

# Flows in Networks: Residual Networks

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# Learning Objectives

- Construct the residual network associated to a flow.
- Understand why edges to reverse existing flow are necessary.

# Last Time

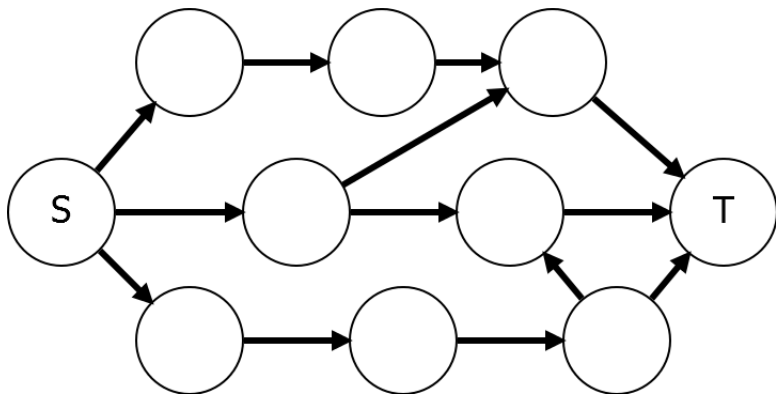
- Defined network.
- Defined flows.
- Defined maxflow problem.

# Technique for Solving Maxflow

Build up flows a little bit at a time.

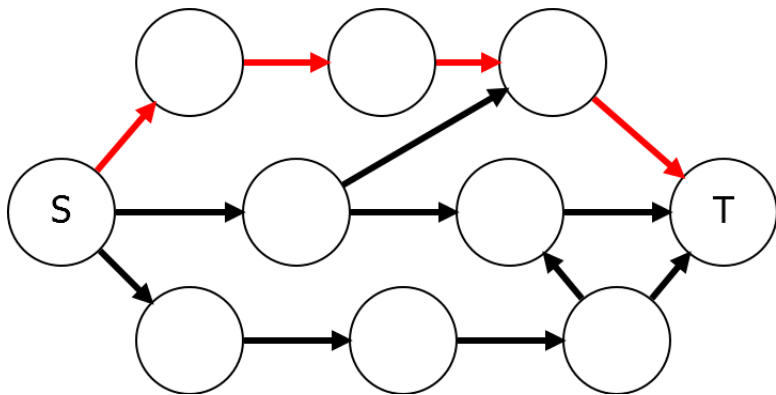
# Example

[All capacities are 1]



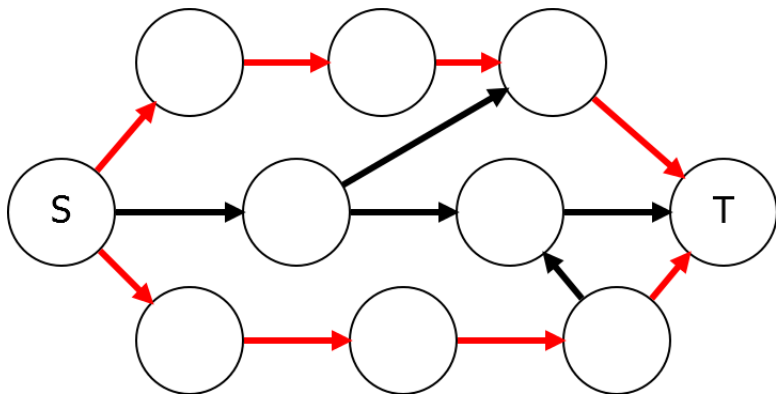
# Example

[All capacities are 1]



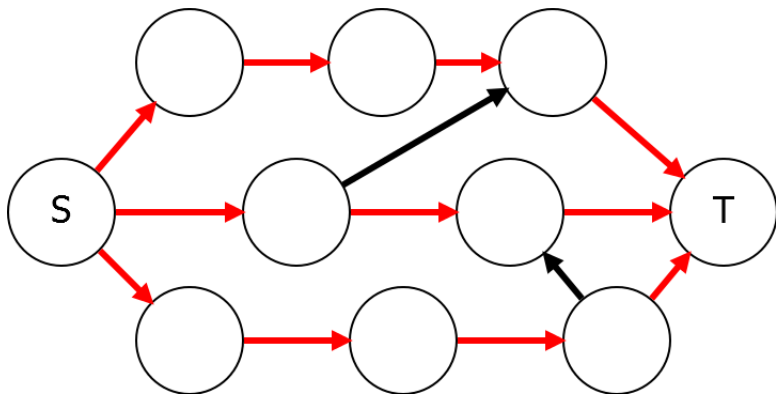
# Example

[All capacities are 1]



# Example

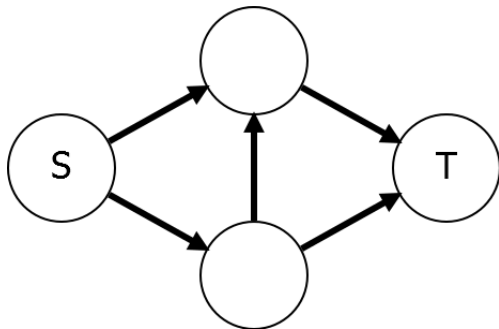
[All capacities are 1]





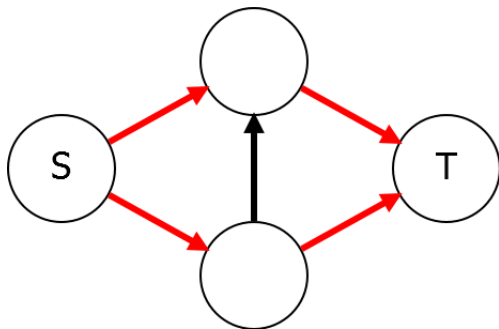
## Example II

Consider another example.



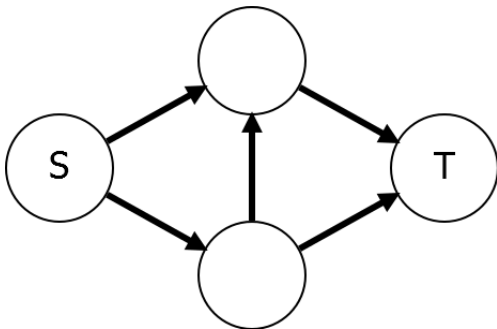
## Example II

Maximum flow of 2.



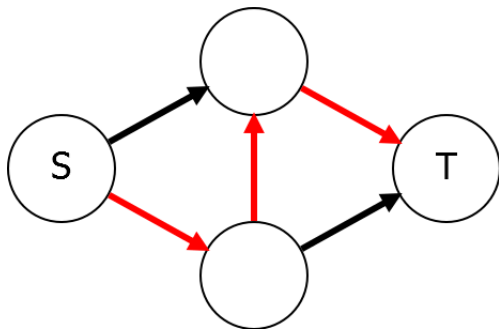
## Example II

Consider adding flow incrementally.



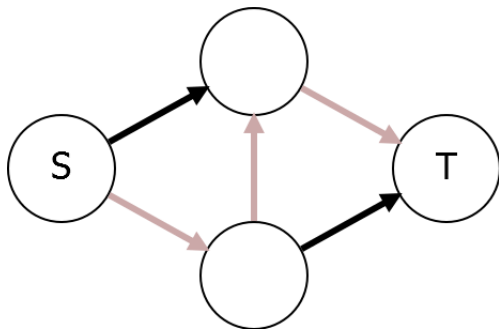
## Example II

Add flow here.



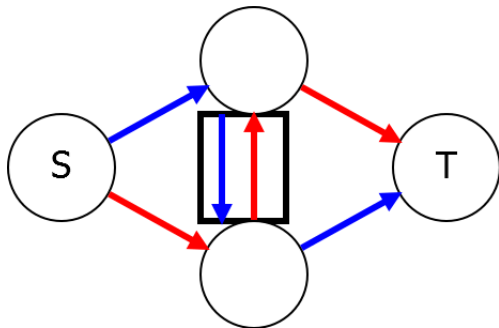
## Example II

Cannot add second unit.



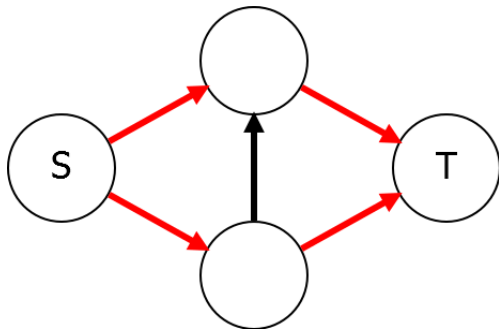
## Example II

Need to add flow here, cancelling flow in the middle.



## Example II

End up with this.



# Residual Network

Given a network  $G$  and flow  $f$ , we construct a residual network  $G_f$ , representing places where flow can still be added to  $f$ , including places where existing flow can be cancelled.

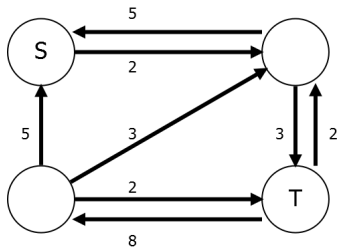
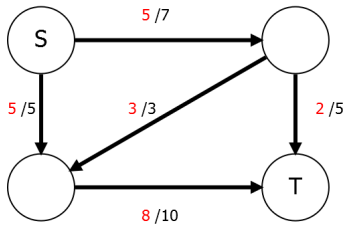


# Residual Network

For each edge  $e$  of  $G$ ,  $G_f$  has edges:

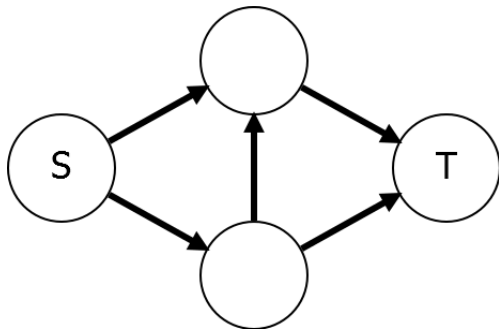
- $e$  with capacity  $C_e - f_e$  (unless  $f_e = C_e$ ).
- opposite  $e$  with capacity  $f_e$  (unless  $f_e = 0$ ).

# Example



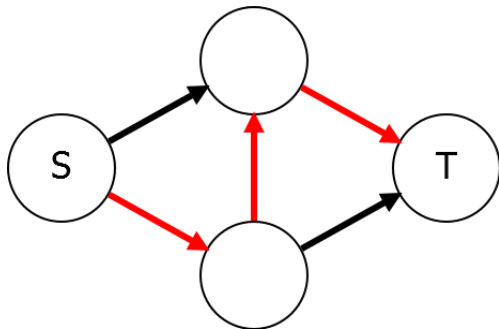
# Review Example

Recall our previous example.



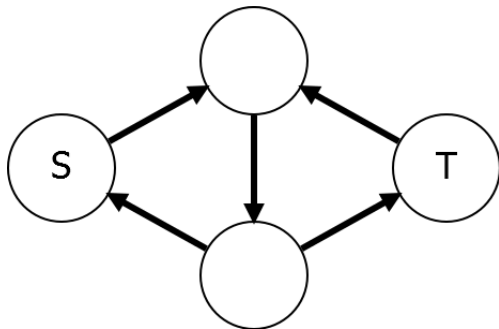
# Review Example

This flow could not be added to directly.



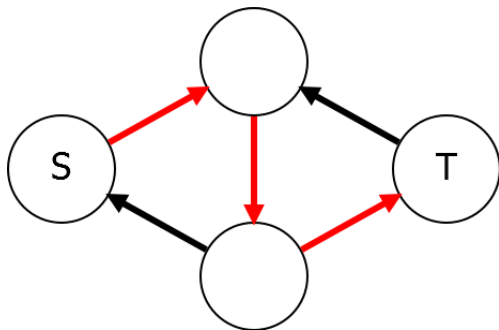
# Review Example

But the residual graph is as shown.



# Review Example

Which can support flow.



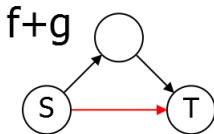
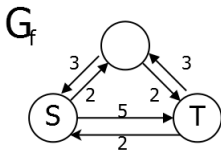
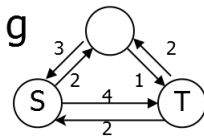
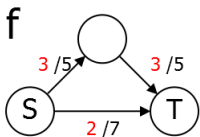
# Residual Flow

Given network  $G$ , flow  $f$ . Any flow,  $g$  on  $G_f$  can be added to  $f$  to get a new flow on  $G$ .

- $g_e$  adds to  $f_e$ .
- $g_{e'}$  subtracts from  $f_e$ .

# Problem

What is the flow of  $f + g$  along the highlighted edge?

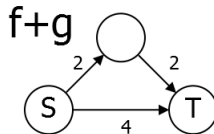
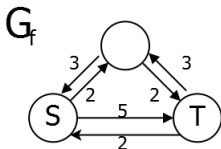
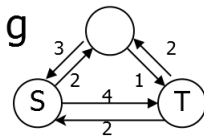
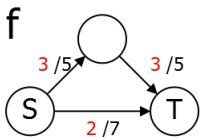




# Solution

Flow is given by:

$$f_e + g_e - g_{e'} = 2 + 4 - 2 = 4.$$



## Theorem

Given  $G$ , a flow  $f$ , and flow  $g$  on  $G_f$ :

- $f + g$  is a flow on  $G$ .
- $|f + g| = |f| + |g|$ .
- All flows on  $G$  are of this form for some  $g$ .

# Proof I

- Conservation of flow of  $f$  and  
Conservation of flow of  $g$  imply  
Conservation of flow of  $f + g$ .
- $f_e + g_e \leq f_e + (C_e - f_e) = C_e$ .
- $f_e - g_{e'} \geq f_e - f_e = 0$ .
- So  $f + g$  is a flow.

## Proof II

- Flow of  $f + g$  out of  $s$  is flow of  $f$  out of  $s$  plus flow of  $g$  out of  $s$ . So
$$|f + g| = |f| + |g|.$$
- For any flow  $h$  for  $G$ , it is not hard to show that  $g := h - f$  is a flow on  $G_f$ .
- So  $h = f + g$ .

# Summary

Flows on  $G_f$  are exactly ways to add flow to  $f$ .