

# Flows in Networks: Network Flows

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# Learning Objectives

- Provide the definitions of a network and a flow.
- Give some examples of real world situations in which network flow problems might arise.

# Last Time

- Last time: Discussed disaster management problem.
- Today: Talk about formalization of this and similar problems.

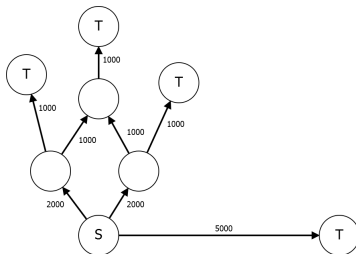
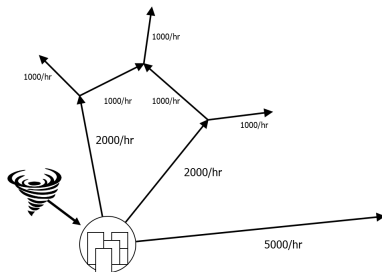
# Network

## Definition

A **network** is a directed graph  $G$  with:

- Each edge,  $e$ , is assigned a positive real capacity,  $C_e$ .
- One (or more) vertex is labelled a **source**.
- One (or more) vertex is labelled a **sink**.

# Example



# Flows

- Next we want to be able to talk about flows (traffic) through a network.
- Rather than talking about where each car goes, we will instead concern ourselves with the total flow,  $f_e$ , through each edge  $e$ .

# Flows

- Next we want to be able to talk about flows (traffic) through a network.
- Rather than talking about where each car goes, we will instead concern ourselves with the total flow,  $f_e$ , through each edge  $e$ .
- This must satisfy two conditions:

# Rate Limitation

For each edge  $e$ ,

$$0 \leq f_e \leq C_e.$$

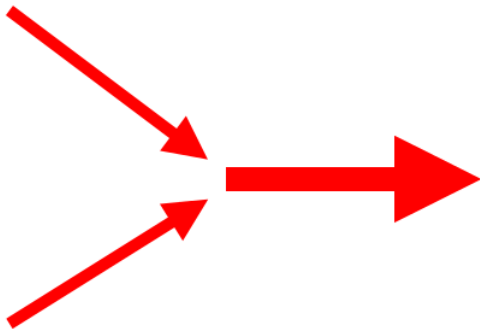




# Conservation of Flow

For all  $v$  not a source or sink:

$$\sum_{e \text{ into } v} f_e = \sum_{e \text{ out of } v} f_e.$$



# Formal Definition

## Definition

A **flow** in a network is an assignment of a real number flow,  $f_e$  to each edge  $e$  so that

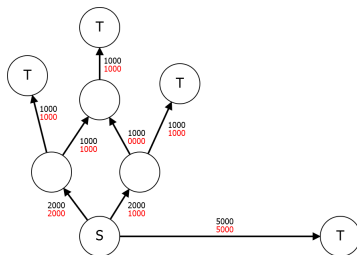
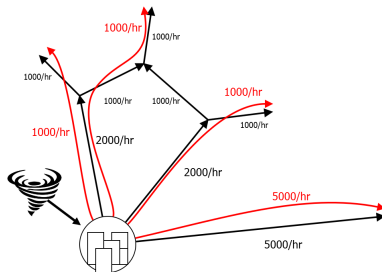
- For all  $e$

$$0 \leq f_e \leq C_e.$$

- For all  $v$  not a source or sink

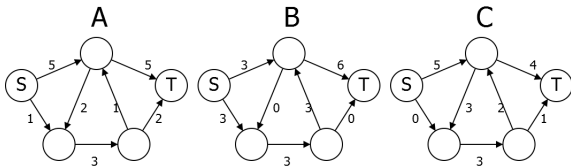
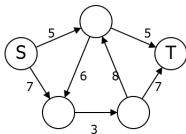
$$\sum_{e \text{ into } v} f_e = \sum_{e \text{ out of } v} f_e.$$

# Example



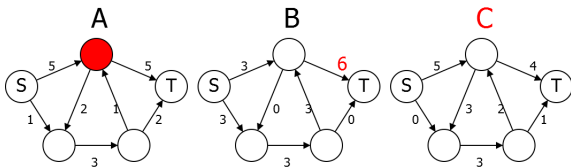
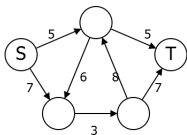
# Problem

Which of the following is a valid flow for the given network?



# Solution

Only flow **C** is valid. **A** fails to conserve flow at one vertex, and **B** exceeds.



# Examples of Network Flows

Network flows are useful to study since they can model a number of real-life phenomena.

# Flows of Goods on a Transportation Network



# Flows of Electricity Through the Power Grid





# Flows of Water Through Pipes



# Flows of Information Through a Communications Network



# Flow Size

One thing to know about a flow is how much stuff is actually flowing.

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## Definition

For a flow,  $f$ , the **size** of the flow is given by

$$|f| := \sum_{e \text{ out of a source}} f_e - \sum_{e \text{ into a source}} f_e.$$

# Sinks

You can also compute this by looking at sinks.

## Lemma

$$|f| = \sum_{e \text{ into a sink}} f_e - \sum_{e \text{ out of a sink}} f_e.$$

# Proof

$$\begin{aligned} 0 &= \sum_e f_e - \sum_e f_e \\ &= \sum_v \left( \sum_{e \text{ into } v} f_e - \sum_{e \text{ out of } v} f_e \right) \\ &= \sum_{v \text{ source or sink}} \left( \sum_{e \text{ into } v} f_e - \sum_{e \text{ out of } v} f_e \right) \end{aligned}$$

# Proof

$$\begin{aligned} &= \sum_{v \text{ source or sink}} \left( \sum_{e \text{ into } v} f_e - \sum_{e \text{ out of } v} f_e \right) \\ &= \sum_{e \text{ into a source}} f_e + \sum_{e \text{ into a sink}} f_e \\ &\quad - \sum_{e \text{ out of a source}} f_e - \sum_{e \text{ out of a sink}} f_e \\ &= -|f| + \left( \sum_{e \text{ into a sink}} f_e - \sum_{e \text{ out of a sink}} f_e \right). \end{aligned}$$

# Problem

How large a flow can we fit through a network?

## Maxflow

Input: A network  $G$

Output: A flow  $f$  for  $G$  with  $|f|$  as large as possible.