

Paths in Graphs: Most Direct Route

Michael Levin

Higher School of Economics

Graph Algorithms
Data Structures and Algorithms

Outline

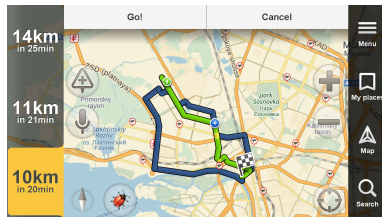
- 1 Paths and Distances
- 2 Breadth-first Search
- 3 Implementation and Analysis
- 4 Proof of Correctness
- 5 Shortest-path Tree

Applications

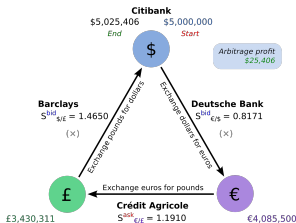
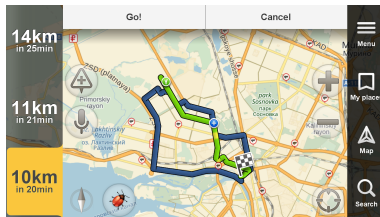
Applications



Applications



Applications



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The most direct route

What is the minimum number of flight segments to get from Hamburg to Moscow?

The most direct route

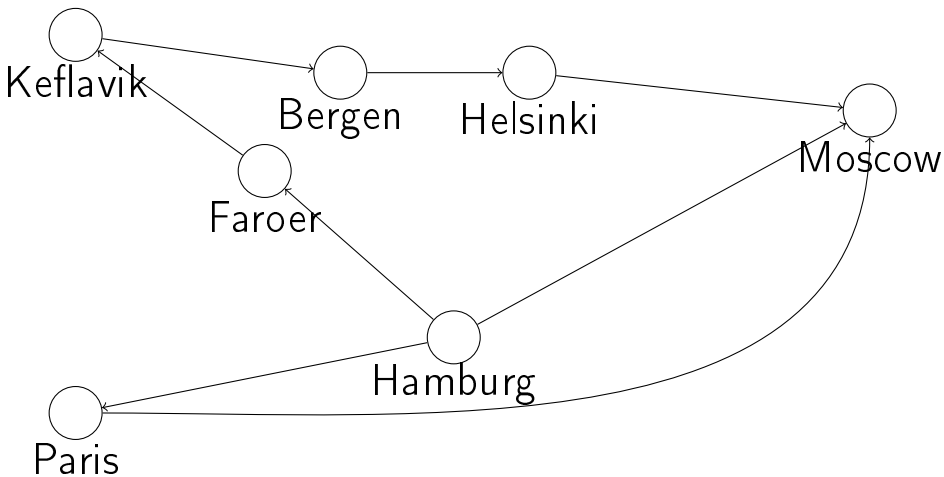
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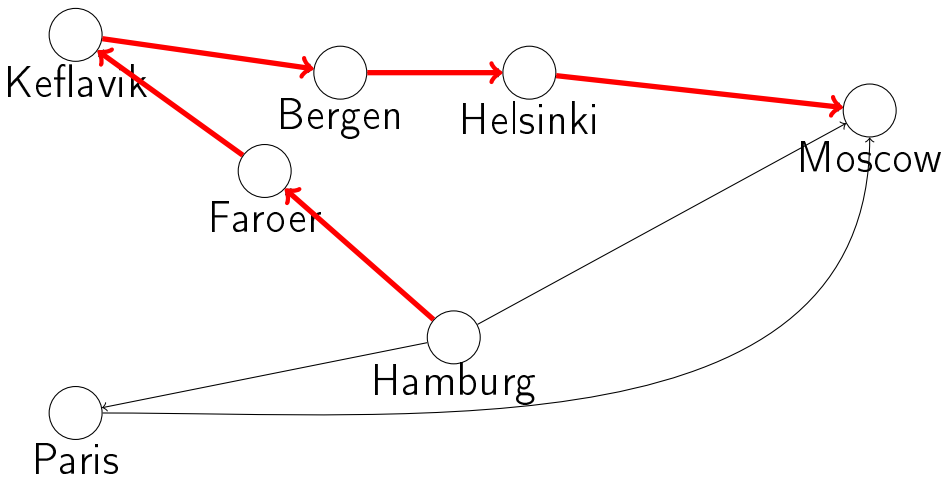


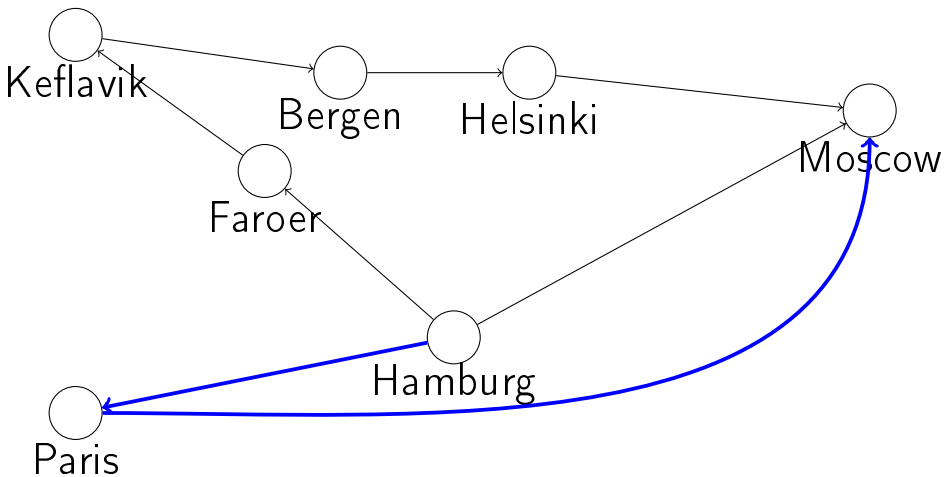
The most direct route

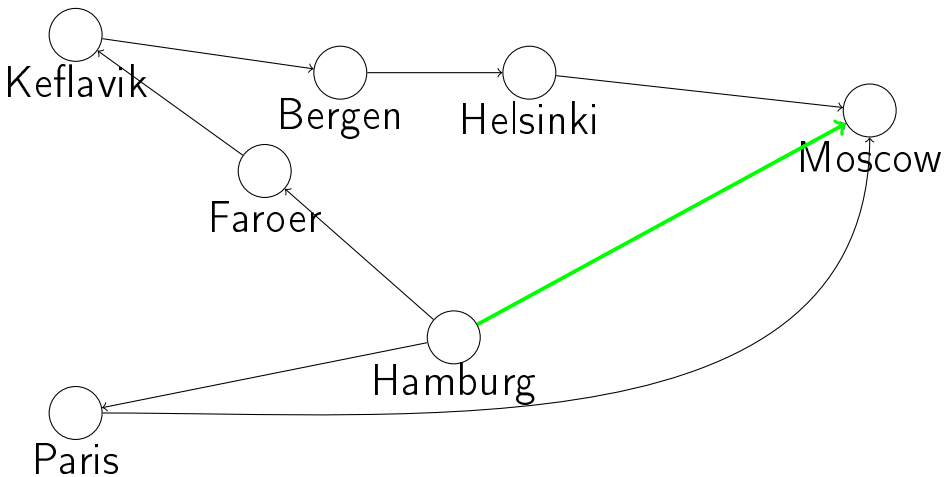
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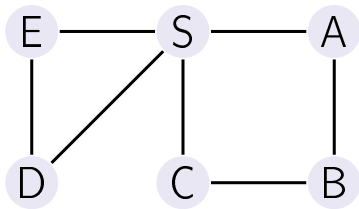






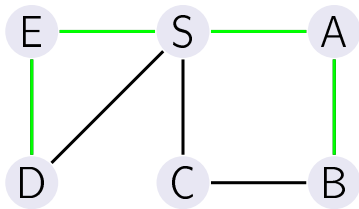
Paths and lengths

Length of the path $L(P)$ is the number of edges in the path.



Paths and lengths

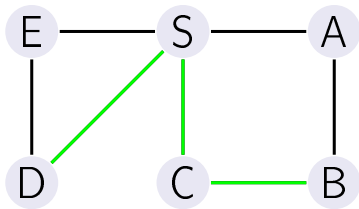
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$$L(D - E - S - A - B) = 4$$

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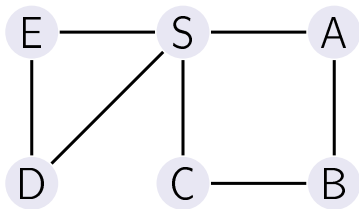


$$L(D - E - S - A - B) = 4$$

$$L(D - S - C - B) = 3$$

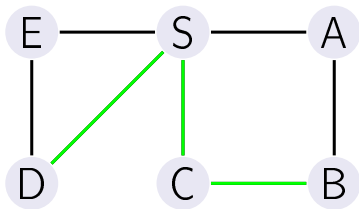
Distances

The **distance** between two vertices is the length of the shortest path between them.



Distances

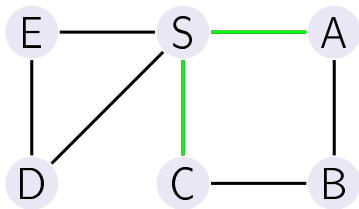
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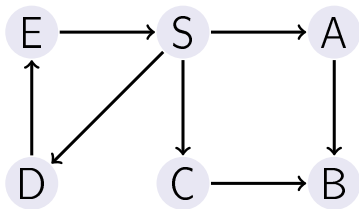


$$d(D, B) = 3$$

$$d(C, A) = 2$$

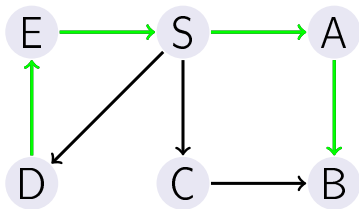
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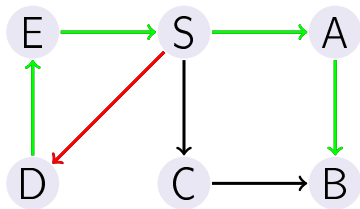
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Distances

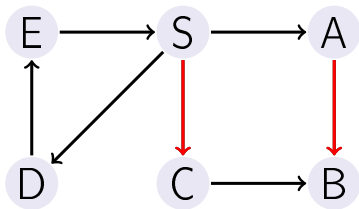
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Distances

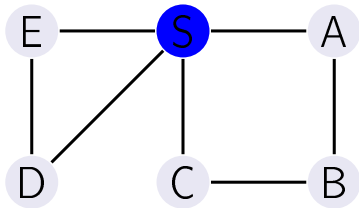
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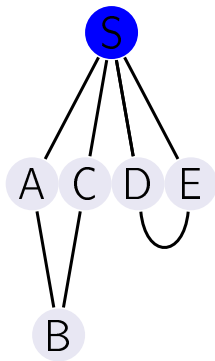
$$d(D, B) = 4$$

$$d(C, A) = \infty$$

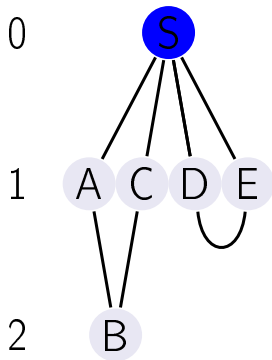
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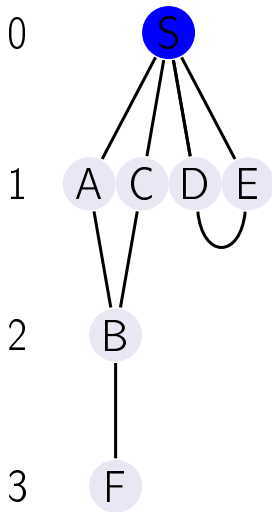
Distance layers



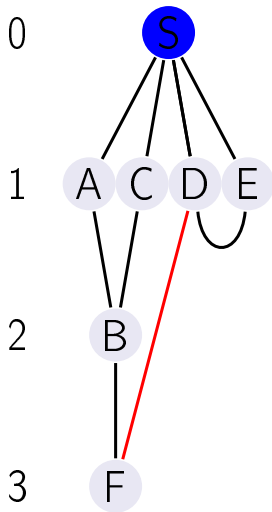
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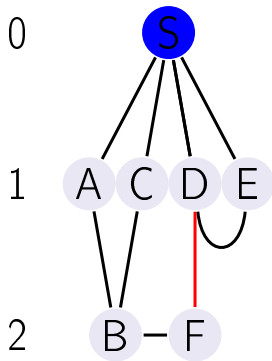
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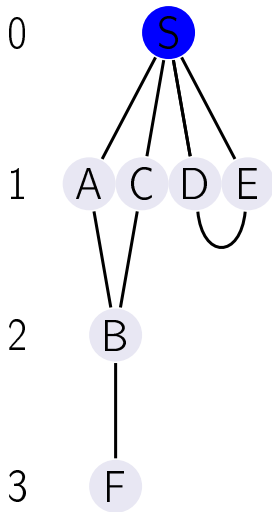
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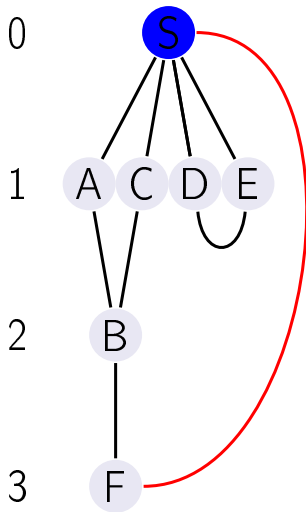
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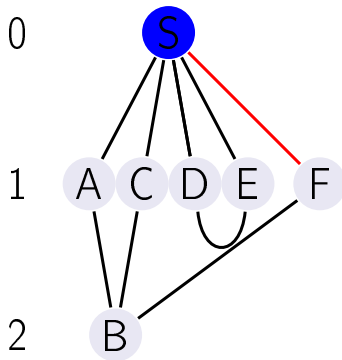
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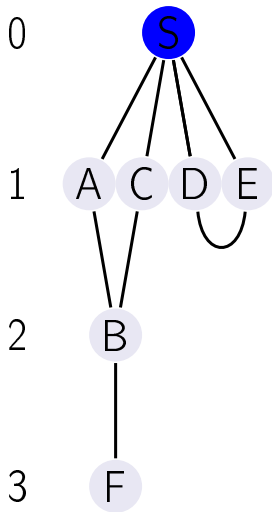
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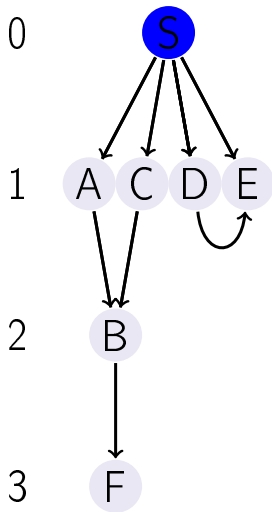
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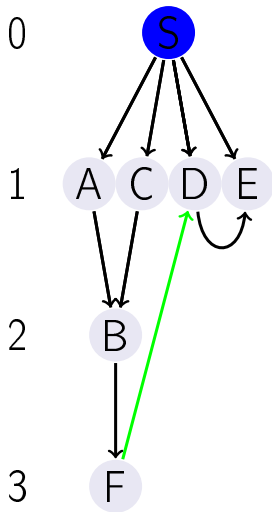
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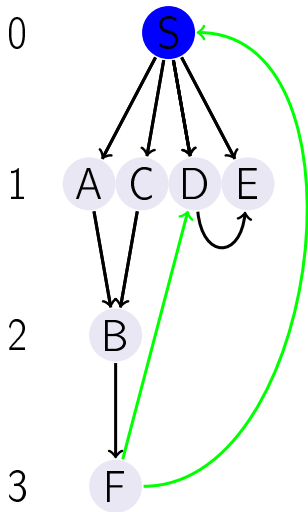
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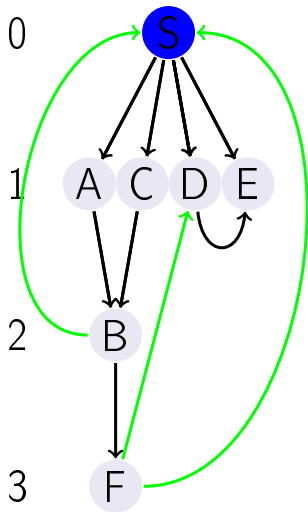
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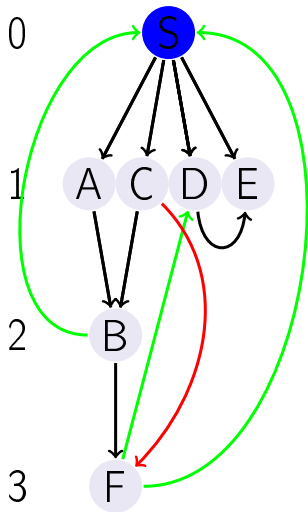
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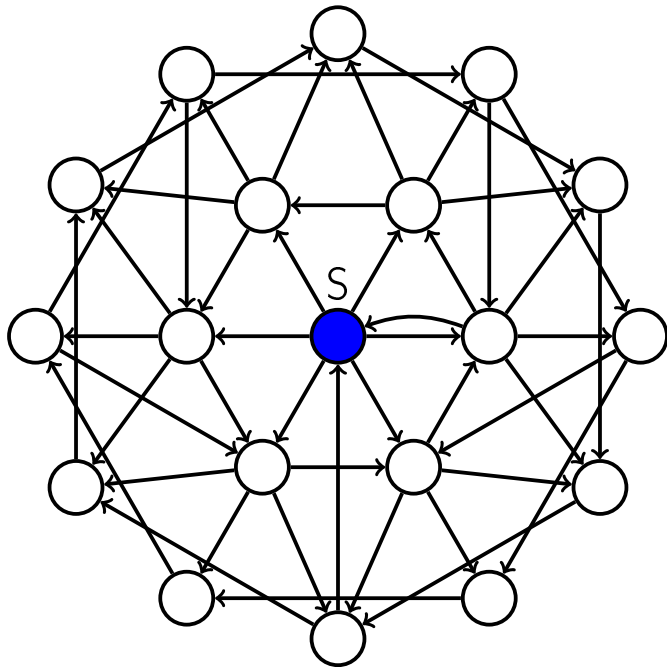


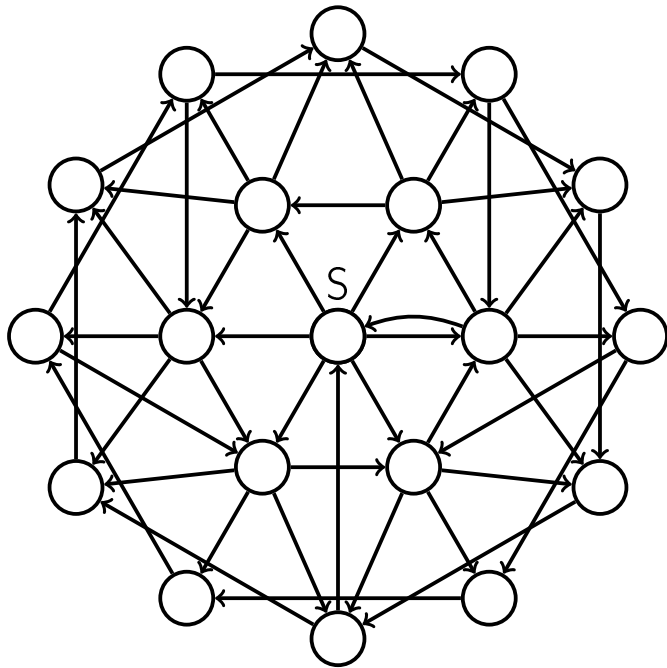
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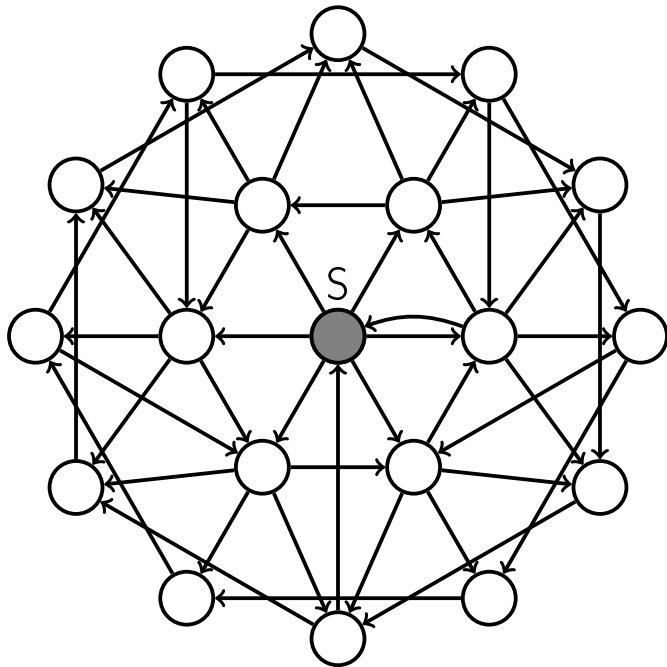


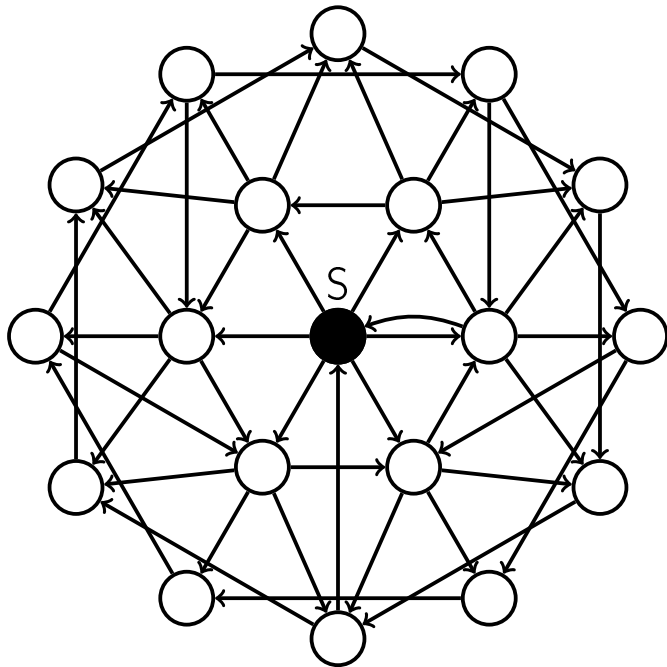
Outline

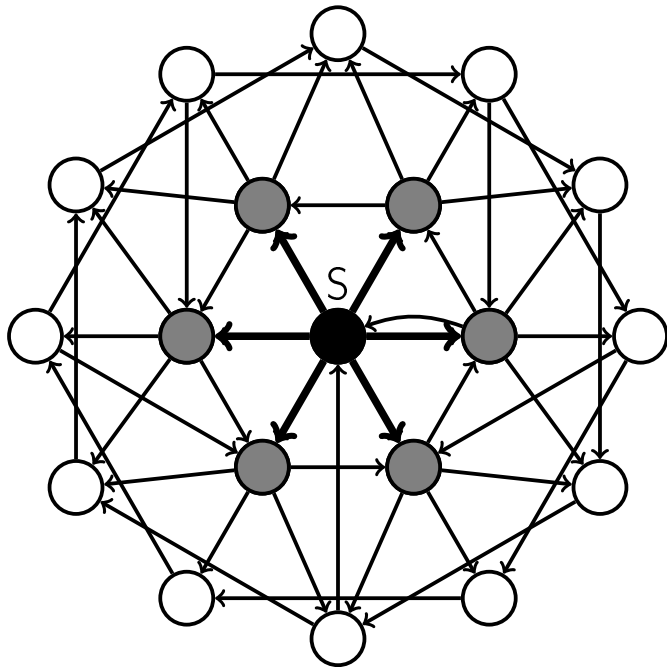
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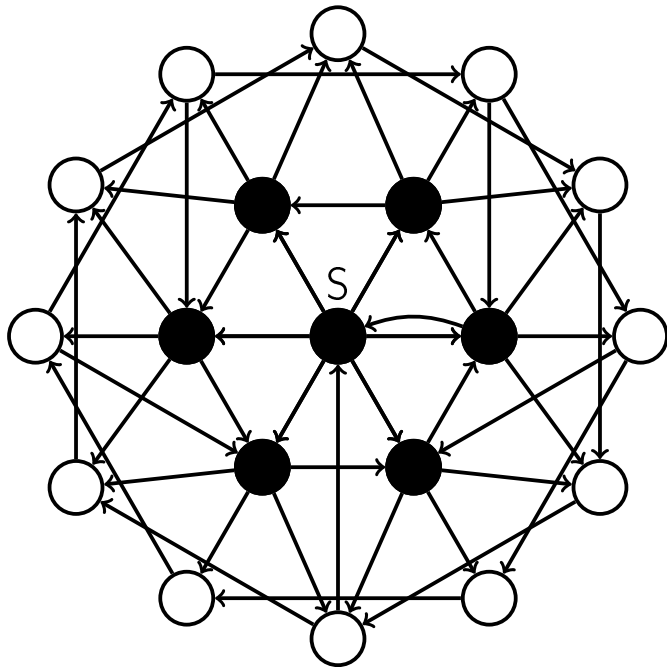


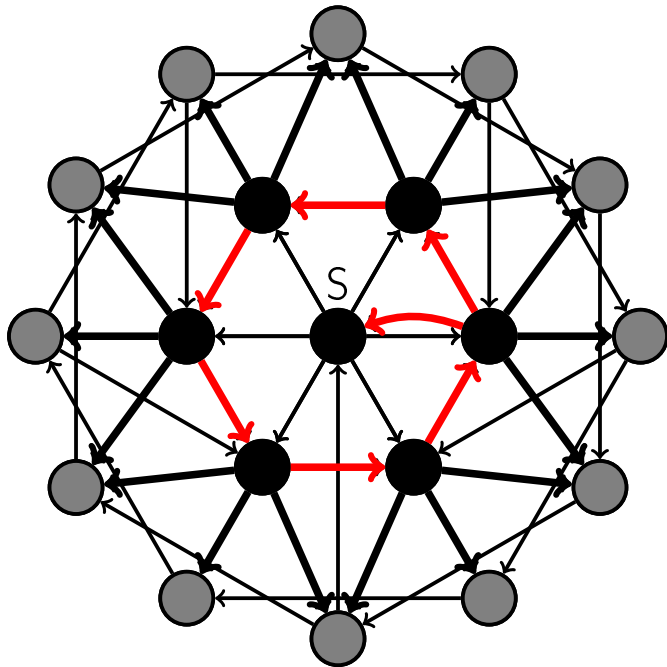


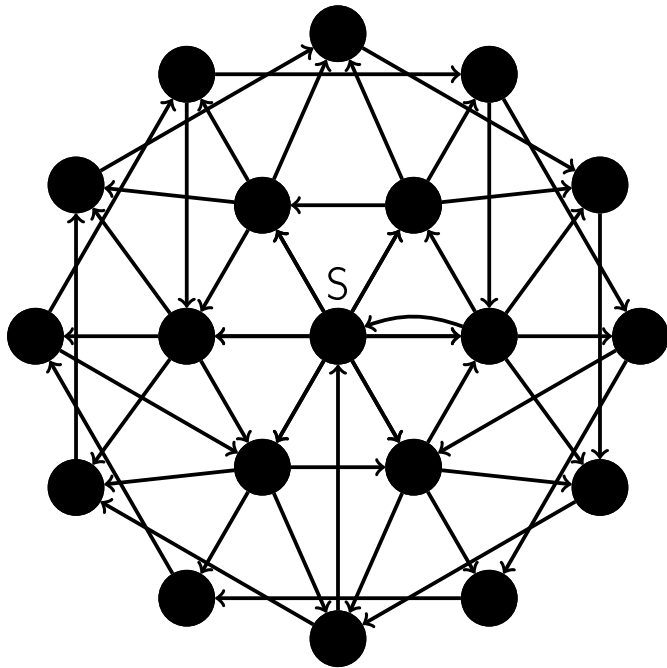


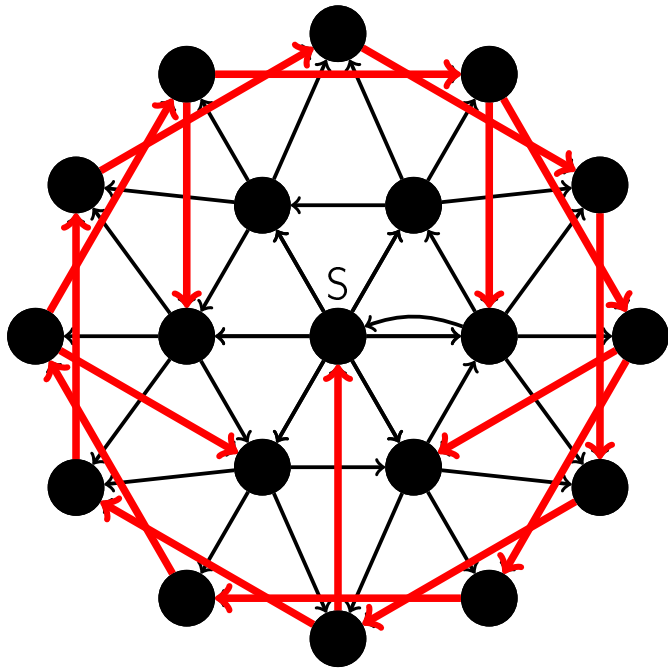


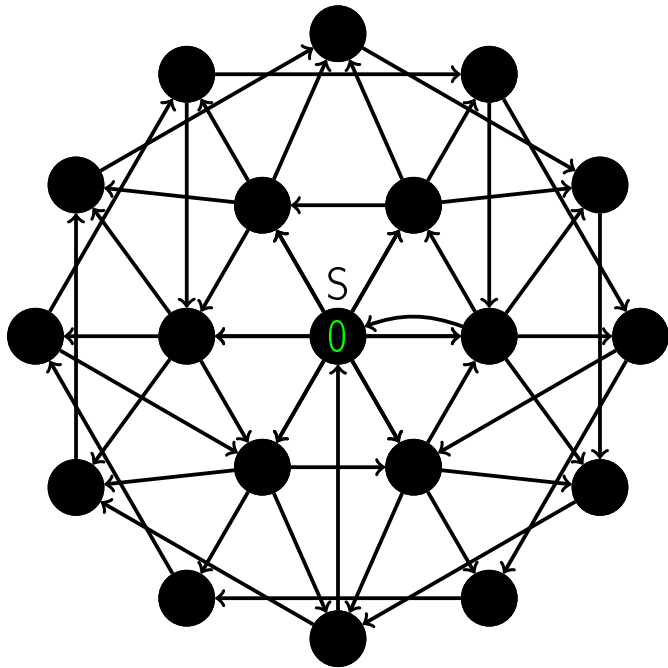


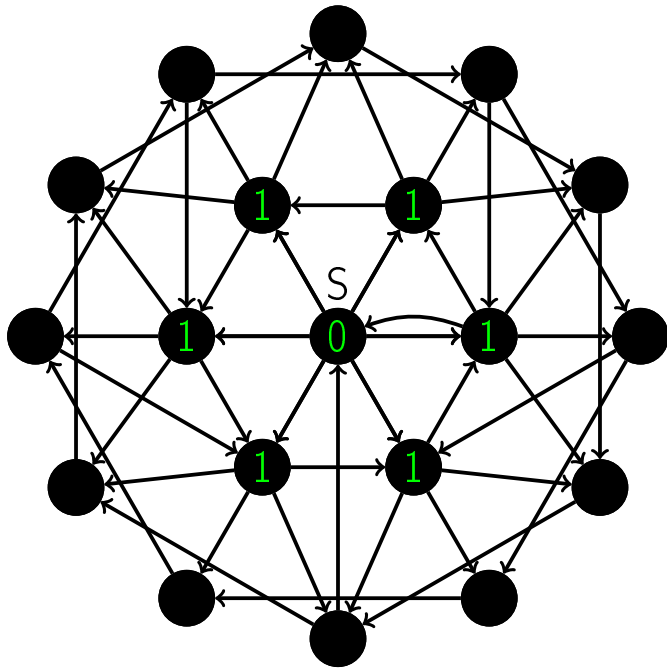


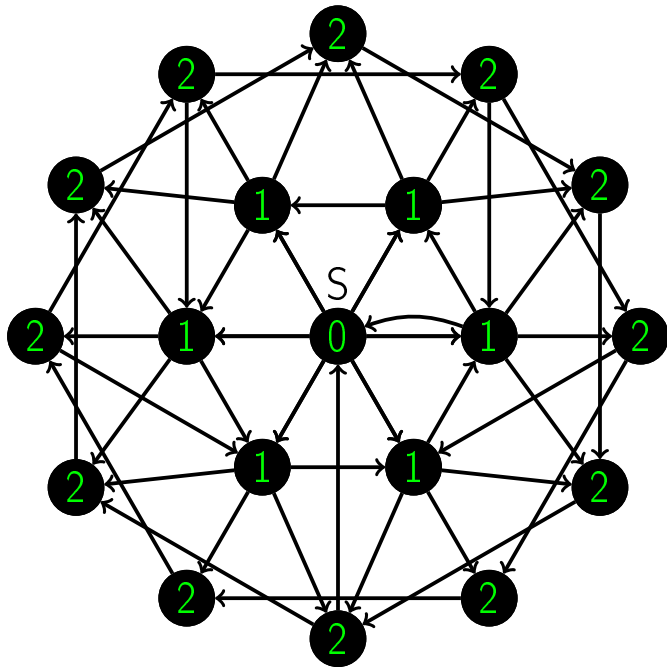


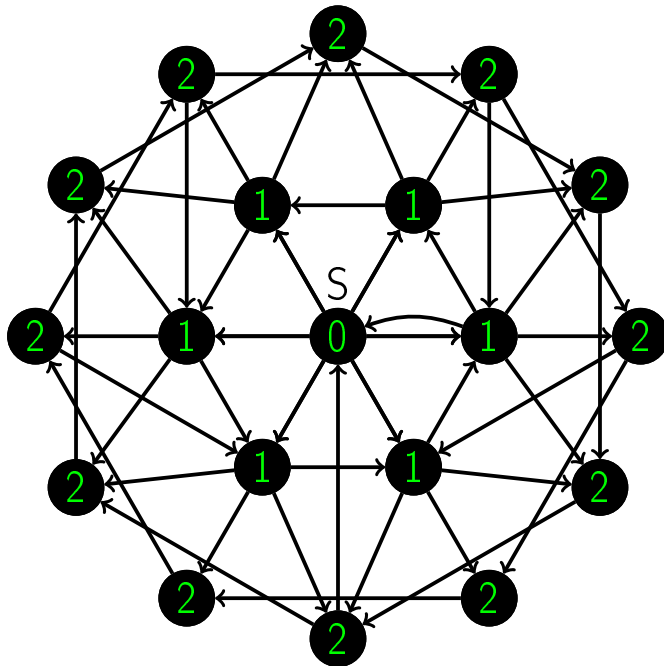


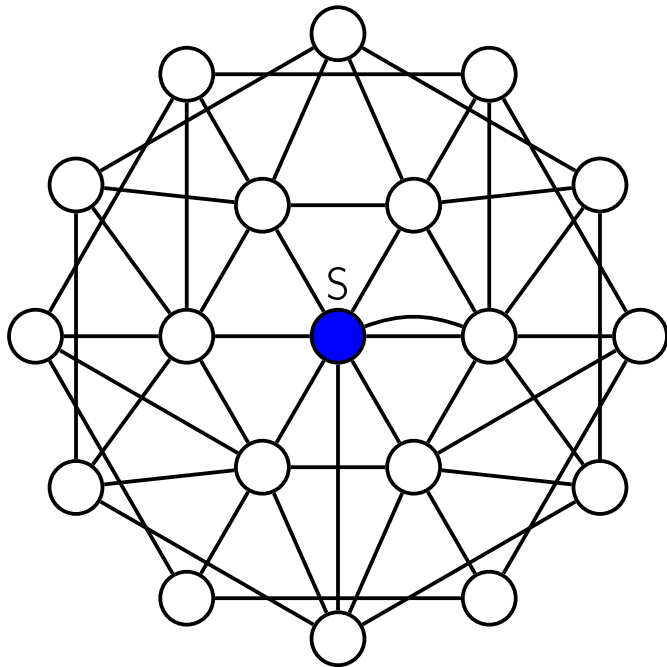


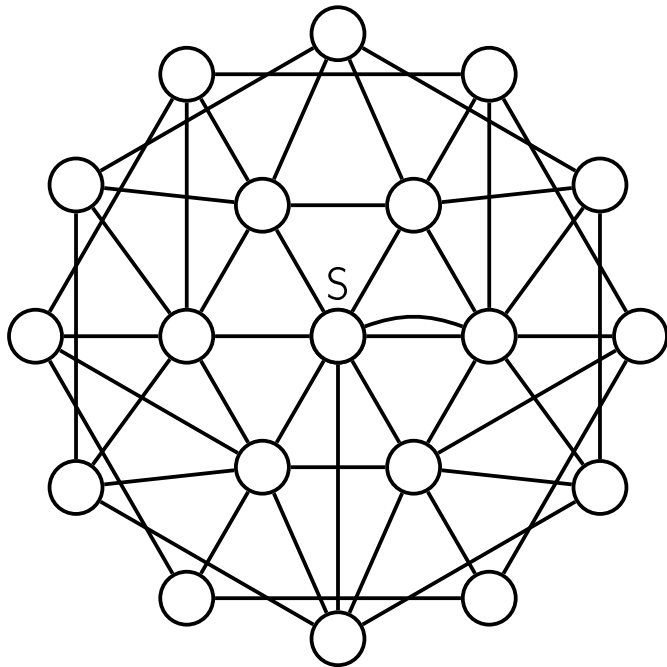


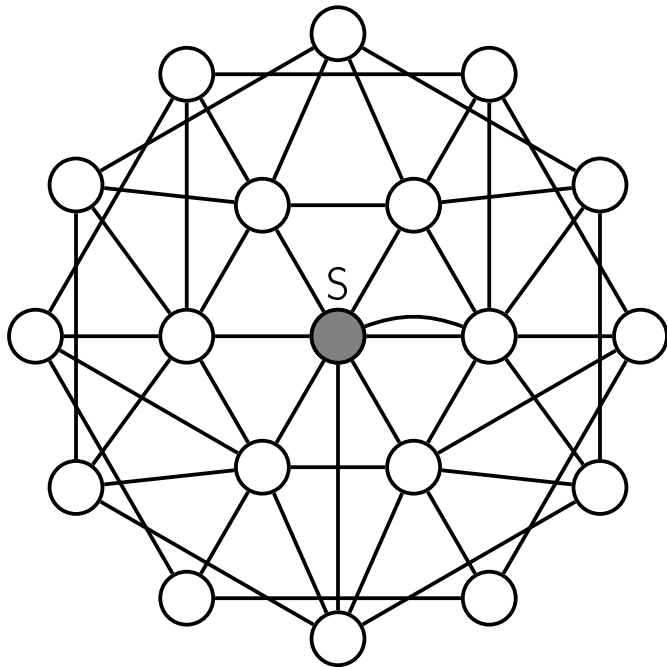


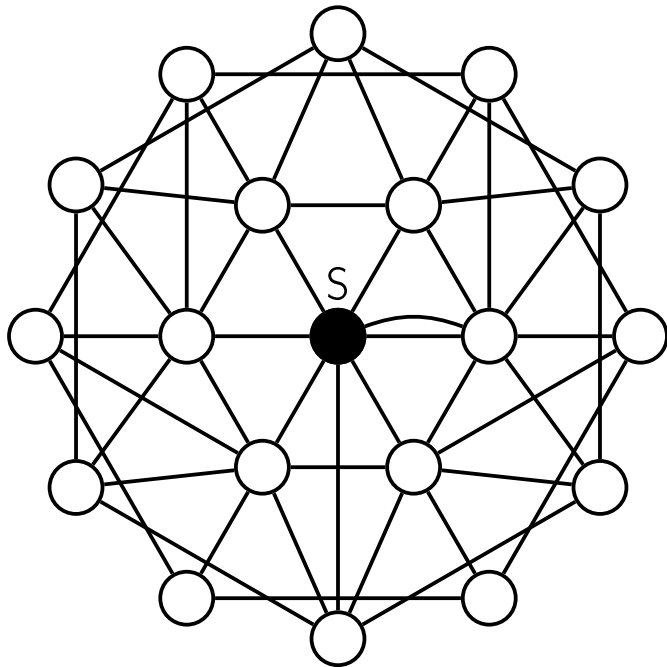


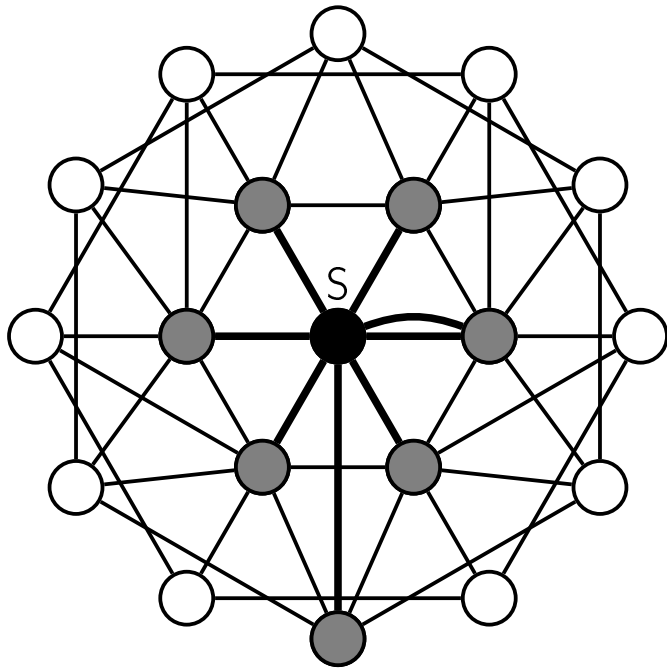


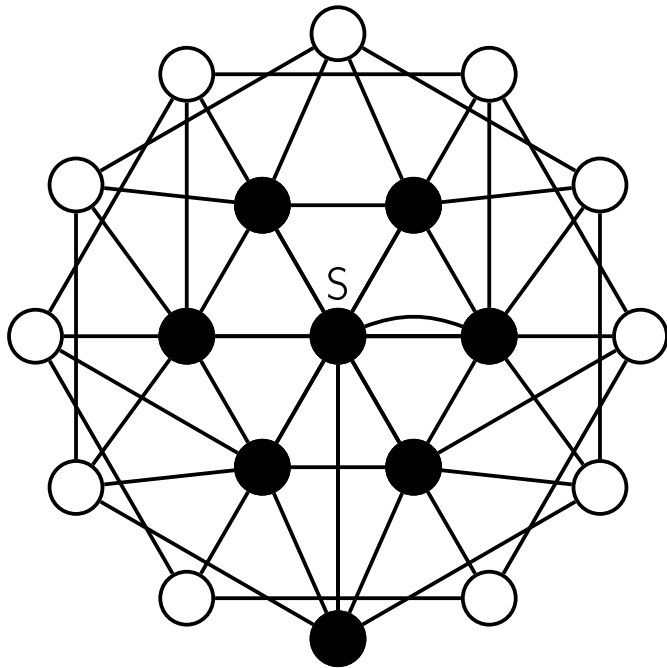


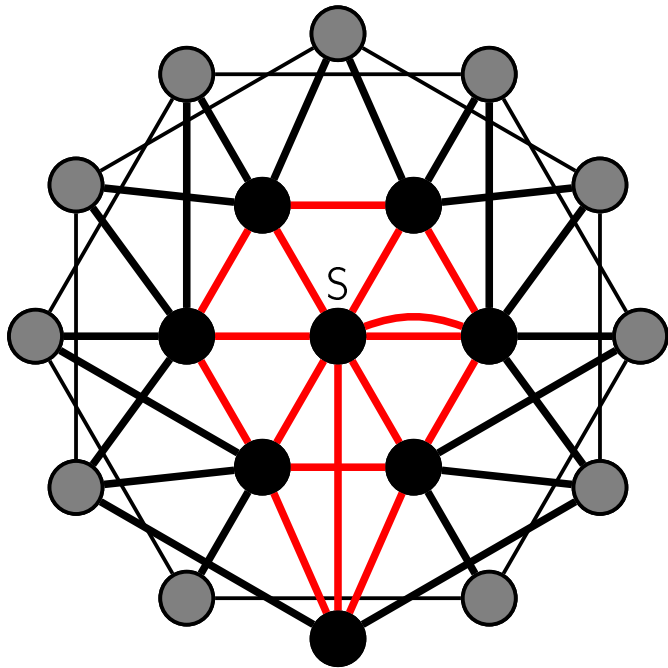


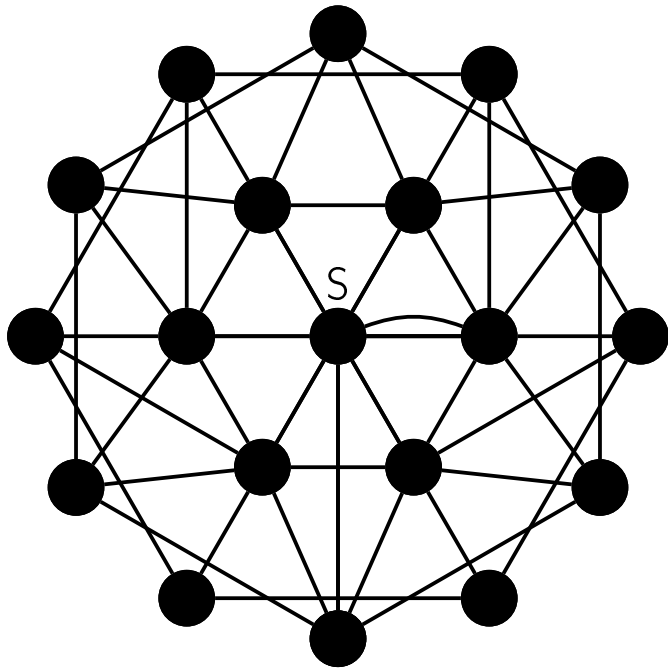


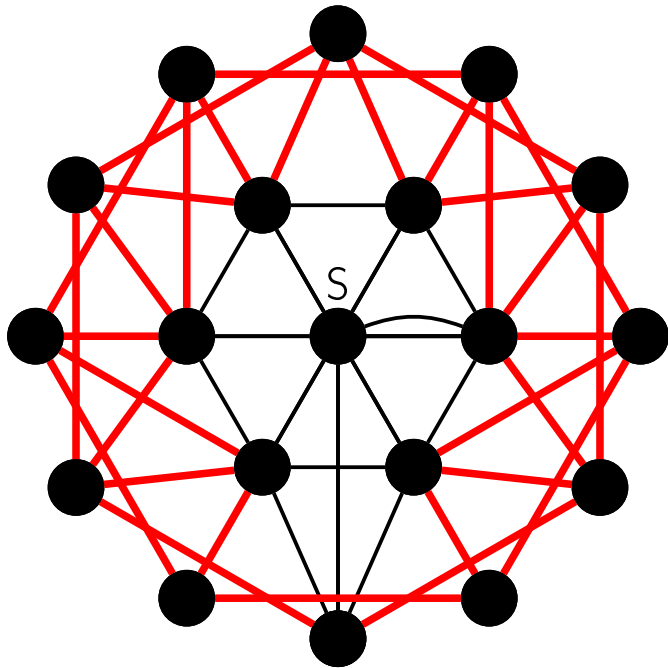


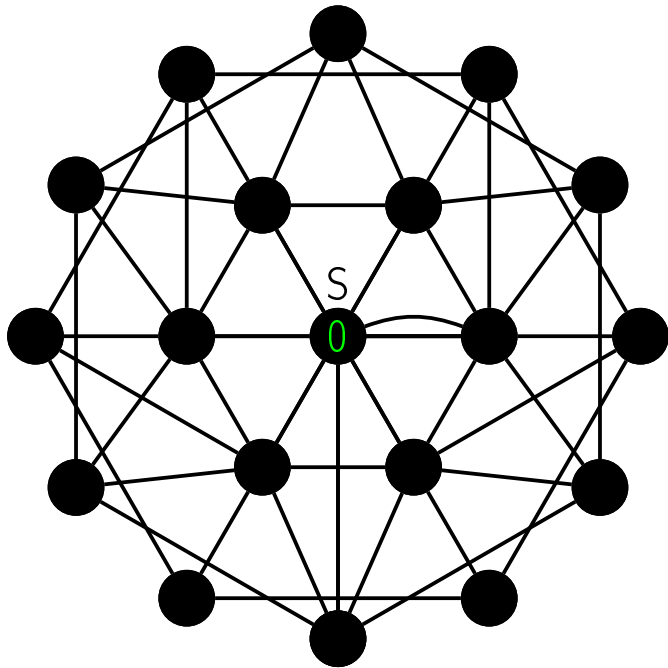


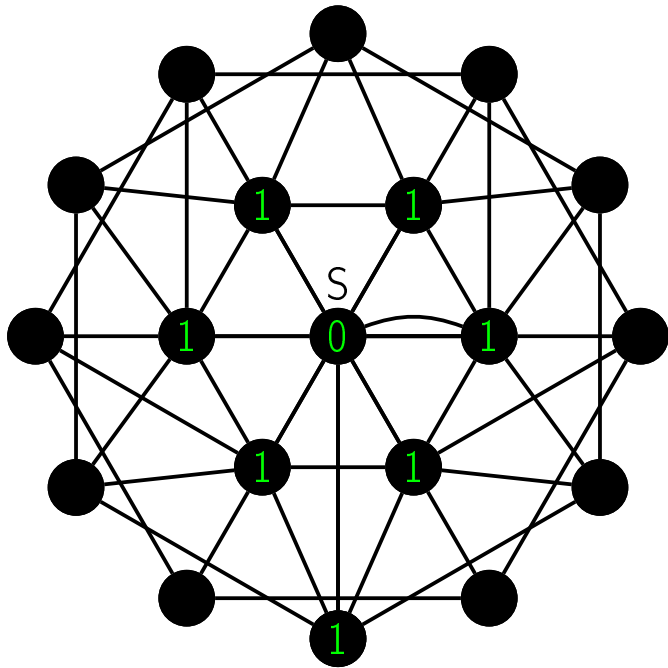


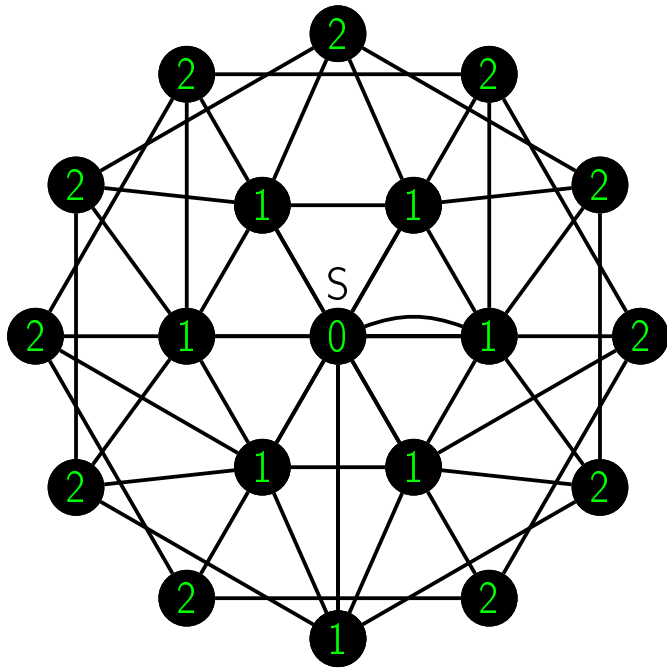


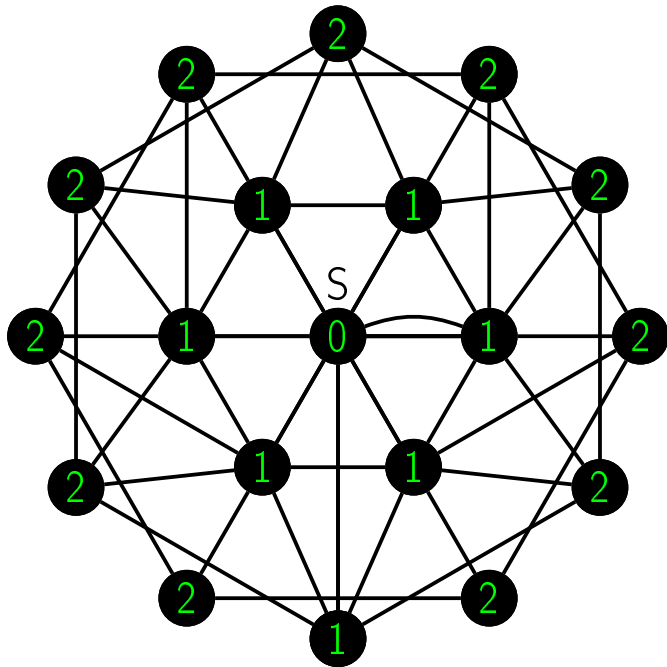


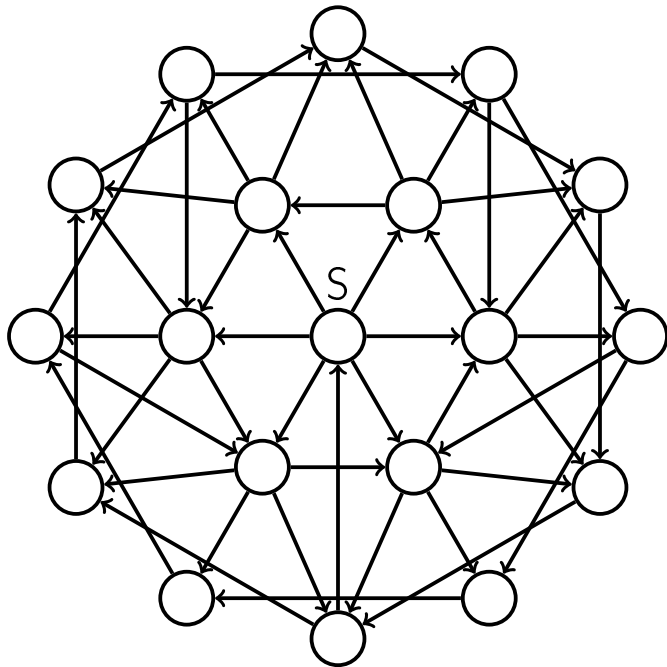


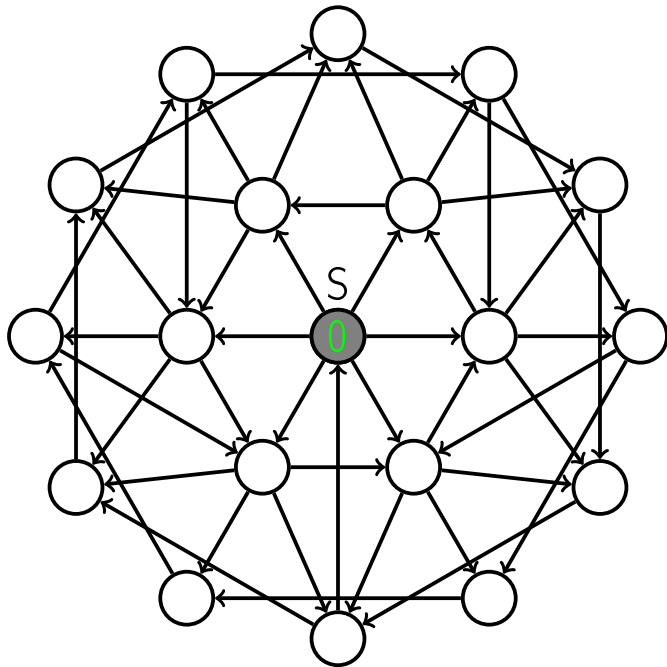


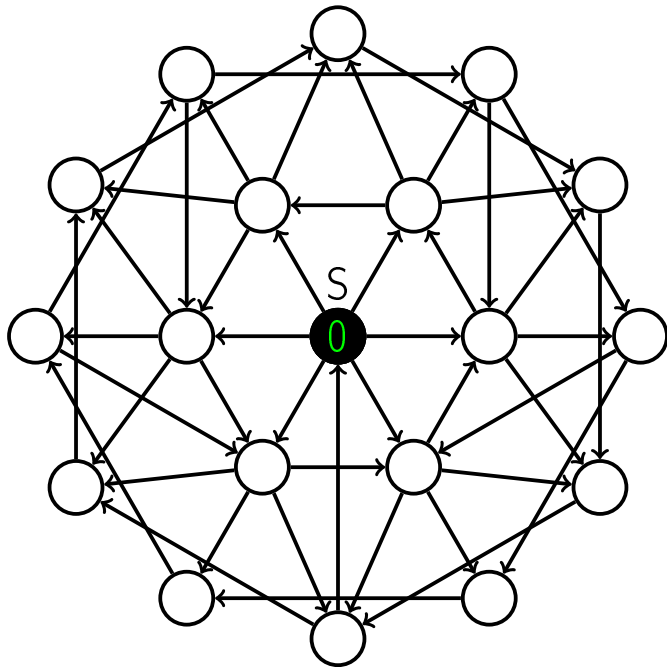


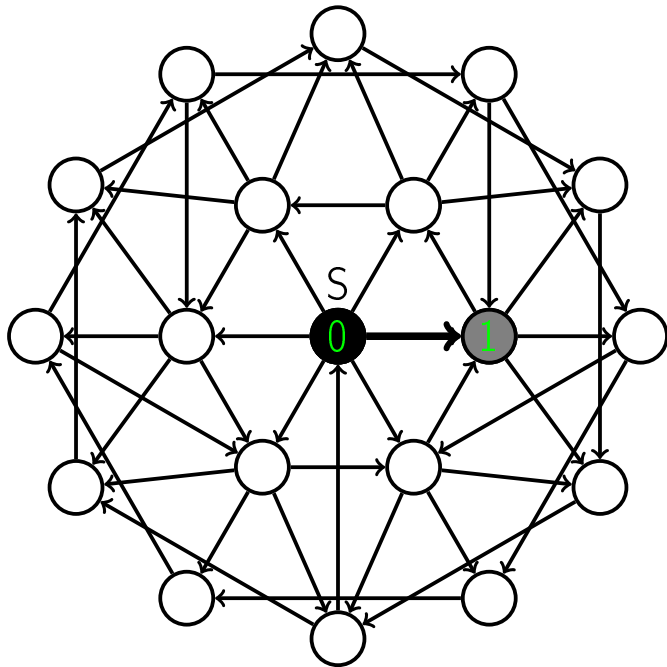


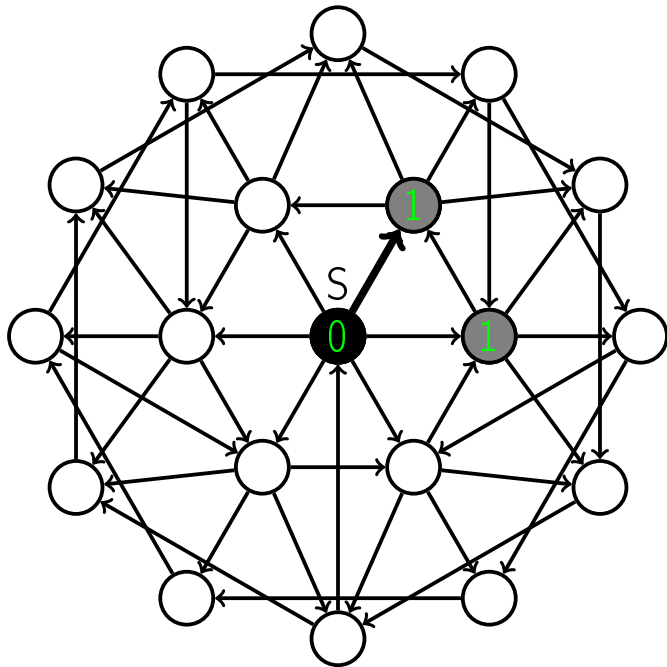


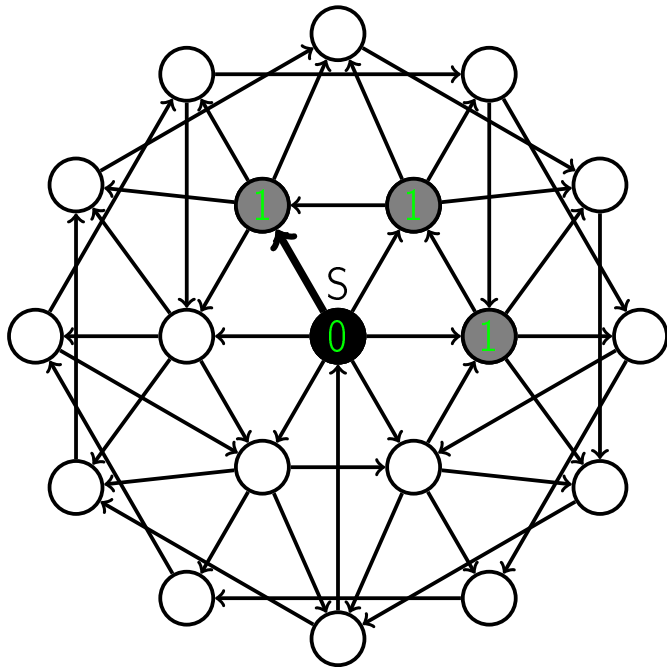


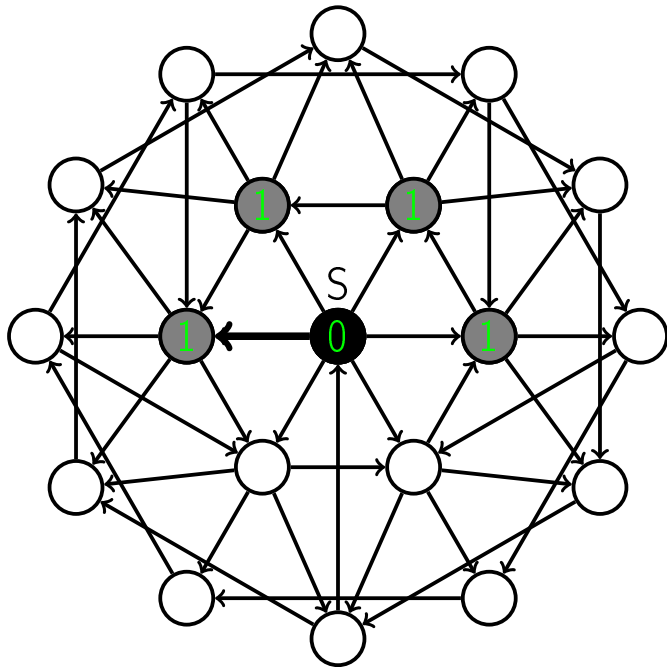


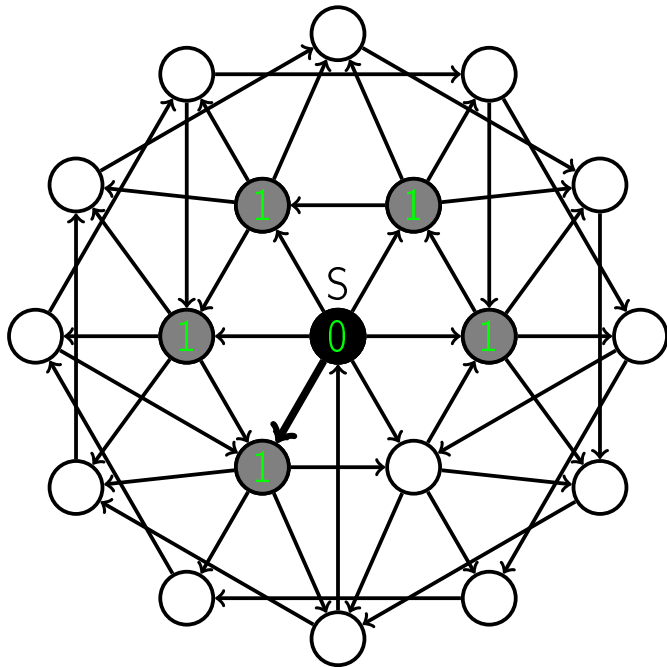


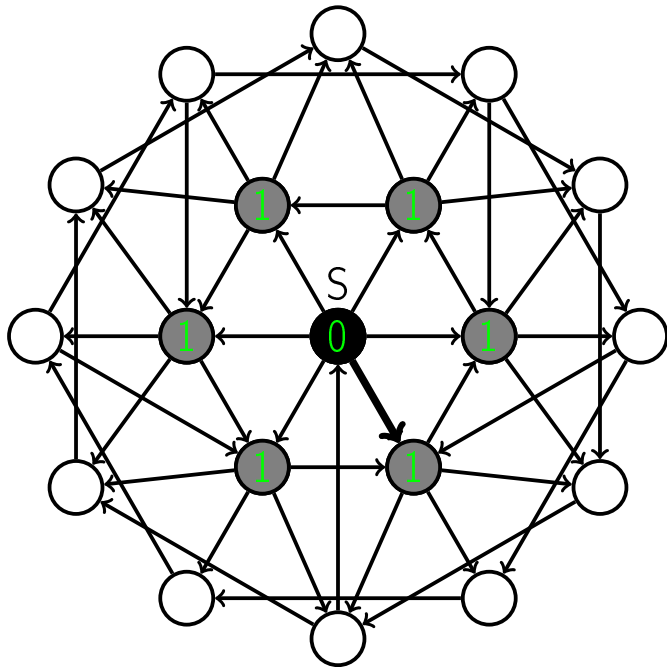


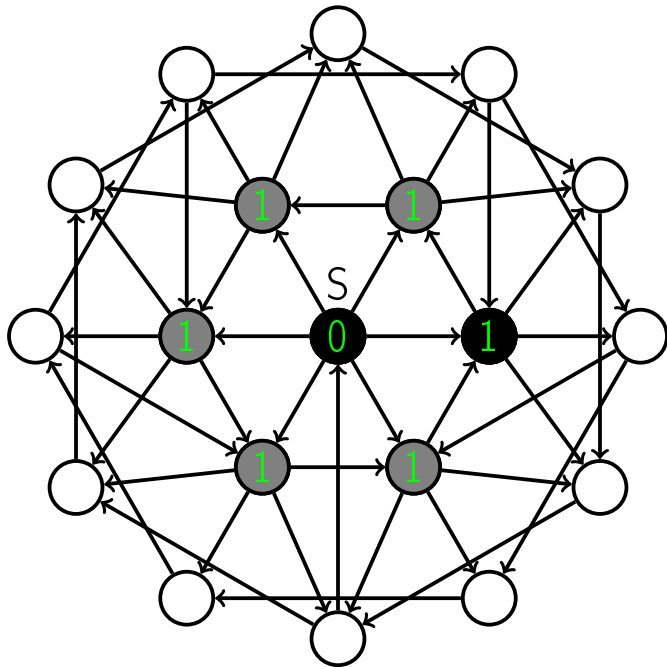


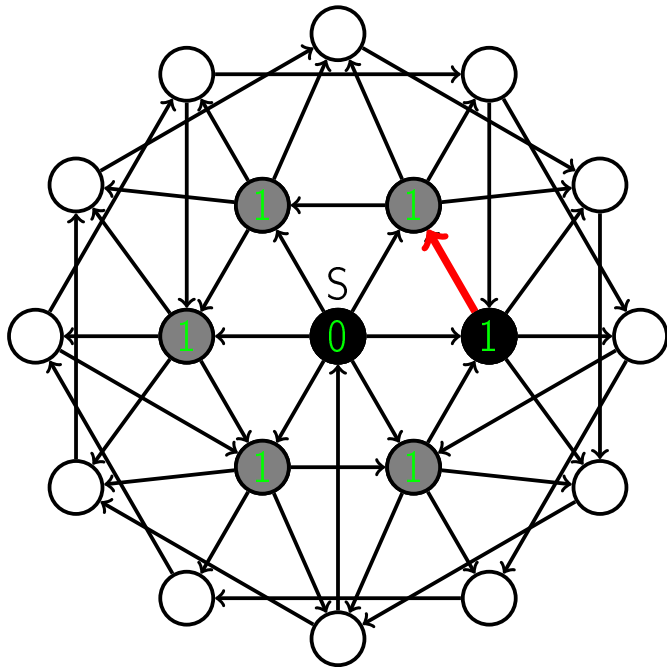


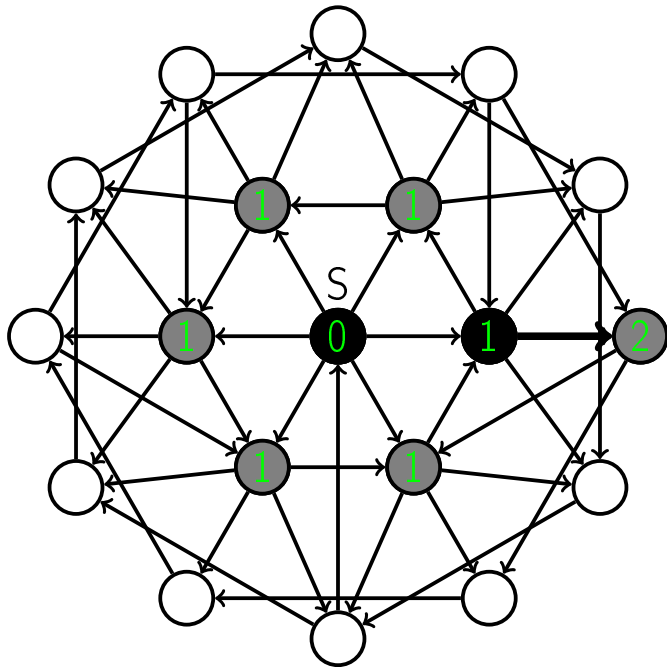


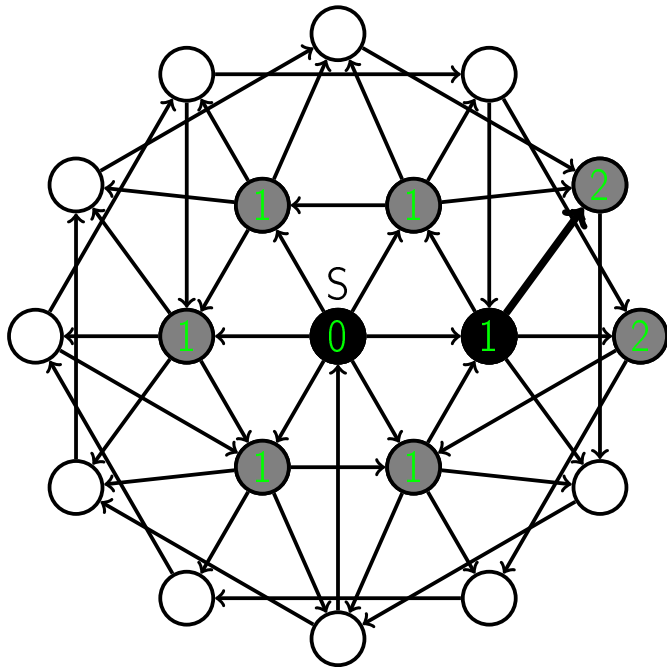


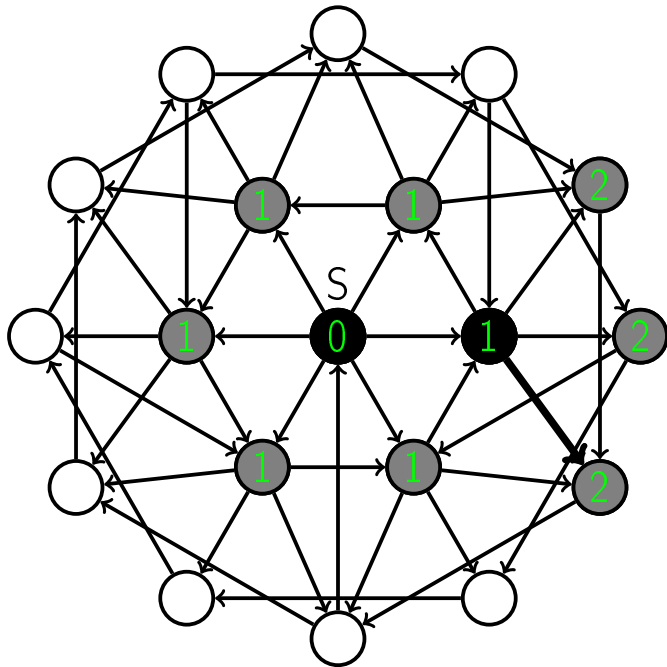


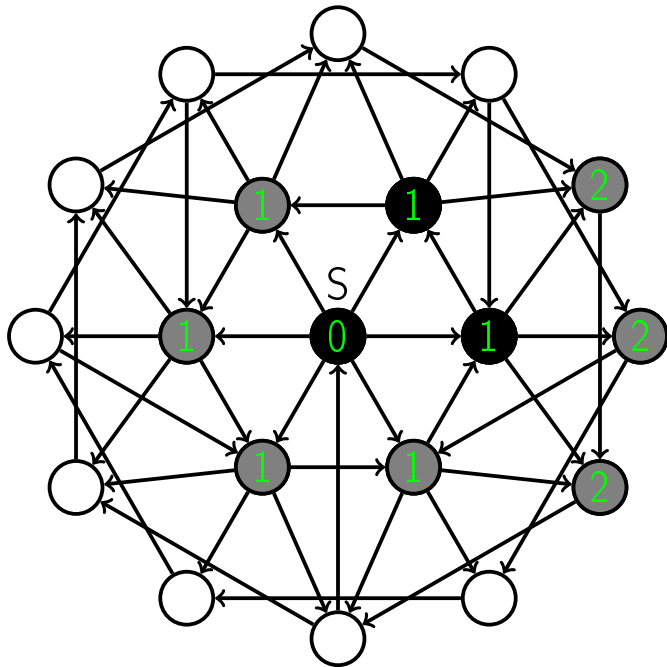


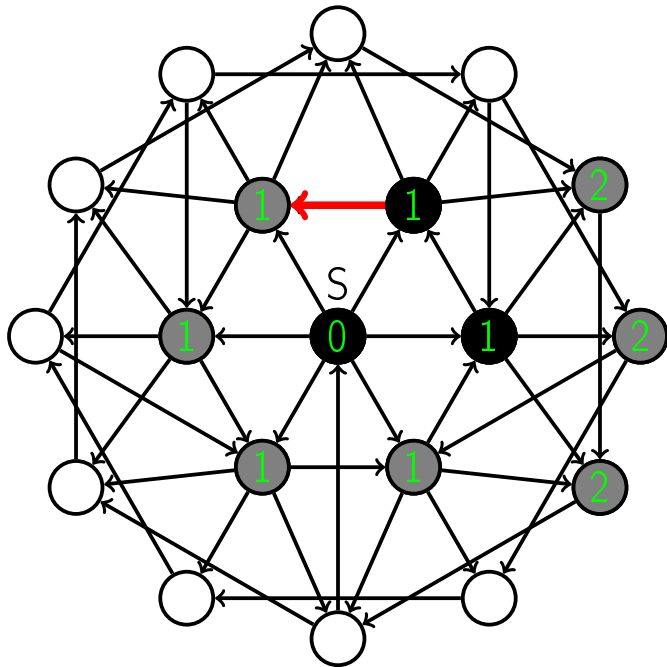


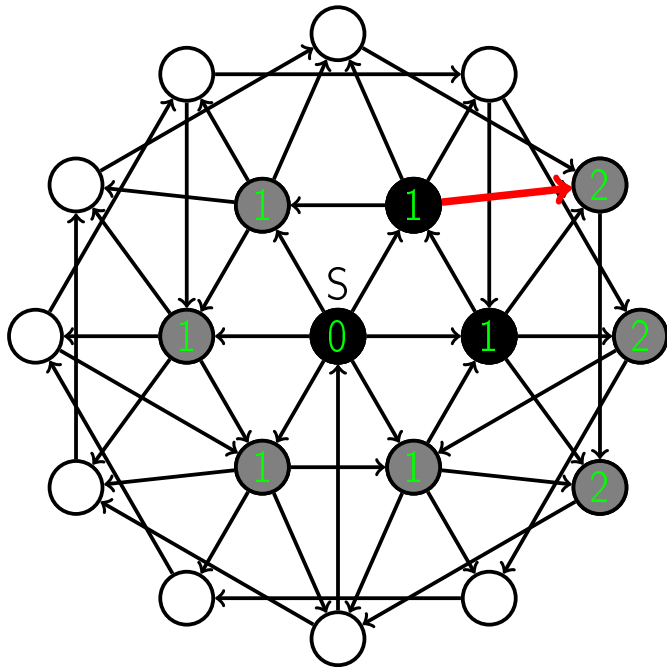


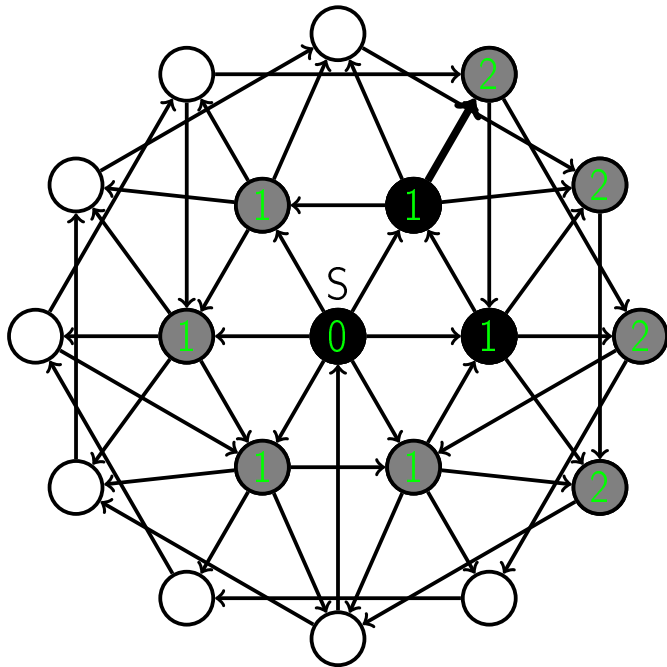


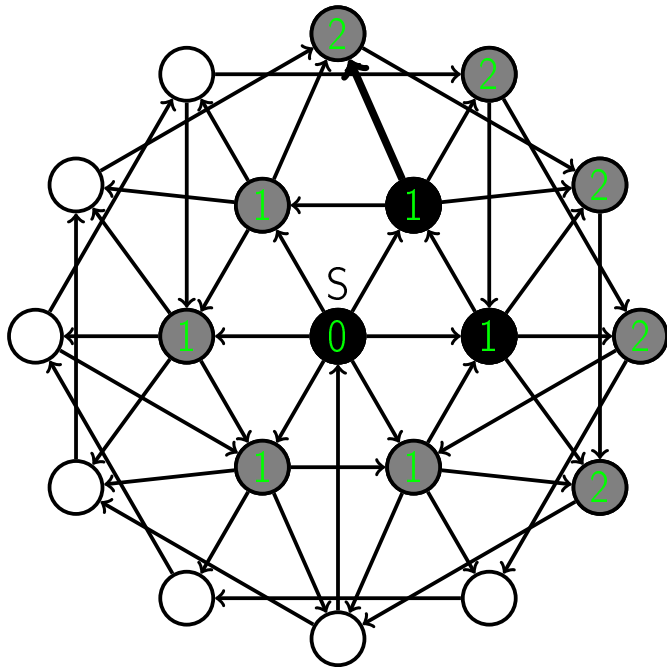


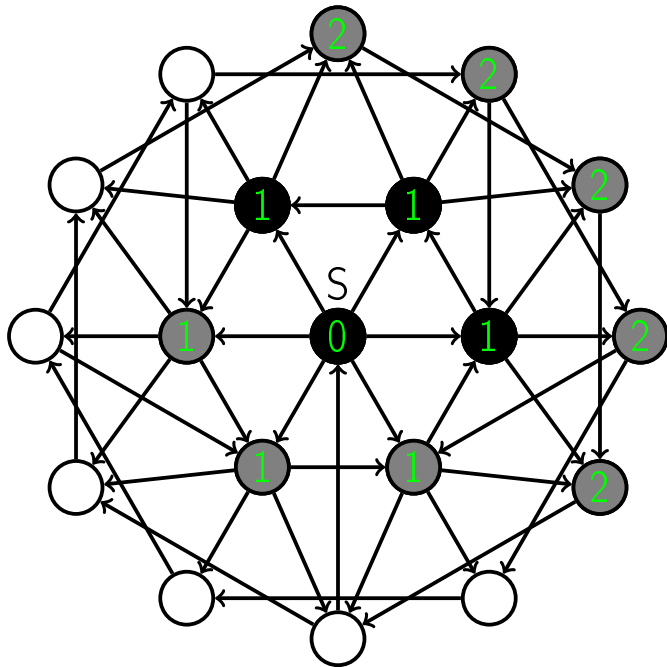


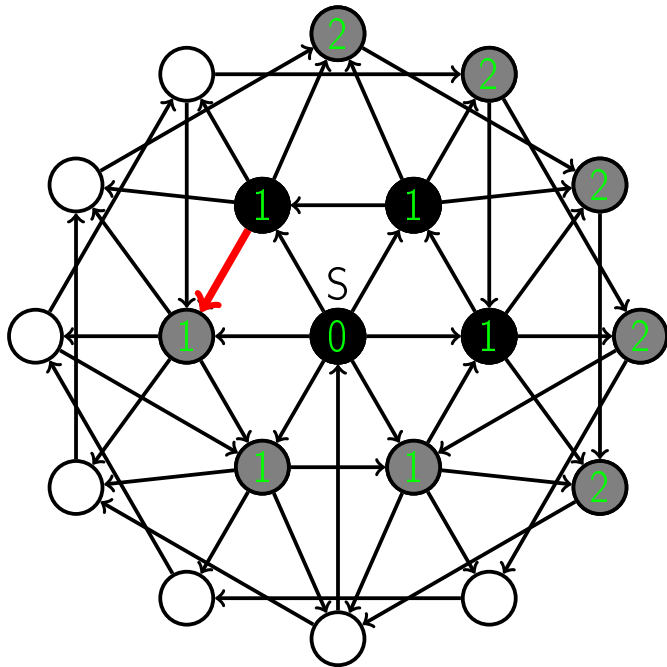


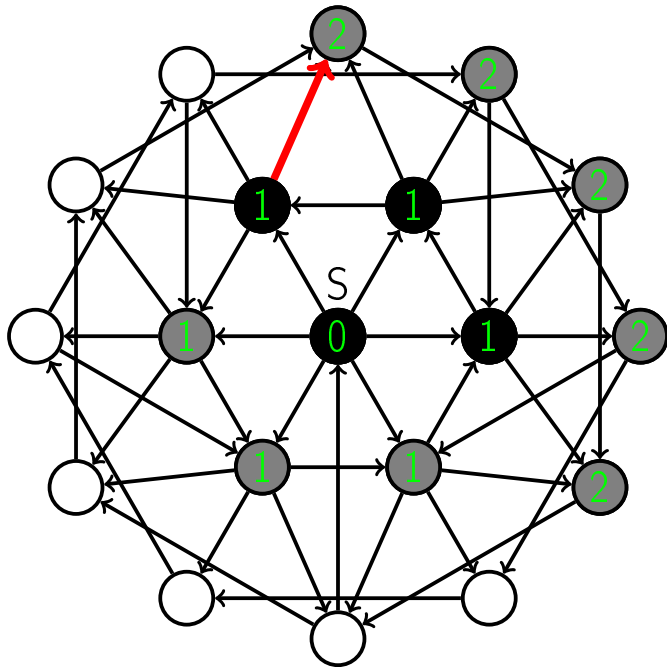


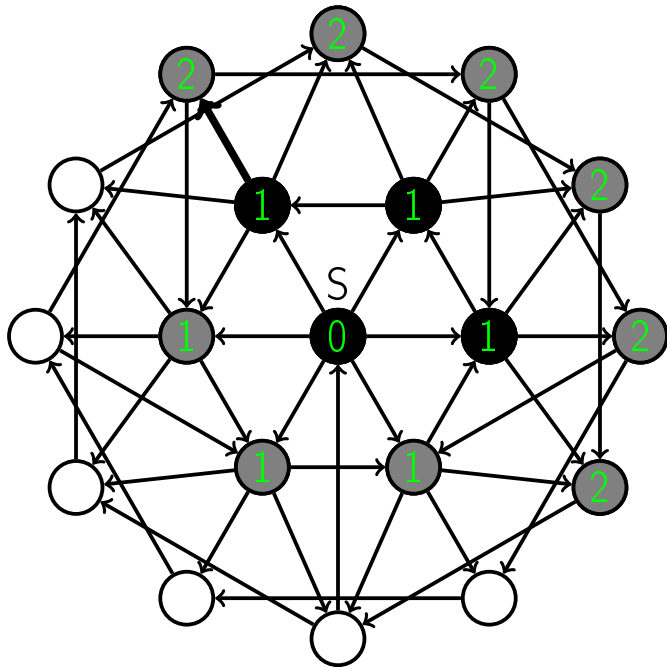


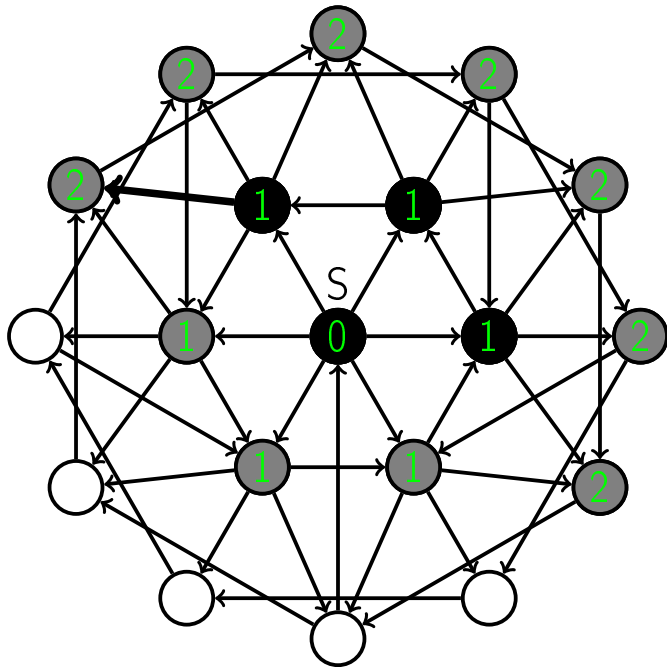


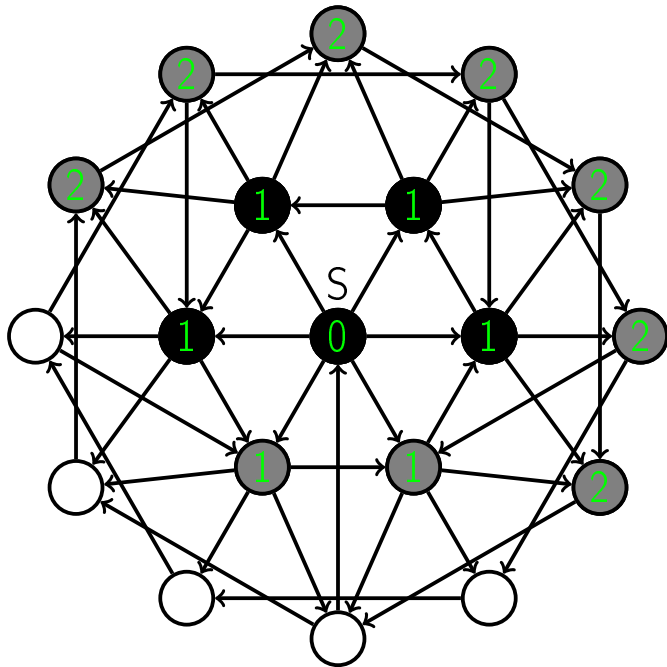


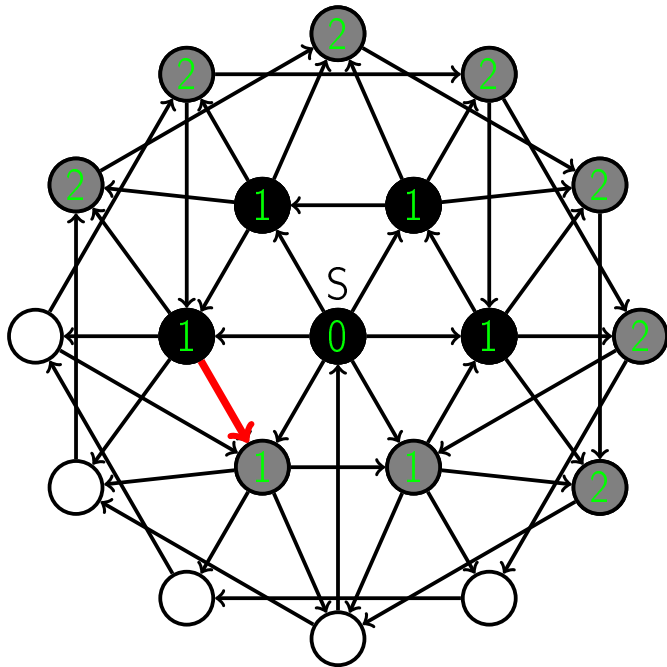


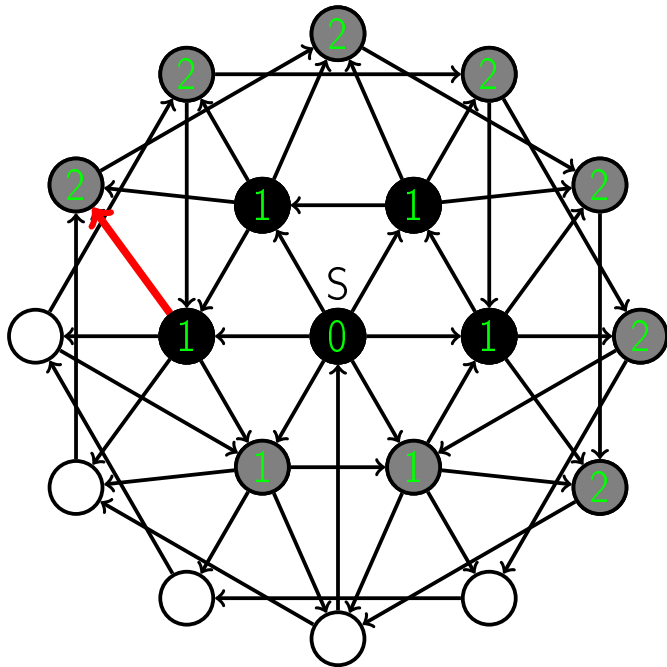


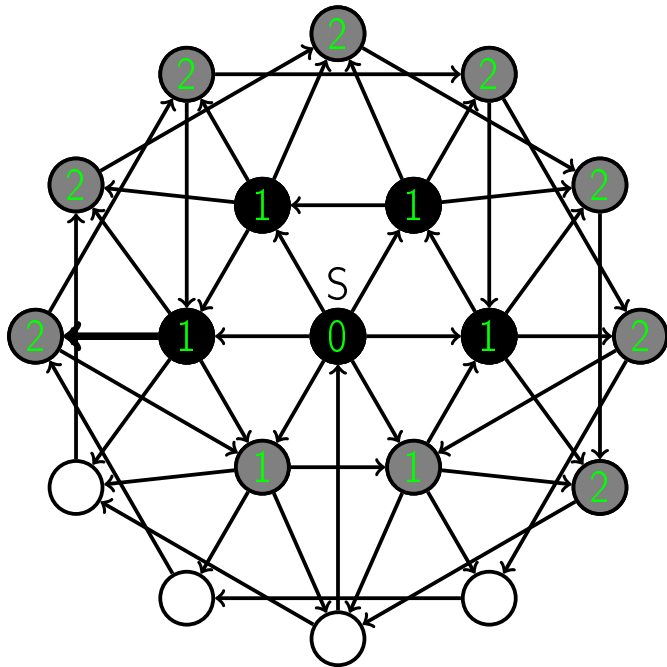


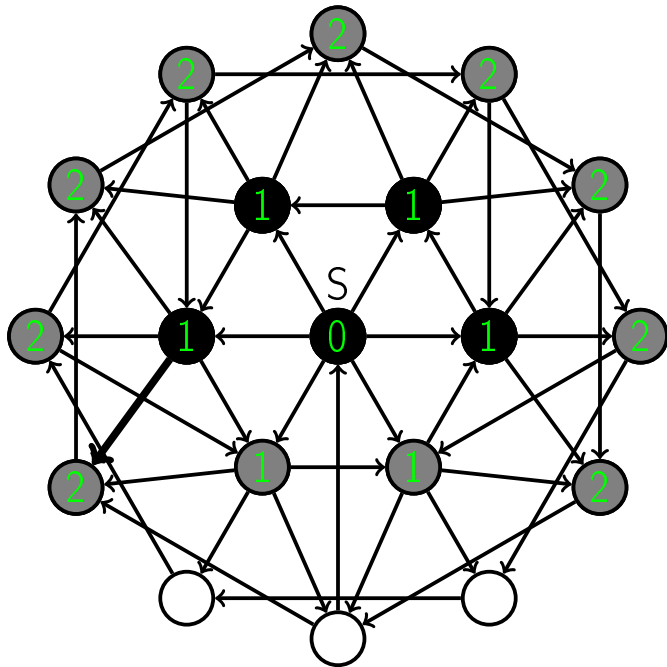


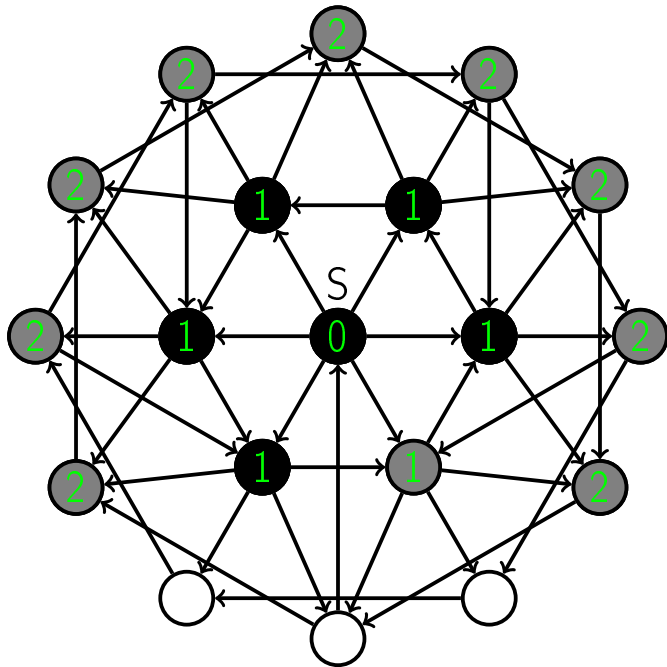


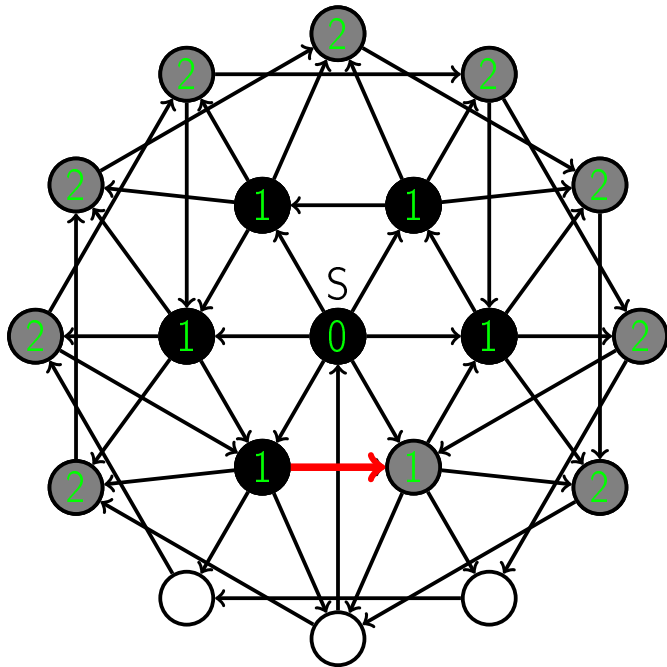


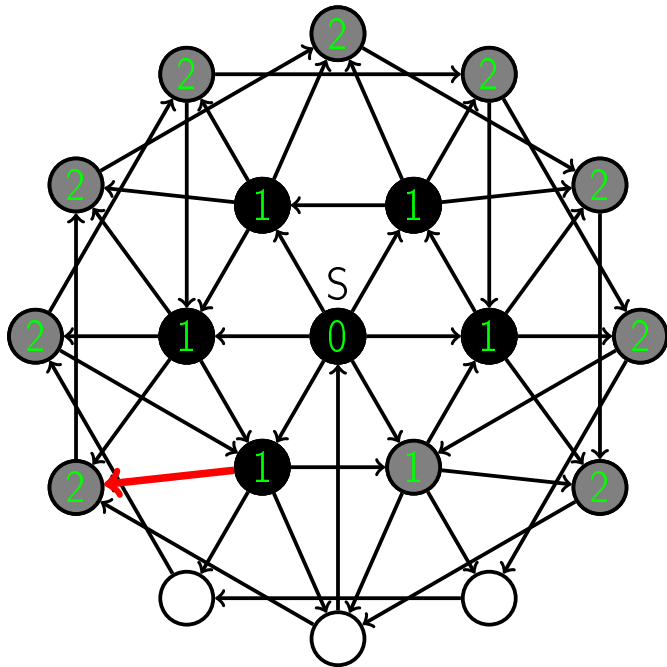


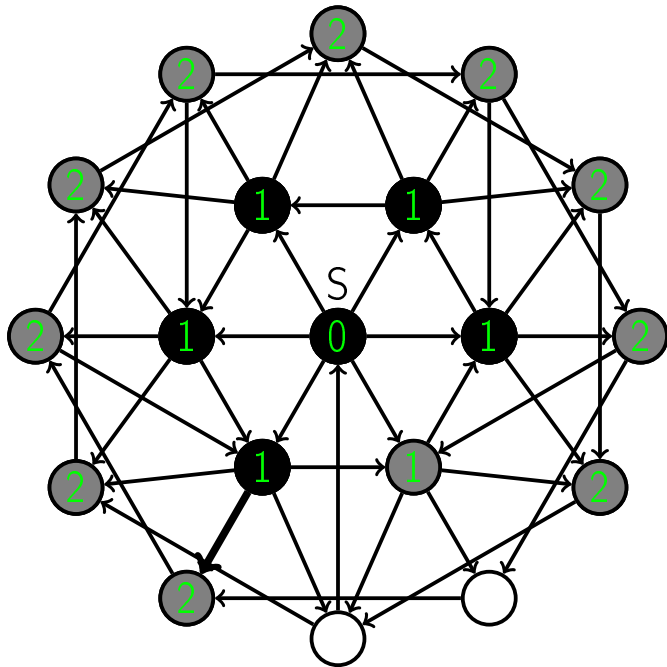


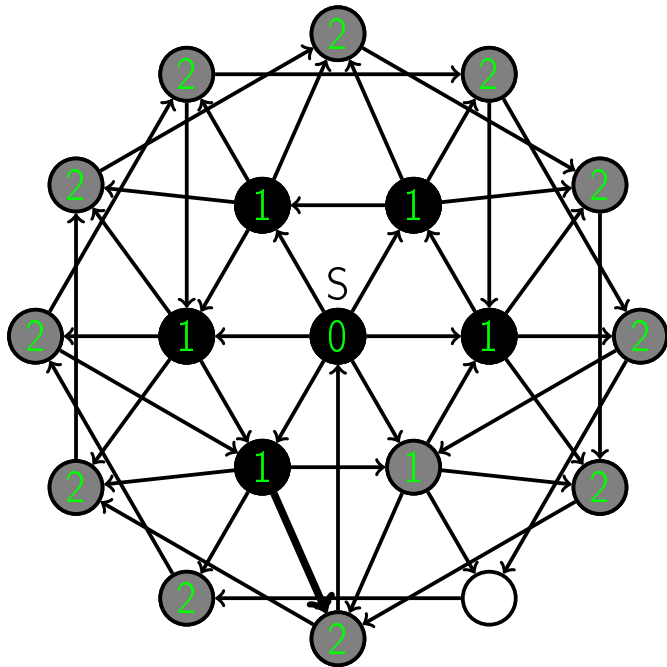


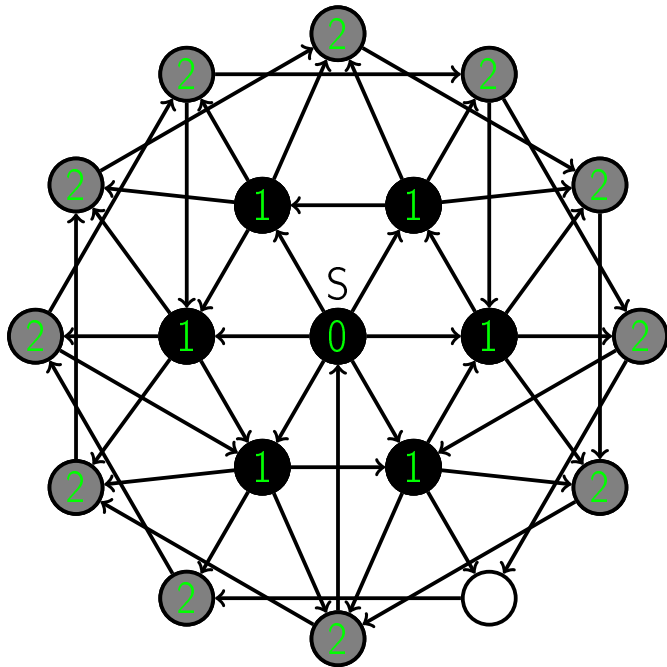


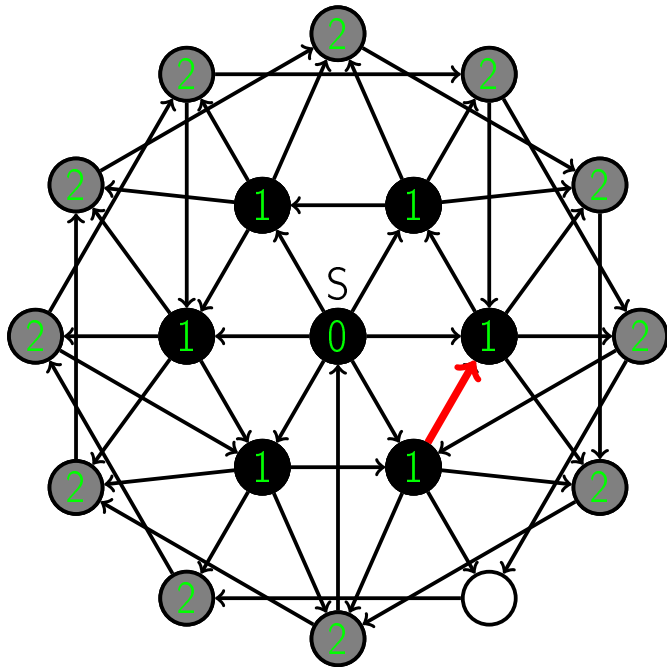


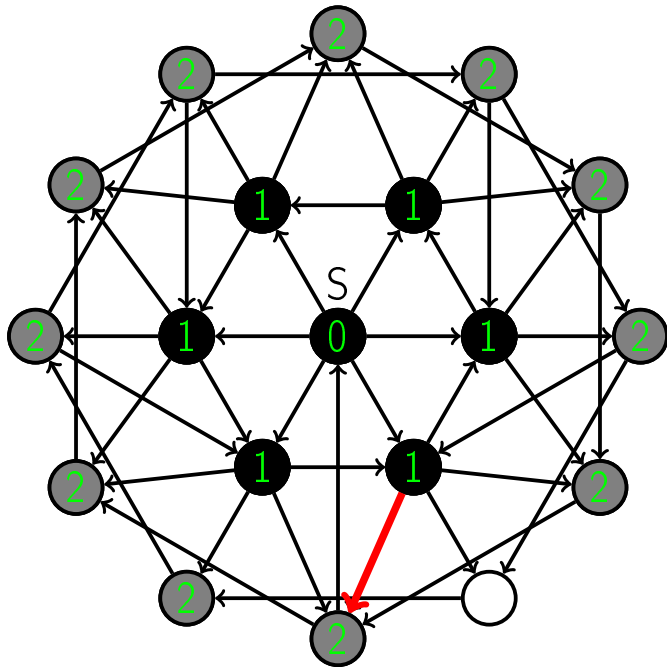


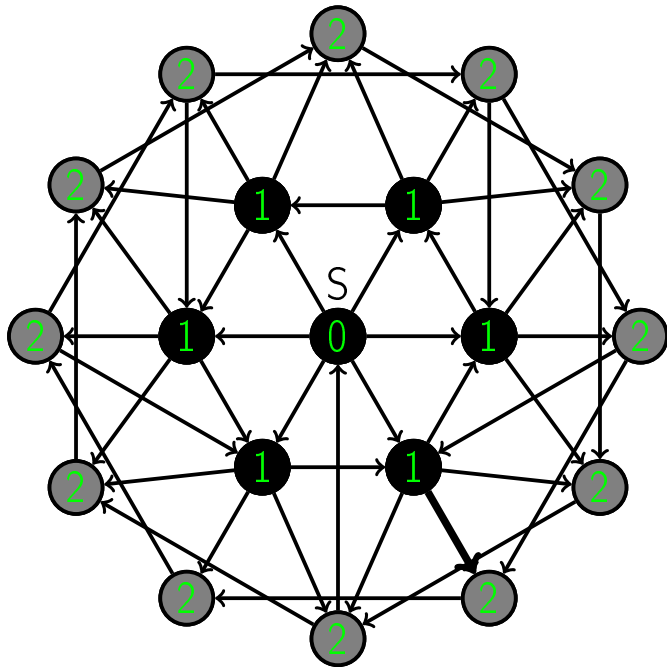


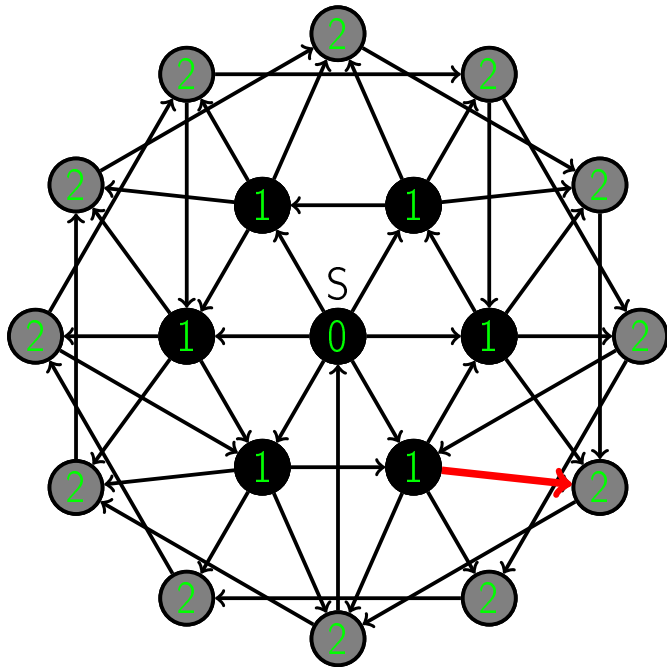


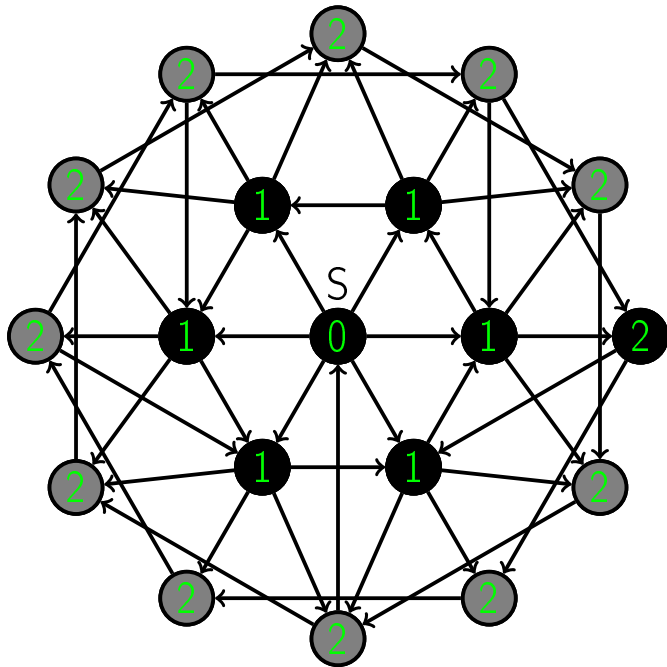


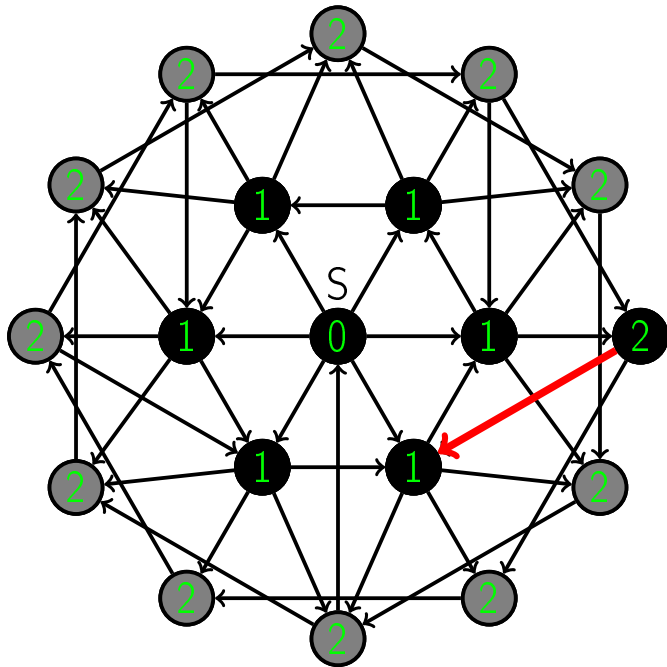


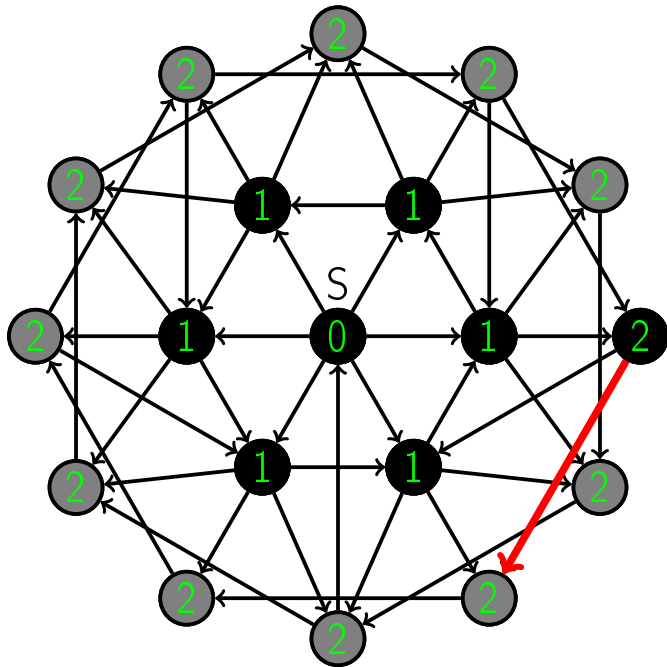


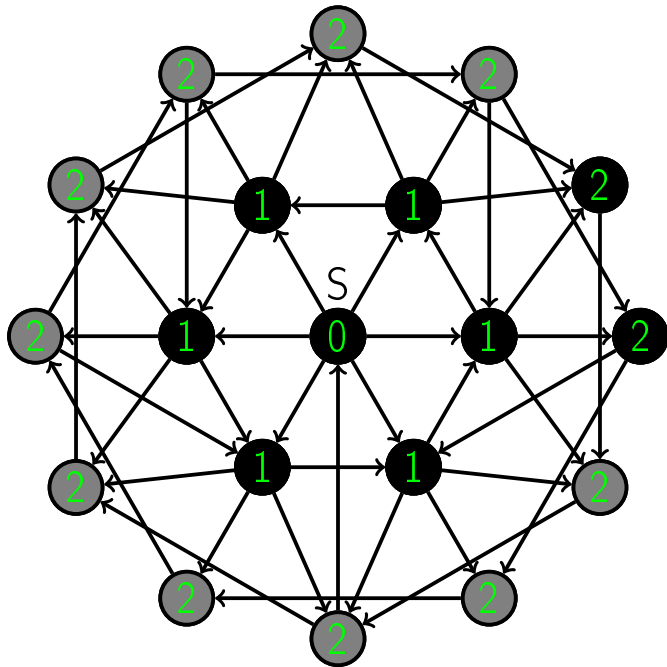


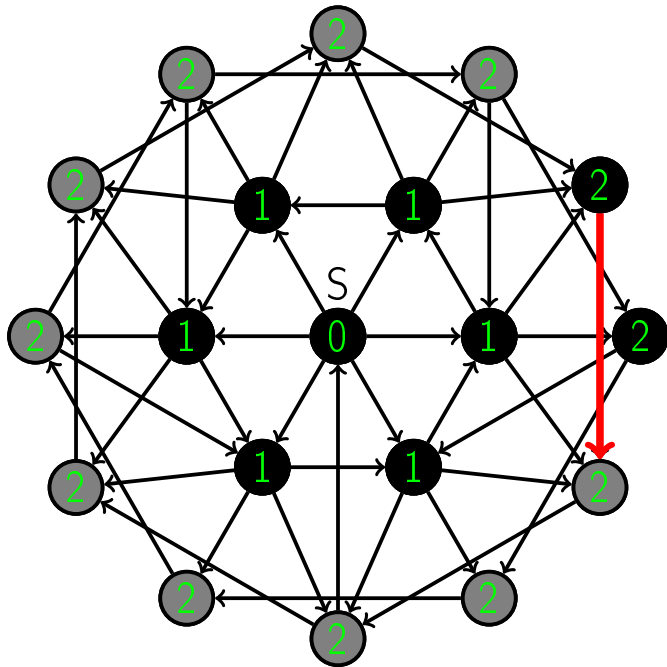


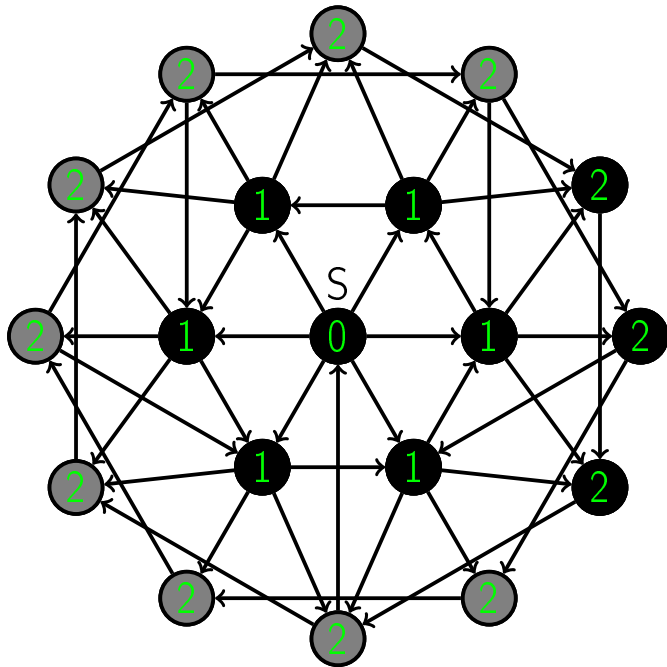


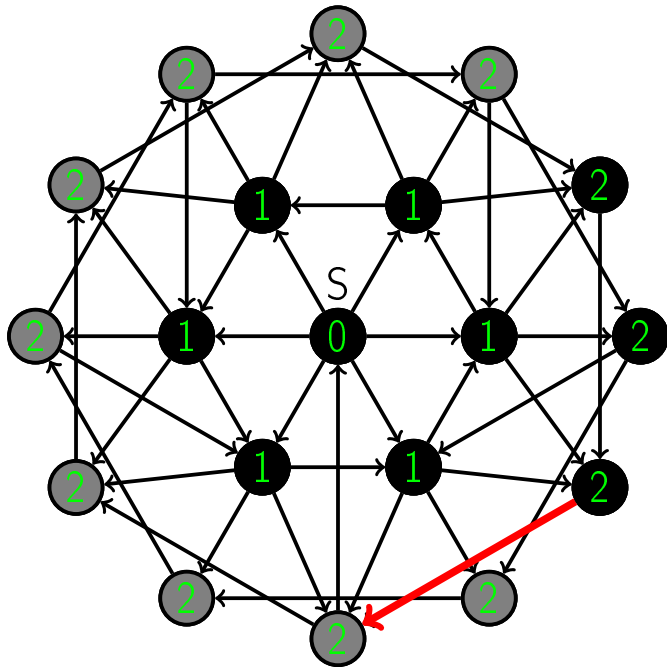


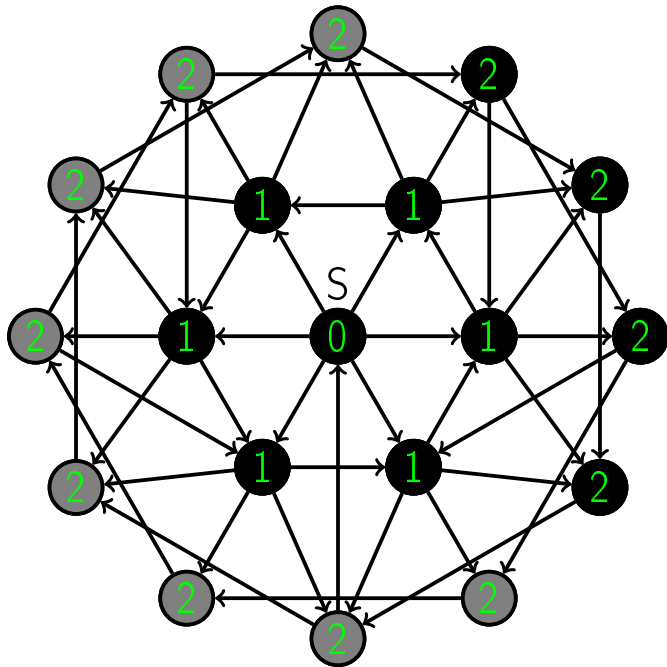


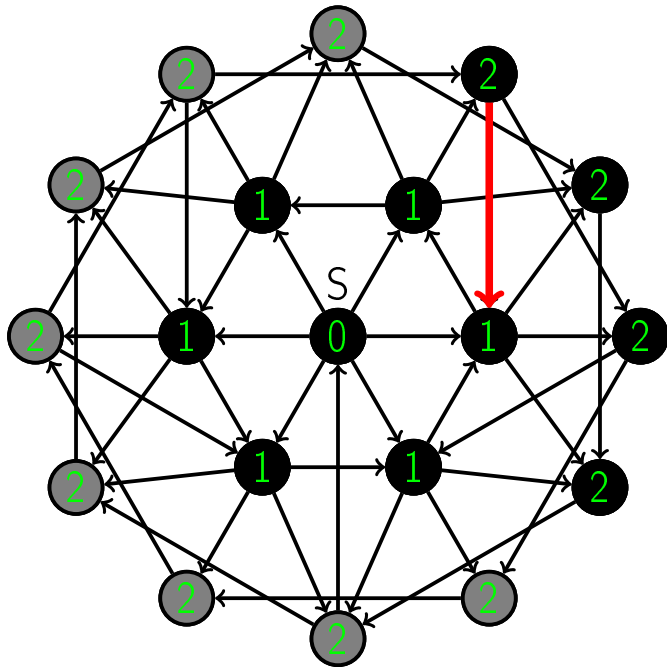


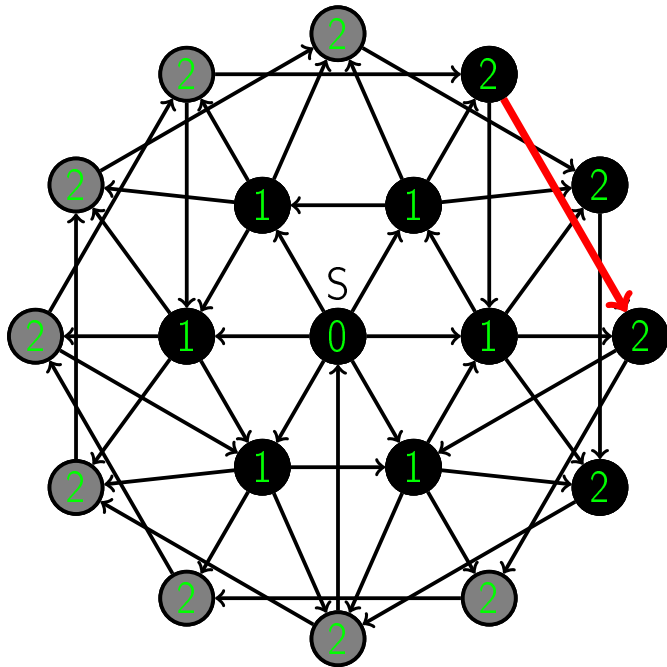


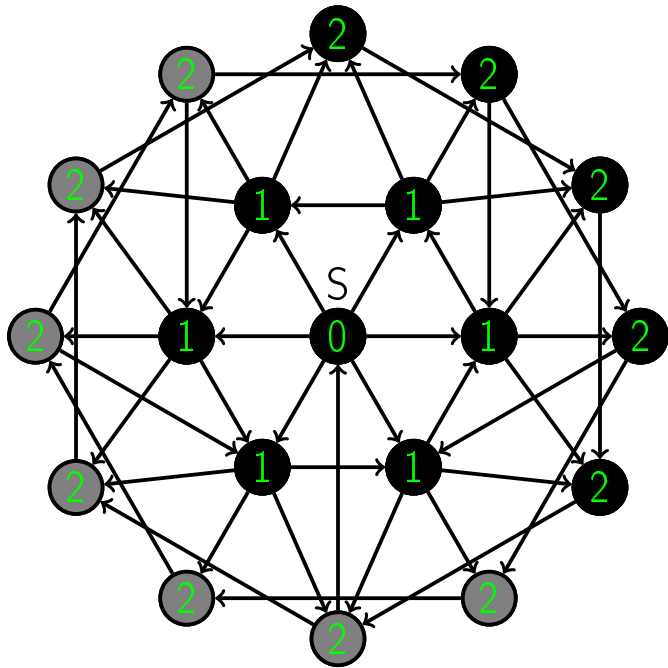


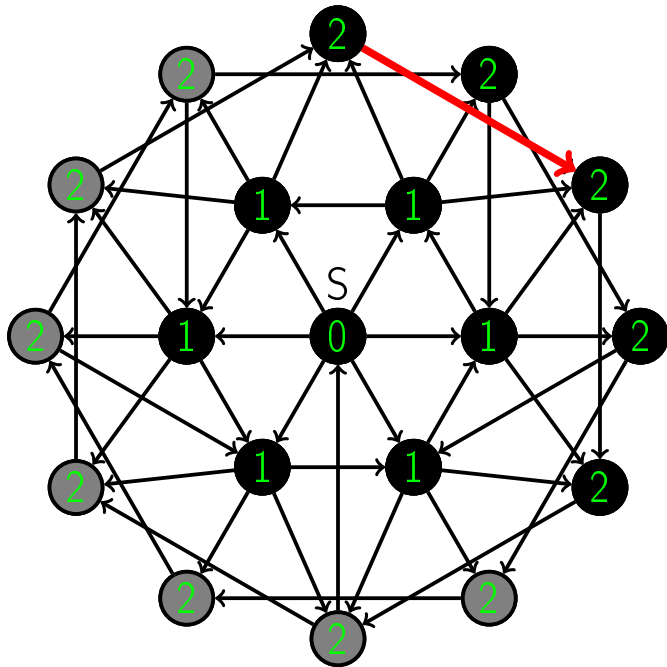


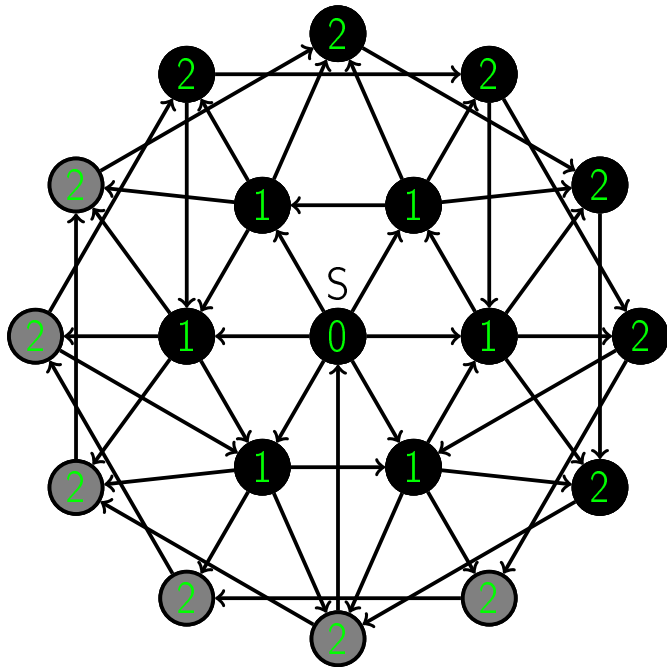


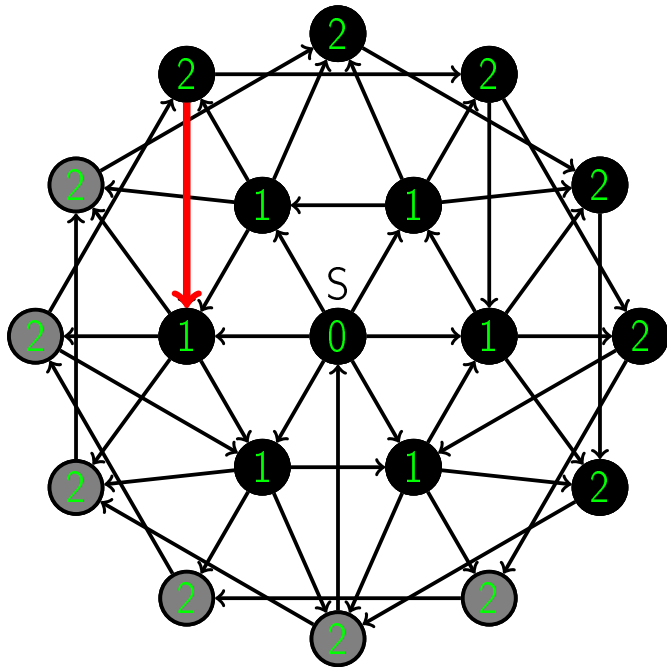


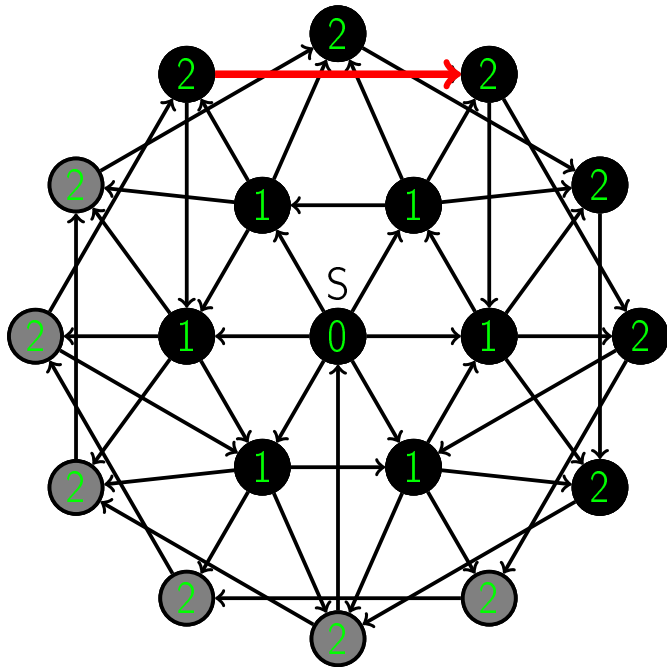


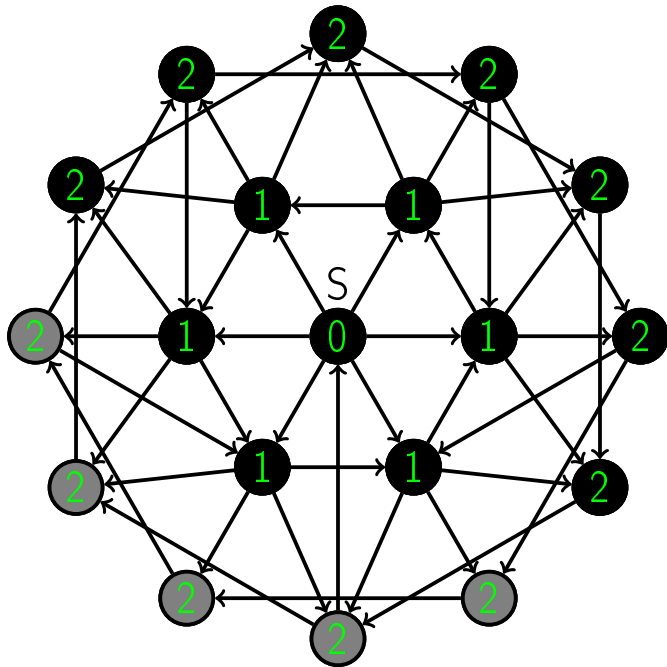


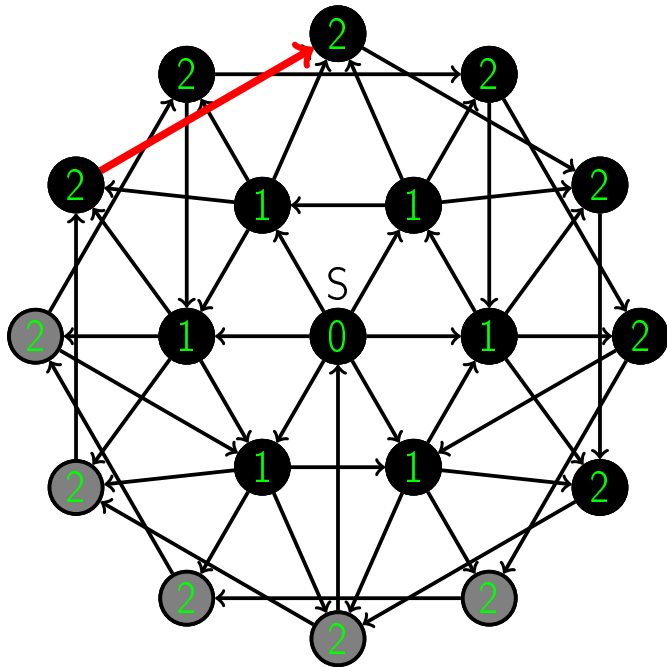


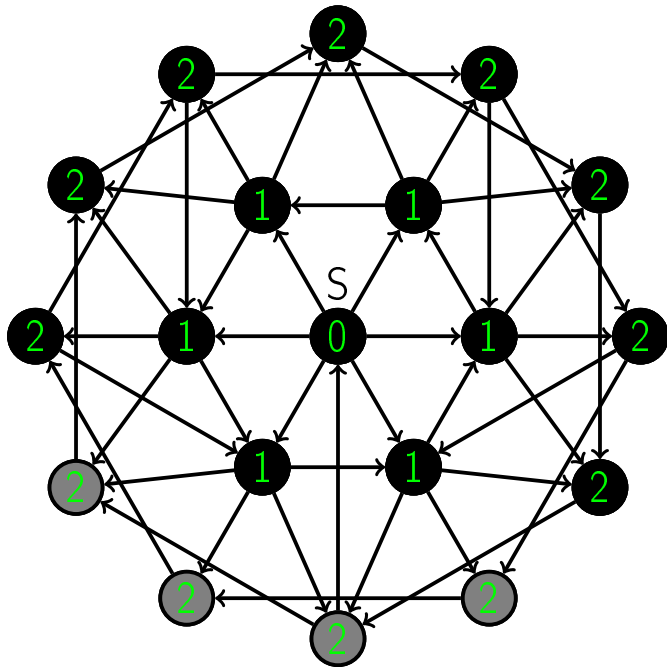


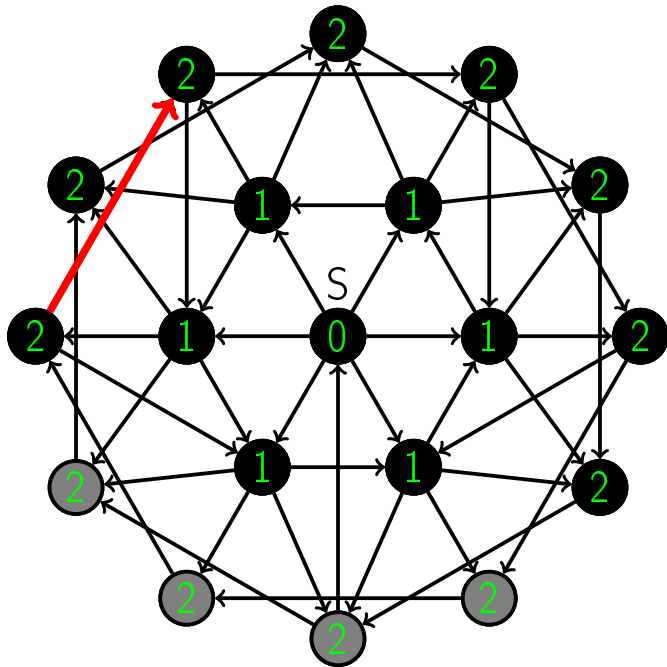


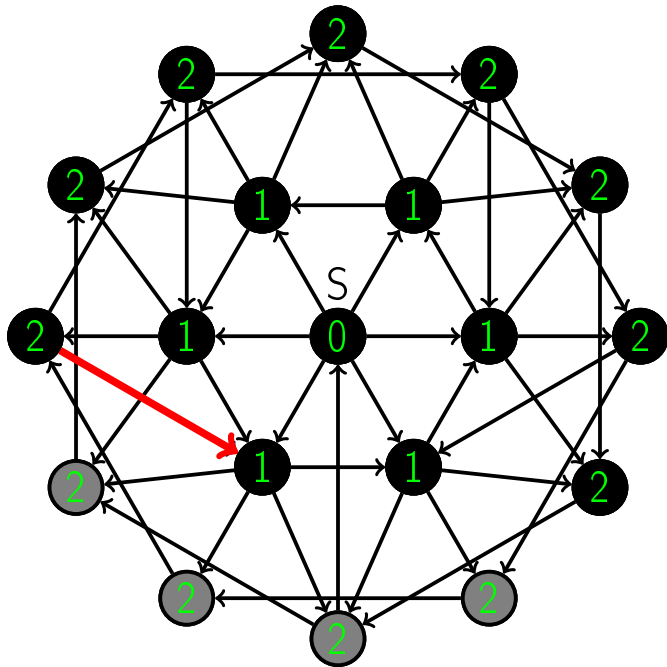


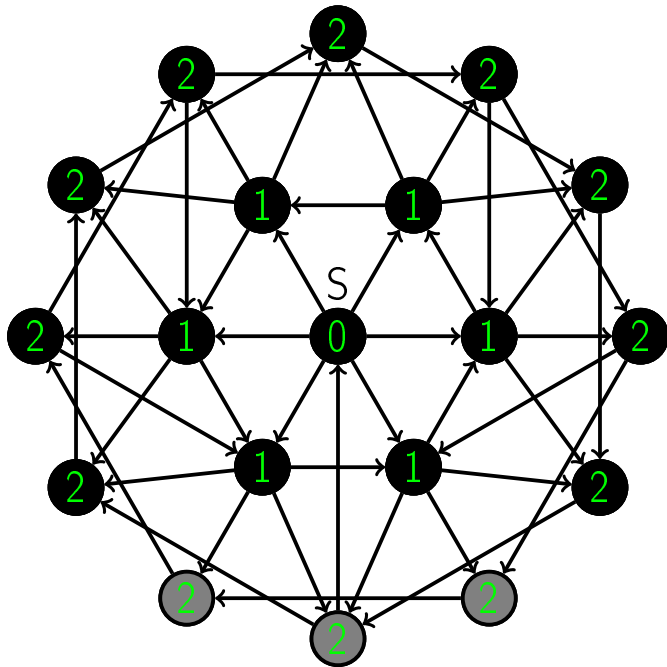


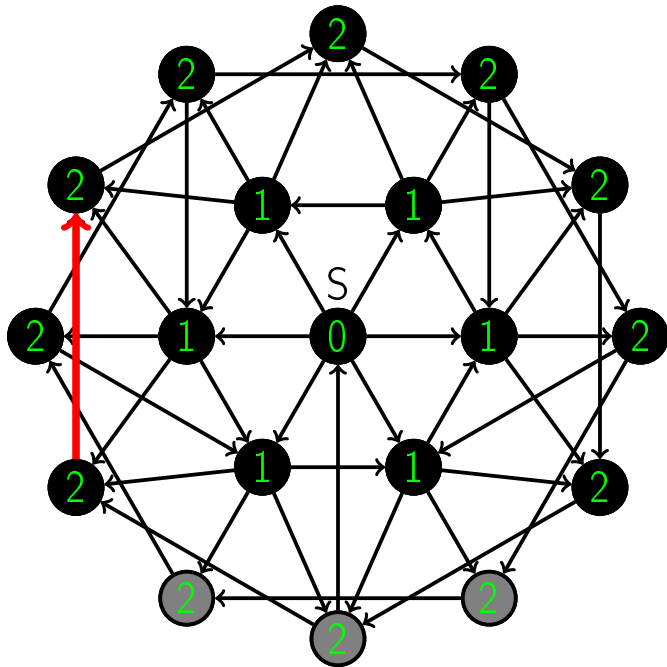


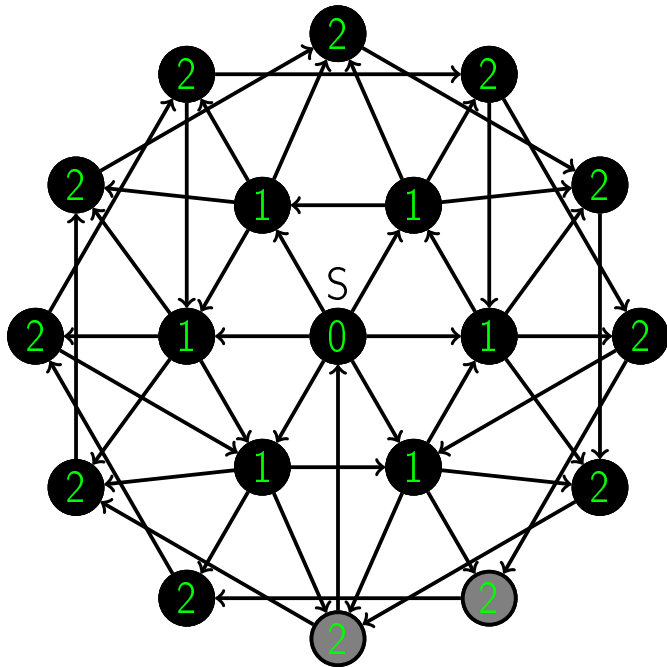


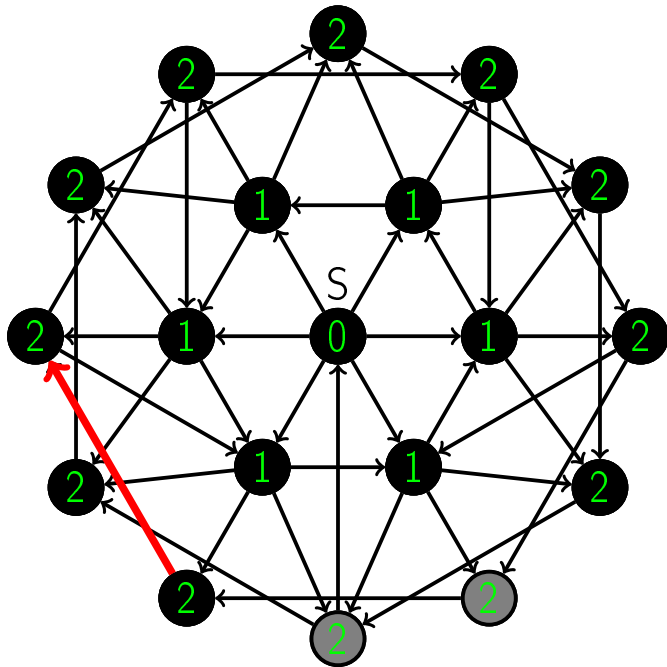


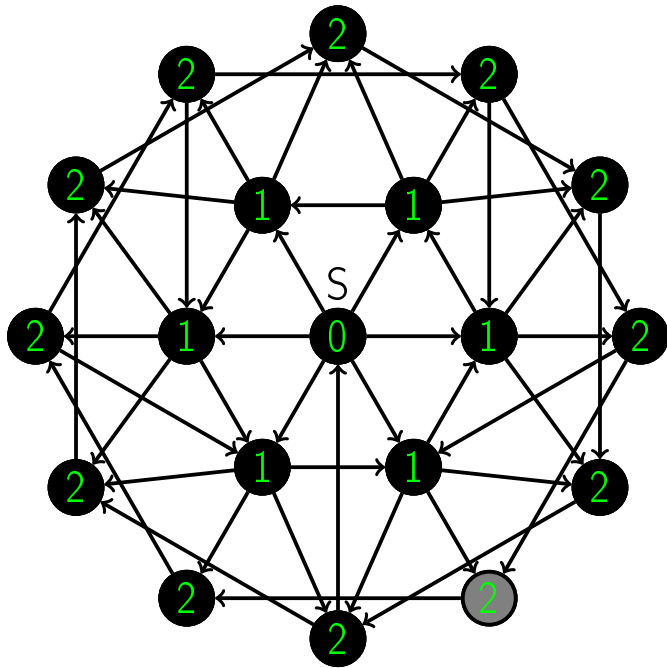


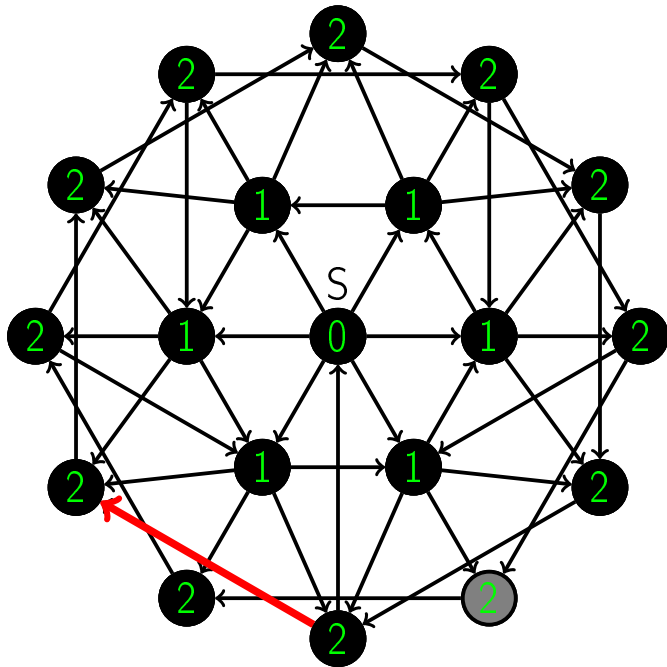


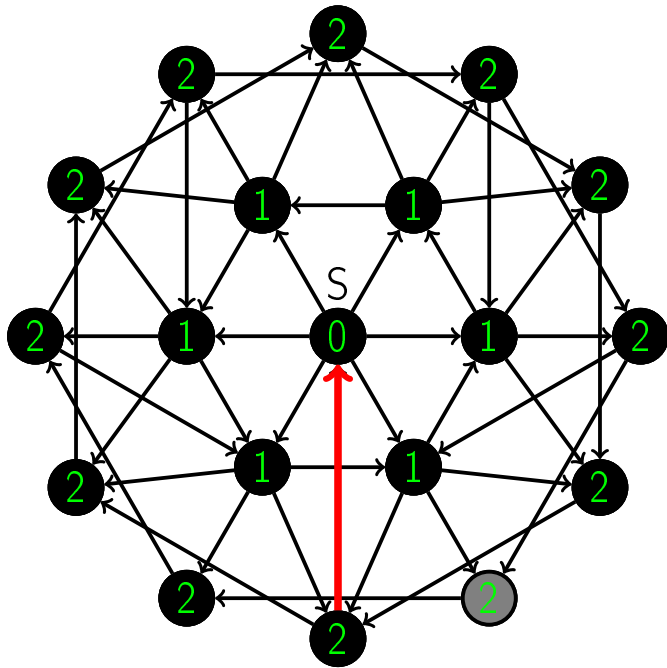


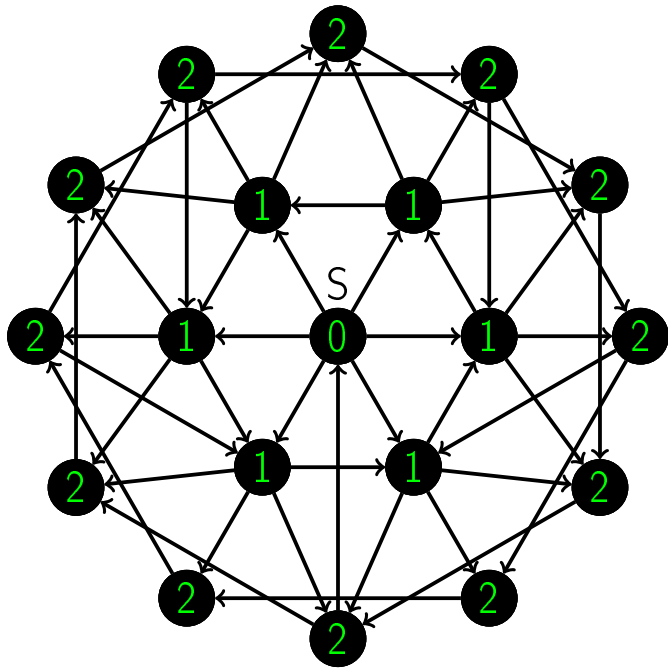


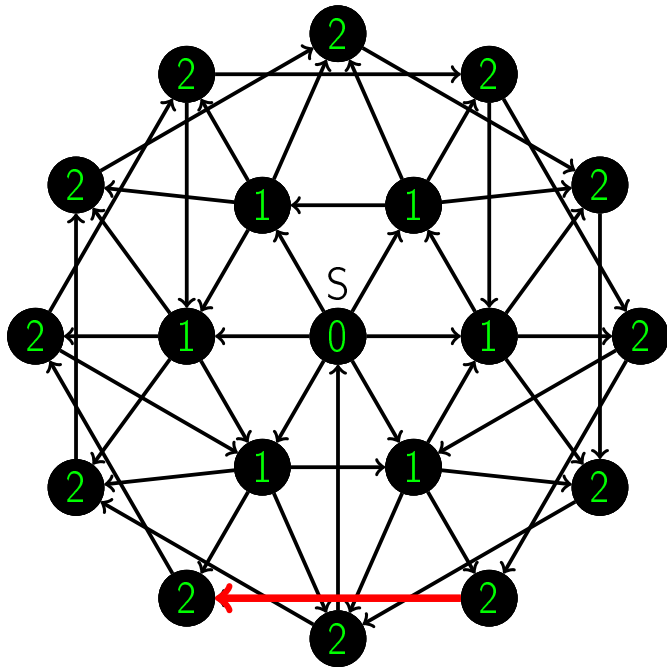


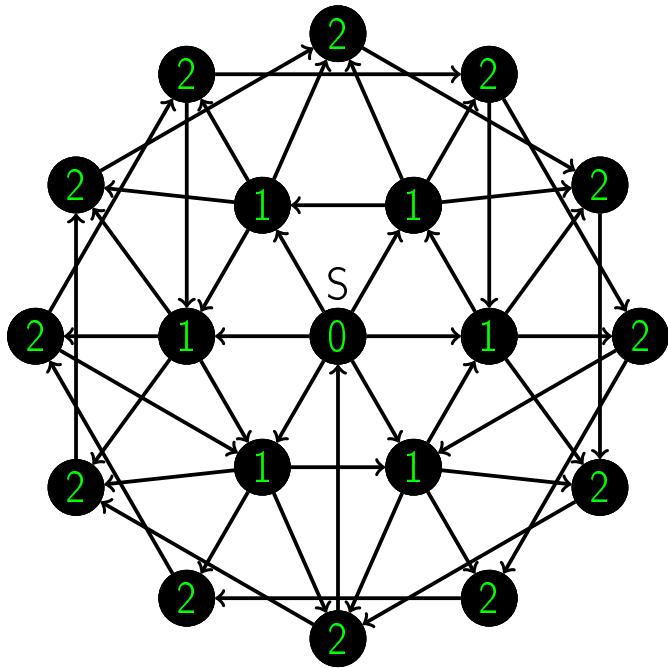


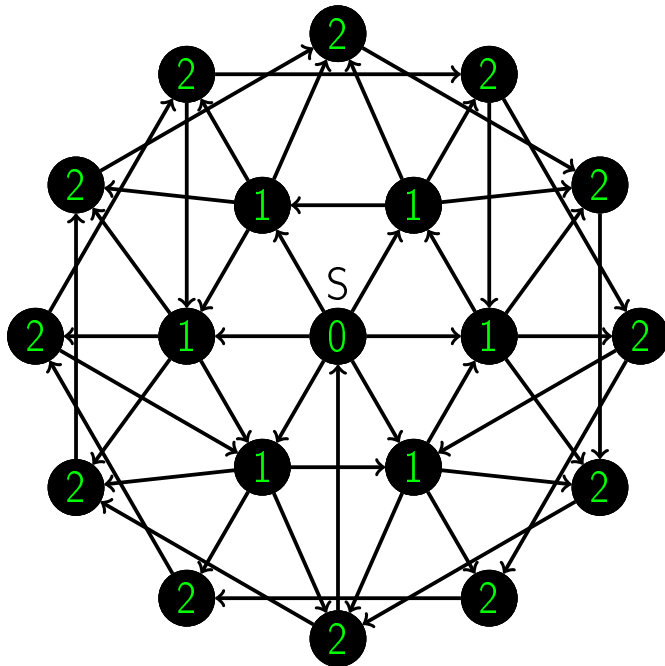












Outline

- 1 Paths and Distances
- 2 Breadth-first Search
- 3 Implementation and Analysis
- 4 Proof of Correctness
- 5 Shortest-path Tree

Breadth-first search

BFS(G, S)

```
for all  $u \in V$ :  
     $\text{dist}[u] \leftarrow \infty$   
 $\text{dist}[S] \leftarrow 0$   
 $Q \leftarrow \{S\}$  {queue containing just  $S$ }  
while  $Q$  is not empty:  
     $u \leftarrow \text{Dequeue}(Q)$   
    for all  $(u, v) \in E$ :  
        if  $\text{dist}[v] = \infty$ :  
             $\text{Enqueue}(Q, v)$   
             $\text{dist}[v] \leftarrow \text{dist}[u] + 1$ 
```

Running time

Lemma

The running time of breadth-first search is $O(|E| + |V|)$.

Proof

Running time

Lemma

The running time of breadth-first search is $O(|E| + |V|)$.

Proof


- Each vertex is enqueued at most once

Running time

Lemma

The running time of breadth-first search is $O(|E| + |V|)$.

Proof

- Each vertex is enqueued at most once
 - Each edge is examined either once (for directed graphs) or twice (for undirected graphs)
- 

Outline

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Reachability

Definition

Node u is **reachable** from node S if there is a path from S to u

Lemma

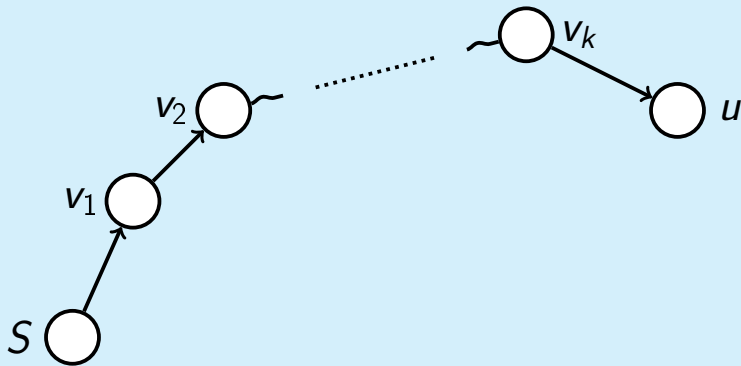
Reachable nodes are discovered at some point, so they get a finite distance estimate from the source. Unreachable nodes are not discovered at any point, and the distance to them stays infinite.

Proof



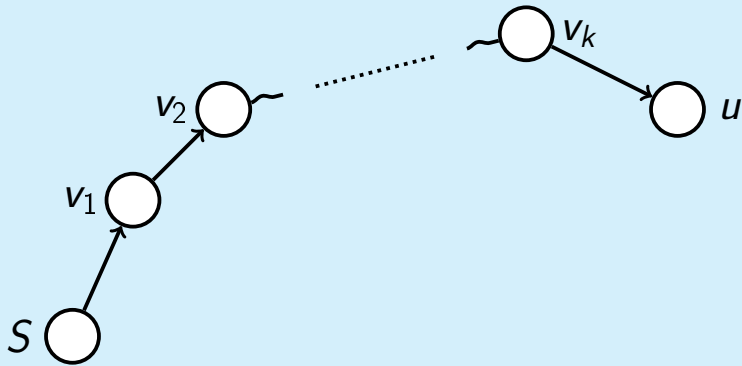
- u — reachable undiscovered closest to S

Proof



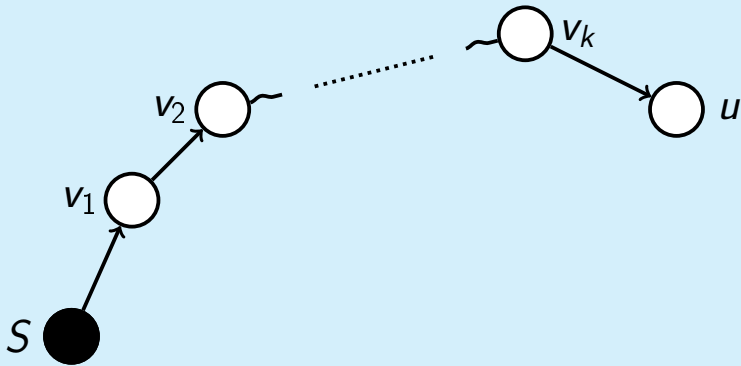
- u — reachable undiscovered closest to S
- $S - v_1 - \dots - v_k - u$ — shortest path

Proof



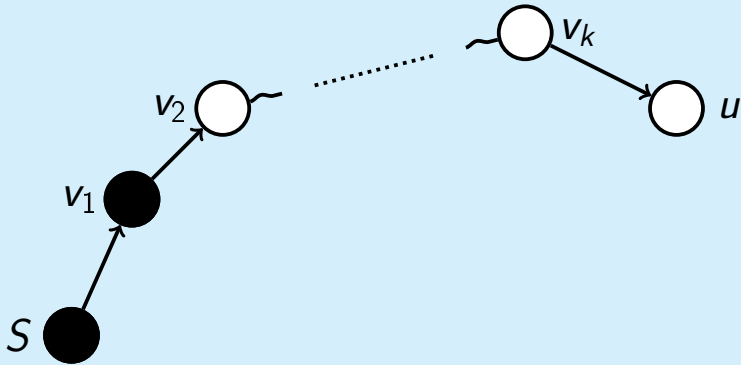
- u — reachable undiscovered closest to S
- $S - v_1 - \dots - v_k - u$ — shortest path
- u is discovered while processing v_k

Proof



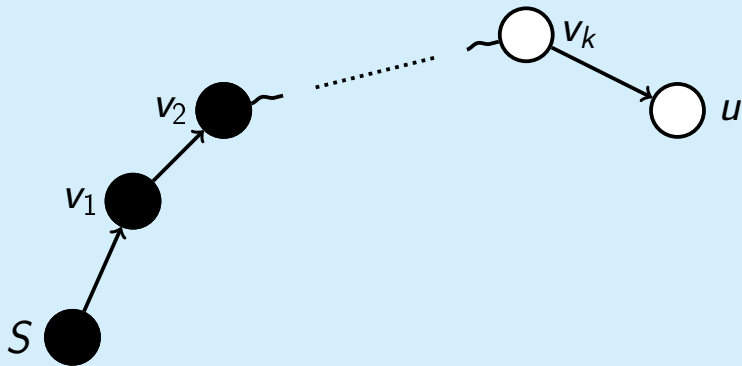
- u — reachable undiscovered closest to S
- $S - v_1 - \dots - v_k - u$ — shortest path
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Proof



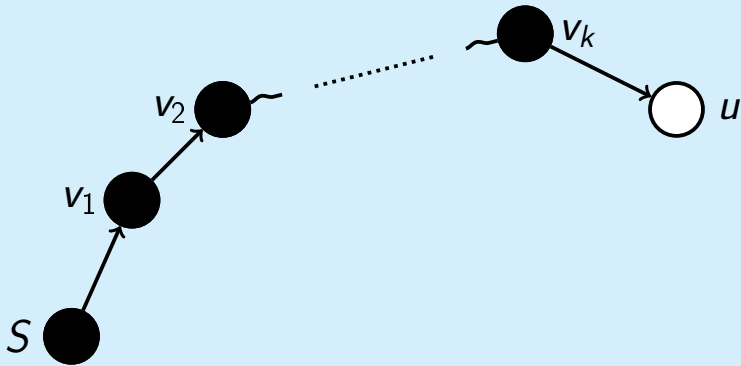
- u — reachable undiscovered closest to S
- $S - v_1 - \dots - v_k - u$ — shortest path
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Proof



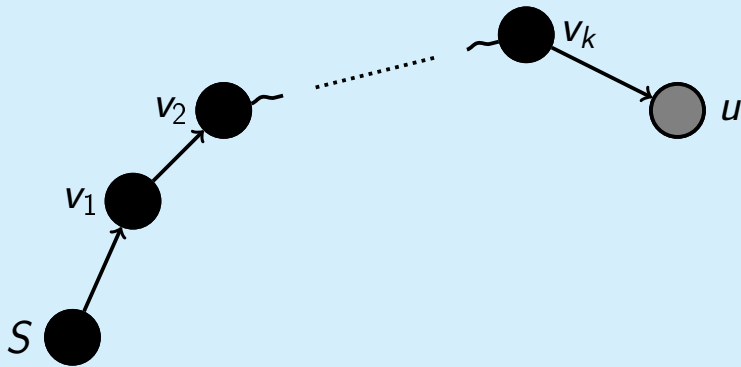
- u — reachable undiscovered closest to S
- $S - v_1 - \dots - v_k - u$ — shortest path
- u is discovered while processing v_k

Proof



- u — reachable undiscovered closest to S
- $S - v_1 - \dots - v_k - u$ — shortest path
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Proof



- u — reachable undiscovered closest to S
- $S - v_1 - \dots - v_k - u$ — shortest path
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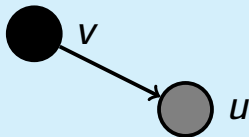
Proof

s ●

● u

- u — first unreachable discovered

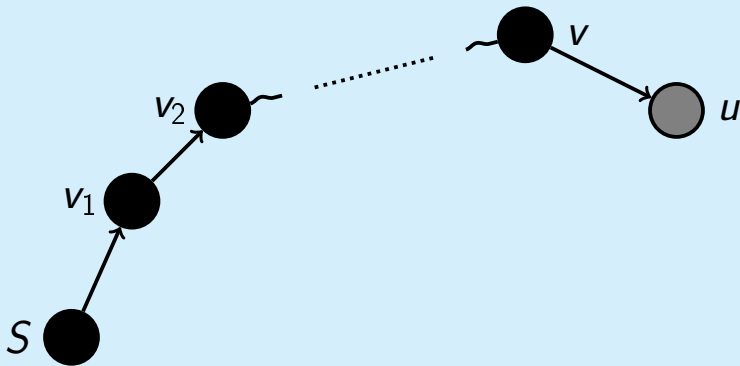
Proof



s ●

- u — first unreachable discovered
- u was discovered while processing v

Proof



- u — first unreachable discovered
- u was discovered while processing v
- u is reachable through v



Order Lemma

Lemma

By the time node u at distance d from S is dequeued, all the nodes at distance at most d have already been discovered (enqueued).

Order Lemma Proof



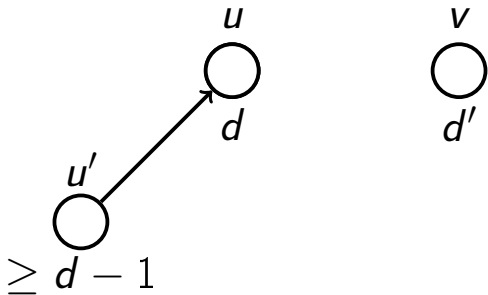
Consider the first time the order was broken

Order Lemma Proof



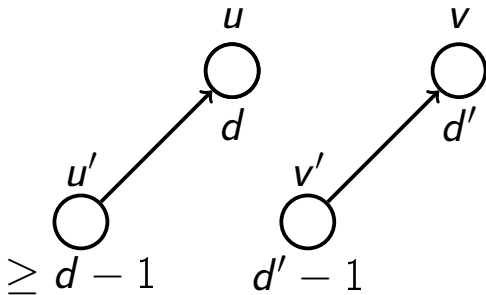
Consider the first time the order was broken
 $d' \leq d$

Order Lemma Proof



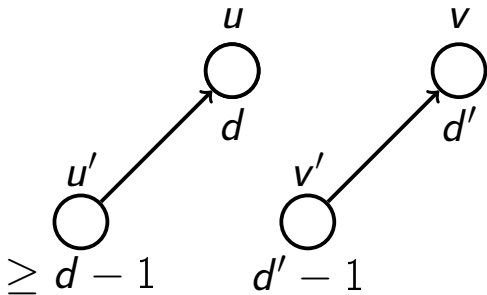
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Order Lemma Proof



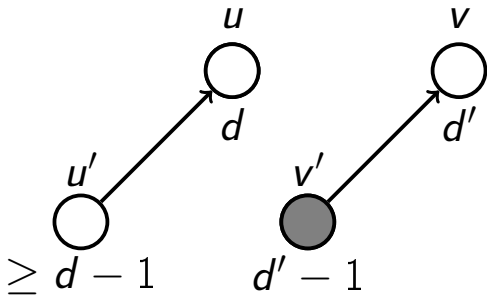
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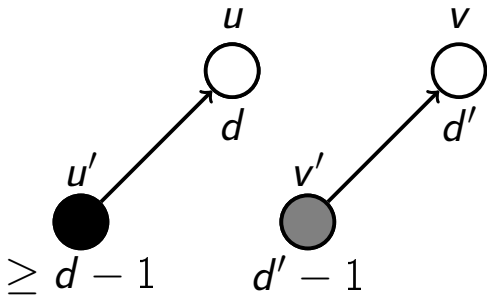
Consider the first time the order was broken
 $d' \leq d \Rightarrow d' - 1 \leq d - 1$, so v' was
discovered before u' was dequeued

Order Lemma Proof



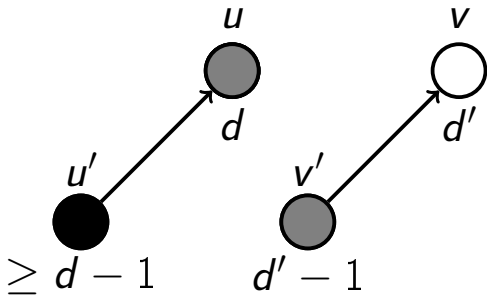
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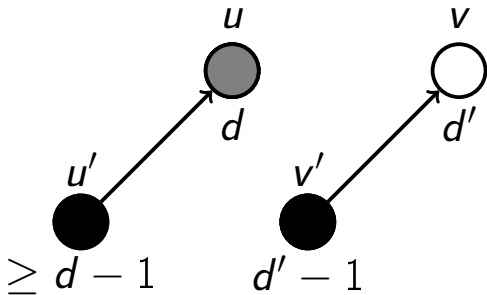
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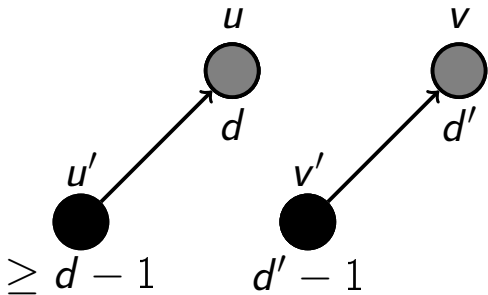
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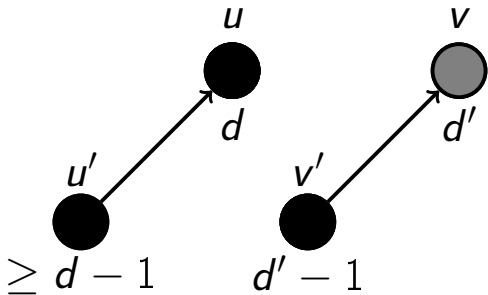
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Order Lemma Proof



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Order Lemma Proof



Consider the first time the order was broken
 $d' \leq d \Rightarrow d' - 1 \leq d - 1$, so v' was
discovered before u' was dequeued

Correct distances

Lemma

When node u is discovered (enqueued), $\text{dist}[u]$ is assigned exactly $d(S, u)$.

Correct distances

Proof

- Use mathematical induction

Correct distances

Proof

- Use mathematical induction
- Base: when S is discovered, $\text{dist}[S]$ is assigned $0 = d(S, S)$

Correct distances

Proof

- Use mathematical induction
- Base: when S is discovered, $\text{dist}[S]$ is assigned $0 = d(S, S)$
- Inductive step: suppose proved for all nodes at distance $\leq k$ from $S \rightarrow$ prove for nodes at distance $k + 1$

Correct distances

Proof

- Take a node v at distance $k + 1$ from S

Correct distances

Proof

- Take a node v at distance $k + 1$ from S
- v was discovered while processing u

Correct distances

Proof

- Take a node v at distance $k + 1$ from S
- v was discovered while processing u
- $d(S, v) \leq d(S, u) + 1 \Rightarrow d(S, u) \geq k$

Correct distances

Proof

- Take a node v at distance $k + 1$ from S
- v was discovered while processing u
- $d(S, v) \leq d(S, u) + 1 \Rightarrow d(S, u) \geq k$
- v is discovered after u is dequeued, so $d(S, u) < d(S, v) = k + 1$

Correct distances

Proof

- Take a node v at distance $k + 1$ from S
- v was discovered while processing u
- $d(S, v) \leq d(S, u) + 1 \Rightarrow d(S, u) \geq k$
- v is discovered after u is dequeued, so $d(S, u) < d(S, v) = k + 1$
- So $d(S, u) = k$, and $\text{dist}[v] \leftarrow \text{dist}[u] + 1 = k + 1$



Queue property

Queue:

d	d	d	\dots	d	d	$d + 1$	$d + 1$	\dots	$d + 1$
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Lemma

At any moment, if the first node in the queue is at distance d from S , then all the nodes in the queue are either at distance d from S or at distance $d + 1$ from S . All the nodes in the queue at distance d go before (if any) all the nodes at distance $d + 1$.

Queue property

Proof

- All nodes at distance d were enqueued before first such node is dequeued, so they go before nodes at distance $d + 1$

Queue property

Proof

- All nodes at distance d were enqueued before first such node is dequeued, so they go before nodes at distance $d + 1$
- Nodes at distance $d - 1$ were enqueued before nodes at d , so they are not in the queue anymore

Queue property

Proof

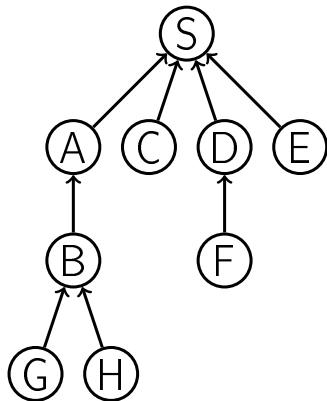
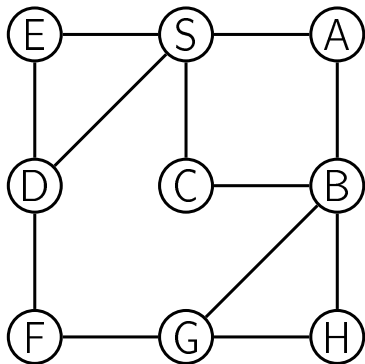
- All nodes at distance d were enqueued before first such node is dequeued, so they go before nodes at distance $d + 1$
- Nodes at distance $d - 1$ were enqueued before nodes at d , so they are not in the queue anymore
- Nodes at distance $> d + 1$ will be discovered when all d are gone



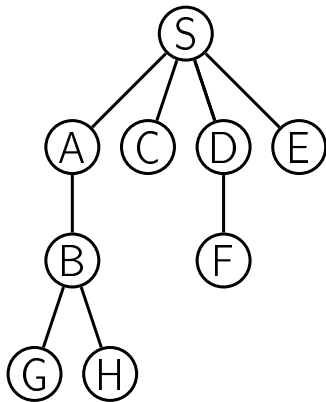
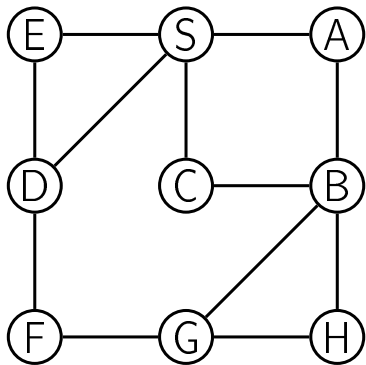
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Shortest-path tree



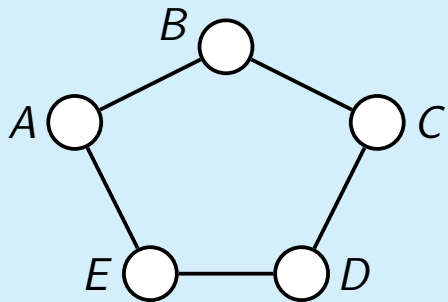
Shortest-path tree



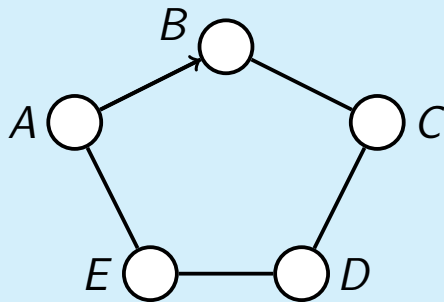
Lemma

Shortest-path tree is indeed a tree, i.e. it doesn't contain cycles (it is a connected component by construction).

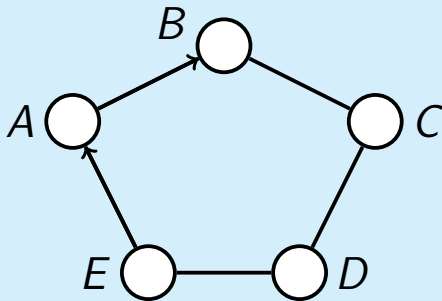
Proof



Proof

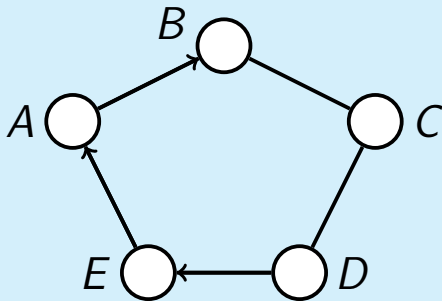


Proof



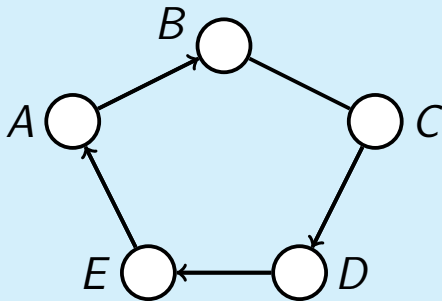
- Only one outgoing edge from each node

Proof



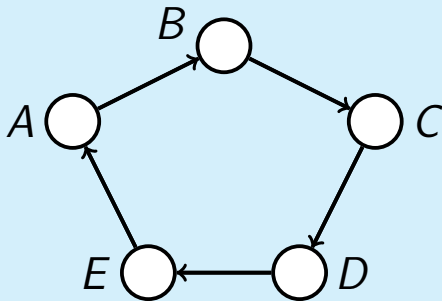
- Only one outgoing edge from each node

Proof



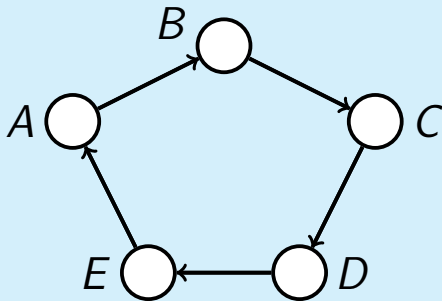
- Only one outgoing edge from each node

Proof



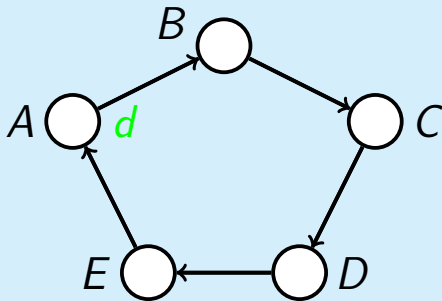
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Proof



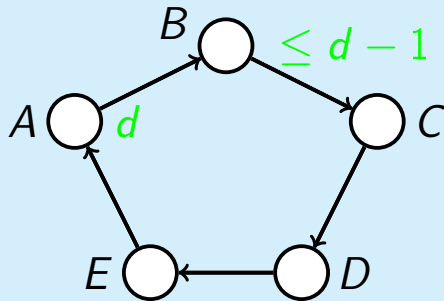
- Only one outgoing edge from each node
- Distance to S decreases after going by edge

Proof



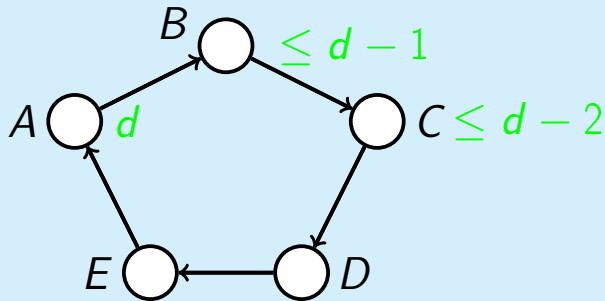
- Only one outgoing edge from each node
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Proof



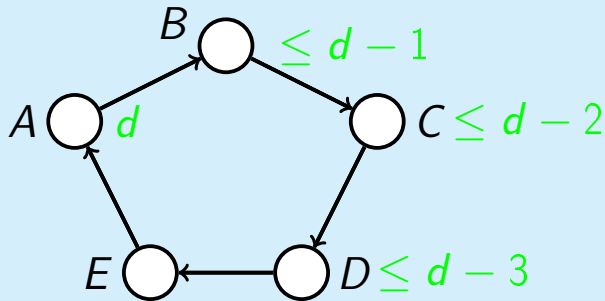
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Proof



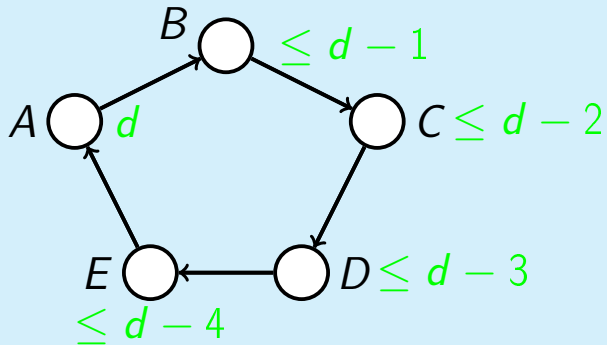
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Proof



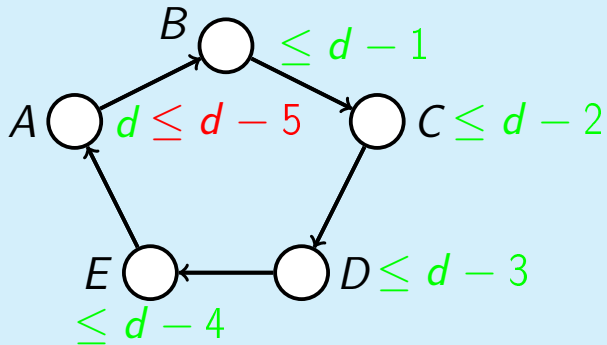
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Constructing shortest-path tree

BFS(G, S)

```
for all  $u \in V$ :  
     $\text{dist}[u] \leftarrow \infty$ ,  $\text{prev}[u] \leftarrow \text{nil}$   
 $\text{dist}[S] \leftarrow 0$   
 $Q \leftarrow \{S\}$  {queue containing just  $S$ }  
while  $Q$  is not empty:  
     $u \leftarrow \text{Dequeue}(Q)$   
    for all  $(u, v) \in E$ :  
        if  $\text{dist}[v] = \infty$ :  
             $\text{Enqueue}(Q, v)$   
             $\text{dist}[v] \leftarrow \text{dist}[u] + 1$ ,  $\text{prev}[v] \leftarrow u$ 
```

Reconstructing Shortest Path

ReconstructPath(S, u, prev)

```
result  $\leftarrow$  empty  
while  $u \neq S$ :  
    result.append( $u$ )  
     $u \leftarrow \text{prev}[u]$   
return Reverse(result)
```

Conclusion

- Can find the minimum number of flight segments to get from one city to another

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- Works in $O(|E| + |V|)$