

Linear Programming: Gaussian Elimination

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Advanced Algorithms and Complexity
Data Structures and Algorithms

Learning Objectives

- Solve a system of linear equations.
- Implement a row reduction algorithm.
- Say something about what the set of solution to a system of linear equations looks like.

Last Time

Linear programming: Dealing with systems of linear inequalities.

Linear Algebra

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For example:

$$\begin{aligned}x + y &= 5 \\ 2x + 4y &= 12.\end{aligned}$$

Method of Substitution

- Use first equation to solve for one variable in terms of the others.
- Substitute into other equations.
- Solve recursively.
- Substitute back in to first equation to get initial variable.

Example

$$x + y = 5$$

$$2x + 4y = 12.$$

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So $y = 1, x = 5 - 1 = 4$.

Problem

What is the value of x in the solution to the following linear system?

$$\begin{aligned}x + 2y &= 6 \\ 3x - y &= -3.\end{aligned}$$

Solution

From the first equation, we get

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Solving gives, $y = 3$, so $x = 6 - 2 \cdot 3 = 0$.

Notation

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$$\begin{aligned}x + y &= 5 \\ 2x + 4y &= 12.\end{aligned}$$

We just store coefficients of equations in an (augmented) matrix, like so:

$$\begin{array}{cc|c} x & y & = 1 \\ \hline 1 & 1 & 5 \\ 2 & 4 & 12 \end{array}$$

Substitution

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By subtracting. Subtracting $2(x + y = 5)$ from $(2x + 4y = 12)$ gives $2y = 2$.

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Subtract twice first row from second.

Basic Row Operations

There are three basic ways to manipulate our matrix. These are called **Basic row operations**. Each of them gives us an equivalent system of equations.

Adding

Add/subtract a multiple of one row to another.

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$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 2 & 4 & 12 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 2 & 2 \end{array} \right]$$

Scaling

Multiply/divide a row by a non-zero constant.

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Multiply/divide a row by a non-zero constant. Dividing the second row by 2:

$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 2 & 2 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 1 \end{array} \right]$$

Swapping

Sometimes you want to change the ordering of rows.

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$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 1 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 1 & 5 \end{array} \right]$$

Row Reduction

Row reduction uses row operations to put a matrix into a simple standard form. The idea is to simulate the substitution method.

Example

Consider the system given by the matrix:

$$\left[\begin{array}{cccc|c} 2 & 4 & -2 & 0 & 2 \\ -1 & -2 & 1 & -2 & -1 \\ 2 & 2 & 0 & 2 & 0 \end{array} \right]$$

Example

Use first for to solve for first variable.

$$\left[\begin{array}{cccc|c} 2 & 4 & -2 & 0 & 2 \\ -1 & -2 & 1 & -2 & -1 \\ 2 & 2 & 0 & 2 & 0 \end{array} \right]$$

Example

Divide first row by 2.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ -1 & -2 & 1 & -2 & -1 \\ 2 & 2 & 0 & 2 & 0 \end{array} \right]$$

Example

Substitute into other equations.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ -1 & -2 & 1 & -2 & -1 \\ 2 & 2 & 0 & 2 & 0 \end{array} \right]$$

Example

Add first row to second.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ 0 & 0 & 0 & -2 & 0 \\ 2 & 2 & 0 & 2 & 0 \end{array} \right]$$

Example

Subtract twice first row from third.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & -2 & 2 & 2 & -2 \end{array} \right]$$

Example

Need to solve for next variable.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & -2 & 2 & 2 & -2 \end{array} \right]$$

Example

Cannot use second row.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & -2 & 2 & 2 & -2 \end{array} \right]$$

Example

Swap second and third rows.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ 0 & -2 & 2 & 2 & -2 \\ 0 & 0 & 0 & -2 & 0 \end{array} \right]$$

Example

Solve for second variable.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ 0 & -2 & 2 & 2 & -2 \\ 0 & 0 & 0 & -2 & 0 \end{array} \right]$$

Example

Divide second row by -2 .

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & -2 & 0 \end{array} \right]$$

Example

Substitute into other equations.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & -2 & 0 \end{array} \right]$$

Example

Subtract twice second row from first.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & -2 & 0 \end{array} \right]$$

Example

Can't solve for third variable.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & -2 & 0 \end{array} \right]$$

Example

Solve for fourth instead.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & -2 & 0 \end{array} \right]$$

Example

Divide last row by -2 .

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

Example

Substitute into other equations.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

Example

Subtract twice third row from first.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

Example

Add third row to second.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

Example

Done.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

Answer

Our matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

corresponds to equations:

$$x + z = -1$$

$$y - z = 1$$

$$w = 0.$$

Solution

So for any value of z , we have solution:

$$x = -1 - z$$

$$y = 1 + z$$

$$w = 0.$$

RowReduce(A)

Leftmost non-zero

Swap row to top

Make entry **pivot**

Rescale to make pivot 1

Subtract row from others to make
other entries in column 0

Repeat

RowReduce(A)

Leftmost non-zero in non-pivot row

Swap row to top of non-pivot rows

Make entry **pivot**

Rescale to make pivot 1

Subtract row from others to make
other entries in column 0

Repeat until no more non-zero
entries outside of pivot rows

Reading off Answer

- Each row has one pivot and a few other non-pivot entries.
- Gives equation writing pivot variable in terms of non-pivot variables.
- If pivot in units column, have equation $0 = 1$, so no solutions.
- Otherwise, set non-pivot variables to anything, gives answer.

Degrees of Freedom

- Your solution set will be a subspace.
- Dimension = number of non-pivot variables.
- Or n minus the number of pivot variables.
- Generally, dimension equals
num. variables — num. equations.

Runtime

- m equations in n variables.
- $\min(n, m)$ pivots.
- For each pivot, need to subtract multiple of row from each other row $O(nm)$ time.
- Total runtime: $O(nm \min(n, m))$.