# Flows in Networks: The Edmonds-Karp Algorithm

Daniel Kane

Department of Computer Science and Engineering
University of California, San Diego

## Advanced Algorithms and Complexity
## Data Structures and Algorithms

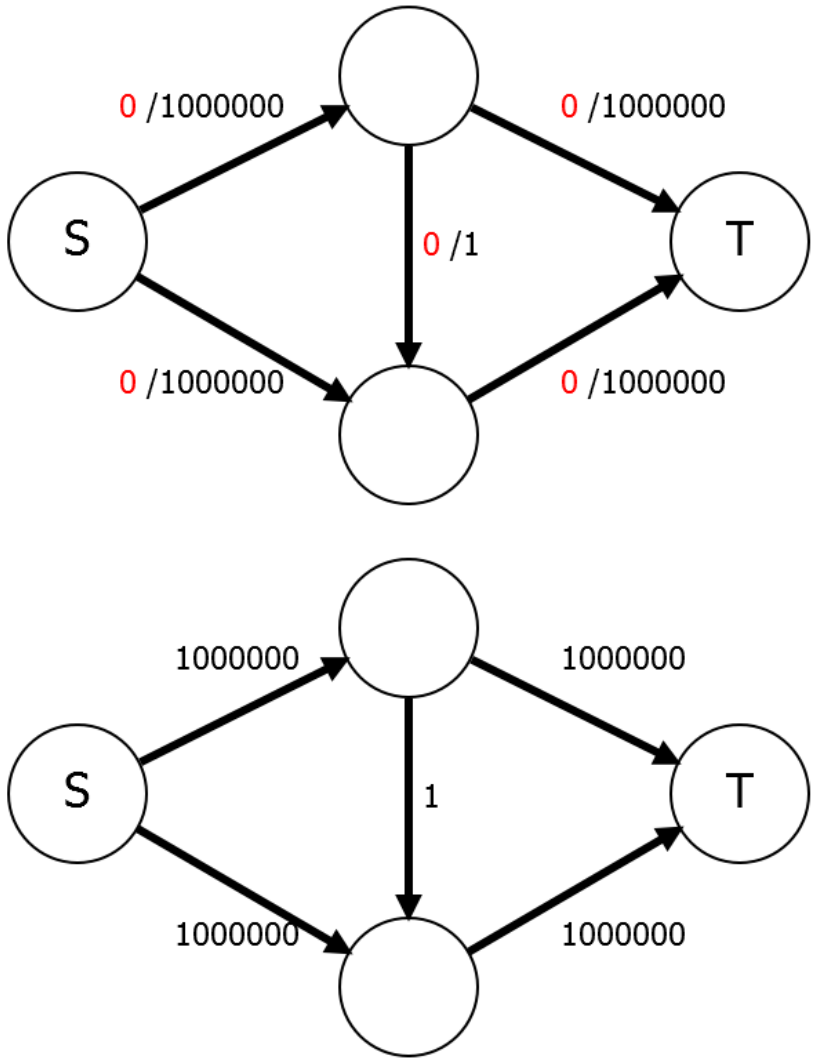
# Learning Objectives

- Implement the Edmonds-Karp algorithm.
- Understand the runtime bound.

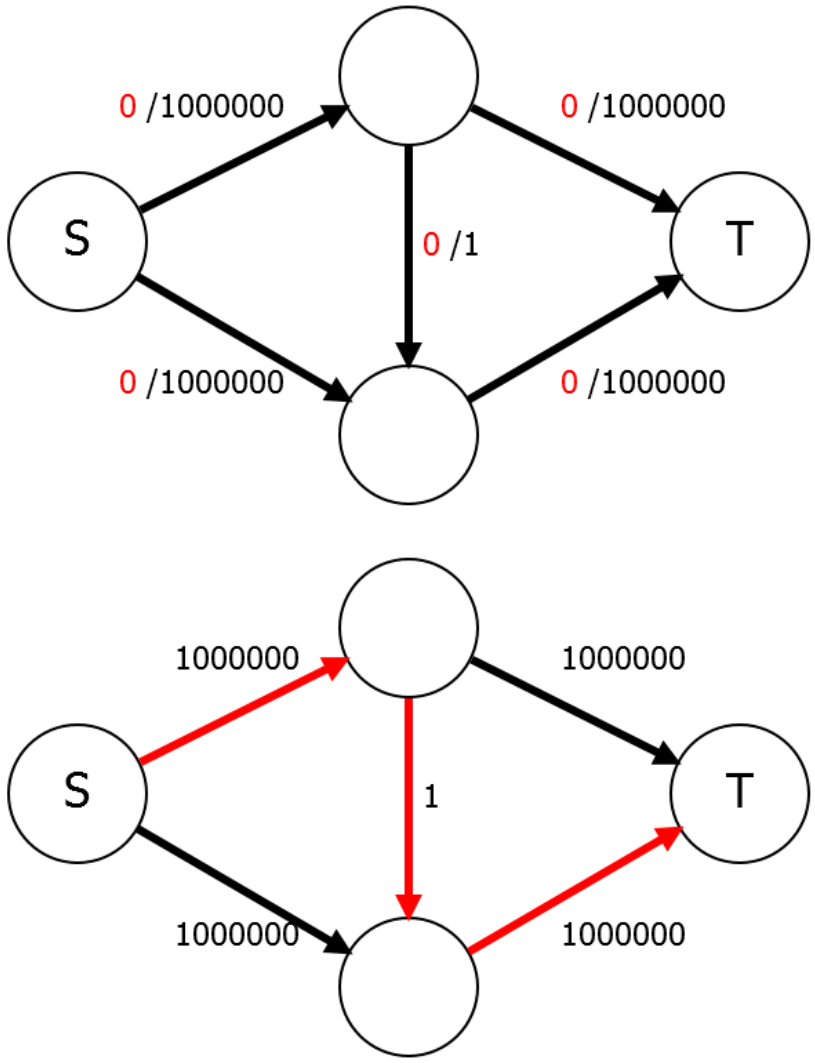
# Last Time

- Ford-Fulkerson algorithm for Maxflow.
- Runtime $O(|E||f|)$.
- Sometimes very slow if graph has large capacities.

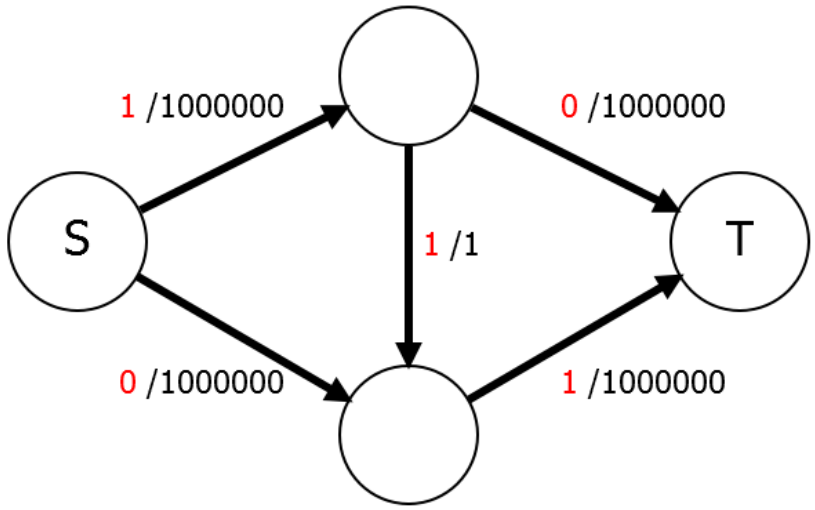
# Bad Example

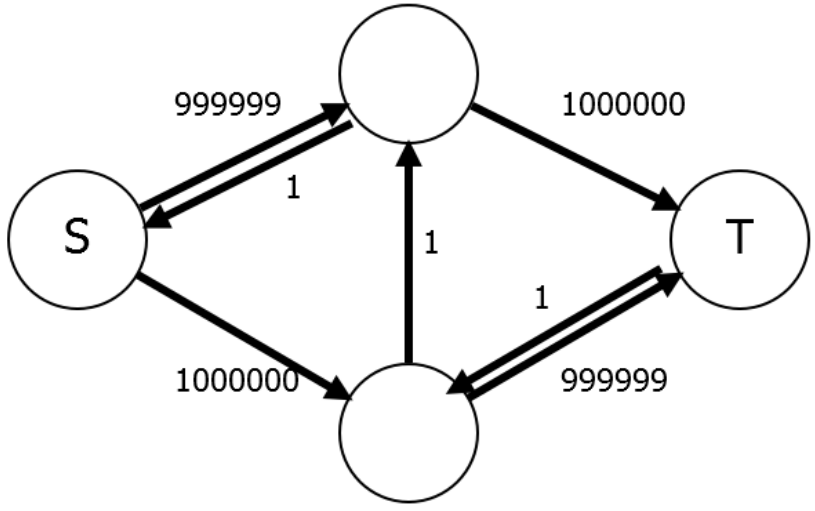

0 /1000000
0 /1000000
S
0 /1
T
0 /1000000
0 /1000000
1000000
1000000
S
1
T
1000000
1000000

# Bad Example

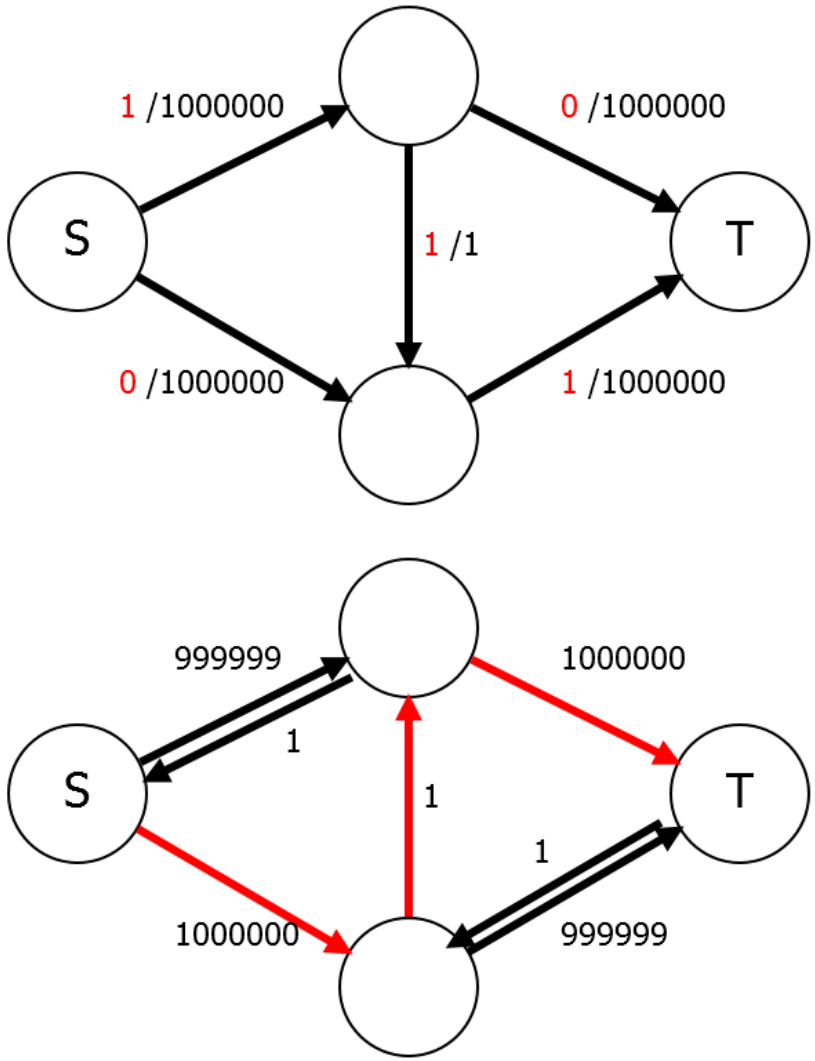

0 /1000000
0 /1000000
S
0 /1
T
0 /1000000
0 /1000000
1000000
1000000
S
1
T
1000000
1000000

# Bad Example

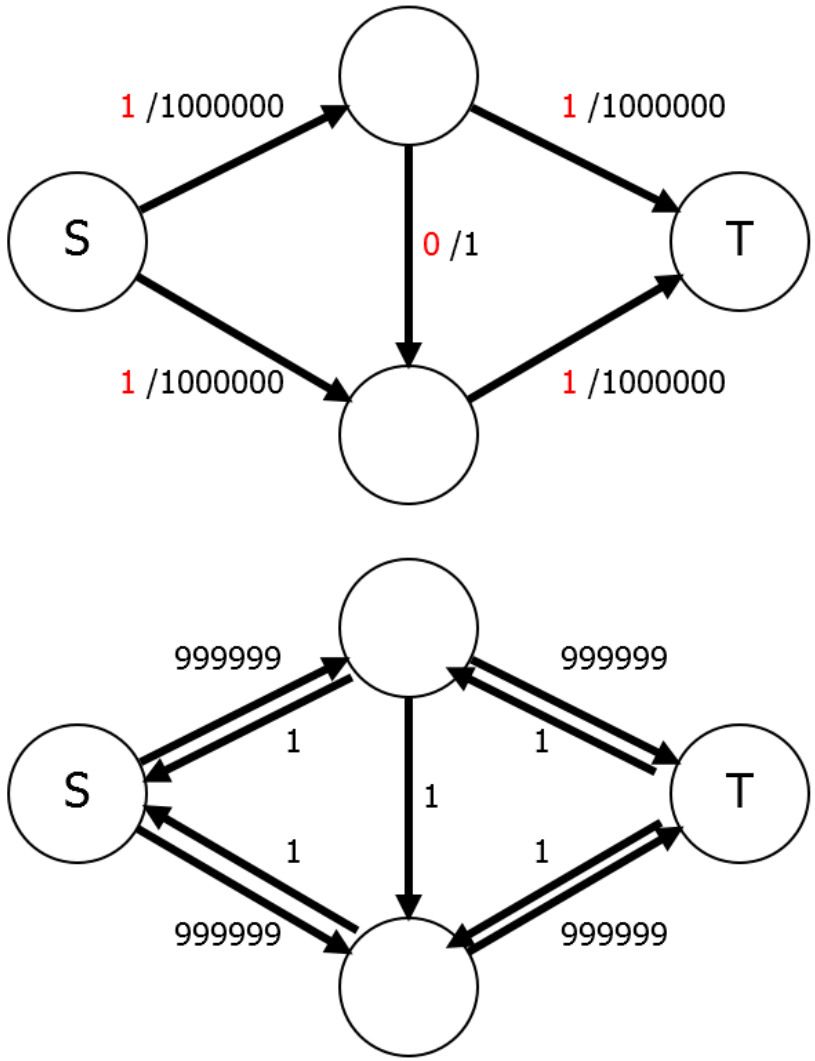

1 /1000000
0 /1000000
S
1 /1
T
0 /1000000
1 /1000000


999999
1000000
1
S
1
T
1
1000000
999999

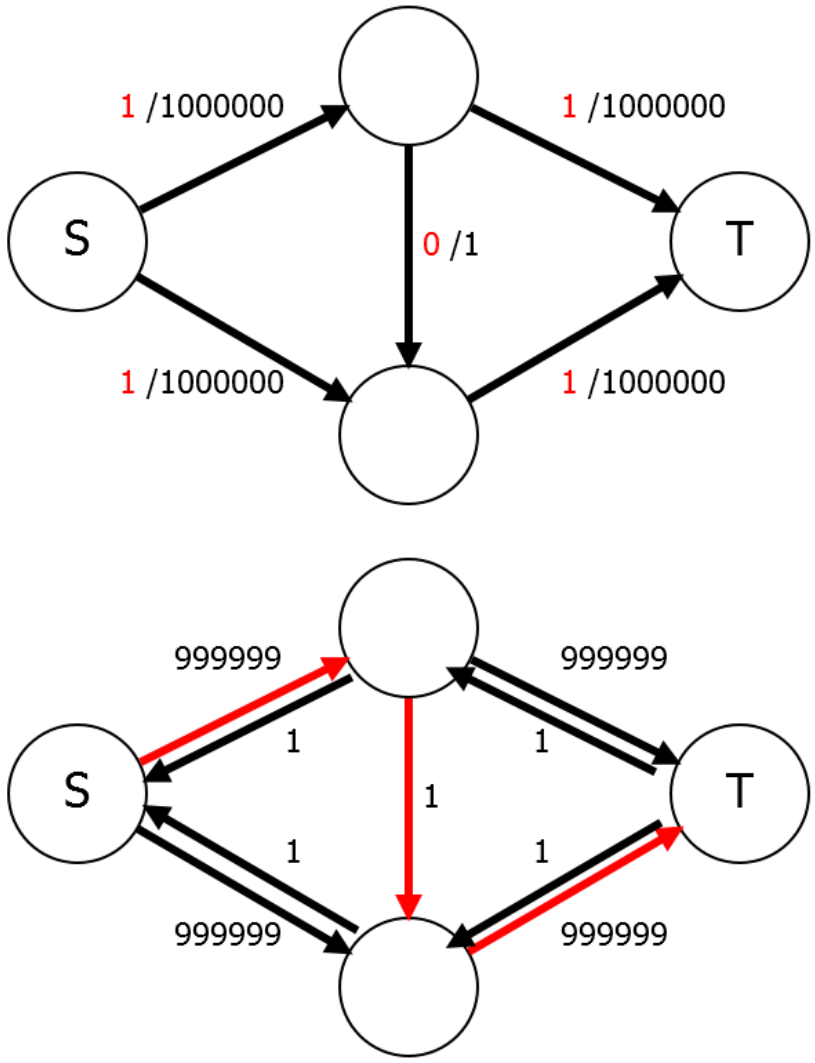
# Bad Example


1 /1000000
0 /1000000
S
1 /1
T
0 /1000000
1 /1000000
999999
1000000
1
S
1
T
1
1000000
999999

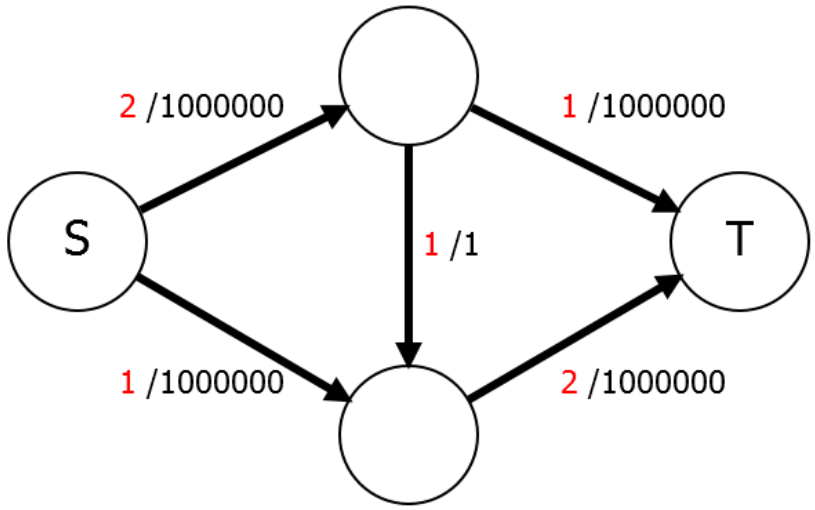
# Bad Example

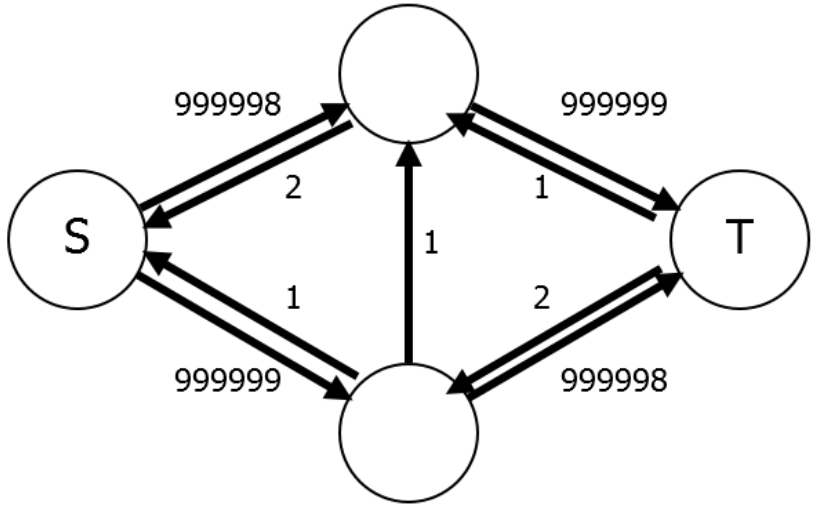

1 /1000000
1 /1000000
S
0 /1
T
1 /1000000
1 /1000000
999999
999999
1
1
S
1
T
1
1
999999
999999

# Bad Example

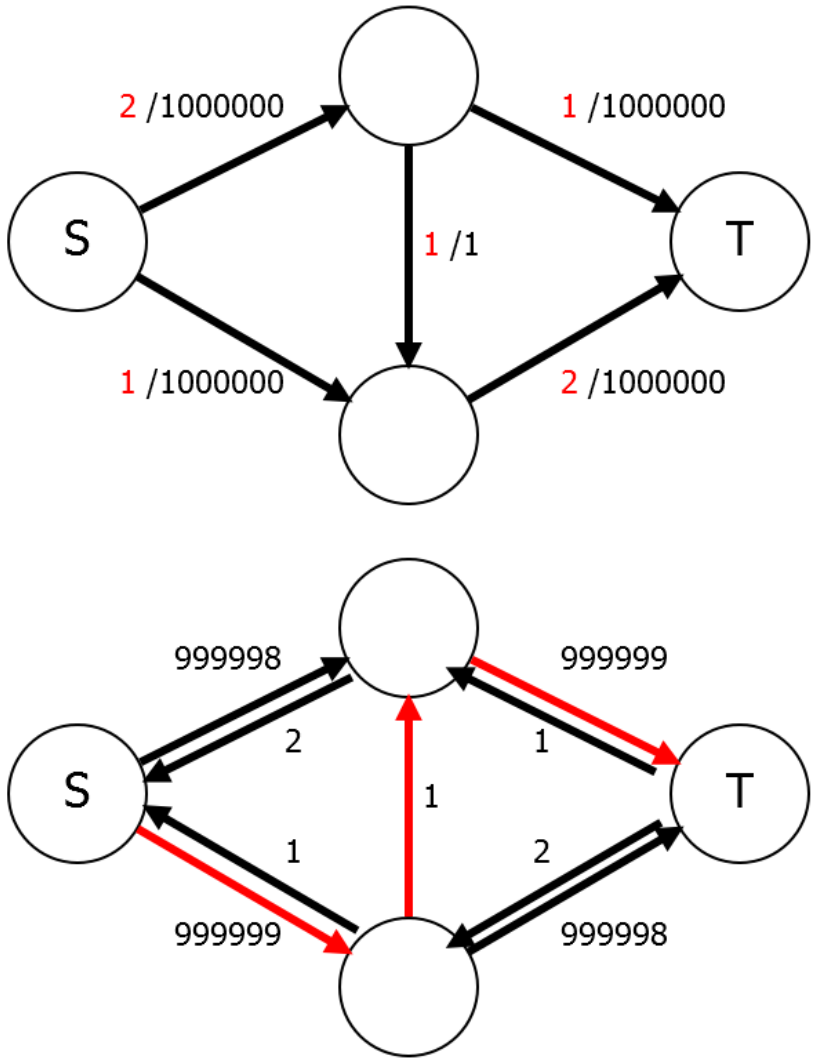

1 /1000000
1 /1000000
S
0 /1
T
1 /1000000
1 /1000000
999999
999999
1
1
S
1
T
1
1
999999
999999

# Bad Example

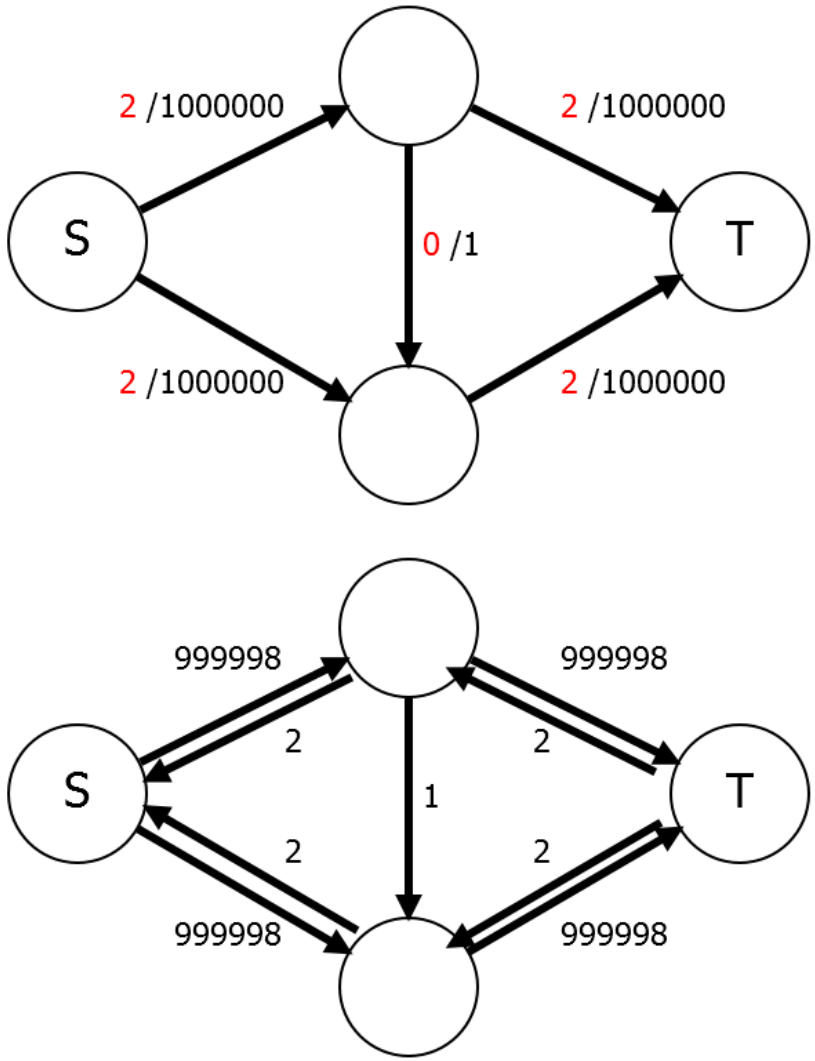

2 /1000000
1 /1000000
S
1 /1
T
1 /1000000
2 /1000000


999998
999999
2
1
S
1
T
1
2
999999
999998

# Bad Example


2 /1000000
1 /1000000
S
1 /1
T
1 /1000000
2 /1000000
999998
999999
2
1
S
1
T
1
2
999999
999998

# Bad Example


2 /1000000
2 /1000000
S
0 /1
T
2 /1000000
2 /1000000
999998
999998
2
2
S
1
T
2
2
999998
999998

# Fix

Fortunately, the Ford-Fulkerson algorithm gives us a choice as to which augmenting path to use.

# Fix

Fortunately, the Ford-Fulkerson algorithm gives us a choice as to which augmenting path to use.

Is there a way to choose augmenting paths to avoid this kind of runtime problem?

# Edmonds-Karp Algorithm

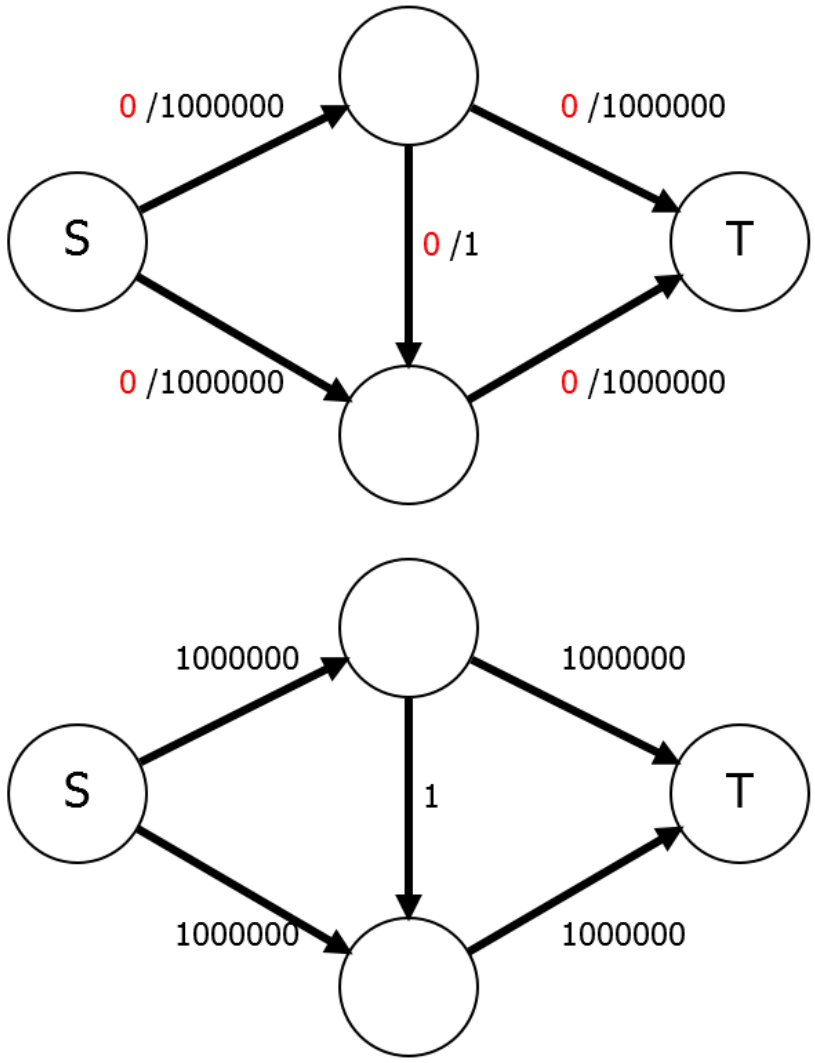
Use the Ford-Fulkerson algorithm, always choosing the shortest (in terms of number of edges) augmenting path.

# Edmonds-Karp Algorithm

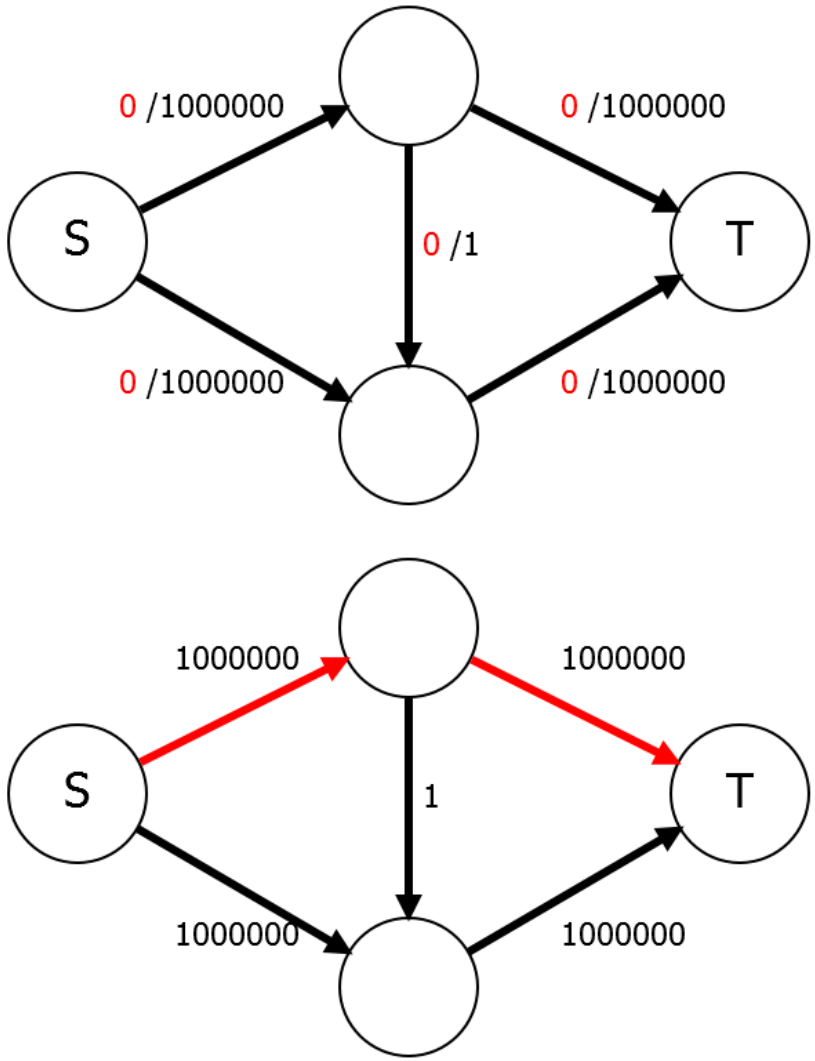
Use the Ford-Fulkerson algorithm, always choosing the shortest (in terms of number of edges) augmenting path.

Find augmenting paths using BFS instead of DFS.

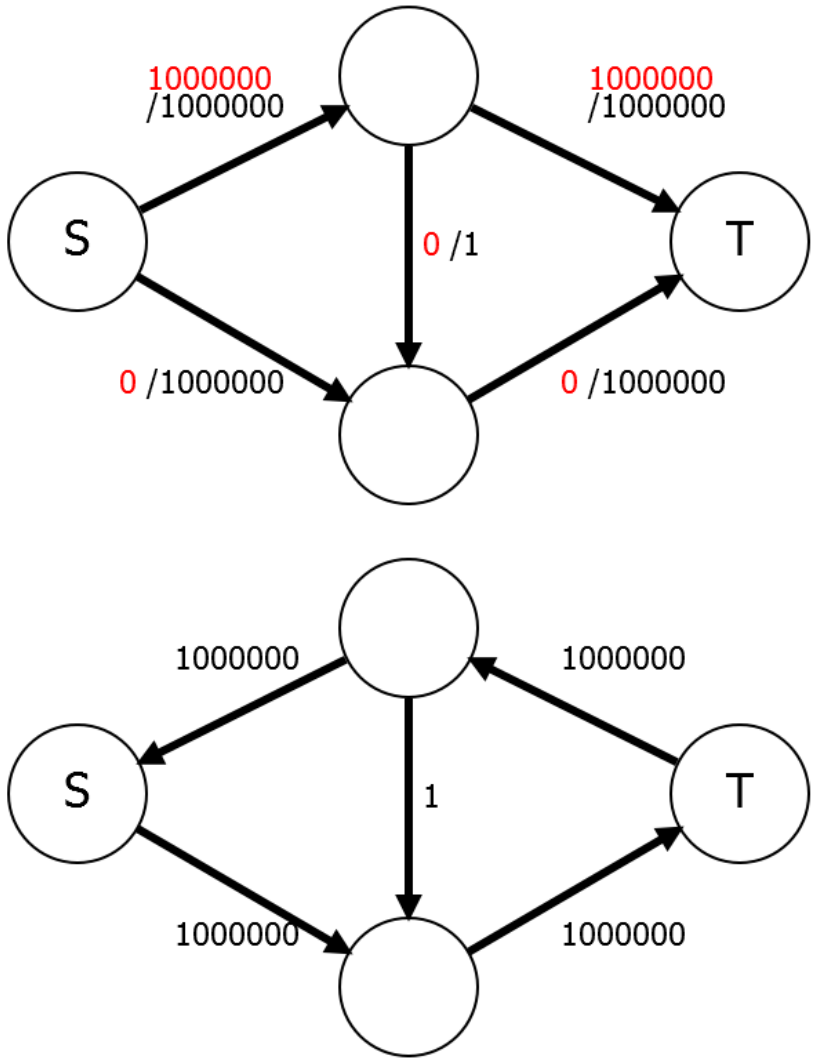
# Edmonds-Karp Execution



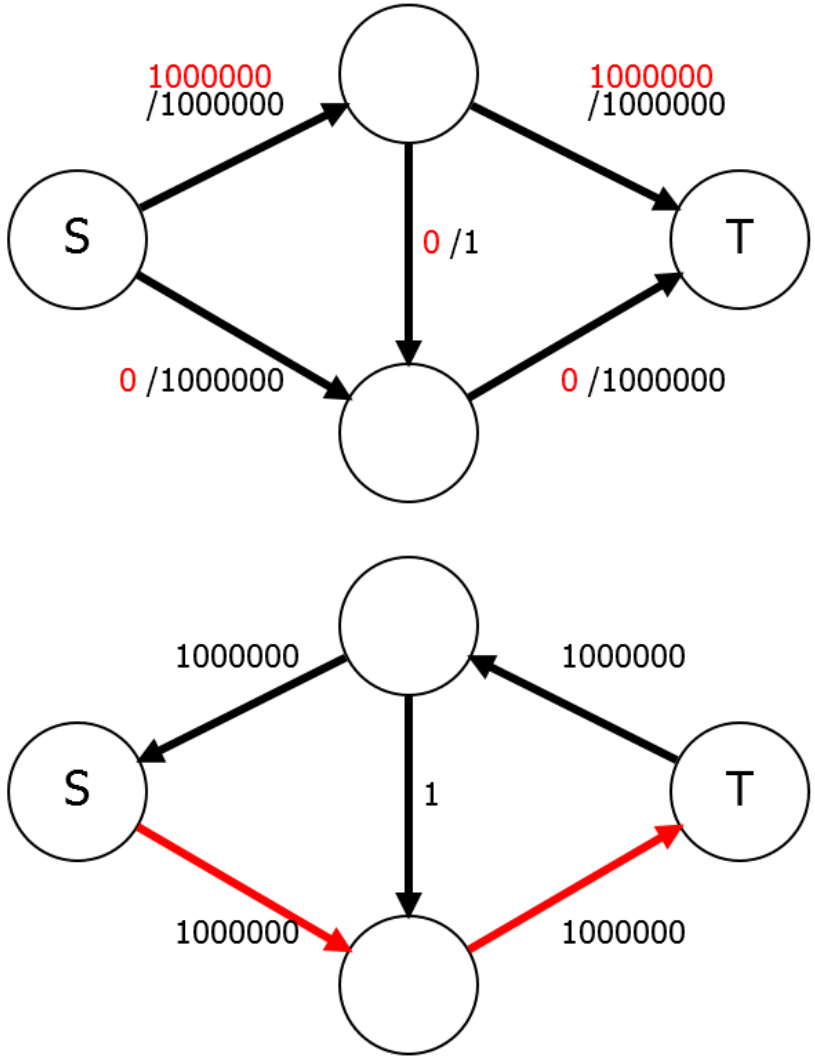
# Edmonds-Karp Execution



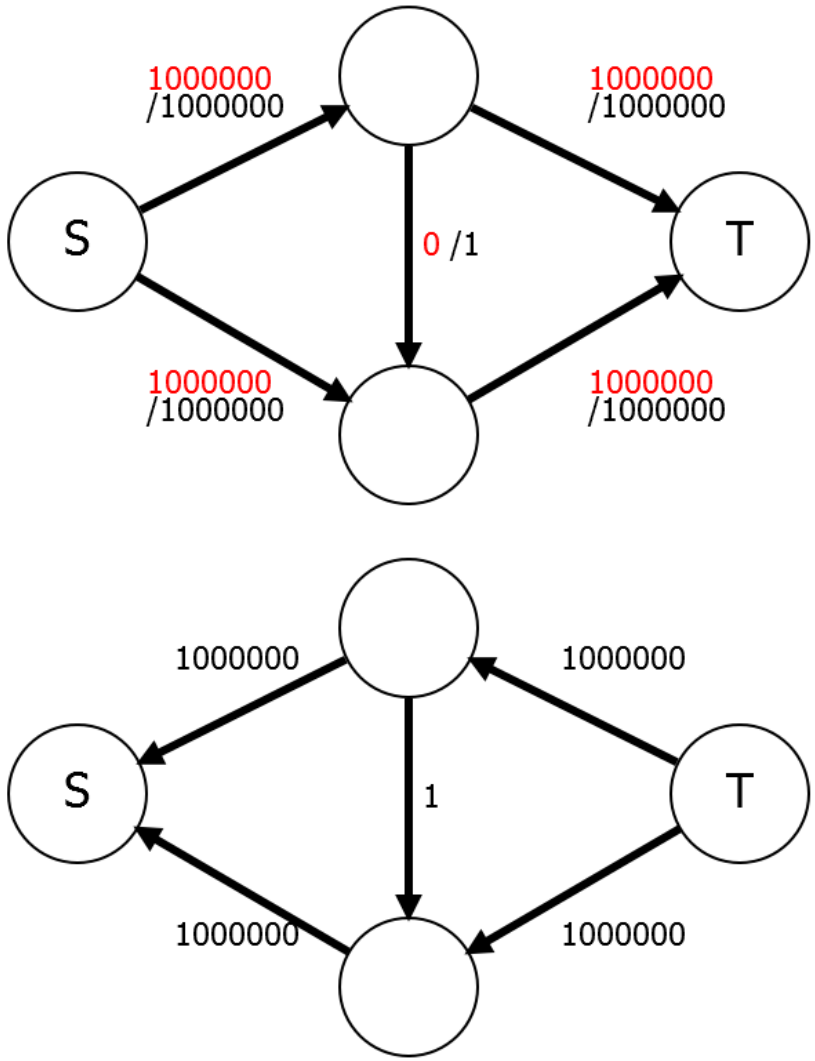
# Edmonds-Karp Execution



# Edmonds-Karp Execution



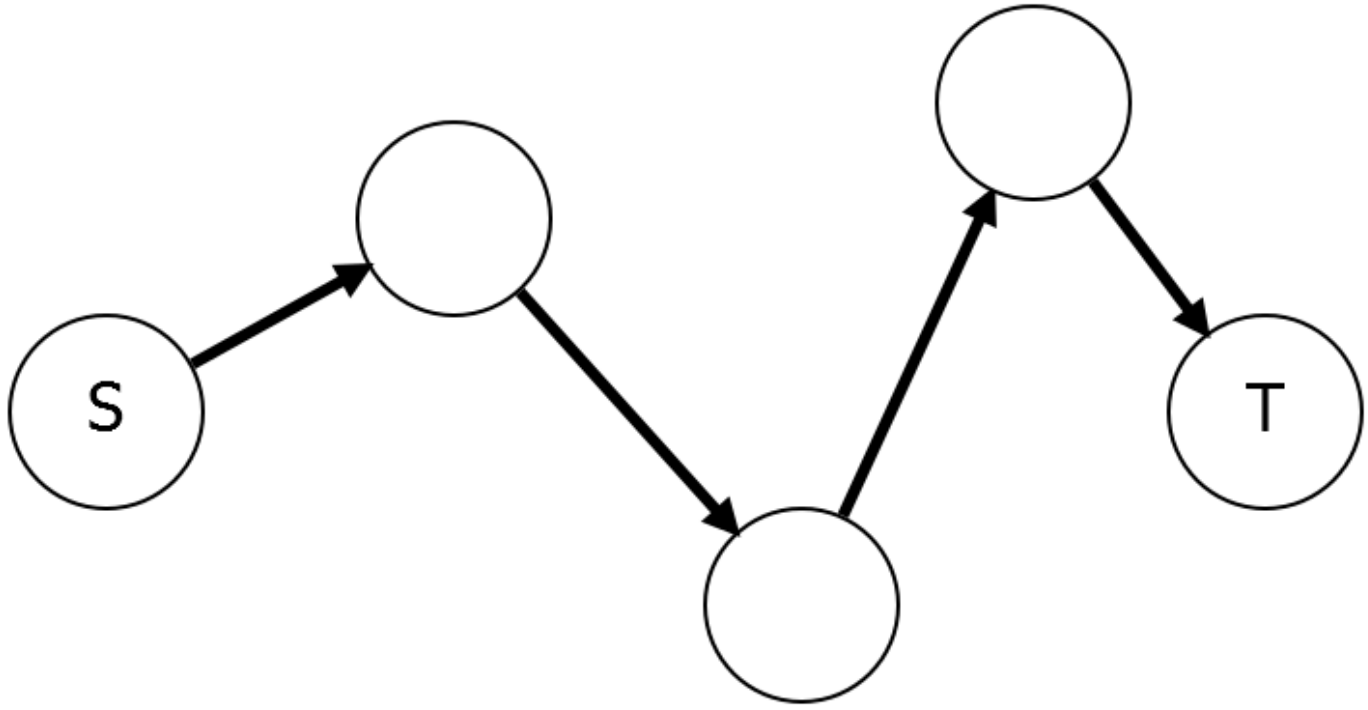
# Edmonds-Karp Execution



# Augmenting Paths

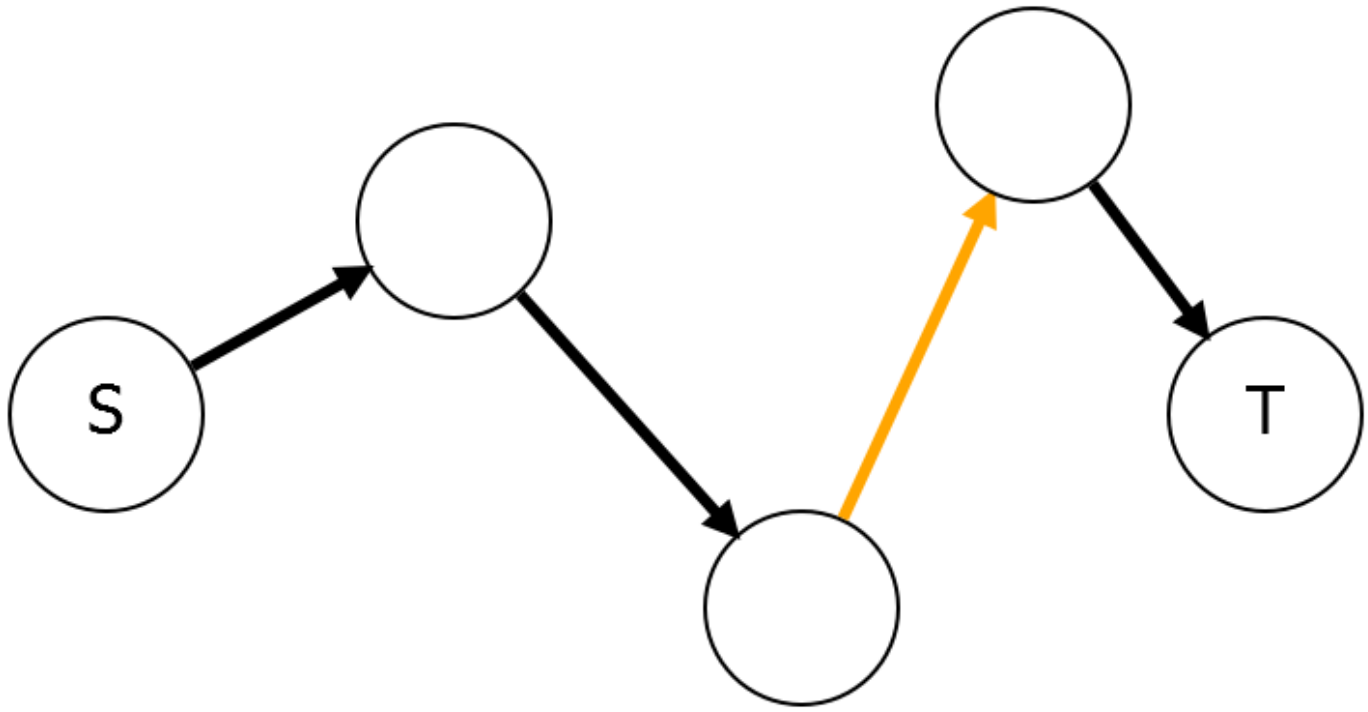
Need to analyze augmenting paths.

.



# Augmenting Paths

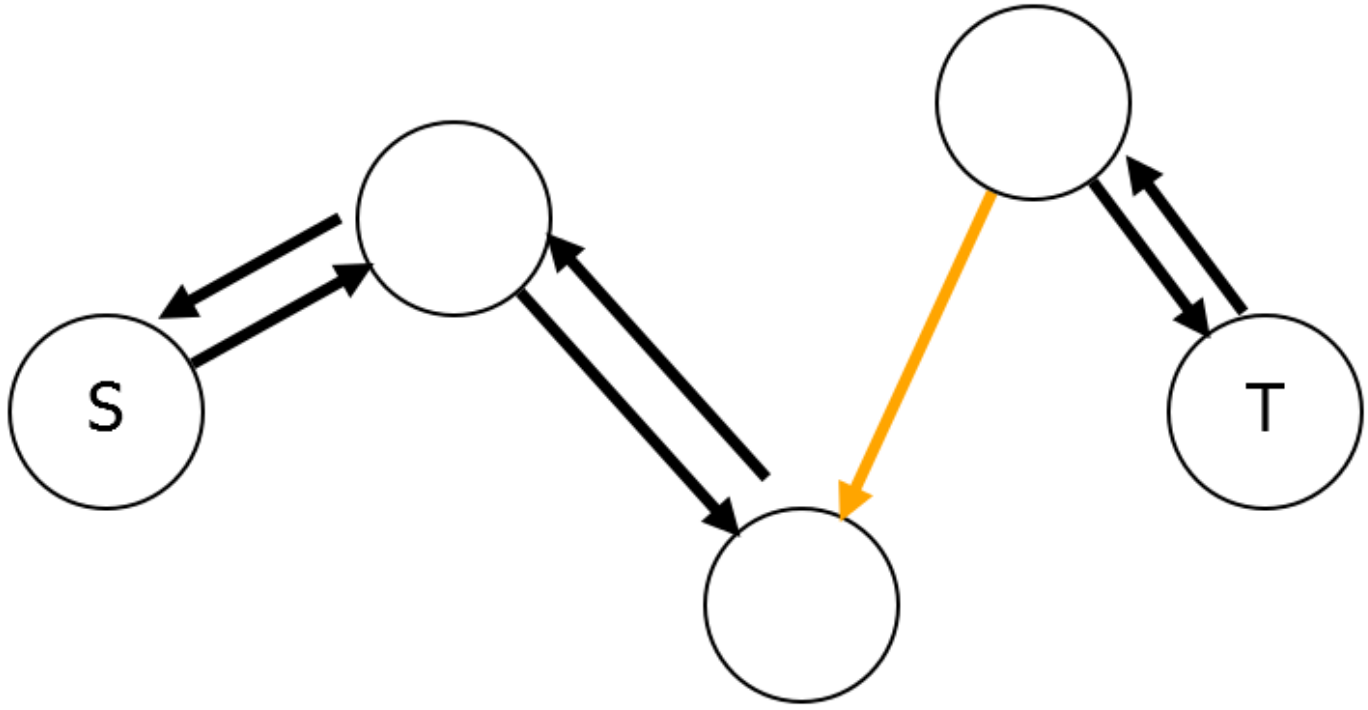
Augmenting flow always saturates (uses all the available flow from) an edge.



# Augmenting Paths

Changes to residual network.

.



# Analysis Idea

- Show that no edge is saturated too many times.
- Fails to hold in the bad case, where the middle edge is repeatedly saturated.

# Increasing Distances

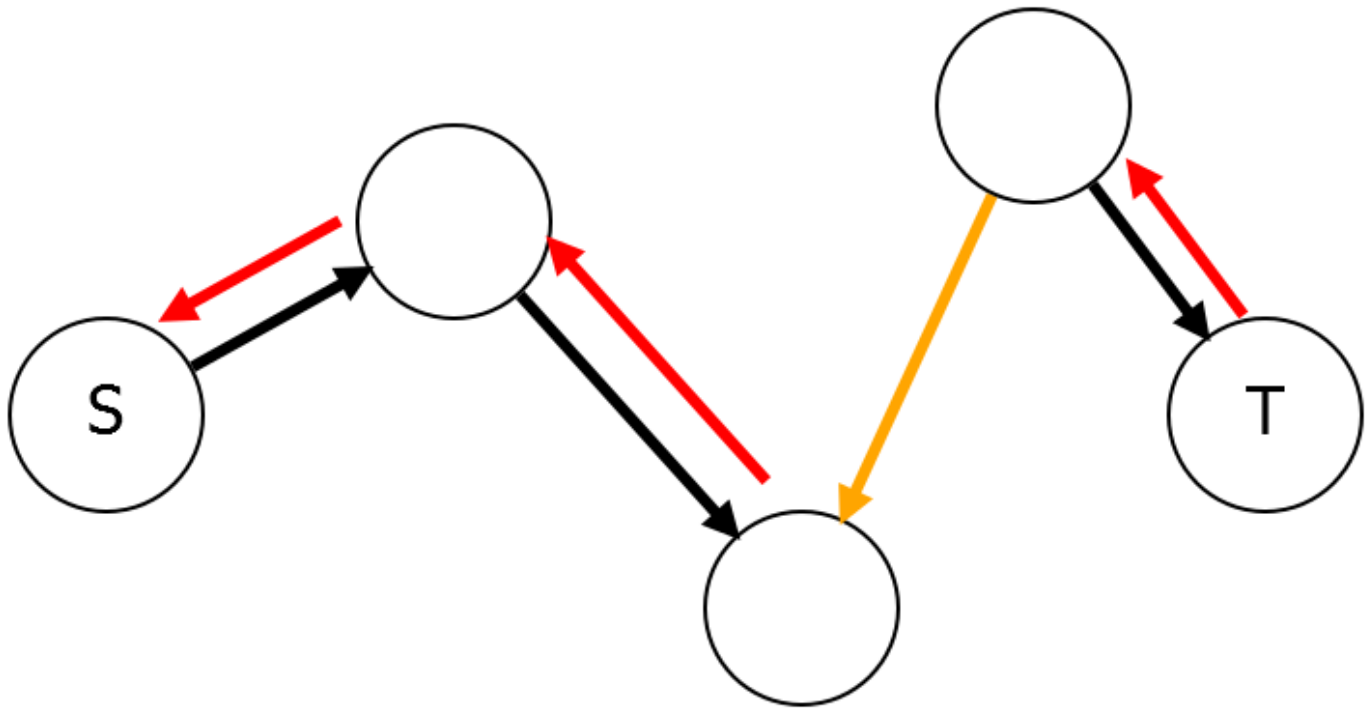
We will need this critical Lemma:

## Lemma

As the Edmonds-Karp algorithm executes, for any vertex $v \in V$ the distance $d_{G_f}(s, v)$ only increases.
Similarly, $d_{G_f}(v, t)$ and $d_{G_f}(s, t)$ can only increase.

# Proof

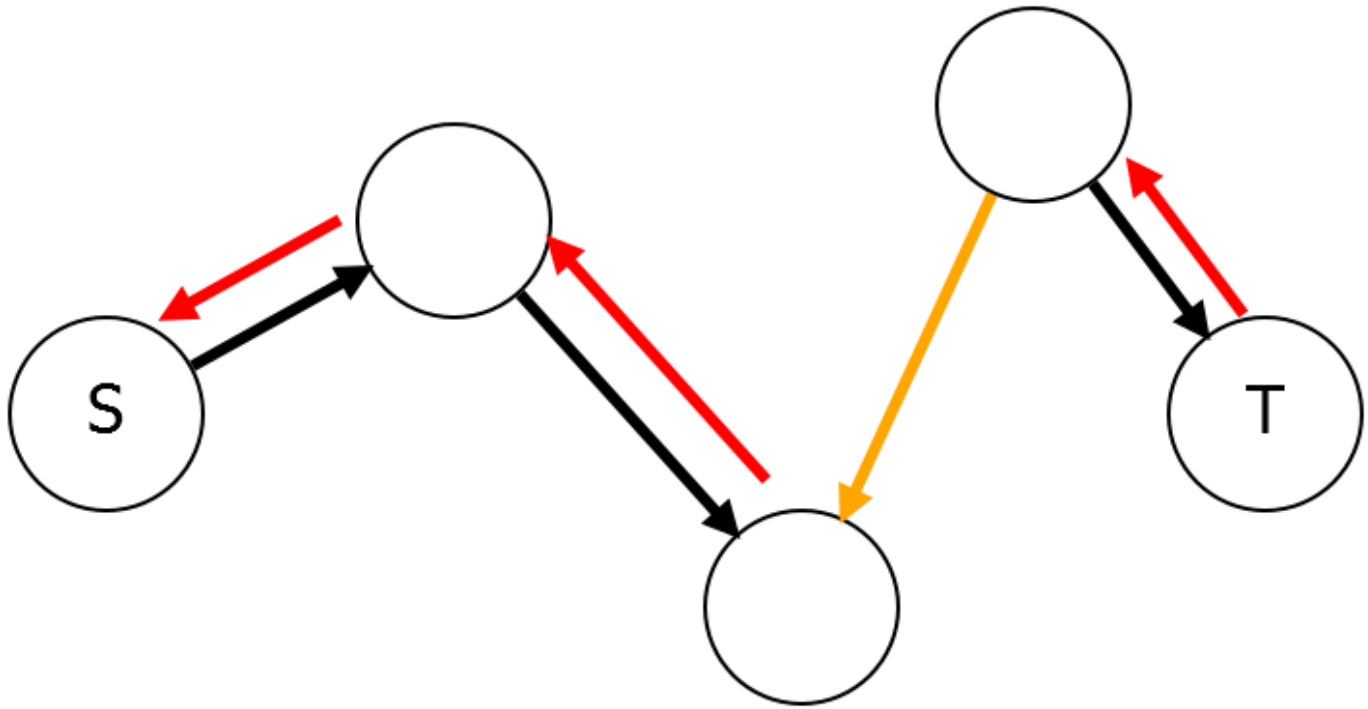
- New edges all point backwards along augmenting path.



# Proof

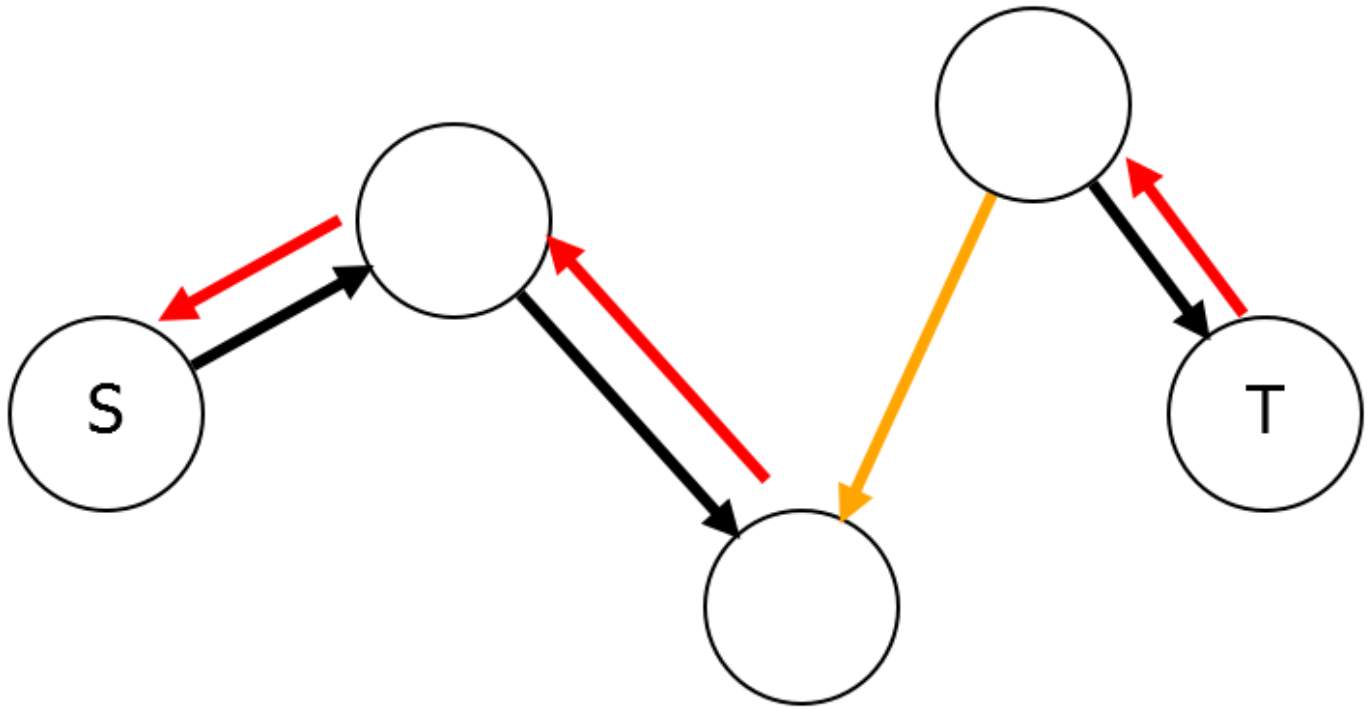
- Augmenting path is a shortest path.

.



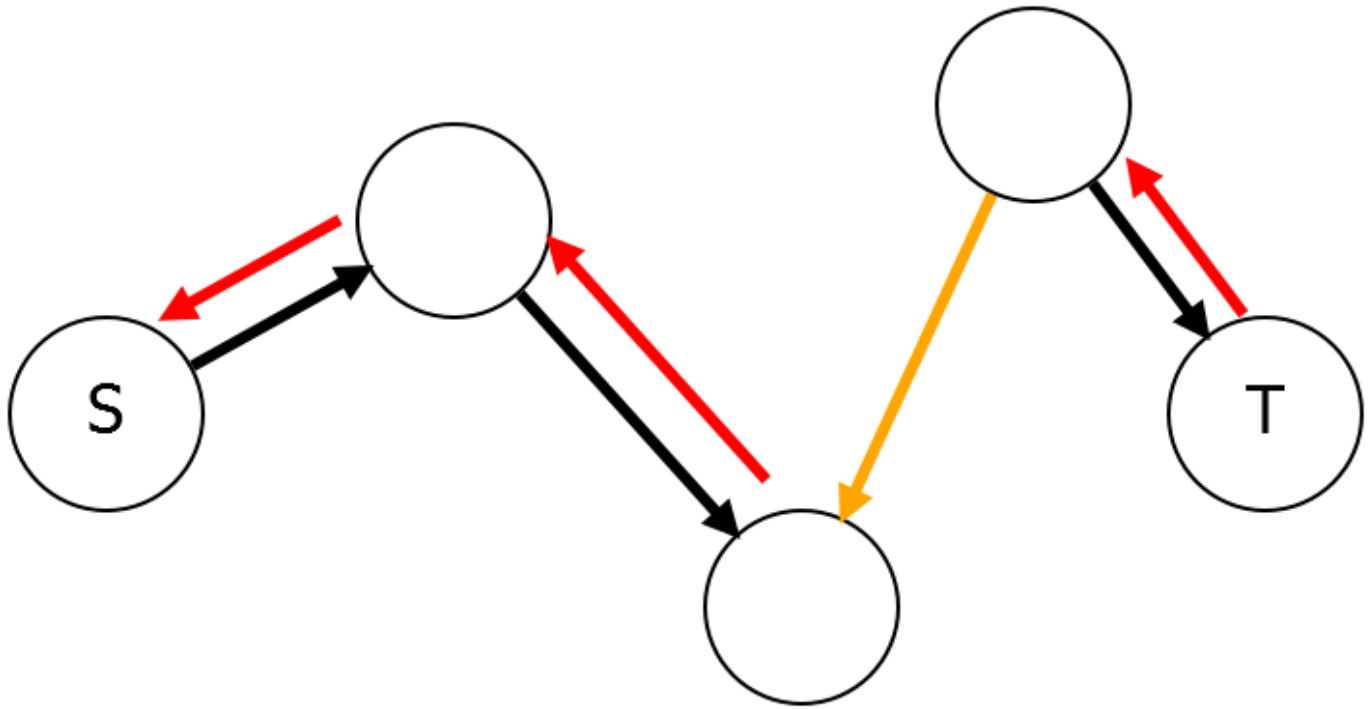
# Proof

- All new edges point from vertices further from $s$ to vertices closer.



# Proof

- New edges do not help you get from $s$ to $v$ faster.



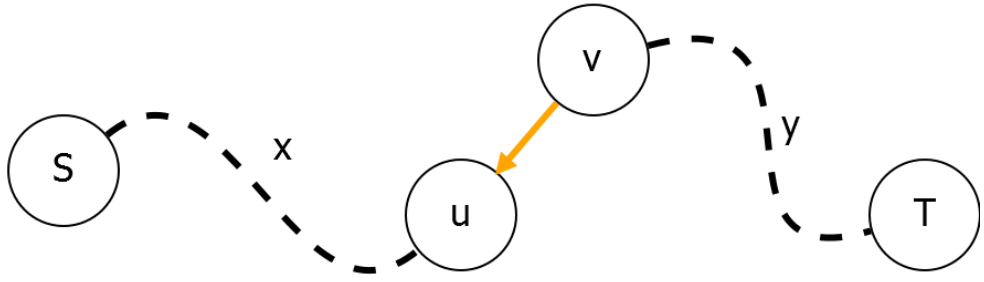
# Reuse Limit

## Lemma

When running the Edmonds-Karp algorithm, if an edge $e$ is saturated, it will not be used in an augmenting path again, until $d_{G_f}(s, t)$ increases.

# Proof

- Initially, $d(s,u) = x, d(v,t) = y, d(s,t) = x + y + 1$.
- When used again, $d(s,v) \geq x + 1$, $d(u,t) \geq y + 1$.
- Therefore, when used again, $d(s,t) \geq (x+1)+(y+1)+1 = x+y+3$.



# Analysis

- $d_{G_f}(s, t)$ can only increase $|V|$ times.
- Each time can only have $O(|E|)$ many saturated edges.
- Therefore, only $O(|V||E|)$ many augmenting path.
- Each path takes $O(|E|)$ time.
- Total runtime: $O(|V||E|^2)$.

# Problem

Which of the following is true about the Edmonds-Karp algorithm:

1. No edge is saturated more than $|V|$ times.
2. The lengths of the augmenting paths decrease as the algorithm progresses.
3. Changing the capacities of edges will not affect the runtime.

# Solution

Which of the following is true about the Edmonds-Karp algorithm:

1. No edge is saturated more than $|V|$ times.
2. The lengths of the augmenting paths decrease as the algorithm progresses.
3. Changing the capacities of edges will not affect the runtime.

# Summary

- Choose augmenting paths based on path length.
- Removes runtime dependence on numerical sizes of capacities.

# Summary

- Choose augmenting paths based on path length.
- Removes runtime dependence on numerical sizes of capacities.
- There are better, much more complicated algorithms.
- State of the art $O(|V||E|)$.

# Next Time

Applications of Maxflow algorithms.