Flows in Networks: Image Segmentation

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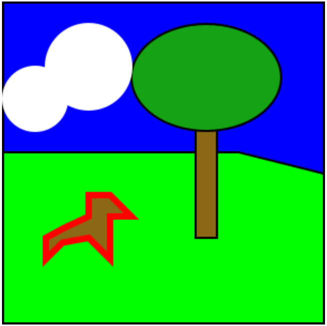
Advanced Algorithms and Complexity
Data Structures and Algorithms

Learning Objectives

- Understand the image segmentation problem.
- Relate this problem to finding minimum cuts in an appropriate network.
- Write and algorithm to solve the image segmentation problem.

Image Segmentation

Given an image separate the foreground from the background.

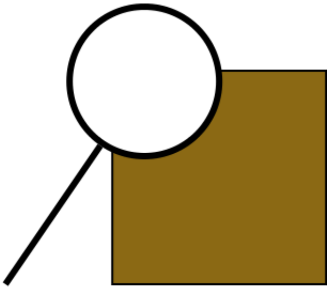


Setup

- Image is a grid of pixels.
- Need to decide which pixels are in the foreground.
- Have some ideas about which pixels are in foreground/background.

Pixel

Some other algorithm judges each pixel to guess whether in foreground or background.



- a_v - likelihood pixel in foreground.
- b_v - likelihood pixel in background.

Simple Version of Problem

Simple Image Segmentation

Input: Values a_v, b_v

Output: Partition pixels into sets $\mathcal{F}$ and $\mathcal{B}$ so that

$$\sum_{v \in \mathcal{F}} a_v + \sum_{v \in \mathcal{B}} b_v$$

is as large as possible.

Problem

What is the best value for the following problem?

v	1	2	3
a_v	3	5	6
b_v	4	3	5

Solution

This version is easy to solve:

- If $a_v > b_v$, put v in $\mathcal{F}$.
- If $b_v > a_v$, put v in $\mathcal{B}$.

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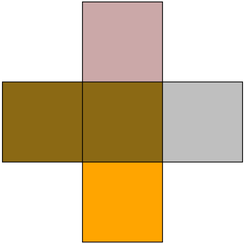
b_v	4	3	5

Answer:

$$4 + 5 + 6 = 15.$$

Nearby Pixels

Also expect that nearby pixels will be on the same size of divide.



Have penalty p_{vw} for putting v in foreground and w in background.

Full Problem

Image Segmentation

Input: Values a_v, b_v, p_{vw}

Output: Partition pixels into sets $\mathcal{F}$ and $\mathcal{B}$ so that

$$\sum_{v \in \mathcal{F}} a_v + \sum_{v \in \mathcal{B}} b_v - \sum_{v \in \mathcal{F}, w \in \mathcal{B}} p_{vw}$$

is as large as possible.

Algebra

Subtracting the sum over all v of $a_v + b_v$, we want to maximize

$$-\left(\sum_{v \in \mathcal{F}} b_v + \sum_{v \in \mathcal{B}} a_v + \sum_{v \in \mathcal{F}, w \in \mathcal{B}} p_{vw} \right).$$

Algebra

Subtracting the sum over all v of $a_v + b_v$, we want to maximize

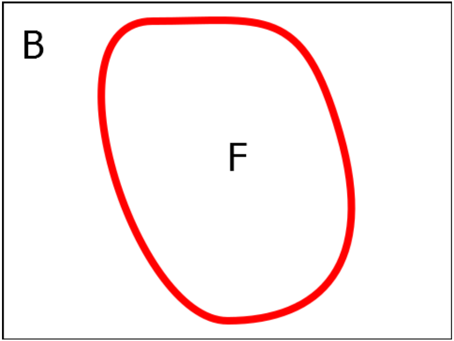
$$-\left(\sum_{v\in\mathcal{F}} b_v + \sum_{v\in\mathcal{B}} a_v + \sum_{v\in\mathcal{F},w\in\mathcal{B}} p_{vw}\right).$$

Equivalently, we can minimize

$$\sum_{v\in\mathcal{F}} b_v + \sum_{v\in\mathcal{B}} a_v + \sum_{v\in\mathcal{F},w\in\mathcal{B}} p_{vw}.$$

Idea

Want to split vertices into two sets. Pay cost based on boundary between sets.



Idea

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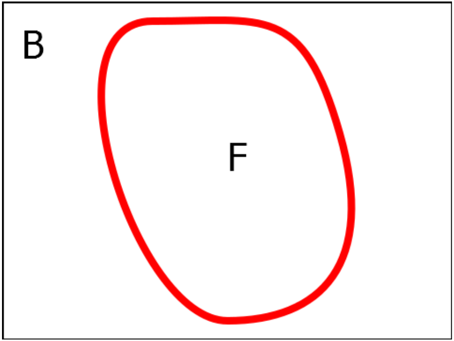
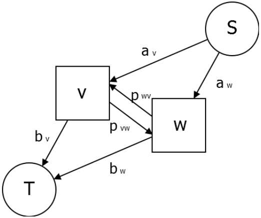


This sounds a lot like computing a mincut!

Network

Create network:

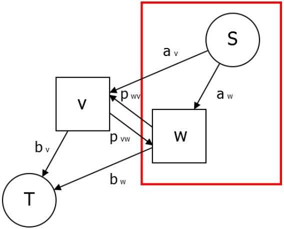
- New vertices s and t.
- Edge s to v with capacity a_v.
- Edge v to t with capacity b_v.
- Edge v to w with capacity p_{vw}.



Cuts

Cut $\mathcal{C}$ has size

$$\sum_{v \in \mathcal{C}} b_v + \sum_{v \notin \mathcal{C}} a_v + \sum_{v \in \mathcal{C}, w \notin \mathcal{C}} p_{vw}.$$



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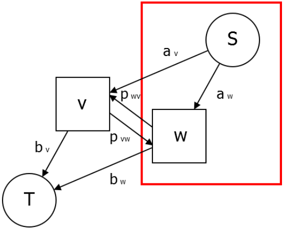

Let $\mathcal{C} = \mathcal{F}$ and $\bar{\mathcal{C}} = \mathcal{B}$!

Algorithm

- Use Maxflow-Mincut!

Algorithm

- Use Maxflow-Mincut!
- Construct network
- Compute Maxflow
- Find corresponding Mincut

Pseudocode

ImageSegmentation(a_v, b_v, p_{vw})

Construct corresponding network G
Compute a maxflow f for G
Compute residual G_f
Let $\mathcal{C}$ be the collection of vertices reachable from s in G_f
return $\mathcal{F} = \mathcal{C}, \mathcal{B} = \bar{\mathcal{C}}$

Summary

- Basic problem in image processing
- Found mathematical formulation
- Looks like a mincut problem
- Used relationship to maxflow to solve