

Chapter 6

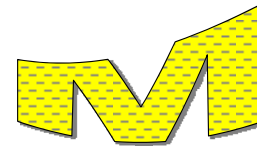
Linear Programming: The Simplex Method

Section 4

Maximization and Minimization with Problem Constraints

Introduction to the Big M Method

In this section, we will present a generalized version of the simplex method that will solve both maximization and minimization problems with any combination of \leq , \geq , $=$ constraints



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Example

Maximize

$$P = 2x_1 + x_2$$

subject to

$$x_1 + x_2 \leq 10$$

$$-x_1 + x_2 \geq 2$$

$$x_1, x_2 > 0$$

To form an equation out of the first inequality, we introduce a **slack variable** s_1 , as before, and write

$$x_1 + x_2 + s_1 = 10.$$

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Example (continued)

To form an equation out of the second inequality we introduce a second variable s_2 and **subtract** it from the left side so that we can write

$$-x_1 + x_2 - s_2 = 2.$$

The variable s_2 is called a **surplus variable**, because it is the amount (surplus) by which the left side of the inequality exceeds the right side.

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Example (continued)

We now express the linear programming problem as a system of equations:

$$x_1 + x_2 + s_1 = 10$$

$$-x_1 + x_2 - s_2 = 2$$

$$-2x_1 - x_2 + P = 0$$

$$x_1, x_2, s_1, s_2 \geq 0$$

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Example (continued)

It can be shown that a basic solution of a system is not feasible if any of the variables (excluding P) are negative. Thus a surplus variable is required to satisfy the nonnegative constraint.

An initial basic solution is found by setting the nonbasic variables x_1 and x_2 equal to 0. That is, $x_1 = 0$, $x_2 = 0$, $s_1 = 10$, $s_2 = -2$, $P = 0$.

This solution is not feasible because the surplus variable s_2 is negative.

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Artificial Variables

In order to use the simplex method on problems with mixed constraints, we turn to a device called an **artificial variable**. This variable has no physical meaning in the original problem and is introduced solely for the purpose of obtaining a basic feasible solution so that we can apply the simplex method.

An artificial variable is a variable introduced into each equation that has a surplus variable. To ensure that we consider only basic feasible solutions, an artificial variable is required to satisfy the nonnegative constraint.

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Example (continued)

Returning to our example, we introduce an artificial variable a_1 into the equation involving the surplus variable s_2 :

$$x_1 + x_2 - s_2 + a_1 = 2$$

To prevent an artificial variable from becoming part of an optimal solution to the original problem, a very large “penalty” is introduced into the objective function. This penalty is created by choosing a positive constant M so large that the artificial variable is forced to be 0 in any final optimal solution of the original problem.

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Example (continued)

We then add the term $-Ma_1$ to the objective function:

$$P = 2x_1 + x_2 - Ma_1$$

We now have a new problem, called the **modified problem**:

Maximize

$$P = 2x_1 + x_2 - Ma_1$$

subject to

$$x_1 + x_2 + s_1 = 10$$

$$x_1 + x_2 - s_2 + a_1 = 2$$

$$x_1, x_2, s_1, s_2, a_1 \geq 0$$

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Big M Method: Form the Modified Problem

- If any problem constraints have negative constants on the right side, multiply both sides by -1 to obtain a constraint with a nonnegative constant. Remember to reverse the direction of the inequality if the constraint is an inequality.
- Introduce a **slack variable** for each constraint of the form \leq .
- Introduce a **surplus variable** and an **artificial variable** in each \geq constraint.
- Introduce an **artificial variable** in each $=$ constraint.
- For each artificial variable a , add $-Ma$ to the objective function. Use the same constant M for all artificial variables.

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Key Steps for Solving a Problem Using the Big M Method

Now that we have learned the steps for finding the modified problem for a linear programming problem, we will turn our attention to the procedure for actually solving such problems. The procedure is called the **Big M Method**.



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Example (continued)

The initial system for the modified problem is

$$x_1 + x_2 + s_1 = 10$$

$$-x_1 + x_2 - s_2 + a_1 = 2$$

$$-2x_1 - x_2 + Ma_1 + P = 0$$

$$x_1, x_2, s_1, s_2, a_1 \geq 0$$

We next write the augmented coefficient matrix for this system, which we call the **preliminary simplex tableau** for the modified problem.

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Example (continued)

$$\begin{array}{cccccc|c}
 x_1 & x_2 & s_1 & s_2 & a_1 & P & \\
 \hline
 1 & 1 & 1 & 0 & 0 & 0 & 10 \\
 -1 & 1 & 0 & -1 & 1 & 0 & 2 \\
 -2 & -1 & 0 & 0 & M & 1 & 0
 \end{array}$$

To start the simplex process we require an **initial simplex tableau**, described on the next slide. The preliminary simplex tableau should either meet these requirements, or it needs to be transformed into one that does.

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Definition: Initial Simplex Tableau

For a system tableau to be considered an **initial simplex tableau**, it must satisfy the following two requirements:

1. The requisite number of basic variables must be selectable. Each basic variable must correspond to a column in the tableau with exactly one nonzero element. Different basic variables must have the nonzero entries in different rows. The remaining variables are then selected as non-basic variables.
2. The basic solution found by setting the non-basic variables equal to zero is feasible.

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Example (continued)

The preliminary simplex tableau from our example satisfies the first requirement, since s_1 , s_2 , and P can be selected as basic variables according to the criterion stated.

However, it does not satisfy the second requirement since the basic solution is not feasible ($s_2 = -2$.)

To use the simplex method, we must first use row operations to transform the tableau into an equivalent matrix that satisfies all initial simplex tableau requirements. **This transformation is not a pivot operation.**

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Example (continued)

$$\begin{array}{cccccc|c}
 x_1 & x_2 & s_1 & s_2 & a_1 & P & \\
 \hline
 1 & 1 & 1 & 0 & 0 & 0 & 10 \\
 -1 & 1 & 0 & -1 & 1 & 0 & 2 \\
 -2 & -1 & 0 & 0 & M & 1 & 0
 \end{array}$$

If you inspect the preliminary tableau, you realize that the problem is that s_2 has a negative coefficient in its column. We need to replace s_2 as a basic variable by something else with a positive coefficient. We choose a_1 .

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Example (continued)

We want to use a_1 as a basic variable instead of s_2 . We proceed to eliminate M from the a_1 column using row operations:

$$(-M)R_2 + R_3 \rightarrow R_3$$

$$\begin{array}{c|cccccc|c} x_1 & x_2 & s_1 & s_2 & a_1 & P & \\ \hline 1 & 1 & 1 & 0 & 0 & 0 & 10 \\ -1 & 1 & 0 & -1 & 1 & 0 & 2 \\ \hline -2 & -1 & 0 & 0 & M & 1 & 0 \end{array}$$

$$\begin{array}{c|cccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 10 \\ -1 & 1 & 0 & -1 & 1 & 0 & 2 \\ \hline M-2 & -M-1 & 0 & M & 0 & 1 & -2M \end{array}$$

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Example (continued)

From the last matrix we see that the basic variables are s_1 , a_1 , and P because those are the columns that meet the requirements for basic variables.

The basic solution found by setting the nonbasic variables x_1 , x_2 , and s_2 equal to 0 is

$$x_1 = 0, x_2 = 0, s_1 = 10, s_2 = 0, a_1 = 2, P = -2M.$$

The basic solution is feasible (P can be negative) and all requirements are met.

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Example (continued)

We now continue with the usual simplex process, using pivot operations. When selecting the pivot columns, keep in mind that M is unspecified, but we know it is a very large positive number.

$$\begin{array}{c|cccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 10 \\ -1 & 1 & 0 & -1 & 1 & 0 & 2 \\ \hline M-2 & -M-1 & 0 & M & 0 & 1 & -2M \end{array}$$

In this example, $M - 2$ and M are positive, and $-M - 1$ is negative. The first pivot column is column 2.

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Example (continued)

If we pivot on the second row, second column, and then on the first row, first column, we obtain:

$$\begin{array}{c|cccccc|c} x_1 & x_2 & s_1 & s_2 & a_1 & P & \\ \hline 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} & 0 & 4 \\ x_2 & 0 & 1 & \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} & 6 \\ \hline P & 0 & 0 & \frac{3}{2} & \frac{1}{2} & M - \frac{1}{2} & 14 \end{array}$$

Since all the indicators in the last row are nonnegative, we have the optimal solution:

$$\text{Max } P = 14 \text{ at } x_1 = 4, x_2 = 6, s_1 = 0, s_2 = 0, a_1 = 0.$$

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Big M Method: Summary

To summarize:

1. Form the preliminary simplex tableau for the modified problem.
2. Use row operations to eliminate the M s in the bottom row of the preliminary simplex tableau in the columns corresponding to the artificial variables. The resulting tableau is the initial simplex tableau.
3. Solve the modified problem by applying the simplex method to the initial simplex tableau found in the second step.

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Big M Method: Summary (continued)

4. Relate the optimal solution of the modified problem to the original problem.
 - A) if the modified problem has no optimal solution, the original problem has no optimal solution.
 - B) if all artificial variables are 0 in the optimal solution to the modified problem, delete the artificial variables to find an optimal solution to the original problem
 - C) if any artificial variables are nonzero in the optimal solution, the original problem has no optimal solution.

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