

Basic Data Structures: Dynamic Arrays and Amortized Analysis

Neil Rhodes

Department of Computer Science and Engineering
University of California, San Diego

Data Structures
Data Structures and Algorithms

Outline

- 1 Dynamic Arrays
- 2 Amortized Analysis—Aggregate Method
- 3 Amortized Analysis—Banker's Method
- 4 Amortized Analysis—Physicist's Method

Problem: static arrays are static!

```
int my_array[100];
```

Problem: static arrays are static!

```
int my_array[100];
```

Semi-solution: dynamically-allocated arrays:

```
int *my_array = new int[size];
```

Problem: might not know max size when allocating an array

Problem: might not know max size when allocating an array

*All problems in computer science
can be solved by another level of
indirection.*

Problem: might not know max size when allocating an array

All problems in computer science can be solved by another level of indirection.

Solution: *dynamic arrays* (also known as *resizable arrays*)

Idea: store a pointer to a dynamically allocated array, and replace it with a newly-allocated array as needed.

Definition

Dynamic Array:

Abstract data type with the following operations (at a minimum):

*must be constant time

Definition

Dynamic Array:

Abstract data type with the following operations (at a minimum):

- $\text{Get}(i)$: returns element at location i^*

*must be constant time

Definition

Dynamic Array:

Abstract data type with the following operations (at a minimum):

- $\text{Get}(i)$: returns element at location i^*
- $\text{Set}(i, val)$: Sets element i to val^*

*must be constant time

Definition

Dynamic Array:

Abstract data type with the following operations (at a minimum):

- $\text{Get}(i)$: returns element at location i^*
- $\text{Set}(i, val)$: Sets element i to val^*
- $\text{PushBack}(val)$: Adds val to the end

*must be constant time

Definition

Dynamic Array:

Abstract data type with the following operations (at a minimum):

- $\text{Get}(i)$: returns element at location i^*
- $\text{Set}(i, val)$: Sets element i to val^*
- $\text{PushBack}(val)$: Adds val to the end
- $\text{Remove}(i)$: Removes element at location i

*must be constant time

Definition

Dynamic Array:

Abstract data type with the following operations (at a minimum):

- $\text{Get}(i)$: returns element at location i^*
- $\text{Set}(i, val)$: Sets element i to val^*
- $\text{PushBack}(val)$: Adds val to the end
- $\text{Remove}(i)$: Removes element at location i
- $\text{Size}()$: the number of elements

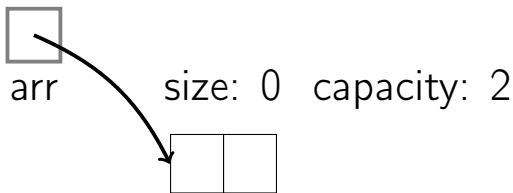
*must be constant time

Implementation

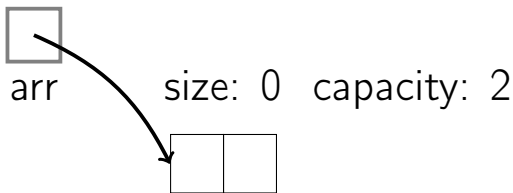
Store:

- `arr`: dynamically-allocated array
- `capacity`: size of the dynamically-allocated array
- `size`: number of elements currently in the array

Dynamic Array Resizing

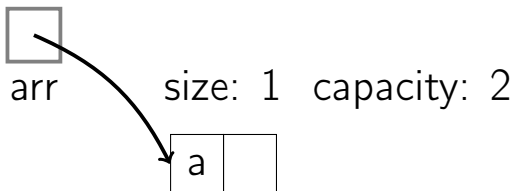


Dynamic Array Resizing



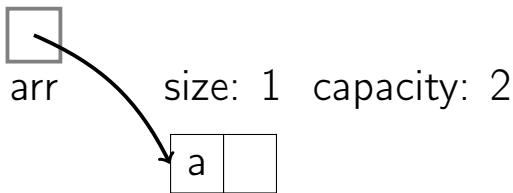
PushBack(a)

Dynamic Array Resizing

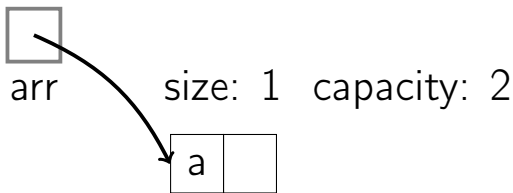


PushBack(a)

Dynamic Array Resizing

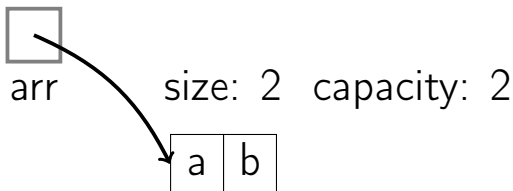


Dynamic Array Resizing



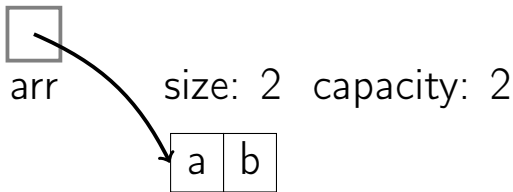
PushBack(b)

Dynamic Array Resizing

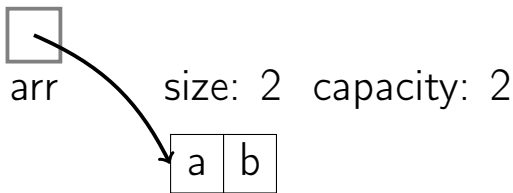


PushBack(b)

Dynamic Array Resizing

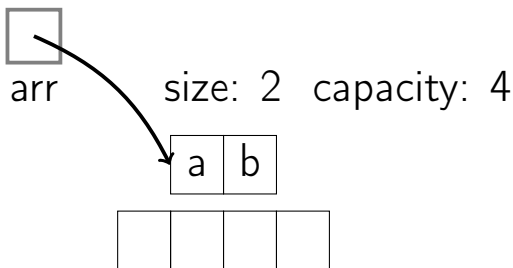


Dynamic Array Resizing



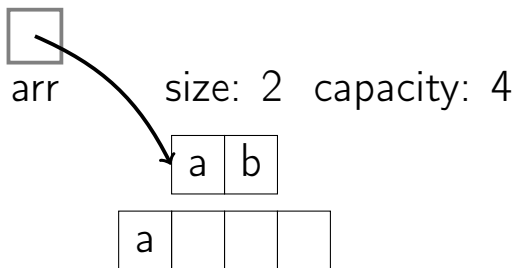
PushBack(c)

Dynamic Array Resizing



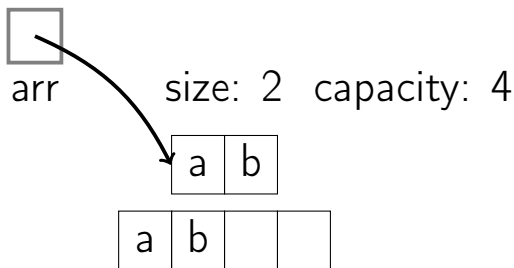
PushBack(c)

Dynamic Array Resizing



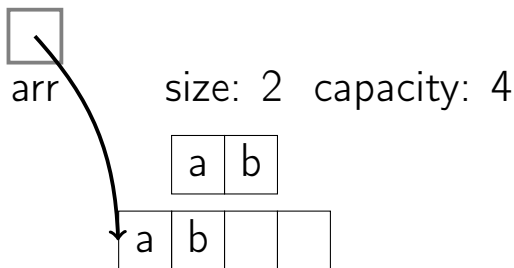
PushBack(c)

Dynamic Array Resizing



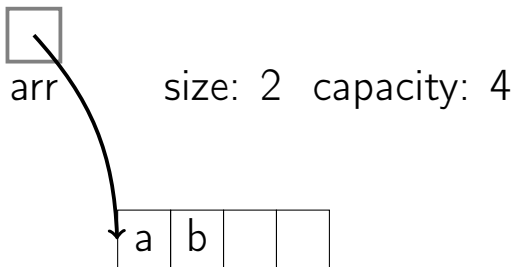
PushBack(c)

Dynamic Array Resizing



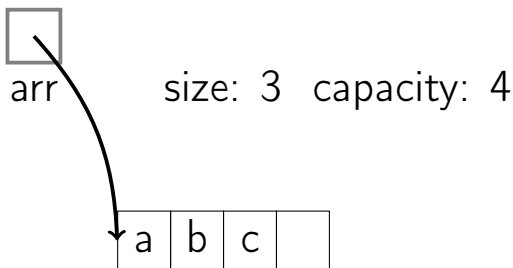
PushBack(c)

Dynamic Array Resizing



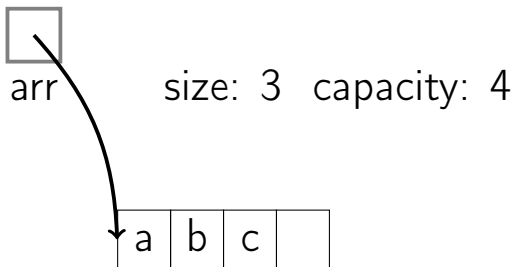
`PushBack(c)`

Dynamic Array Resizing

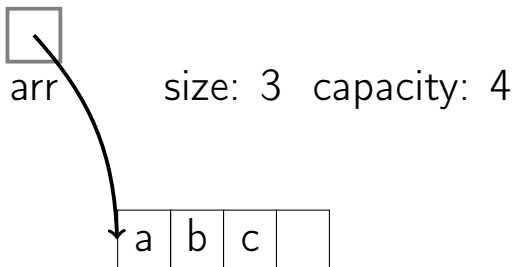


`PushBack(c)`

Dynamic Array Resizing

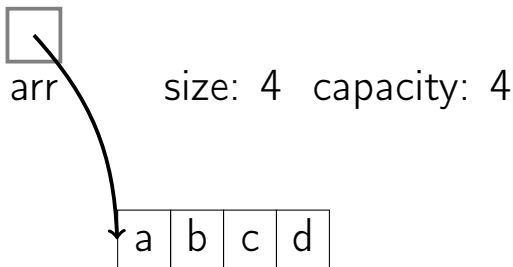


Dynamic Array Resizing



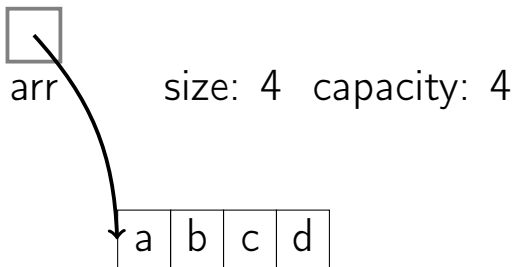
PushBack(d)

Dynamic Array Resizing

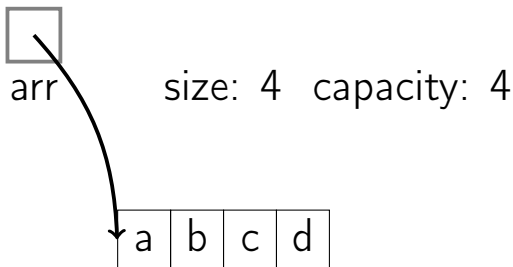


`PushBack(d)`

Dynamic Array Resizing

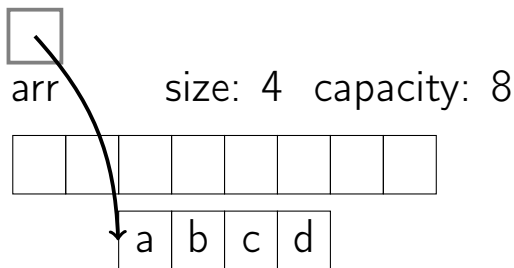


Dynamic Array Resizing



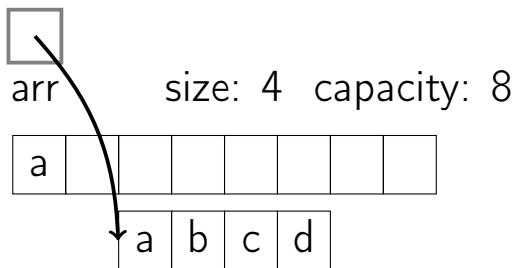
`PushBack(e)`

Dynamic Array Resizing



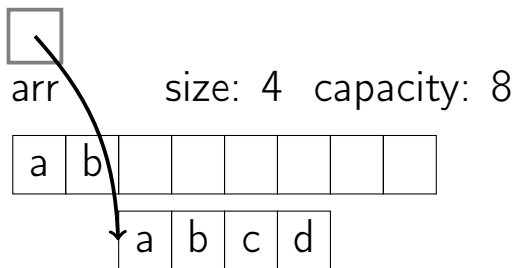
PushBack(e)

Dynamic Array Resizing



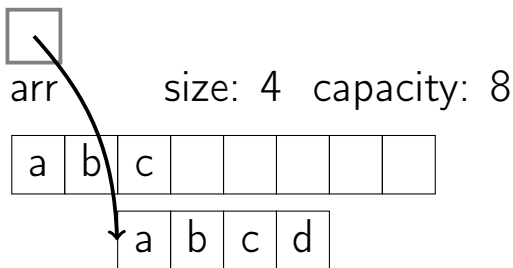
PushBack(e)

Dynamic Array Resizing



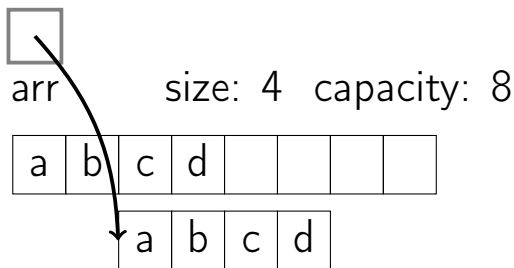
PushBack(e)

Dynamic Array Resizing



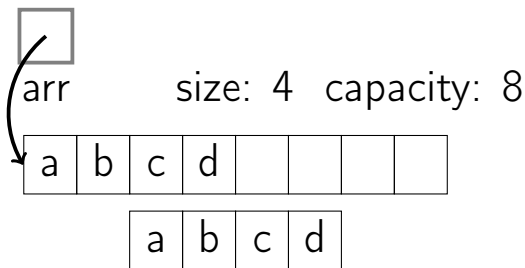
PushBack(e)

Dynamic Array Resizing



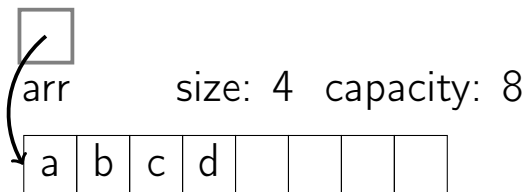
PushBack(e)

Dynamic Array Resizing



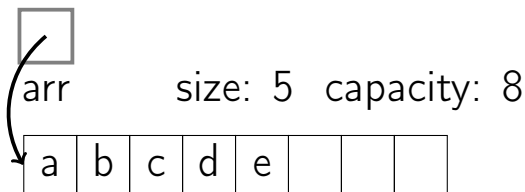
PushBack(e)

Dynamic Array Resizing



PushBack(e)

Dynamic Array Resizing



PushBack(e)

Get(*i*)

```
if  $i < 0$  or  $i \geq size$ :
```

```
    ERROR: index out of range
```

```
return arr[i]
```

Set(i , val)

if $i < 0$ or $i \geq size$:

 ERROR: index out of range

arr[i] = val

PushBack(*val*)

```
if size = capacity:  
    allocate new_arr[ $2 \times \textit{capacity}$ ]  
    for i from 0 to size - 1:  
        new_arr[i]  $\leftarrow$  arr[i]  
    free arr  
    arr  $\leftarrow$  new_arr; capacity  $\leftarrow 2 \times \textit{capacity}$   
arr[size]  $\leftarrow$  val  
size  $\leftarrow$  size + 1
```

Remove(*i*)

if $i < 0$ or $i \geq \text{size}$:

 ERROR: index out of range

for j from i to $\text{size} - 2$:

$\text{arr}[j] \leftarrow \text{arr}[j + 1]$

$\text{size} \leftarrow \text{size} - 1$

```
Size()
```

```
return size
```

Common Implementations

- C++: `vector`
- Java: `ArrayList`
- Python: `list` (the only kind of array)

Runtimes

Get(i) | $O(1)$

Runtimes

Get(<i>i</i>)	$O(1)$
Set(<i>i</i> , <i>val</i>)	$O(1)$

Runtimes

Get(i)	$O(1)$
Set(i, val)	$O(1)$
PushBack(val)	$O(n)$

Runtimes

Get(i)	$O(1)$
Set(i, val)	$O(1)$
PushBack(val)	$O(n)$
Remove(i)	$O(n)$

Runtimes

Get(i)	$O(1)$
Set(i, val)	$O(1)$
PushBack(val)	$O(n)$
Remove(i)	$O(n)$
Size()	$O(1)$

Summary

- Unlike static arrays, dynamic arrays can be resized.

Summary

- Unlike static arrays, dynamic arrays can be resized.
- Appending a new element to a dynamic array is often constant time, but can take $O(n)$.

Summary

- Unlike static arrays, dynamic arrays can be resized.
- Appending a new element to a dynamic array is often constant time, but can take $O(n)$.
- Some space is wasted

Summary

- Unlike static arrays, dynamic arrays can be resized.
- Appending a new element to a dynamic array is often constant time, but can take $O(n)$.
- Some space is wasted

Summary

- Unlike static arrays, dynamic arrays can be resized.
- Appending a new element to a dynamic array is often constant time, but can take $O(n)$.
- Some space is wasted—at most half.

Outline

- ① Dynamic Arrays
- ② Amortized Analysis—Aggregate Method
- ③ Amortized Analysis—Banker's Method
- ④ Amortized Analysis—Physicist's Method

Sometimes, looking at the individual worst-case may be too severe. We may want to know the total worst-case cost for a sequence of operations.

Dynamic Array

We only resize every so often.

Many $O(1)$ operations are followed by an $O(n)$ operations.

What is the total cost of inserting many elements?

Definition

Amortized cost: Given a sequence of n operations, the amortized cost is:

$$\frac{\text{Cost}(n \text{ operations})}{n}$$

Aggregate Method

Dynamic array: n calls to PushBack

Aggregate Method

Dynamic array: n calls to PushBack

Let $c_i =$ cost of i 'th insertion.

Aggregate Method

Dynamic array: n calls to PushBack

Let $c_i = \text{cost of } i\text{'th insertion.}$

$$c_i = 1 + \left\{ \right.$$

Aggregate Method

Dynamic array: n calls to PushBack

Let $c_i = \text{cost of } i\text{'th insertion.}$

$$c_i = 1 + \begin{cases} i - 1 & \text{if } i - 1 \text{ is a power of 2} \end{cases}$$

Aggregate Method

Dynamic array: n calls to PushBack

Let $c_i =$ cost of i 'th insertion.

$$c_i = 1 + \begin{cases} i - 1 & \text{if } i - 1 \text{ is a power of 2} \\ 0 & \text{otherwise} \end{cases}$$

Aggregate Method

Dynamic array: n calls to PushBack

Let $c_i = \text{cost of } i\text{'th insertion.}$

$$c_i = 1 + \begin{cases} i - 1 & \text{if } i - 1 \text{ is a power of 2} \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\sum_{i=1}^n c_i}{n}$$

Aggregate Method

Dynamic array: n calls to PushBack

Let $c_i = \text{cost of } i\text{'th insertion.}$

$$c_i = 1 + \begin{cases} i - 1 & \text{if } i - 1 \text{ is a power of 2} \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\sum_{i=1}^n c_i}{n} = \frac{n + \sum_{j=1}^{\lfloor \log_2(n-1) \rfloor} 2^j}{n}$$

Aggregate Method

Dynamic array: n calls to PushBack

Let $c_i =$ cost of i 'th insertion.

$$c_i = 1 + \begin{cases} i - 1 & \text{if } i - 1 \text{ is a power of 2} \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\sum_{i=1}^n c_i}{n} = \frac{n + \sum_{j=1}^{\lfloor \log_2(n-1) \rfloor} 2^j}{n} = \frac{O(n)}{n}$$

Aggregate Method

Dynamic array: n calls to PushBack

Let $c_i =$ cost of i 'th insertion.

$$c_i = 1 + \begin{cases} i - 1 & \text{if } i - 1 \text{ is a power of 2} \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\sum_{i=1}^n c_i}{n} = \frac{n + \sum_{j=1}^{\lfloor \log_2(n-1) \rfloor} 2^j}{n} = \frac{O(n)}{n} = O(1)$$

Outline

- 1 Dynamic Arrays
- 2 Amortized Analysis—Aggregate Method
- 3 Amortized Analysis—Banker's Method
- 4 Amortized Analysis—Physicist's Method

Banker's Method

- Charge extra for each cheap operation.

Banker's Method

- Charge extra for each cheap operation.
- Save the extra charge as tokens in your data structure (conceptually).

Banker's Method

- Charge extra for each cheap operation.
- Save the extra charge as tokens in your data structure (conceptually).
- Use the tokens to pay for expensive operations.

Banker's Method

- Charge extra for each cheap operation.
- Save the extra charge as tokens in your data structure (conceptually).
- Use the tokens to pay for expensive operations.

Like an amortizing loan.

Banker's Method

Dynamic array: n calls to PushBack

Banker's Method

Dynamic array: n calls to PushBack

Charge 3 for each insertion: 1 token is the raw cost for insertion.

Banker's Method

Dynamic array: n calls to PushBack

Charge 3 for each insertion: 1 token is the raw cost for insertion.

- Resize needed: To pay for moving the elements, use the token that's present on each element that needs to move.

Banker's Method

Dynamic array: n calls to PushBack

Charge 3 for each insertion: 1 token is the raw cost for insertion.

- Resize needed: To pay for moving the elements, use the token that's present on each element that needs to move.
- Place one token on the newly-inserted element, and one token $\frac{\text{capacity}}{2}$ elements prior.

Dynamic Array Resizing



arr

size: 0 capacity: 0

Dynamic Array Resizing



arr

size: 0 capacity: 0

PushBack(a)

Dynamic Array Resizing



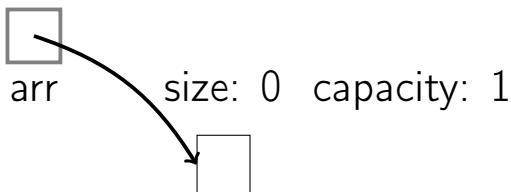
arr

size: 0 capacity: 1



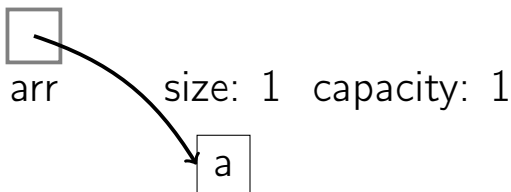
PushBack(a)

Dynamic Array Resizing



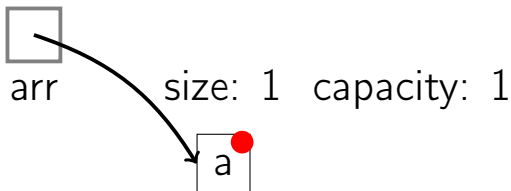
PushBack(a)

Dynamic Array Resizing



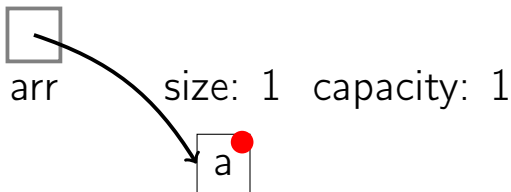
PushBack(a)

Dynamic Array Resizing

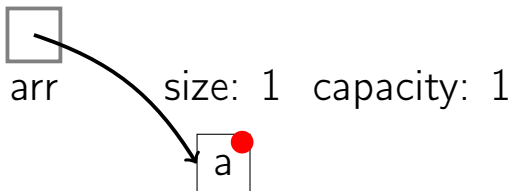


PushBack(a)

Dynamic Array Resizing

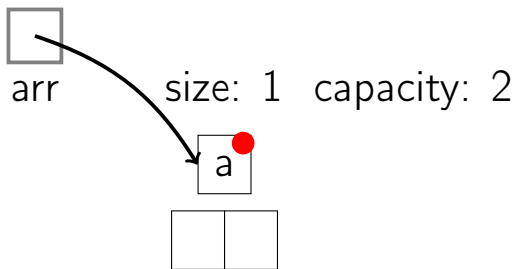


Dynamic Array Resizing



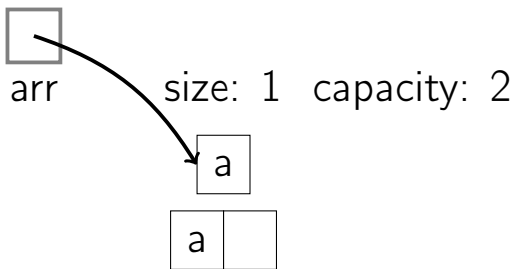
PushBack(b)

Dynamic Array Resizing



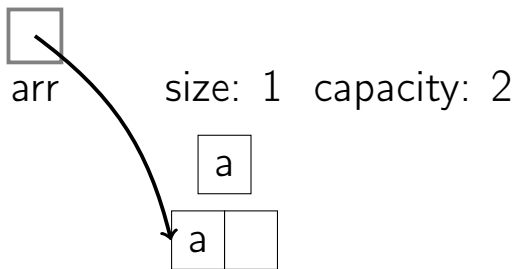
`PushBack(b)`

Dynamic Array Resizing



PushBack(b)

Dynamic Array Resizing



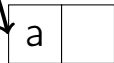
`PushBack(b)`

Dynamic Array Resizing



arr

size: 1 capacity: 2



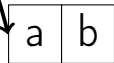
PushBack(b)

Dynamic Array Resizing



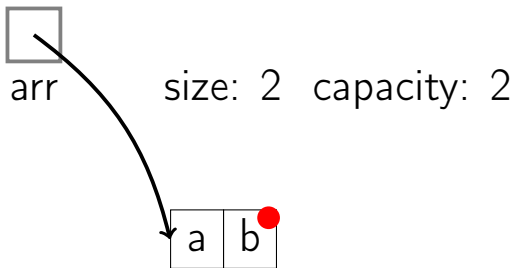
arr

size: 2 capacity: 2



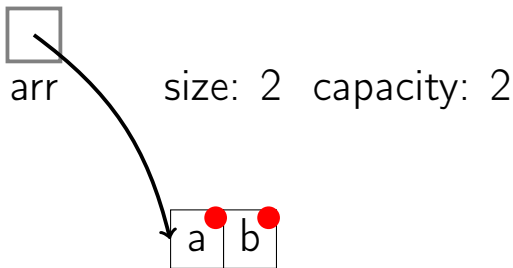
PushBack(b)

Dynamic Array Resizing



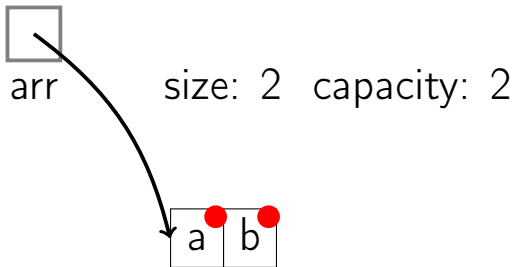
PushBack(b)

Dynamic Array Resizing

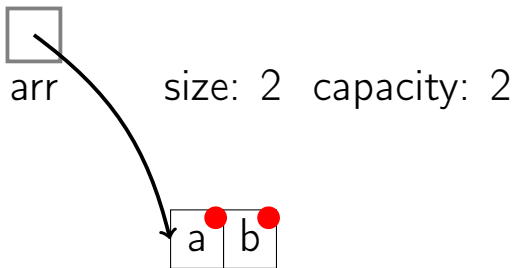


PushBack(b)

Dynamic Array Resizing

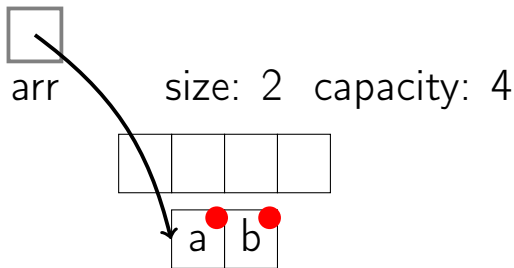


Dynamic Array Resizing



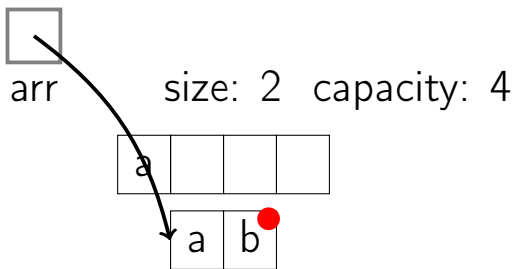
PushBack(c)

Dynamic Array Resizing



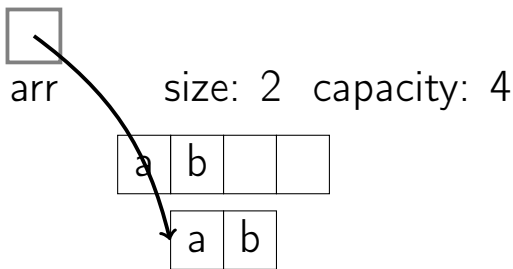
PushBack(c)

Dynamic Array Resizing



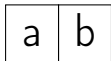
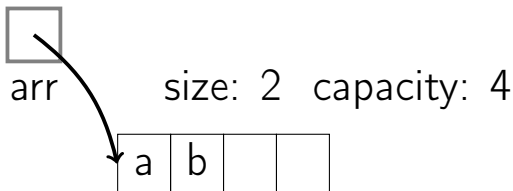
PushBack(c)

Dynamic Array Resizing



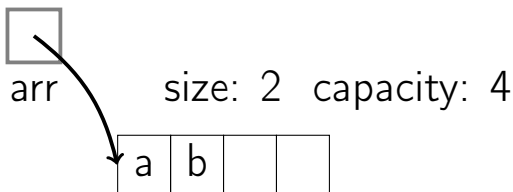
PushBack(c)

Dynamic Array Resizing



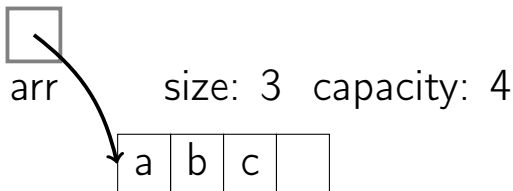
PushBack(c)

Dynamic Array Resizing



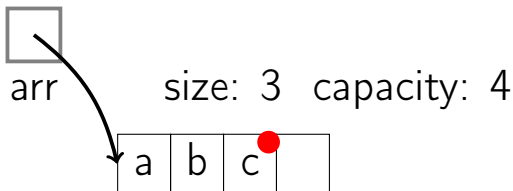
PushBack(c)

Dynamic Array Resizing



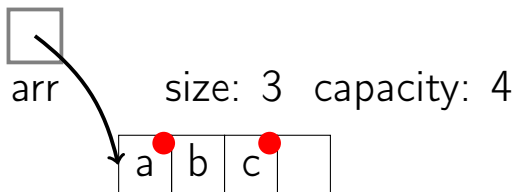
`PushBack(c)`

Dynamic Array Resizing



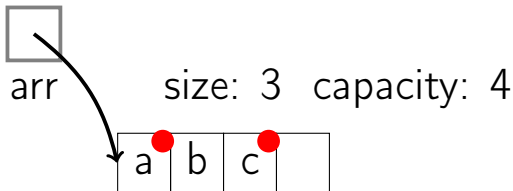
PushBack(c)

Dynamic Array Resizing

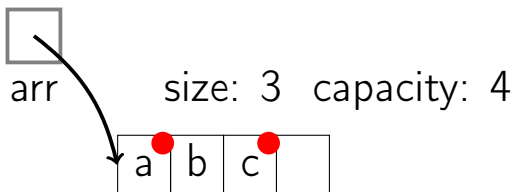


`PushBack(c)`

Dynamic Array Resizing

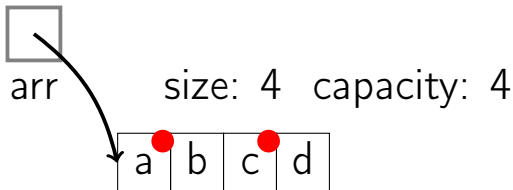


Dynamic Array Resizing



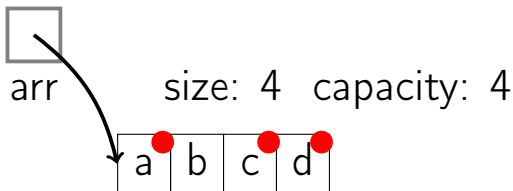
PushBack(d)

Dynamic Array Resizing



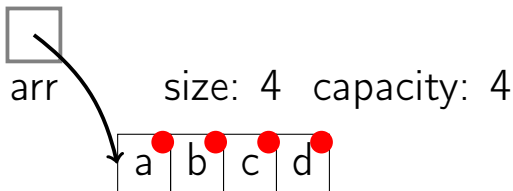
PushBack(d)

Dynamic Array Resizing



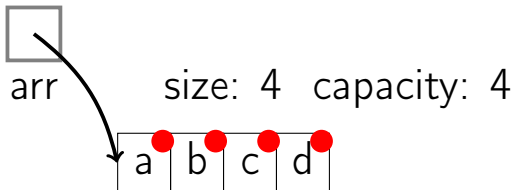
PushBack(d)

Dynamic Array Resizing

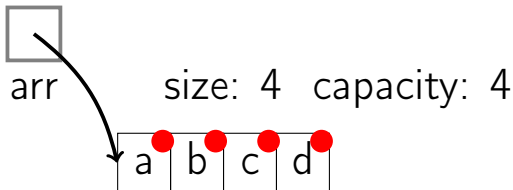


`PushBack(d)`

Dynamic Array Resizing

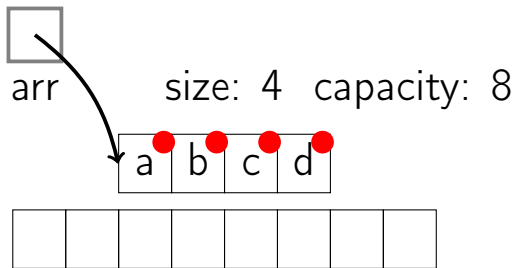


Dynamic Array Resizing



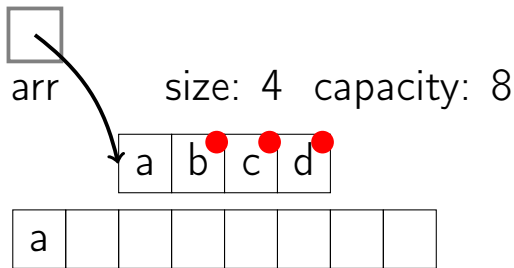
`PushBack(e)`

Dynamic Array Resizing



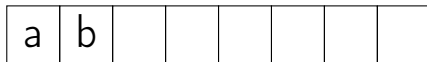
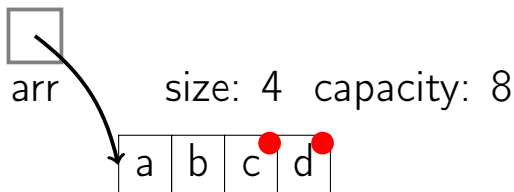
PushBack(e)

Dynamic Array Resizing



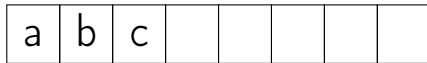
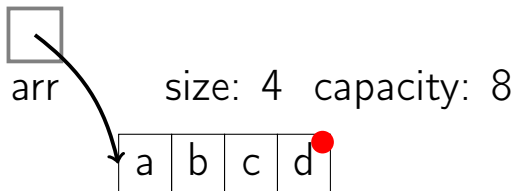
PushBack(e)

Dynamic Array Resizing



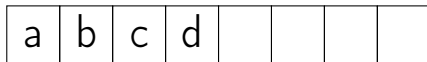
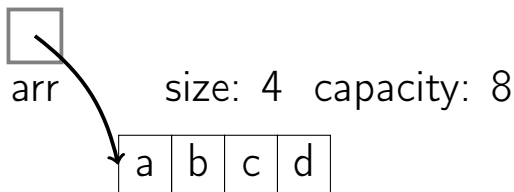
PushBack(e)

Dynamic Array Resizing



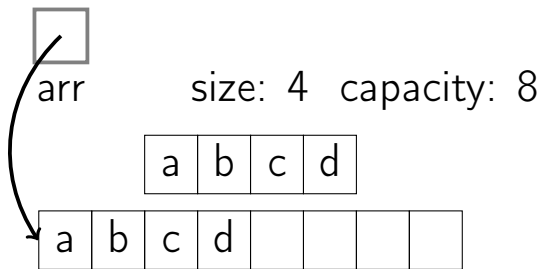
PushBack(e)

Dynamic Array Resizing



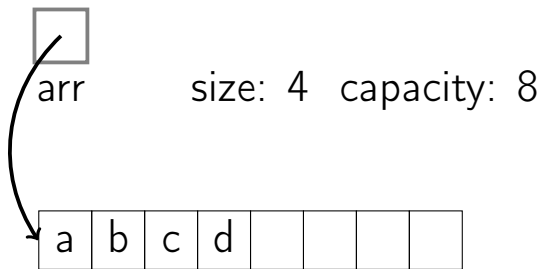
PushBack(e)

Dynamic Array Resizing



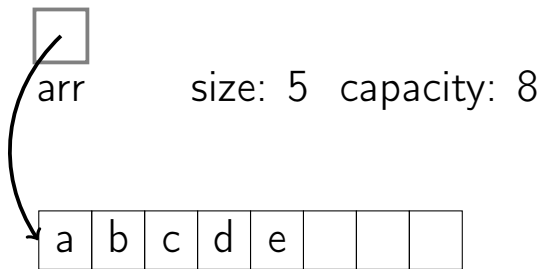
PushBack(e)

Dynamic Array Resizing



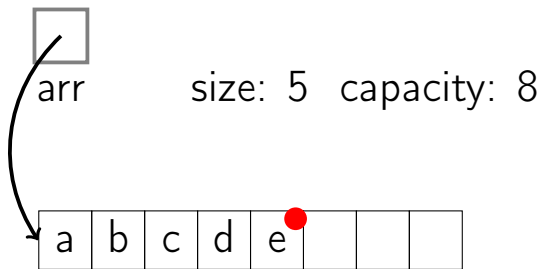
PushBack(e)

Dynamic Array Resizing



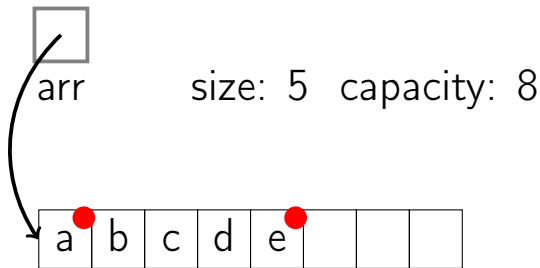
PushBack(e)

Dynamic Array Resizing



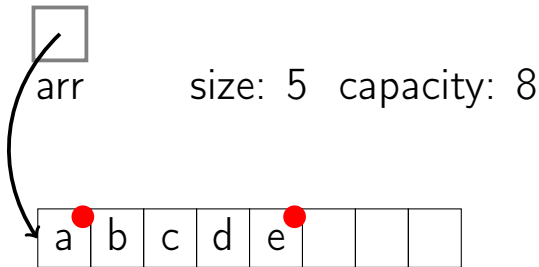
PushBack(e)

Dynamic Array Resizing

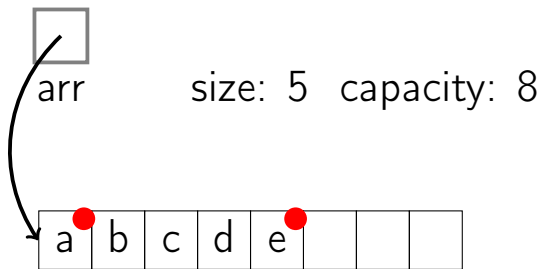


PushBack(e)

Dynamic Array Resizing



Dynamic Array Resizing



$O(1)$ amortized cost for each PushBack

Banker's Method

Dynamic array: n calls to PushBack

Charge 3 for each insertion. 1 coin is the raw cost for insertion.

- Resize needed: To pay for moving the elements, use the coin that's present on each element that needs to move.
- Place one coin on the newly-inserted element, and one coin $\frac{\text{capacity}}{2}$ elements prior.

Outline

- ① Dynamic Arrays
- ② Amortized Analysis—Aggregate Method
- ③ Amortized Analysis—Banker's Method
- ④ Amortized Analysis—Physicist's Method

Physicist's Method

- Define a *potential function*, Φ which maps states of the data structure to integers:

Physicist's Method

- Define a *potential function*, Φ which maps states of the data structure to integers:
 - $\Phi(h_0) = 0$

Physicist's Method

- Define a *potential function*, Φ which maps states of the data structure to integers:
 - $\Phi(h_0) = 0$
 - $\Phi(h_t) \geq 0$

Physicist's Method

- Define a *potential function*, Φ which maps states of the data structure to integers:
 - $\Phi(h_0) = 0$
 - $\Phi(h_t) \geq 0$
- amortized cost for operation t :
 $c_t + \Phi(h_t) - \Phi(h_{t-1})$

Physicist's Method

- Define a *potential function*, Φ which maps states of the data structure to integers:
 - $\Phi(h_0) = 0$
 - $\Phi(h_t) \geq 0$
- amortized cost for operation t :
$$c_t + \Phi(h_t) - \Phi(h_{t-1})$$

Choose Φ so that:

Physicist's Method

- Define a *potential function*, Φ which maps states of the data structure to integers:
 - $\Phi(h_0) = 0$
 - $\Phi(h_t) \geq 0$
- amortized cost for operation t :
$$c_t + \Phi(h_t) - \Phi(h_{t-1})$$

Choose Φ so that:

- if c_t is small, the potential increases

Physicist's Method

- Define a *potential function*, Φ which maps states of the data structure to integers:
 - $\Phi(h_0) = 0$
 - $\Phi(h_t) \geq 0$
- amortized cost for operation t :
$$c_t + \Phi(h_t) - \Phi(h_{t-1})$$

Choose Φ so that:

- if c_t is small, the potential increases
- if c_t is large, the potential decreases by the same scale

Physicist's Method

- The cost of n operations is: $\sum_{i=1}^n c_i$

Physicist's Method

- The cost of n operations is: $\sum_{i=1}^n c_i$
- The sum of the amortized costs is:

$$\sum_{i=1}^n (c_i + \Phi(h_i) - \Phi(h_{i-1}))$$

Physicist's Method

- The cost of n operations is: $\sum_{i=1}^n c_i$
- The sum of the amortized costs is:

$$\begin{aligned} & \sum_{i=1}^n (c_i + \Phi(h_i) - \Phi(h_{i-1})) \\ &= c_1 + \Phi(h_1) - \Phi(h_0) + \\ & \quad c_2 + \Phi(h_2) - \Phi(h_1) \cdots + \\ & \quad c_n + \Phi(h_n) - \Phi(h_{n-1}) \end{aligned}$$

Physicist's Method

- The cost of n operations is: $\sum_{i=1}^n c_i$
- The sum of the amortized costs is:

$$\begin{aligned} & \sum_{i=1}^n (c_i + \Phi(h_i) - \Phi(h_{i-1})) \\ &= c_1 + \Phi(h_1) - \Phi(h_0) + \\ & \quad c_2 + \Phi(h_2) - \Phi(h_1) \cdots + \\ & \quad c_n + \Phi(h_n) - \Phi(h_{n-1}) \\ &= \Phi(h_n) - \Phi(h_0) + \sum_{i=1}^n c_i \end{aligned}$$

Physicist's Method

- The cost of n operations is: $\sum_{i=1}^n c_i$
- The sum of the amortized costs is:

$$\begin{aligned} & \sum_{i=1}^n (c_i + \Phi(h_i) - \Phi(h_{i-1})) \\ &= c_1 + \Phi(h_1) - \Phi(h_0) + \\ & \quad c_2 + \Phi(h_2) - \Phi(h_1) \cdots + \\ & \quad c_n + \Phi(h_n) - \Phi(h_{n-1}) \\ &= \Phi(h_n) - \Phi(h_0) + \sum_{i=1}^n c_i \geq \sum_{i=1}^n c_i \end{aligned}$$

Physicist's Method

Dynamic array: n calls to PushBack

Physicist's Method

Dynamic array: n calls to PushBack

Let $\Phi(h) = 2 \times \textit{size} - \textit{capacity}$

Physicist's Method

Dynamic array: n calls to PushBack

Let $\Phi(h) = 2 \times \textit{size} - \textit{capacity}$

- $\Phi(h_0) = 2 \times 0 - 0 = 0$

Physicist's Method

Dynamic array: n calls to PushBack

Let $\Phi(h) = 2 \times \textit{size} - \textit{capacity}$

- $\Phi(h_0) = 2 \times 0 - 0 = 0$
- $\Phi(h_i) = 2 \times \textit{size} - \textit{capacity} > 0$
(since $\textit{size} > \frac{\textit{capacity}}{2}$)

Dynamic Array Resizing

Without resize when adding element i

Amortized cost of adding element i :

Dynamic Array Resizing

Without resize when adding element i

Amortized cost of adding element i :

Dynamic Array Resizing

Without resize when adding element i

Amortized cost of adding element i :

$$c_i + \Phi(h_i) - \Phi(h_{i-1})$$

Dynamic Array Resizing

Without resize when adding element i

Amortized cost of adding element i :

$$\begin{aligned} & c_i + \Phi(h_i) - \Phi(h_{i-1}) \\ &= 1 + 2 \times \textit{size}_i - \textit{cap}_i - (2 \times \textit{size}_{i-1} - \textit{cap}_{i-1}) \end{aligned}$$

Dynamic Array Resizing

Without resize when adding element i

Amortized cost of adding element i :

$$\begin{aligned} & c_i + \Phi(h_i) - \Phi(h_{i-1}) \\ &= 1 + 2 \times \textit{size}_i - \textit{cap}_i - (2 \times \textit{size}_{i-1} - \textit{cap}_{i-1}) \\ &= 1 + 2 \times (\textit{size}_i - \textit{size}_{i-1}) \end{aligned}$$

Dynamic Array Resizing

Without resize when adding element i

Amortized cost of adding element i :

$$\begin{aligned} & c_i + \Phi(h_i) - \Phi(h_{i-1}) \\ &= 1 + 2 \times \textit{size}_i - \textit{cap}_i - (2 \times \textit{size}_{i-1} - \textit{cap}_{i-1}) \\ &= 1 + 2 \times (\textit{size}_i - \textit{size}_{i-1}) \\ &= 3 \end{aligned}$$

Dynamic Array Resizing

With resize when adding element i

Dynamic Array Resizing

With resize when adding element i

Let $k = \text{size}_{i-1} = \text{cap}_{i-1}$

Dynamic Array Resizing

With resize when adding element i

Let $k = \text{size}_{i-1} = \text{cap}_{i-1}$

Then:

$$\Phi(h_{i-1}) = 2\text{size}_{i-1} - \text{cap}_{i-1} = 2k - k = k$$

Dynamic Array Resizing

With resize when adding element i

Let $k = \text{size}_{i-1} = \text{cap}_{i-1}$

Then:

$$\Phi(h_{i-1}) = 2\text{size}_{i-1} - \text{cap}_{i-1} = 2k - k = k$$

$$\Phi(h_i) = 2\text{size}_i - \text{cap}_i = 2(k+1) - 2k = 2$$

Dynamic Array Resizing

With resize when adding element i

Let $k = \text{size}_{i-1} = \text{cap}_{i-1}$

Then:

$$\Phi(h_{i-1}) = 2\text{size}_{i-1} - \text{cap}_{i-1} = 2k - k = k$$

$$\Phi(h_i) = 2\text{size}_i - \text{cap}_i = 2(k+1) - 2k = 2$$

Amortized cost of adding element i :

Dynamic Array Resizing

With resize when adding element i

Let $k = \text{size}_{i-1} = \text{cap}_{i-1}$

Then:

$$\Phi(h_{i-1}) = 2\text{size}_{i-1} - \text{cap}_{i-1} = 2k - k = k$$

$$\Phi(h_i) = 2\text{size}_i - \text{cap}_i = 2(k+1) - 2k = 2$$

Amortized cost of adding element i :

$$c_i + \Phi(h_i) - \Phi(h_{i-1})$$

Dynamic Array Resizing

With resize when adding element i

Let $k = \text{size}_{i-1} = \text{cap}_{i-1}$

Then:

$$\Phi(h_{i-1}) = 2\text{size}_{i-1} - \text{cap}_{i-1} = 2k - k = k$$

$$\Phi(h_i) = 2\text{size}_i - \text{cap}_i = 2(k+1) - 2k = 2$$

Amortized cost of adding element i :

$$\begin{aligned} & c_i + \Phi(h_i) - \Phi(h_{i-1}) \\ &= (\text{size}_i) + 2 - k \end{aligned}$$

Dynamic Array Resizing

With resize when adding element i

Let $k = \text{size}_{i-1} = \text{cap}_{i-1}$

Then:

$$\Phi(h_{i-1}) = 2\text{size}_{i-1} - \text{cap}_{i-1} = 2k - k = k$$

$$\Phi(h_i) = 2\text{size}_i - \text{cap}_i = 2(k+1) - 2k = 2$$

Amortized cost of adding element i :

$$\begin{aligned} & c_i + \Phi(h_i) - \Phi(h_{i-1}) \\ &= (\text{size}_i) + 2 - k \\ &= (k+1) + 2 - k \end{aligned}$$

Dynamic Array Resizing

With resize when adding element i

Let $k = \text{size}_{i-1} = \text{cap}_{i-1}$

Then:

$$\Phi(h_{i-1}) = 2\text{size}_{i-1} - \text{cap}_{i-1} = 2k - k = k$$

$$\Phi(h_i) = 2\text{size}_i - \text{cap}_i = 2(k+1) - 2k = 2$$

Amortized cost of adding element i :

$$\begin{aligned} & c_i + \Phi(h_i) - \Phi(h_{i-1}) \\ &= (\text{size}_i) + 2 - k \\ &= (k+1) + 2 - k \\ &= 3 \end{aligned}$$

Alternatives to Doubling the Array Size

We could use some different growth factor (1.5, 2.5, etc.).

Could we use a constant amount?

Cannot Use Constant Amount

If we expand by 10 each time, then:

Let c_i = cost of i 'th insertion.

Cannot Use Constant Amount

If we expand by 10 each time, then:

Let c_i = cost of i 'th insertion.

Cannot Use Constant Amount

If we expand by 10 each time, then:

Let c_i = cost of i 'th insertion.

$$c_i = 1 + \left\{ \right.$$

Cannot Use Constant Amount

If we expand by 10 each time, then:

Let c_i = cost of i 'th insertion.

$$c_i = 1 + \begin{cases} i - 1 & \text{if } i - 1 \text{ is a multiple of } 10 \end{cases}$$

Cannot Use Constant Amount

If we expand by 10 each time, then:

Let c_i = cost of i 'th insertion.

$$c_i = 1 + \begin{cases} i - 1 & \text{if } i - 1 \text{ is a multiple of } 10 \\ 0 & \text{otherwise} \end{cases}$$

Cannot Use Constant Amount

If we expand by 10 each time, then:

Let c_i = cost of i 'th insertion.

$$c_i = 1 + \begin{cases} i - 1 & \text{if } i - 1 \text{ is a multiple of } 10 \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\sum_{i=1}^n c_i}{n}$$

Cannot Use Constant Amount

If we expand by 10 each time, then:

Let c_i = cost of i 'th insertion.

$$c_i = 1 + \begin{cases} i - 1 & \text{if } i - 1 \text{ is a multiple of } 10 \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\sum_{i=1}^n c_i}{n} = \frac{n + \sum_{j=1}^{(n-1)/10} 10j}{n}$$

Cannot Use Constant Amount

If we expand by 10 each time, then:

Let c_i = cost of i 'th insertion.

$$c_i = 1 + \begin{cases} i - 1 & \text{if } i - 1 \text{ is a multiple of } 10 \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\sum_{i=1}^n c_i}{n} = \frac{n + \sum_{j=1}^{(n-1)/10} 10j}{n} = \frac{n + 10 \sum_{j=1}^{(n-1)/10} j}{n}$$

Cannot Use Constant Amount

If we expand by 10 each time, then:

Let c_i = cost of i 'th insertion.

$$c_i = 1 + \begin{cases} i - 1 & \text{if } i - 1 \text{ is a multiple of } 10 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \frac{\sum_{i=1}^n c_i}{n} &= \frac{n + \sum_{j=1}^{(n-1)/10} 10j}{n} = \frac{n + 10 \sum_{j=1}^{(n-1)/10} j}{n} \\ &= \frac{n + 10O(n^2)}{n} \end{aligned}$$

Cannot Use Constant Amount

If we expand by 10 each time, then:

Let c_i = cost of i 'th insertion.

$$c_i = 1 + \begin{cases} i - 1 & \text{if } i - 1 \text{ is a multiple of } 10 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \frac{\sum_{i=1}^n c_i}{n} &= \frac{n + \sum_{j=1}^{(n-1)/10} 10j}{n} = \frac{n + 10 \sum_{j=1}^{(n-1)/10} j}{n} \\ &= \frac{n + 10O(n^2)}{n} = \frac{O(n^2)}{n} \end{aligned}$$

Cannot Use Constant Amount

If we expand by 10 each time, then:

Let c_i = cost of i 'th insertion.

$$c_i = 1 + \begin{cases} i - 1 & \text{if } i - 1 \text{ is a multiple of } 10 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \frac{\sum_{i=1}^n c_i}{n} &= \frac{n + \sum_{j=1}^{(n-1)/10} 10j}{n} = \frac{n + 10 \sum_{j=1}^{(n-1)/10} j}{n} \\ &= \frac{n + 10O(n^2)}{n} = \frac{O(n^2)}{n} = O(n) \end{aligned}$$

Summary

- Calculate amortized cost of an operation in the context of a sequence of operations.

Summary

- Calculate amortized cost of an operation in the context of a sequence of operations.
- Three ways to do analysis:

Summary

- Calculate amortized cost of an operation in the context of a sequence of operations.
- Three ways to do analysis:
 - Aggregate method (brute-force sum)

Summary

- Calculate amortized cost of an operation in the context of a sequence of operations.
- Three ways to do analysis:
 - Aggregate method (brute-force sum)
 - Banker's method (tokens)

Summary

- Calculate amortized cost of an operation in the context of a sequence of operations.
- Three ways to do analysis:
 - Aggregate method (brute-force sum)
 - Banker's method (tokens)
 - Physicist's method (potential function, Φ)

Summary

- Calculate amortized cost of an operation in the context of a sequence of operations.
- Three ways to do analysis:
 - Aggregate method (brute-force sum)
 - Banker's method (tokens)
 - Physicist's method (potential function, Φ)
- Nothing changes in the code: runtime analysis only.