Priority Queues: Binary Heaps

Alexander S. Kulikov

Steklov Institute of Mathematics at St. Petersburg
Russian Academy of Sciences

Data Structures
Data Structures and Algorithms

Outline

1. Binary Trees
2. Basic Operations
3. Complete Binary Trees
4. Pseudocode
5. Heap Sort
6. Final Remarks

Definition

Binary max-heap is a binary tree (each node has zero, one, or two children) where the value of each node is at least the values of its children.

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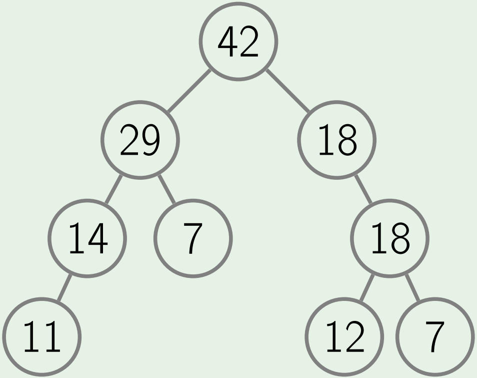
Binary max-heap is a binary tree (each node has zero, one, or two children) where the value of each node is at least the values of its children.

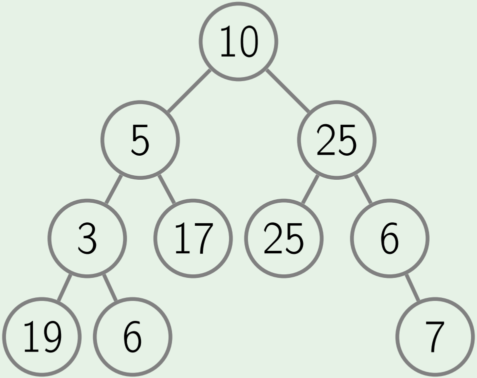
In other words

For each edge of the tree, the value of the parent is at least the value of the child.

Example: heap



Example: not a heap



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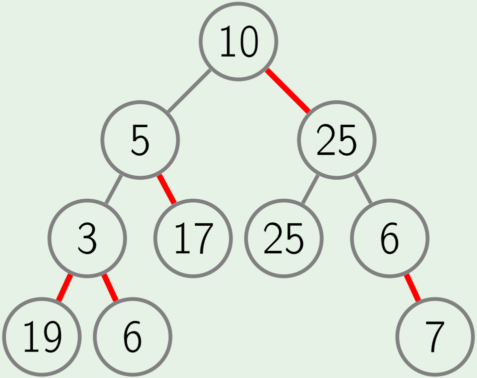
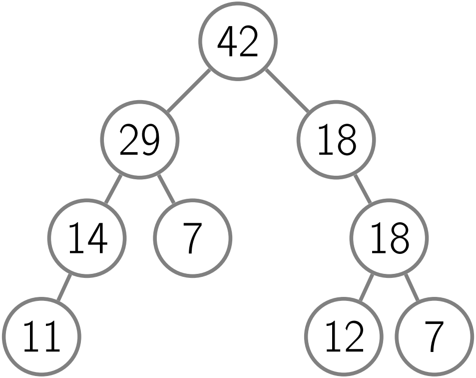


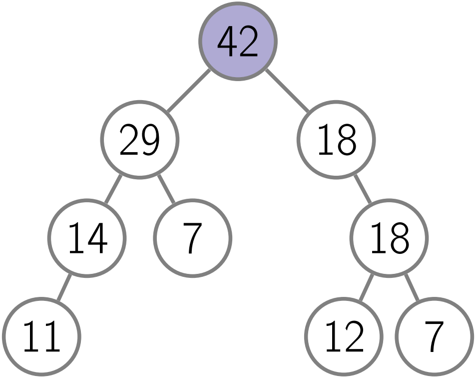
Outline

GetMax



GetMax

return the root value



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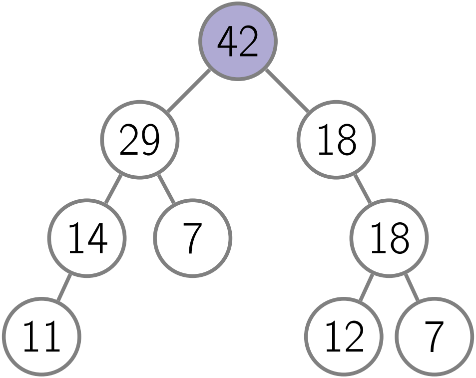
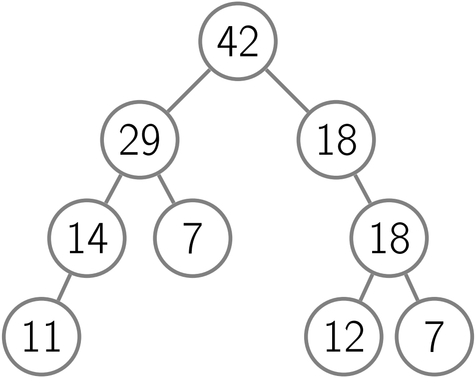


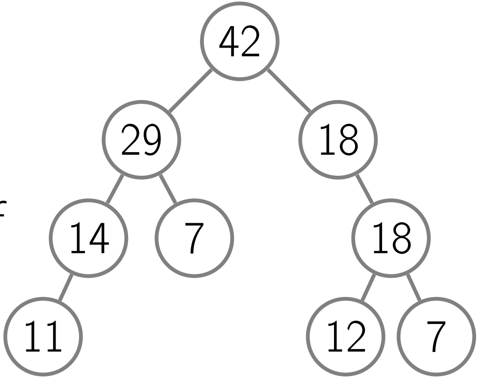
running time: $O(1)$

Insert



Insert

attach a new node to any leaf



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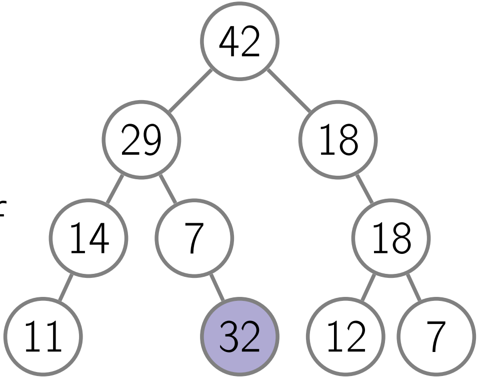
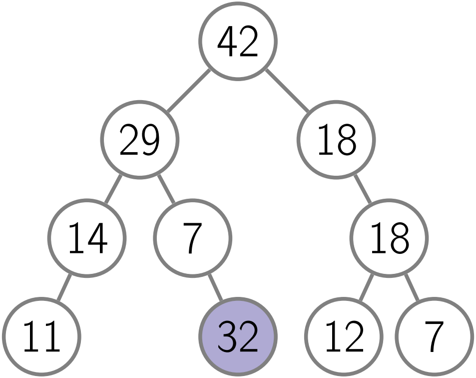


Insert

this may violate the heap property



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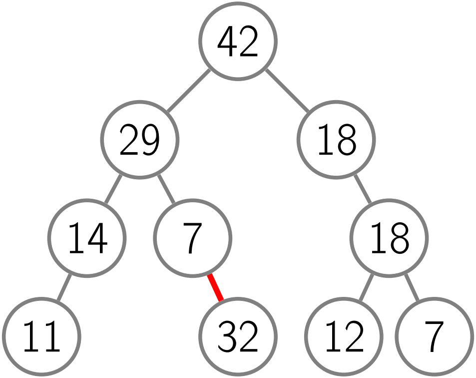
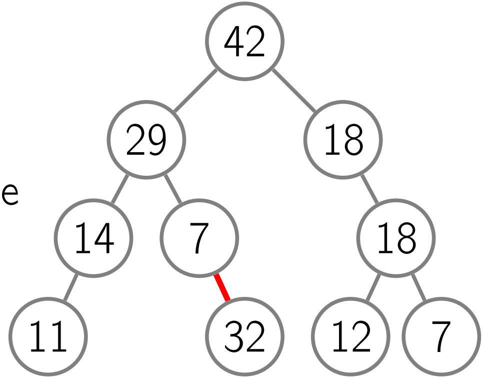


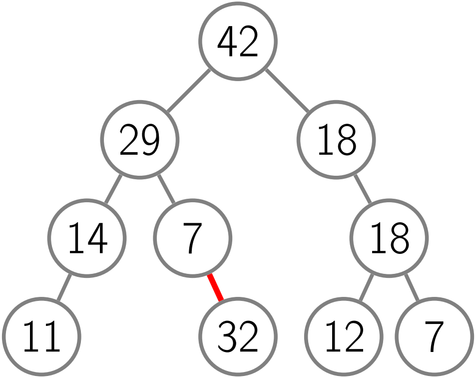
Insert

to fix this, we let the new node sift up



SiftUp

for this, we swap the problematic node with its parent until the property is satisfied



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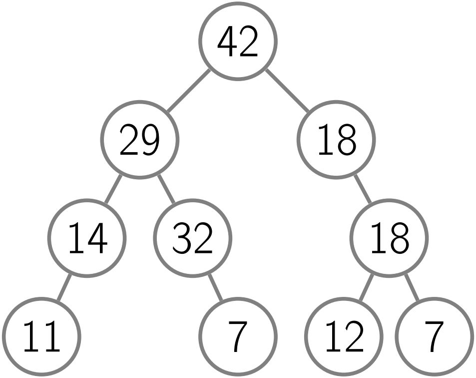

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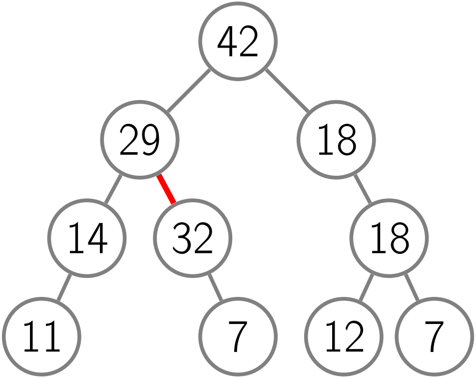

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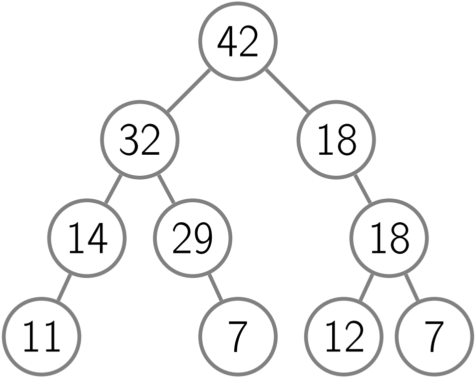
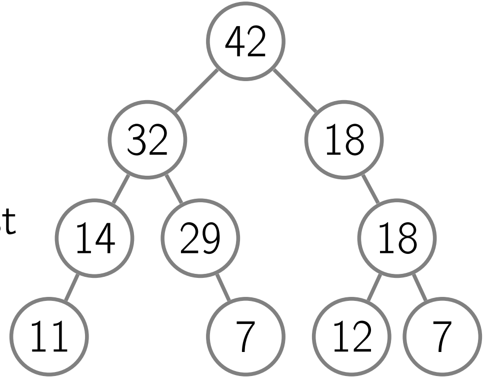


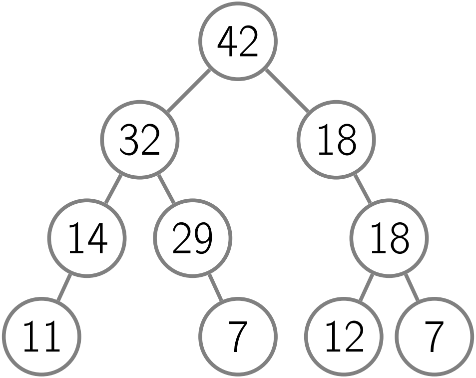
SiftUp

invariant: heap property is violated on at most one edge

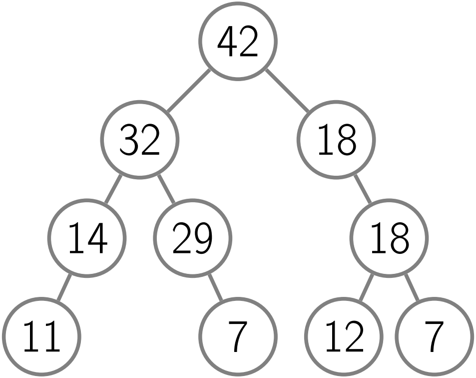


SiftUp

this edge gets closer to the root while sifting up

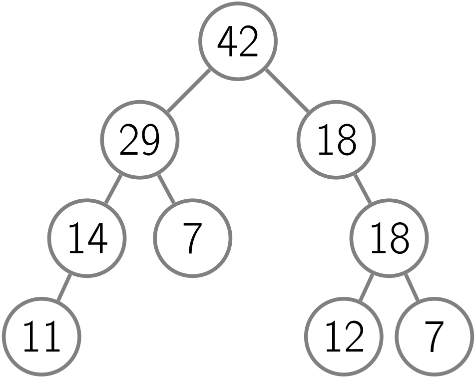


`SiftUp`



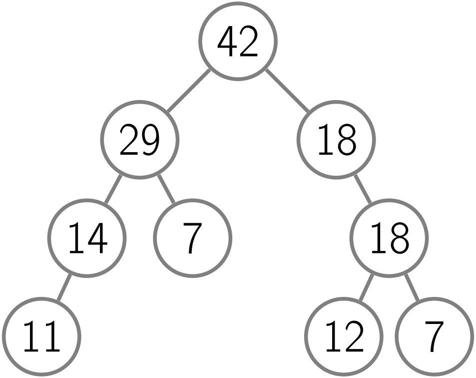
running time: $O(\text{tree height})$

ExtractMax



ExtractMax

replace the root
with any leaf



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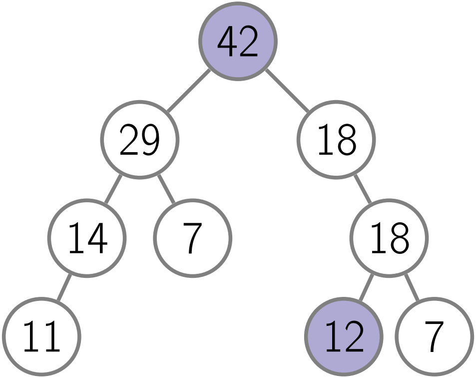

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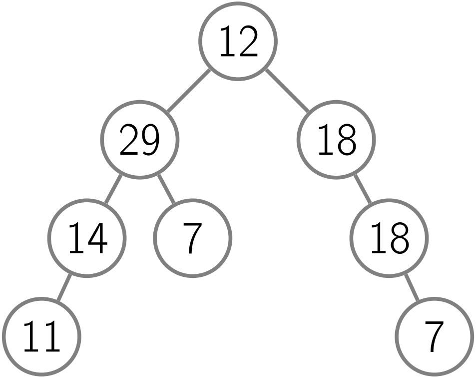
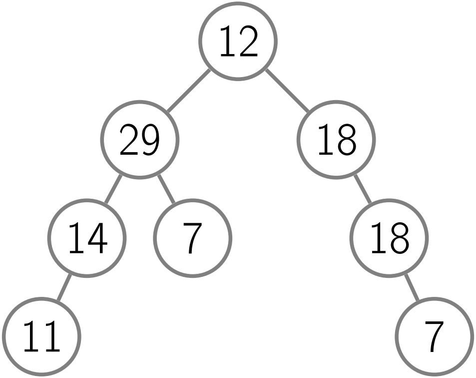


ExtractMax

again, this may violate the heap property



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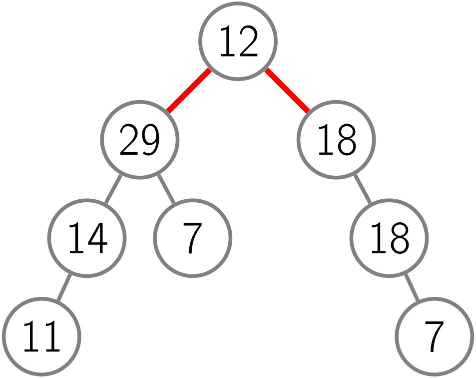
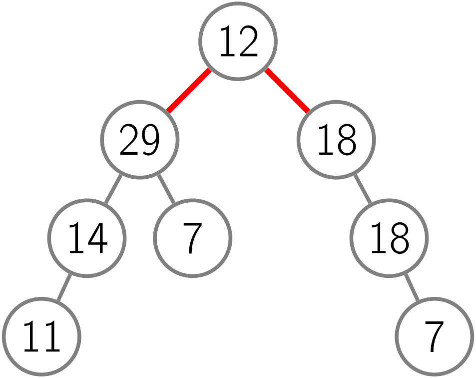


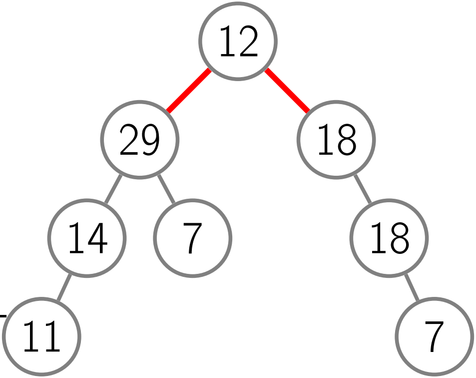
ExtractMax

to fix it, we let the problematic node sift down



SiftDown

for this, we swap the problematic node with larger child until the heap property is satisfied



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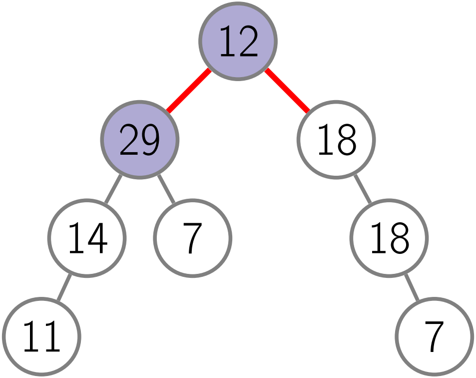

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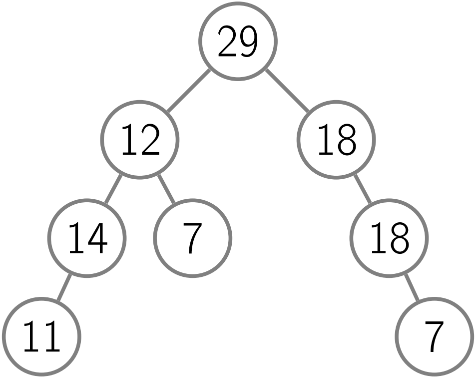

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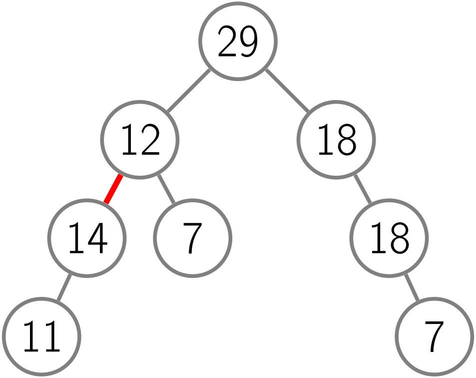

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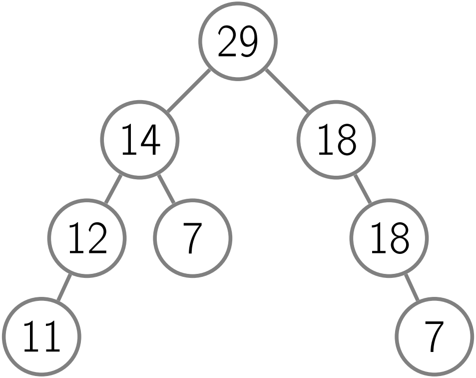
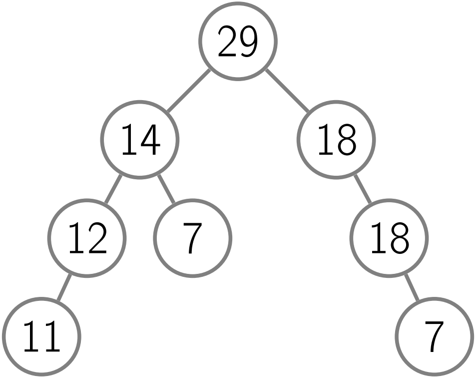


SiftDown

we swap with the larger child which automatically fixes one of the two bad edges



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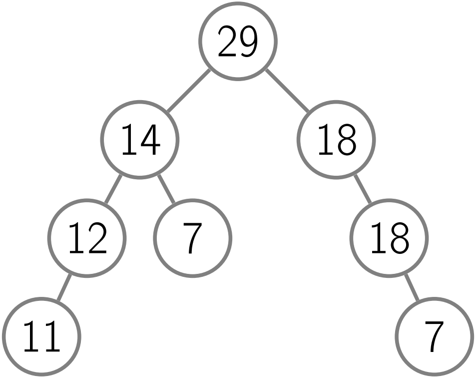
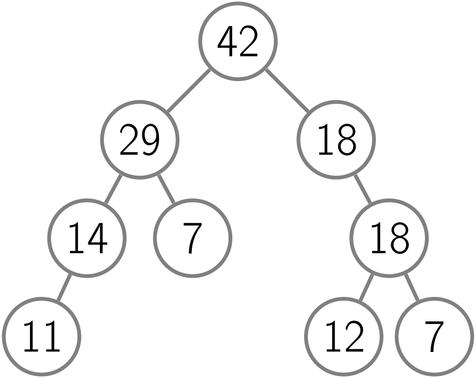


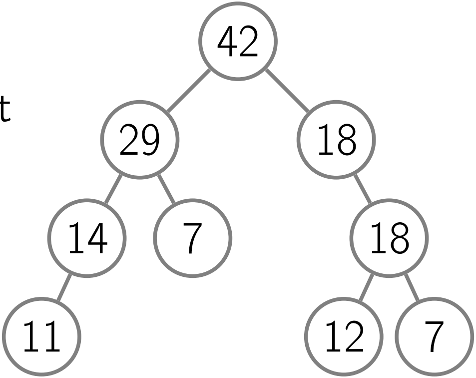
running time: O(tree height)

ChangePriority



ChangePriority

change the priority and let the changed element sift up or down depending on whether its priority decreased or increased



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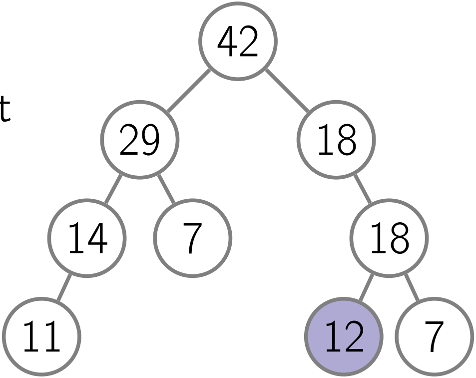

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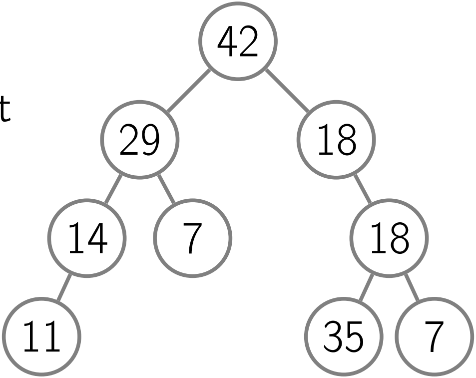

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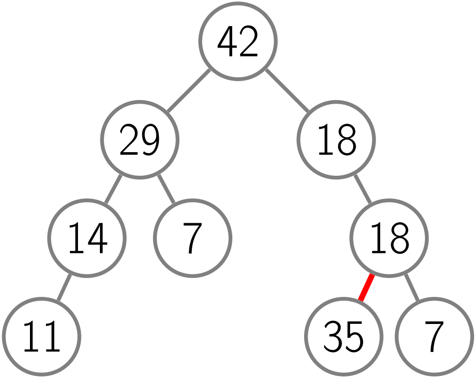

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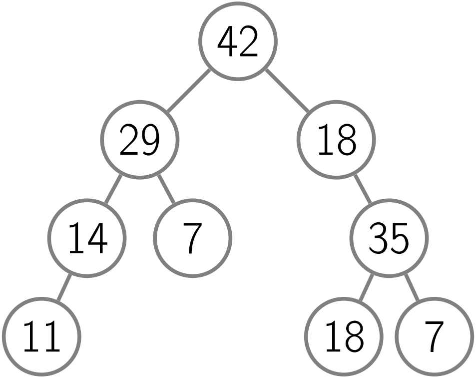

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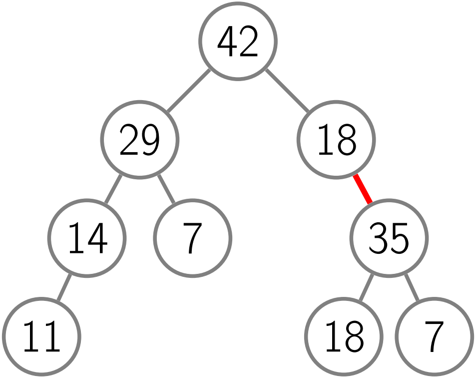

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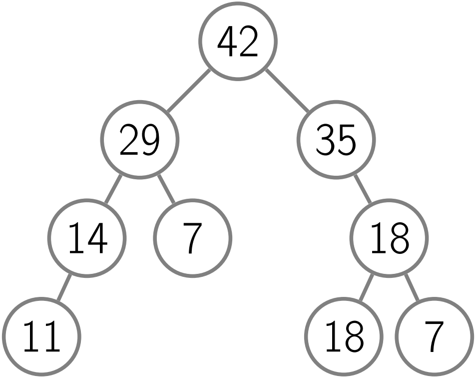

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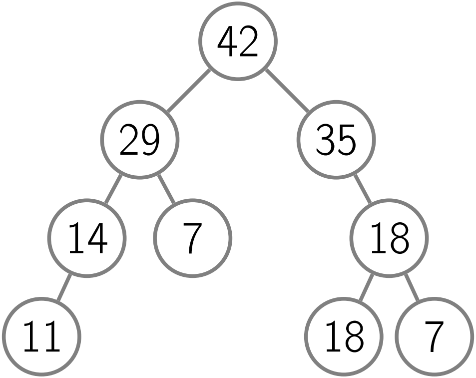
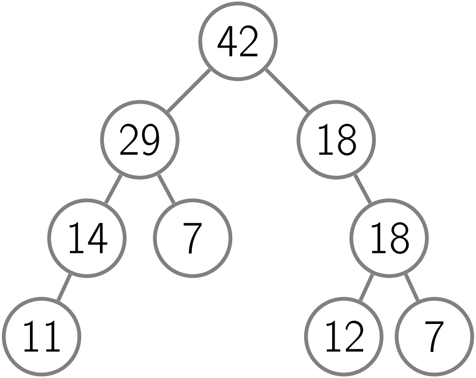


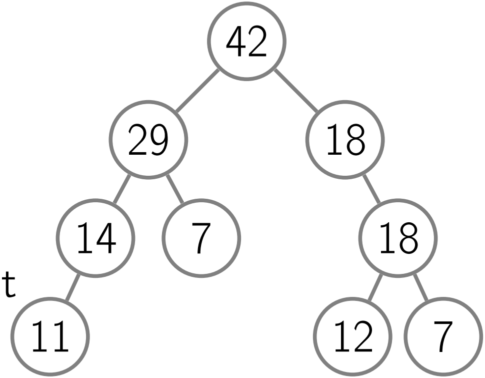
running time: $O(\text{tree height})$

Remove



Remove

change the priority of the element to ∞, let it sift up, and then extract maximum



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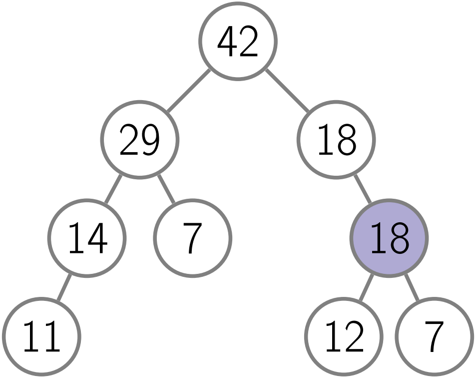

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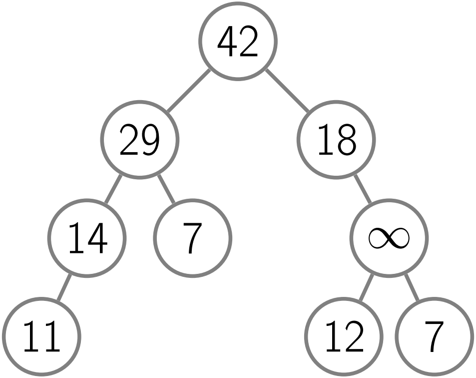

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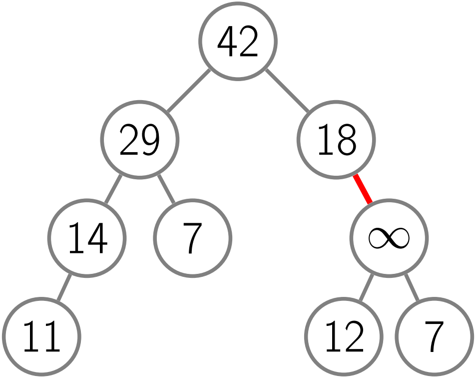

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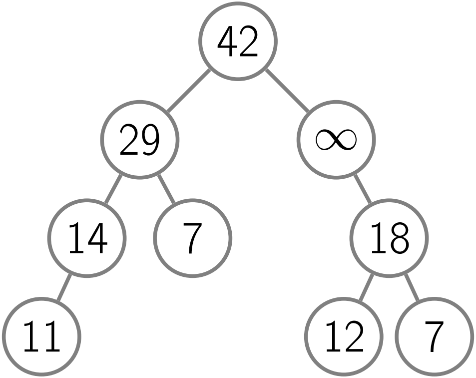

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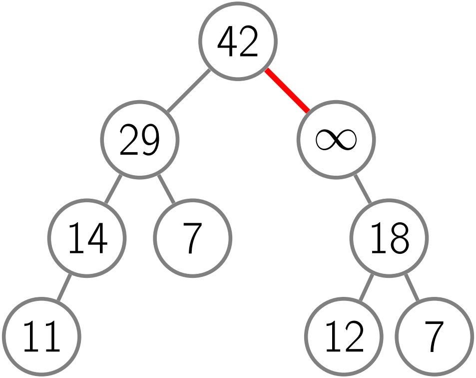

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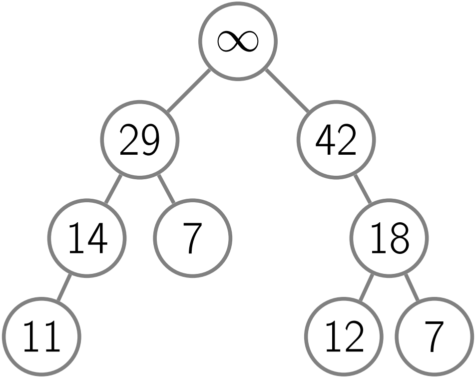
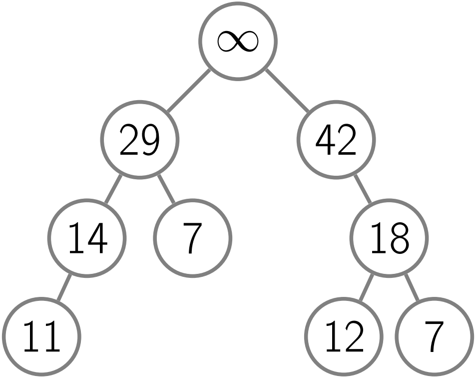


Remove

now, call
`ExtractMax()`



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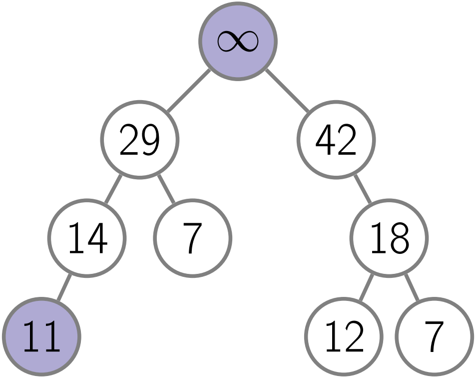

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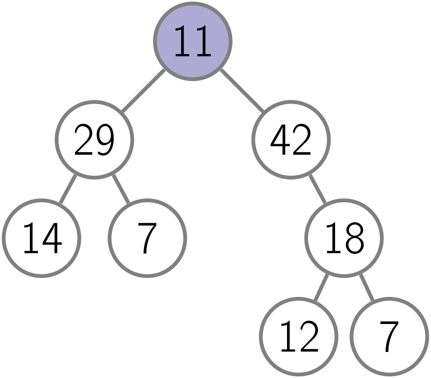

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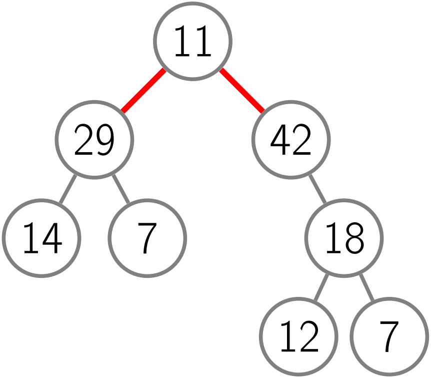

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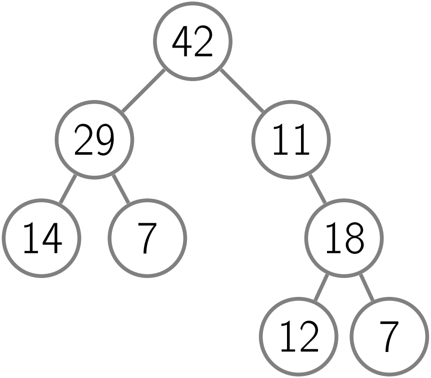

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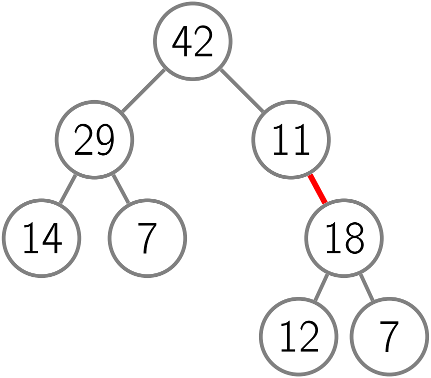

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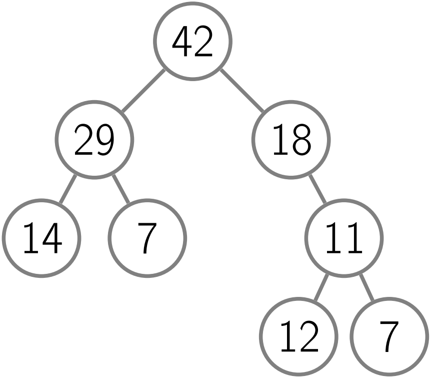

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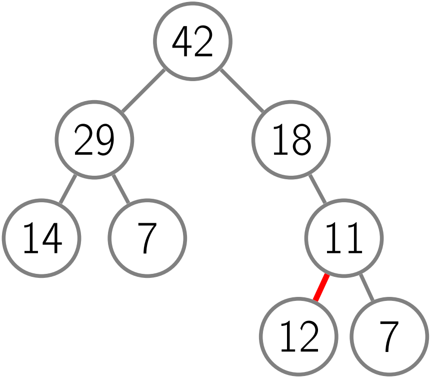

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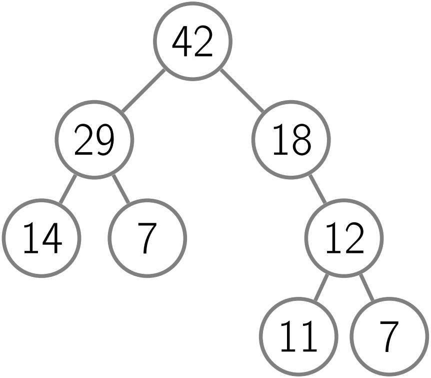

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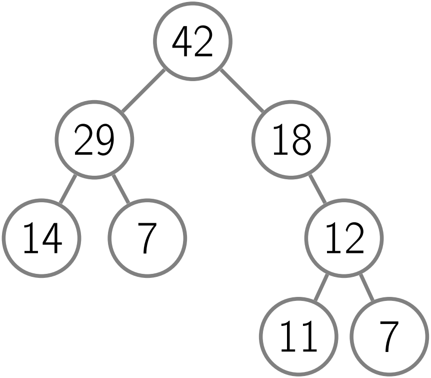

running time: $O(\text{tree height})$

Summary

- `GetMax` works in time $O(1)$, all other operations work in time $O(\text{tree height})$

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- `GetMax` works in time $O(1)$, all other operations work in time $O(\text{tree height})$
- we definitely want a tree to be shallow

Outline

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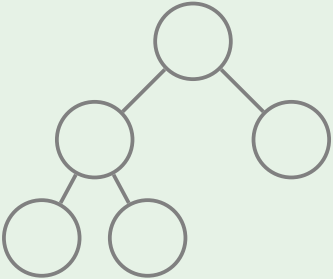
3. Complete Binary Trees
4. Pseudocode
5. Heap Sort
6. Final Remarks

How to Keep a Tree Shallow?

Definition

A binary tree is complete if all its levels are filled except possibly the last one which is filled from left to right.

Example: complete binary tree



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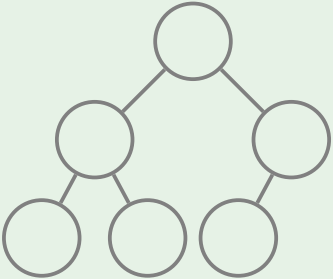

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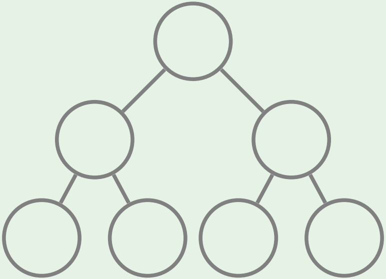

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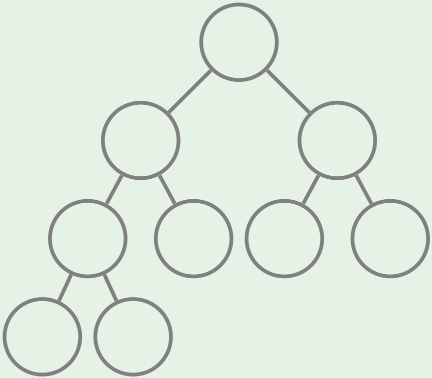
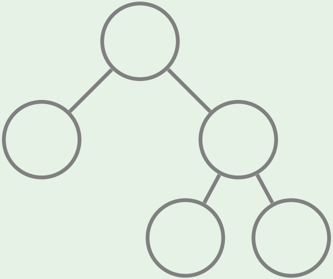


Example: not complete binary tree



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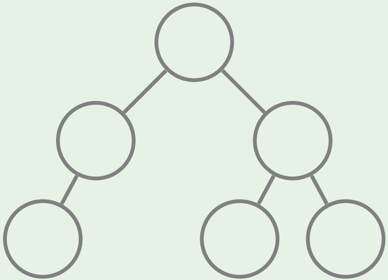

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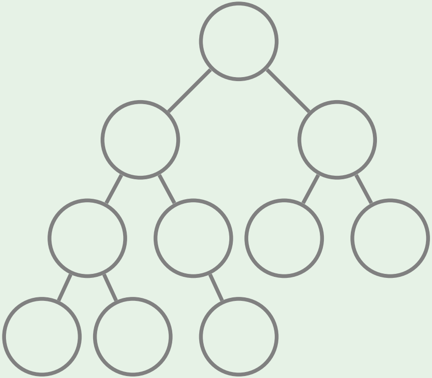

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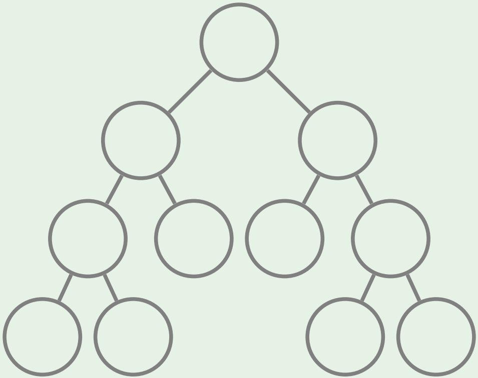


First Advantage: Low Height

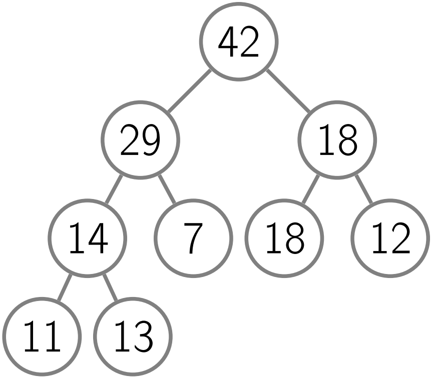
Lemma

A complete binary tree with n nodes has height at most $O(\log n)$.

Proof

- Complete the last level to get a full binary tree on $n' \geq n$ nodes and the same number of levels ℓ.
- Note that $n' \leq 2n$.
- Then $n' = 2^\ell - 1$ and hence $\ell = \log_2(n' + 1) \leq \log_2(2n + 1) = O(\log n)$. $\square$

Second Advantage: Store as Array



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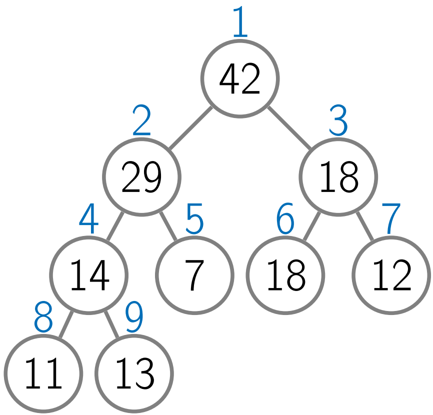

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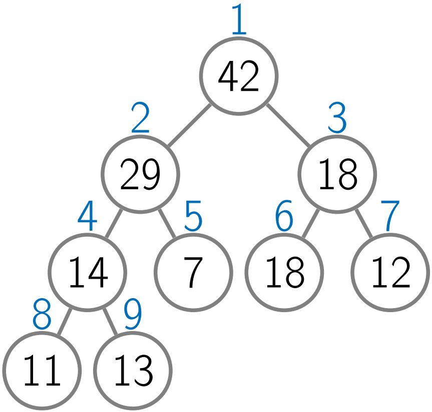

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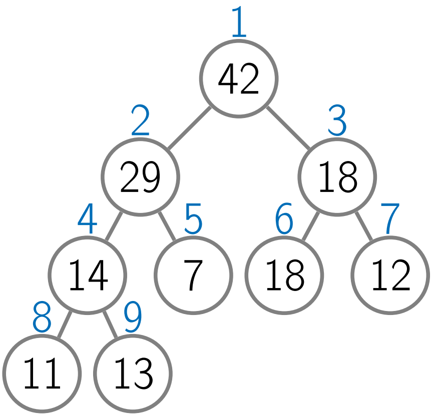

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$$\text{leftchild}(i) = 2i$$

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1	2	3	4	5	6	7	8	9
42	29	18	14	7	18	12	11	13

- What do we pay for these advantages?

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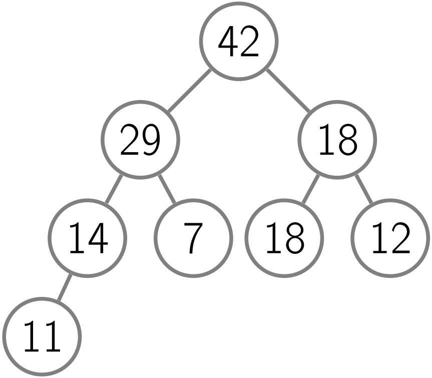
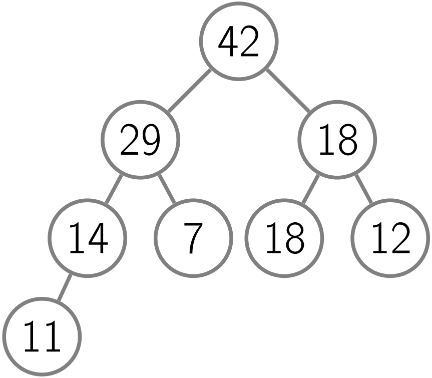
- We need to keep the tree complete.
- Which binary heap operations modify the shape of the tree?
- Only `Insert` and `ExtractMax` (`Remove` changes the shape by calling `ExtractMax`).

Keeping the Tree Complete



Keeping the Tree Complete

to insert an element, insert it as a leaf in the leftmost vacant position in the last level and let it sift up



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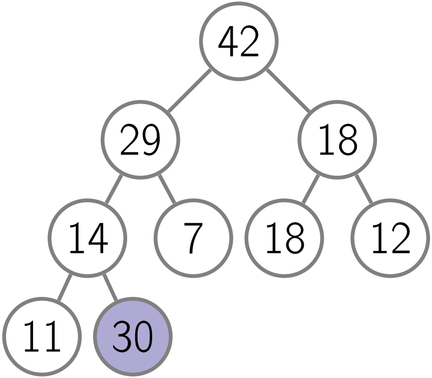

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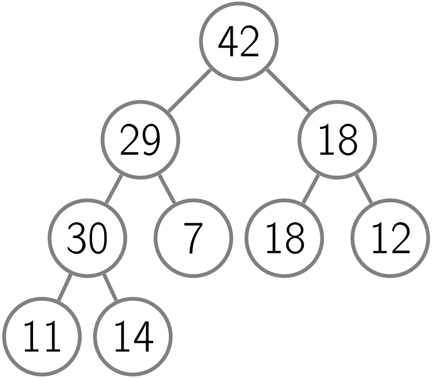

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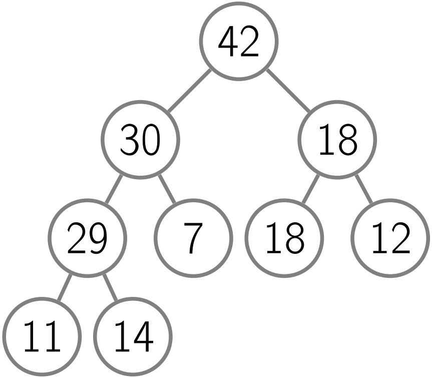
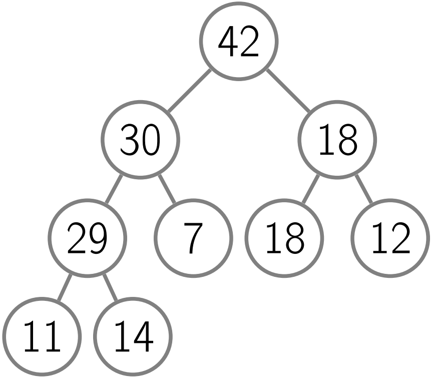


Keeping the Tree Complete

to extract the maximum value, replace the root by the last leaf and let it sift down



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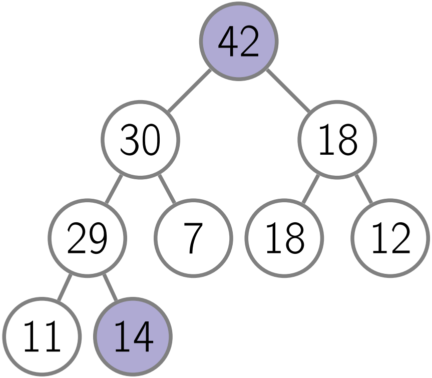

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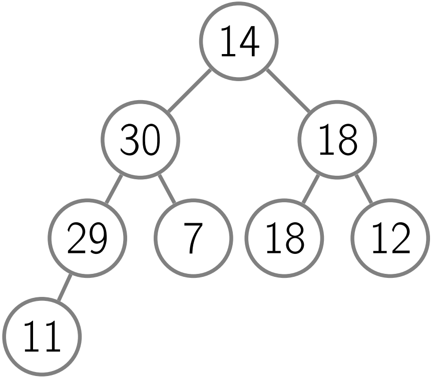

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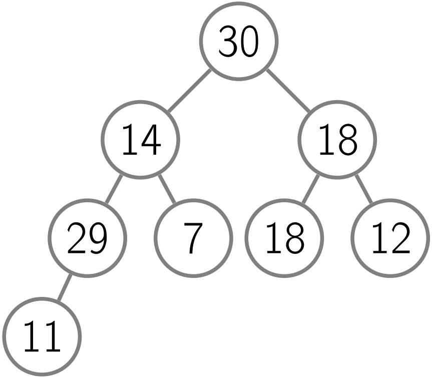

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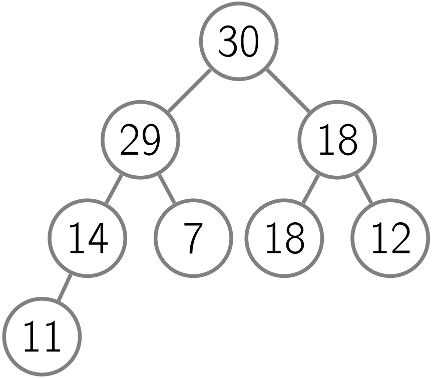

Outline

1. Binary Trees
2. Basic Operations
3. Complete Binary Trees
4. Pseudocode
5. Heap Sort
6. Final Remarks

General Setting

- *maxSize* is the maximum number of elements in the heap

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- *size* is the size of the heap

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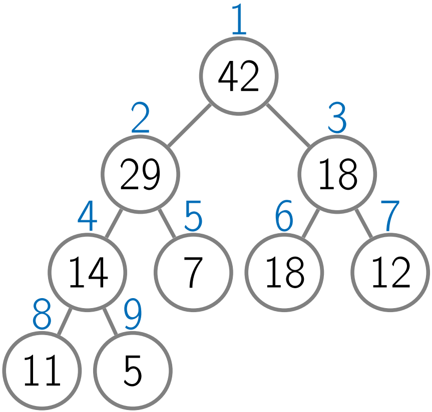
- *size* is the size of the heap
- $H[1 \dots maxSize]$ is an array of length *maxSize* where the heap occupies the first *size* elements

Example



$size = 9$

$maxSize = 13$

	1	2	3	4	5	6	7	8	9	10	11	12	13
H	42	29	18	14	7	18	12	11	5	30	29	2	8

Parent(i)

return $\lfloor \frac{i}{2} \rfloor$

LeftChild(i)

return $2i$

RightChild(i)

return $2i + 1$

SiftUp(i)

```
while i > 1 and H[Parent(i)] < H[i]:
    swap H[Parent(i)] and H[i]
    i ← Parent(i)
```

SiftDown(i)

$maxIndex \leftarrow i$
$\ell \leftarrow$ LeftChild(i)
if $\ell \leq size$ and $H[\ell] > H[maxIndex]$:
  $maxIndex \leftarrow \ell$
$r \leftarrow$ RightChild(i)
if $r \leq size$ and $H[r] > H[maxIndex]$:
  $maxIndex \leftarrow r$
if $i \neq maxIndex$:
  swap $H[i]$ and $H[maxIndex]$
  SiftDown($maxIndex$)

Insert(p)

```
if size = maxSize:
  return ERROR
size ← size + 1
H[size] ← p
SiftUp(size)
```

ExtractMax()

$result \leftarrow H[1]$
$H[1] \leftarrow H[size]$
$size \leftarrow size - 1$
SiftDown(1)
return $result$

Remove(i)

$H[i] \leftarrow \infty$
SiftUp(i)
ExtractMax()

`ChangePriority`(i, p)

```
oldp ← H[i]
H[i] ← p
if p > oldp:
  SiftUp(i)
else:
  SiftDown(i)
```

Summary

The resulting implementation is

- fast: all operations work in time $O(\log n)$ (`GetMax` even works in $O(1)$)

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- fast: all operations work in time $O(\log n)$ (`GetMax` even works in $O(1)$)
- space efficient: we store an array of priorities; parent-child connections are not stored, but are computed on the fly
- easy to implement: all operations are implemented in just a few lines of code

Outline

Sort Using Priority Queues

```
HeapSort(A[1...n])

create an empty priority queue
for i from 1 to n:
   Insert(A[i])
for i from n downto 1:
   A[i] ← ExtractMax()
```

- The resulting algorithms is comparison-based and has running time $O(n \log n)$ (hence, asymptotically optimal!).

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- Natural generalization of selection sort: instead of simply scanning the rest of the array to find the maximum value, use a smart data structure.
- Not in-place: uses additional space to store the priority queue.

This lesson

In-place heap sort algorithm. For this, we will first turn a given array into a heap by permuting its elements.

Turn Array into a Heap

BuildHeap($A[1 \ldots n]$)

```
size ← n
for i from ⌊n/2⌋ downto 1:
  SiftDown(i)
```

- We repair the heap property going from bottom to top.

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- When we reach the root, the heap property is satisfied in the whole tree.
- Online visualization
- Running time: $O(n \log n)$

In-place Heap Sort

HeapSort($A[1 \ldots n]$)

```
BuildHeap(A)                      {size = n}
repeat (n − 1) times:
   swap A[1] and A[size]
   size ← size − 1
   SiftDown(1)
```

Building Running Time

- The running time of `BuildHeap` is $O(n \log n)$ since we call `SiftDown` for $O(n)$ nodes.

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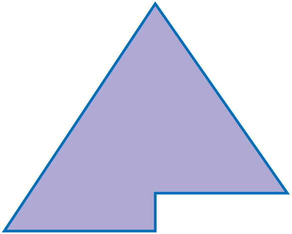
Building Running Time

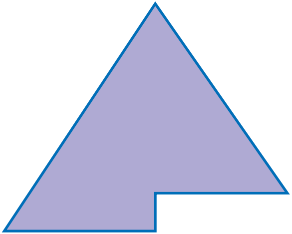
- The running time of `BuildHeap` is $O(n \log n)$ since we call `SiftDown` for $O(n)$ nodes.
- If a node is already close to the leaves, then sifting it down is fast.
- We have many such nodes!
- Was our estimate of the running time of `BuildHeap` too pessimistic?

Building Running Time



# nodes	$T(\texttt{SiftDown})$
1	$\log_2 n$
2	
$\vdots$	$\vdots$
$\leq n/4$	2
$\leq n/2$	1

Building Running Time



# nodes	$T(\texttt{SiftDown})$
1	$\log_2 n$
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$\vdots$	$\vdots$
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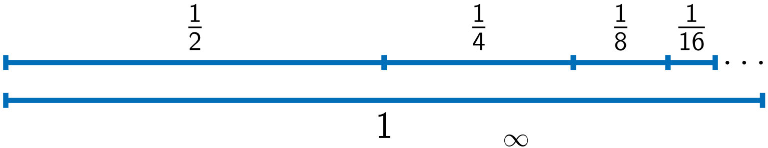
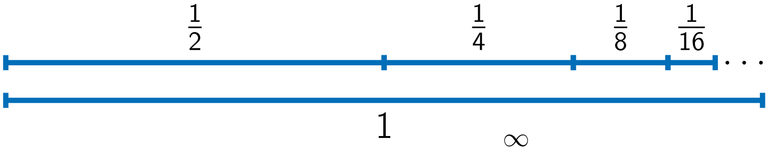
$$
\begin{aligned}
T(\texttt{BuildHeap}) &\leq \frac{n}{2}\cdot 1 + \frac{n}{4}\cdot 2 + \frac{n}{8}\cdot 3 + \dots \\
&\leq n \cdot \sum_{i=1}^{\infty} \frac{i}{2^i} = 2n
\end{aligned}
$$

Estimating the Sum



$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$$

Estimating the Sum



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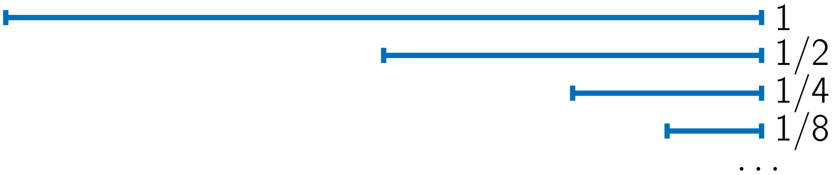

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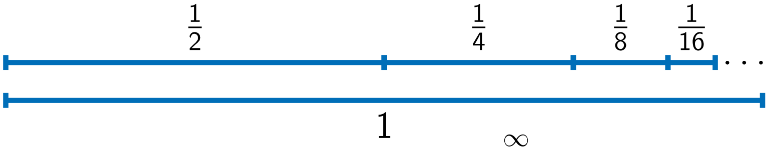

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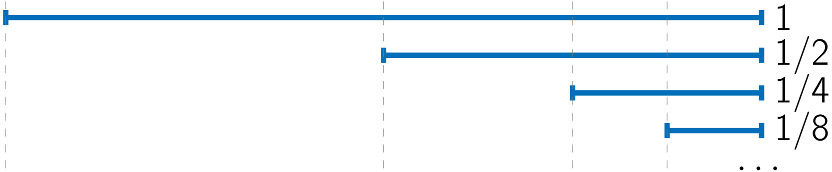

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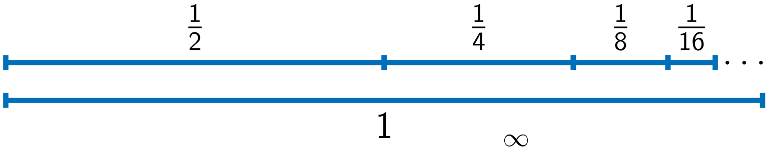
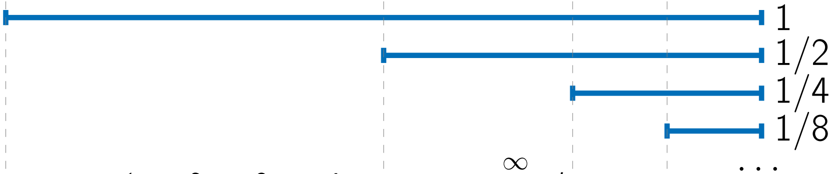


$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots=\sum_{k=1}^{\infty}\frac{1}{2^k}=1$$



$$\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\ldots=\sum_{k=1}^{\infty}\frac{k}{2^k}=2$$

Partial sorting

Input: An array $A[1 \dots n]$, an integer $1 \leq k \leq n$.

Output: The last k elements of a sorted version of A.

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Input: An array $A[1 \ldots n]$, an integer $1 \leq k \leq n$.

Output: The last k elements of a sorted version of A.

Can be solved in $O(n)$ if $k = O(\frac{n}{\log n})$!

```
PartialSorting(A[1...n], k)

BuildHeap(A)
for i from 1 to k:
  ExtractMax()
```

$\texttt{PartialSorting}(A[1 \dots n], k)$

```
BuildHeap(A)
for i from 1 to k:
  ExtractMax()
```

Running time: $O(n + k \log n)$

Summary

Heap sort is a time and space efficient comparison-based algorithm: has running time $O(n \log n)$, uses no additional space.

Outline

0-based Arrays

Parent(i)

return $\left\lfloor \frac{i-1}{2} \right\rfloor$

LeftChild(i)

return $2i + 1$

RightChild(i)

return $2i + 2$

Binary Min-Heap

Definition

Binary min-heap is a binary tree (each node has zero, one, or two children) where the value of each node is at most the values of its children.

Can be implemented similarly.

d-ary Heap

- In a d-ary heap nodes on all levels except for possibly the last one have exactly d children.

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d-ary Heap

- In a d-ary heap nodes on all levels except for possibly the last one have exactly d children.
- The height of such a tree is about $\log_d n$.
- The running time of `SiftUp` is $O(\log_d n)$.
- The running time of `SiftDown` is $O(d \log_d n)$: on each level, we find the largest value among d children.

Summary

- Priority queue supports two main operations: `Insert` and `ExtractMax`.

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- In an array/list implementation one operation is very fast ($O(1)$) but the other one is very slow ($O(n)$).

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Summary

- Priority queue supports two main operations: `Insert` and `ExtractMax`.
- In an array/list implementation one operation is very fast ($O(1)$) but the other one is very slow ($O(n)$).
- Binary heap gives an implementation where both operations take $O(\log n)$ time.
- Can be made also space efficient.