Disjoint Sets: Naive Implementations

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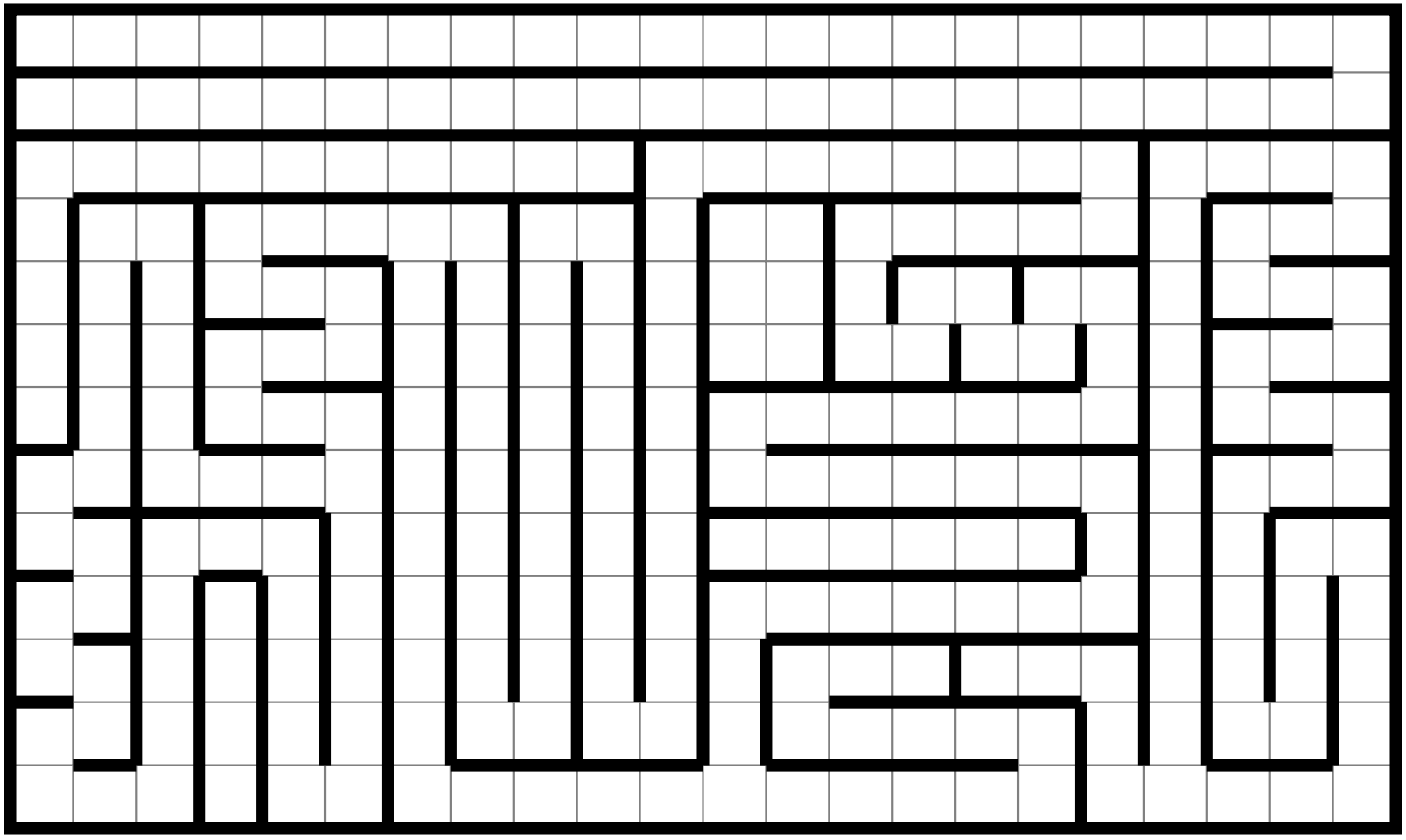
Data Structures
Data Structures and Algorithms

Outline

Maze: Is B Reachable from A?



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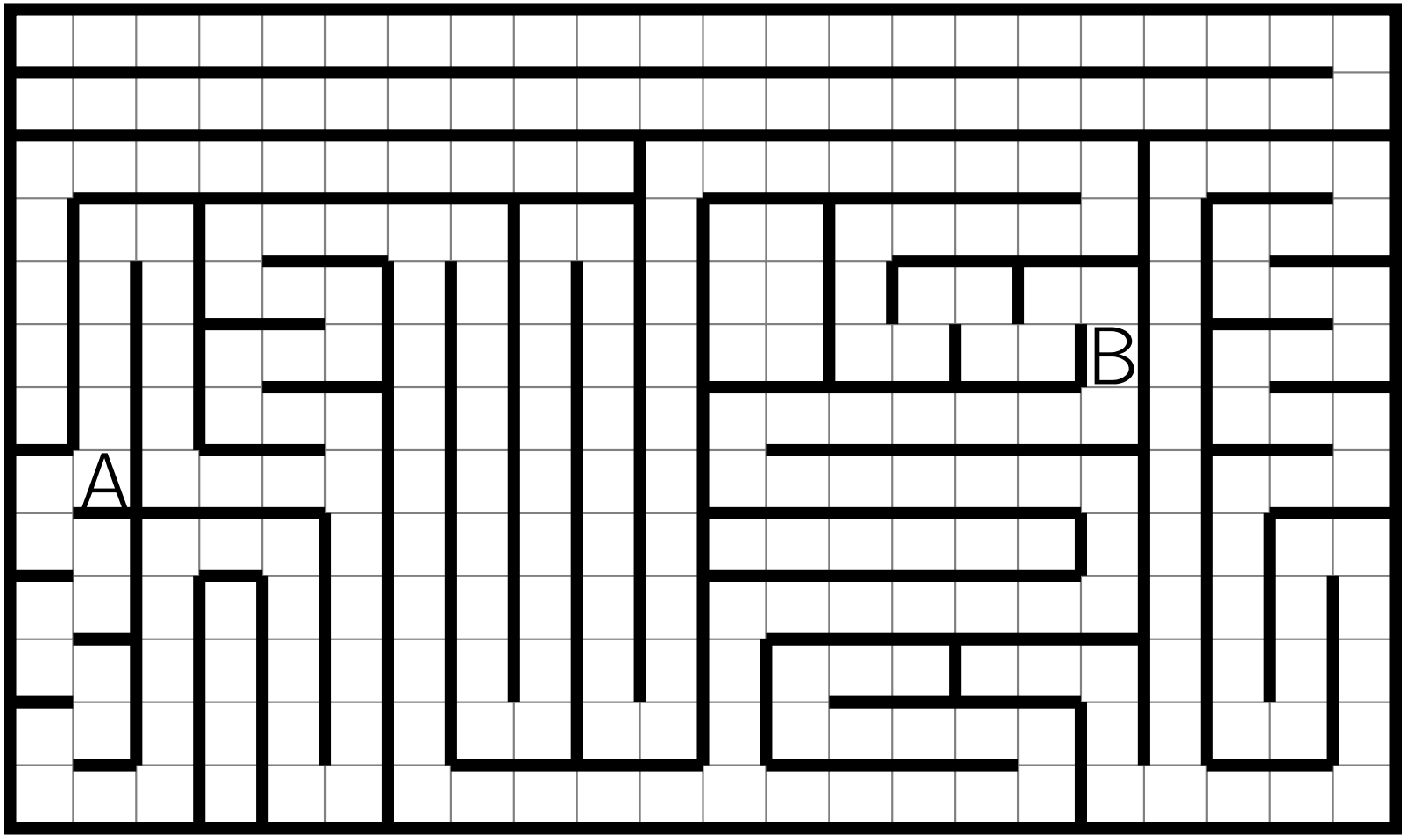

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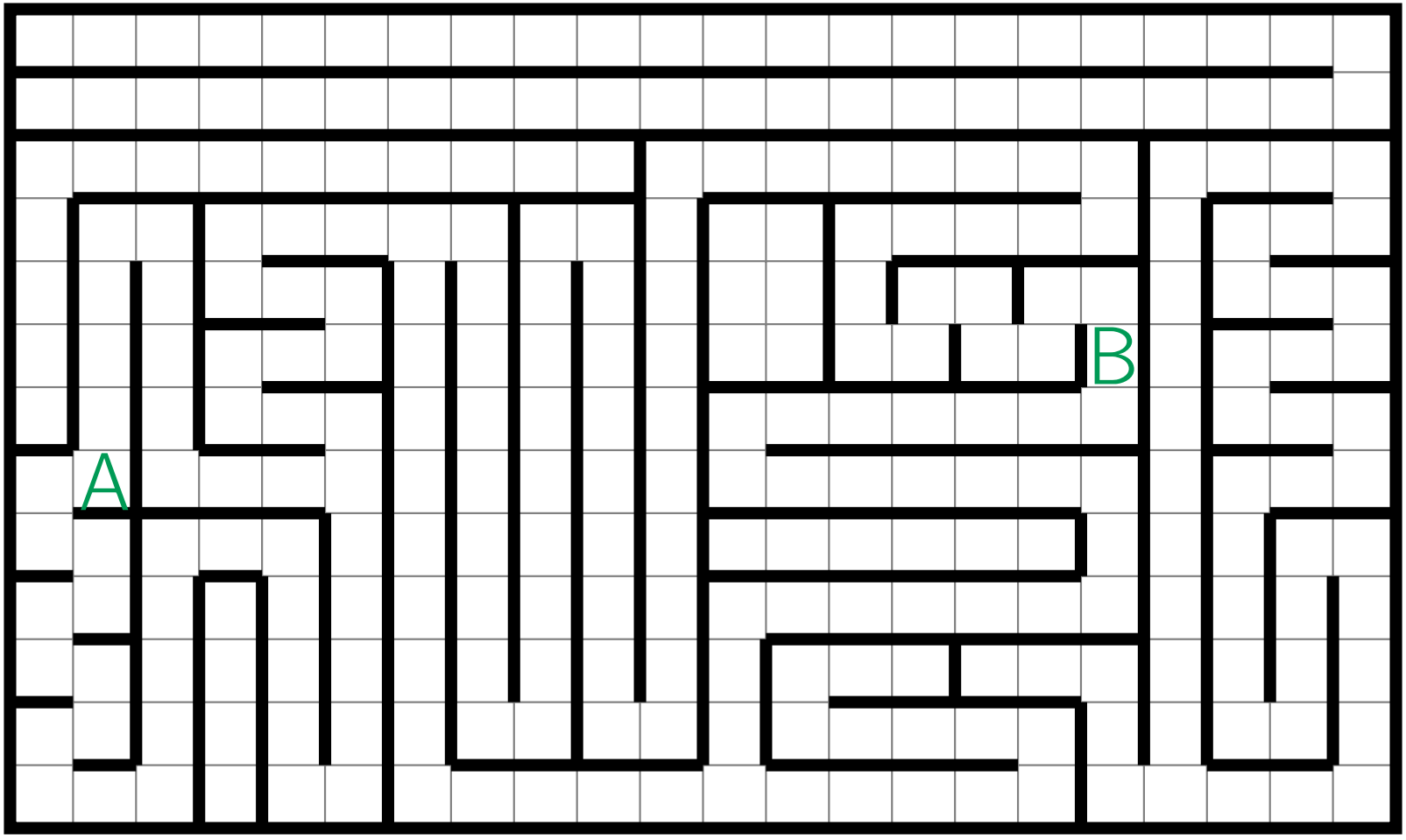

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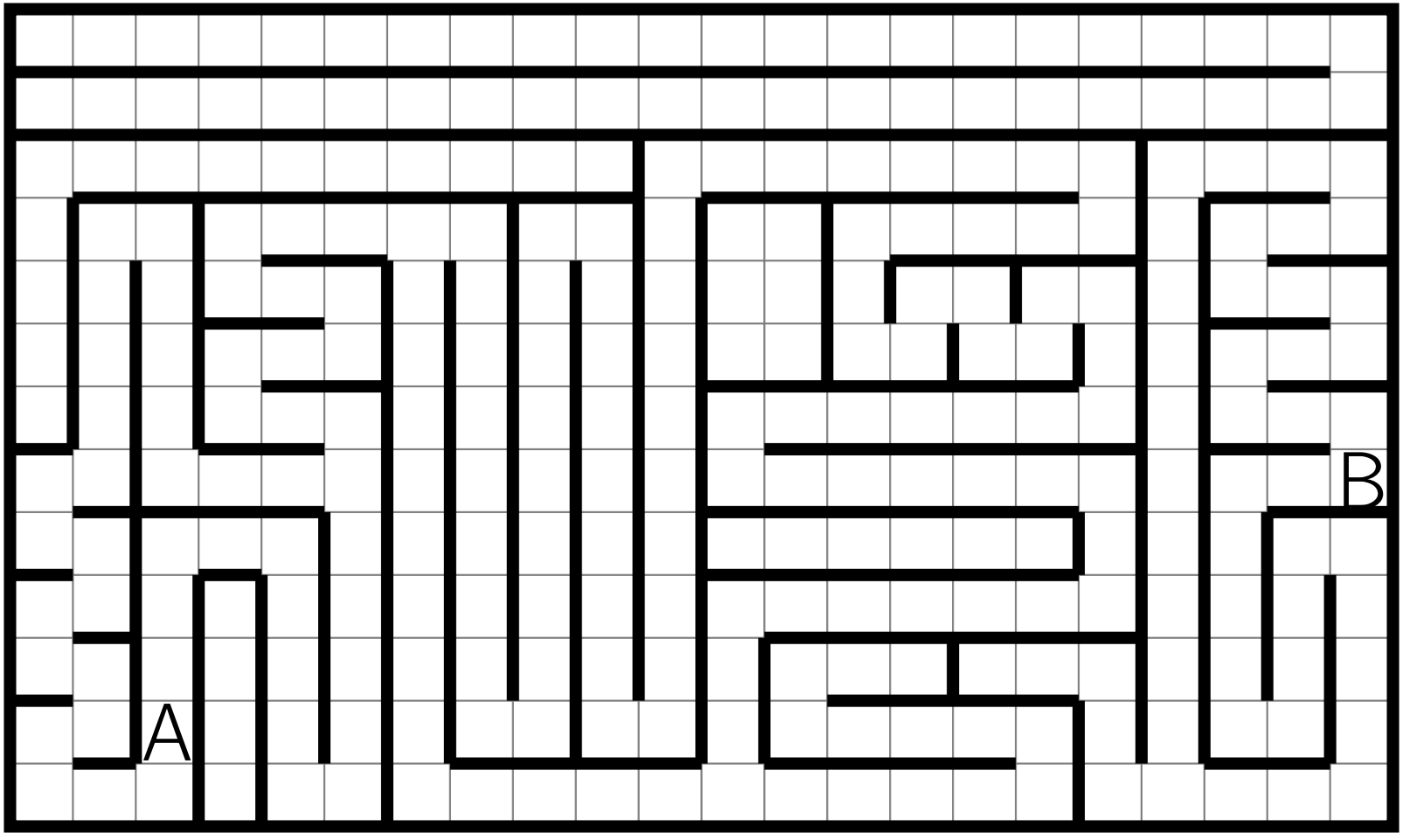

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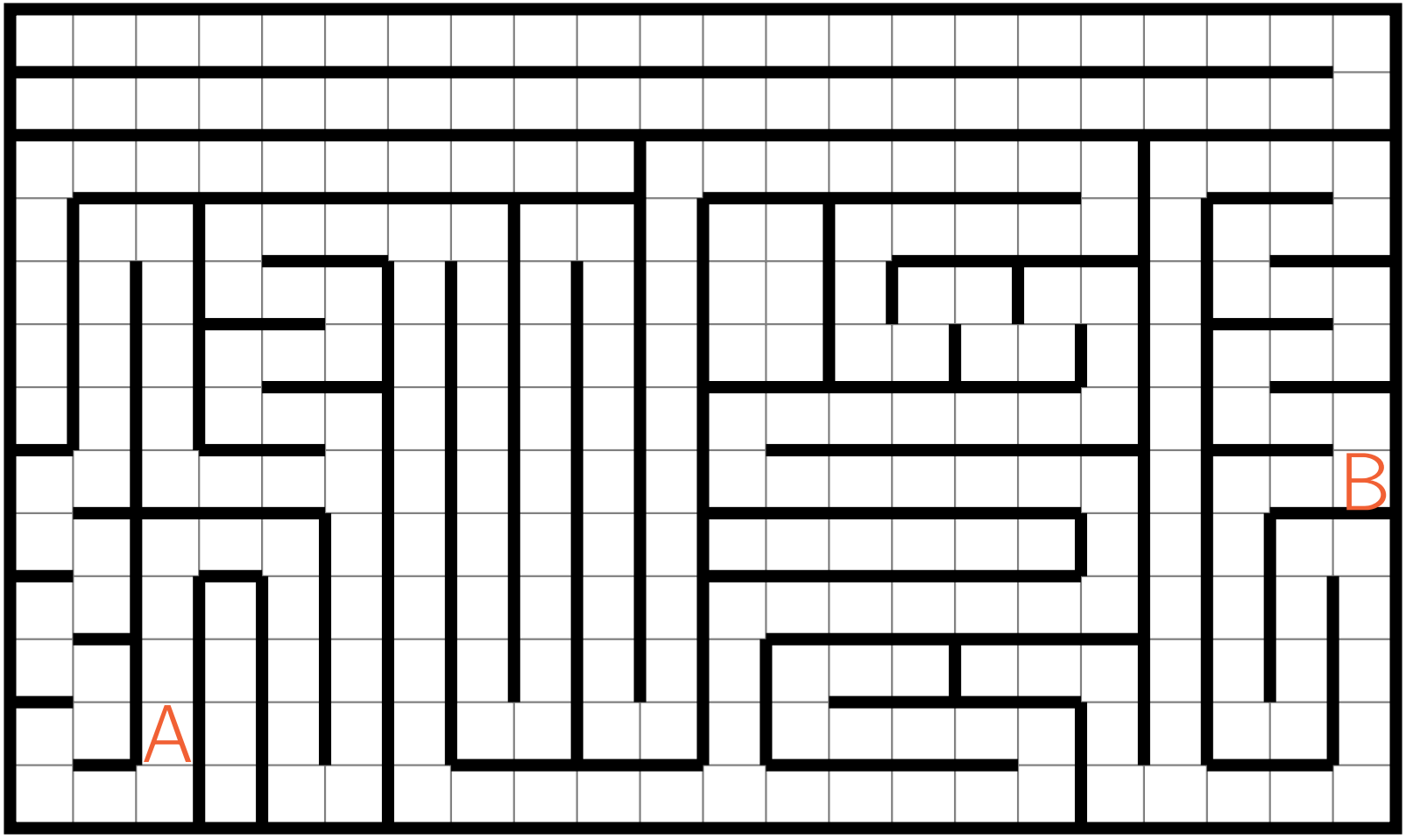

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  - if x and y lie in the same set, then `Find`(x) = `Find`(y)
  - otherwise, `Find`(x) $\neq$ `Find`(y)
- `Union`(x, y) merges two sets containing x and y

Preprocess(*maze*)

```
for each cell c in maze:
  MakeSet(c)
for each cell c in maze:
  for each neighbor n of c:
    Union(c, n)
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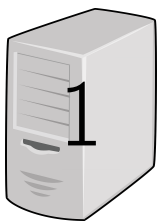
IsReachable(*A*, *B*)

```
return Find(A) = Find(B)
```

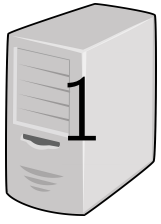
Building a Network

Building a Network



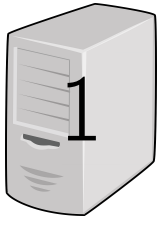
`MakeSet(1)`

Building a Network



`MakeSet(2)`

Building a Network



`MakeSet(3)`

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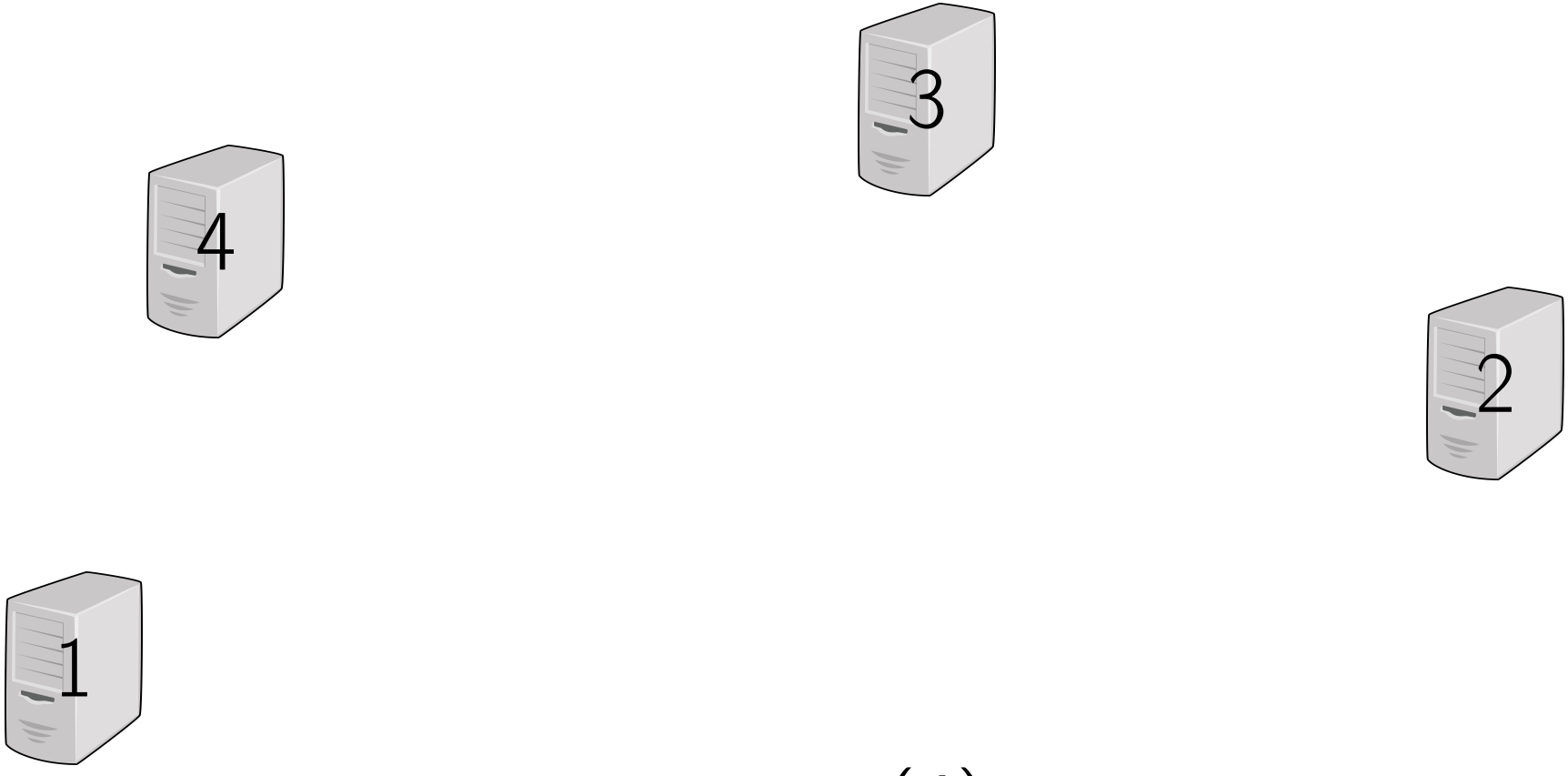
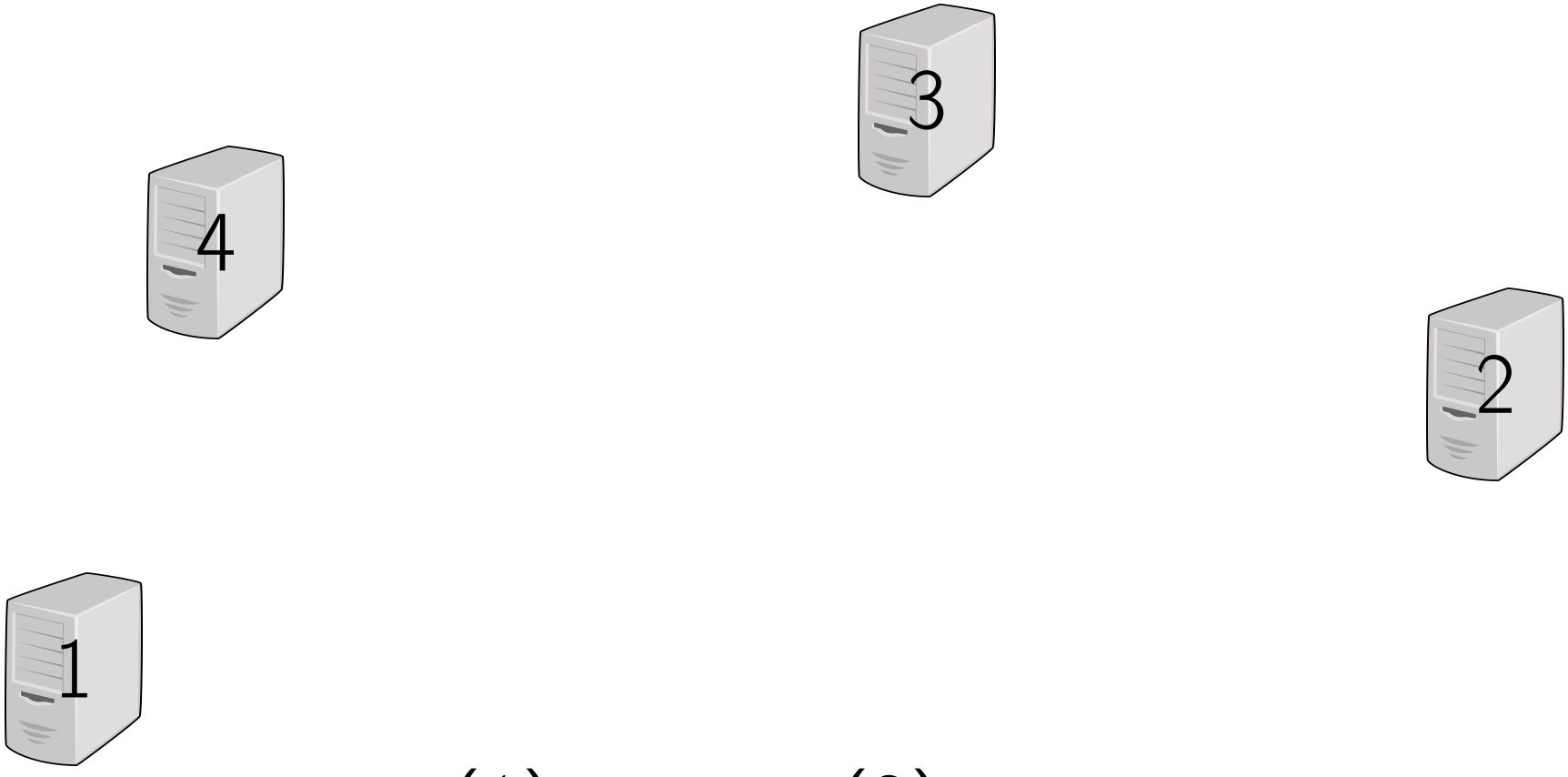


`MakeSet(4)`

Building a Network



$$\texttt{Find}(1) = \texttt{Find}(2) \rightarrow \texttt{False}$$

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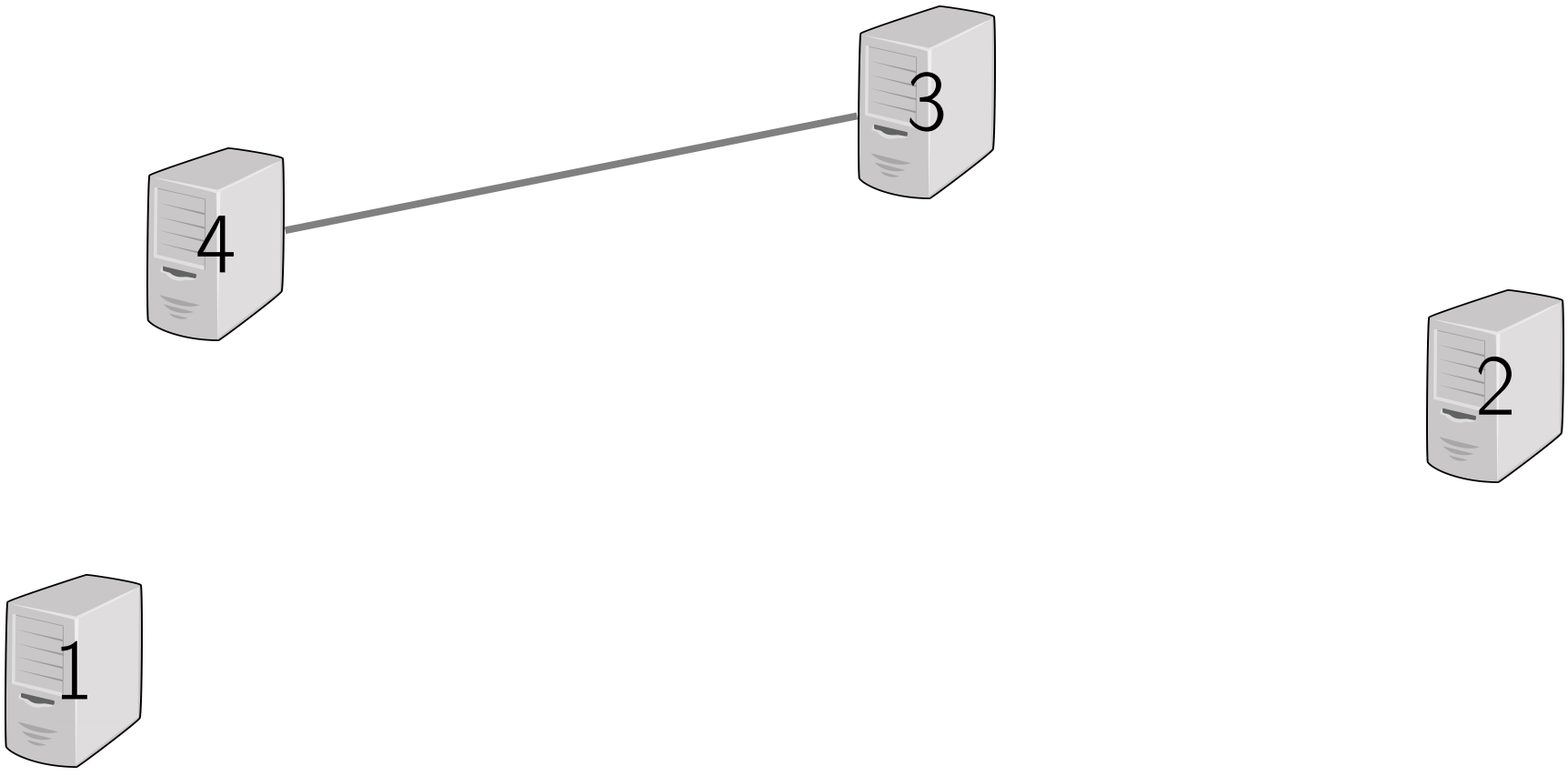

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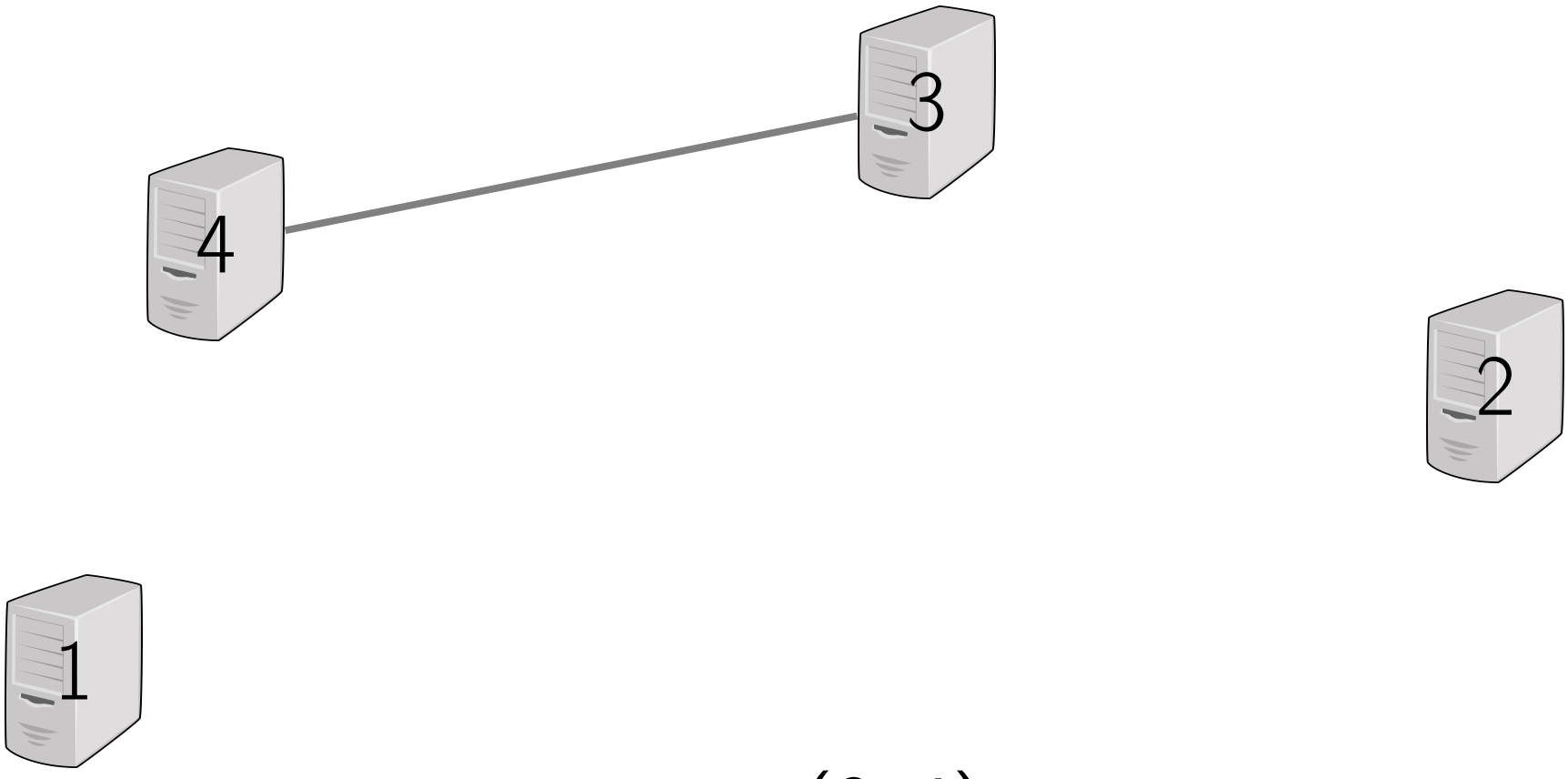
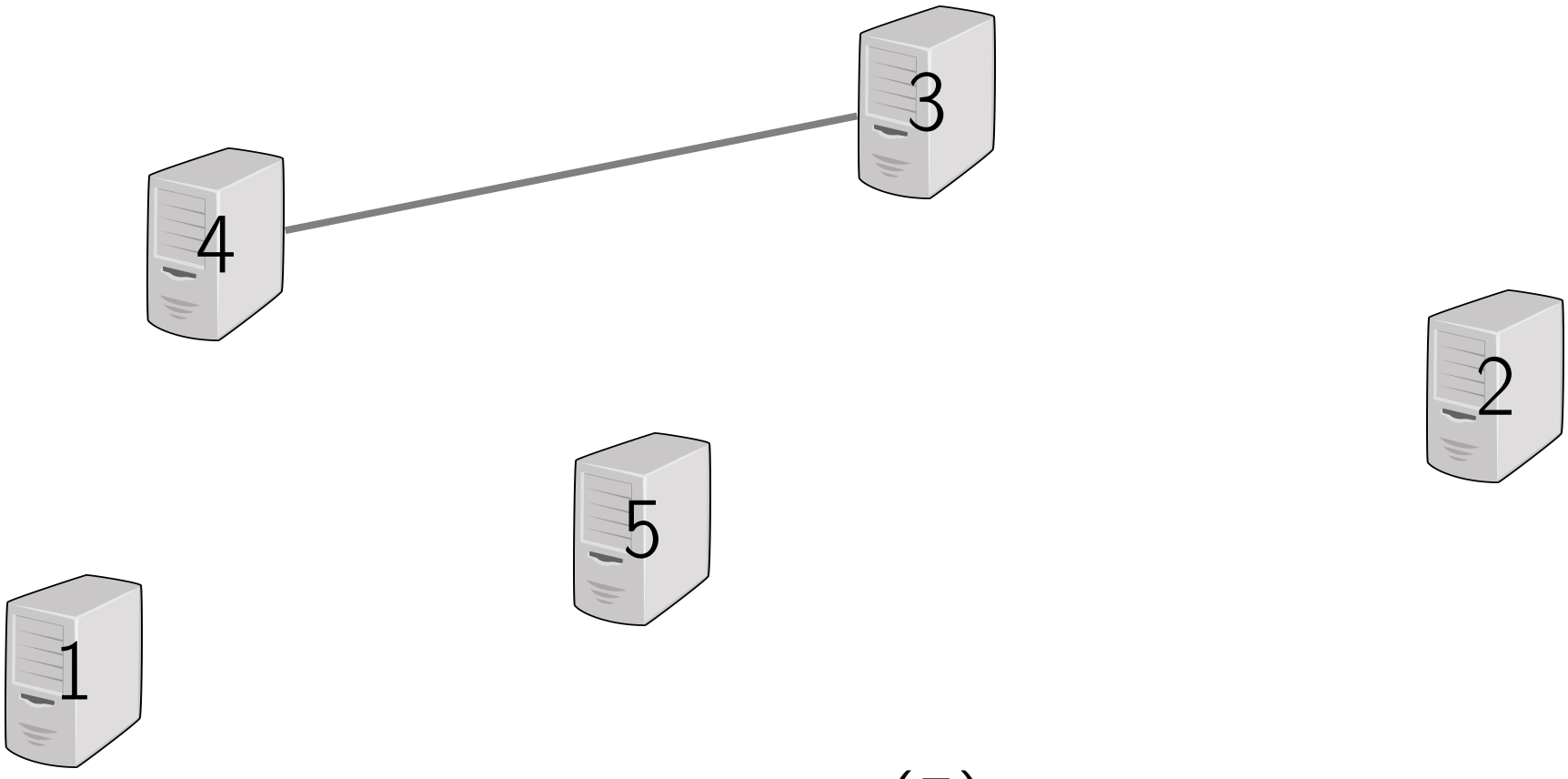


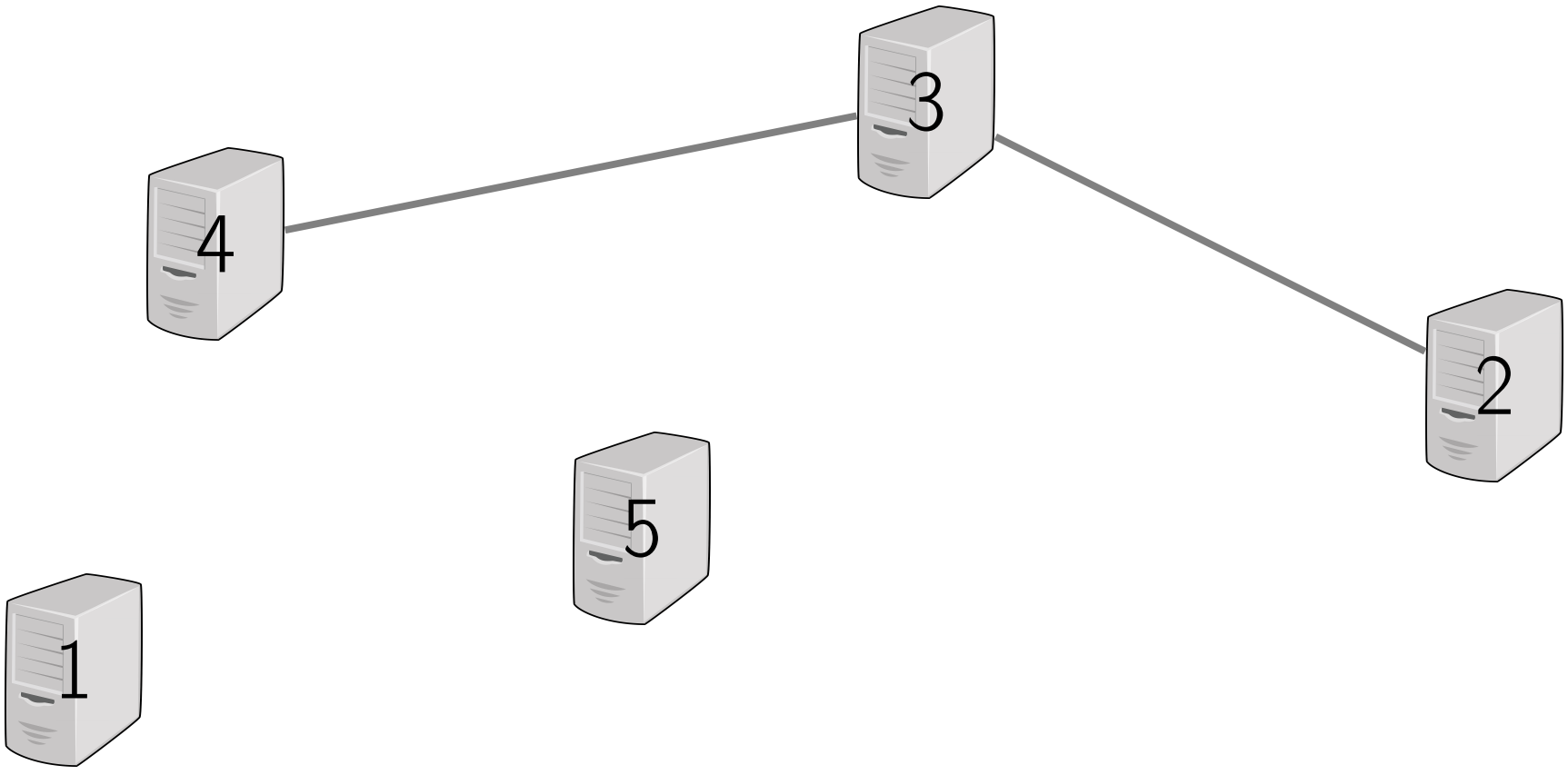
`Union`$(3, 4)$

Building a Network



`MakeSet(5)`

Building a Network



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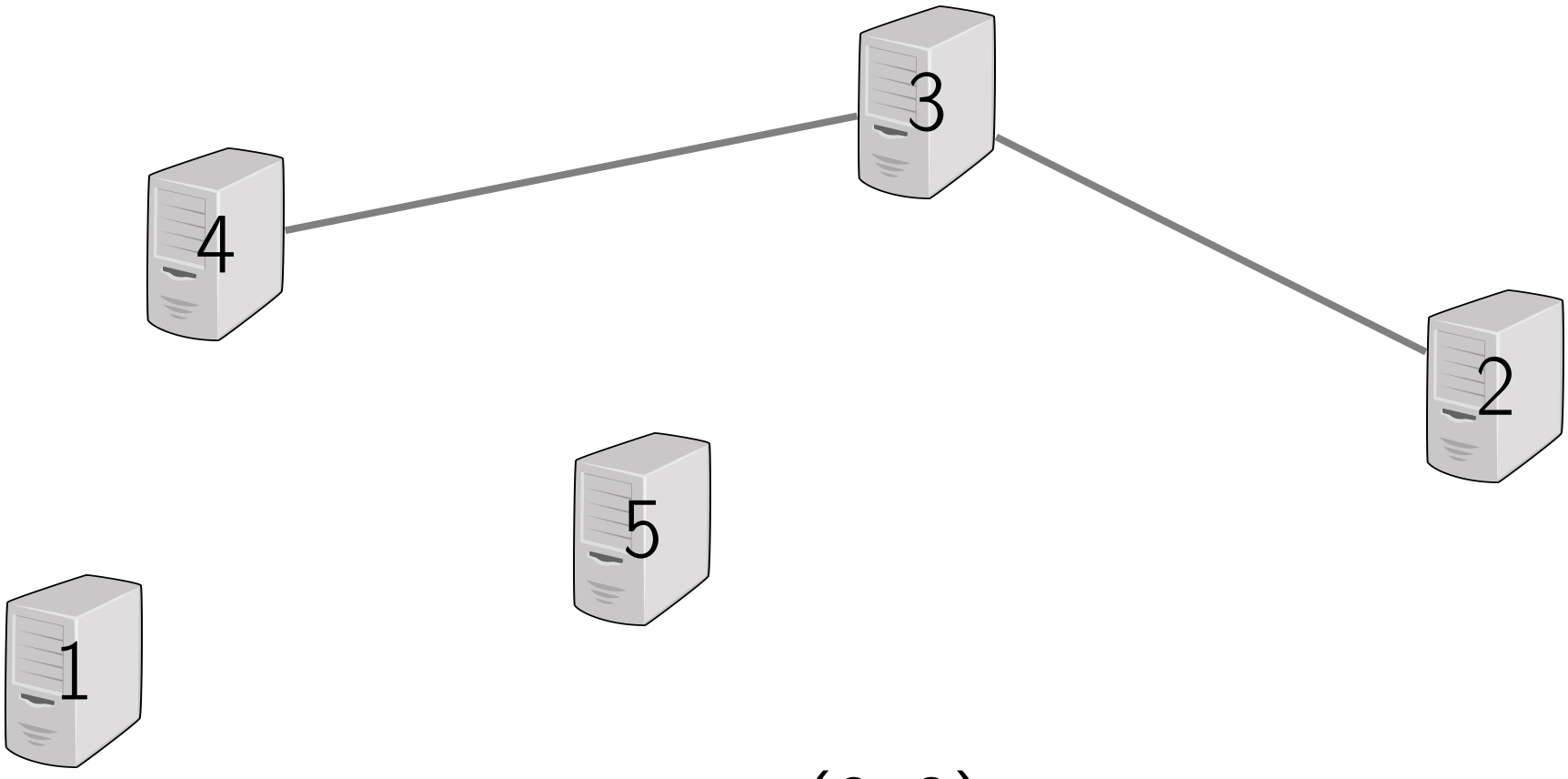

`Union`$(3, 2)$

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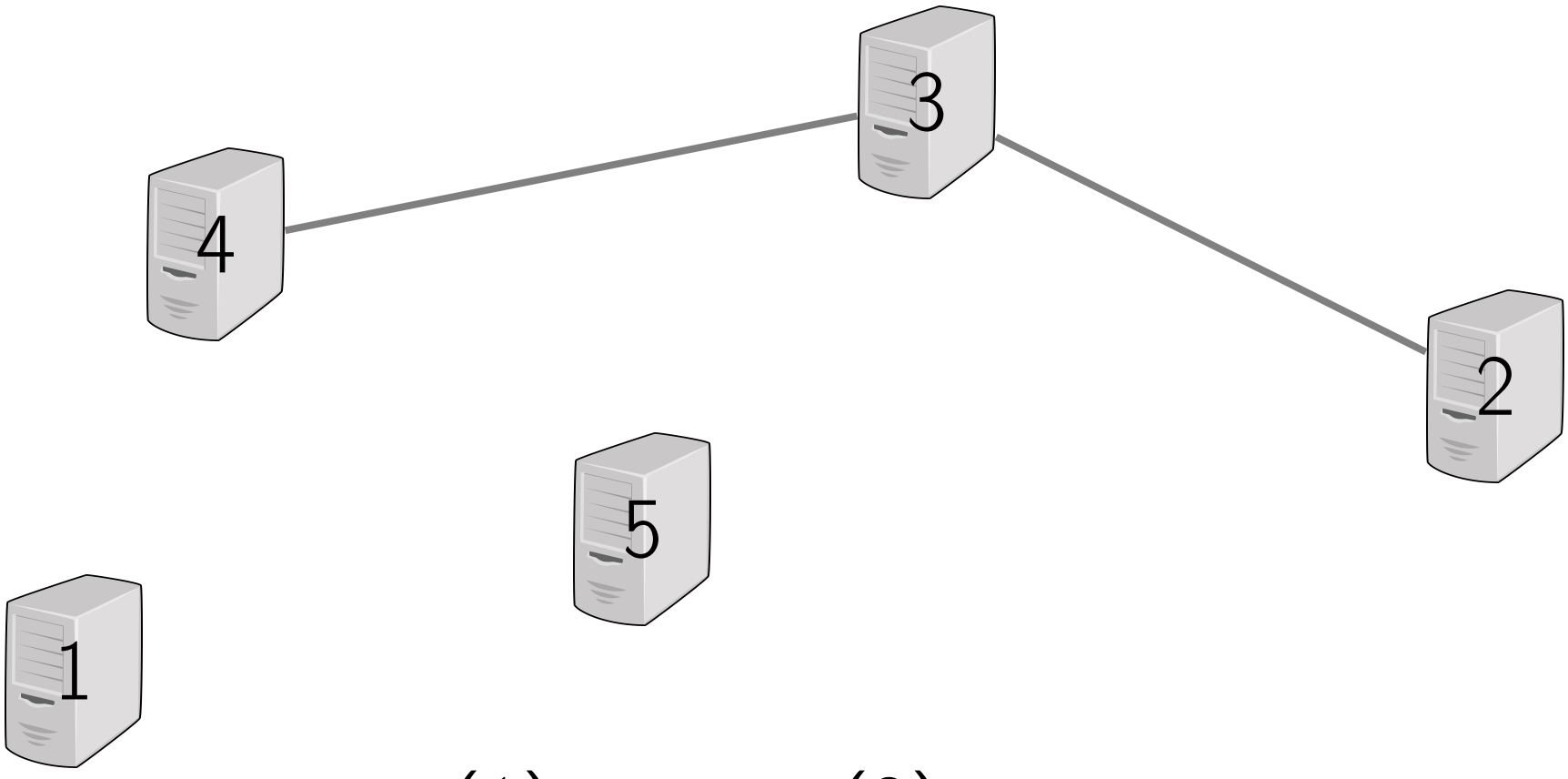

$$\texttt{Find}(1) = \texttt{Find}(2) \rightarrow \texttt{False}$$

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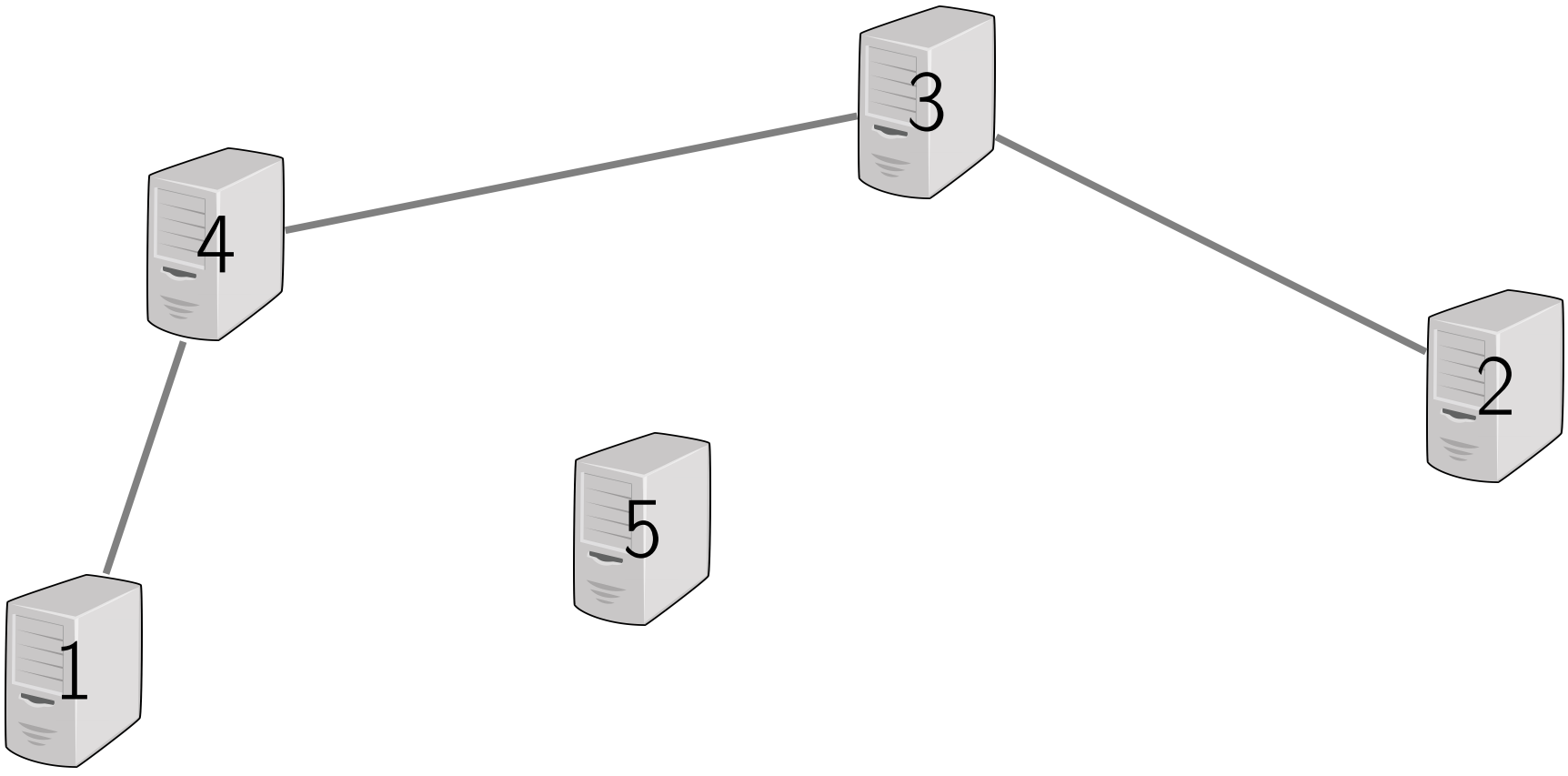

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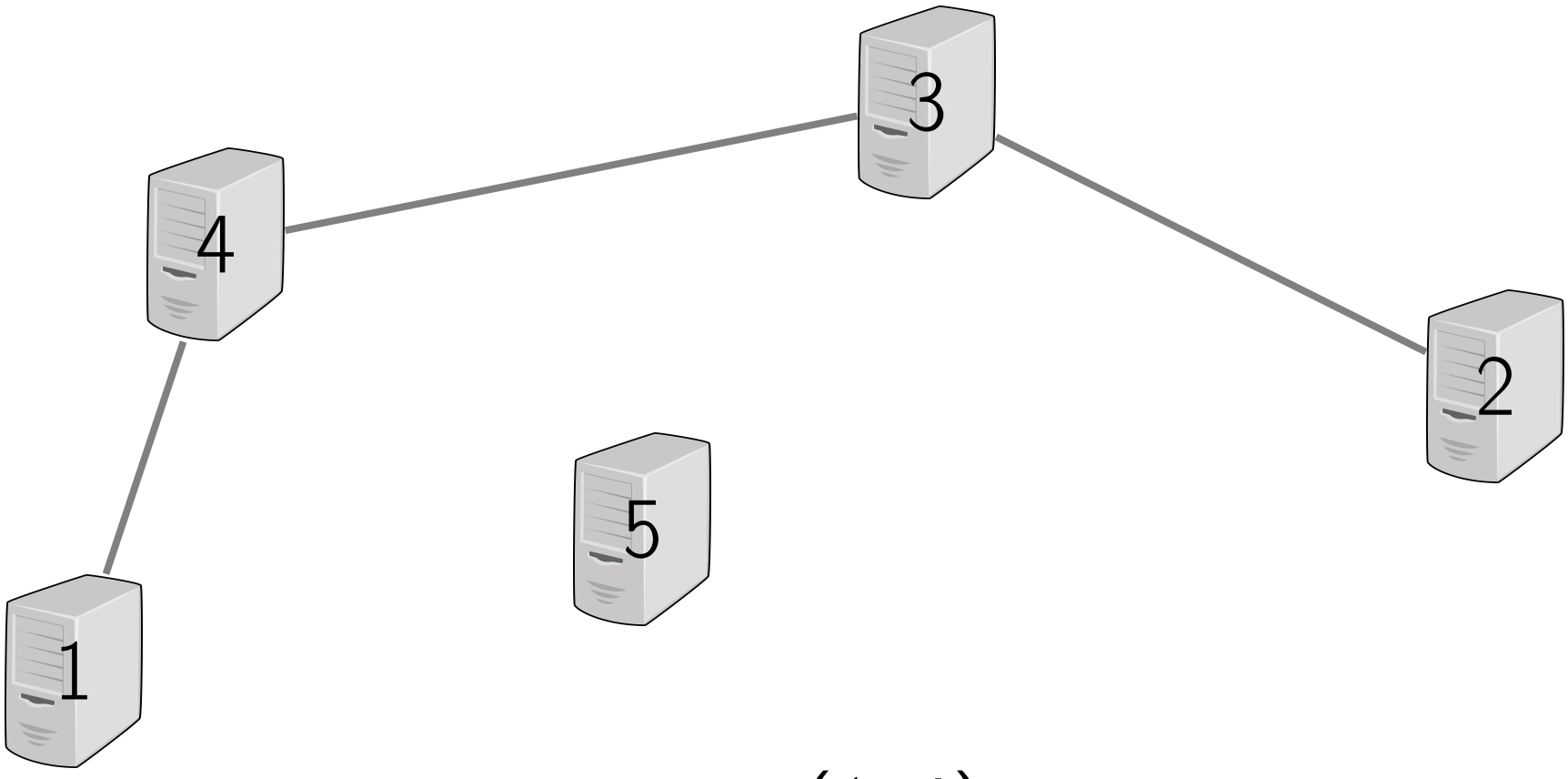

`Union(1, 4)`

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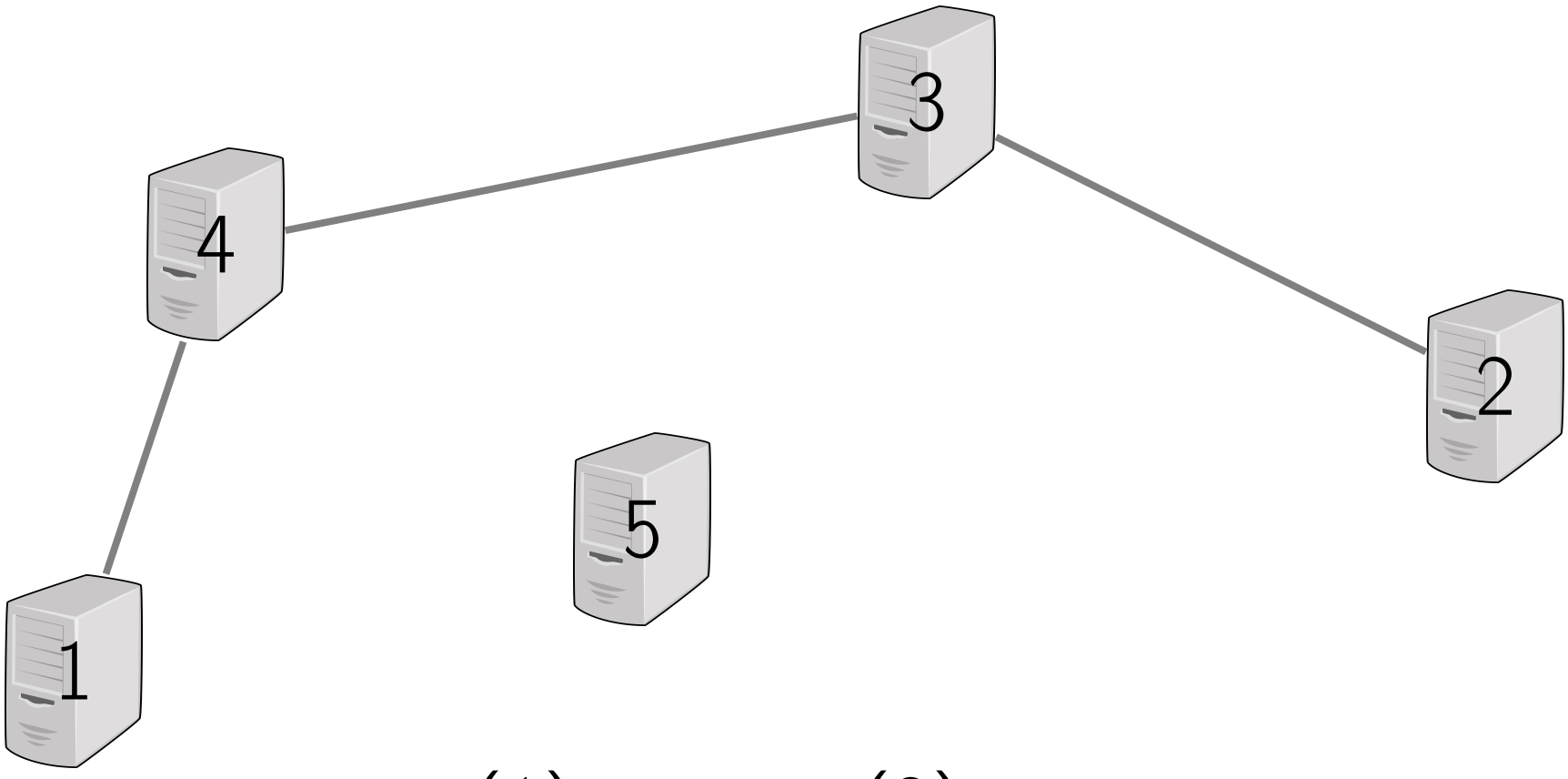

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Outline

For simplicity, we assume that our n objects are just integers $1, 2, \ldots, n$.

Using the Smallest Element as ID

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- Use array `smallest`$[1 \ldots n]$: `smallest`$[i]$ stores the smallest element in the set i belongs to

Example

$$\{9, 3, 2, 4, 7\} \quad \{5\} \quad \{6, 1, 8\}$$

	1	2	3	4	5	6	7	8	9
`smallest`	1	2	2	2	5	1	2	1	2

MakeSet(i)

$\texttt{smallest}[i] \leftarrow i$

Find(i)

$\texttt{return smallest}[i]$

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Find(i)

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return smallest[i]
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Running time: $O(1)$

Union(i, j)

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i_id ← Find(i)
j_id ← Find(j)
if i_id = j_id:
    return
m ← min(i_id, j_id)
for k from 1 to n:
    if smallest[k] in {i_id, j_id}:
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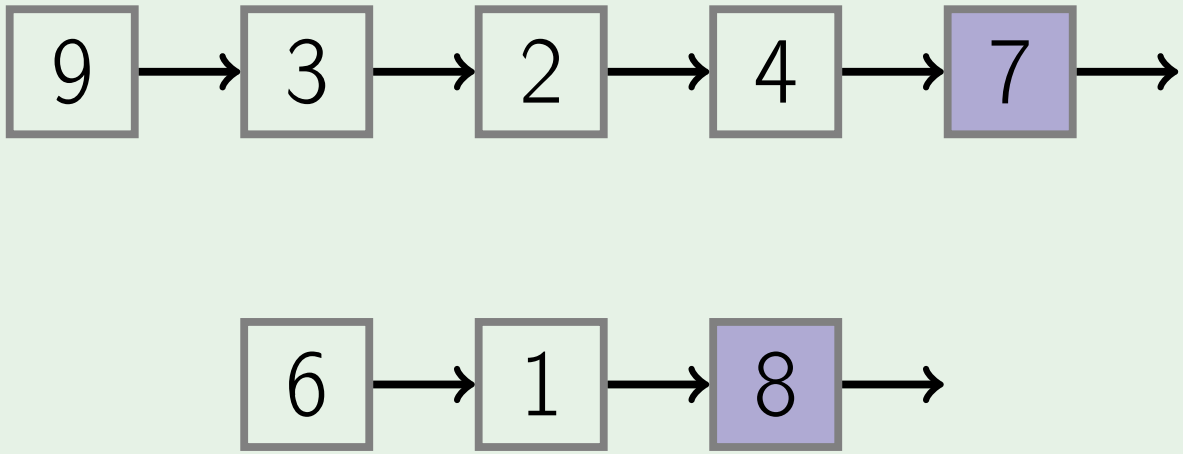
- Linked list!
- Idea: represent a set as a linked list, use the list tail as ID of the set

Example: merging two lists



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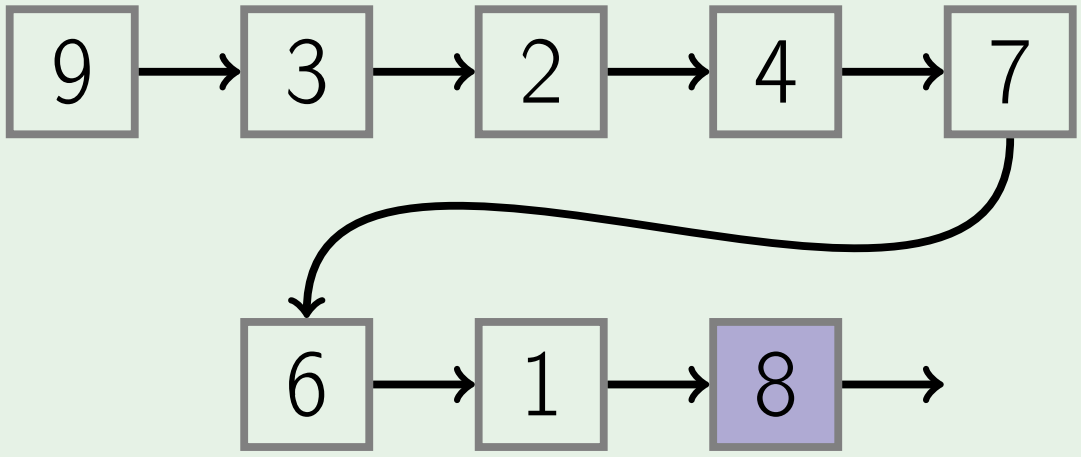

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- Cons:
  - Running time of `Find` is $O(n)$ as we need to traverse the list to find its tail
  - `Union`(x, y) works in time $O(1)$ only if we can get the tail of the list of x and the head of the list of y in constant time!

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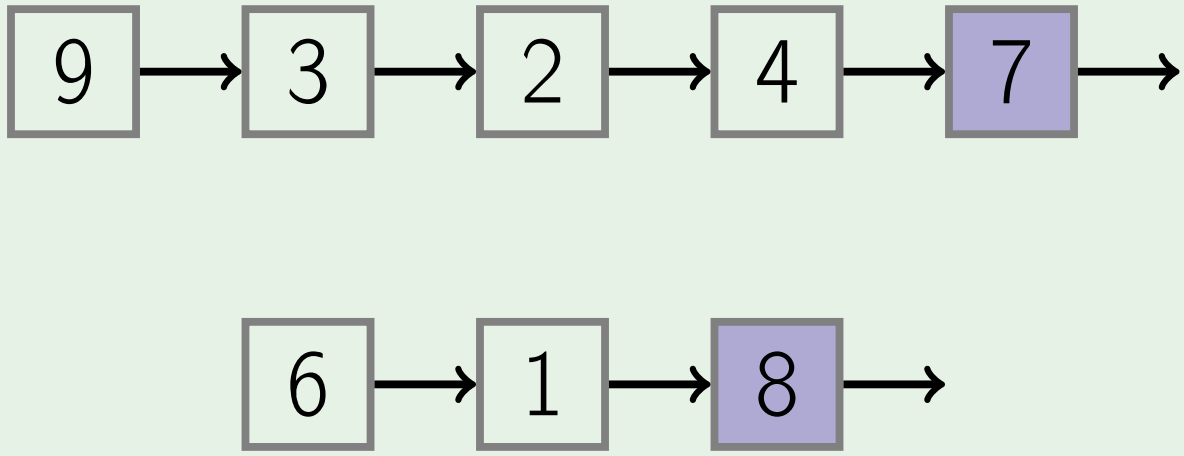

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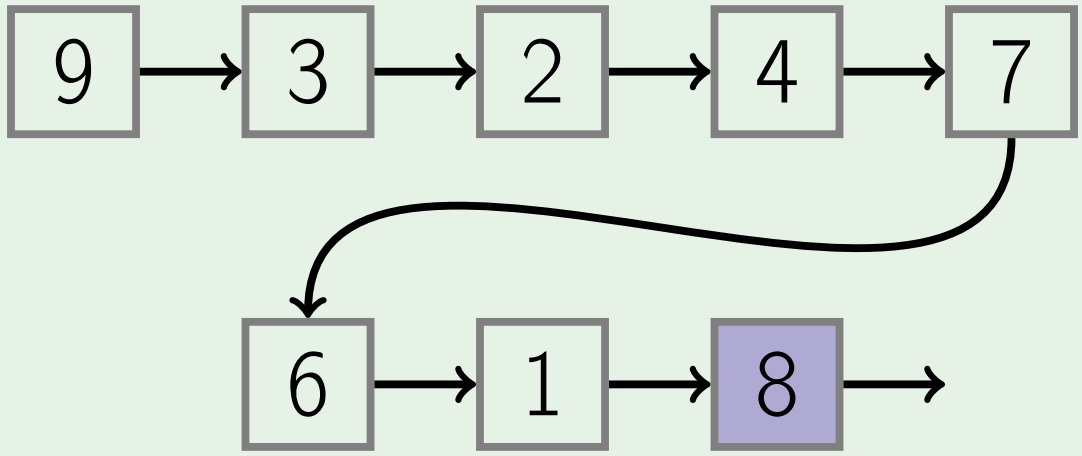

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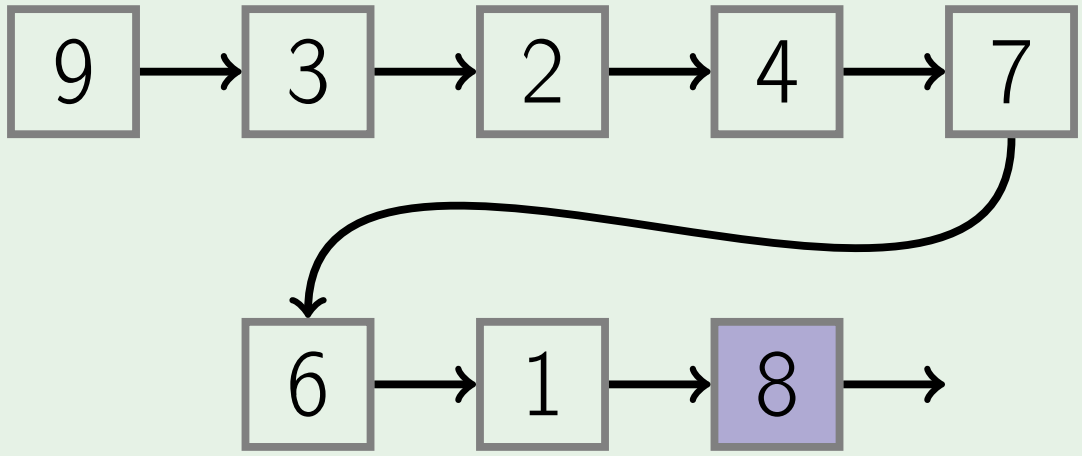

`Find`(9) goes through all elements

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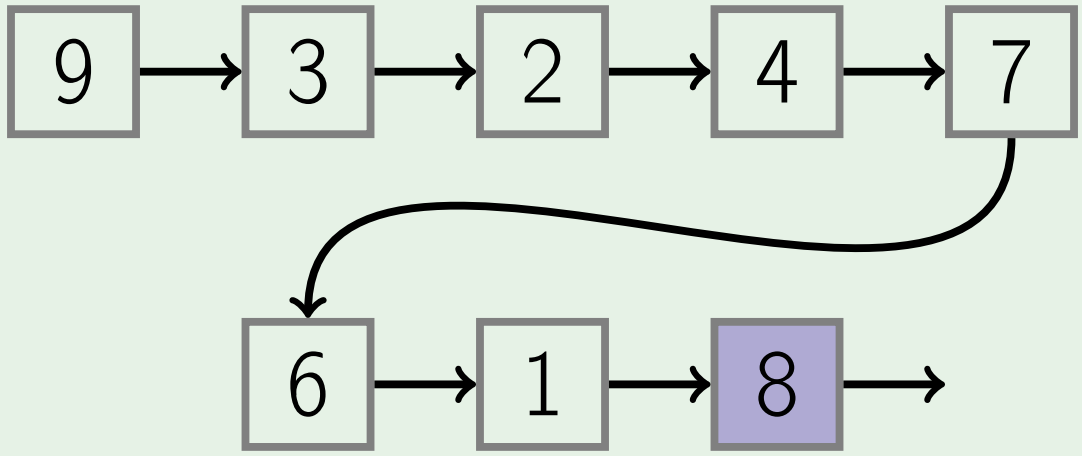
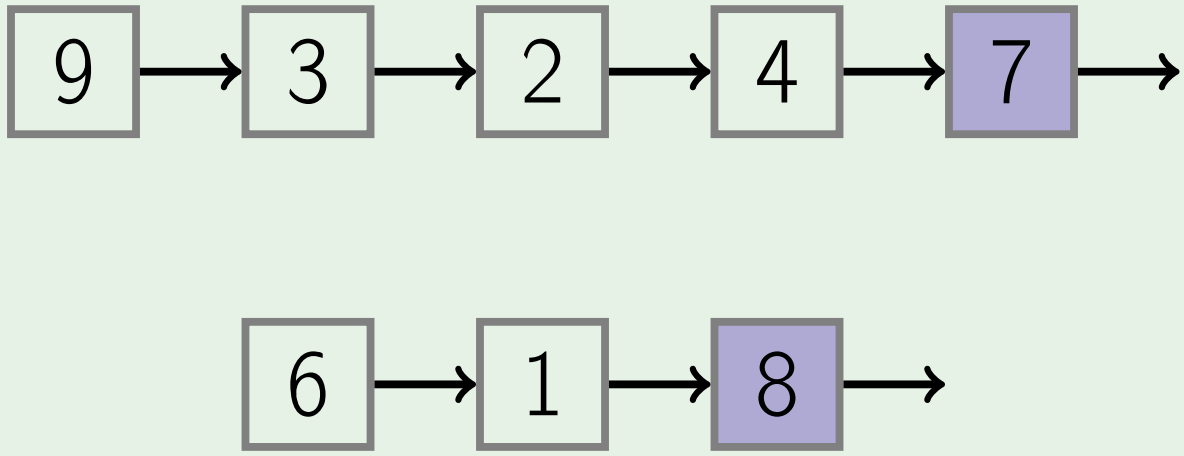
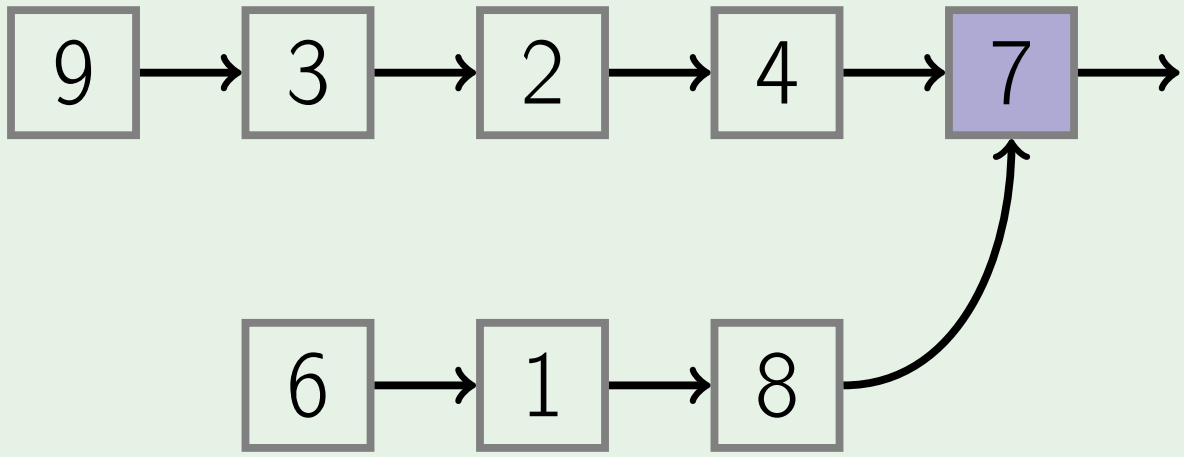


can we merge in a different way?

Example: merging two lists



Example: merging two lists



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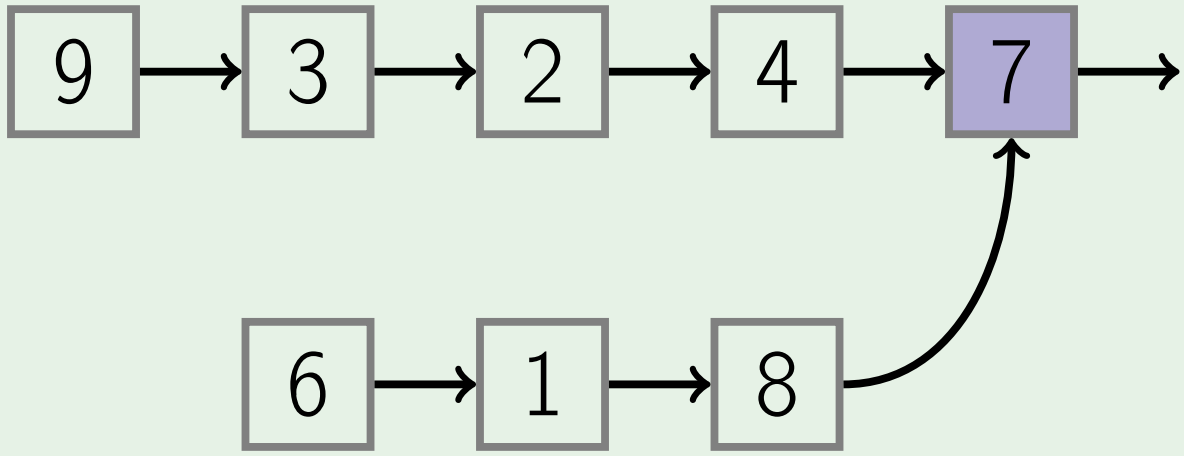
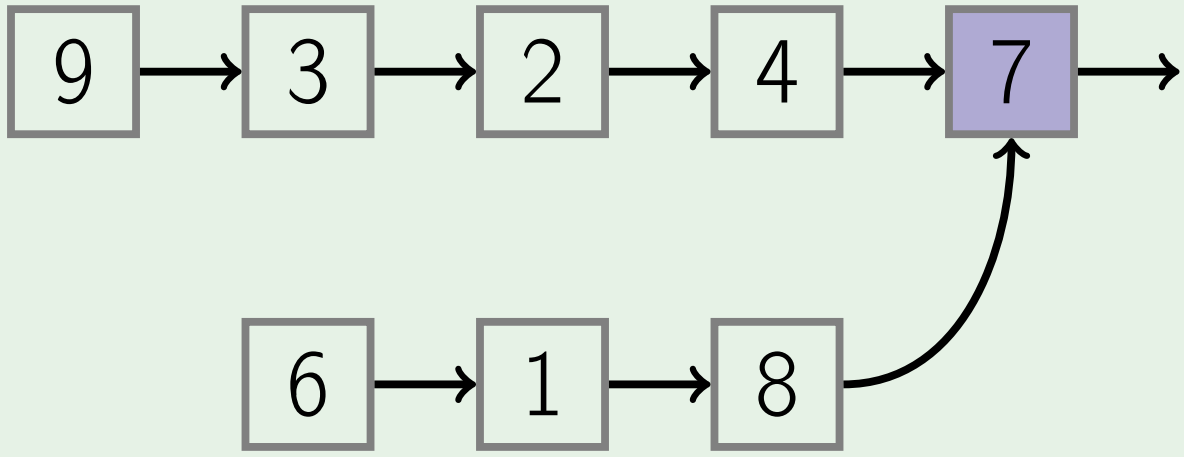


instead of a list we get a tree

Example: merging two lists



we'll see that representing sets as trees gives a very efficient implementation: nearly constant amortized time for all operations