Information Bottleneck

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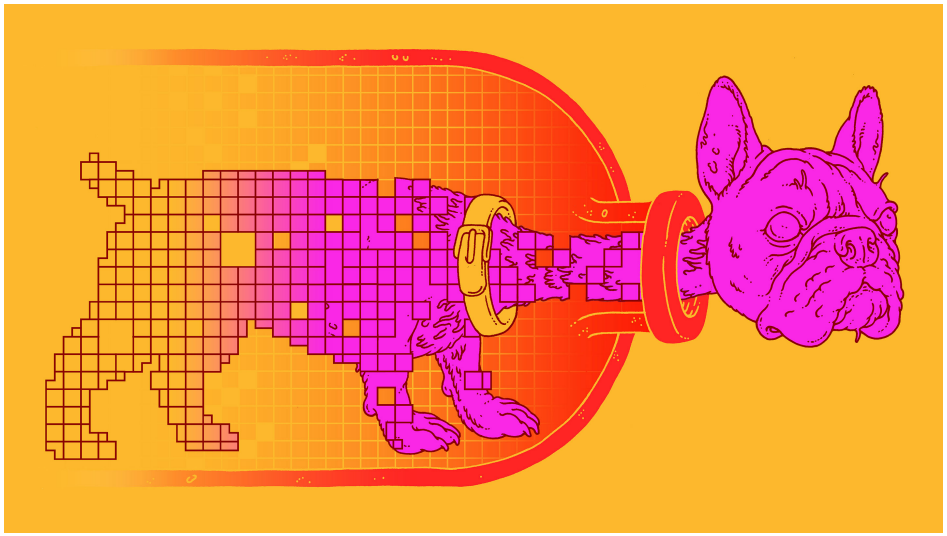
15/10/2025

Outline

- Background of the Information Bottleneck (IB) framework
- The IB cracks open the black box of deep learning
  - Two phases of the information dynamics during training
  - An information compression bounds
- Ongoing debates on the IB framework



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- Deep learning (DL) has shown its great potential in learning critical information for certain tasks
- The theoretical understanding of DL remains unsatisfied
- Information theory plays an important role in explaining the DL mechanism

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The magic of DNN is its generalization ability: learning from special cases, while showing intelligence toward general concepts.DNN is a black box.DL is actually extracting patterns from a large scale complex data, it involves data commpression, has similar concept of encoding and decoding

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- Learning Theory: data-dependent but architecture-independent bound
- Information Theory: using the same notations and techniques
- Stochastic dynamics of the training: convergence of SGD

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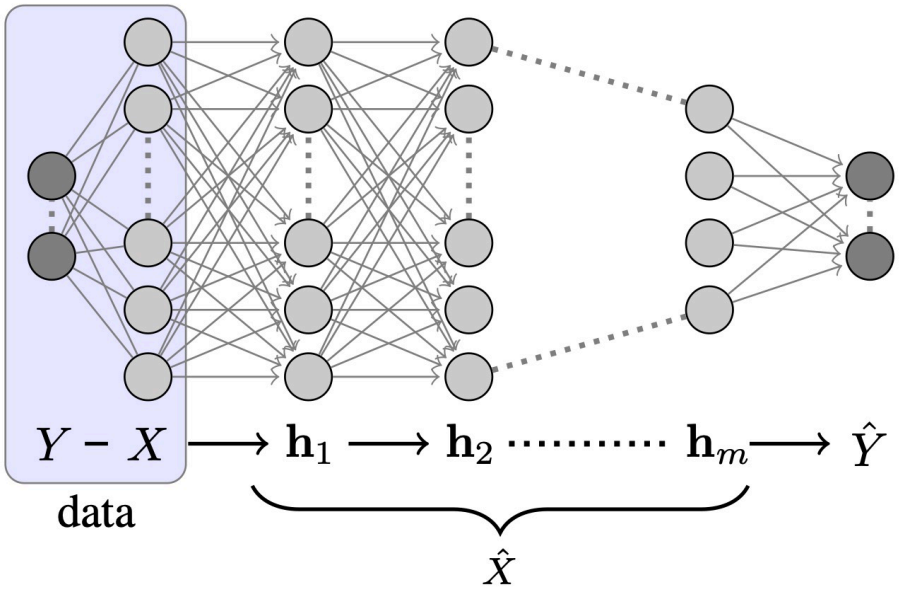
from worst case results to a typical case result

Review of DNN



- Y: target label
- X: input data
- $\mathbf{h}_i$: features
- $\hat{Y}$: prediction

- The input data goes through a series of transformations

$$\mathbf{h}_{i+1} = \sigma(W_i \mathbf{h}_i + b_i), \quad \mathbf{h}_0 = X \tag{1}$$

- The cascade network forms a **Markov chain**

$$Y \to X \to \hat{X} \to \hat{Y} \tag{2}$$

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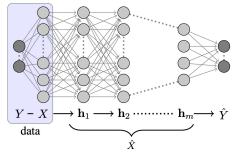

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Each layer generates a new representation of the input dataThe prediction Markov chain is incorrect, $X \to \hat{X} \to YX$ is a high entropy random variable, Y is a simple low dimensional variable. The last layer generate a random variable, as close as Y.Each representation is calculated by the previous one, and only affect the next one.If the training works well, this series of refinement of data, lead to a good prediction.The IB framework focuses on what really happens for these intermediate representations.

Information Theory Basics

- The KL divergence: for any two distributions p and q over $\mathcal{X}$:

$$D\left(p\|q\right) = \sum_{x\in\mathcal{X}} p(x)\log\frac{p(x)}{q(x)} \geq 0$$

- The mutual information: for any two r.v.s X and Y:

$$I(X;Y) = D\left(p(x,y)\|p(x)p(y)\right) = H(X) - H(X|Y)$$

- Data Processing Inequality (DPI): for any Markov chain $X \to Y \to Z$

$$I(X;Y) \geq I(X;Z)$$

The IB Method (Tishby, Pereira, and Bialek 1999)

- Standard compression

$$X \to \hat{X}$$

- A constraint through a distance function:

$$\mathcal{L}\left[p\left(\hat{x}|x\right)\right] = I(X;\hat{X}) + \beta \mathbb{E}\left[d\left(x,\hat{x}\right)\right]$$

- Relevance compression

$$Y \to X \to \hat{X}$$

- A constraint on the meaningful information:

$$\mathcal{L}\left[p\left(\hat{x}|x\right)\right] = I(X;\hat{X}) - \beta I(\hat{X};Y)$$

- Using the Lagrange multipliers method, the optimal solution is

$$p^*(\hat{x}|x) = \frac{p(\hat{x})}{Z(x,\beta)} \exp\left[-\beta D\left(p(y|x) \| p(y|\hat{x})\right)\right], \tag{3}$$

where $Z(x,\beta)$ is the normalization function

- Iteratively solved by Blauht-Arimoto algorithm

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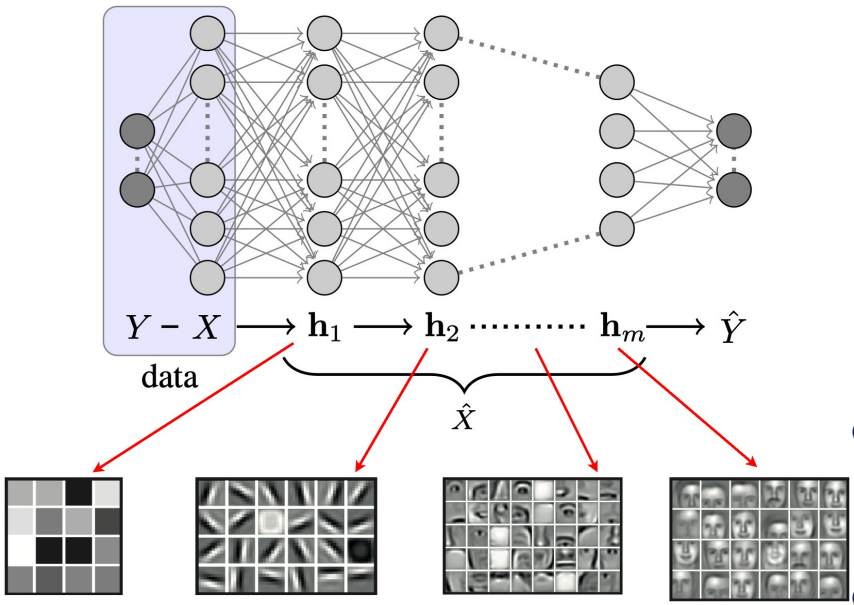
The IB method was introduced long ago, which originated from the Rate-Distortion problem. The key difference is the introduction of an additional variable that dtermines what is relevant.These two methods are equivalent when we choose a KL divergence distortion $d_{IB}\left(x,\hat{x}\right) = D\left(p(y|x) \| p(y|\hat{x})\right)$.

What do the DNN layers represent?

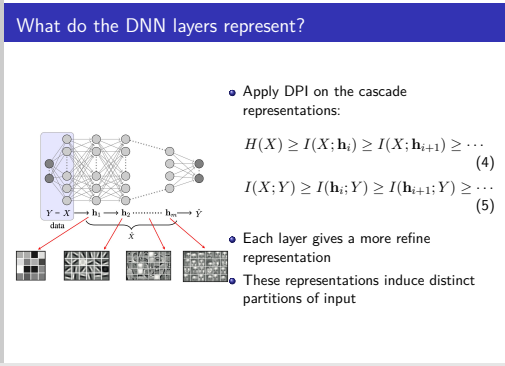


- Apply DPI on the cascade representations:

$$H(X) \geq I(X;\mathbf{h}_i) \geq I(X;\mathbf{h}_{i+1}) \geq \cdots \tag{4}$$

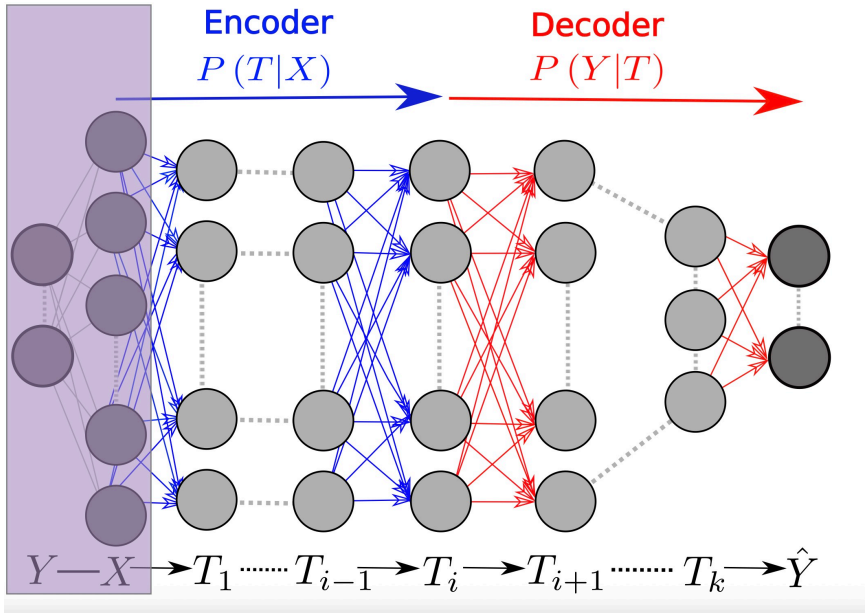
$$I(X;Y) \geq I(\mathbf{h}_i;Y) \geq I(\mathbf{h}_{i+1};Y) \geq \cdots \tag{5}$$

- Each layer gives a more refine representation
- These representations induce distinct partitions of input



Now, let's go back to the DNN, and take a look at what do the layers represent and how IB method related to DNN?When we look back to these cascade representation, we would find out two chains of inequalities. We will calculate the mutual information of each layer between the input data X and target label Y.The equality in (5) is achievable if each layer representation is a sufficient minimal statistics.

Consider a DNN as a pair of encoder and decoder



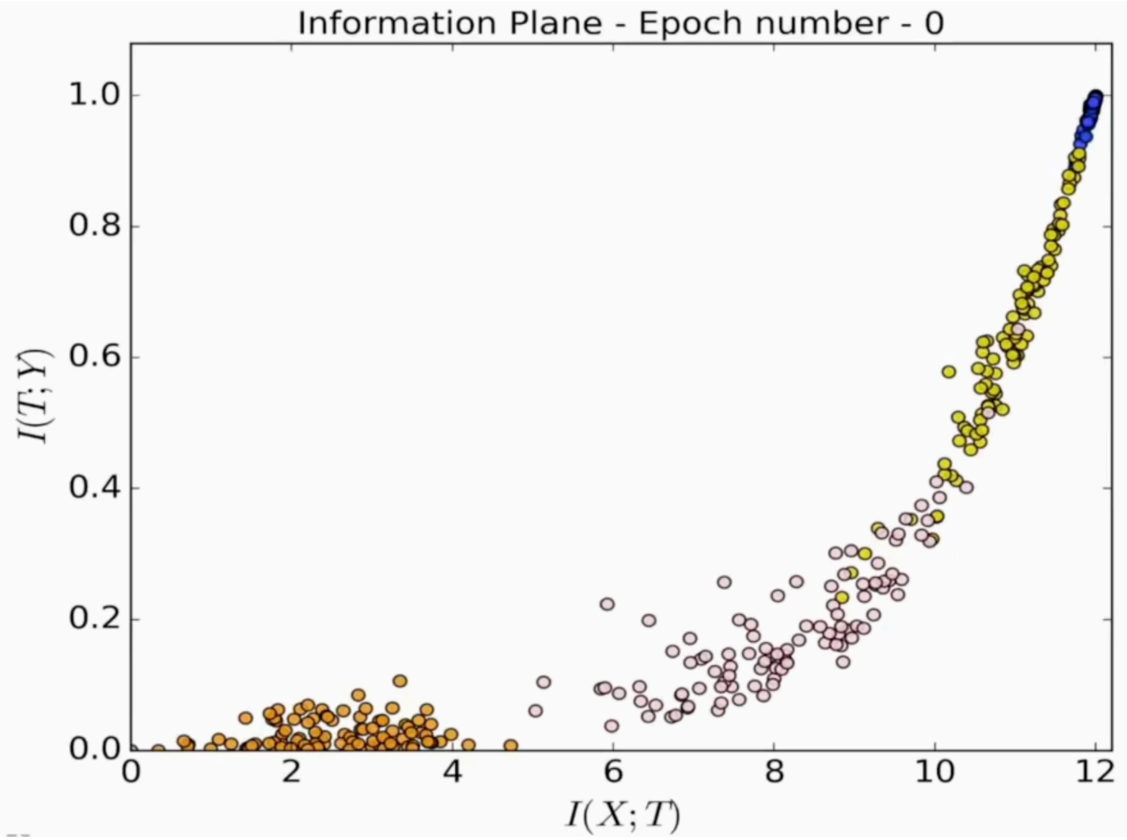
- A resemblance between DNN and relevance compression
- Learn a minimal sufficient statistics T (**bottleneck**): minimal description length $I(X;T)$ and maximal relevant information $I(T;Y)$

Statement (Tishby and Zaslavsky 2015)

The sample complexity of DNN is determined by the encoder MI $I(X;T)$, and the accuracy is determined by the decoder MI $I(T;Y)$

The Information Planes

- IB framework formulates a trade-off between $I(X;T)$ and $I(T;Y)$
- The set of all possible pairs of MIs for any encoder $p(T|X)$ is called the **information plane**



- Simulations demonstrate a two phases dynamics during training

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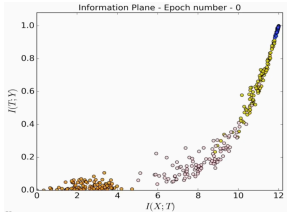
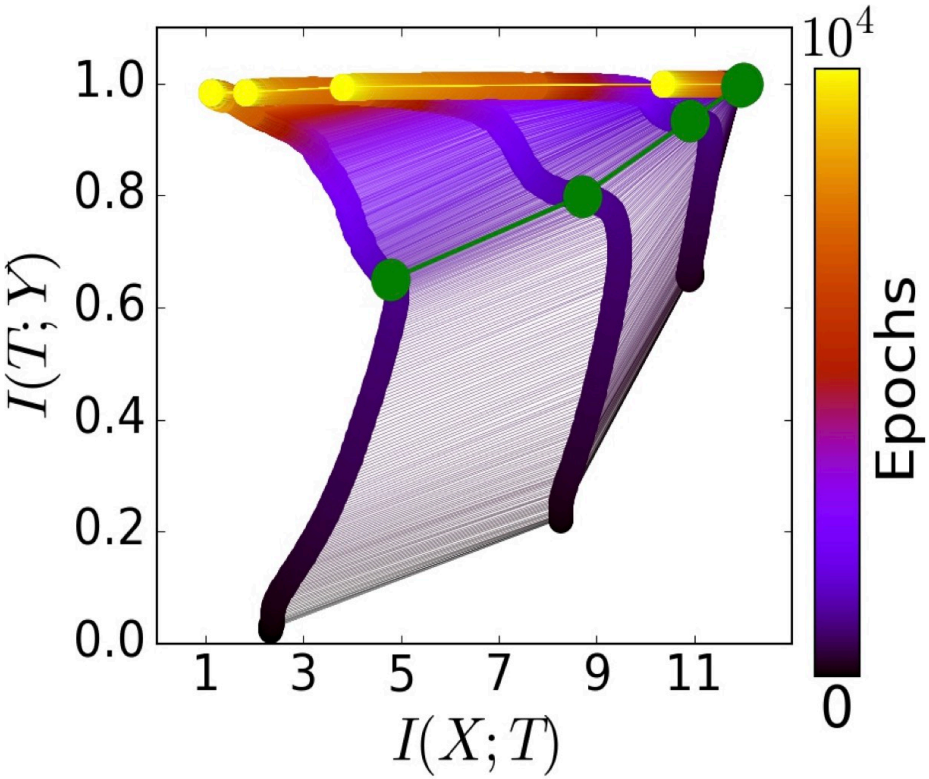


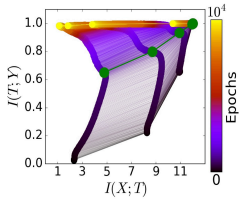
- Simulations demonstrate a two phases dynamics during training

Each point represents a different initial condition. The points in the same color come from the same layer. The first layer is on the very right side, and the last layer is on the left. A simple binary decision task, input are 12 binary bits. A 7-layer DNN, with 12-10-7-5-4-3-2 neutrons each layer. The rule makes sure that the mutual information is close to 1 ($p(x)$ is uniform, and $p(y = 1) \approx 0.5$). The training uses the traditional SGD algorithm without any regularization or dropout.

Information Dynamics



- **Fitting Phase:** the layers increase the information on the labels
- **Compression Phase:** the layers lose irrelevant information about data



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a fast process, the representations learn information from both the labels and the data, while preserving the DPI order.a slow process until convergenceThe first phase is reasonable since the ERM algorithm, but for the second phase requires an explanation.

Rethinking Learning Theory (Intuitively)

- "Old" Generalization bound:

$$\epsilon^2 \leq \frac{\log |H| + \log \frac{2}{\delta}}{2m}$$

- ϵ: generalization error
- $|H|$: size of hypothesis space
- m: number of training samples
- δ: confidence
- **Don't work for DL!**

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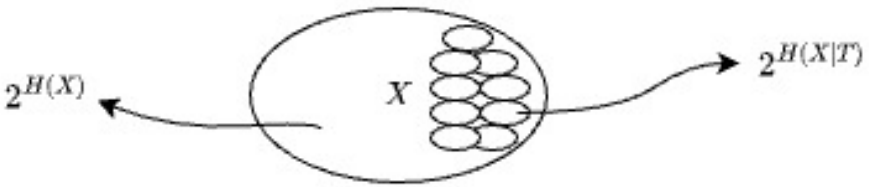

- Input Compression bound:
- $|H| \sim 2^{|X|} \to 2^{|T|}$
- Typical set and typical partition

$$p(x_1, \cdots, x_n) \approx 2^{-nH(X)}$$
$$p(x_1, \cdots, x_n|T) \approx 2^{-nH(X|T)}$$

- The size of partition $|T| \sim 2^{I(X;T)}$

$$\epsilon^2 \leq \frac{2^{I(X;T)} + \log \frac{2}{\delta}}{2m}$$

- every bit of compression is like doubling the training samples.

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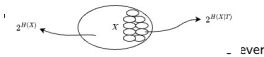

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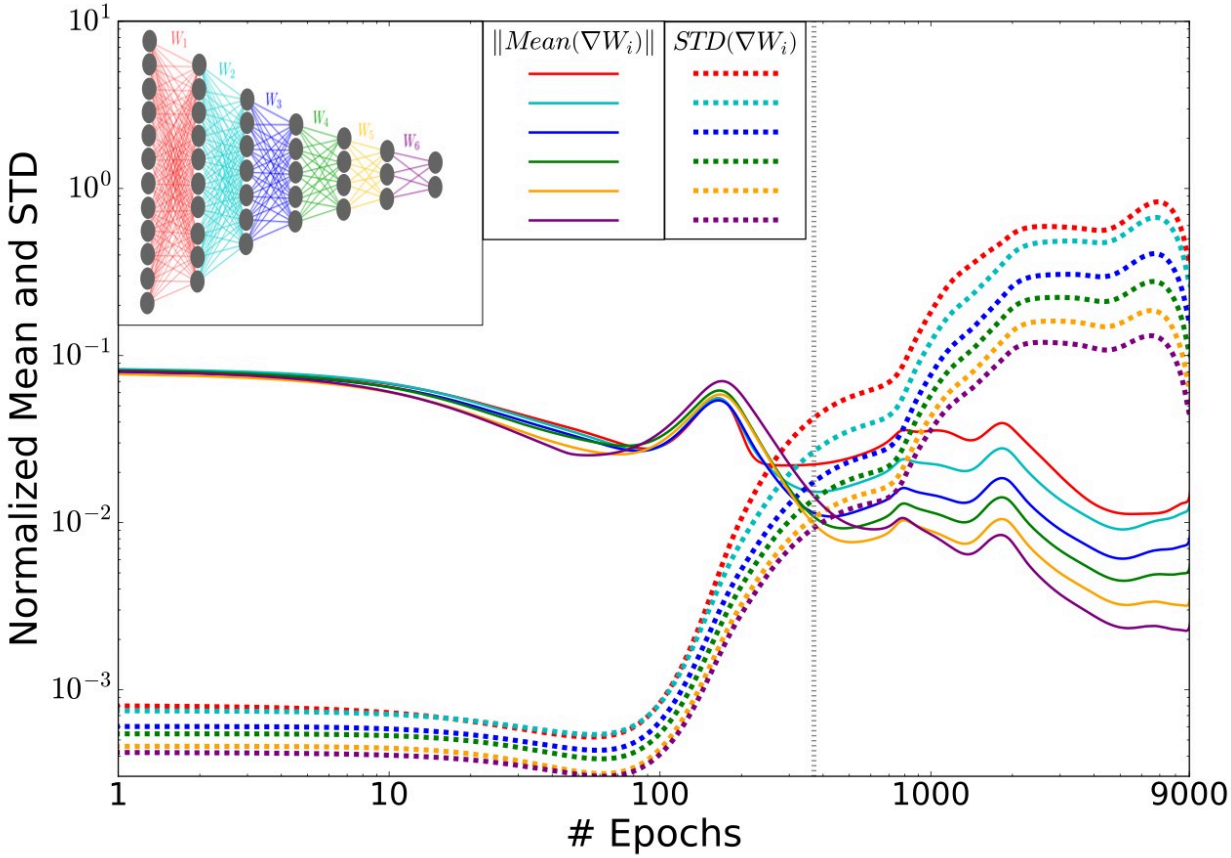
Interesting Results



Two clear phases during training:

- High SNR phase: memorization (drift)
- Low SNR phase: forgetting (diffusion)

Claim: These two stochastic gradient phases correspond the fitting and compression phases

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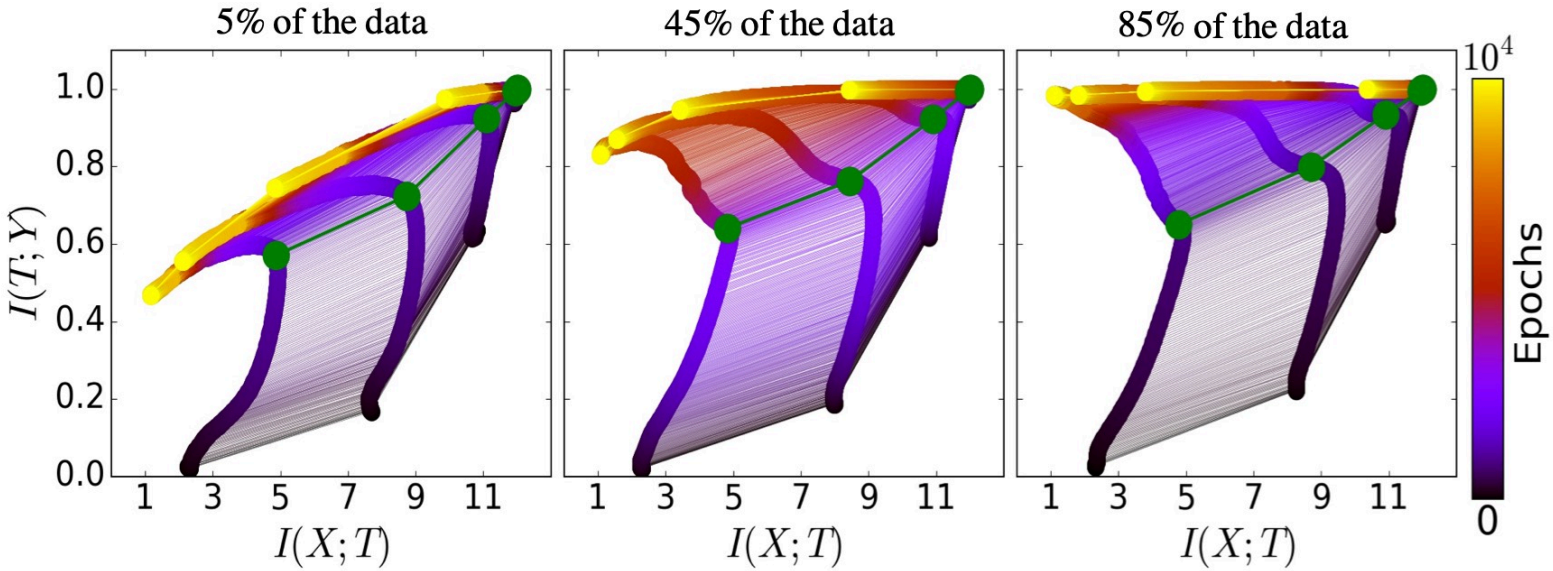

The information dynamics of the layers, for different training samples

- The initial fitting phases are similar
- Lack of training data causes over-compression (over-fitting) in the compression phase

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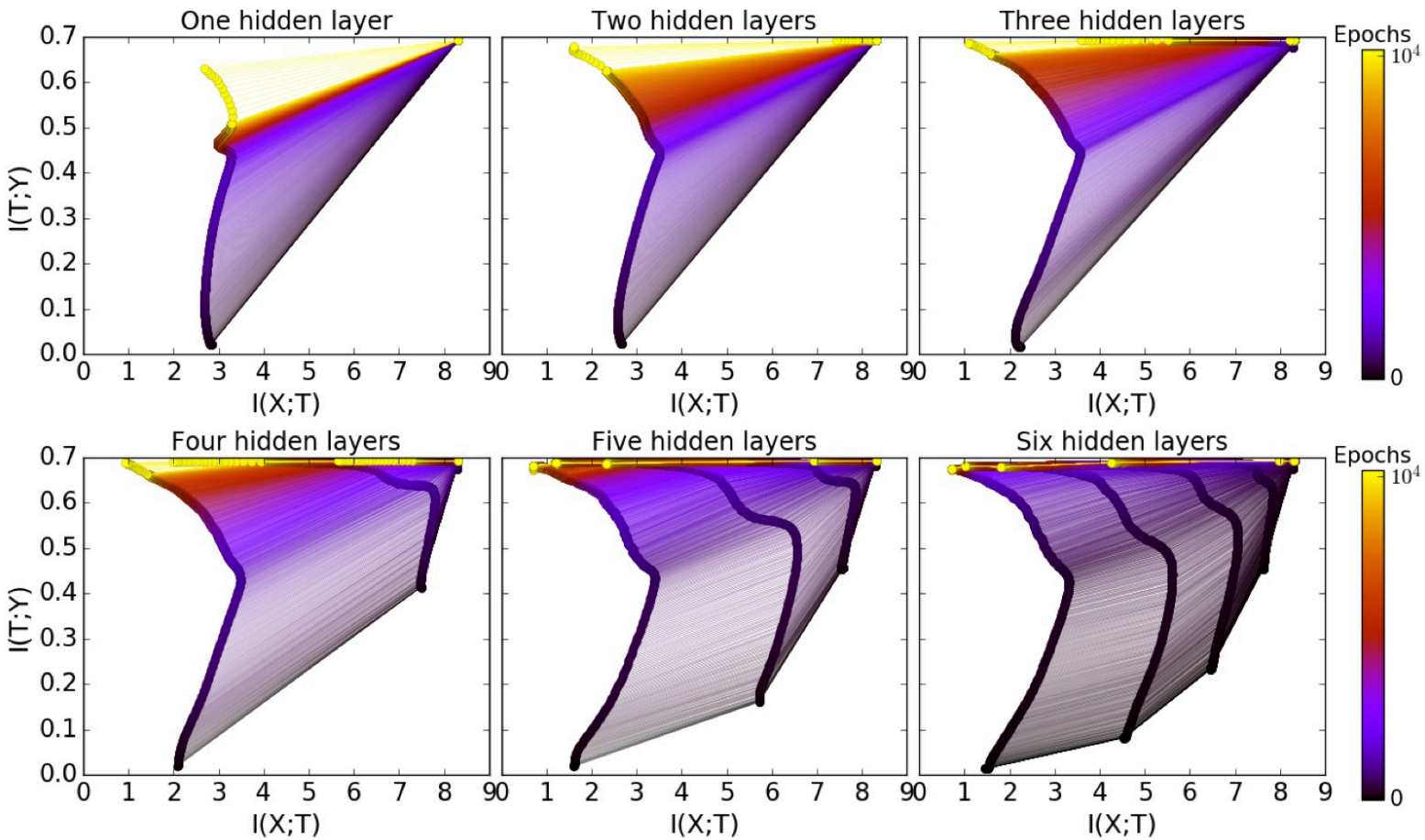

The information dynamics of the layers, for different architectures

- Adding layers reduce the training epochs
- The compression is faster for deeper layers

Review of IB

- DNN learns a bottleneck representation that optimizes the trade-off between compression and prediction
- Training DNN undergoes two distinct phases consisting of a fitting phase and a compresssion phase
- The compression phase contributes to the generalization performance of DNN
- The compression phase occurs due to the diffusion behaviour of SGD

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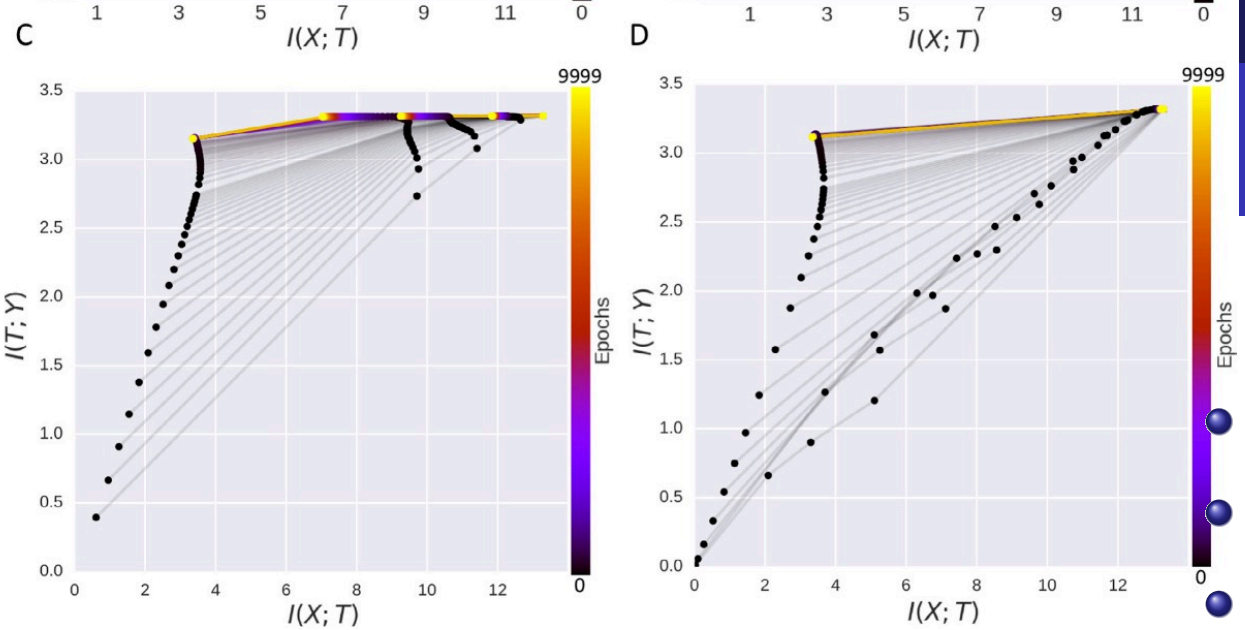
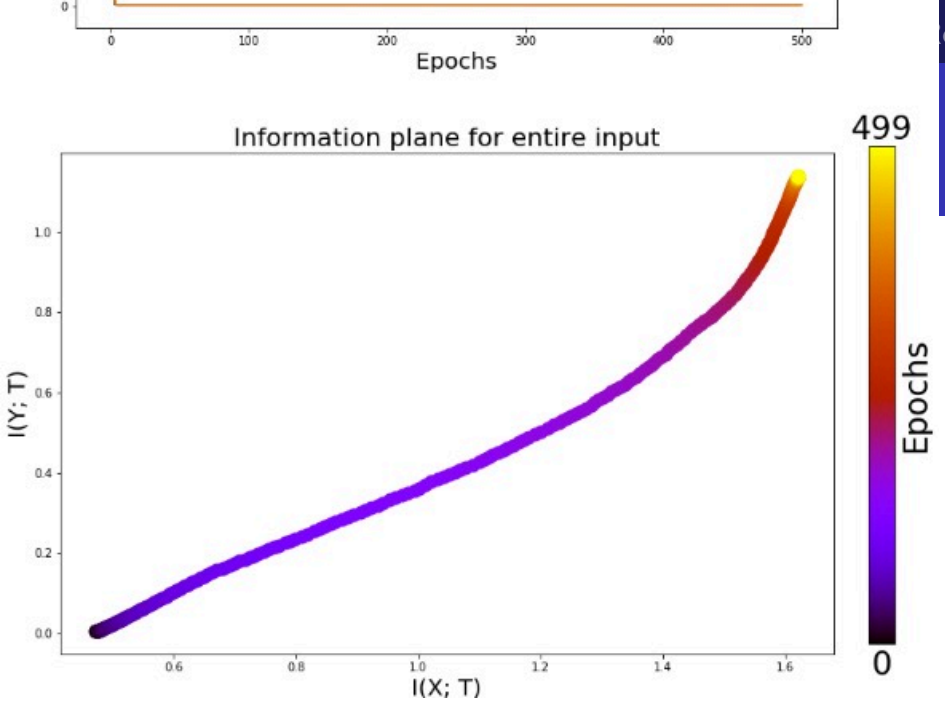
None of them hold true in the general case! (Saxe et al. 2019)



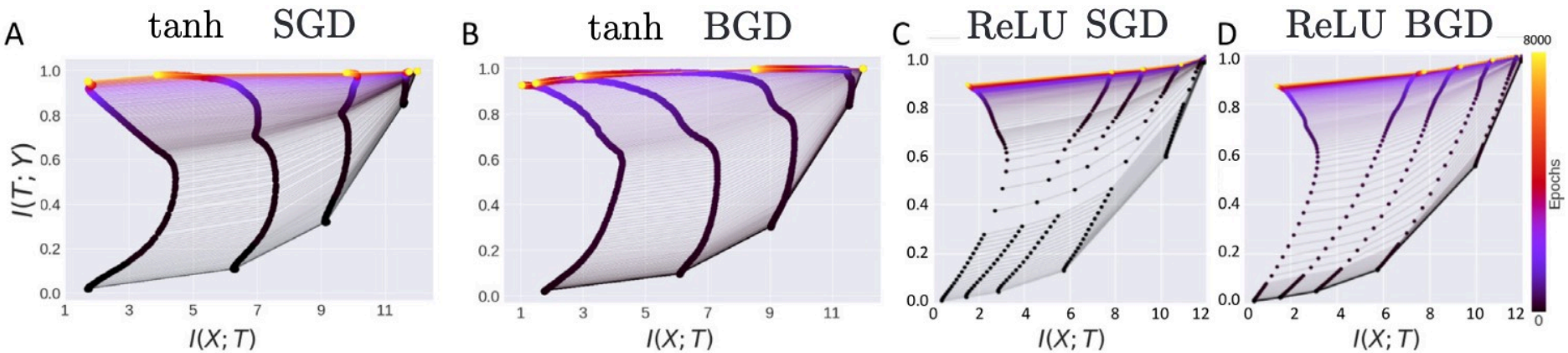
- (A)(B): binary task
- (C)(D): MNIST dataset
- (A)(C): DNN with tanh
- (B)(D): DNN with ReLU
- No compression is observed except the final classification layer using sigmoid neurons
- **Compression arises from the saturation in non-linear activation**

C



- Generalization and information plane in a deep linear network
- No compression is observed
- The network generalizes well

Challenge on Diffusion Behaviour



- SGD: stochastic gradient descent; BGD: full batch gradient descent (no randomness)
- The randomness in the training does not contribute to compression of information

Reference I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