# Linear Programming: Gaussian Elimination

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## Advanced Algorithms and Complexity
## Data Structures and Algorithms

## Learning Objectives

- Translate between systems of equations and augmented matrices.
- Row reduce a matrix.
- Write an algorithm to solve linear systems.

# Last Time

Solving systems of linear equations by substitution.

# Notation

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$$\begin{aligned} x + y &= 5 \\ 2x + 4y &= 12. \end{aligned}$$

We just store coefficients of equations in an (augmented) matrix, like so:

$$\begin{array}{c} \begin{array}{ccc} x & y & = 1 \end{array} \\ \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 2 & 4 & 12 \end{array}\right] \end{array}$$

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Subtract twice first row from second.

# Basic Row Operations

There are three basic ways to manipulate our matrix. These are called Basic row operations. Each of them gives us an equivalent system of equations.

# Adding

Add/subtract a multiple of one row to another.

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$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 2 & 4 & 12 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 2 & 2 \end{array}\right]$$

# Scaling

Multiply/divide a row by a non-zero constant.

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$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 2 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 1 \end{array}\right]$$

# Swapping

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Sometimes you want to change the ordering of rows. For example, swapping the first and second rows we get

$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 1 & 5 \end{array}\right]$$

# Row Reduction

Row reduction uses row operations to put a matrix into a simple standard form. The idea is to simulate the substitution method.

# Example

Consider the system given by the matrix:

$$\left[\begin{array}{rrrr|r} 2 & 4 & -2 & 0 & 2 \\ -1 & -2 & 1 & -2 & -1 \\ 2 & 2 & 0 & 2 & 0 \end{array}\right]$$

# Example

Use first for to solve for first variable.

$$\left[\begin{array}{cccc|c} 2 & 4 & -2 & 0 & 2 \\ -1 & -2 & 1 & -2 & -1 \\ 2 & 2 & 0 & 2 & 0 \end{array}\right]$$

# Example

Divide first row by 2.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ -1 & -2 & 1 & -2 & -1 \\ 2 & 2 & 0 & 2 & 0 \end{array}\right]$$

# Example

Substitute into other equations.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ -1 & -2 & 1 & -2 & -1 \\ 2 & 2 & 0 & 2 & 0 \end{array}\right]$$

# Example

Add first row to second.

$$\left[\begin{array}{rrrr|r} 1 & 2 & -1 & 0 & 1 \\ 0 & 0 & 0 & -2 & 0 \\ 2 & 2 & 0 & 2 & 0 \end{array}\right]$$

# Example

Subtract twice first row from third.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & -2 & 2 & 2 & -2 \end{array}\right]$$

# Example

Need to solve for next variable.

$$\left[\begin{array}{rrrr|r} 1 & 2 & -1 & 0 & 1 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & -2 & 2 & 2 & -2 \end{array}\right]$$

# Example

Cannot use second row.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & -2 & 2 & 2 & -2 \end{array}\right]$$

# Example

Swap second and third rows.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ 0 & -2 & 2 & 2 & -2 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right]$$

# Example

Solve for second variable.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ 0 & -2 & 2 & 2 & -2 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right]$$

# Example

Divide second row by $-2$.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right]$$

# Example

Substitute into other equations.

$$\left[\begin{array}{rrrr|r} 1 & 2 & -1 & 0 & 1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right]$$

# Example

Subtract twice second row from first.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right]$$

# Example

Can't solve for third variable.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right]$$

# Example

Solve for fourth instead.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right]$$

# Example

Divide last row by $-2$.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

# Example

Substitute into other equations.

$$\left[\begin{array}{rrrr|r} 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

# Example

Subtract twice third row from first.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

# Example

Add third row to second.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

# Example

Done.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

# Answer

Our matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

corresponds to equations:

$$\begin{aligned} x + z &= -1 \\ y - z &= 1 \\ w &= 0. \end{aligned}$$

# Solution

So for any value of $z$, we have solution:

$$\begin{aligned} x &= -1 - z \\ y &= 1 + z \\ w &= 0. \end{aligned}$$

## RowReduce($A$)

Leftmost non-zero

Swap row to top

Make entry pivot

Rescale to make pivot 1

Subtract row from others to make other entries in column 0

Repeat

## RowReduce($A$)

Leftmost non-zero in non-pivot row
Swap row to top of non-pivot rows
Make entry pivot
Rescale to make pivot 1
Subtract row from others to make
other entries in column 0
Repeat until no more non-zero
entries outside of pivot rows

# Reading off Answer

- Each row has one pivot and a few other non-pivot entries.
- Gives equation writing pivot variable in terms of non-pivot variables.
- If pivot in units column, have equation $0 = 1$, so no solutions.
- Otherwise, set non-pivot variables to anything, gives answer.

# Runtime

- $m$ equations in $n$ variables.
- $\min(n, m)$ pivots.
- For each pivot, need to subtract multiple of row from each other row $O(nm)$ time.
- Total runtime: $O(nm \min(n, m))$.