order), and $s''(o)$ is the number of votes that place o as one of the bottom $f(p)$ votes in the upper half (we omit references to the profile to simplify the notation).

Let us denote by A the set of alternatives o such that $s''(o) > 0$. As before, this set can be computed in time bounded by a polynomial in the size of the profile. Let $t = |A|$. If every alternative is in A (that is, $t = 2^p$), then, we compute an alternative with the highest k-approval score by computing the scores of all alternatives in A and selecting the one with the highest score. Since the size of A is polynomial in the size of the profile, the task takes polynomial time (in the size of the profile).

The case when $t < 2^p$ is harder. To address it, let us assume that we have computed the set B of top $t + 1$ alternatives according to their s'-score (the 2^{p-1}-approval score). Next, let o be an alternative in B with the maximum k-approval score $s_k(o)$.

We claim that o is also an alternative with the maximum k-approval score over all alternatives. Indeed, consider an arbitrary alternative o'. If $o' \in B$, then $s_k(o) \geq s_k(o)$ (by the way o was selected). Thus, let us assume that $o' \notin B$. Since $|B| > |A|$, there is at least one alternative $o'' \in B \setminus A$. As $o'' \in B$, $s_k(o) \geq s_k(o'')$. Moreover, as $o'' \notin A$, $s_k(o'') = s'(o'') - s''(o'') = s'(o'')$. Finally, since $o'' \in B$ and $o' \notin B$, $s'(o'') \geq s'(o')$. Combining these three inequalities, we obtain that $s_k(o) \geq s'(o')$. Since $s'(o') \geq s'(o') - s'(o'') = s_k(o')$, we get $s_k(o) \geq s_k(o')$. Thus, the claim follows.

Clearly, $t + 1$ is bounded by a polynomial in the size of the profile. Thus, once B is computed, finding an alternative in B with the highest k-approval score can be done in time polynomial in the size of the profile. To complete the proof, it suffices then to show how to compute B in polynomial time.

To this end, for each $i = 1, \ldots, p$, we set d_i to the absolute value of the difference between the numbers of trees in the profile with the root labeled with X_i and with 0 (respectively, with 1) as the preferred value. We also select any alternative that has the highest s'-score (we explained in the previous proof how to compute it in polynomial time) and denote it by o. Finally, we compute the score of o and denote it by W' (to use the notation from the previous proof).

Let $S \subseteq \{1, \ldots, p\}$ be a set of attribute indices, and let o_S be an alternative obtained from o by "flipping" its values in positions in S. Every alternative can be described in these terms. This is useful as the s'-score of o_S is easy to compute. Namely, we have

$$s'(o_S) = W' - w(S),$$

where $w(S) = \sum_{i \in S} d_i$ is the *weight* of S.

It follows that B is determined by $t+1$ smallest-weight subsets of $\{1, \ldots, p\}$. We will now show that given a list $D = \{d_1, d_2, \ldots, d_p\}$ and an integer t, the $t+1$ smallest-weight subsets of $\{1, \ldots, p\}$ can be computed in time bounded by a polynomial in p and t.

Let r be an integer such that $2^r \geq t+1$. Let us assume that L_r is the set of $t+1$ smallest-weight subsets of $\{1,\ldots,r\}$. Let

$$L'_{r+1} = L_r \cup \{S \cup \{r+1\} \colon S \in L_r\}$$

and let L_{r+1} be the collection of $t+1$ smallest-weight subsets S of L'_{r+1}. We will show that L_{r+1} contains $t+1$ smallest-weight subsets S of $\{1,\ldots,r+1\}$. Indeed, let us consider $S \subseteq \{1,\ldots,r+1\}$ such that $S \notin L'_{r+1}$. If $S \subseteq \{1,\ldots,r\}$, then $S \notin L_r$. Thus, $w(S) \geq w(S')$, for every $S' \in L_r$. If $r+1 \in S$, then $S = R \cup \{r+1\}$, for some $R \subseteq \{1,\ldots,r\}$. Since $S \notin L'_{r+1}$, $R \notin L_r$. Thus, $w(R) \geq w(R')$, for every $R' \in L_r$ and so, $w(S) \geq w(R' \cup \{r+1\})$ for all $R' \in L_r$. In each case, it follows that there are at least $t+1$ sets S' in L'_{r+1} such that $w(S) \geq w(S')$. Thus for every $S' \in L_{r+1}$, $w(S) \geq w(S')$.

Clearly, the list L_p consists of $t+1$ smallest weight subsets of $\{1,\ldots,p\}$. Thus, it can be taken for B. To compute it, we first find the smallest r such that $2^r \geq t+1$ (such an r exists as we are now considering the case when $t < 2^p$). We then construct the collection U of all subsets of $\{1,\ldots,r\}$ (this collection has no more than $2t$ elements and can be constructed in time bounded by a polynomial in p and t). Next, we construct L_r by selecting from U its $t+1$ smallest-weight elements. Since $|U| \leq 2t$, this task also can be accomplish in polynomial time (in p and t).

From now on, we construct $L_{r+1}, L_{r+2}, \ldots L_p$ recursively, as described above. Since each step of the construction can be accomplished by the same polynomial-time algorithm (form the collection L', select its $t+1$ smallest-weight elements to form the next L), and since the number of steps is bounded by p, the total time needed to construct B (L_p) is bounded by a polynomial in p and t. $\square$

For the k-approval rule, we summarize the results in Table 7.1, where Table 7.1a presents our results as discussed above, and results in Table 7.1b were obtained by

Table 7.1: k-Approval

	UP	CP
UI	P	P
CI	P	P

(a) $k = 2^{p-1} \pm f(p)$

	UP	CP
UI	NPC	NPC
CI	NPC	NPC

(b) $k = c \cdot 2^{p-M}$ and $k \neq 2^{p-1}$

others [56].

(k, l)-Approval

To the best of our knowledge, the complexity of the 2-valued (k, l)-approval rule has not been studied. It is evident that (k, l)-approval is an effective implicit positional scoring rule. It turns out that, as with the k-approval rule, for some values of the parameters, the evaluation problem for (k, l)-approval is NP-complete. Preliminary results we obtained have been published [59]. We describe cases where $k = l = 2^{p-c}$, where c is a constant and $1 < c < p$. If $a = 2$ and $b = 1$, we refer to the rule $(2^{p-2}, 2^{p-2})$-approval as $2K$-*approval*. We show proof of NP-completeness of the evaluation problem for $(2^{p-2}, 2^{p-2})$-approval.

Theorem 18. *The following problem is NP-complete: decide for a given UI-UP profile $\mathcal{V}$ and an integer R whether there is an alternative o such that $s_w(\mathcal{V}, o) \geq R$, where w is the scoring vector of the $(2^{p-2}, 2^{p-2})$-approval rule.*

Proof. We can guess in polynomial time an alternative $o \in \mathcal{X}$ and verify in polynomial time that $S_w(\mathcal{V}, o) \geq R$ (this is possible because (k, l)-approval is an effective implicit scoring rule; the score of an alternative in a vote can be computed in polynomial time once its position is known, and the position can be computed in polynomial time be traversing the tree representing the vote). So membership in NP follows. Hardness follows from a polynomial reduction from the problem 2-*MINSAT*[2] [51], which is

[2]Let N be an integer ($N > 1$), the N-MINSAT problem is defined as follows. Given a set Φ of n N-clauses $\{c_1, \ldots, c_n\}$ over a set of propositional variables $\{X_1, \ldots, X_p\}$, and a positive integer l

NP-complete. Given an instance $\langle \Phi, l \rangle$ of the 2-MINSAT problem, we construct the set of attributes $\mathcal{I}$, the set of alternatives $\mathcal{X}$, the profile $\mathcal{V}$ and the threshold R.

Important observations are that o is among the top first quarter of alternatives in an LP-tree $\mathcal{L}$ if and only if the top two most important attributes in $\mathcal{L}$ are both assigned the preferred values; and that o is among the second top quarter of alternatives if and only if the most important attribute is assigned the preferred value and the second most important one is assigned the non-preferred one.

(1). We define $\mathcal{I} = \{X_1, \ldots, X_p\}$, where X_is are all propositional letters occurring in Φ. Clearly, the set $\mathcal{X}$ of all alternatives over $\mathcal{I}$ coincides with the set of truth assignments of variables in $\mathcal{I}$.

(2). Let Ψ be the set of formulas $\{\neg c_i : c_i \in \Phi\}$. For each $\neg c_i \in \Psi$, we build $a+b$ UI-UP trees. For instance, if $\neg c_i = X_2 \wedge \neg X_4$, then we proceed as follows. Firstly, we build $a-b$ duplicate trees shown in Figure 7.1a. Secondly, we construct b duplicate trees shown in Figure 7.1b. Thirdly, we build another b duplicate trees shown in Figure 7.1c. (In all three figures we only indicate the top two attributes since the other attributes can be ordered arbitrarily.) Denote by $\mathcal{V}_i$ the set of these $a+b$ UI-UP trees for formula $\neg c_i$. Then $\mathcal{V} = \bigcup_{1 \leq i \leq n} \mathcal{V}_i$ and has $n * (a+b)$ votes.

(3). Finally, we set $R = (n-l) * (a^2 - ab + b^2) + l * ab$.

Note that the construction of $\mathcal{V}$ ensures that if $o \models \neg c_i$, $S_w(\mathcal{V}_i, o) = a^2 - ab + b^2$; otherwise if $o \not\models \neg c_i$, $S_w(\mathcal{V}_i, o) = ab$. We have $a^2 - ab + b^2 > ab$ since $(a-b)^2 > 0$. Hence, there is an assignment satisfying at most l clauses in Φ if and only if there is an assignment satisfying at least $n-l$ formulas in Ψ if and only if there is an alternative with the $(2^{p-2}, 2^{p-2})$-approval score of at least R given the profile $\mathcal{V}$.

Since the first equivalence is clear, it suffices to show the second. Let o be an assignment satisfying l' formulas in Ψ. We have $S_w(\mathcal{V}, o) - R = (l' + l - n) * (a^2 - ab + b^2) + (n - l' - l) * ab = (l' + l - n) * (a^2 - 2ab + b^2) = (l' + l - n) * (a-b)^2$. It

$(l \leq n)$, decide whether there is a truth assignment that satisfies at most l clauses in Φ.

follows that $S_w(\mathcal{V}, o) \geq R$ if and only if $l' + l - n \geq 0$ if and only if $l' \geq n - l$. $\square$

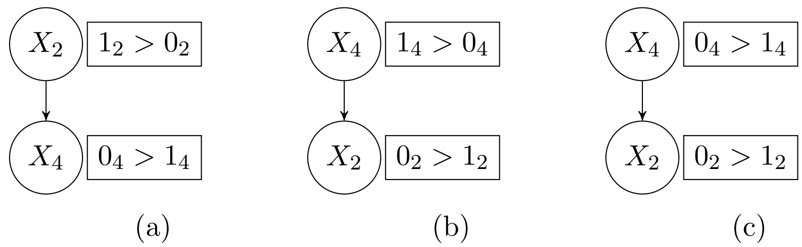


Figure 7.1: UI-UP LP-trees

This hardness proof applies to more general classes of LP-trees, namely UI-CP, CI-UP and CI-CP, and the winner problem for those cases is NP-hard. Below we show the proof of NP-completeness of the evaluation problem for $(2^{p-3}, 2^{p-3})$-approval.

Theorem 19. *Let w be the scoring vector $(a, \ldots, a, b, \ldots, b, 0 \ldots, 0)$ with the numbers of a's and b's each equal to 2^{p-3}. The problem of deciding for a given UI-UP profile V and an integer R whether there is an alternative o such that $s_w(V, o) \geq R$ is NP-complete.*

Proof. We can guess in polynomial time an alternative $o \in \mathcal{X}$ and verify in polynomial time that $S_w(V, o) \geq R$. So membership in NP follows.

Hardness follows from a polynomial reduction from the NP-complete problem 3-*MAXSAT* [74]. Let Φ be a set of n 3-clauses $\{c_1, \ldots, c_n\}$ over $\{X_1, \ldots, X_p\}$, l an integer such that $0 \leq l \leq n$. Given an instance of 3-*MAXSAT* $I = \langle \Phi, l \rangle$, we construct the set of attributes X, the set of alternatives $\mathcal{X}$, the profile V and the threshold R as follows.

(1) $X = \{X_1, \ldots, X_p\}$. $\mathcal{X}$ is then the set of all alternatives over X.

(2) Let Ψ be the set of formulas $\{\neg c_i : c_i \in \Phi\}$. For each $\neg c_i \in \Psi$, we build multiple UI-UP LP-trees. Assume there is $c_i = \neg X_1 \vee \neg X_2 \vee \neg X_3 \in \Phi$. Then we have $\neg c_i = X_1 \wedge X_2 \wedge X_3 \in \Psi$. For $\neg c_i$, we build a^2 duplicate trees of type 7.2a, a^2 duplicate trees of type 7.2b, a^2 duplicate trees of type 7.2c, a^2 duplicate trees of

Table 7.2: (k, l)-Approval

	UP	CP
UI	P	P
CI	P	P

(a) $k = l = 2^{p-1}$

	UP	CP
UI	NPC	NPC
CI	NPC	NPC

(b) $k = l = 2^{p-c}$ and $1 < c < p$

type 7.2d, $a^2 - ab$ duplicate trees of type 7.2e, $a^2 - ab$ duplicate trees of type 7.2f and $(a-b)^2$ duplicate trees of type 7.2g. Denote by V_i the set of $7a^2 - 4ab + b^2$ UI-UP LP-trees for formula $\neg c_i$. Then $V = \bigcup_{1 \leq i \leq n} V_i$ and has $n * (7a^2 - 4ab + b^2)$ votes.

(3) We set $R = a^3 * l + (3a^2b - 3ab^2 + b^3) * (n - l)$.

Note that the construction of V ensures that if $o \models \neg c_i$, $S_w(V_i, o) = 3a^2b - 3ab^2 + b^3$; otherwise if $o \not\models \neg c_i$, $S_w(V_i, o) = a^3$. We have $a^3 > 3a^2b - 3ab^2 + b^3$ since $a^3 - (3a^2b - 3ab^2 + b^3) = (a-b)^3 > 0$. Therefore, there is an assignment satisfying at least l clauses in Φ *iff* there is an assignment falsifying at least l formulas in Ψ *iff* there is an alternative scoring at least R with respect to profile V and our scoring vector w. Since the first equivalence is obvious, it suffices to show the second one.

($\Rightarrow$) Assume o is the assignment that falsifies l' ($l' \geq l$) formulas in Ψ, its score $S_w(V, o) = a^3 * l' + (3a^2b - 3ab^2 + b^3) * (n - l')$. Then $S_w(V, o) - R = a^3 * (l' - l) + (3a^2b - 3ab^2 + b^3) * (l - l') = a^3 * (l' - l) - (3a^2b - 3ab^2 + b^3) * (l' - l) = (a-b)^3 * (l' - l) \geq 0$. Thus, $S_w(V, o) \geq R$.

($\Leftarrow$) Suppose o is the alternative such that $S_w(V, o) \geq R$. Prove by contradiction. Assume o falsifies l' formulas in Ψ and $l' < l$. Then $S_w(V, o) - R = (a-b)^3 * (l' - l) < 0$, which implies that $S_w(V, o) < R$. Contradiction! Therefore, it must be that $l' \geq l$.

$\square$

For the (k, l)-approval rule, we capture our results as Table 7.2.

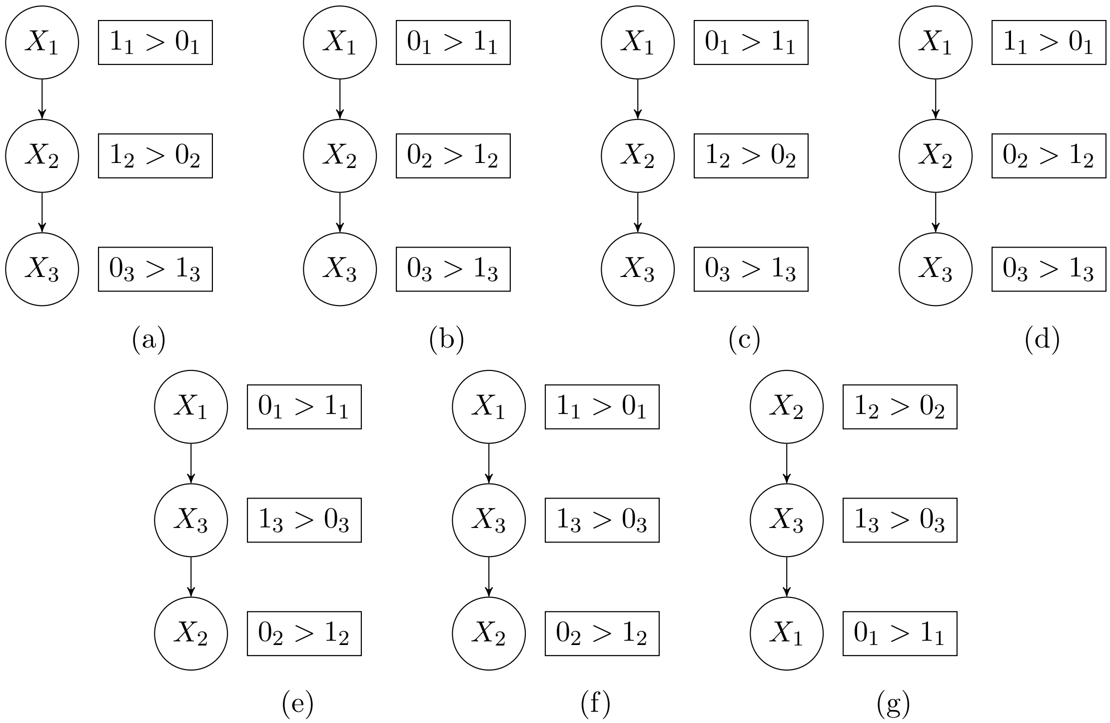


Figure 7.2: UI-UP LP-trees

b-Borda

By *b-Borda* we mean a positional scoring rule with the scoring vector $\langle b, b{-}1, b{-}2, \ldots \rangle$. Let $m = 2^p$ denote the number of alternatives in $\mathcal{X}(\mathcal{I})$ (where, as always, $\mathcal{I} = \{X_1, \ldots, X_p\}$). If $b \geq 2^p - 1$, b-Borda can be reduced to the (standard) Borda rule. In the most restrictive case of UI-UP profiles, the evaluation problem for the Borda rule is in P, and it is NP-complete for the three other classes of profiles [56].

When $b < 2^p - 1$, we show that for some values of b, the winner and the evaluation problems under the b-Borda rules are NP-hard and NP-complete, respectively, no matter what the type of LP-trees used in profiles. The cases of UI-CP, CI-UP and CI-CP trees are handled by a fairly direct reduction from the corresponding problems under the Borda rule. The case of UI-UP profiles requires a different argument (the winner and the evaluation problems under the standard Borda rule are, as we noted, in P). We start with the latter.

We denote by *half-Borda* the b-Borda rule with $b = 2^{p-1}-1$, where p is the number of attributes in $\mathcal{I}$. We have the following theorem on half-Borda.

Theorem 20. *The evaluation and the winner problems under half-Borda for UIUP-profiles are NP-complete and NP-hard, respectively.*

Proof. We show that the evaluation problem is NP-complete. The membership in NP is obvious. The NP-hardness follows from a polynomial reduction from the 2-MINSAT problem.

Given a 2-MINSAT instance (Φ, l), where Φ consists of 2-clauses $C_1, \dots, C_m$ over variables $X_1, \dots, X_p$, we construct an instance of our problem as follows.

First, we introduce a new binary variable X_q and define the set of attributes $\mathcal{I}$ by setting $\mathcal{I} = \{X_1, \dots, X_p, X_q\}$.

Second, for each $C_i \in \Phi$, we now build a set P_i of 12 UI-UP LP-trees over $\mathcal{I}$. As an example, let C_i be $\neg X_2 \vee X_4$[3]. The fragment of the profile determined by C_i is given by the multi-set

$$P_i = \{B_{i_1}, B_{i_2}, B_{i_1}, B_{i_2}, B_{i_1}, B_{i_2}, B'_{i_1}, B'_{i_2}, B''_{i_1}, B''_{i_2}, B''_{i_1}, B''_{i_2}\},$$

where the trees $B_{i_1}, B_{i_2}, B'_{i_1}, B'_{i_2}, B''_{i_1}$, and B''_{i_2} are shown in Figure 7.3. In other words, the profile P_i contains three copies of B_{i_1} and B_{i_2}, one copy of B'_{i_1} and B'_{i_2}, and two copies of B''_{i_1} and B''_{i_2}. We define the overall profile P as the collection of all profiles P_i, $1 \leq i \leq m$. That is, $P = \bigcup_{1 \leq i \leq m} P_i$. Clearly, we have $12 \cdot m$ UI-UP LP-trees in the profile P.

Finally, we set the threshold value $R = 15a \cdot (m - l) + 3a \cdot l$, where we use a to denote 2^{p-1}.

Let o be an outcome over $\mathcal{I}$. Let B be a UIUP tree over $\mathcal{I}$, X_j the most important attribute of B. We define the half-Borda score of o in tree B, denoted by $s_{HB}(B, o)$,

[3] We will build P_i according to what C_i contains: the two atoms in C_i are the labels of the top two levels of trees, and whether the atom is negated affects the preference on that atom.

to be 0 if outcome o has the non-preferred value on X_j; $s_{Borda}(B_{|\mathcal{I}\setminus\{X_j\}}, o_{|\mathcal{I}\setminus\{X_j\}})$, otherwise. We now compute the half-Borda score of o according to whether it satisfies X_q and C_i. If $o \models X_q \wedge \neg C_i$, that is, $o \models X_q \wedge X_2 \wedge \neg X_4$, we have

$$\begin{aligned} s_{HB}(P_i, o) &= \underbrace{(2^p - 1 + 2^{p-1} + 1) * 3}_{\text{three copies of } B_{i_1} \text{ and } B_{i_2}} + \underbrace{(0)}_{B'_{i_1} \text{ and } B'_{i_2}} + \underbrace{(2^p - 1 + 2^{p-1} + 1) * 2}_{\text{two copies of } B''_{i_1} \text{ and } B''_{i_2}} \\ &= 15a. \end{aligned}$$

If $o \models X_q \wedge C_i$, we need to consider three cases:

(1). If $o \models X_q \wedge \neg X_2 \wedge X_4$, we have

$$\begin{aligned} s_{HB}(P_i, o) = & \underbrace{(0) * 3}_{\text{three copies of } B_{i_1} \text{ and } B_{i_2}} + \underbrace{(2^p - 1 + 2^{p-1} + 1)}_{B'_{i_1} \text{ and } B'_{i_2}} + \underbrace{(0) * 2}_{\text{two copies of } B''_{i_1} \text{ and } B''_{i_2}} \\ = & 3a. \end{aligned}$$

(2). If $o \models X_q \wedge \neg X_2 \wedge \neg X_4$, we have

$$\begin{aligned} s_{HB}(P_i, o) = & \underbrace{(0) * 3}_{\text{three copies of } B_{i_1} \text{ and } B_{i_2}} + \underbrace{(2^{p-1} - 1 + 1)}_{B'_{i_1} \text{ and } B'_{i_2}} + \underbrace{(2^{p-1} - 1 + 1) * 2}_{\text{two copies of } B''_{i_1} \text{ and } B''_{i_2}} \\ = & 3a. \end{aligned}$$

(3). If $o \models X_q \wedge X_2 \wedge X_4$, we have

$$\begin{aligned} s_{HB}(P_i, o) = & \underbrace{(2^{p-1} - 1 + 1) * 3}_{\text{three copies of } B_{i_1} \text{ and } B_{i_2}} + \underbrace{(0)}_{B'_{i_1} \text{ and } B'_{i_2}} + \underbrace{(0)}_{\text{two copies of } B''_{i_1} \text{ and } B''_{i_2}} \\ = & 3a. \end{aligned}$$

Thus, for $o \models X_q \wedge C_i$, we have $s_{HB}(P_i, o) = 3a$.

Similarly, we can compute that $s_{HB}(P_i, o) < 15a$, if $o \models \neg X_q \wedge \neg C_i$; and $s_{HB}(P_i, o) <$

$3a$, if $o \models \neg X_q \wedge C_i$.

We now show that there exists an outcome over $\mathcal{I}$ with score at least R if and only if there exists an assignment over I that satisfies at most l clauses in Φ.

($\Leftarrow$) We assume there is an assignment v over I satisfying at most l clauses in Φ. Define an outcome $o = (v, 1_q)$. It is clear that $s_{HB}(P, o) \geq R$.

($\Rightarrow$) We assume there is an outcome o over $\mathcal{I}$ such that $s_{HB}(P, o) \geq R$. If $o \models \neg X_q$, we could flip the value on X_q from 0_q to 1_q, and obtain o' such that $s_{HB}(P, o') > s_{HB}(P, o) \geq R$. Assuming $o'|_I$ satisfies l' ($l' > l$) clauses in Φ, we have that $s_{HB}(P, o') = 15a \cdot (m - l') + 3a \cdot l' > R$; thus, $l' < l$. A contradiction! Otherwise, if $o \models X_q$, we are done. □

Corollary 21. *Theorem Theorem 20 holds for b-Borda when $b = 2^{p-c} - 1$, where c is a constant and $1 \leq c < p$.*

Corollary 21 holds because we can construct c to be 1 and then the proof of Theorem 20 follows.

Theorem 22. *Let $b = 2^{p-c} - 1$, where p is the number of attributes and c a fixed integer such that $1 \leq c < p$. The evaluation and the winner problems under b-Borda for profiles consisting of CI-UP trees (UI-CP and CI-CP trees, respectively) are NP-complete and NP-hard, respectively.*

Proof. We only show an argument for the class CI-UP. The reasoning for other two types of profiles is similar. Moreover, we only show that the evaluation problem (under the restriction to profiles consisting of CI-UP trees) is NP-complete. Indeed, it directly implies that the corresponding variant of the winner problem is NP-hard.

As in other arguments before, the membership in the class NP is evident. Thus, we focus on the hardness part of the argument. To show NP-hardness, we construct a reduction from the evaluation problem under Borda when profiles consist of CI-UP trees ($Borda^{ev}_{CI-UP}$, for short). That problem is known to be NP-complete [56].

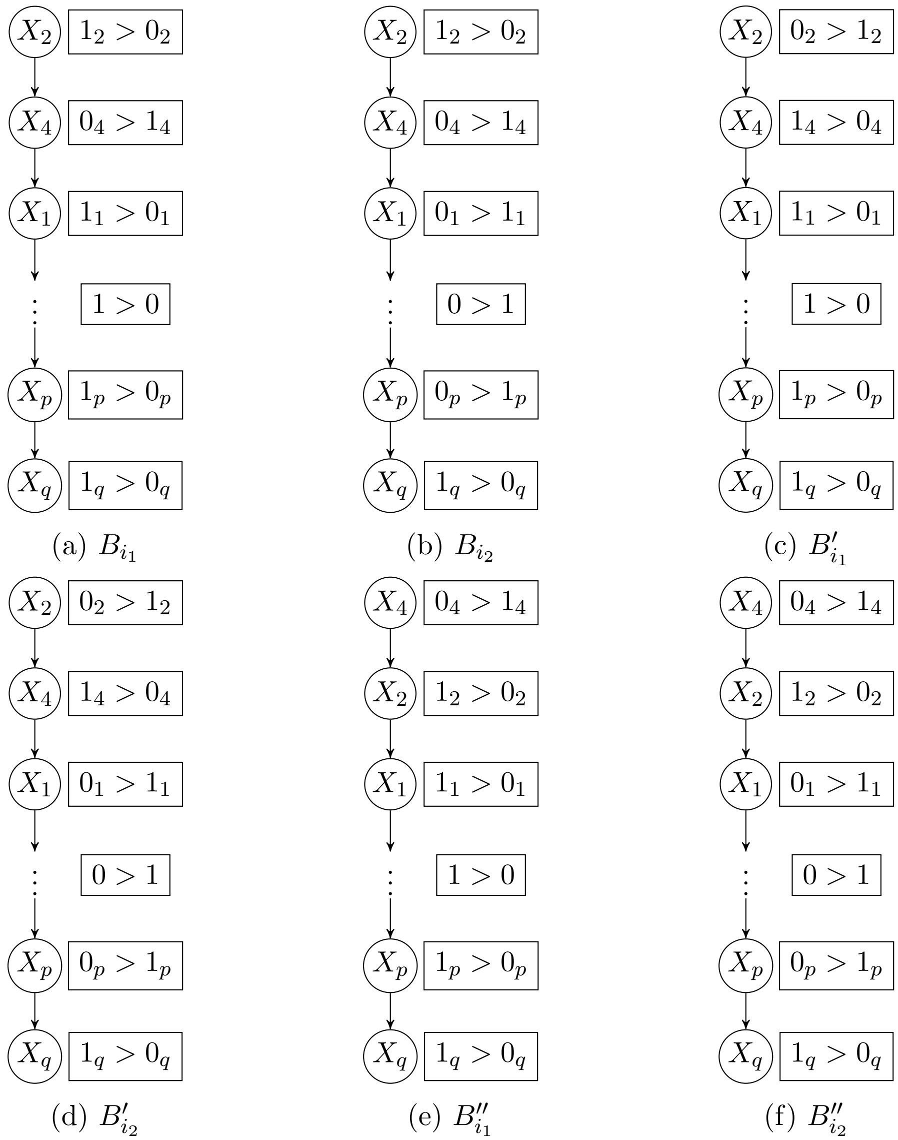


Figure 7.3: UI-UP LP-trees

Given an instance $\langle I, P, l \rangle$ of $Borda^{ev}_{CI-UP}$, where I is a set of p attributes $X_1, \ldots, X_p$, $P = \langle T_1, \ldots, T_m$ is a profile of m CI-UP trees over I, and l is a positive integer, we construct an instance $\langle \mathcal{I}, \mathcal{P}, \ell \rangle$ of our problem as follows.

First, we define $\mathcal{I} = \{Y_1, \ldots, Y_c, X_1, \ldots, X_p\}$, where $Y_1, \ldots, Y_c$ are new attributes. Second, we construct a UI-UP tree T built of c nodes labeled $Y_1, \ldots Y_c$ (from top to bottom), with the node labeled with Y_i having a local preference $1 > 0$. Then, for each $T_i \in P$, $1 \leq i \leq m$, we form a CI-UP tree T'_i by connecting the bottom node

Table 7.3: b-Borda

	UP	CP
UI	P	NPC
CI	NPC	NPC

(a) $b = 2^p - 1$

	UP	CP
UI	NPC	NPC
CI	NPC	NPC

(b) $b = 2^{p-c} - 1$ and $1 \leq c < p$

of T (the one labeled wit Y_c) by a "straight-down" edge to the root of T_i. We define $\mathcal{V} = \{T'_1, \ldots, T'_,\}$. Finally, we set $\ell = l$.

It is simple to verify that under the profile P there is an alternative with the Borda score of at least l if and only if under the profile $\mathcal{P}$ there is an alternative with the b-Borda score of at least ℓ. □

For the b-Borda rule, we include the complexity results in Table 7.3, where Table 7.3b shows existing results by others [56], and Table 7.3a presents results obtained by us.

7.4 The Problems in Answer-Set Programming

The winner and the evaluation problems are in general intractable in the setting we consider. Yet, they arise in practice and computational tools to handle them are needed. We develop and evaluate a computational approach based on *answer-set programming* (ASP) [67]. We propose several ASP encodings for both problems for the Borda, k-approval, and (k, l)-approval rules (for the lack of space only the encodings for Borda are discussed). The encodings are adjusted to two ASP solvers for experiments: *clingo* [42], and *clingcon* [72] and demonstrate the effectiveness of ASP in modeling problems related to preference aggregation.

Encoding LP Trees As Logic Programs

In the winner and evaluation problems, we use LP-trees only to compute the ranking of an alternative. Therefore, we encode trees as program rules in a way that enables that computation for a given alternative. In the encoding, an alternative o is represented by a set of ground atoms $eval(i, x_i)$, $i = 1, 2, \ldots, p$ and $x_i \in \{0, 1\}$. An atom $eval(i, x_i)$ holds precisely when the alternative o has value x_i on attribute X_i.

If X_i is the attribute labeling a node t in vote v at depth d_i^v, $CPT(t)$ determines which of the values 0_i and 1_i is preferred there. Let us assume $\mathcal{P}(t) = \{t_1, \ldots, t_j\}$ and $Inst(t) = \{t_{j+1}, \ldots, t_\ell\}$, where each t_q is labeled by X_{i_q}. The location of t is determined by its depth d_i^v and by the set of values $x_{i_{j+1}}, \ldots, x_{i_\ell}$ of the attributes labeling $Inst(t)$ (they determine whether we descend to the left or to the right child as we descend down the tree). Thus, $CPT(t)$ can be represented by program rules as follows. For each row $u : 1_i > 0_i$ in $CPT(t)$, where $u = x_{i_1}, \ldots, x_{i_j}$, we include in the program the rule

$$\begin{aligned} vote(v, d_i^v, i, 1) \;\text{:-}\; & eval(i_1, x_{i_1}), \ldots, eval(i_j, x_{i_j}), \\ & eval(i_{j+1}, x_{i_{j+1}}), \ldots, eval(i_\ell, x_{i_\ell}) \end{aligned} \tag{7.1}$$

(and similarly, in the case when that row has the form $u : 0_i > 1_i$).

In this representation, the property $vote(v, d_i^v, i, a_i)$ will hold true for an alternative o represented by ground atoms $eval(i, x_i)$ *precisely when* (or *if*, denoted by ":-" in our encodings) that alternative takes us to a node in v at depth d_i^v labeled with the attribute X_i, for which at that node the value a_i is preferred. Since, in order to compute the score of an alternative on a tree v all we need to know is whether $vote(v, d_i^v, i, a_i)$ holds (cf. our discussion below), this representation of trees is sufficient for our purpose.

For example, the LP-tree v in Figure 3.3 is translated into the logic program in

Figure 7.4 (*voteID(v)* identifies the id of the vote (LP-tree)).

```
voteID(1).
vote(1,1,1,1).
vote(1,2,2,1) :- eval(1,1).
vote(1,3,3,1) :- eval(2,1), eval(1,1).
vote(1,3,3,0) :- eval(2,0), eval(1,1).
vote(1,2,3,0) :- eval(1,0).
vote(1,3,2,0) :- eval(1,0).
```

Figure 7.4: Translation of v in logic rules

Encoding Positional Scoring Rules In ASP

Encoding the Borda evaluation problem in *clingo*

The evaluation and the winner problems for Borda can be encoded in terms of rules on top of those that represent an LP profile. Given a representation of an alternative and of the profile, the rules evaluate the score of the alternative and maximize it or test if it meets or exceeds the threshold.

We first show the encoding of the Borda evaluation problem in *clingo* (Figure 7.5). Parameters in the evaluation problem are defined as facts (lines 1-4): predicates *attribute/1*s representing three attributes, *numIss/1* the number of attributes, *threshold/1* the threshold value, together with *val/1*s the two values in the attributes' binary domains. Line 5 generates the search space of all alternatives over three binary attributes. It expresses that if X is an attribute, exactly one of *eval(X,Y)* holds for all *val(Y)*, i.e., exactly one value Y is assigned to X.

Let o be an alternative represented by a set of ground atoms $eval(i, x_i)$, one atom for each attribute X_i. Based on the representation of trees described above, for

```
1 attribute(1). attribute(2). attribute(3).
2 numIss(3).
3 val(0). val(1).
4 threshold(5).
5 1{ eval(I,M) : val(M) }1 :- attribute(I).
6 wform(V,I,W) :- vote(V,D,I,A), eval(I,A), numIss(P), W=#pow(2,P-D).
7 wform(V,I,0) :- vote(V,D,I,A), eval(I,M), A != M.
8 goal :- S = #sum [ wform(V,I,W) = W ], threshold(TH), S >= TH.
9 :- not goal.
```

Figure 7.5: Borda evaluation problem encoding in *clingo*

every tree v we get the set of ground atoms $vote(v, d_i^v, i, a_i)$. The Borda score of an alternative in that tree corresponds to the rank of the leaf the alternative leads to (in a "non-collapsed" tree), which is determined by the direction of descent (left or right) at each level. Roughly speaking, these directions give the binary representation of that rank, that is, the Borda score of the alternative. Let us define $s_B(v, o)$ as a function that computes the Borda score of alternative o given one vote v. Then one can check that

$$s_B(v, o) = \sum_{i=1}^{p} 2^{p-d_i^v} \cdot f(a_i, x_i), \tag{7.2}$$

where $f(a_i, x_i)$ returns 1 if $a_i = x_i$, 0 otherwise. Thus, to compute the Borda score with regard to a profile $\mathcal{V}$, we have

$$s_B(\mathcal{V}, o) = \sum_{v=1}^{n} \sum_{i=1}^{p} 2^{p-d_i^v} \cdot f(a_i, x_i). \tag{7.3}$$

In the program in Figure 7.5, lines 6 and 7 introduce predicate *wform/3* which computes $2^{p-d_i^v} \cdot f(a_i, x_i)$ used to compute Borda score. According to equation (7.3), if attribute I appears in vote V at depth D and A is its preferred value, and if the value of I is indeed A in an alternative o, then the weight W on I in V is 2^{P-D}, where P is the number of attributes; if attribute I is assigned the less preferred value in o, then the weight W on I in V is 0. The Borda score of the alternative is then equal to the sum of all the weights on every attribute in every vote, and this is computed using

```
$domain(1..4).
attribute(1). attribute(2). attribute(3).
numIss(3).
val(0). val(1).
threshold(5).
1{ eval(I,M) : val(M) }1 :- attribute(I).
wform(V,I,W) :- vote(V,D,I,A), eval(I,A), numIss(P), W=#pow(2,P-D).
wform(V,I,0) :- vote(V,D,I,A), eval(X,M), A != M.
weight(V,I) $== W :- wform(V,I,W).
$sum{ weight(V,I) : voteID(V) : var(I) } $>= TH :- threshold(TH).
```

Figure 7.6: Borda evaluation problem encoding using *clingcon*

the aggregate function *#sum* built in the input language of *clingo* (rule 8). Rule 9 is an *integrity constraint* stating that contradiction is reached if predicate *goal/0* does not hold in the solution. Together with rule 8, it is ensured that the Borda evaluation problem is satisfiable if and only if there is an answer set in which *goal/0* holds.

The encoding for the Borda winner problem for *clingo* replaces rules 7 and 8 in Figure 7.5 with the following single rule:

```
#maximize[ wform(V,I,W) = W ].
```

The *#maximize* statement is an optimization statement that maximizes the sum of all weights (W's) for which *wform(V,I,W)* holds.

Encoding the Borda evaluation problem in *clingcon*

In this encoding, we exploit *clingcon*'s ability to handle some numeric constraints by specialized constraint solving techniques (by means of the CP solver *Gecode* [77]). In Figure 7.6 we encode the Borda evaluation problem in *clingcon*.

Lines 2-8 are same as lines 1-7 in Figure 7.5. Line 9 defines the constraint variable *weight(V,I)* that assigns weight W to each pair (V,I) and line 10 defines a global constraint by use of *\$sum* declares that the Borda score must be at least the threshold. Line 1 restricts the domain of all constraint variables (only *weight/2* in this case) to [1,4] as weights of attributes in an LP-tree of 3 attributes are 2^0, 2^1 and 2^2.

The encoding for the Borda winner problem for *clingcon* replaces rules 10 in Figure 7.6 with the following one rule:

```
$maximize{weight(V,I):voteID(V):attribute(I)}.
```

The *\$maximize* statement is an optimization statement that maximizes the sum over the set of constraint variables *weight(V,I)*.

Encoding the k-approval evaluation problem in *clingo*

One method to aggregate LP-trees according to k-approval can be designed reusing the Borda encodings for both problems and solvers. Given an alternative o, we can first compute $s_B(v,o)$ in every vote v and then compare $s_B(v,o)$ with $m-k$. If $s_B(v,o) \leq m-k$, $s_k(v,o)=1$; otherwise, $s_k(v,o)=0$. This method, however, is later turned out not quite effective for *clingo* in the sense that the rules to calculate Borda scores using aggregating predicate *#sum* result in large ground propositional theories that is hard for *clingo* to solve. We managed to work around this ineffectiveness by coming up with encodings using a heuristic that reduce the size of the ground programs for *clingo*. The heuristic is described in Theorem 23.

Theorem 23. *Given an LP-tree v and a positive integer k, we can construct in $O(p^2)$ time a Boolean formula ϕ of length $O(p^2)$ such that $s_k(v,o)=1$ for an alternative o*

iff o satisfies ϕ*.*

Proof. The algorithm is as follows.

1. ϕ is a disjunction of conjunctions of literals over attributes built as follows.

2. Compute the k-th preferred alternative $\vec{d_k}$ in time linear in p. Denote by $IO_{\vec{d_k}}$ the importance order $\vec{d_k}$ induces. Assume $IO_{\vec{d_k}} = X_{i_1} \rhd X_{i_2} \rhd \ldots \rhd X_{i_p}$.

3. The first conjunction $C_1 = l_{i_1} \wedge \ldots \wedge l_{i_j}$, where each l_{i_k}, $1 \leq k \leq j$, is X_{i_k} (resp. $\neg X_{i_k}$) if $\vec{d_k}(X_{i_j}) = 1_{i_j}$ (resp. $\vec{d_k}(X_{i_j}) = 0_{i_j}$).

4. For every attribute $X_{i_j} \in IO_{\vec{d_k}}$ such that $\vec{d_k}$ assigns it with its less preferred value (e.g., if $1_{i_j} > 0_{i_j}$, $\vec{d_k}(X_{i_j}) = 0_{i_j}$), we have a conjunction $C_{i_j} = l_{i_1} \wedge \ldots \wedge l_{i_{j-1}} \wedge l_{i_j}$, where each l_{i_k}, $1 \leq k \leq j-1$, is X_{i_k} (resp. $\neg X_{i_k}$) if $\vec{d_k}(X_{i_j}) = 1_{i_j}$ (resp. $\vec{d_k}(X_{i_j}) = 0_{i_j}$) and l_{i_j} is X_{i_k} (resp. $\neg X_{i_k}$) if $\vec{d_k}(X_{i_j}) = 0_{i_j}$ (resp. $\vec{d_k}(X_{i_j}) = 1_{i_j}$).

□

In order to compute the k-th preferred alternative $\vec{d_k}$, we need some auxiliary predicates to help the computation. We define predicates *voteK/4* and *evalK/4* that are basically copies of *vote/4* and *eval/4* in the logic representation of LP-trees except that *evalK/4* describes $\vec{d_k}$. A predicate *evalK(V,D,I,M)* means that in vote V the k-th ranked alternative assigns value M to attribute I at depth D. For the example LP-tree in Figure 3.3, we have the follow ancillary logic program in Figure 7.7.

```
voteK(1,1,1,1).

voteK(1,2,2,1) :- evalK(1,1,1,1).

voteK(1,3,3,1) :- evalK(1,2,2,1), evalK(1,1,1,1).

voteK(1,3,3,0) :- evalK(1,2,2,0), evalK(1,1,1,1).

voteK(1,2,3,0) :- evalK(1,1,1,0).

voteK(1,3,2,0) :- evalK(1,1,1,0).
```

Figure 7.7: Auxiliary data in logic rules for computing $\vec{d_k}$

We now present the encoding of the k-Approval evaluation problem in *clingo* (Figure 7.8), where $k = 5$.

```
attribute(1). attribute(2). attribute(3).
numIss(3).
val(0). val(1).
k(1,1). k(2,0). k(3,0).
threshold(5).
evalK(V,D,I,M) :- vK(V,D,I,M), k(D,0).
evalK(V,D,I,1-M) :- vK(V,D,I,M), k(D,1).
1{ eval(I,M) : val(M) }1 :- attribute(I).
rank(V,1) :- vote(V), numIss(N),
             N{eval(I,M) : evalK(VV,D,I,M) : V==VV}N.
rank(V,1) :- vote(V), k(D,1),
             D-1{eval(I,M) : evalK(VV,DD,I,M) : A==AA : DD<=D-1}D-1,
             1{eval(I,M) : evalK(V,D,I,MM) : M!=MM}1.
goal :- S = #sum [ rank(V,Y) = Y ], threshold(TH), S >= TH.
:- not goal.
```

Figure 7.8: k-Approval evaluation problem encoding in *clingo*

7.5 The Problems in Weighted Partial Maximum Satisfiability

In this section, we call "the evaluation and the winner problems based on a positional scoring rule r" by "the r problems." We show an algorithm that translate the positional scoring rule problems into Weighted Partial Maximum Satisfiability instances. We first translate the positional scoring rule problems to Weighted Terms Maximum Satisfiability instances, which are then transformed into Weighted Partial Maximum Satisfiability instances.

Weighted Partial Maximum Satisfiability

Definition 34. Let X be a set of Boolean variables $\{X_1, \ldots, X_q\}$, Ψ a set of weighted terms of the form

$$\{(t_1, w_1), \ldots, (t_n, w_n)\},$$

where each term is a conjunction of literals on X. The *Weighted Terms Maximum Satisfiability* (WTM) problem is to find an assignment of X that maximizes the sum of the weights of the satisfied terms in Ψ.

Definition 35. Let X be a set of Boolean variables $\{X_1, \ldots, X_p\}$, a *weighted partial formula* Φ [4] is a multi-set of weighted clauses over X of the form

$$\{(c_1, w_1), \ldots, (c_n, w_n), (c_{n+1}, w_{n+1}), \ldots, (c_{n+m}, w_{n+m})\},$$

where each w_i, $1 \leq i \leq n$, is a positive integer and $w_{n+1} = \ldots = w_{n+m} = \sigma = 1 + \sum_{i=1}^{n} w_i$. Clause c_j is *hard* if $w_j = \sigma$; *soft*, otherwise.

Definition 36. Let X be a set of Boolean variables $\{X_1, \ldots, X_p\}$, Φ a weighted partial formula, the *Weighted Partial Maximum Satisfiability* (WPM) problem is to find an assignment of X that maximizes the sum SW of weights of satisfied clauses in Φ. If $SW < m * \sigma$, it means that at least one hard clause is falsified and we say that Φ is *unsatisfiable*.

Clearly, the WPM problem generalizes the SAT problem [40], the MAXSAT problem [24] and the Partial MAXSAT problem [24].

[4] This definition is slightly adapted of the commonly used [4, 3].

Translating a WTM instance into an equivalent WPM instance

Now we show how a WTM instance Ψ can be translated into a WPM instance Φ in polynomial time such that a solution to Φ projected onto Ψ's alphabet X_Ψ is a solution to Ψ.

Theorem 24. *Given a WTM instance Ψ, a WPM instance Φ can be computed in time $O(nq)$ such that any solution to Φ restricted to X_Ψ is a solution to Ψ.*

Proof. The translation algorithm is detailed in Algorithm 5.

Let X_Ψ be $\{X_1, \ldots, X_q\}$, X_Φ be $\{X_1, \ldots, X_q, C_1, \ldots, C_n\}$. Assume v is a solution to the WPM instance Φ over X_Φ, we show that the restriction, $v|_{X_\Psi}$, is a solution to the original WTM instance Ψ. Let $S = \{t_{i_1}, \ldots, t_{i_s}\}$ be the set of terms in Ψ satisfied by v (or, equivalently, $v|_{X_\Psi}$). It is clear that v satisfies $\{C_{i_k} : t_{i_k} \in S\}$ and falsifies $\{C_{i_k} : t_{i_k} \notin S\}$; since, otherwise, v would not have the maximal sum SW for Φ. Denote by x_i the number of literals in term t_i. Let v' be an arbitrary assignment such that $SW_{v'} < SW_v$, and $S' = \{t_{j_1}, \ldots, t_{j_r}\}$ the set of terms in Ψ satisfied by v'.

According to Algorithm 5, we have $SW_v = \sum_{k=1}^{s} w_{i_k} + \sum_{k=1}^{n} \sigma + \sum_{k=1}^{n}(\sum_{o=1}^{x_k} \sigma)$ and $SW_{v'} = \sum_{k=1}^{r} w_{j_k} + \sum_{k=1}^{n} \sigma + \sum_{k=1}^{n}(\sum_{o=1}^{x_k} \sigma)$. Then, we have $SW_v - SW_{v'} = \sum_{k=1}^{s} w_{i_k} - \sum_{k=1}^{r} w_{j_k} > 0$. Thus, we know $v|_{X_\Psi}$ is a solution to the original WTM instance Ψ. $\square$

Encoding Borda problems in WTM and WPM

The LP-tree in Figure 3.3 under Borda is translated to a WTM instance in Figure 7.9.

Then the WTM instance in Figure 7.9 is transformed into a WPM instance in Figure 7.10.

Algorithm 5: Compute equivalent WPM instances from WTM instances

Input: a WTM instance Ψ
Output: an equivalent WPM instance Φ

```
Φ ← ∅;
σ ← 1 + Σ_{i=1}^{n} w_i;
foreach (t_i, w_i) ∈ Ψ do
    introduce a new variable C_i and Φ ← Φ ∪ (C_i, w_i);
    Φ ← Φ ∪ (C_i ∨ ⋁_{l_j ∈ t_i} ¬l_j, σ);
    foreach l_j ∈ t_i do
        Φ ← Φ ∪ (¬C_i ∨ l_j, σ);
    end
end
return Φ
```

$(X_1, 4)$
$(X_1 \wedge X_2, 2)$
$(X_1 \wedge X_2 \wedge X_3, 1)$
$(X_1 \wedge \neg X_2 \wedge \neg X_3, 1)$
$(\neg X_1 \wedge \neg X_3, 2)$
$(\neg X_1 \wedge \neg X_2, 1)$

Figure 7.9: The WTM instance of the LP-tree v

Encoding k-approval problems in WTM and WPM

The LP-tree in Figure 3.3 under 5-Approval is translated to a WTM instance in Figure 7.11.

$(\neg X_1 \wedge \neg X_2 \wedge \neg X_3, 1)$

$(X_1, 1)$

Figure 7.11: The WTM instance of the LP-tree v under 5-Approval

Then the WTM instance in Figure 7.11 is transformed into a WPM instance in Figure 7.12.

$(C_1, 4)$
$(\neg C_1 \vee X_1, 12)$
$(\neg X_1 \vee C_1, 12)$
$(C_2, 2)$
$(\neg C_2 \vee X_1, 12)$
$(\neg C_2 \vee X_2, 12)$
$(\neg X_1 \vee \neg X_2 \vee C_2, 12)$
$(C_3, 1)$
$(\neg C_3 \vee X_1, 12)$
$(\neg C_3 \vee X_2, 12)$
$(\neg C_3 \vee X_3, 12)$
$(\neg X_1 \vee \neg X_2 \vee \neg X_3 \vee C_3, 12)$
$(C_4, 1)$
$(\neg C_4 \vee X_1, 12)$
$(\neg C_4 \vee \neg X_2, 12)$
$(\neg C_4 \vee \neg X_3, 12)$
$(\neg X_1 \vee X_2 \vee X_3 \vee C_4, 12)$
$(C_5, 2)$
$(\neg C_5 \vee \neg X_1, 12)$
$(\neg C_5 \vee \neg X_3, 12)$
$(X_1 \vee X_3 \vee C_5, 12)$
$(C_6, 1)$
$(\neg C_6 \vee \neg X_1, 12)$
$(\neg C_6 \vee \neg X_2, 12)$
$(X_1 \vee X_2 \vee C_6, 12)$

Figure 7.10: The WPM instance of the LP-tree v

$(C_1, 1)$

$(\neg C_1 \vee \neg X_1, 3)$

$(\neg C_1 \vee \neg X_2, 3)$

$(\neg C_1 \vee \neg X_3, 3)$

$(X_1 \vee X_2 \vee X_3 \vee C_1, 3)$

$(C_2, 1)$

$(\neg C_2 \vee X_1, 3)$

$(\neg X_1 \vee C_2, 3)$

Figure 7.12: The WPM instance of the LP-tree v under 5-Approval

7.6 Experiments

Here we present and analyze the experimental results from solving the Winner problem and the Evaluation problem using two Answer Set Programming solvers *clingo* (version 4.2.1) and *clingcon* (version 2.0.3) and one Constraint Satisfaction Problem solver *toulbar* (version 0.9.6.0-dev).

All our experiments were performed on a machine with an Intel(R) Core(TM) i7 CPU @ 2.67GHz and 8 GB RAM running Ubuntu 12.04 LTS.

We first consider the winner problem. In the study, we consider the computation time with a fixed number of attributes (5/10/20) and for each number of attributes we range the number of votes in a profile up to 3000 for $\{Borda, 2^{p-2}\text{-}approval, 2K\text{-}approval\} \times \{clingcon, clingo, toulbar\}$. Then we fix the number of votes (1000) and vary the number of attributes up to 20, again for same set of settings. Each time result in seconds is computed as the mean of 20 tests over different randomly generated profiles of LP-trees.

Structure of the Simple LP Trees

To experiment with the programs presented above and with *clingo* and *clingcon* solvers, we generate logic programs that represent random LP-trees and profiles of random LP-trees. Our algorithm generates encodings of trees from the most general class CI-CP under the following restrictions: (1) Each LP-tree has exactly two paths with the splitting node appearing at depth $d_s = \lfloor \frac{p}{2} \rfloor$; (2) Each non-root node at depth $\leq d_s + 1$ has exactly one parent; (3) Each node at depth $> d_s + 1$ has exactly two parents, one of which is at depth $< d_s$.[5]

The algorithm starts by randomly selecting attributes to label the nodes on the path from the root to the splitting node and then, similarly, labels the nodes on each

[5]The restrictions are motivated by the size of the representation considerations. They ensure that the size of generated LP-trees is linear in the number of attributes.

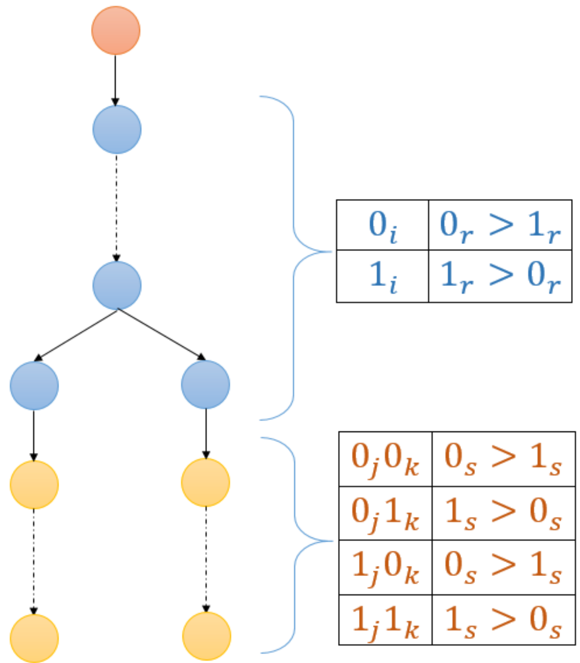


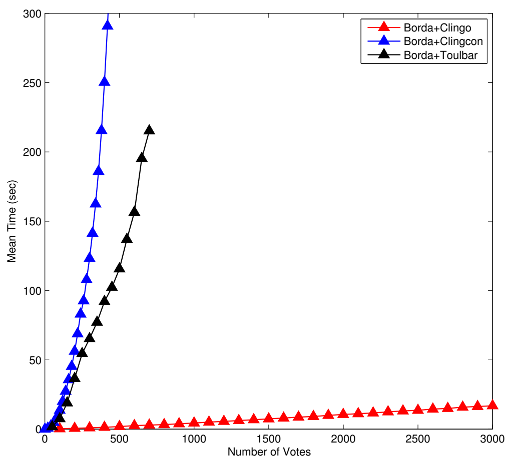
Figure 7.13: *Simple* CI-CP tree

of the two paths (different labeling can be produced for each of them). Then, for each non-root node, the algorithm selects at random one or two parent nodes (as appropriate based on the location of the node). Finally, the algorithm decides local preferences (for each combination of values of the parent attributes) randomly picking one over the other. In each step, all possible choices are equally likely. We call CI-CP LP-trees satisfying these restrictions *simple*. Each simple LP-tree has size linear in p. Figure 7.13 depicts a CI-CP tree of 4 attributes in this class.

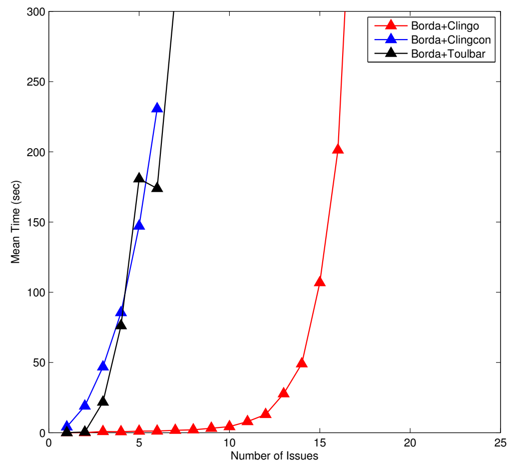
Solving the Winner Problem

We refer to 7.14 for the empirical results on solving the Winner problem. Each point in a figure represents an average of computation time spent on solving 20 different Winner problem instances given randomly generated LP profiles.

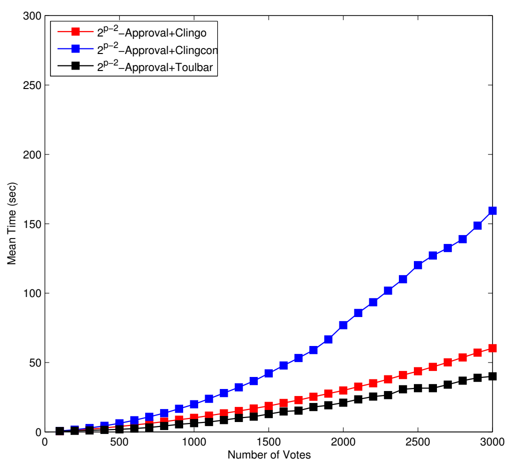
It is clear that our experiments on the winner problem for the three voting rules with fixed number of attributes are consistent with the property that the problem is solvable in polynomial time. All three solvers scale up well. Figures 7.14(a),(c)



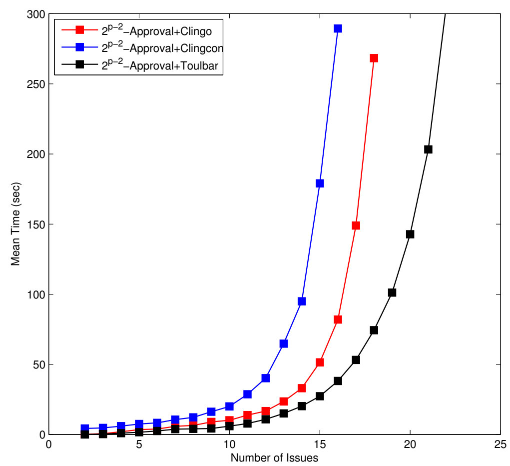
(a) Borda, fixed #attributes(10)



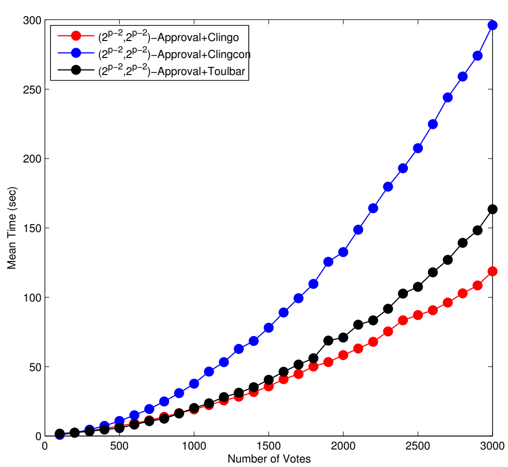
(b) Borda, fixed #votes(1000)



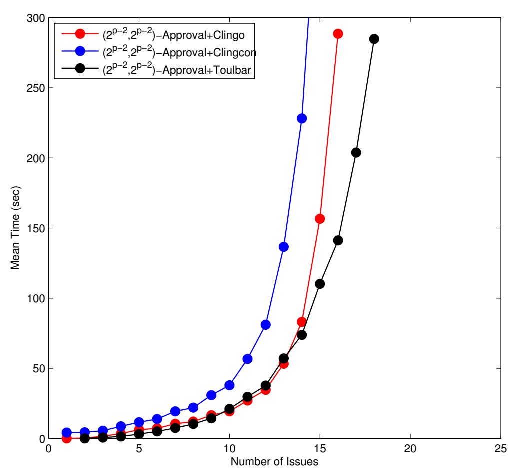
(c) 2^{p-2}-Approval, fixed #attributes(10)



(d) 2^{p-2}-Approval, fixed #votes(1000)



(e) $2K$-Approval, fixed #attributes(10)



(f) $2K$-Approval, fixed #votes(1000)

Figure 7.14: Solving the winner problem given *simple* LP-trees

and (e) depict the result for the cases with 10 attributes. When we fix the number of votes and vary the number of attributes the time grows exponentially with p (cf. Figures 7.14(b),(d) and (f)), again consistently with the computational complexity of the problems (NP-hardness).

Generally *clingo* is better compared to *clingcon* in solving the winner problem for the three scoring rules. Moreover, *clingo* outperforms *toulbar* in solving the winner problem for the Borda rule, while *toulbar* performs better than *clingo* for the two approval rules.

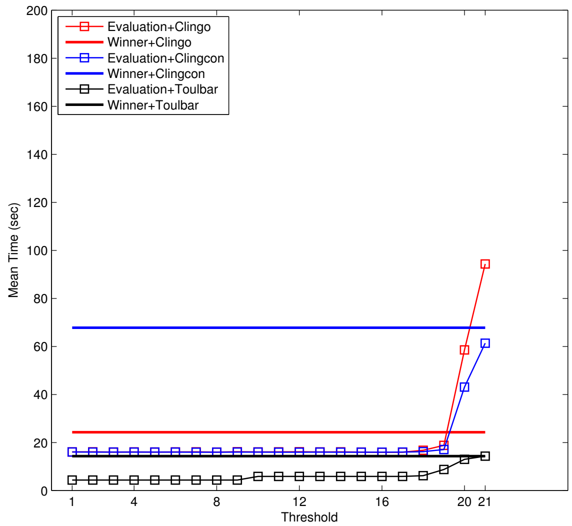
For the winner problems, our experiments demonstrates that profiles of LP-trees of practical sizes can be effectively handled by our solvers, up to 3000 votes per profile over up to 20 attributes. But going beyond 20 attributes remains a challenge.

Solving the Evaluation Problem

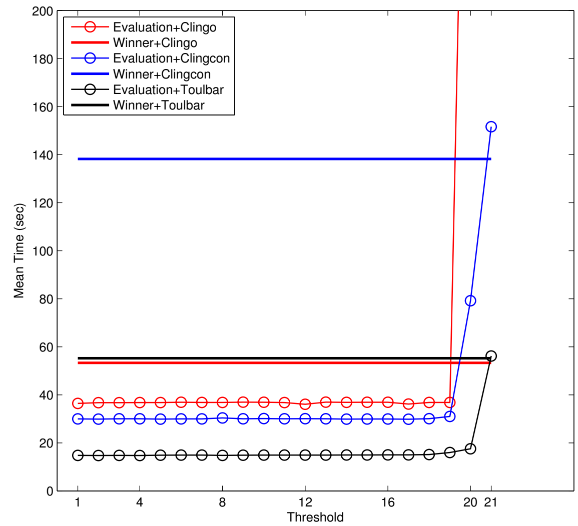
The evaluation problem can be reduced to the winner problem, as an evaluation problem instance has an answer *YES* if and only if the score of the winner equals or exceeds the threshold. Thus, the evaluation problem is at most as complex as the winner problem.

For the evaluation problem, we compare its experimental complexity with that of the winner problem. For each of the 20 randomly generated profiles of 1000 votes, we compute the winning score WS and set the threshold for the evaluation problem with a percentage of WS, starting with 5% and incremented by 5% for the following tests until we reach the full value of WS. We run one more test with the threshold $WS + 1$ (there is no solution then and the overall method allows for the experimental comparison of the hardness of the winner and evaluation problems). That allows us to study the effectiveness of solvers. We again present and compare average time results.

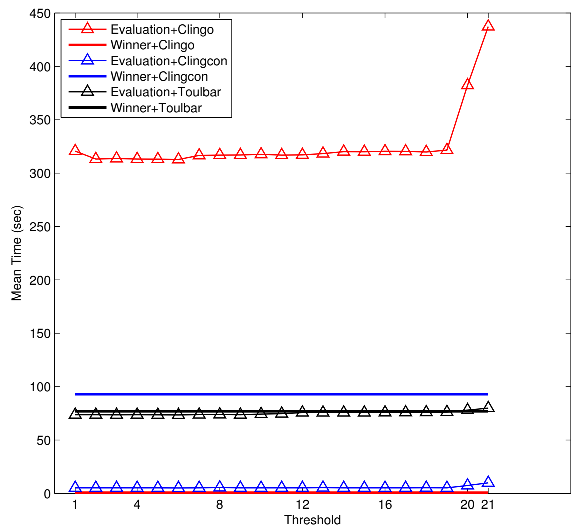
First, we note that for *clingo*, the evaluation problem is harder than the winner



(a) 2^{p-2}-Approval (1000votes/13attributes)



(b) $2K$-Approval (1000votes/13attributes)



(c) Borda (1000votes/4attributes)

Figure 7.15: Solving the evaluation problem given *simple* LP-trees

problem in the entire range for Borda (Figure 7.15(c)). We attribute that to the fact that the encodings of the evaluation problem have to model the threshold constraint with the #*sum* rule which, in *clingo*, leads to large ground theories that it finds hard to handle. In the winner problem encodings, the #*sum* rule is replaced with an optimization construct, which allows us to keep the size of the ground theory low.

Second, we notice that, except for Borda and *clingo*, the evaluation problem is easier than the winner problem when the threshold values are smaller than the winning score and the evaluation problem becomes harder when the thresholds are close to it. We refer to Figures 7.15(a), (b) and (c).

Thirdly, in all cases *clingcon* outperforms *clingo* on the evaluation problems (cf. Figures 7.15(a), (b) and (c)). It is especially clear for Borda, where the range of scores is much larger than in the case of approval rules. That poses a challenge for *clingo* that instantiates the *#sum* rule over that large range, which *clingcon* is able to avoid.

Finally, we compare the effectiveness of *clingcon* and *toulbar* on solving the evaluation problems. Generally, *clingcon* performs better for the Borda rule, whereas *toulbar* is better for the two approval rules. Again, to see this, we refer to Figures 7.15(a), (b) and (c).

7.7 Conclusions

Aggregating votes expressed as LP-trees is a rich source of interesting theoretical and practical problems. In particular, the complexity of the winner and evaluation problems for scoring rules is far from being fully understood. First results on the topic were provided by Lang et al. [56]; our work exhibited another class of positional scoring rules for which the problems are NP-hard and NP-complete, respectively. However, a full understanding of what makes a positional scoring rule hard remains an open problem.

Importantly, our results show that ASP tools are effective in modeling and solving the winners and the evaluation problems for some positional scoring rules such as Borda, 2^{p-2}-approval and $2K$-approval. When the number of attributes is fixed the ASP tools scale up consistently with the polynomial time complexity. In general, the tools are practical even if the number of attributes is up to 15 and the number of votes is as high as 500. This is remarkable as 15 binary attributes determine the space of over 30,000 alternatives.

Finally, the preference aggregation problems form interesting benchmarks for ASP tools that stimulate advances in ASP solver development. As the preference aggregation problems involve large domains, they put to the test those features of ASP tools that attempt to get around the problem of grounding programs over large domains. Our results show that the optimization statements in *clingo* in general perform well. When they cannot be used, as in the evaluation problem, it is no longer the case. The solver *clingcon*, which reduces grounding and preprocessing work by delegating some tasks to a constraint solver, performs well in comparison to *clingo* on the evaluation problem, especially for the Borda rule (and we conjecture, for all rules that result in large score ranges).

In the future work we will expand our experimentation by developing methods to generate richer classes of randomly generated LP-trees. We will also consider the use of ASP tools to aggregate votes given in other preference systems such as CP-nets [12] and answer set optimization (ASO) preferences [22].

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Chapter 8 Conclusion and Future Work

The research in my dissertation is about various aspects of preferences: preference modeling, preference learning, and preference reasoning. Preferences is a major research component studied in artificial intelligence (AI) and decision theory, and is closely related to the social choice theory considered by economists and political scientists. In my dissertation, I explore emerging connections between preferences in AI and social choice theory. Most of my research is on qualitative preference representations that extend and combine existing formalisms such as lexicographic preference trees (LP-trees) [11], answer-set optimization theories (ASO-theories) [22], possibilistic logic [29], and conditional preference networks (CP-nets) [13]; on learning problems that aim at discovering qualitative preference models and predictive preference information from practical data; and on preference reasoning problems centered around qualitative preference optimization and aggregation methods. Applications of my research include recommender systems, decision support tools, multi-agent systems, and Internet trading and marketing platforms.

I introduced partial lexicographic preference trees (PLP-trees) extending the language of lexicographic preference trees (LP-trees). I also proposed preference trees (P-trees) as a generalization of PLP-trees. Both PLP-trees and P-trees are intuitive qualitative preference languages over combinatorial domains, and often compactly represent total preorders over outcomes in such large domains. I studied the expressive power of the two languages and showed that they are closely related to existing preference formalisms.

For preference learning, my research focused on learning PLP-trees. I studied various learning problems for PLP-trees and obtained results on these problems both theoretically and experimentally. My results showed that PLP-trees are highly accurate in modeling preferences arising in practice, and can be effectively learned. To reduce the overfitting of PLP-trees, I introduced the formalism of PLP-forests, collections of PLP-trees. My empirical results on learning PLP-forests showed that PLP-forests are more expressive and accurate than PLP-trees.

Finally, for preference reasoning, I studied preference aggregation problems (e.g., winner determination) in the setting of LP-trees, a special case of PLP-trees that represent total orders. Applying aggregation methods in social choice theory, I showed that the aggregation problems are generally NP-hard. For these hard problems, my empirical study using answer-set programming (ASP) tools, designed specifically for solving NP-hard problems, showed that ASP solvers are effective on large instances.

8.1 Future Work

My long-term research goal is to study computational problems related to preferences, and develop applications that help people or software agents make better decisions. Particularly, I intend to embed theories and practices on preferences into areas including data science, and automated planning and scheduling.

Data science Discovering preference models from large data sets and reasoning about them can be of great value when decisions need to be customized for individual users. For instance, e-Commerce companies want to make quality marketing decisions on what customers would be interested in purchasing at a future time. I propose to introduce contextual information and human-in-the-loop into existing learning methods (e.g., collaborative filtering and content-based filtering used in recommender systems), in order to provide context-aware and user-centered predictions. On the collective level, I mean to leverage social science methods (e.g., voting rules) to combine individual models for joint decisions. I plan to build preferential data sets and develop predictive systems, with collaborators or sponsors from fields such as machine learning, computer vision, psychology, cognitive science, and behavioral science.

Automated planning and scheduling In planning and scheduling, constraints and preferences of agents may be more faceted than simply "fastest" or "cheapest." I propose to design mathematical models allowing intuitive representations of these individual accommodations. Furthermore, I intend to implement systems that automate the acquisition of user constraints and preferences, and the computation of optimal plans or schedules based on these user-specific information. This line of research potentially promotes collaboration with researchers of expertise in travel scheduling, manufacturing, and traffic control.

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Bibliography

[1] Kenneth J. Aarrow. *Social Choice and Individual Values*. John Wiley and Sons, 1951.

[2] D Allouche, S de Givry, and T Schiex. Toulbar2, an open source exact cost function network solver. Technical report, Technical report, INRIA, 2010.

[3] Carlos Ansótegui, María Luisa Bonet, and Jordi Levy. Solving (weighted) partial maxsat through satisfiability testing. In *Theory and Applications of Satisfiability Testing*, pages 427–440. 2009.

[4] Carlos Ansótegui, Maria Luisa Bonet, and Jordi Levy. A new algorithm for weighted partial maxsat. In *AAAI*, 2010.

[5] Sanjeev Arora and Boaz Barak. *Computational Complexity - A Modern Approach*. Cambridge University Press, 2009.

[6] K.J. Arrow, A. Sen, and K. Suzumura. *Handbook of Social Choice and Welfare, Vol 1*. Elsevier, 2002.

[7] Fahiem Bacchus and Adam J. Grove. Graphical models for preference and utility. In Philippe Besnard and Steve Hanks, editors, *UAI*, pages 3–10. Morgan Kaufmann, 1995.

[8] J.J. Bartholdi, C.A. Tovey, and M. A. Trick. The computational difficulty of manipulating an election. *Social Choice and Welfare*, 6:227–241, 1989.

[9] J.J. Bartholdi, C.A. Tovey, and M. A. Trick. Voting schemes for which it can be difficult to tell who won the election. *Social Choice and Welfare*, 6:157–165, 1989.

[10] J.J. Bartholdi, C.A. Tovey, and M. A. Trick. How hard is it to control an election? *Mathl. Comput. Modelling (Special Issue on Formal Theories of Politics*, 16(8/9):27–40, 1992.

[11] Richard Booth, Yann Chevaleyre, Jérôme Lang, Jérôme Mengin, and Chattrakul Sombattheera. Learning conditionally lexicographic preference relations. In *ECAI*, pages 269–274, 2010.

[12] C. Boutilier, R. Brafman, C. Domshlak, H. Hoos, and D. Poole. CP-nets: A tool for representing and reasoning with conditional ceteris paribus preference statements. *Journal of Artificial Intelligence Research*, 21:135–191, 2004.

[13] Craig Boutilier, Ronen I Brafman, Carmel Domshlak, Holger H Hoos, and David Poole. Cp-nets: a tool for representing and reasoning with conditional ceteris paribus preference statements. *Journal of Artificial Intelligence Research*, 21(1):135–191, 2004.

[14] Ronen I. Brafman and Carmel Domshlak. Introducing variable importance tradeoffs into cp-nets. In *UAI*, pages 69–76, 2002.

[15] Steven J. Brams and Peter C. Fishburn. *Approval Voting*. Springer-Verlag, 2007.

[16] Felix Brandt, Vincent Conitzer, , and Ulle Endriss. *Computational social choice.* MIT Press., 2012.

[17] Michael Bräuning and H Eyke. Learning conditional lexicographic preference trees. *Preference learning: problems and applications in AI*, 2012.

[18] Leo Breiman. Random forests. *Machine learning*, 45(1):5–32, 2001.

[19] Gerhard Brewka. Complex preferences for answer set optimization. In *KR*, pages 213–223, 2004.

[20] Gerhard Brewka, Salem Benferhat, and Daniel Le Berre. Qualitative choice logic. *Artificial Intelligence*, 157(1):203–237, 2004.

[21] Gerhard Brewka, Thomas Eiter, and Miroslaw Truszczynski. Answer set programming at a glance.

[22] Gerhard Brewka, Ilkka Niemelä, and Miroslaw Truszczynski. Answer set optimization. In *IJCAI*, pages 867–872, 2003.

[23] Gerhard Brewka, Ilkka Niemela, and Miroslaw Truszczynski. Answer set optimization. In *PROC. IJCAI-03*, pages 867–872. Morgan Kaufmann, 2003.

[24] David Cohen, Martin Cooper, and Peter Jeavons. A complete characterization of complexity for boolean constraint optimization problems. In *Principles and Practice of Constraint Programming*, pages 212–226. 2004.

[25] Florence Dupin De Saint-Cyr, Jérôme Lang, and Thomas Schiex. Penalty logic and its link with dempster-shafer theory. In *Proceedings of the Tenth international conference on Uncertainty in artificial intelligence*, 1994.

[26] Yannis Dimopoulos, Loizos Michael, and Fani Athienitou. Ceteris paribus preference elicitation with predictive guarantees. In *IJCAI*, volume 9, pages 1–6. Citeseer, 2009.

[27] József Dombi, Csanád Imreh, and Nándor Vincze. Learning lexicographic orders. *European Journal of Operational Research*, 183:748–756, 2007.

[28] Carmel Domshlak, Eyke Hllermeier, Souhila Kaci, and Henri Prade. Preferences in ai: An overview. *Artificial Intelligence*, 175(7-8):1037 – 1052, 2011.

[29] Didier Dubois, Jérôme Lang, and Henri Prade. A brief overview of possibilistic logic. In *ECSQARU*, pages 53–57, 1991.

[30] J. Duggan and T. Schwartz. Strategic manipulability without resoluteness or shared beliefs: Gibbard-satterthwaite generalized. *Social Choice and Welfare*, 17:85–93, 2000.

[31] Hélène Fargier, Vincent Conitzer, Jérôme Lang, Jérôme Mengin, and Nicolas Schmidt. Issue-by-issue voting: an experimental evaluation. In *MPREF*, 2012.

[32] Peter C. Fishburn. *The Theory of Social Choice*. Princeton University Press, 1973.

[33] Peter C. Fishburn. Axioms for lexicographic preferences. *The Review of Economic Studies*, 42:415–419, 1975.

[34] Niall M Fraser. Applications of preference trees. In *Proceedings of IEEE Systems Man and Cybernetics Conference*, pages 132–136. IEEE, 1993.

[35] Niall M Fraser. Ordinal preference representations. *Theory and Decision*, 36(1):45–67, 1994.

[36] Niall M Fraser. Ordinal preference representations. *Theory and Decision*, 36(1):45–67, 1994.

[37] Johannes Fürnkranz and Eyke Hüllermeier. *Preference learning*. Springer, 2011.

[38] Johannes Fürnkranz and Eyke Hüllermeier. Preference learning: An introduction. In *Preference Learning*, pages 1–17. Springer, 2011.

[39] M. Garey and D. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman and Company, New York, 1979.

[40] Michael R. Garey and David S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman & Co., New York, NY, USA, 1979.

[41] William I Gasarch. The P =? NP poll. *Sigact News*, 33(2):34–47, 2002.

[42] M. Gebser, R. Kaminski, B. Kaufmann, M. Ostrowski, T. Schaub, and M. Schneider. Potassco: The Potsdam answer set solving collection. *AI Communications*, 24(2):105–124, 2011.

[43] Martin Gebser, Benjamin Kaufmann, Roland Kaminski, Max Ostrowski, Torsten Schaub, and Marius Schneider. Potassco: The potsdam answer set solving collection. *AI Communications*, 24(2):107–124, 2011.

[44] A. Gibbard. Manipulation of voting schemes: A general result. *Econometrica*, 41:587–601, 1973.

[45] Judy Goldsmith, Jrme Lang, Miroslaw Truszczyski, and Nic Wilson. The computational complexity of dominance and consistency in cp-nets. In *In Proceedings of IJCAI-05*, pages 144–149, 2005.

[46] Christophe Gonzales and Patrice Perny. GAI Networks for Utility Elicitation. In *Proceedings of the 9th International Conference on the Principles of Knowledge Representation and Reasoning*, pages 224–234, 2004. INT LIP6 DECISION.

[47] Joshua T Guerin, Thomas E Allen, and Judy Goldsmith. Learning cp-net preferences online from user queries. In *Algorithmic Decision Theory*, pages 208–220. Springer, 2013.

[48] Peter Haddawy and Steve Hanks. Representations for decision-theoretic planning: Utility functions for deadline goals. *KR*, 92:71–82, 1992.

[49] James L Hein. *Discrete Structures, Logic, and Computability.* Jones and Bartlett Publishers, LLC, 2010.

[50] Souhila Kaci. *Working with Preferences: Less Is More.* Cognitive Technologies. Springer, 2011.

[51] Rajeev Kohli, Ramesh Krishnamurti, and Prakash Mirchandani. The minimum satisfiability problem. *SIAM J. Discrete Math.*, 7(2):275–283, 1994.

[52] Frédéric Koriche and Bruno Zanuttini. Learning conditional preference networks. *Artificial Intelligence*, 174(11):685–703, 2010.

[53] Mark W. Krentel. The complexity of optimization problems. *J. Comput. Syst. Sci.*, 36(3):490–509, 1988.

[54] Jérôme Lang and Jérôme Mengin. The complexity of learning separable ceteris paribus preferences. In *IJCAI*, pages 848–853, 2009.

[55] Jérôme Lang, Jérôme Mengin, and Lirong Xia. Aggregating conditionally lexicographic preferences on multi-issue domains. In *CP*, pages 973–987, 2012.

[56] Jérôme Lang, Jérôme Mengin, and Lirong Xia. Aggregating conditionally lexicographic preferences on multi-issue domains. In *CP*, pages 973–987, 2012.

[57] Harry R. Lewis and Christos H. Papadimitriou. *Elements of the Theory of Computation.* Prentice Hall PTR, Upper Saddle River, NJ, USA, 2nd edition, 1997.

[58] Xudong Liu. Aggregating lexicographic preference trees using answer set programming: Extended abstract. In *The 23rd International Joint Conference on Artificial Intelligence Doctoral Consortium, (IJCAI-DC)*, 2013.

[59] Xudong Liu and Miroslaw Truszczynski. Aggregating conditionally lexicographic preferences using answer set programming solvers. In *Algorithmic Decision Theory*, volume 8176, pages 244–258. Springer, 2013.

[60] Xudong Liu and Miroslaw Truszczynski. Aggregating conditionally lexicographic preferences using answer set programming solvers. In *ADT*, pages 244–258, 2013.

[61] Xudong Liu and Miroslaw Truszczynski. Preference trees: A language for representing and reasoning about qualitative preferences. In *Workshops at the Twenty-Eighth AAAI Conference on Artificial Intelligence*, 2014.

[62] Xudong Liu and Miroslaw Truszczynski. Learning partial lexicographic preference trees over combinatorial domains. In *Proceedings of the 29th AAAI Conference on Artificial Intelligence (AAAI)*, pages 1539–1545. AAAI Press, 2015.

[63] Xudong Liu and Miroslaw Truszczynski. Reasoning with preference trees over combinatorial domains. In *Proceedings of the 4th International Conference on Algorithmic Decision Theory (ADT)*, volume 9346, pages 19–34. Springer, 2015.

[64] Xudong Liu and Miroslaw Truszczynski. Reasoning with preference trees over combinatorial domains. In *Algorithmic Decision Theory*, pages 19–34. Springer, 2015.

[65] Carsten Lund and Mihalis Yannakakis. On the hardness of approximating minimization problems. *Journal of the ACM (JACM)*, 41(5):960–981, 1994.

[66] Victor W Marek and Miroslaw Truszczyński. Stable models and an alternative logic programming paradigm. In *The Logic Programming Paradigm*, pages 375–398. Springer, 1999.

[67] V.W. Marek and M. Truszczynski. Stable models and an alternative logic programming paradigm. In K.R. Apt, V.W. Marek, M. Truszczynski, and D.S. Warren, editors, *The Logic Programming Paradigm: a 25-Year Perspective*, pages 375–398. Springer, Berlin, 1999.

[68] Nicholas Mattei, James Forshee, and Judy Goldsmith. An empirical study of voting rules and manipulation with large datasets. *Proceedings of COMSOC*, 2012.

[69] K.O. May. A Set of Independent Necessary and Sufficient Conditions for Simple Majority Decision. *Econometrica*, 20(4):680–684, 1952.

[70] Ilkka Niemelä. Logic programs with stable model semantics as a constraint programming paradigm. *Annals of Mathematics and Artificial Intelligence*, 25(3-4):241–273, 1999.

[71] Ilkka Niemelä. Logic programs with stable model semantics as a constraint programming paradigm. *Annals of Mathematics and Artificial Intelligence*, 25:241–273, 1999.

[72] Max Ostrowski and Torsten Schaub. ASP modulo CSP: The clingcon system. *TPLP*, 12(4-5):485–503, 2012.

[73] C. Papadimitriou. *Computational Complexity.* Addison-Wesley Publishing Company, Inc., 1994.

[74] Christos Papadimitriou and Mihalis Yannakakis. Optimization, approximation, and complexity classes. In *Proceedings of the twentieth annual ACM symposium on Theory of computing*, STOC '88, pages 229–234, New York, NY, USA, 1988. ACM.

[75] M. Satterthwaite. Strategy-proofness and Arrow's Conditions: Existence and correspondence theorems for voting procedures and social welfare functions. *Journal of Economic Theory*, 10:187–216, 1975.

[76] Michael Schmitt and Laura Martignon. On the complexity of learning lexicographic strategies. *The Journal of Machine Learning Research*, 7:55–83, 2006.

[77] Christian Schulte, Guido Tack, and Mikael Z. Lagerkvist. Modeling and programming with gecode, 2010.

[78] Wikipedia. Social choice theory, 2013.

[79] Nic Wilson. Extending cp-nets with stronger conditional preference statements. In *in Proceedings of AAAI-04*, pages 735–741, 2004.

[80] Lirong Xia and Vincent Conitzer. Approximating common voting rules by sequential voting in multi-issue domains. In *ISAIM*, 2012.

[81] Fusun Yaman, Thomas J Walsh, Michael L Littman, and Marie Desjardins. Democratic approximation of lexicographic preference models. In *Proceedings of the 25th international conference on Machine learning*, pages 1200–1207. ACM, 2008.

[82] Ying Zhu and Miroslaw Truszczynski. On optimal solutions of answer set optimization problems. In *Logic Programming and Nonmonotonic Reasoning*, pages 556–568, 2013.

Vita

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Student member: AAAI
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Refereed Publications

1. **Xudong Liu**. *Modeling, Learning and Reasoning with Qualitative Preferences.* In Proceedings of the 4th International Conference on Algorithmic Decision Theory (ADT), volume 9346, pages 587-592, 2015. Springer
2. **Xudong Liu** and Miroslaw Truszczynski. *Reasoning with Preference Trees over Combinatorial Domains.* In Proceedings of the 4th International

Conference on Algorithmic Decision Theory (ADT), volume 9346, pages 19-34, 2015. Springer

3. **Xudong Liu** and Miroslaw Truszczynski. *Learning Partial Lexicographic Preference Trees over Combinatorial Domains*. In Proceedings of the 29th AAAI Conference on Artificial Intelligence (AAAI), pages 1539-1545, 2015. AAAI Press
4. **Xudong Liu** and Miroslaw Truszczynski. *Preference Trees: A Language for Representing and Reasoning about Qualitative Preferences*. In Proceedings of the 8th AAAI Multidisciplinary Workshop on Advances in Preference Handling (MPREF), pages 55-60, 2014. AAAI Press
5. **Xudong Liu** and Miroslaw Truszczynski. *Aggregating Conditionally Lexicographic Preferences Using Answer Set Programming Solvers*. In Proceedings of the 3rd International Conference on Algorithmic Decision Theory (ADT), volume 8176, pages 244-258, 2013. Springer
6. Matthew Spradling, Judy Goldsmith, **Xudong Liu**, Chandrima Dadi and Zhiyu Li. *Roles and Teams Hedonic Game*. In Proceedings of the 3rd International Conference on Algorithmic Decision Theory (ADT), volume 8176, pages 351-362, 2013. Springer
7. **Xudong Liu**. *Aggregating Lexicographic Preference Trees Using Answer Set Programming: Extended Abstract*. In 23rd International Joint Conference on Artificial Intelligence Doctoral Consortium (IJCAI DC), 2013.

Honors and Awards

Verizon Fellowship	Fall 2015 - Spring 2016
Graduate Teaching Assistantship	Fall 2014 - Spring 2015, Fall 2012 - Spring 2013
AAAI-15 Student Volunteer and Scholarship Award	Jan. 2015
International Student Tuition Scholarship	Jan. 2015
Nominee of the Dissertation Year Fellowship	Dec. 2014
Harrison D. Brailsford Graduate Scholarship	Oct. 2014
Kentucky Opportunity Fellowship Awards	July 2013 - June 2014
Nominee of the ACM Award for Outstanding Teaching Assistant	2013
NSF Student Travel Award	Aug. 2013
IJCAI-13 Travel Grant Award	Aug. 2013
Graduate Research Assistantship	Fall 2010 - Spring 2013
Daniel R. Reedy Quality Achievement Fellowship	Aug. 2010 - May 2013
UK Student Travel Funding Awards	2013 (IJCAI, ADT), 2014 (AAAI)
Chinese National Endeavor Scholarship	Fall 2008
Outstanding Student Scholarships	Fall 2006 - Spring 2010

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