# Linear Programming: Convex Polytopes

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Advanced Algorithms and Complexity
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## Learning Objectives

- Understand what a convex polytope is and why it is relevant to linear programming.
- Get a feel for what a convex polytope looks like.
- Prove some basic facts about convex polytopes.

# Linear Programs

Optimize linear function given linear inequality constraints.

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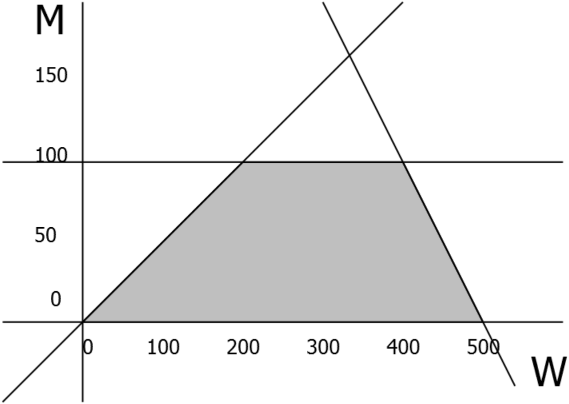
Optimize linear function given linear inequality constraints.

Want to understand region of points defined by inequalities.

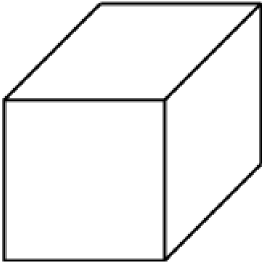
# Example

From factory example



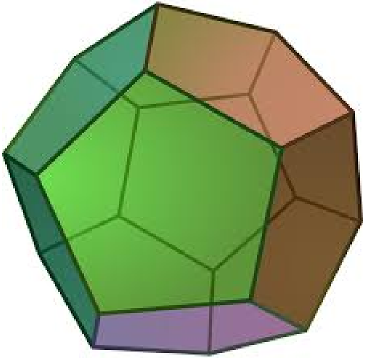
# Example II

Equations: $0 \leq x, y, z \leq 1$ give cube.



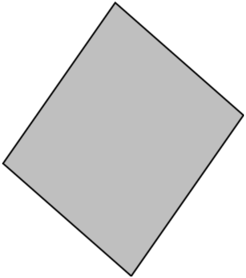
# In General

Get what's called a convex polytope



# Hyperplanes

A single linear equation defines a hyperplane.



# Hyperplanes

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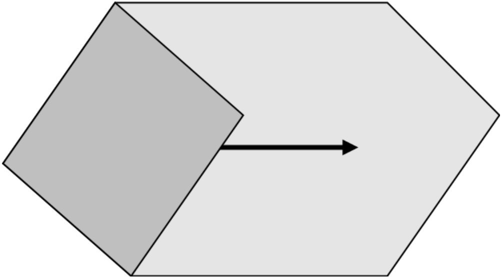


An inequality, defines a halfspace.

# Polytopes

So a system of linear inequalities, defines a region bounded by a bunch of hyperplanes.

**Definition**

A polytope is a region in $\mathbb{R}^n$ bounded by finitely many flat surfaces.

# Polytopes

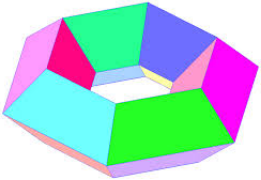
So a system of linear inequalities, defines a region bounded by a bunch of hyperplanes.

## Definition

A polytope is a region in $\mathbb{R}^n$ bounded by finitely many flat surfaces. These surfaces may intersect in lower dimensional facets (like edges), with zero-dimensional facets called vertices.

# More Conditions

But not every polytope is possible.

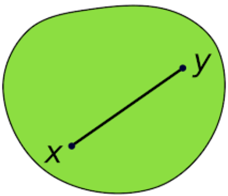


Everything must be on one side of each face.

# Convexity

## Definition

A region $\mathcal{C} \subset \mathbb{R}^n$ is convex, if for any $x, y \in \mathcal{C}$, the line segment connecting $x$ and $y$ is contained in $\mathcal{C}$.



# Convexity

**Lemma**

An intersection of halfspaces is convex.

# Proof

- Defined by $Ax \geq b$.
- Need for $x, y \in \mathcal{C}$ and $t \in [0, 1]$, $tx + (1 - t)y \in \mathcal{C}$.

$$
\begin{aligned}
A(tx + (1 - t)y) &= tAx + (1 - t)Ay \\
&\geq tb + (1 - t)b \\
&= b.
\end{aligned}
$$

# Convex Polytope

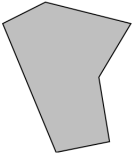
**Theorem**

The region defined by a system of linear inequalities is always a convex polytope.

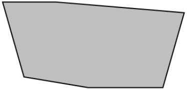
# Problem

Which of these figures is a convex polytope?

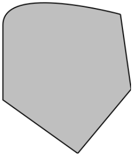
A



B

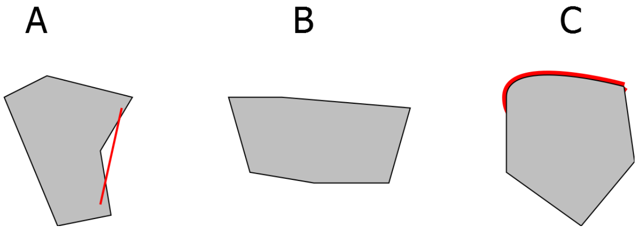


C



# Solution

Only B.



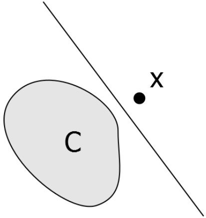
# Lemmas

We will conclude with a couple important lemmas about convex polytopes.

# Separation

**Lemma**

Let $\mathcal{C}$ be a convex region and $x \notin \mathcal{C}$ a point. Then there is a hyperplane $H$ separating $x$ from $\mathcal{C}$.



# Separation

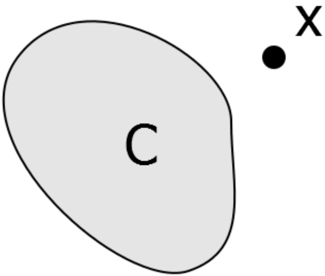
**Lemma**

Let $\mathcal{C}$ be a convex region and $x \notin \mathcal{C}$ a point. Then there is a hyperplane $H$ separating $x$ from $\mathcal{C}$.

Note that if $\mathcal{C}$ is given by a system of linear inequalities, we can just find one of the defining inequalities that $x$ violates.

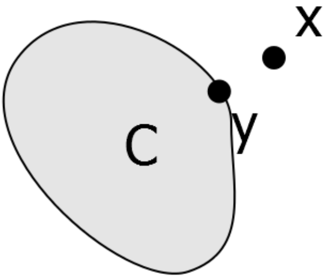
# Proof (Optional)

Start with $x$.



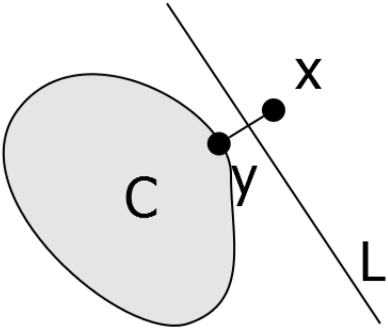
# Proof (Optional)

Let $y$ be closest point in $\mathcal{C}$



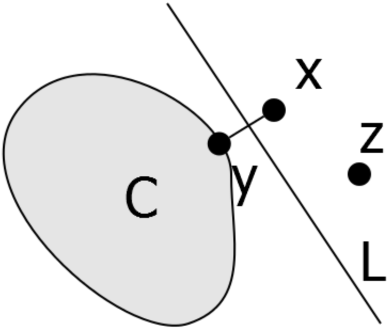
# Proof (Optional)

Let $L$ be the perpendicular bisector of $xy$.



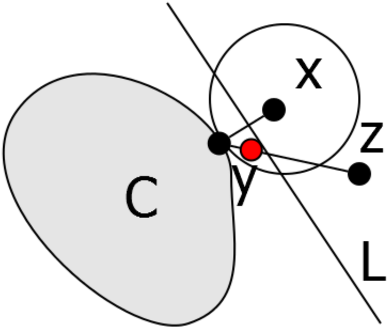
# Proof (Optional)

If $z \in \mathcal{C}$ on wrong side of $L$,



# Proof (Optional)

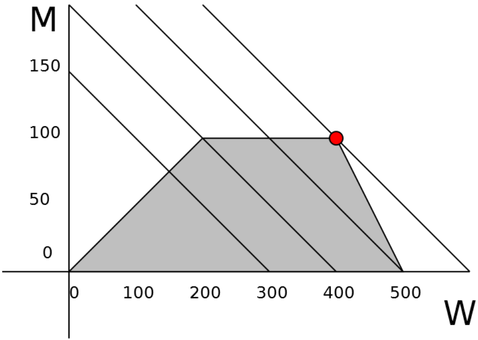
$yz$ contains point closer to $x$. Contradiction.



# Extreme Points

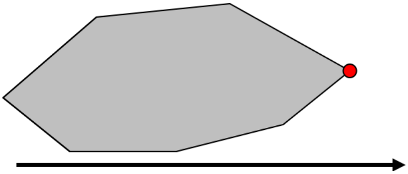
## Lemma

A linear function on a polytope takes its minimum/maximum values on vertices.



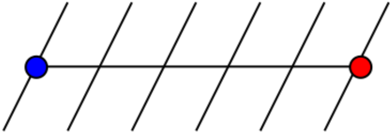
# Intuition

The corners are the only extreme points.
Optima must be there.



# Idea

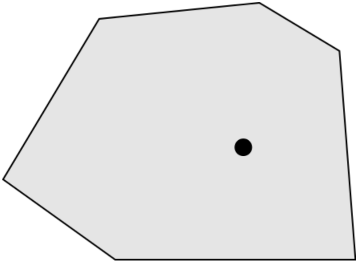
Linear function on segment takes extreme values on ends.



Use to push towards corners.

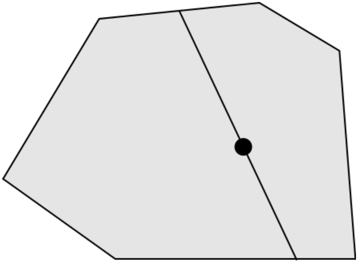
# Proof

Start at any point.



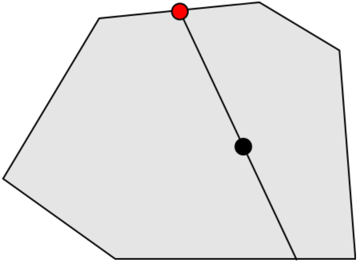
# Proof

Pick line through point.



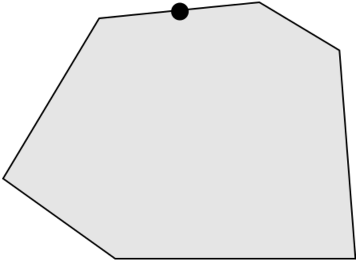
# Proof

Extreme values at endpoint.



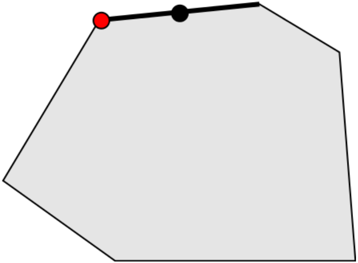
# Proof

Point is on a facet.



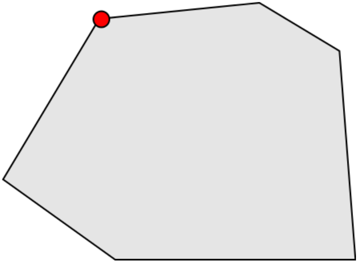
# Proof

Repeat to push to lower dimensional facet.



# Proof

Eventually on a vertex.



# Summary

- Region determined by LP always convex polytope.
- Optimum always at vertex.
- Can separate from outside points by hyperplanes.