

Flows in Networks: Maxflow-Mincut

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Advanced Algorithms and Complexity
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Learning Objectives

- Understand the relationship between flows and cuts.
- Produce a cut with size matching that of a maximum flow.
- Identify when a flow is maximum.

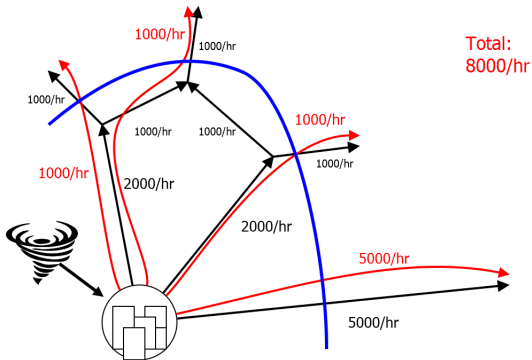
Problem

In order to find maxflows, we need a way of verifying that they are optimal.

In particular, we need techniques for bounding the size of the maxflow.

Idea

Recall our original example:



Idea

Find a **bottleneck** for the flow. All flow needs to cross the bottleneck.

Cuts

Definition

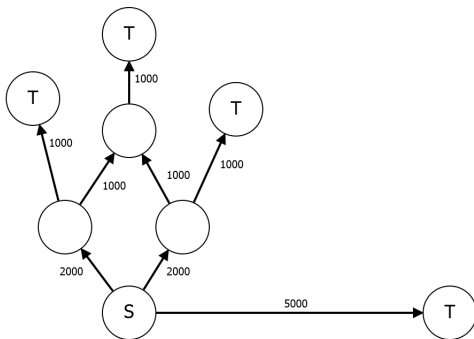
Given a network G , a **cut** \mathcal{C} , is a set of vertices of G so that \mathcal{C} contains all sources of G and no sinks of G .

The **size** of a cut is given by

$$|\mathcal{C}| := \sum_{e \text{ out of } \mathcal{C}} c_e$$

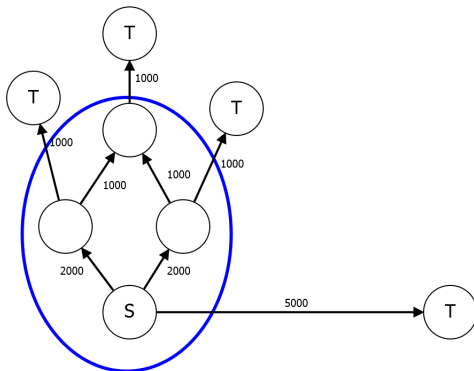
Example

Network G .



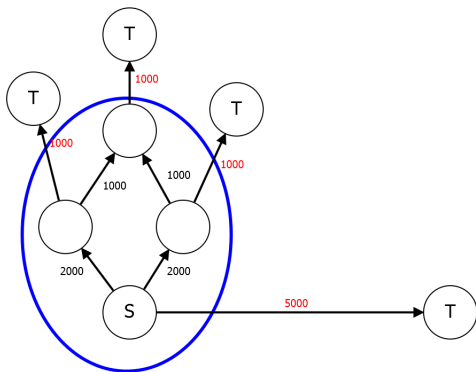
Example

Cut \mathcal{C} .



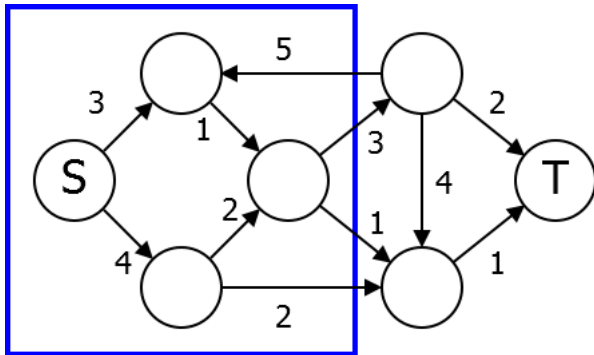
Example

Edges cut. Total size 8000.



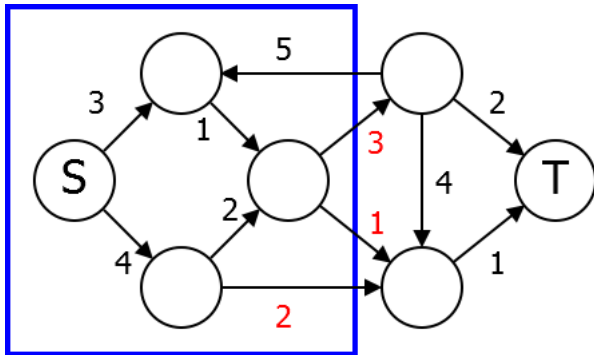
Problem

What is the size of the cut below?



Solution

$$1 + 2 + 3 = 6.$$



Bound

Lemma

Let G be a network. For any flow f and any cut \mathcal{C} ,

$$|f| \leq |\mathcal{C}|.$$

Proof

$$\begin{aligned} |f| &= \sum_{v \text{ source}} \left(\sum_{e \text{ out of } v} f_e - \sum_{e \text{ into } v} f_e \right) \\ &= \sum_{v \in \mathcal{C}} \left(\sum_{e \text{ out of } v} f_e - \sum_{e \text{ into } v} f_e \right) \\ &= \sum_{e \text{ out of } \mathcal{C}} f_e - \sum_{e \text{ into } \mathcal{C}} f_e \\ &\leq \sum_{e \text{ out of } \mathcal{C}} C_e = |\mathcal{C}|. \end{aligned}$$

Bounds

In other words, for any cut \mathcal{C} , we get an upper bound on the maxflow. In particular,

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Question: Is this bound good enough?
Surprisingly, it is...

Maxflow-Mincut

Theorem

For any network G ,

$$\max_{\text{flows } f} |f| = \min_{\text{cuts } \mathcal{C}} |\mathcal{C}|.$$

Maxflow-Mincut

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$$\max_{\text{flows } f} |f| = \min_{\text{cuts } \mathcal{C}} |\mathcal{C}|.$$

In other words, there is always a cut small enough to give the correct upper bound.

A Special Case

What happens when $\text{Maxflow} = 0$?

- There is **no** path from source to sink.
- Let \mathcal{C} be the set of vertices reachable from sources.
- There are no edges out of \mathcal{C} .
- So $|\mathcal{C}| = 0$.

The General Case

- Let f be a maxflow for G .
- Note that G_f has maxflow 0.
- There is a cut \mathcal{C} with size 0 in G_f .
- Claim: $|\mathcal{C}| = |f|$.

Proof

$$\begin{aligned}|f| &= \sum_{e \text{ out of } \mathcal{C}} f_e - \sum_{e \text{ into } \mathcal{C}} f_e \\ &= \sum_{e \text{ out of } \mathcal{C}} C_e - \sum_{e \text{ into } \mathcal{C}} 0 \\ &= |\mathcal{C}|.\end{aligned}$$

Conclusion

- We have found an f and \mathcal{C} with $|f| = |\mathcal{C}|$.
- By Lemma, cannot have larger $|f|$ or smaller $|\mathcal{C}|$.
- So $\max |f| = \min |\mathcal{C}|$.

Summary

- Can always check if flow is maximal by finding matching cut.
- f a maxflow only if there is no source-sink path in G_f .
- We will use this in an algorithm next time.