ELECTROMAGNETIC WAVE THEORY

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PREFACE

This book presents a unified macroscopic theory of electromagnetic waves in accordance with the principle of special relativity from the point of view of the form invariance of the Maxwell equations and the constitutive relations. Great emphasis is placed on the fundamental importance of the $\overline{k}$ vector in electromagnetic wave theory. We introduce a fundamental unit $\mathrm{K}_o = 2\pi$ meter^{-1} for the spatial frequency, which is cycle per meter in spatial variation. This is similar to the fundamental unit for temporal frequency Hz, which is cycle per second in time variation. The unit K_o is directly proportional to the unit Hz; one K_o in spatial frequency corresponds to 300 MHz in temporal frequency.

This is a textbook on electromagnetic wave theory, and topics essential to the understanding of electromagnetic waves are selected and presented. Chapter 1 presents fundamental laws and equations for electromagnetic theory. Chapter 2 is devoted to the treatment of transmission line theory. Electromagnetic waves in media are studied in Chapter 3 with the kDB system developed to study waves in anisotropic and bianisotropic media. Chapter 4 presents a detailed treatment of reflection, transmission, guidance, and resonance of electromagnetic waves. Starting with Čerenkov radiation, we study radiation and antenna theory in Chapter 5. Chapter 6 then elaborates on the various theorems and limiting cases of Maxwell's theory important to the study of electromagnetic wave behavior. Scattering by spheres, cylinders, rough surfaces, and volume inhomogeneities are treated in Chapter 7. In Chapter 8, we present Maxwell's theory from the point of view of Lorentz covariance in accordance with the principle of special relativity. The problem section at the end of each section provides useful exercise and applications.

The various topics in the book can be taught independently, and the material is organized in the order of increasing complexity in mathematical techniques and conceptual abstraction and sophistication. This book has been used in several undergraduate and graduate courses that I have been teaching at the Massachusetts Institute of Technology.

The first version of the book was published in 1975 by Wiley Interscience, New York, entitled *Theory of Electromagnetic Waves,* which was based on my 1968 Ph.D. thesis, where the concept of bianisotropic media was introduced. The book was expanded and published in 1986 with the present title and its second edition appeared in 1990. Since 1998, it has been published by EMW Publishing Company, Massachusetts. The development of the various concepts in the book relies heavily on published work. I have not attempted the task of referring to all relevant publications. The list of books and journal articles in the Reference Section at the end of the book is at best representative and by no means exhaustive. Some of the results contained in the book are taken from many of my research projects, which have been supported by grants and contracts from the National Science Foundation, the National Aeronautics and Space Administration, the Office of Naval Research, the Army Research Office, the Jet Propulsion Laboratory of the California Institute of Technology, the MIT Lincoln Laboratory, the Schlumberger-Doll Research Center, the Digital Equipment Corporation, the IBM Corporation, and the funding support associated with the award of the S. T. Li prize for the year 2000.

During the writing and preparation of the book, many people helped. In particular, I would like to acknowledge Chi On Ao for formulating the TEX macros, and Zhen Wu for editing the text and constructing the index. Over the years, many of my teaching and research assistants provided useful suggestions and proofreading, notably Leung Tsang, Michael Zuniga, Weng Chew, Tarek Habashy, Robert Shin, Shun-Lien Chuang, Jay Kyoon Lee, Apo Sezginer, Soon Yun Poh, Eric Yang, Michael Tsuk, Hsiu Chi Han, Yan Zhang, Henning Braunisch, Bae-Ian Wu, Xudong Chen, and Baile Zhang. I would like to express my gratitude to them and to the students whose enthusiastic response and feedback continuously give me joy and satisfaction in teaching.

J. A. Kong

Cambridge, Massachusetts
December 2007

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1.1 Maxwell's Theory

A. Maxwell's Equations

The laws of electricity and magnetism were established in 1873 by James Clerk Maxwell (1831–1879). In three-dimensional vector notation, the Maxwell equations are

$$\nabla \times \overline{H} = \frac{\partial}{\partial t}\overline{D} + \overline{J} \tag{1.1.1}$$

$$\nabla \times \overline{E} = -\frac{\partial}{\partial t}\overline{B} \tag{1.1.2}$$

$$\nabla \cdot \overline{D} = \rho \tag{1.1.3}$$

$$\nabla \cdot \overline{B} = 0 \tag{1.1.4}$$

where $\overline{E}$, $\overline{B}$, $\overline{H}$, $\overline{D}$, $\overline{J}$, and ρ are real functions of position and time.

$\overline{E}$ =	electric field strength	(volts/m)
$\overline{B}$ =	magnetic flux density	(webers/m^2)
$\overline{H}$ =	magnetic field strength	(amperes/m)
$\overline{D}$ =	electric displacement	(coulombs/m^2)
$\overline{J}$ =	electric current density	(amperes/m^2)
ρ =	electric charge density	(coulombs/m^3)

Equation (1.1.1) is Ampère's law or the generalized Ampère circuit law. Equation (1.1.2) is Faraday's law or Faraday's magnetic induction law. Equation (1.1.3) is Coulomb's law or Gauss' law for electric fields. Equation (1.1.4) is Gauss' law or Gauss' law for magnetic fields. We generally refer to $\overline{E}$ and $\overline{D}$ as electric fields, and $\overline{H}$ and $\overline{B}$ as magnetic fields.

Maxwell's contribution to the laws of electricity and magnetism is the term $\partial\overline{D}/\partial t$, which is called the displacement current. The addition of the displacement current to the electric current density $\overline{J}\,(\overline{r}, t)$ in the original Ampère's law has at least three major consequences. First, in a capacitor which is an open circuit for direct current, the displacement current insures the continuity of alternating currents in electric circuits. Secondly, the continuity law

$$\nabla \cdot \overline{J} = -\frac{\partial}{\partial t}\rho \tag{1.1.5}$$

follows from (1.1.1) and (1.1.3) by making use of the vector identity $\nabla \cdot (\nabla \times \overline{H}) = 0$. It is the displacement term that guarantees the conservation of electric current and charge densities. Eq. (1.1.5) states that the electric current and charge densities are conserved at all time. Thirdly, Faraday's law in (1.1.2) states that surrounding a time-varying magnetic field, electric fields are produced, and are also time-varying. With the displacement term in (1.1.1), Ampère's law states that around time-varying electric fields, time-varying magnetic fields are produced. This interrelationship between the time-varying electric and magnetic fields constitutes the foundation of electromagnetic wave theory and led Maxwell to the prediction of electromagnetic waves.

In developing his theory for the electromagnetic fields in space and time, Maxwell conceived of a substance filling the whole space called aether. In the aether, the electric fields $\overline{D}$ and $\overline{E}$ are related by a dielectric permittivity ϵ_o, and the magnetic fields $\overline{B}$ and $\overline{H}$ are related by a magnetic permeability μ_o.

$$\overline{D} = \epsilon_o \overline{E} \tag{1.1.6a}$$

$$\overline{B} = \mu_o \overline{H} \tag{1.1.6b}$$

where

$$\epsilon_o \approx 8.85 \times 10^{-12} \qquad \text{farad/meter}$$

$$\mu_o = 4\pi \times 10^{-7} \qquad \text{henry/meter}$$

where the numerical values for ϵ_o and μ_o are expressed in MKS units. We now call (1.1.6) the constitutive relations for free space.

With Equations (1.1.1)–(1.1.6), Maxwell's theory of electromagnetic fields is completely expressed. Originally written in Cartesian component form, Maxwell's equations were cast in the current vector form by Oliver Heaviside (1850–1925). In 1888, Heinrich Rudolf Hertz (1857–1894) demonstrated the generation of radio waves and experimentally verified Maxwell's theory. Since then, electromagnetic theory has played a central role in the development of radio, television, wireless communications, radar, microwave heating, remote sensing, and numerous other practical applications. The special theory of relativity developed by Albert Einstein (1879–1955) in 1905 further asserted the rigorousness and elegance of Maxwell's theory. As a well-established scientific discipline, this sophisticated theoretical structure embodies many principles and concepts which serve as fundamental rules of nature and vital links for all scientific disciplines.

James Clerk Maxwell (13 June 1831 – 5 November 1879)

James Clerk Maxwell attended University of Edinburgh (1847–1850), and studied under William Hopkins at Cambridge University (1850–1854). He was a fellow of Trinity (1855–1856), Professor of Natural Philosophy at Marischal College of the University of Aberdeen (1856–1860), and at King's College (1860–1865). He was the first Cavendish Professor of Experimental Physics at Cambridge University to build and direct the Cavendish Laboratory (1871–1879). He published four books and about 100 papers starting at age 14, including 'On Faraday's Lines of Forces' in 1855, 'On Physical Lines of Force' in 1861, and 'A Dynamical Theory of the Electromagnetic Field' in 1864. In 1865, at age 33, he retired to his country home estate to write his monumental book *A Treatise of Electricity and Magnetism* (Constable and Company, London, 1873; Dover Publications, New York, 1006 pages, 1954).

Michael Faraday (22 September 1791 – 25 August 1867)

Faraday became an assistant to Sir Humphry Davy at the Royal Institution on 1 March 1813. In September 1821, his experimentation demonstrated electro-magnetic rotation, initiated the concept of electric motor. In August 1831, he discovered electro-magnetic induction, and that magnetism produced electricity through movement, the principle behind the electric transformer and generator. He became professor of chemistry in 1833. Faraday published many of his results in the three-volume *Experimental Researches in Electricity* (1839–1855).

Johann Carl Friedrich Gauss (30 April 1777 – 23 February 1855)

Gauss studied mathematics at the University of Göttingen from 1795 to 1798, and received his doctoral degree from the University of Helmstedt in 1799. In 1807 he took the position of director of the Göttingen Observatory. In 1832 he presented a systematic use of absolute units (length, mass, time) to measure nonmechanical quantities. From 1831 to 1837 he worked closely with Wilhelm Eduard Weber (24 October 1804 – 23 June 1891) on terrestrial magnetism and organized a system of stations for systematic observations.

André-Marie Ampère (20 January 1775 – 10 June 1836)

Ampère was appointed professor at Bourg Ecole Centrale in 1802, at the Ecole Polytechnique in 1809, and at Université de France in 1826. In September 1820, Ampère showed that two parallel conductors attract each other if they carry currents that flow in the same direction and repel if the currents flow in opposite directions. In 1823–1826, he completed his memoir on the 'Mathematical Theory of Electrodynamic Phenomena, Uniquely Deduced from Experience'.

Charles-Augustin de Coulomb (14 June 1736 – 23 August 1806)

Coulomb worked in the Corps du Génie until he retired in 1791. In 1777 he invented the torsion balance, which enabled him to establish the fundamental laws of electricity by measuring the force between two small spheres charged with electricity. Between 1785 and 1791, he published seven treatises on electricity and magnetism.

B. Vector Analysis

A vector $\overline{A}$ has a magnitude and a direction, which can be represented graphically by a straight-line element of length proportional to the magnitude of $\overline{A}$ and with an arrow pointing in the direction of $\overline{A}$. In a Cartesian coordinate system (also called the rectangular coordinate system), we write in terms of the three Cartesian components A_x, A_y, and A_z [Fig. 1.1.1].

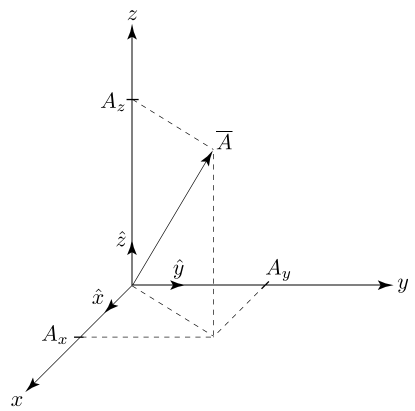


Figure 1.1.1 Projection of $\overline{A}$ in rectangular coordinate system.

$$\overline{A} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z$$

where A_x, A_y, A_z are the projections of $\overline{A}$ onto the x, y, z axes. We denote the directions of the x, y, z axes with $\hat{x}$, $\hat{y}$, $\hat{z}$ each of them has unit magnitude with the scalar product $\hat{x} \cdot \hat{x} = \hat{y} \cdot \hat{y} = \hat{z} \cdot \hat{z} = 1$. They are called the unit vectors. Furthermore $\hat{x} \cdot \hat{y} = \hat{y} \cdot \hat{z} = \hat{z} \cdot \hat{x} = 0$. We use a hat instead of an overbar to represent the vector with unit magnitude.

Rene Descartes (31 March 1596 – 11 February 1650)

Rene Descartes originated the Cartesian coordinates and founded analytic geometry. His philosophy is called Cartesianism (from Cartesius, the Latin form of his name), with the famous statement 'I think, therefore I am.' He preached universal doubt; only one thing cannot be doubted: doubt itself.

Vector Addition and Subtraction

Two vectors $\overline{A}$ and $\overline{B}$, when they are not in the same direction or in opposite directions, determine a plane. In Cartesian components, we write

$$\begin{aligned}\overline{A} &= \hat{x}A_x + \hat{y}A_y + \hat{z}A_z \\ \overline{B} &= \hat{x}B_x + \hat{y}B_y + \hat{z}B_z\end{aligned}$$

It follows that

$$\overline{A} \pm \overline{B} = \hat{x}(A_x \pm B_x) + \hat{y}(A_y \pm B_y) + \hat{z}(A_z \pm B_z)$$

Scalar Dot Product

The scalar or dot product of two vectors $\overline{A}$ and $\overline{B}$, denoted by $\overline{A} \cdot \overline{B}$, is a scalar number,

$$\overline{A} \cdot \overline{B} = A_x B_x + A_y B_y + A_z B_z$$

Vector Cross Product

The vector or cross product of two vectors $\overline{A}$ and $\overline{B}$, denoted by $\overline{A} \times \overline{B}$, is a vector. In terms of their Cartesian components,

$$\begin{aligned}\overline{A} \times \overline{B} &= \hat{x}(A_y B_z - A_z B_y) + \hat{y}(A_z B_x - A_x B_z) + \hat{z}(A_x B_y - A_y B_x) \\ &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}\end{aligned}$$

For the three orthogonal unit vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$ it is seen that $\hat{x} = \hat{y} \times \hat{z}$, $\hat{y} = \hat{z} \times \hat{x}$, $\hat{z} = \hat{x} \times \hat{y}$.

The direction of $\overline{A} \times \overline{B}$ follows the right-hand rule, i.e., when the fingers of the right hand rotate from $\overline{A}$ to $\overline{B}$, the thumb of the right hand points in the direction of $\overline{A} \times \overline{B}$. Thus the vector $\overline{A} \times \overline{B}$ is perpendicular to both $\overline{A}$ and $\overline{B}$ and the plane containing $\overline{A}$ and $\overline{B}$. It is seen that for $\overline{A} = \hat{x}A_x + \hat{y}A_y$ and $\overline{B} = \hat{x}B_x + \hat{y}B_y$ both in the xy-plane, $\overline{A} \times \overline{B} = \hat{z}(A_x B_y - A_y B_x)$ is in the $\hat{z}$ direction perpendicular to both $\overline{A}$ and $\overline{B}$.

Division by a vector is not defined; thus $\overline{B}/\overline{A}$ and $1/\overline{A}$ are meaningless expressions. If none of the operations of addition, subtraction, dot product, or cross product is imposed on $\overline{A}$ and $\overline{B}$, the entity $\overline{A}\,\overline{B}$ is called a dyad. In the language of tensor analysis, a dyad is a tensor of second rank, while all vectors are tensors of first rank.

Operation of Three Vectors

For three vectors $\overline{A}$, $\overline{B}$, and $\overline{C}$, we have

$$\overline{C}\cdot(\overline{A}\times\overline{B})=\overline{A}\cdot(\overline{B}\times\overline{C})=\overline{B}\cdot(\overline{C}\times\overline{A}) \tag{1.1.7}$$

$$=\begin{vmatrix} C_x & C_y & C_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}=\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}=\begin{vmatrix} B_x & B_y & B_z \\ C_x & C_y & C_z \\ A_x & A_y & A_z \end{vmatrix}$$

$$\begin{aligned}\overline{C}\times(\overline{A}\times\overline{B}) &= \hat{x}\ [C_y\ (A_xB_y-A_yB_x)-C_z(A_zB_x-A_xB_z)] \\ &\quad+\hat{y}\ [C_z(A_yB_z-A_zB_y)-C_x(A_xB_y-A_zB_y)] \\ &\quad+\hat{z}\ [C_x(A_zB_x-A_xB_z)-C_y(A_xB_y-A_yB_x)] \\ &= (\hat{x}A_x+\hat{y}A_y+\hat{z}A_z)(C_xB_x+C_yB_y+C_zB_z) \\ &\quad-(C_xA_x+C_yA_y+C_zA_z)(\hat{x}B_x+\hat{y}B_y+\hat{z}B_z) \\ &= \overline{A}(\overline{C}\cdot\overline{B})-(\overline{C}\cdot\overline{A})\overline{B} \end{aligned} \tag{1.1.8}$$

Notice that the vector $\overline{C}\times(\overline{A}\times\overline{B})$ is perpendicular to $\overline{C}$ and lies in the plane determined by $\overline{A}$ and $\overline{B}$.

Operation with the del Operator

The del operator ∇ is a vector differential operator written as

$$\nabla=\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}$$

The following can be proved in Cartesian coordinates or in vector form:

$$\nabla\cdot(\overline{E}\times\overline{H})=\overline{H}\cdot(\nabla\times\overline{E})-\overline{E}\cdot(\nabla\times\overline{H}) \tag{1.1.9}$$

$$\nabla\cdot(\nabla\times\overline{A})=0 \tag{1.1.10}$$

$$\nabla\times(\nabla\Phi)=0 \tag{1.1.11}$$

$$\nabla\times(\nabla\times\overline{E})=\nabla(\nabla\cdot\overline{E})-\nabla^2\overline{E} \tag{1.1.12}$$

where

$$\nabla^2=\nabla\cdot\nabla=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} \tag{1.1.13}$$

is the Laplacian operator in the rectangular coordinate system.

Pierre-Simon Laplace (28 March 1749 – 5 March 1827)

Pierre-Simon Laplace was appointed to a chair of mathematics at the École Militaire in Paris at the age of 19. During the French Revolution he helped to establish the metric system. The Laplace equation $\nabla^2\cdot\Phi=0$ was published in 1813.

Gradient of a Scalar

When the del operator operates on a scalar function $\Phi(x, y, z)$, the result is a vector

$$\nabla\Phi = \hat{x}\frac{\partial}{\partial x}\Phi + \hat{y}\frac{\partial}{\partial y}\Phi + \hat{z}\frac{\partial}{\partial z}\Phi \tag{1.1.14}$$

called the gradient of $\Phi(x, y, z)$. The differential form of the gradient of Φ as defined states that

$$\begin{aligned}
\nabla\Phi &= \hat{x} \lim_{\Delta x\to 0} \frac{\Delta\Phi}{\Delta x} + \hat{y} \lim_{\Delta y\to 0} \frac{\Delta\Phi}{\Delta y} + \hat{z} \lim_{\Delta z\to 0} \frac{\Delta\Phi}{\Delta z} \\
&= \hat{x} \lim_{\Delta x\to 0} \frac{1}{\Delta x}\left[\left(\Phi(x+\frac{\Delta x}{2}, y, z) - \Phi(x-\frac{\Delta x}{2}, y, z)\right)\right] \\
&\quad + \hat{y} \lim_{\Delta y\to 0} \frac{1}{\Delta y}\left[\left(\Phi(x, y+\frac{\Delta y}{2}, z) - \Phi(x, y-\frac{\Delta y}{2}, z)\right)\right] \\
&\quad + \hat{z} \lim_{\Delta z\to 0} \frac{1}{\Delta z}\left[\left(\Phi(x, y, z+\frac{\Delta z}{2}) - \Phi(x, y, z-\frac{\Delta z}{2})\right)\right]
\end{aligned} \tag{1.1.15}$$

When $\Phi(x, y, z) = \Phi(x)$ is a function of x only, $\nabla\Phi(x)$ is a vector pointing in the direction of increasing x with the magnitude equal to the slope of the function at x.

EXAMPLE 1.1.1 Electric field vector as gradient of a potential function.

When there is no time variation, we may write the electric field vector $\overline{E}$ as

$$\overline{E} = -\nabla\Phi \tag{E1.1.1.1}$$

and call Φ a potential function. As the gradient $\nabla\Phi$ points in the direction of increasing potential Φ, the electric field $\overline{E}$ points from high potential towards low potential, similar to water flowing from a high altitude to lower ground.

Giving the potential of a point charge Q is

$$\Phi = \frac{Q}{4\pi r}$$

the electric field is

$$\overline{E} = -\frac{\partial}{\partial r}\Phi = \frac{Q}{4\pi r^2}$$

Thus the electric field points from high potential to low potential.

— END OF EXAMPLE 1.1.1 —

Divergence of a Vector

The divergence of a vector function is a scalar, defined as

$$\begin{aligned}\nabla \cdot \overline{D} &= \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \cdot (\hat{x}D_x + \hat{y}D_y + \hat{z}D_z) \\ &= \frac{\partial}{\partial x}D_x + \frac{\partial}{\partial y}D_y + \frac{\partial}{\partial z}D_z \end{aligned} \tag{1.1.16}$$

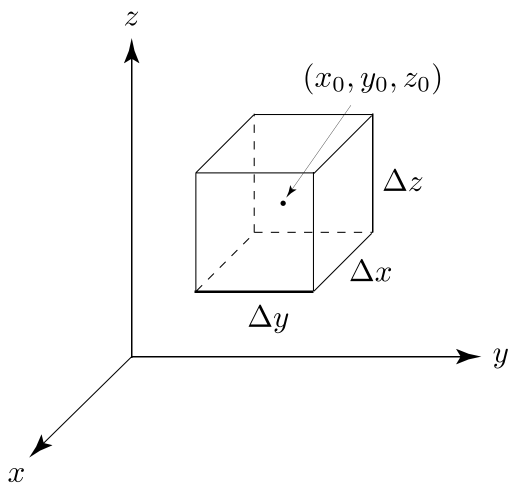


Figure 1.1.2 Differential volume $\Delta x \Delta y \Delta z$.

Consider a differential volume with sides $\Delta x, \Delta y, \Delta z$ centered around a point (x_0, y_0, z_0) [Fig. 1.1.2]. The divergence as defined states that

$$\begin{aligned}\nabla \cdot \overline{D} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0 \\ \Delta z \to 0}} \frac{1}{\Delta x \Delta y \Delta z} \Bigg\{ &\Delta y \Delta z \left[D_x(x_0 + \frac{\Delta x}{2}, y_0, z_0) - D_x(x_0 - \frac{\Delta x}{2}, y_0, z_0) \right] \\ &+ \Delta z \Delta x \left[D_y(x_0, y_0 + \frac{\Delta y}{z}, z_0) - D_y\ (x_0, y_0 - \frac{\Delta y}{z}, z_0) \right] \\ &+ \Delta x \Delta y \left[D_z(x_0, y_0, z_0 + \frac{\Delta z}{2}) - D_z(x_0, y_0, z_0 - \frac{\Delta z}{2}) \right] \Bigg\} \end{aligned} \tag{1.1.17}$$

Gauss Theorem or Divergence Theorem

The first term in the braces is equal to the field component D_x at the surface at $x = x_0 + \frac{\Delta x}{2}$ multiplied by the surface area $\Delta y \Delta z$. We define a surface normal vector $d\overline{S}$ pointing outward of the volume such that at the surface at $x = x_0 + \frac{\Delta x}{2}$, $d\overline{S} = \hat{x}\Delta y \Delta z$ and at the surface at $x = x_0 - \frac{\Delta x}{2}$, $d\overline{S} = -\hat{x}\Delta y \Delta z$. Then the negative sign in the second term is due to $\overline{D}$ dot multiplied by $d\overline{S}$. All six terms account for the six differential areas bounding the differential volume $\Delta V = \Delta x \Delta y \Delta z$ with a surface normal $d\overline{S}$. We thus express the divergence of $\overline{D}$ as

$$\nabla \cdot \overline{D} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oiint d\overline{S} \cdot \overline{D} \tag{1.1.18}$$

Applying (1.1.18) to a large volume V containing an infinite number of such infinitesimal differential volumes [Fig. 1.1.3], we note that integrating the divergence over the volume surfaces shared by adjacent differential volumes will have no contribution because the surface normals point in opposite directions and thus cancel. The result is the divergence theorem or Gauss theorem

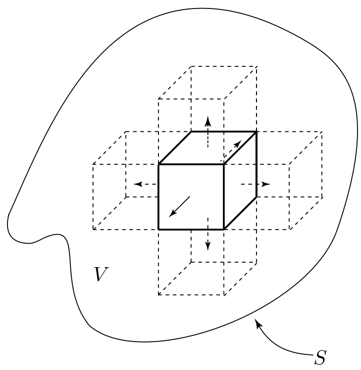


Figure 1.1.3 Derivation of divergence theorem.

$$\iiint_V dV\, \nabla \cdot \overline{D} = \oiint_S d\overline{S} \cdot \overline{D} \tag{1.1.19}$$

The divergence theorem states that the volume integral of the divergence of the vector field $\overline{D}$ is equal to the total outward flux $\overline{D}$ through the surface S enclosing the volume.

Curl of a Vector

The curl of a vector field $\overline{H}$ is a vector defined as

$$\nabla\times\overline{H}=\left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right)\times\overline{H}=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ H_x & H_y & H_z\end{vmatrix}$$
$$=\hat{x}\left(\frac{\partial}{\partial y}H_z-\frac{\partial}{\partial z}H_y\right)+\hat{y}\left(\frac{\partial}{\partial z}H_x-\frac{\partial}{\partial x}H_z\right)+\hat{z}\left(\frac{\partial}{\partial x}H_y-\frac{\partial}{\partial y}H_x\right) \tag{1.1.20}$$

Consider a differential volume of sides $\Delta x, \Delta y, \Delta z$ centered around a point (x_0, y_0, z_0). In the Cartesian coordinate system, the differential form of the curl of $\overline{H}$ as defined states that

$$\begin{aligned}\nabla\times\overline{H}=&\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0\\ \Delta z\to 0}}\left\{\frac{1}{\Delta x}\left[\hat{x}\times\left(\overline{H}(x_0+\frac{\Delta x}{2},y_0,z_0)-\overline{H}(x_0-\frac{\Delta x}{2},y_0,z_0)\right)\right]\right.\\
&+\frac{1}{\Delta y}\left[\hat{y}\times\left(\overline{H}(x_0,y_0+\frac{\Delta y}{2},z_0)-\overline{H}(x_0,y_0-\frac{\Delta y}{2},z_0)\right)\right]\\
&\left.+\frac{1}{\Delta z}\left[\hat{z}\times\left(\overline{H}(x_0,y_0,z_0+\frac{\Delta z}{2})-\overline{H}(x_0,y_0,z_0-\frac{\Delta z}{2})\right)\right]\right\}\\
=&\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0\\ \Delta z\to 0}}\frac{1}{\Delta x\Delta y\Delta z}\times\\
&\left\{\hat{x}\left[\Delta x\Delta z\left(H_z(x_0,y_0+\frac{\Delta y}{2},z_0)-H_z(x_0,y_0-\frac{\Delta y}{2},z_0)\right)\right.\right.\\
&\left.-\Delta x\Delta y\left(H_y(x_0,y_0,z_0+\frac{\Delta z}{2})-H_y(x_0,y_0,z_0-\frac{\Delta z}{2})\right)\right]\\
&+\hat{y}\left[\Delta x\Delta y\left(H_x(x_0,y_0,z_0+\frac{\Delta z}{2})-H_x(x_0,y_0,z_0-\frac{\Delta z}{2})\right)\right.\\
&\left.-\Delta y\Delta z\left(H_z(x_0+\frac{\Delta x}{2},y_0,z_0)-H_z(x_0-\frac{\Delta x}{2},y_0,z_0)\right)\right]\\
&+\hat{z}\left[\Delta y\Delta z\left(H_y(x_0+\frac{\Delta x}{2},y_0,z_0)-H_y(x_0-\frac{\Delta x}{2},y_0,z_0)\right)\right.\\
&\left.\left.-\Delta x\Delta z\left(H_x(x_0,y_0+\frac{\Delta y}{2},z_0)-H_x(x_0,y_0-\frac{\Delta y}{2},z_0)\right)\right]\right\}\end{aligned} \tag{1.1.21}$$

Stokes Theorem

The $\hat{z}$ component of (1.1.21) is

$$\begin{aligned}\hat{z}\cdot(\nabla\times\overline{H}) &= (\nabla\times\overline{H})_z = \frac{\partial}{\partial x}H_y - \frac{\partial}{\partial y}H_x \\ &= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{1}{\Delta x\Delta y}\Big\{\Delta y\left[H_y(x_0+\frac{\Delta x}{2},y_0,z_0) - H_y(x_0-\frac{\Delta x}{2},y_0,z_0)\right] \\ &\qquad - \Delta x\left[H_x(x_0,y_0+\frac{\Delta y}{2},z_0) - H_x(x_0,y_0-\frac{\Delta y}{2},z_0)\right]\Big\}\end{aligned}$$

The first term in the bracket is equal to the component H_y at $x = x_0 + \frac{\Delta x}{2}$ multiplied by the differential length Δy. We define a vector differential length $d\bar{l}$ [Fig. 1.1.4] such that for the side Δy at $x = x_0 + \frac{\Delta x}{2}$, $d\bar{l} = \hat{y}dy$; for the side Δx at $y_0 + \frac{\Delta y}{2}$, $d\bar{l} = -\hat{x}dx$; for the side Δy at $x = x_0 - \frac{\Delta x}{2}$, $d\bar{l} = -\hat{y}dy$; and for the side Δx at $y = y_0 - \frac{\Delta y}{2}$, $d\bar{l} = \hat{x}dx$. If we use the fingers of the right hand to trace the direction of $d\bar{l}$ along the loop, the right-hand thumb points in the surface normal direction $\hat{z}$. Thus

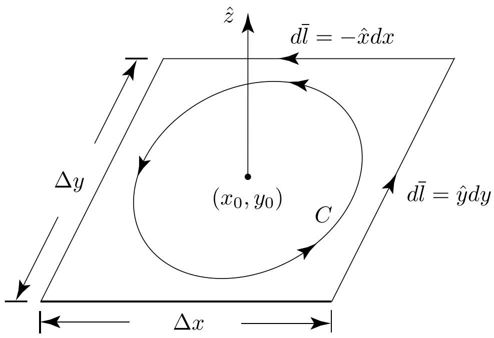


Figure 1.1.4 Derivation of $\hat{z}$-component of the curl of a vector field.

$$\hat{z}\cdot(\nabla\times\overline{H}) = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{1}{\Delta S}\oint_C d\bar{l}\cdot\overline{H} \tag{1.1.22}$$

where C denotes the contour circulating the area $\Delta S = \Delta x\Delta y$. Similar results are derivable for the $\hat{x}$ and $\hat{y}$ components of $\nabla\times\overline{H}$. For a differential area ΔS with a surface normal in the direction of $\hat{s}$, we have

$$\hat{s}\cdot(\nabla\times\overline{H}) = \lim_{\Delta S\to 0} \frac{1}{\Delta S}\oint_C d\bar{l}\cdot\overline{H} \tag{1.1.23}$$

We now apply (1.1.21) to an open surface S, subdivide into N differential areas [Fig. 1.1.5]. For a differential area ΔS_j bounded by a contour C_j and with a surface normal $\hat{s_j}$, we have $\Delta\overline{S}_j = \hat{s}_j \Delta S_j$ and

$$\Delta\overline{S}_j \cdot (\nabla \times \overline{H})_j = \oint_{C_j} d\bar{l} \cdot \overline{H}$$

Adding the contributions of all N differential areas [Fig. 1.1.5], we find

$$\lim_{\substack{\Delta S_j \to 0 \\ N \to \infty}} \sum_{j=1}^{N} \Delta\overline{S}_j \cdot (\nabla \times \overline{H})_j = \oint_C d\bar{l} \cdot \overline{H}$$

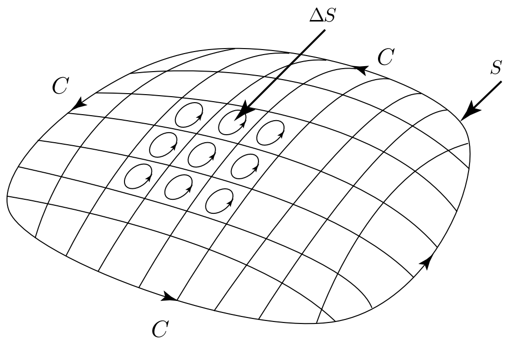


Figure 1.1.5 Derivation of Stokes' theorem.

Since the common part of the contours in two adjacent elements is traversed in opposite directions by the two contours, the net contribution of all the common parts in the interior sums to zero and only the contribution from the external contour C bounding the open surface S remains in the line integral on the right-hand side. The left-hand side becomes a surface integral, and the result is Stokes' theorem:

$$\iint d\overline{S} \cdot (\nabla \times \overline{H}) = \oint_C d\bar{l} \cdot \overline{H} \qquad (1.1.24)$$

Stokes' theorem states that the surface integral of the curl of the vector field $\overline{H}$ over an open surface S is equal to the closed line integral of the vector along the contour enclosing the open surface.

George Gabriel Stokes (13 August 1819 – 1 February 1903) was appointed Lucasian Professor of Mathematics at Cambridge University in 1849. His mathematical and physical papers were published in 5 volumes, the first 3 of which Stokes edited himself in 1880, 1883 and 1891. The last 2 were edited by Joseph Larmor in 1887 and 1891.

Maxwell Equations in Integral Form

Applying Stokes theorem to the Ampére's law and Faraday's law and applying the divergence theorem to Gauss' and continuity laws, we find

$$\oint_C d\bar{l} \cdot \overline{H} = \iint d\overline{S} \cdot \overline{J} + \iint d\overline{S} \cdot \frac{\partial}{\partial t}\overline{D} \tag{1.1.25}$$

$$\oint_C d\bar{l} \cdot \overline{E} = -\iint d\overline{S} \cdot \frac{\partial}{\partial t}\overline{B} \tag{1.1.26}$$

$$\oiint_S d\overline{S} \cdot \overline{D} = \iiint_V dV\, \nabla \cdot \overline{D} = \iiint_V dV\, \rho \tag{1.1.27}$$

$$\oiint_S d\overline{S} \cdot \overline{B} = \iiint_V dV\, \nabla \cdot \overline{B} = 0 \tag{1.1.28}$$

$$\oiint_S d\overline{S} \cdot \overline{J} = -\iiint_V dV \frac{\partial}{\partial t}\rho \tag{1.1.29}$$

These are the integral form of Maxwell equations.

Oliver Heaviside (18 May 1850 – 3 February 1925)

The year after the publication of Maxwell's *Treatise of Electricity and Magnetism* in 1873, Heaviside resigned from his job at age 24 and devoted all his time to the study of Maxwell's theory. Despite of the criticism from all the disbelievers, he remained the faithful decipher and declared himself a Maxwellian. He refuted the quaternion notation initiated by Hamilton and Tait and developed the vector notation to cast Maxwell's equation into the form as we show in this book.

Cylindrical and Spherical Coordinate Systems

In addition to the rectangular coordinates with unit vectors $\hat{x}, \hat{y}, \hat{z}$, the cylindrical coordinate system with unit vectors $\hat{\rho}, \hat{\phi}, \hat{z}$, and the spherical coordinate system with unit vectors $\hat{\rho}, \hat{\theta}, \hat{\phi}$ are often used in this book.

In a general orthogonal coordinate system, we use $\hat{u}_i$ $(i = 1, 2, 3)$ to denote the three basis vectors, $dl_i = h_i du_i$ to denote a differential length, where h_i is called a metric coefficient. The basis vectors are perpendicular to one another $\hat{u}_i \cdot \hat{u}_j = 0$ for $i \neq j$ but they are not necessarily of unit length. In Table 1.1.1 we summarize the basis vectors and the metric coefficients for the rectangular (or Cartesian), cylindrical, and spherical coordinate systems.

Orthogonal Coordinate System	Rectangular Coordinates (x, y, z)	Cylindrical Coordinates (ρ, ϕ, z)	Spherical Coordinates (r, θ, ϕ)
Base Vectors $(\hat{u}_1, \hat{u}_2, \hat{u}_3)$	$\hat{x}, \hat{y}, \hat{z}$	$\hat{\rho}, \hat{\phi}, \hat{z}$	$\hat{r}, \hat{\theta}, \hat{\phi}$
Metric Coefficients (h_1, h_2, h_3)	$1, 1, 1$	$1, \rho, 1$	$1, r, r\sin\theta$
Differential Volume $(h_1 h_2 h_3 du_1 du_2 du_3)$	$dxdydz$	$\rho d\rho d\phi dz$	$r^2 \sin\theta dr d\theta d\phi$

Table 1.1.1 Orthogonal coordinate systems.

In terms of the general orthogonal coordinate system, the gradient, the divergence, the curl, and the Laplacian operators are defined as

$$\nabla\Phi = \hat{u}_1 \frac{\partial\Phi}{h_1 \partial u_1} + \hat{u}_2 \frac{\partial\Phi}{h_2 \partial u_2} + \hat{u}_3 \frac{\partial\Phi}{h_3 \partial u_3}$$

$$\nabla \cdot \overline{D} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1}(h_2 h_3 D_1) + \frac{\partial}{\partial u_2}(h_3 h_1 D_2) + \frac{\partial}{\partial u_3}(h_1 h_2 D_3) \right]$$

$$\nabla \times \overline{H} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{u}_1 & h_2 \hat{u}_2 & h_3 \hat{u}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1 H_1 & h_2 H_2 & h_3 H_3 \end{vmatrix}$$

$$\nabla^2\Phi = \nabla \cdot \nabla\Phi$$

$$= \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} h_2 h_3 \frac{\partial\Phi}{h_1 \partial u_1} + \frac{\partial}{\partial u_2} h_3 h_1 \frac{\partial\Phi}{h_2 \partial u_2} + \frac{\partial}{\partial u_3} h_1 h_2 \frac{\partial\Phi}{h_3 \partial u_3} \right]$$

Identifying the metrics h_1, h_2, h_3 with those as listed in Table 1.1.1, we readily obtain the expressions in cylindrical and spherical coordinates.
In the cylindrical coordinate system [Fig. 1.1.6],

Vector differential length $d\bar{l} = \hat{\rho} d\rho + \hat{\phi} \rho d\phi + \hat{z} dz$

Differential area $d\overline{S} = \hat{\rho} \rho d\phi dz + \hat{\phi} d\rho dz + \hat{z} \rho d\rho d\phi$

Differential volume $dV = \rho d\rho d\phi dz$

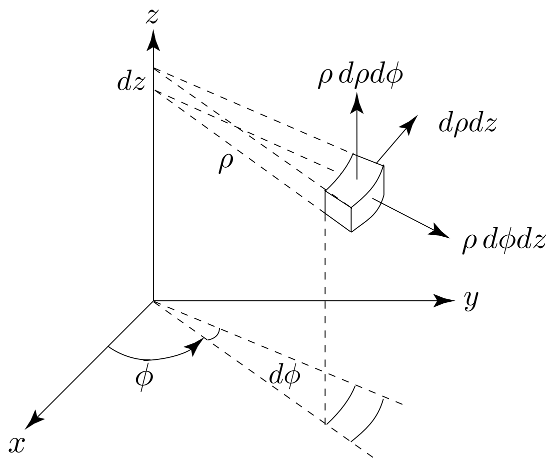


Figure 1.1.6 Cylindrical coordinate system.

$$\nabla\Phi = \hat{\rho}\frac{\partial\Phi}{\partial\rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial\Phi}{\partial\phi} + \hat{z}\frac{\partial\Phi}{\partial z}$$

$$\nabla\cdot\overline{D} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial}{\partial\phi}D_\phi + \frac{\partial}{\partial z}D_z$$

$$\nabla\times\overline{H} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\phi} & \dfrac{\partial}{\partial z} \\ H_\rho & \rho H_\phi & H_z \end{vmatrix}$$

$$\begin{aligned}\nabla^2\Phi &= \nabla\cdot\nabla\Phi \\ &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\frac{\partial\Phi}{\partial\rho}\right] + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}\end{aligned}$$

In the spherical coordinate system [Fig. 1.1.7],

Vector differential length $\quad d\bar{l} = \hat{r}dr + \hat{\theta}rd\theta + \hat{\phi}r\sin\theta d\phi$

Differential area $\quad d\overline{S} = \hat{r}r^2\sin\theta d\theta d\phi + \hat{\theta}r\sin\theta dr d\phi + \hat{\phi}r dr d\theta$

Differential volume $\quad dV = r^2\sin\theta dr d\theta d\phi$

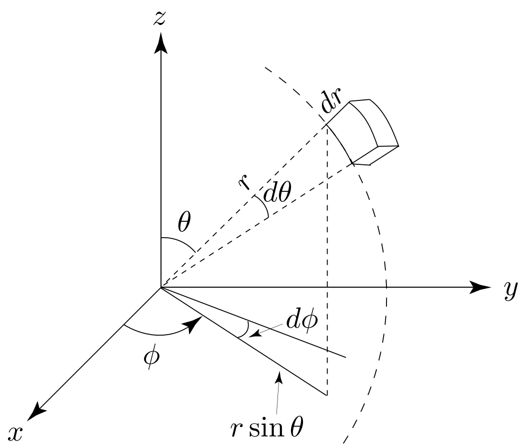


Figure 1.1.7 Spherical coordinate system.

$$\nabla\Phi = \hat{r}\frac{\partial\Phi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial\Phi}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}$$

$$\nabla\cdot\overline{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta D_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}D_\phi$$

$$\nabla\times\overline{H} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ H_r & rH_\theta & r\sin\theta H_\phi \end{vmatrix}$$

$$\begin{aligned} \nabla^2\Phi &= \nabla\cdot\nabla\Phi \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial\Phi}{\partial r}\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial\Phi}{\partial\theta}\right] + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} \\ &= \frac{1}{r}\frac{\partial^2}{\partial r^2}\left[r\Phi\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial\Phi}{\partial\theta}\right] + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} \end{aligned}$$

Index Notation

A vector in the Cartesian coordinate system can be represented by its three components. Thus, A_j with $j = 1, 2, 3$ represents A_1, A_2, A_3 of the vector $\overline{A}$. The dot product $\overline{A}\cdot\overline{B}$ is written as A_jB_j where the repeated index j implies summation over j from 1 to 3:

$$A_jB_j = \sum_{j=1}^{3} A_jB_j = A_1B_1 + A_2B_2 + A_3B_3$$

To express cross products in index notation we need to define a Levi-Cevita symbol ε_{ijk} where i, j, k take values from 1 to 3. When any of the two indices are equal the Levi-Cevita symbol is zero. Otherwise, ε_{ijk} is either $+1$ or -1. It is $+1$ if ijk is an even permutation of 1,2,3; -1 if ijk is an odd permutation of 1,2,3. Thus $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$ and $\varepsilon_{213} = \varepsilon_{132} = \varepsilon_{321} = -1$ and all others equal to 0. Let $\overline{C} = \overline{A} \times \overline{B}$. In index notation, we write $C_i = \varepsilon_{ijk} A_j B_k$. Thus, $C_1 = \varepsilon_{123} A_2 B_3 + \varepsilon_{132} A_3 B_2 = A_2 B_3 - A_3 B_2$. The dyad $\overline{A}\,\overline{B}$ is $A_j B_k$, no summation implied because no index is repeated. The identities (1.1.7) and (1.1.8) are

$$
\begin{aligned}
C_i \varepsilon_{ijk} A_j B_k &= A_j \varepsilon_{jki} B_k C_i = B_k \varepsilon_{kij} C_i A_j \\
\varepsilon_{ijk} C_j \varepsilon_{klm} A_l B_m &= (\varepsilon_{ijk} \varepsilon_{klm}) C_j A_l B_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{lj}) C_j A_l B_m \\
&= A_i C_m B_m - C_l A_l B_i
\end{aligned}
$$

where $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ when $i \neq j$.

In index notation, divergence of D_j, $\nabla \cdot D_j$, is $\partial_j D_j$.

In index notation, ∇ is represented by ∂_i and $\nabla\phi$ by $\partial_i \phi$.

In index notation, curl of H_i, $\nabla \times H_i$, is written as $\varepsilon_{ijk} \partial_j H_k$.

The identities (1.1.9)–(1.1.12) are, in index notation

$$
\begin{aligned}
\partial_i (\varepsilon_{ijk} E_j H_k) &= \varepsilon_{ijk} H_k \partial_i E_j + \varepsilon_{ijk} E_j \partial_i H_k = H_k \varepsilon_{kij} \partial_i E_j - E_j \varepsilon_{jik} \partial_i H_k \\
\partial_i \varepsilon_{ijk} \partial_j A_k &= -\varepsilon_{jik} \partial_i \partial_j A_k = 0 \\
\varepsilon_{ijk} \partial_j \partial_k \phi &= -\varepsilon_{ikj} \partial_j \partial_k \phi = -\varepsilon_{ikj} \partial_k \partial_j \phi = 0 \\
\varepsilon_{ijk} \partial_j \varepsilon_{klm} \partial_l E_m &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \partial_j \partial_l E_m = \partial_m \partial_i E_m - \partial_j \partial_j E_i
\end{aligned}
$$

Maxwell equations, when written in index notation, take the form:

$$
\begin{aligned}
\varepsilon_{ijk} \partial_j H_k &= \partial_t D_i + J_i \\
\varepsilon_{ijk} \partial_j E_k &= \partial_t B_i \\
\partial_j D_j &= \rho \\
\partial_j B_j &= -\partial_t \rho
\end{aligned}
$$

where ∂_t denotes partial derivative with respect to time.

EXAMPLE 1.1.2 Poisson equation and Laplace equation.

In (E1.1.1.1), we wrote the electric field vector as the gradient of a potential function Φ:

$$\overline{E} = -\nabla\Phi \tag{E1.1.2.1}$$

By virtue of (1.1.11), we see that $\nabla \times \overline{E} = 0$. Thus the above definition for the electric field is true only when the term $\partial\overline{B}/\partial t$ in Faraday's law can be neglected, i.e., when there is no time variation. We may refer to the above electric field as the static electric field. Derive an equation for Φ .

SOLUTION:

Coulomb's law (or Gauss' law for electricity) in free space is

$$\nabla \cdot \overline{E} = \rho / \epsilon_o$$

In terms of the potential function, we obtain the Poisson equation

$$\nabla^2 \Phi = -\rho / \epsilon_o \tag{E1.1.2.2}$$

In places where there is no charge density, $\rho = 0$, we have the Laplace equation $\nabla^2 \Phi = 0$.

— END OF EXAMPLE 1.1.2 —

Siméon Denis Poisson (21 June 1781 – 25 April 1840) studied mathematics at the Ecole Polytechnique and was student of Pierre-Simon Laplace and Joseph-Louis Lagrange. His memoir on finite differences was written at age 18. His well-known contributions include Poisson's equation in potential theory was developed in 1829–1835.

EXAMPLE 1.1.3

The voltage V_{ab} is defined as the integration of $\overline{E}$ along a line segment of $\overline{\ell}$ from point a to point b .

$$V_{ab} = \int_a^b d\overline{\ell} \cdot \overline{E} \tag{E1.1.3.1}$$

Thus V_{ab} is the potential difference between points a and b . For positive V_{ab} , the electric field vector points from a to b . Point a is at a higher potential Φ_a than Φ_b at point b , $\Phi_b < \Phi_a$ and $V_{ab} = \Phi_a - \Phi_b$.

— END OF EXAMPLE 1.1.3 —

EXAMPLE 1.1.4

Maxwell's equations were originally written in the form of scalar partial differential equations. Written in terms of all field components, we find that for Ampère's law,

$$\frac{\partial}{\partial y} H_z - \frac{\partial}{\partial z} H_y = \frac{\partial}{\partial t} D_x + J_x \tag{E1.1.4.1a}$$

$$\frac{\partial}{\partial z} H_x - \frac{\partial}{\partial x} H_z = \frac{\partial}{\partial t} D_y + J_y \tag{E1.1.4.1b}$$

$$\frac{\partial}{\partial x} H_y - \frac{\partial}{\partial y} H_x = \frac{\partial}{\partial t} D_z + J_z \tag{E1.1.4.1c}$$

for Faraday's law,

$$\frac{\partial}{\partial y}E_z - \frac{\partial}{\partial z}E_y = -\frac{\partial}{\partial t}B_x \qquad \text{(E1.1.4.2}a\text{)}$$

$$\frac{\partial}{\partial z}E_x - \frac{\partial}{\partial x}E_z = -\frac{\partial}{\partial t}B_y \qquad \text{(E1.1.4.2}b\text{)}$$

$$\frac{\partial}{\partial x}E_y - \frac{\partial}{\partial y}E_x = -\frac{\partial}{\partial t}B_z \qquad \text{(E1.1.4.2}c\text{)}$$

for Coulomb's law

$$\frac{\partial}{\partial x}D_x + \frac{\partial}{\partial y}D_y + \frac{\partial}{\partial z}D_z = \rho \qquad \text{(E1.1.4.3)}$$

and for Gauss' law

$$\frac{\partial}{\partial x}B_x + \frac{\partial}{\partial y}B_y + \frac{\partial}{\partial z}B_z = 0 \qquad \text{(E1.1.4.4)}$$

Taking the sum of $\partial(\text{E1.1.4.1}a)/\partial x$, $\partial(\text{E1.1.4.1}b)/\partial y$, $\partial(\text{E1.1.4.1}c)/\partial z$, and making use of (E1.1.4.3), we obtain

$$\frac{\partial}{\partial x}J_x + \frac{\partial}{\partial y}J_y + \frac{\partial}{\partial z}J_z = -\frac{\partial}{\partial t}\rho \qquad \text{(E1.1.4.5)}$$

which is the continuity law. Given (E1.1.4.5), Coulomb's law can be derived from Ampère's law. Likewise, Gauss' law can be derived from Faraday's law, $\nabla \cdot \overline{B} = Const$, noticing that no static magnetic monopole is found to exist and that $Const = 0$. Thus (E1.1.4.3) and (E1.1.4.4) are not independent scalar equations, they can be derived from (E1.1.4.1) and (E1.1.4.2).

— End of Example 1.1.4 —

Problems

P1.1.1

Three vectors $\overline{A}$, $\overline{B}$, and $\overline{C}$ drawn in a head-to-tail fashion form the three sides of a triangle. What is $\overline{A}+\overline{B}+\overline{C}$ and what is $\overline{A}+\overline{B}-\overline{C}$?

P1.1.2

Prove $|\overline{A}\times\overline{B}|^2 = A^2B^2 - (\overline{A}\cdot\overline{B})^2$ by using $\overline{C}\times(\overline{A}\times\overline{B}) = \overline{A}(\overline{C}\cdot\overline{B}) - (\overline{C}\cdot\overline{A})\overline{B}$.

P1.1.3

A position vector $\overline{r} = \hat{x}\sqrt{2} + \hat{y}\sqrt{2} + \hat{z}2$. Determine its spherical components r, θ, ϕ and its cylindrical components ρ, ϕ, z .

P1.1.4

Find a unit vector $\hat{c}$ that is perpendicular to both $\overline{A} = \hat{x}4 + \hat{y}5 - \hat{z}3$ and $\overline{B} = \hat{x}2 - \hat{y}7 - \hat{z}1.5$.

P1.1.5

Let $\overline{A} = \hat{x}A$, and the projection of another vector $\overline{B}$ on $\overline{A}$ be $B_x = B\cos\theta_{AB}$. What is $\overline{A}\cdot\overline{B}$ in terms of the angle θ_{AB} between $\overline{A}$ and $\overline{B}$?

P1.1.6

Assume $A > B$ and draw a line projecting $\overline{B}$ on $\overline{A}$. The line length $h = B\sin\theta_{AB}$, which is also related to A from $h^2 = |\overline{A}-\overline{B}|^2 - (A - B\cos\theta_{AB})^2$ by the cosine law in geometry. Show that $\overline{A}\cdot\overline{B} = AB\cos\theta_{AB}$

P1.1.7

The direction of $\overline{A}\times\overline{B}$ follows the right-hand rule, i.e., when the fingers of the right hand rotate from $\overline{A}$ to $\overline{B}$, the thumb of the right hand points in the direction of $\overline{A}\times\overline{B}$. Thus the vector $\overline{A}\times\overline{B}$ is perpendicular to both $\overline{A}$ and $\overline{B}$ and the plane containing $\overline{A}$ and $\overline{B}$. Let $\overline{A} = \hat{x}A_x + \hat{y}A_y$ and $\overline{B} = \hat{x}B_x + \hat{y}B_y$ both in the xy -plane, find $\overline{A}\times\overline{B}$.

P1.1.8

Using $\cos\theta_{AB} = \overline{A}\cdot\overline{B}/AB$, show that $|\overline{A}\times\overline{B}| = |AB\sin\theta_{AB}|$.

P1.1.9

For $\Phi(x) = x^2$, and $\Phi(x) = -x^3$, what are their gradients?

P1.1.10

The function $\Phi = x^2 + 2y^2$ describes a family of ellipses. Find its gradient and show that $\nabla\Phi$ is normal to the ellipse and pointing in the directions of an expanding ellipse.

P1.1.11

Consider the function $\Phi = x + y$. Find the gradient of the function.

P1.1.12

Prove the following identities:

$$\nabla\cdot(\overline{E}\times\overline{H}) = \overline{H}\cdot(\nabla\times\overline{E}) - \overline{E}\cdot(\nabla\times\overline{H}) \tag{1.1.9}$$

$$\nabla\cdot(\nabla\times\overline{A}) = 0 \tag{1.1.10}$$

$$\nabla\times(\nabla\Phi) = 0 \tag{1.1.11}$$

$$\nabla\times(\nabla\times\overline{E}) = \nabla(\nabla\cdot\overline{E}) - \nabla^2\overline{E} \tag{1.1.12}$$

P1.1.13

The six terms in (1.1.21) are associated with the six differential surfaces bounding (x_0, y_0, z_0) . For the first term, the surface normal is in the $\hat{x}$ direction; we write $d\overline{S} = \hat{x}\Delta y\Delta z$. For the second term $d\overline{S} = -\hat{x}\Delta y\Delta z$. For the third term $d\overline{S} = \hat{y}\Delta z\Delta x$, etc. Derive a curl theorem by integrating over the volume similar to the divergence theorem.

P1.1.14

What is the result if the surface integral of $\nabla \times \overline{H}$ is carried out over a closed surface? Compare with Stokes Theorem in (1.1.24) and the curl theorem in P1.1.13 for the curl integrated over a volume V enclosed by a surface S.

P1.1.15

For the vector $\overline{A} = \hat{\rho}\rho^2 + \hat{z}2z$, verify the divergence theorem for the circular cylindrical region enclosed by $\rho = 5, z = 0$, and $z = 3$.

P1.1.16

Prove that $[\overline{A} \times (\nabla \times \overline{B})]_i = A_j \partial_i B_j - [(\overline{A} \cdot \nabla)\overline{B}]_i$.

P1.1.17

Prove that $\nabla(\overline{A} \cdot \overline{B}) = (\overline{A} \cdot \nabla)\overline{B} + (\overline{B} \cdot \nabla)\overline{A} + \overline{A} \times (\nabla \times \overline{B}) + \overline{B} \times (\nabla \times \overline{A})$.

P1.1.18

Show that $\nabla(\overline{A} \cdot \overline{A}) = 2(\overline{A} \cdot \nabla)\overline{A} + 2\overline{A} \times (\nabla \times \overline{A})$.

P1.1.19

Express static electric field vector as the gradient of a potential function

$$\Phi = \frac{C}{\sqrt{x^2 + y^2 + z^2}}$$

and find the electric field of a charge q from Maxwell equations.

1.2 Electromagnetic Waves

A. Wave Equation and Wave Solution

The Maxwell equations in differential form are valid at all times for every point in space. First we shall investigate solutions to the Maxwell equations in regions devoid of source, namely in regions where $\overline{J} = 0$ and $\rho = 0$. This of course does not mean that there is no source anywhere in all space. Sources must exist outside the regions of interest in order to produce fields in these regions. Thus in source-free regions in free space, the Maxwell equations become

$$\nabla \times \overline{H} = \epsilon_o \frac{\partial}{\partial t} \overline{E} \tag{1.2.1}$$

$$\nabla \times \overline{E} = -\mu_o \frac{\partial}{\partial t} \overline{H} \tag{1.2.2}$$

$$\nabla \cdot \overline{E} = 0 \tag{1.2.3}$$

$$\nabla \cdot \overline{H} = 0 \tag{1.2.4}$$

To derive an equation for the vector field $\overline{E}$, we take curl of (1.2.2), substitute (1.2.1)

$$\nabla \times \nabla \times \overline{E} = -\mu_o \frac{\partial}{\partial t} \nabla \times \overline{H} = -\mu_o \epsilon_o \frac{\partial^2}{\partial t^2} \overline{E} \tag{1.2.5}$$

and make use of the vector identity $\nabla \times \nabla \times \overline{E} = \nabla\nabla \cdot \overline{E} - \nabla^2 \overline{E}$. Noticing from (1.2.3) that $\nabla \cdot \overline{E} = 0$, we have

$$\boxed{\nabla^2 \overline{E} - \mu_o \epsilon_o \frac{\partial^2}{\partial t^2} \overline{E} = 0} \tag{1.2.6}$$

This is known as the Helmholtz wave equation. Solutions to the wave equation (1.2.6) that satisfy all Maxwell equations are electromagnetic waves.

Hermann Ludwig Ferdinand von Helmholtz (31 August 1821 – 8 September 1894) was a professor of anatomy and physiology at the University of Bonn in 1858, then became a professor of physics at the University of Berlin in 1871, and the first director of the Physico-Technical Institute of Berlin in 1888. His 3-volume Handbook of Physiological Optics appeared between 1856 and 1867.

Wave Solution

We shall now study a solution to (1.2.6) assuming $E_y = E_z = 0$. Let E_x be a function only of z and t and independent of x and y. The electric field vector can be written as

$$\overline{E} = \hat{x} E_x(z,t)$$

The wave equation it satisfies follows from (1.2.6) which becomes

$$\frac{\partial^2}{\partial z^2} E_x - \mu_o \epsilon_o \frac{\partial^2}{\partial t^2} E_x = 0 \tag{1.2.7}$$

The simplest solution to (1.2.7) takes the form

$$\overline{E} = \hat{x} E_x(z,t) = \hat{x} E_0 \cos(kz - \omega t) \tag{1.2.8}$$

Substituting (1.2.8) in (1.2.7) we find that the following equation, called the dispersion relation, must be satisfied:

$$\boxed{k^2 = \omega^2 \mu_o \epsilon_o} \tag{1.2.9}$$

The dispersion relation provides an important connection between the spatial frequency k and the temporal frequency ω.

There are two points of view useful in the study of a space-time varying quantity such as $E_x(z,t)$. The temporal view point is to examine the time variation at fixed points in space. The spatial view point is to examine spatial variation at fixed times, a process that amounts to taking a series of pictures.

From the temporal view point, we first fix our attention on one particular point in space, say $z = 0$. We then have the electric field $E_x(z = 0, t) = E_0 \cos \omega t$. Plotted as a function of time in Fig. 1.2.1, we find that the waveform repeats itself in time as $\omega t = 2m\pi$ for any integer m. The period is defined as the time T for which $\omega T = 2\pi$. The number of periods in a time of one second is the frequency f defined as $f = 1/T$, which gives

$$f = \frac{\omega}{2\pi} \tag{1.2.10}$$

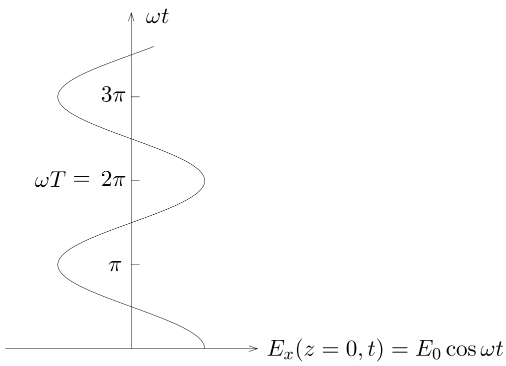


Figure 1.2.1 Electric field strength as a function of ωt at $z = 0$.

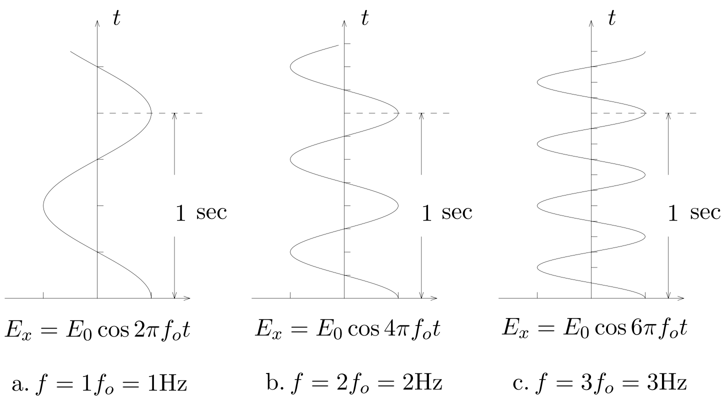


Figure 1.2.2 Electric field strength vs. t for different frequencies ω.

The unit for frequency f is Hertz (Hz) with 1 Hz = 1 s^{-1}, which is equal to the number of cycles per second. Since $\omega = 2\pi f$, ω is the angular frequency of the wave.

In this book, we often refer to ω as the frequency, simply because ω is more commonly encountered than f. The temporal frequency ω characterizes the variation of the wave in time. We plot in Fig. 1.2.2a $E_x(z=0,t)$ as a function of t instead of ωt. Let there be one period within the time interval of 1 second. Thus, $f = f_o = 1\,\text{Hz}$, and we let $\omega = \omega_o = 2\pi$ rad/s. In Fig. 1.2.2b, we plot $\omega = 2\omega_o$; there are two periods in a time interval of one second and the period in time is 0.5 seconds. In Fig. 1.2.2c, $\omega = 3\omega_o$ and there are three periods in one second.

B. Unit for Spatial Frequency k

To examine wave behavior from the spatial view point, let $\omega t = 0$. The electric field becomes

$$E_x(z, t = 0) = E_0 \cos kz \tag{1.2.11}$$

The electric field thus varies periodically in space. We plot $E_x(z, t = 0)$ as a function of kz in Fig. 1.2.3. The waveform repeats itself periodically in space when $kz = 2m\pi$ for integer values of m. The period of one spatial variation is the wavelength λ defined as the distance for which $k\lambda = 2\pi$. The number of spatial variations per unit distance is

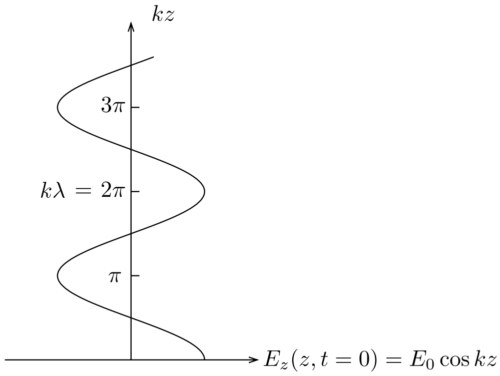


Figure 1.2.3 Electric field strength as a function of kz at $t = 0$.

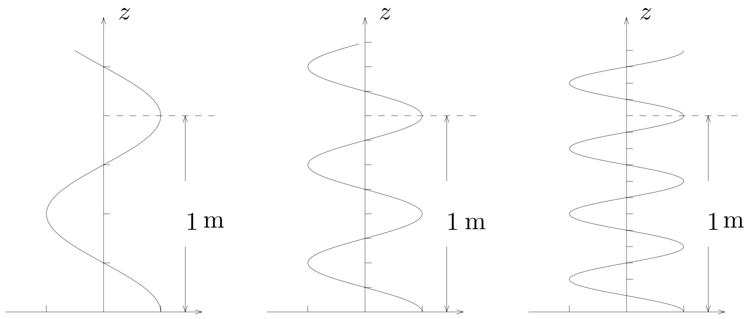
$$k = \frac{2\pi}{\lambda} \tag{1.2.12}$$

We call k the spatial frequency, which characterizes the spatial variations of the field strength, similar to the temporal frequency which characterized the temporal variations of the field strength. The spatial frequency is also called the wavenumber as it is equal to the number of wavelengths in a distance of 2π and has the dimension of inverse length.

Let me define for the spatial frequency k a fundamental unit K_o:

$$\boxed{1\,\mathrm{K}_o = 2\pi \text{ rad/m}} \tag{1.2.13}$$

Similar to the unit Hz which is cycles per second in temporal variation, K_o is cycles per meter in spatial variation. For a wave that has a spatial



$E_x = E_0 \cos 1k_o z$ $\quad$ $E_x = E_0 \cos 2k_o z$ $\quad$ $E_x = E_0 \cos 3k_o z$

a. $k = 1k_o = 1\mathrm{K}_o$ $\quad$ b. $k = 2k_o = 2\mathrm{K}_o$ $\quad$ c. $k = 3k_o = 3\mathrm{K}_o$

Figure 1.2.4 Electric field strength vs. distance z with different spatial frequency k.

frequency of one period of spatial variation in one meter distance, we have $k = 1\,\mathrm{K}_o$. An electromagnetic wave in free space with $k = 3\,\mathrm{K}_o$ has three spatial variations in a distance of one meter.

We plot in Fig. 1.2.4a $E_x(z, t = 0)$ as a function of z instead of kz. There is one cycle of spatial variation within the wavelength of 1 meter. Since $\mathrm{K}_o = 2\pi$ rad/m, we have $k = 1\,\mathrm{K}_o = 2\pi$ rad/m. In Fig. 1.2.4b, we plot $k = 2\,\mathrm{K}_o$; there are two variations in a spatial distance of one meter and the wavelength is 0.5 meters. In Fig. 1.2.4c, $k = 3\,\mathrm{K}_o$ and there are three variations in one meter.

From the dispersion relation for electromagnetic waves (1.2.9), we see that the spatial frequency and the temporal frequency are related by the velocity of light. Thus for a spatial frequency of $1\,\mathrm{K}_o$, the corresponding temporal frequency is $f = 300\,\mathrm{MHz}$. With k expressed in unit K_o, we find

$$\boxed{f = 3 \times 10^8\ k \quad \mathrm{Hz}\ ; \qquad \lambda = 1/k \quad \mathrm{m}} \tag{1.2.14}$$

Within the spatial frequency range of $0.01\,\mathrm{K}_o$ to $100\,\mathrm{K}_o$ electromagnetic waves are used for microwave heating, radar, navigation, and carrying signals from radio, television, and satellite communications. The visible light has a spatial frequency band between $1.4 \times 10^6 \sim 2.6 \times 10^6\,\mathrm{K}_o$. In Fig. 1.2.5 we illustrate the electromagnetic wave spectrum according to the spatial frequency in K_o and corresponding wavelength in meters, frequency in Hz, and energy in electron-volts (eV).

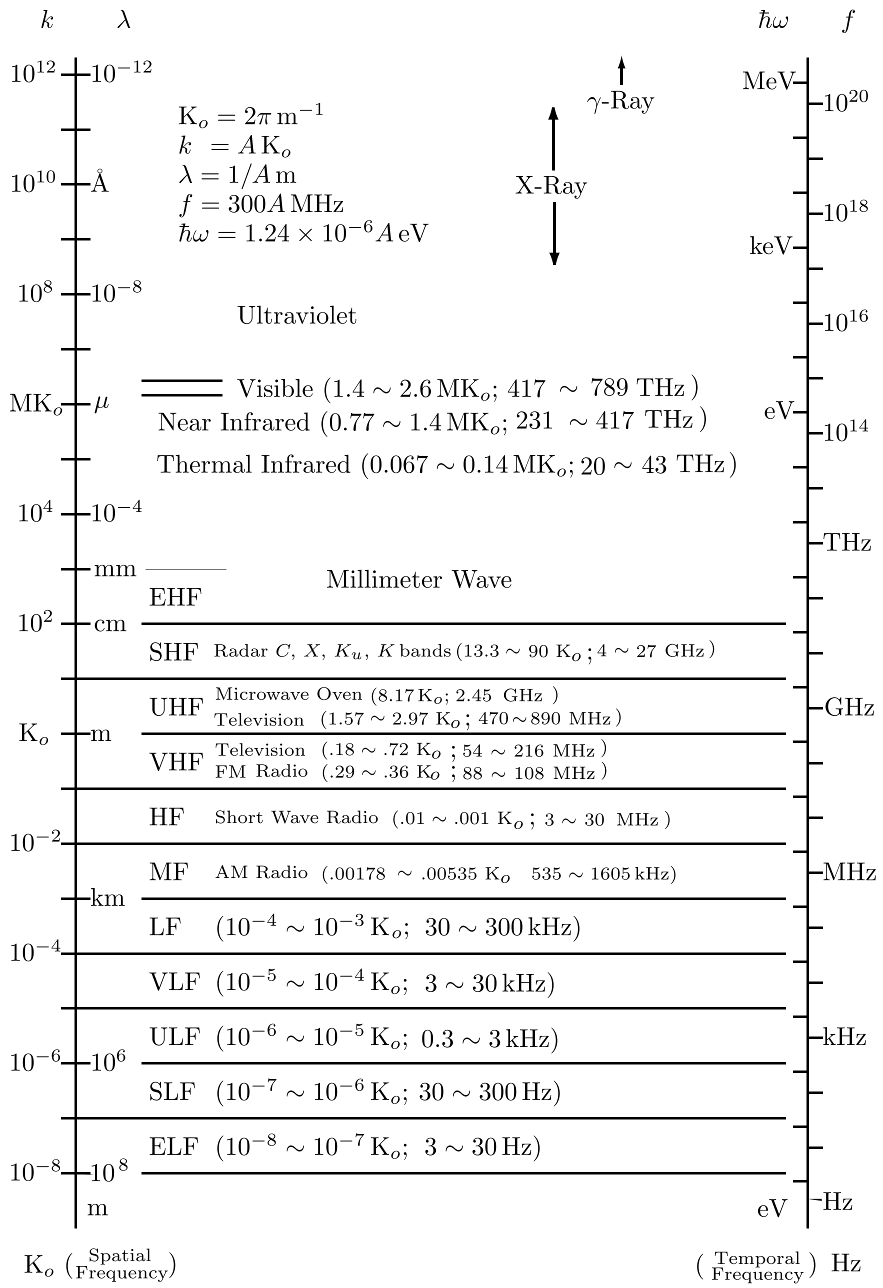


Figure 1.2.5 Electromagnetic wave spectrum.

In this book I shall place great emphasis on the use of k, which is of more fundamental importance in electromagnetic wave theory than both of the more popular concepts of wavelength λ and frequency f. The corresponding values of wavelength λ and frequency f are, for $k = A\,\mathrm{K}_o$,

$$\lambda = 2\pi/k = 2\pi/(A\mathrm{K}_o) = \frac{1}{A}\,\mathrm{m}; \quad f = ck/2\pi = cA\mathrm{K}_o/2\pi = 3\times10^8 A\,\mathrm{Hz}$$

The photon energy in electron-volts (eV) is calculated from

$$\hbar\omega = (\hbar c A \mathrm{K}_o/q)\,\mathrm{eV} \approx 1.24 \times 10^{-6} A\,\mathrm{eV} = \hbar c k/q \;\; \mathrm{eV}$$

where $q = 1.6 \times 10^{-19}$ coulombs is the electron charge, and $\hbar = 1.05 \times 10^{-34}$ Joule-second is Planck's constant $h = 6.626 \times 10^{-34}$ J-sec divided by 2π.

Max Karl Ernst Ludwig Planck (23 April 1858 – 4 October 1947)

Max Planck entered the University of Munich in 1874. He taught at the University of Munich in 1880–1885, Kiel 1885–1889. After the death of Kirchhoff in 1887, Planck succeeded his chair of theoretical physics at the University of Berlin in 1889 until his retirement in 1927. In 1900 he announced a formula now known as Planck's radiation formula and introduced the quanta of energy.

EXAMPLE 1.2.1 Operating frequencies of common devices:

Device	*Temporal frequency* (Hz)	*Spatial frequency* (K_o)
AM Radio	535 – 1605 kHz	0.00178 – 0.00535 K_o
Shortwave Radio	3 – 30 MHz	0.01 – 0.1 K_o
FM Radio	88 – 108 MHz	0.293 – 0.36 K_o
Airport ILS	108 – 112 MHz	0.35 – 0.373 K_o
Commercial Television		
Channels 2-4	54 – 72 MHz	0.18 – 0.24 K_o
Channels 5-6	76 – 88 MHz	0.253 – 0.293 K_o
Channels 7-13	174 – 216 MHz	0.58 – 0.72 K_o
Channels 14-83	470 – 890 MHz	1.57 – 2.97 K_o
Microwave Oven	2.45 GHz	8.17 K_o
Communication Satellite		
Downlink	3.70 – 4.20 GHz	12.3 – 14 K_o
Uplink	5.925 – 6.425 GHz	19.75 – 21.4 K_o

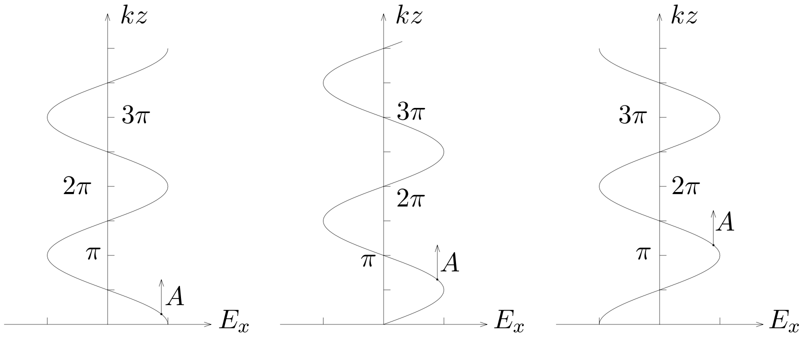
— END OF EXAMPLE 1.2.1 —

Phase Velocity and Phase Delay

In Figs. 1.2.6b and 1.2.6c we plot $E_x(z,t)$ at two progressive times $\omega t = \pi/2$ and $\omega t = \pi$. We observe that the electric field vector at A appears to be propagating along the $\hat{z}$ direction as time progresses. The velocity of propagation V_p is determined from $kz - \omega t =$ constant which gives

$$V_p = \frac{dz}{dt} = \frac{\omega}{k} \tag{1.2.15}$$

We call V_p the phase velocity. By virtue of the dispersion relation (1.2.9), we see that $V_p = (\mu_o \epsilon_o)^{-1/2}$, which is equal to the velocity of light in free space c.



a. $\omega t = 0$ b. $\omega t = \dfrac{\pi}{2}$ c. $\omega t = \pi$

$E_x = E_0 \cos kz$ $E_x = E_0 \sin kz$ $E_x = -E_0 \cos kz$

Figure 1.2.6 Electric field strength vs. kz at different times.

The spatial frequency k is, according to the dispersion relation, directly related to the temporal frequency ω by the phase delay

$$\Lambda_p = \frac{k}{\omega} = \sqrt{\mu_o \epsilon_o} \tag{1.2.16}$$

which determines how much time it takes for the wave to propagate a unit distance. In free space $\Lambda_p = 10^{-8}/3$ s/m or it takes 3.33 nanoseconds for an electromagnetic wave to travel the distance of one meter.

EXAMPLE 1.2.2 Electric field vector $\overline{E}$ and magnetic field vector $\overline{H}$.

A wave equation similar to (1.2.6) can be derived for the magnetic field vector $\overline{H}$. Wave solutions for $\overline{E}$ and $\overline{H}$ can be written as

$$\overline{E} = \hat{x} E_x(z,t) = \hat{x} E_0 \cos(kz - \omega t) \tag{E1.2.2.1}$$

$$\overline{H} = \hat{y} H_y(z,t) = \hat{y} H_0 \cos(kz - \omega t) \tag{E1.2.2.2}$$

It is seen that $\overline{E}$ and $\overline{H}$ satisfy (1.2.3) and (1.2.4). From (1.2.1), we find

$$\begin{aligned} \nabla \times \overline{H} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & H_y & 0 \end{vmatrix} = \hat{x}\ kH_0 \sin(kz - \omega t) \\ &= \epsilon_o \frac{\partial}{\partial t} \overline{E} = \hat{x}\ \omega \epsilon_o E_0 \sin(kz - \omega t) \end{aligned}$$

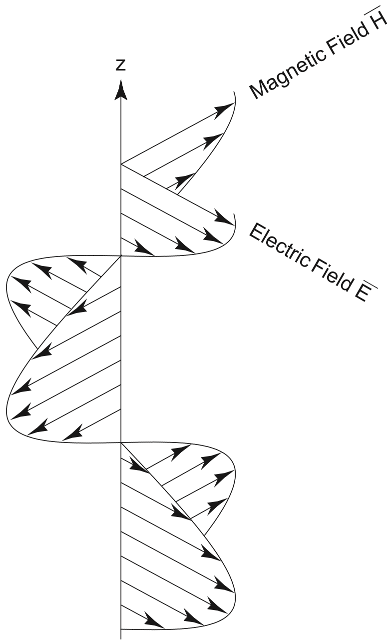


Figure E1.2.2.1 Electric and magnetic fields of an electromagnetic wave.

The magnitudes E_0 and H_0 are related by $E_0/H_0 = k/\omega\epsilon_o = \sqrt{\mu_o/\epsilon_o} = \eta$, where $\eta = \sqrt{\mu_o/\epsilon_o}$ is called the free-space impedance. The same result is obtained by substituting (E1.2.2.1) and (E1.2.2.2) into (1.2.2). The electromagnetic wave is propagating in the positive $\hat{z}$ direction. The field vectors of the electromagnetic wave are transversal to the direction of propagation and lie in the xy-plane, on which the phase $kz - \omega t$ of the wave is a constant. Since the phase front of the wave is the xy-plane, we call the electromagnetic wave as represented by (E1.2.2.1) and (E1.2.2.2) a plane wave.

— END OF EXAMPLE 1.2.2 —

C. Polarization

The polarization of a wave is conventionally defined by the time variation of the tip of the electric field vector $\overline{E}$ at a fixed point in space. If the tip moves along a straight line, the wave is linearly polarized. When the locus of the tip is a circle, the wave is circularly polarized. For an elliptically polarized wave, the tip of $\overline{E}$ describes an ellipse. If the right-hand thumb points in the direction of propagation while the fingers point in the direction of the tip motion, the wave is defined as right-hand polarized. The wave is left-hand polarized when it is described by the left-hand thumb and fingers.

Consider the following wave solution:

$$\begin{aligned}\overline{E}(z,t) &= \hat{x}E_x + \hat{y}E_y \\ &= \hat{x}\cos(kz-\omega t) + \hat{y}A\cos(kz-\omega t+\psi) \qquad (1.2.17)\end{aligned}$$

with $A > 0$. The wave propagates in the $+\hat{z}$ direction. From the temporal view point,

$$\overline{E}(t) = \hat{x}\cos(\omega t) + \hat{y}A\cos(\omega t-\psi)$$

We now study polarization for the following special cases:
Case 1) $\psi = 2m\pi$, where $m = 0, 1, 2, ...$ is an integer. We have

$$\overline{E}(t) = \hat{x}\cos(\omega t) + \hat{y}A\cos(\omega t)$$

The tip of the electric field vector moves along a line as shown in Fig. 1.2.7a. The wave is linearly polarized.
Case 2) $\psi = (2m+1)\pi$, we have

$$\overline{E}(t) = \hat{x}\cos(\omega t) - \hat{y}A\cos(\omega t)$$

The tip of the electric field vector moves along a line as shown in Fig. 1.2.7b. The wave is linearly polarized.
Case 3) $\psi = \pi/2$ and $A = 1$, we have

$$\overline{E}(t) = \hat{x}\cos(\omega t) + \hat{y}\sin(\omega t) \qquad (1.2.18)$$

It can be seen that while the x component is at its maximum the y component is zero. As time progresses, the y component increases and the x component decreases. The tip of $\overline{E}$ rotates from the positive E_x

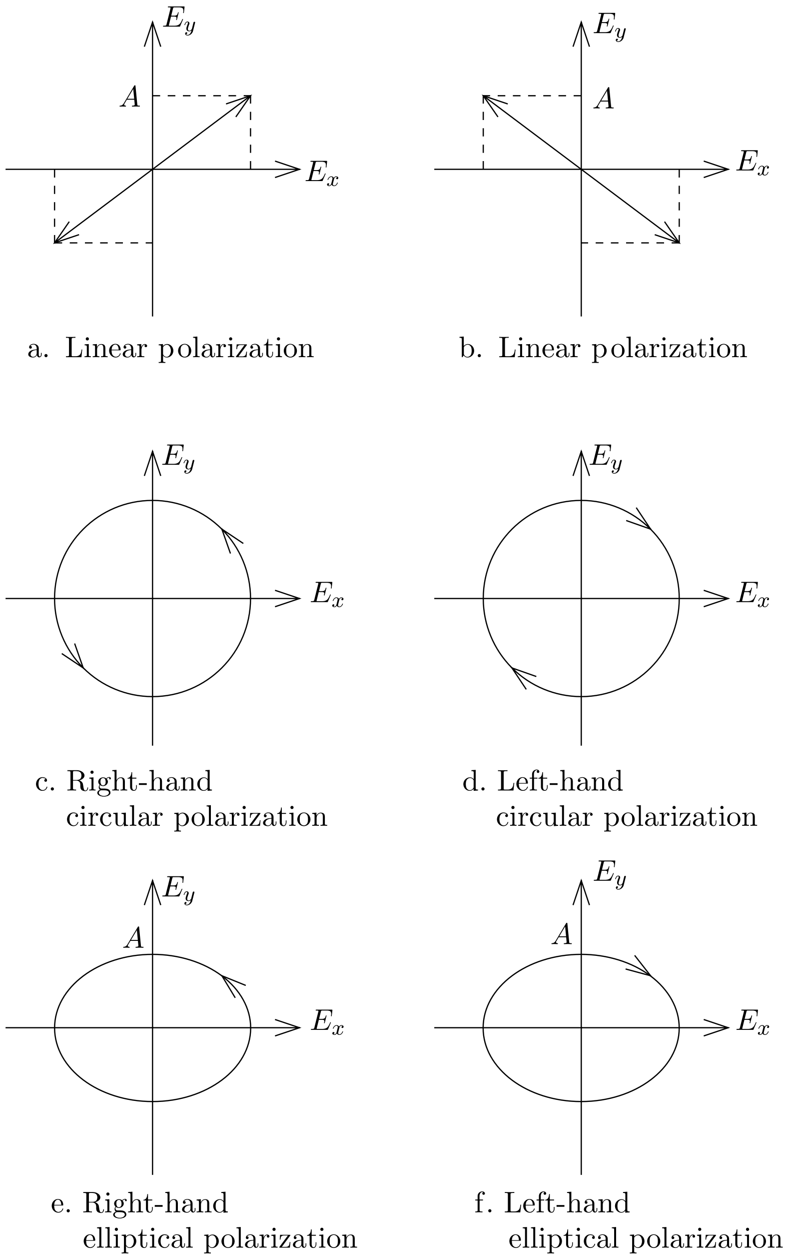


Figure 1.2.7 Polarizations.

axis to the positive E_y axis [Fig. 1.2.7c]. Elimination of t from the x and y components in (1.2.18) yields a circle of radius 1, $E_x^2 + E_y^2 = 1$. Thus the wave is right-hand circularly polarized.

Case 4) $\psi = -\pi/2$ and $A = 1$, we have

$$\overline{E}(t) = \hat{x}\cos(\omega t) - \hat{y}\sin(\omega t) \tag{1.2.19}$$

As time progresses, the y component increases and the x component decreases. The tip of $\overline{E}$ rotates from the positive E_x axis to the negative E_y axis. Thus the wave is left-hand circularly polarized [Fig. 1.2.7d].

Case 5) $\psi = \pm\pi/2$, we have

$$\overline{E}(t) = \hat{x}\cos(\omega t) \pm \hat{y}A\sin(\omega t) \tag{1.2.20}$$

The wave is right-hand elliptically polarized for $\psi = \pi/2$ [Fig. 1.2.7e] and left-hand elliptically polarized for $\psi = -\pi/2$ [Fig. 1.2.7f].

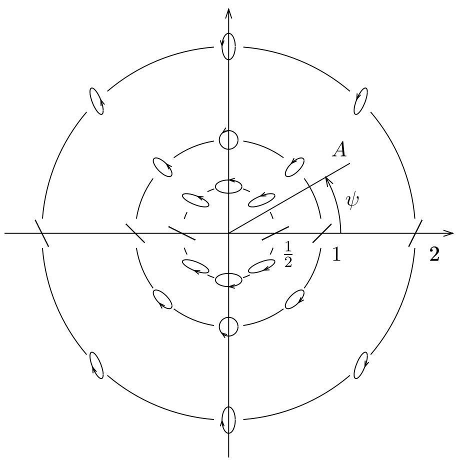


Figure 1.2.8 Polarizations for various values of ψ and A.

The above discussion can be summarized in Fig. 1.2.8 where we illustrate the polarization for different values of A and ψ. On the horizontal axis, $\psi = 0$, or π, the wave is linearly polarized. If $A = 1$ and $\psi = \pi/2$, the wave is right-hand circularly polarized. For $A = 1$ and $\psi = -\pi/2$, the wave is left-hand circularly polarized. Otherwise the wave is elliptically polarized. The polarization is right-handed if the phase difference is between zero and π, and left-handed if ψ is between π and 2π.

EXAMPLE 1.2.3 Polarization from the spatial view point.

Wave polarization can be viewed by either taking a series of still pictures at several fixed times, called the spatial view point or by making observations at a fixed point in space, called the temporal view point. The definition of polarization so far has been discussed from the temporal view point. Let us now look at polarization from the spatial view point.

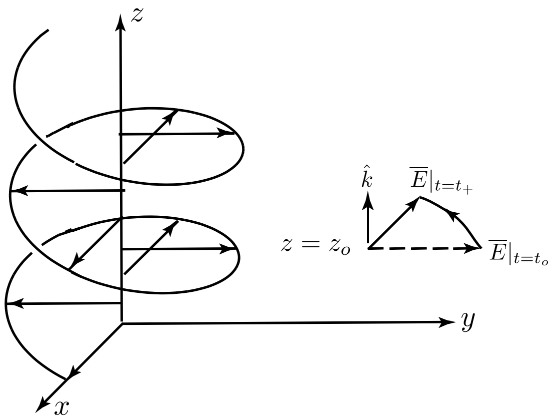


Figure E1.2.3.1 Spatial view of polarization.

Consider the right-hand circularly polarized wave with $\psi = \pi/2$ and $A = 1$ in case 3), setting $t = 0$ in wave solution (1.2.17), we have

$$\overline{E}(z, t = 0) = \hat{x}\cos(kz) - \hat{y}\sin(kz) = \hat{x}E_x(z) - \hat{y}E_y(z)$$

This is a left-handed helix as shown below.

$$E_x = E_0 \cos\left(\frac{2\pi}{\lambda} z\right) \qquad E_y = E_0 \sin\left(\frac{2\pi}{\lambda} z\right)$$

The parametric equation of a helix is

$$x = R\cos\left(\frac{2\pi}{p} z\right) \qquad y = R\sin\left(\frac{2\pi}{p} z\right) r$$

where p is the pitch of the helix. Thus, the locus of the tip point of the electric field vector measured along the z axis is a left-handed helix with the pitch $p = \lambda$. The helix advances along $+\hat{z}$ without rotating. At $z = z_0 = 3\lambda/4$, electric field vector is at $\overline{E}|_{t=t_o}$ when $t_o = 0$, it is shown as $\overline{E}|_{t=t_+}$ when $t_+ = \pi/4\omega$.

— END OF EXAMPLE 1.2.3 —

Poincaré Sphere and Stokes Parameters

We now use the ellipse as shown in Fig. 1.2.9 to illustrate all polarization states by introducing two parameters: polarization angle α and orientation angle β. We let the major axis of the ellipse be e_1 and the minor axis $e_2 \leq e_1$. The shape of the ellipse can be specified by the ellipticity angle α defined as

$$\tan\alpha = \pm\frac{e_2}{e_1} \tag{1.2.21}$$

where the plus sign corresponds to right-hand polarization for which $0 \leq \alpha \leq \pi/4$ and the negative sign to left-hand polarization for which $-\pi/4 \leq \alpha \leq 0$. We see that for linearly polarized wave $\alpha = 0$. For as is evident from the defining equation for Fig. 1.2.9.

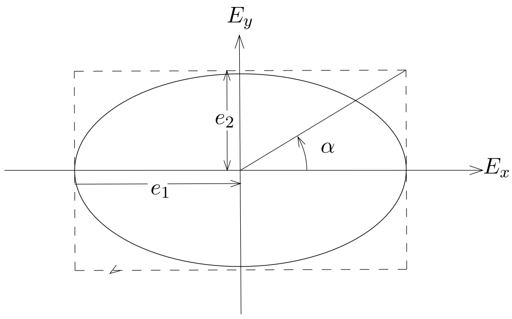


Figure 1.2.9 Elliptical polarization.

$$\overline{E}(t) = \hat{x}\cos\omega t + \hat{y}A\sin\omega t \tag{1.2.22}$$

For right-hand circularly polarized waves, $\alpha = -\pi/4$ and $e_2 = e_1$, for left-hand circularly polarized waves, $\alpha = -\pi/4$ and $e_2 = e_1$. For right-hand polarization, $\alpha \geq 0$, for left-hand polarization, $\alpha \leq 0$.

The orientation angle β is introduced with Fig. 1.2.10 by rotating the ellipse in Fig. 1.2.9. The major axis of the ellipse is rotated and makes the angle β with the E_x axis with $0 \leq \beta \leq \pi$. Thus for a linearly polarized wave along the E_y- axis, $\beta = \pi/2$.

Instead of the planar representation of polarization states as shown in Fig. 1.2.8, we shall now discuss representation of polarization states with a sphere called Poincaré sphere as shown in Fig. 1.2.11. The radius

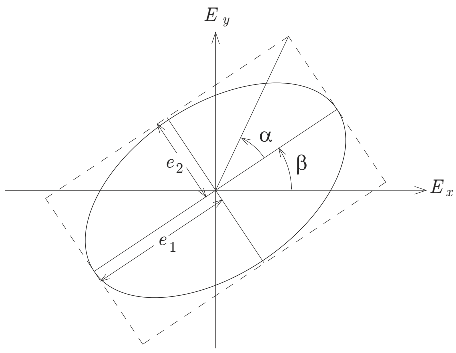


Figure 1.2.10 Elliptical polarization.

of the sphere is I, and the three axes are Q, U, V as shown below:

$$\begin{aligned} Q &= I \cos 2\alpha \cos 2\beta \\ U &= I \cos 2\alpha \sin 2\beta \\ V &= I \sin 2\alpha \end{aligned}$$

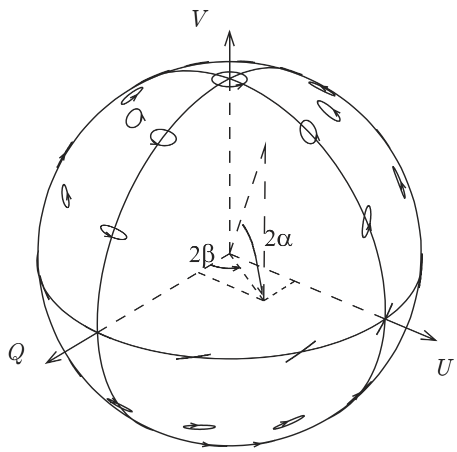


Figure 1.2.11 Poincare sphere.

We see that $I^2 = Q^2 + U^2 + V^2$. When the wave is right-hand circularly polarized $Q = U = 0$, $V = I$, as $\alpha = \pi/4$. When the wave is left-hand circularly polarized, $Q = U = 0$, $V = -I$, as $\alpha = -\pi/4$. When the wave is linearly polarized, $V = 0$, as $\alpha = 0$. With a rigorous mathematical derivation, I, Q, U, V can be derived from e_x and e_y, and they are called Stokes parameters, which are useful in characterizing polarized as well as unpolarized electromagnetic waves.

Jules Henri Poincaré (29 April 1854 – 17 July 1912)

Henri Poincaré entered the Ecole Polytechnique in 1873, graduating in 1875, and received his doctorate in mathematics from the University of Paris in 1879. In 1886 he was appointed to a chair of mathematical physics and probability at the Sorbonne and also at the Ecole Polytechnique. In 1894, he published the first of his six papers on algebraic topology.

Example 1.2.4

To facilitate a mathematical discussion of polarization, we decompose the $\overline{E}$ vector of a wave into two components perpendicular to the direction of propagation. For a specific point in space, we write

$$\overline{E}(t) = \hat{x}E_x + \hat{y}E_y = \hat{x}e_x \cos(\omega t - \psi_x) + \hat{y}e_y \cos(\omega t - \psi_y) \qquad \text{(E1.2.4.1)}$$

where $\hat{x}$, $\hat{y}$, and the direction of propagation are mutually perpendicular and thus form an orthogonal system. We assume the amplitudes e_y and e_x are both positive. The locus of the tip $\overline{E}(t)$ is determined by eliminating the time t dependence between the two components E_x and E_y.

In general, a polarized wave has elliptical polarization; that is, when time is eliminated from the two components of $\overline{E}$, the resultant equation describes an ellipse. Consider the case $\psi_x = \psi_0$, $\psi_y - \psi_x = \pm\pi/2$ in (E1.2.4.1) and let $e_x = e_1 > e_2 = e_y$. We have

$$\overline{E}'(t) = \hat{x}'E'_x + \hat{y}'E'_y = \hat{x}'e_1 \cos(\omega t - \psi_0) + \hat{y}'e_2 \sin(\omega t - \psi_0) \qquad \text{(E1.2.4.2)}$$

with e_1 denoting the major axis and e_2 the minor axis, we write

$$\tan\alpha = \frac{e_2}{e_1} \qquad \text{(E1.2.4.3)}$$

where $-\pi/4 \leq \alpha \leq \pi/4$.

The general polarization states are more popularly described with the Poincaré sphere as discussed below. Consider the elliptical polarization as given by (E1.2.4.1), which describes a tilted ellipse as plotted in Fig. 1.2.10. The major axis of the ellipse described in (E1.2.4.2) is rotated and makes the angle β with the E_h axis with $0 \leq \beta \leq \pi$. We call β the orientation angle.

In view of (E1.2.4.2) and Fig. 1.2.9, we have from coordinate transformation

$$E'_x = E_x \cos\beta + E_y \sin\beta$$
$$E'_y = -E_x \sin\beta + E_y \cos\beta$$

leading to

$$e_1 \cos(\omega t - \psi_0) = E_x \cos\beta + E_y \sin\beta \tag{E1.2.4.4a}$$
$$e_2 \sin(\omega t - \psi_0) = -E_x \sin\beta + E_y \cos\beta \tag{E1.2.4.4b}$$

Substituting the components E_h and E_v of (E1.2.4.1) in (E1.2.4.4) and comparing the coefficients of $\cos\omega t$ and $\sin\omega t$, we obtain

$$e_1 \cos\psi_0 = e_x \cos\psi_x \cos\beta + e_y \cos\psi_y \sin\beta \tag{E1.2.4.5a}$$
$$e_1 \sin\psi_0 = e_x \sin\psi_x \cos\beta + e_y \sin\psi_y \sin\beta \tag{E1.2.4.5b}$$
$$e_2 \cos\psi_0 = -e_x \sin\psi_x \sin\beta + e_y \sin\psi_y \cos\beta \tag{E1.2.4.5c}$$
$$e_2 \sin\psi_0 = e_x \cos\psi_x \sin\beta - e_y \cos\psi_y \cos\beta \tag{E1.2.4.5d}$$

Eliminating ψ_0 from (E1.2.4.5a) and (E1.2.4.5b), we find

$$e_1^2 = e_x^2 \cos^2\beta + e_y^2 \sin^2\beta + e_x e_y \sin 2\beta \cos\psi \tag{E1.2.4.6a}$$

Similarly from (E1.2.4.5c) and (E1.2.4.5d), we have

$$e_2^2 = e_x^2 \sin^2\beta + e_y^2 \cos^2\beta - e_h e_v \sin 2\beta \cos\psi \tag{E1.2.4.6b}$$

Multiplying (E1.2.4.5a) by (E1.2.4.5c), (E1.2.4.5b) by (E1.2.4.5d) and then adding, we again eliminate ψ_0 and obtain

$$e_1 e_2 = e_x e_y \sin\psi \tag{E1.2.4.6c}$$

Finally we multiply (E1.2.4.5a) by (E1.2.4.5d) and subtract from the product of (E1.2.4.5b) and (E1.2.4.5c), which yields

$$2 e_x e_y \cos\psi = (e_x^2 - e_y^2) \tan 2\beta \tag{E1.2.4.6d}$$

Equation (E1.2.4.6) will be used in the following discussion on Stokes parameters and the Poincaré sphere.

To facilitate the discussion of various polarization states of electromagnetic waves, the four Stokes parameters pertaining to $\overline{E}(t)$ given in (E1.2.4.1) are defined as follows :

$$I = \frac{1}{\eta}\left(e_x^2 + e_y^2\right) \tag{E1.2.4.7a}$$
$$Q = \frac{1}{\eta}\left(e_x^2 - e_y^2\right) \tag{E1.2.4.7b}$$
$$U = \frac{2}{\eta} e_x e_y \cos\psi \tag{E1.2.4.7c}$$
$$V = \frac{2}{\eta} e_x e_y \sin\psi \tag{E1.2.4.7d}$$

Notice that $I^2 = Q^2 + U^2 + V^2$.

Adding (E1.2.4.6a) and (E1.2.4.6b) yields $e_1^2 + e_2^2 = e_x^2 + e_y^2 = \eta I$. Making use of (E1.2.4.3), we have

$$e_1^2 = \eta I \cos^2 \alpha \tag{E1.2.4.8}$$

Subtracting (E1.2.4.6b) from (E1.2.4.6a) and making use of (E1.2.4.6d), we find $e_1^2 - e_2^2 = (e_x^2 - e_y^2)/\cos 2\beta$. Making use of (E1.2.4.3) and (E1.2.4.8), we find

$$Q = \frac{1}{\eta}(e_x^2 - e_y^2) = I \cos 2\alpha \cos 2\beta \tag{E1.2.4.9a}$$

In terms of I, we find from (E1.2.4.7c), (E1.2.4.6d) and (E1.2.4.9a)

$$U = I \cos 2\alpha \sin 2\beta \tag{E1.2.4.9b}$$

and from (E1.2.4.7d), (E1.2.4.6c) and (E1.2.4.8)

$$V = I \sin 2\alpha \tag{E1.2.4.9c}$$

Equation (E1.2.4.9) suggests a simple geometrical representation of all states of polarization by recognizing that Q, U, and V can be regarded as the rectangular components of a point on a sphere with radius I, known as the Poincaré sphere. We define, in the spherical coordinate system, $\theta = \pi/2 - 2\alpha$ and $\phi = 2\beta$. As seen from (E1.2.4.3), positive α is for right-hand polarization which is represented by points on the upper hemisphere. On the lower hemisphere, the points correspond to left-hand polarization. The north pole represents right-hand circular polarization and the south pole represents left-hand circular polarization. The sphere is called the *Poincaré sphere*. Fig. 1.2.8 is seen to be a planar projection of the Poincaré sphere with the plane and the sphere touching each other at $Q = I$. The equator is mapped into the horizontal axis.

— End of Example 1.2.4 —

Example 1.2.5 Partial polarization.

Radiation from many natural and man-made sources consists of field components that fluctuate with time. We write

$$E_h = e_h(t) \cos\Big(\omega t - \psi_h(t)\Big)$$
$$E_v = e_v(t) \cos\Big(\omega t - \psi_v(t)\Big)$$

The wave is quasi-monochromatic when $e_h(t)$, $e_v(t)$, $\psi_h(t)$, and $\psi_v(t)$ are slowly varying compared with $\cos \omega t$. The Stokes parameters are defined by a time-average procedure over a large time interval T, denoted with the brackets $<>$:

$$< E_h^2(t) >= \frac{1}{T}\int_0^T dt\,[E_h(t)]^2$$

The Stokes parameters are

$$\begin{aligned}
I &= I_h + I_v = \frac{1}{\eta}\left(< E_h^2 > + < E_v^2 >\right) \\
Q &= I_h - I_v = \frac{1}{\eta}\left(< E_h^2 > - < E_v^2 >\right) = I < \cos 2\alpha \cos 2\beta > \\
U &= \frac{2}{\eta} < E_h E_v \cos\psi > = I < \cos 2\alpha \sin 2\beta > \\
V &= \frac{2}{\eta} < E_h E_v \sin\psi > = I < \sin 2\alpha >
\end{aligned}$$

For completely unpolarized waves, E_h and E_v are uncorrelated and we have I = total Poynting power and $Q = U = V = 0$. For completely polarized waves we have $I^2 = Q^2 + U^2 + V^2$. For partially polarized waves it can be shown that $I^2 \geq Q^2 + U^2 + V^2$ [Example 1.2A.2]. With the Poincaré sphere of radius I, the partially polarized waves correspond to points inside the sphere.

In concluding this section on wave polarization, we remark that the polarization is defined according to the time variations of the $\overline{E}$ vector. As we shall see in Chapter 3, it is imperative that we define polarization in terms of $\overline{D}$ when anisotropic and bianisotropic media are involved. This is because in isotropic media $\overline{E}$ is perpendicular to $\overline{k}$, $\overline{k}\cdot\overline{E} = 0$, while in non-isotropic media $\overline{k}\cdot\overline{D} = 0$. This also suggests that wave polarization can be defined in terms of the field vector $\overline{B}$.

— End of Example 1.2.5 —

Problems

P1.2.1

Electromagnetic waves satisfy all of the Maxwell equations. Consider, in free space, the following electric field vectors:

$$\begin{aligned}
\overline{E}_1 &= \hat{x}\cos(\omega t - kz) \\
\overline{E}_2 &= \hat{z}\cos(\omega t - kz) \\
\overline{E}_3 &= (\hat{x} + \hat{z})\cos(\omega t + ky) \\
\overline{E}_4 &= (\hat{x} + \hat{z})\cos\left(\omega t + k|x + z|/\sqrt{2}\right)
\end{aligned}$$

Do these electric field vectors satisfy the wave equation and all Maxwell equations? Which of the four fields qualify as electromagnetic waves? For those not qualified as electromagnetic waves, state which of the Maxwell equations are violated.

P1.2.2

The electric field vector

$$\overline{E} = \hat{x} E_0 \cos(kz - \omega t)$$

represents an electromagnetic wave propagating in the $+\hat{z}$ direction. What is the expression if the wave is propagating in the $-\hat{z}$ direction?

P1.2.3

An electromagnetic wave has spatial frequency $k_o = 100\,\mathrm{K}_o$. Determine the wavelength in meters and the temporal frequency in GHz.

Determine the spatial frequency in unit of K_o for a laser light at wavelength $\lambda = 0.6328\,\mu\mathrm{m}$.

Determine the spatial frequency in unit of K_o for a microwave oven at frequency 2.4 GHz.

P1.2.4

The known spectrum of electromagnetic waves covers a wide range of frequencies. Electromagnetic phenomena are all described by Maxwell's equations and, by convention, are generally classified according to wavelengths or frequencies. Radio waves, television signals, radar beams, visible light, X rays, and gamma rays are examples of electromagnetic waves.

(a) Give in meters the wavelengths corresponding to the following frequencies:
  (i) 60 Hz
  (ii) AM radio (535–1605 kHz)
  (iii) FM radio (88–108 MHz)
  (iv) Visible light ($\sim 10^{14}$ Hz)
  (v) X-rays ($\sim 10^{18}$ Hz)

(b) Give in Hertz the temporal frequencies corresponding to the wavelengths:
  (i) 1 km, (ii) 1 m, (iii) 1 mm, (iv) 1 μm, (v) 1 Å.

(c) Give in K_o the spatial frequencies corresponding to the wavelengths in (b).

(d) Give in eV the spatial frequencies corresponding to the wavelengths in (b).

P1.2.5

Consider the electric field amplitude

$$E_x(z,t) = E_0 \cos(kz - \omega t)$$

Find the phase velocity $v_p = \omega/k$ and the group velocity $v_g = d\omega/dk$.

P1.2.6

Consider an electromagnetic wave propagating in the $\hat{z}$-direction with

$$\overline{E} = \hat{x} e_x \cos(kz - \omega t + \psi_x) + \hat{y} e_y \cos(kz - \omega t + \psi_y)$$

where e_x, e_y, ψ_x, and ψ_y are all real numbers.

(a) Let $e_x = 2$, $e_y = 1$, $\psi_x = \pi/2$, $\psi_y = \pi/4$. What is the polarization?
(b) Let $e_x = 1$, $e_y = \psi_x = 0$. This is a linearly polarized wave. Prove that it can be expressed as the superposition of a right-hand circularly polarized wave and a left-hand circularly polarized wave.
(c) Let $e_x = 1$, $\psi_x = \pi/4$, $\psi_y = -\pi/4$, $e_y = 1$. This is a circularly polarized wave. Prove that it can be decomposed into two linearly polarized waves.

P1.2.7

Wave polarization can be viewed by either taking a series of still pictures at several fixed times, called the spatial view point or by making observations at a fixed point in space, called the temporal view point. We define polarization from the temporal view point. Let us now look at polarization from the spatial view point.

Consider an electromagnetic wave with $k = 100\,\mathrm{K}_o$ propagating in the $\hat{z}$ direction.

$$\overline{E}(\overline{r}, t) = E_0[\hat{x}\cos(kz - \omega t) - \hat{y}\sin(kz - \omega t)]$$

What are the wavelength and the polarization of this wave?

From the spatial point of view, by taking a picture at $t = 0$, the tips of the electric field vectors form a helix. Is the helix right-handed or left-handed? What is the pitch of this helix?

Observing at a fixed point in space, show that the tip of the electric field describes the same polarization as in the temporal view point when the helix advances without turning.

P1.2.8

For polarized waves

$$\begin{aligned} I &= I_h + I_v \\ Q &= I_h - I_v = I\cos 2\alpha \cos 2\beta \\ U &= I\cos 2\alpha \sin 2\beta \\ V &= I\sin 2\alpha \end{aligned}$$

Show that when the wave is right-handed circularly polarized $Q = U = 0$ and $V = I$, when it is left-hand circularly polarized, $Q = U = 0$ and $V = -I$, and when the wave is linearly polarized, $V = 0$.

1.3 Force, Power, and Energy

A. Lorentz Force Law

The interaction of the electric and magnetic fields with the current and charge densities are governed by the Lorentz force law

$$\overline{f} = \rho\overline{E} + \overline{J} \times \overline{B} \tag{1.3.1}$$

where $\overline{f}$ is the force density (with unit $\mathrm{N/m^3}$). The Lorentz force law relates electromagnetism to mechanics. The manifestation of the electric field vector $\overline{E}$ and the magnetic field vector $\overline{B}$ can be demonstrated with the forces exerted on the charge density ρ and the current density $\overline{J}$. It can thus be used to define the fields $\overline{E}$ and $\overline{B}$.

Hendrik Antoon Lorentz (18 July 1853 – 4 February 1928)

Hendrik Lorentz entered the University of Leyden in 1870, obtained his B.Sc. degree in 1871, and in 1875, his doctor's degree for his thesis on the reflection and refraction of light. Three years later he was appointed to the Professor of Physics at Leyden. In 1904 he developed the Lorentz transformation formula that form the basis for the special theory of relativity .

Example 1.3.1 Coulomb's law.

For static electric fields in the absence of magnetic fields, the Lorentz force law becomes $\overline{f} = \rho\overline{E}$. Acting on a charged particle q, the total force is $\overline{F} = q\overline{E}$. Assuming that the electric field $\overline{E}$ is generated by another charged particle Q situated at the origin, we have

$$\overline{E} = \hat{r}\frac{Q}{4\pi\epsilon_o r^2}$$

Thus the total force acting on the charged particle q is

$$\overline{F} = \hat{r}\frac{qQ}{4\pi\epsilon_o r^2}$$

which is proportional to the squared inverse distance. This is the well-known Coulomb's law.

— End of Example 1.3.1 —

Issac Newton (25 December 1642 – 20 March 1727)

Newton attended Cambridge University at the age of 19 and entered Trinity College in 1661. After receiving his B.A. degree in 1664, he returned to his birth place Woolsthorpe, England. In the next two years, he extended the binomial theorem, invented calculus, discovered the law of universal gravitation, and experimentally proved that white light is composed of all colors, all these great accomplishments in scientific history before his 25th birthday.

EXAMPLE 1.3.2 Cyclotron frequency.

Consider a particle with charge q and mass m moving with velocity $\overline{v}$ in a uniform static magnetic field in the $-\hat{z}$ direction, $\overline{B} = -\hat{z}B_0$. In the absence of electric fields, if the velocity v has no component in the $\hat{z}$ direction, the Lorentz force is perpendicular to the direction of the velocity and the charge particle moves in the x-y plane. Let $\overline{v} = \hat{x}v_x + \hat{y}v_y$, we have

$$\overline{F} = q\overline{v} \times \overline{B} = -\hat{x}qv_yB_0 + \hat{y}qv_xB_0$$

Equating to Newton's law

$$\overline{F} = m\frac{d\overline{v}}{dt} = \hat{x}m\frac{dv_x}{dt} + \hat{y}m\frac{dv_y}{dt}$$

we find

$$m\frac{dv_x}{dt} = -qv_yB_0 \qquad \text{(E1.3.2.1}a)$$

$$m\frac{dv_y}{dt} = qv_xB_0 \qquad \text{(E1.3.2.1}b)$$

Eliminating v_y from the above two equations, we find

$$\frac{d^2v_x}{dt^2} = -\omega_c^2 v_x$$

where

$$\omega_c = \frac{qB_0}{m} \qquad \text{(E1.3.2.2)}$$

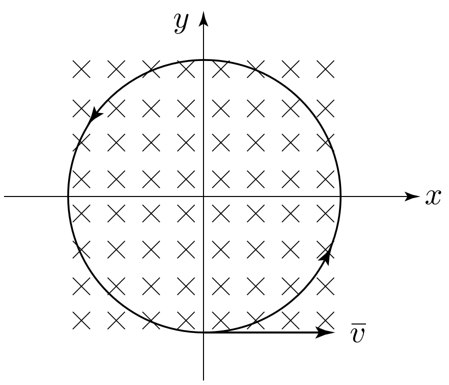


Figure E1.3.2.1 Cyclotron frequency.

is called the cyclotron frequency, which is proportional to the magnitude of the magnetic field and is independent of the velocity of the particle.

The solution to (E1.3.2.1) can be written as

$$v_x = \frac{dx}{dt} = v \cos \omega_c t \tag{E1.3.2.3a}$$

$$v_y = \frac{dy}{dt} = v \sin \omega_c t \tag{E1.3.2.3b}$$

To find the trajectory of the particle, we write the solution of (E1.3.2.3) as

$$x = \frac{v}{\omega_c} \sin \omega_c t = R \sin \omega_c t \tag{E1.3.2.4a}$$

$$y = -\frac{v}{\omega_c} \cos \omega_c t = -R \cos \omega_c t \tag{E1.3.2.4b}$$

The trajectory of the particle is thus a circle with radius

$$R = (x^2 + y^2)^{1/2} = \frac{v}{\omega_c} \tag{E1.3.2.5}$$

In terms of the applied magnetic field, we find from (E1.3.2.2)

$$R = \frac{mv}{qB_0} \tag{E1.3.2.6}$$

It is seen that the larger the magnetic field, the smaller the radius. If the charged particle has a velocity component in the $\hat{z}$ direction, the trajectory of the particle will follow a helical path.

— End of Example 1.3.2 —

Exercise 1.3.1 Centrifugal force.

In cylindrical coordinate system, $\overline{\rho}$ is the radial vector and $\hat{\rho}$ is in the radial direction. The force acting on the charge in the above example is

$$\begin{aligned}\overline{F} &= \hat{x} m \frac{d^2 x}{dt^2} + \hat{y} m \frac{d^2 y}{dt^2} = m \frac{d^2}{dt^2} \overline{\rho} \\ &= m R \omega_c^2 (-\hat{x} \sin \omega_c t + \hat{y} \cos \omega_c t) = -m \omega_c^2 (\hat{x} x + \hat{y} y) \\ &= -m \hat{\rho} R \omega_c^2 = -\hat{\rho} m \frac{v^2}{R}\end{aligned} \tag{Ex1.3.1.1}$$

which is equal to the negative of the centrifugal force pointing in the $\hat{\rho}$ direction, whose magnitude is equal to the Lorentz force $\hat{\rho} q v B_o$.

— End of Exercise 1.3.1 —

EXAMPLE 1.3.3 Cyclotron.

A cyclotron [Fig. E1.3.3.1] is an accelerator for charged particles. The a.c. source provides an alternating voltages at the cyclotron frequency and a charged particle is repeatedly accelerated every time it passes through the voltage drop.

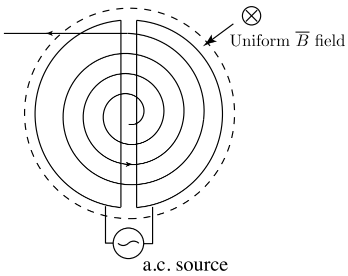


Figure E1.3.3.1 Cyclotron.

— END OF EXAMPLE 1.3.3 —

EXAMPLE 1.3.4 Isotope separation.

To separate the isotope Uranium 235 from Uranium 238, the isotopes are first vaporized and then ionized by electric discharge. Accelerated through a voltage drop V , they acquire a kinetic energy $qV = mv^2/2$. Passing through [Fig. E1.3.4.1] a uniform magnetic field, the isotopes move along circular paths of different radii.

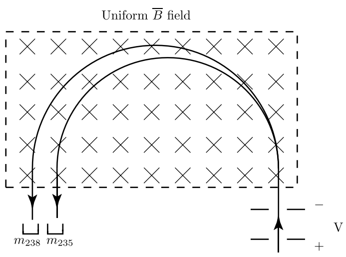


Figure E1.3.4.1 Isotope separation.

$$\frac{R_{235}}{R_{238}} = \frac{m_{235}v_{235}}{m_{238}v_{238}} = \frac{m_{235}}{m_{238}}\sqrt{\frac{m_{238}}{m_{235}}} = \sqrt{\frac{m_{235}}{m_{238}}}$$

Thus Uranium 235 can be obtained in a collector with a smaller radius.

— END OF EXAMPLE 1.3.4 —

EXAMPLE 1.3.5

The two rods attract each other when their currents are in the same direction and are repulsive when their currents are in the opposite directions.

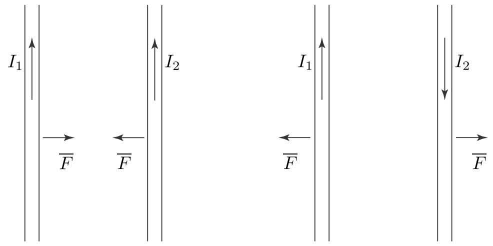


Figure E1.3.5.1 Attractive and repulsive forces.

— END OF EXAMPLE 1.3.5 —

EXAMPLE 1.3.6 Linear motor.

In Fig. E1.3.6.1, we show a sliding bar with length l moving perpendicular to a DC magnetic field $\overline{B} = \hat{z}B_0$ in the $\hat{z}$ direction. According to the Lorentz force law, a force

$$\overline{F}_m = \hat{y}Il \times \hat{z}B_0 = \hat{x}IlB_0$$

is produced that moves the sliding bar in the $\hat{x}$ direction.

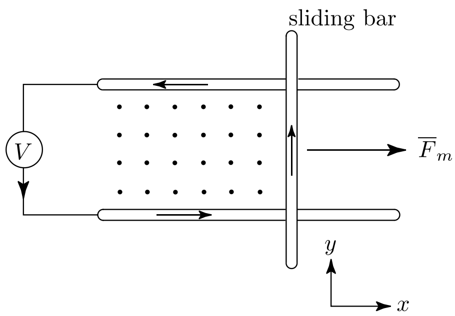


Figure E1.3.6.1 Linear motor.

If a force is applied to move the sliding bar with velocity $v = -\hat{x}$, an induced voltage $V = vlB_0$ will be generated across the resistor.

— END OF EXAMPLE 1.3.6 —

EXAMPLE 1.3.7 Magnetic moment and magnetic torque.

A rectangular loop [Figure E1.3.7.1] carrying a static current I is placed in a static magnetic field $\overline{B} = \hat{x}B_0$. The magnetic moment of the current loop is $\overline{M} = \hat{m}M$. Its direction $\hat{m}$ follows from the right-hand rule: with the fingers pointing in the direction of the current, the thumb of the right hand is pointing in the direction of $\hat{m}$. Its magnitude M is equal to the area of the loop A times the current I, $M = AI$. If the rectangular loop has lengths l_x and l_y, the area of the loop is $A = l_x l_y$.

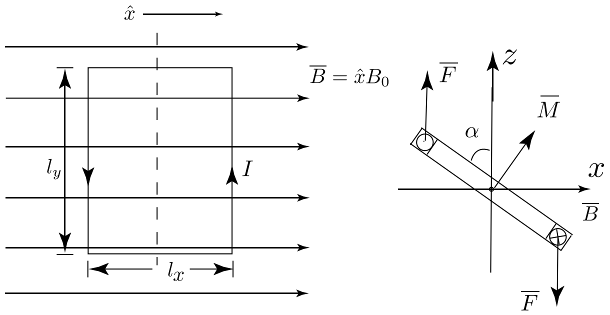


Figure E1.3.7.1 Torque on a loop.

The loop is on the x-y plane with two sides aligned with the x-axis and two sides aligned with the y-axis. Since the static magnetic field is in the $\hat{x}$ direction, there is no force acting on the two sides with length l_x aligned with the x-axis. The forces acting on the two sides with length l_y aligned with the y axis are in the positive and negative $\hat{y}$ directions. Thus the loop is rotating around the y-axis following the right-hand rule; with the fingers pointing in the direction of the rotation, the thumb of the right hand is pointing in the $\hat{y}$ direction.

The torque acting on the loop is calculated as

$$\overline{T} = \frac{1}{2}l_x\hat{x} \times (\hat{y} \times \hat{x}Il_yB_0) - \frac{1}{2}l_x\hat{x} \times (-\hat{y} \times \hat{x}Il_yB_0) = \hat{y}IAB_0$$

For the current configuration, $\overline{M} = \hat{z}IA$ and $\overline{B} = \hat{x}B_0$. In general, the magnetic torque is

$$\overline{T} = \overline{M} \times \overline{B} \tag{E1.3.7.1}$$

Thus there is no torque acting on the component of $\overline{M}$ in the direction of the magnetic field.

— END OF EXAMPLE 1.3.7 —

Example 1.3.8

A simple DC motor [Fig. E1.3.8.1] consists of a loop of area A with N turns, called an armature, which is immersed in a uniform magnetic field, either produced by a permanent magnet or an electromagnet. The armature is connected to a commutator which is a divided slip ring. A DC current I is supplied through a pair of brushes resting against the commutator such that the torque

$$T = NB_o IA \sin\alpha$$

produced by the current on the armature always acts in the same direction.

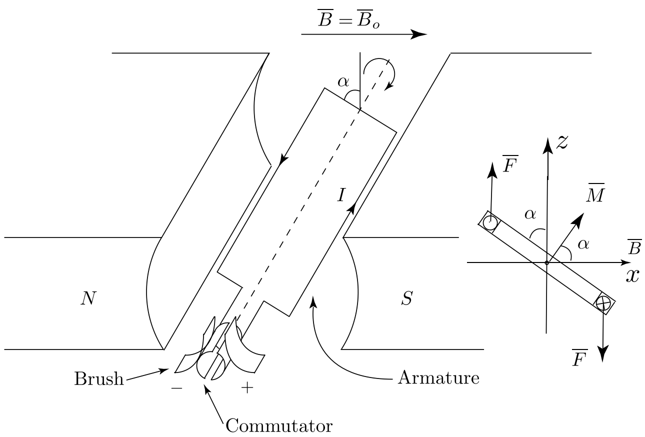


Figure E1.3.8.1a DC motor.

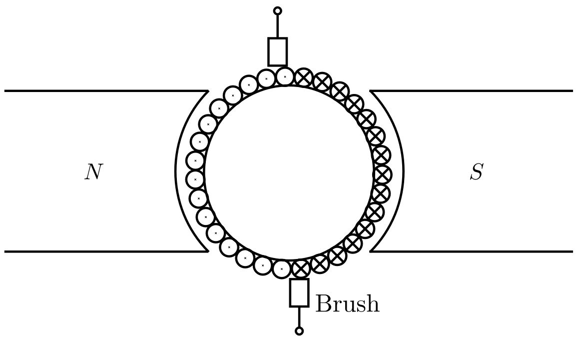


Figure E1.3.8.1b Side view of a DC motor.

— End of Example 1.3.8 —

Alessandro Volta (18 February 1745 – 5 March 1827)

Alessandro Volta was appointed to the chair of physics at the University of Pavia in 1775. In 1800, Volta built the first electric battery, consisting of alternating zinc and silver disks separated by layers of paper or cloth soaked in a solution of either sodium hydroxide or brine, called the 'voltaic pile'.

Hans Christian Oersted (14 August 1777 – 9 March 1851)

Oersted became a professor at the University of Copenhagen in 1806. In April 1820, during an evening lecture to a few advanced students, he discovered that a wire connecting the ends of a voltaic battery deflected a magnet in its vicinity. This discovery was published on 21 July 1820.

EXAMPLE 1.3.9

In October of 1821, Faraday demonstrated the principle of electric motor with a dish of mercury. When he connected a battery to form a circuit with the mercury pool, using a fixed wire carrying current and a dangling magnet with one end fixed and the other end moving around the surface of the pool of mercury. Let the magnet be designated as a magnetic moment $\overline{M}$ placed in a magnetic field $\overline{B}$. The torque acting on the magnet is $\overline{T} = \overline{M} \times \overline{B}$. Show that the magnet rotates around the wire in a circular trajectory.

SOLUTION:

To find the magnetic field $\overline{H}$ at the position of the loop due to the straight wire carrying current I_0 in the $\hat{z}$ direction, we use the integral form of Ampère's law,

$$\oint_C \overline{H} \cdot d\bar{l} = \int_0^{2\pi} H_\phi d \; d\phi = 2\pi d H_\phi = \int_s \overline{J} \cdot ds = I_0$$

which gives the magnetic field $\overline{B}$ at the loop's position

$$\overline{B} = \mu_0 \overline{H} = \hat{\phi} \frac{I_0 \mu_0}{2\pi d}$$

To calculate the torque, we apply,

$$\overline{T} = \overline{M} \times \overline{B} = \hat{z} \frac{M I_0 \mu_0}{2\pi d}$$

which means that the current loop will move about the z-axis in a counter-clockwise direction.

— END OF EXAMPLE 1.3.9 —

B. Lenz' Law and Electromotive Force (EMF)

We apply Stokes theorem to Faraday's law and define the line integral of $\overline{E}$ as the electromotive force (EMF):

$$\begin{aligned}\text{EMF} &= \oint_C d\bar{l} \cdot \overline{E} = -\frac{\partial}{\partial t} \iint d\overline{S} \cdot \overline{B} \\ &= -\frac{\partial}{\partial t} \Psi \end{aligned} \tag{1.3.2}$$

where

$$\Psi = \iint_A d\overline{S} \cdot \overline{B} \tag{1.3.3}$$

is the magnetic flux linking a loop with area A bounded by a closed contour C [Fig. 1.3.1]. Equation (1.3.2) states that the EMF is equal to the negative time derivative of the magnetic flux linking the loop. Thus the EMF always produces a flux in the loop to oppose the direction of change of the flux linking the loop; if Ψ is increasing, the EMF decreases the flux, and vice versa. This is known as Lenz' law.

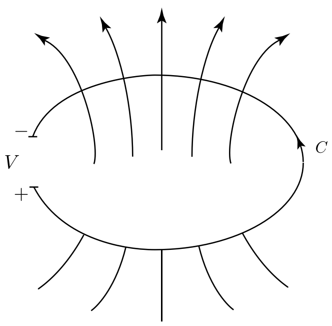


Figure 1.3.1 Flux linking a loop.

Heinrich Lenz (12 February 1804 – 10 February 1865)

Heinrich Lenz was scientific assistant at the St. Petersburg Academy of Science, becoming full Academician in 1834. From 1835 to 1841, he served as lecturer in physics at the Naval Military School. He was dean of mathematics and physics (1840–1863) at the University of St. Petersburg. He began his investigation of electromagnetism in 1831 and in 1833 discovered Lenz' law, which is fundamental to electrical machinery.

Notice that the EMF has unit of voltage (Volt) and not unit of force. The voltage drop across the loop V is equal to the negative of the induced EMF.

$$V = -\text{EMF} = \frac{d}{dt}\Psi \tag{1.3.4}$$

Thus in the presence of a time varying magnetic field linking a loop, a voltage is generated to oppose the time change of the magnetic field. The voltage generated across the loop V is equal to the negative of the induced EMF.

LeChatelier's Principle (Henri Louis Le Chatelier, 8 October 1850 – 17 September 1936) is the chemist's version of Lenz' law, which states that when an external stress (pressure, concentration, or temperature change) is applied to a chemical system that is in a state of equilibrium, the system will automatically respond so as to undo the stress applied externally.

In Physics, this same phenomenon is embodied in the Third Law of Motion, that is, for every action there is an equal and opposite reaction. In biology, a condition in an organism known as homeostasis means that when a stress is applied to an organism, the organism's bodily functions automatically respond so as to remove the stress.

Example 1.3.10 Linear generator.

If a force is applied to move the sliding bar with velocity $v = -dx/dt$ as shown in Fig. E1.3.10.1, the total magnetic field $\Psi = xlB_o$ linking the loop will be decreasing at the rate of vlB_0. According to Lenz' law, a current in the bar must be produced to oppose the decreasing of the magnetic flux. Thus an induced voltage $V = vlB_0$ is generated across the resistor.

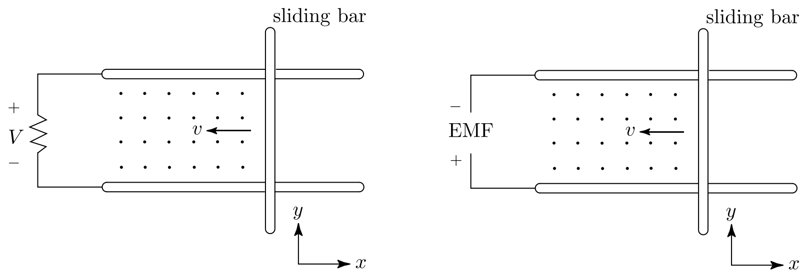


Figure E1.3.10.1 Linear generator.

— End of Example 1.3.10 —

EXAMPLE 1.3.11 AC generator.

An AC generator can be made of the DC motor by replacing the DC current source with a load resistance R and providing an external rotatory force on the armature. Applying a torque that makes the loop turn in the direction as shown in Fig. E1.3.11.1, a motional EMF

$$V = \int d\bar{l} \cdot \overline{E} = \int d\bar{l} \cdot \overline{F}/q = \int d\bar{l} \cdot \overline{v} \times \overline{B} = \omega AB \sin\alpha$$

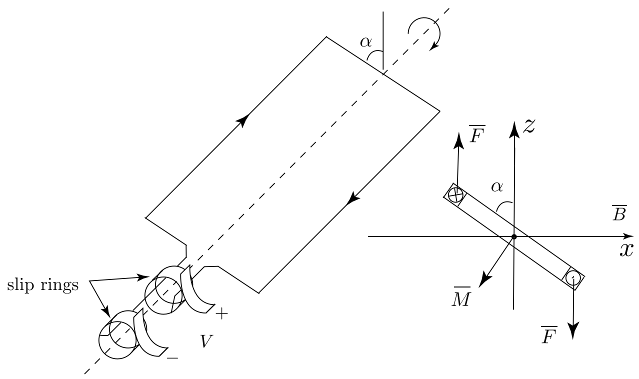


Figure E1.3.11.1 AC generator.

is produced. For the armature rotating with an angular frequency ω , we have $\overline{v} \times \overline{B} = \hat{l}\omega BA \sin\alpha$ and $\alpha = \omega t$.

The same result can be derived by using Lenz' law

$$\text{EMF} = -d\Psi/dt$$

where

$$\Psi = \iint_A d\overline{S} \cdot \overline{B} = -AB\cos\alpha \qquad \text{(E1.3.11.1)}$$

We find the generated AC voltage

$$V = -EMF = \omega AB \sin\omega t$$

— END OF EXAMPLE 1.3.11 —

C. Poynting's Theorem and Poynting Vector

Energy conservation immediately follows from the Maxwell equations. Dot-multiply Faraday's law (1.1.2) by $\overline{H}$, Ampère's law (1.1.1) by $\overline{E}$ and subtract. By making use of the vector identity $\nabla \cdot (\overline{E} \times \overline{H}) = \overline{H} \cdot \nabla \times \overline{E} - \overline{E} \cdot \nabla \times \overline{H}$, we obtain Poynting's theorem

$$\nabla \cdot (\overline{E} \times \overline{H}) + \overline{H} \cdot \frac{\partial \overline{B}}{\partial t} + \overline{E} \cdot \frac{\partial \overline{D}}{\partial t} = -\overline{E} \cdot \overline{J} \tag{1.3.5}$$

The Poynting vector

$$\overline{S} = \overline{E} \times \overline{H} \tag{1.3.6}$$

is interpreted as the power flow density with the dimension watts/m^2, and $\overline{H} \cdot (\partial \overline{B}/\partial t) + \overline{E} \cdot (\partial \overline{D}/\partial t)$ represents the time rate of change of the stored electric and magnetic energy density. On the right-hand side of (1.3.5), $-\overline{E} \cdot \overline{J}$ is the power supplied by the current $\overline{J}$.

John Henry Poynting (9 September 1852 – 30 March 1914)

John Henry Poynting was educated at Liverpool and Cambridge and was one of Maxwell's students. He was professor of physics at Mason Science College (later the University of Birmingham) from 1880 until his death. In 1884–1885, he established Poynting's theorem.

Example 1.3.12

Consider the simple wave solution

$$\overline{E} = \hat{x} E_0 \cos(kz - \omega t) \tag{E1.3.12.1a}$$

$$\overline{H} = \hat{y} H_0 \cos(kz - \omega t) \tag{E1.3.12.1b}$$

where $H_0 = E_0/\eta_o$ and $\eta_o = \sqrt{\mu_o/\epsilon_o}$ is called the characteristic impedance of free space. Substituting (E1.3.12.1) in (1.3.5) we see that Poynting's theorem is satisfied.

The Poynting vector is calculated to be Poynting's vector

$$\overline{S} = \overline{E} \times \overline{H} = \hat{z} \sqrt{\frac{\epsilon_o}{\mu_o}} E_0^2 \cos^2(kz - \omega t) \tag{E1.3.12.2}$$

In free space, we find

$$\overline{H} \cdot \frac{\partial}{\partial t}(\mu_o \overline{H}) = \frac{\partial}{\partial t}\left[\frac{1}{2}\mu_o \overline{H} \cdot \overline{H}\right] = \frac{\partial}{\partial t} W_m$$

and

$$\overline{E} \cdot \frac{\partial}{\partial t}(\epsilon_o \overline{E}) = \frac{\partial}{\partial t}\left[\frac{1}{2}\epsilon_o \overline{E} \cdot \overline{E}\right] = \frac{\partial}{\partial t} W_e$$

In the source-free region we also have $\overline{J} = 0$. Poynting's theorem becomes

$$\nabla \cdot (\overline{E} \times \overline{H}) + \frac{\partial}{\partial t}(W_e + W_m) = 0 \tag{E1.3.12.3}$$

where

$$W_e = \frac{1}{2}\,\epsilon_o \left|\overline{E}\right|^2 = \frac{1}{2}\epsilon_o E_0^2 \cos^2(kz - \omega t) \tag{E1.3.12.4}$$

is the stored electric energy density and

$$W_m = \frac{1}{2}\,\mu_o \left|\overline{H}\right|^2 = \frac{1}{2}\mu_o H_0^2 \cos^2(kz - \omega t) \tag{E1.3.12.5}$$

is the stored magnetic energy density. It is seen that the stored electric energy is equal to the stored magnetic energy, $W_e = W_m$.

— End of Example 1.3.12 —

James Watt (19 January 1736 – 25 August 1819)

James Watt was a Scottish engineer who played an important part in the development of the steam engine as a practical power source and a key stimulus to the Industrial Revolution. Watt is the unit of power.

James Prescott Joule (24 December 1818 – 11 October 1889)

Joule attended the University of Manchester in 1835 and in 1840 he published his paper On the Production of Heat by Voltaic Electricity. He experimentally verified the law of conservation of energy in his study of the transfer of mechanical energy into heat energy. Joule is the unit of energy.

William Thomson (Lord Kelvin) (26 June 1824 – 17 December 1907)

At age 22, William Thomson became professor of physics at the University of Glasgow where he remained for 53 years until his retirement in 1899. He first defined the absolute temperature scale in 1847. In 1851 he published the paper, 'On the Dynamical Theory of Heat.' In 1866 he was Knighted by Queen Victoria. In 1892 he became Lord Kelvin of Largs. Kelvin is the unit of absolute temperature.

EXAMPLE 1.3.13 Power, energy, force, and radiation pressure.

The time-average Poynting vector power density is given by

$$<\overline{S}>=\frac{1}{T}\int_0^T dt\,\overline{S}=\hat{z}\frac{E_0^2}{2\eta_o}=\hat{z}\frac{1}{2}\eta_o H_0^2=\hat{z}P \tag{E1.3.13.1}$$

where

$$P=\frac{E_0^2}{2\eta_o}=\frac{1}{2}\eta_o H_0^2$$

is the power density of the wave with unit of Watts/m^2. The total time-average electromagnetic energy density (with unit J/m^3) is equal to the sum of the electric energy density and the magnetic energy density,

$$W=<W_e>+<W_m>=\frac{1}{2}\epsilon_o E_0^2=\frac{1}{2}\mu_o H_0^2 \tag{E1.3.13.2}$$

We may define an energy velocity v_e equal to the ratio of power density to energy density. We find $P/W=v_e=1/\sqrt{\mu_o\epsilon_o}$ which is the velocity of light.

Radiation pressure is force per unit area $F=P/v_e$ (with unit N/m^2). Thus the radiation pressure of the wave is

$$F=P/v_e=W=\frac{1}{2}\epsilon_o E_0^2=\frac{1}{2}\mu_o H_0^2 \tag{E1.3.13.3}$$

which is equal to the time-average total energy density in the wave and acts in the direction of propagation of the wave. The radiation pressure, although generally very small, can lead to large scale effects. For example, comet tails are forced to point away from the Sun due to the radiation pressure from the Sun.

— END OF EXAMPLE 1.3.13 —

Applying the divergence theorem to Poynting's theorem (1.3.5), we write

$$\oiint_S d\overline{S}\cdot\overline{E}\times\overline{H}=-\frac{\partial}{\partial t}\iiint_V dV\left(\frac{1}{2}\epsilon_o E^2+\frac{1}{2}\mu_o H^2\right)-\iiint_V dV\,\overline{E}\cdot\overline{J} \tag{1.3.7}$$

The left-hand side represents power flow out of the surface enclosing the volume V. The first term on the right-hand side represents the depletion of the electric energy and the magnetic energy inside the volume in order to supply the outflow of the Poynting power. The last term represents the power generated by the source J inside the volume V.

Momentum Conservation Theorem

Substituting the Maxwell equations for ρ and $\overline{J}$ in the Lorentz force law

$$\overline{f} = \rho \overline{E} + \overline{J} \times \overline{B} \tag{1.3.8}$$

we find that

$$\overline{f} = -\frac{\partial}{\partial t}(\overline{D} \times \overline{B}) - \nabla \cdot \left[\frac{1}{2}(\overline{D} \cdot \overline{E} + \overline{B} \cdot \overline{H})\overline{\overline{I}} - \overline{D}\,\overline{E} - \overline{B}\,\overline{H}\right] \tag{1.3.9}$$

where $\overline{\overline{I}}$ is a unit dyad with diagonal elements equal to 1 and all off-diagonal elements equal to zero.

The interpretation of the terms is

$$\overline{G} = \overline{D} \times \overline{B} = \text{momentum density vector} \tag{1.3.10}$$

$$\begin{aligned}\overline{\overline{T}} &= \frac{1}{2}(\overline{D} \cdot \overline{E} + \overline{B} \cdot \overline{H})\overline{\overline{I}} - \overline{D}\,\overline{E} - \overline{B}\,\overline{H} \\ &= \text{Maxwell stress tensor}\end{aligned} \tag{1.3.11}$$

Thus we have the theorem

$$\nabla \cdot \overline{\overline{T}} + \frac{\partial \overline{G}}{\partial t} = -\overline{f} \tag{1.3.12}$$

which expresses conservation of momentum. This is in a form similar to Poynting's theorem in (1.3.5) except that it is now a vector equation. In fact, (1.3.5) and (1.3.12) combine to become a four-dimensional conservation theorem in relativity.

Problems

P1.3.1

According to the classical model of an atom as proposed by Niels Bohr (1885–1962), electrons revolve around the nucleus in quantized orbits with radii $R = n\hbar/mv$ where n is an integer, m is the electron mass and v is the electron velocity. Letting the nucleus be a positive charge of Ze, calculate R by equating the centrifugal force with the Lorentz force. Estimate the radius for a hydrogen atom with $Z = 1$.

P1.3.2

The Earth receives over all frequency bands about $1.5\,\mathrm{kW/m^2}$ of power from the Sun.

(a) The Earth-Sun distance is 150×10^9 m . How long does it take the sunlight to reach the Earth?
(b) The Earth radius is 6400 km. What is the total power received by the Earth?
(c) The Sun radiates 10^{-20} W m^{-2} Hz^{-1} at 3 GHz. Assuming constant power level over 1 GHz bandwidth, what is the Poynting power density and the corresponding electric field amplitude?

P1.3.3

For an electromagnetic wave with electric field with $E_0 = 3 \times 10^6$ V/m (which is the breakdown electric field strength for air), find the power density and radiation pressure. What is the area required in order to supply the electric power of 2.4×10^{11} W for use by a nation?

P1.3.4

In cylindrical coordinate system, $\overline{\rho} = \hat{\rho}\rho = \hat{x}x + \overline{y}y$ is the radial vector. Show that the force acting on the charge in Example 1.3.2 is

$$\overline{F} = -\hat{\rho}m\frac{v^2}{R}$$

which is equal to the negative of the centrifugal force pointing in the $\hat{\rho}$ direction, whose magnitude is equal to the Lorentz force $\hat{\rho}qvB_o$.

P1.3.5

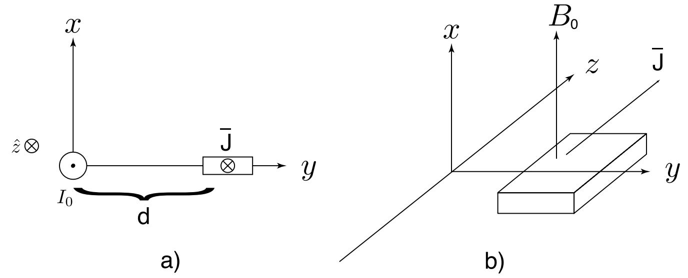


Figure P1.3.5.1

(a) Consider an infinitely long wire with current I_0 flowing along the $-\hat{z}$ direction as shown in Fig. P1.3.5.1a. Find the $\overline{B}$ field at $y = d$ generated by the current.
(b) Consider a slab of semiconductor with positive charge carriers of density N so that there is a uniform current density of $\overline{J} = \hat{z}Nqv$ flowing in the $+\hat{z}$ direction as shown in Figure P1.3.5.1b. Calculate the force density $\overline{F}$ acting on the charges if a static magnetic field $\overline{B} = \hat{x}B_0$ is applied.

P1.3.6

For a charged particle q moving with velocity v in a constant magnetic field B_0 , the trajectory is a circle. Set the Lorentz force equal to the centrifugal force and derive the cyclotron frequency and the radius of the circle.

P1.3.7

The solar wind is a high-conductivity plasma which is emitted radially from the surface of the Sun. Let us calculate the flux of electromagnetic energy in the solar wind at the orbit of the Earth.

In the plane of the Earth's orbit, the magnetic field of the Sun is approximately radial, pointing outward in certain regions and inwards in others. This field is "frozen" in the high-conductivity plasma. Since the Sun rotates (with a period of 27 days), and the plasma has a radial velocity, the lines of $\overline{B}$ are in fact Archimedes spirals ($r = a\theta$ in polar coordinates) and, at the Earth, they form an angle of about $45°$ with the Sun-Earth direction. This is the so-called garden hose effect.

At the orbit of the Earth the solar wind has a density of about 10^7 proton-masses/m^3 and a velocity of about 4×10^5 m/sec, while the magnetic field of the Sun is about 5×10^{-9}(webers/m^2).

(a) First show that, in an electrically neutral $(\rho = 0)$ and nonmagnetic fluid of conductivity σ and velocity $\overline{v}$, the Maxwell equations become

$$\nabla\cdot\overline{D} = 0 \qquad \nabla\cdot\overline{B} = 0 \qquad \nabla\times\overline{E} = -\frac{\partial\overline{B}}{\partial t}$$

$$\nabla\times\overline{B} = \mu_0\left\{\sigma(\overline{E} + \overline{v}\times\overline{B}) + \epsilon_0\frac{\partial\overline{E}}{\partial t}\right\}$$

the polarization currents being negligibly small compared to the conduction currents. Note that, for an infinite conductivity,

$$\overline{E} = -\overline{v}\times\overline{B}$$

This is a satisfactory approximation for the solar wind.

(b) Show that the component of $\overline{v}$ which is normal to $\overline{B}$ is $\overline{v}_n = \frac{1}{B^2}\overline{B}\times(\overline{v}\times\overline{B})$, and that the Poynting vector of the solar wind is

$$\overline{S} = \frac{B^2}{\mu_o}\overline{v}_n$$

Numerically it is approximately equal to 4×10^{-9} times the average value of the Poynting vector of the solar radiation, which is about $1.4\,\mathrm{kW/m^2}$. The Poynting vector of the solar wind is normal to the local $\overline{B}$ and it points at an angle of $45°$ away from the Sun-Earth direction.

(c) Compare the relative magnitudes of the kinetic, electric, and magnetic energy densities. Which is the largest?

P1.3.8

Particles excited by an electromagnetic wave may be modeled as harmonic oscillators with a characteristic frequency and damping. For electrons,

$$\frac{\partial^2 \overline{x}}{\partial t^2} + \delta \frac{\partial \overline{x}}{\partial t} + \omega_0^2 \overline{x} + \frac{q\overline{E}}{m} = 0$$

which is just an expression for momentum conservation $(F = MA)$.

(a) Assume $\delta = 0$. Show that for $\omega > \omega_0$ (ω is defined as the frequency of the $\overline{E}$-field), the electrons vibrate in phase with the E-field while for $\omega < \omega_0$, they are $180°$ out of phase. Can you explain opacity of certain substances in terms of this effect? (see Scientific American, Sept. 1968, p. 60 ff.)

(b) Derive a Poynting theorem and show that

$$\nabla \cdot \overline{S} + \frac{\partial W}{\partial t} + P_D = 0$$

$$S \equiv E \times H \quad \text{(Electromagnetic power density)}$$

Determine W and P_D.

P1.3.9

Consider two infinite parallel metal plates separated by a distance d along the $\hat{x}$ direction. Initially the system is at rest, and the top plate has a uniform surface charge density of σ while the bottom plate has a uniform surface charge density of $-\sigma$. At time $t = 0$ a uniformly decaying magnetic field is applied parallel to the plane of the plates, that is,

$$\overline{B}(t) = \hat{y} B_0 e^{-\gamma t}$$

(a) Calculate the Poynting vector, $\overline{S}$, for the system and the momentum density vector, $\overline{g}_f$, of the field for $t > 0$ using the relation,

$$\overline{g}_f = \mu_0 \epsilon_0 \overline{S}$$

(b) As the magnetic field begins to decay, it will induce an electric field. By the Lorentz force law, this induced field exerts a force on the two charged metal plates. Determine the strength and direction of this induced electric field and the resulting force density vector exerted on the two plates.

(c) From mechanics, the force and momentum vectors are related by

$$\overline{F} = \frac{d}{dt}\,\overline{p}.$$

Using this relation, calculate the mechanical momentum density vector that results from the induced electric field.

(d) As the magnetic field decays, the momentum of the field is transferred to the plate in the form of mechanical momentum. Using the results of parts (a) and (c), show that for $t > 0$, the total momentum of the system is conserved.

P1.3.10

The magnetic moment of a particle with charge q at position $\overline{r}$ with velocity $\overline{v}$ is defined as

$$\overline{M} = \frac{1}{2} q \overline{r} \times \overline{v}$$

Show that the magnetic moment of a plane loop with area A carrying current I is

$$\overline{M} = \hat{m} I A$$

where $\hat{m}$ is the normal to the plane loop following the right-hand rule: with the fingers following the direction of the current, the thumb of the right hand is pointing in the direction of $\hat{n}$.

P1.3.11

In mechanics, the classical equations of motion are $\overline{T} = d\overline{L}/dt$, where $\overline{L}$ is the angular momentum. The magnetic moment $\overline{M}$ is analogous to the expression for the mechanical angular momentum $\overline{L}$ in terms of the velocity of mass distributions instead of the charge distributions. The magnetic moment of a particle with charge q at position $\overline{r}$ with velocity $\overline{v}$ is defined as

$$\overline{M} = \frac{1}{2} q \overline{r} \times \overline{v}$$

If the charged particle has mass m, the mechanical angular momentum is

$$\overline{L} = m \overline{r} \times \overline{v}$$

We set $\overline{M} = \gamma \overline{L}$ and called γ the gyromagnetic ratio. From (E1.3.7.1), we see that applying to the magnetic moment, we have $d\overline{M}/dt = \gamma d\overline{L}/dt = \gamma \overline{T} = \gamma \overline{M} \times \overline{B}$.

(a) Determine the gyromagnetic ratio γ for the charged particle.

(b) Consider a nucleus with magnetic moment $\overline{M}$ placed in a dc magnetic field in the $\hat{z}$-direction, $\overline{B} = \hat{z} B_0$. The nucleus is precessing about the $\hat{z}$ axis. Determine the frequency of precession, which is called the Larmor frequency.

(c) Place the magnetic moment $\overline{M}$ of a nucleus precesses in a static magnetic field $\overline{B} = \hat{x} x + \hat{z} B_z$. Show that $B_z = B_0 - z$ to satisfy Maxwell equations.

(d) When the nucleus is placed on the z axis where $x = 0$, $\overline{B} = \hat{z}B_z$. Determine the Larmor frequency of precession and show that it is a function of z.

(e) An induced voltage with the angular frequency ω due to $\overline{M}$ can be picked up from an RF (radio frequency) coil placed on the x-z plane. Assume that the magnetic dipoles are spinning protons of water at room temperature, with $\gamma = 2.7 \times 10^8\,\mathrm{T}^{-1}\mathrm{s}^{-1}$. Let there be two protons precessing on the z axis with a separation of δ_z. Calculate the difference of Larmor frequency in kHz of the pick-up coil if $\delta_z = 1\,\mathrm{mm}$.

P1.3.12

Consider a loop carrying a current of I_l with normal $\hat{n} = \hat{x} - \hat{y}$ is placed a distance d above a straight wire, which is carrying a current of I_0. Calculate the magnetic moment of the current loop and the magnetic field generated by straight wire at the loop's position. Using these two values, calculate the torque vector, $\overline{T}$, of loop. In what direction does the loop move due to the torque?

P1.3.13

Joule's law, $P_d = \overline{J} \cdot \overline{E}$, determines power dissipation per volume due to Ohmic loss. Derive Joule's law by using the Lorentz force law, $\overline{f} = \rho\overline{E}$, and assuming an average constant drifting velocity $\overline{v}$ due to collision of the conduction electrons.

P1.3.14

Consider the simple wave solution

$$\overline{E} = \hat{x}E_0 \cos(kz - \omega t) \qquad \text{(P1.3.14.1}a\text{)}$$

$$\overline{H} = \hat{y}H_0 \cos(kz - \omega t) \qquad \text{(P1.3.14.1}b\text{)}$$

where $H_0 = E_0/\eta_o$ and $\eta_o = \sqrt{\mu_o/\epsilon_o}$ is the characteristic impedance of free space. Substituting (P1.3.14.1) in (1.3.5) to show that Poynting's theorem is satisfied. Derive the associated Lorentz force.

P1.3.15

Use Maxwell's equations to show that for $\overline{J} = 0$ and $\rho = 0$,

$$\frac{\partial}{\partial t}(\overline{D} \times \overline{B}) + \nabla \cdot \left(W\overline{\overline{I}} - \overline{D}\,\overline{E} - \overline{B}\,\overline{H}\right) = 0$$

where the total stored energy density $W = \left(\overline{D} \cdot \overline{E} + \overline{B} \cdot \overline{H}\right)/2$. Consider $\overline{D} = \epsilon_o\overline{E}$ and $\overline{B} = \mu_o\overline{H}$ and use index notation.

1.4 Hertzian Waves

A. Hertzian Dipole

A Hertzian dipole is made of two opposite charges $\pm q$ separated by an infinitesimally small distance ℓ. The dipole moment $p = q\ell$ has an angular frequency ω such that each point charge changes from $+q$ to $-q$ and vice versa in a period of $2\pi/\omega$. Mathematically, p is defined as the product of $\ell \to 0$ and $q \to \infty$ such that p is a constant. Assume that the two charges are situated at $z = \pm\ell/2$ on the z-axis [Fig. 1.4.1]. Hertz solved for all the electromagnetic fields with the use of a potential function known as the Hertzian potential Π

$$\Pi = \frac{q\ell}{4\pi r}\cos(kr - \omega t) \tag{1.4.1}$$

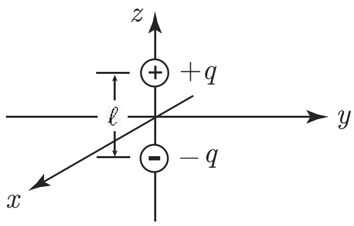


Figure 1.4.1 Hertzian dipole.

The solution to the wave equation for Π that Hertz studied for his Hertzian dipole assumes spherical symmetry. Substituting the Hertian potential Π (1.4.1) into the wave equation in spherical coordinate system, we find

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Pi) - \mu_o\epsilon_o\frac{\partial^2}{\partial t^2}\Pi = 0$$

and obtain the dispersion relation $k^2 = \omega^2\mu_o\epsilon_o$.

Heinrich Rudolf Hertz (22 February 1857 – 1 January 1894)

Heinrich Rudolf Hertz attended Dresden Polytechnic (1876), University of Munich (1877), and Berlin Academy (1878–80). He studied under Professors Hermann von Helmholtz and Gustav Kirchhoff, and his doctoral thesis was on Electromagnetic Induction in Rotating Conductors. He was employed as an Assistant to Helmholtz (1880–83) at the Berlin Academy, Privatdozent at the University of Kiel (1883–85), Professor of Physics at the Karlsruhe Technische Hochschule (1885–89), and University of Bonn (1889–94).

To derive the electromagnetic fields $\overline{E}$ and $\overline{H}$, we write $\overline{\Pi} = \hat{z}\Pi$ and define a vector potential $\overline{A}$ and a scalar potential Φ such that

$$\overline{A} = \mu_o \frac{\partial \overline{\Pi}}{\partial t} \tag{1.4.2}$$

$$\Phi = -\frac{1}{\epsilon_o} \nabla \cdot \overline{\Pi} \tag{1.4.3}$$

In terms of Φ and $\overline{A}$, the magnetic field $\overline{H}$ and the electric field $\overline{E}$ are

$$\overline{H} = \frac{1}{\mu_o} \nabla \times \overline{A} \tag{1.4.4}$$

$$\overline{E} = -\frac{\partial \overline{A}}{\partial t} - \nabla \Phi \tag{1.4.5}$$

Notice that (1.4.4) satisfies Gauss' law of $\nabla \cdot \overline{B} = 0$ and (1.4.5) follows from Faraday's law.

It is seen from (1.4.2) and (1.4.3) that

$$\nabla \cdot \overline{A} + \mu_o \epsilon_o \frac{\partial \Phi}{\partial t} = 0 \tag{1.4.6}$$

which is known as the Lorenz gauge condition relating the scalar and vector potentials. Making use of (1.4.4), (1.4.5), and (1.4.6), we can derive from Ampère's law and Gauss' law of $\nabla \cdot \overline{D} = \rho$ the following inhomogeneous Helmholtz equations:

$$\nabla^2 \overline{A} - \mu_o \epsilon_o \frac{\partial^2}{\partial t^2} \overline{A} = -\mu_o \overline{J} \tag{1.4.7}$$

$$\nabla^2 \Phi - \mu_o \epsilon_o \frac{\partial^2}{\partial t^2} \Phi = -\rho / \epsilon_o \tag{1.4.8}$$

The Hertzian potential provides a solution to the above equations.

Ludvig Valentin Lorenz (18 January 1829 – 9 June 1891)

Ludvig Lorenz graduated from the Technical University of Denmark and taught at the Danish Military Academy. The Lorenz gauge condition and the retarded potentials were contained in the article 'On the Identity of the Vibrations of Light with Electrical Currents', published in the Philosophical Magazine and Journal of Science in July–December 1867. In 1869, Lorenz arrived at the result of a dielectric mixing formula, which was also obtained by H. A. Lorentz in 1878, now known as the 'Lorenz-Lorentz' formula.

Example 1.4.1

In spherical coordinates, the unit vectors are [Fig. E1.4.1.1]

$$\hat{r} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta$$
$$\hat{\theta} = \hat{x}\cos\theta\cos\phi + \hat{y}\cos\theta\sin\phi - \hat{z}\sin\theta$$
$$\hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi$$
$$\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$$

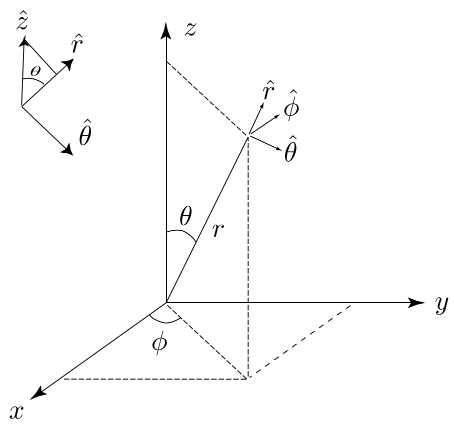


Figure E1.4.1.1 Unit vectors in spherical coordinates.

The vector del operators in spherical coordinate system are

$$\nabla\Phi = \hat{r}\frac{\partial\Phi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial\Phi}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}$$

$$\nabla\cdot\overline{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}A_\phi$$

$$\nabla\times\overline{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial^2}{\partial r^2}\left[r\Phi\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial\Phi}{\partial\theta}\right] + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$

— End of Example 1.4.1 —

B. Electric and Magnetic Fields

The magnetic field $\overline{H}$ is obtained from (1.4.7) with

$$\overline{A} = \mu_o \frac{\partial \overline{\Pi}}{\partial t} = \hat{z}\mu_o \frac{\partial \Pi}{\partial t} = (\hat{r}\cos\theta - \hat{\theta}\sin\theta)\frac{\omega \mu_o q\ell}{4\pi r}\sin(kr - \omega t)$$

$$\begin{aligned}
\overline{H} &= \frac{1}{\mu_o}\nabla \times \overline{A} = \frac{1}{\mu_o r^2 \sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & 0 \end{vmatrix} \\
&= \hat{\phi}\frac{1}{\mu_o r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial}{\partial \theta}A_r\right] \\
&= \hat{\phi}\frac{\omega k q\ell}{4\pi r}\sin\theta\left[\frac{1}{kr}\sin(kr-\omega t) - \cos(kr - \omega t)\right]
\end{aligned} \tag{1.4.9}$$

To obtain the electric field $\overline{E}$ from (1.4.8), noticing that $\partial r/\partial z = z/r = \cos\theta$. We find

$$\Phi = -\frac{1}{\epsilon_o}\frac{\partial \Pi}{\partial z} = \frac{k\ q\ell}{4\pi\epsilon_o r}\cos\theta\left[\frac{1}{kr}\cos(kr-\omega t) + \sin(kr - \omega t)\right]$$

$$\begin{aligned}
\overline{E} &= -\frac{\partial \overline{A}}{\partial t} - \nabla\Phi \\
&= (\hat{r}\cos\theta - \hat{\theta}\sin\theta)\frac{\omega^2\mu_o q\ell}{4\pi r}\cos(kr-\omega t) - \left[\hat{r}\frac{\partial \Phi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right] \\
&= \frac{k^2 q\ell}{4\pi\epsilon_o r}\Bigg\{\hat{r}2\cos\theta\left[\frac{1}{kr}\sin(kr-\omega t) + \frac{1}{k^2r^2}\cos(kr-\omega t)\right] \\
&\qquad + \hat{\theta}\sin\theta\left[\frac{1}{kr}\sin(kr-\omega t) + (\frac{1}{k^2r^2} - 1)\cos(kr - \omega t)\right]\Bigg\}
\end{aligned} \tag{1.4.10}$$

Consider the following special cases:

Case A) When $kr \gg 1$, we are in the far field as $r \gg \lambda/2\pi$; or at a fixed r, the frequency $\omega = ck \gg c/r$. We only keep terms of the order of $1/r$. The field vectors are

$$\overline{E} = -\hat{\theta}\frac{k^2 q\ell}{4\pi\epsilon_o r}\sin\theta\cos(kr - \omega t) \tag{1.4.11}$$

$$\overline{H} = -\hat{\phi}\frac{\omega k q\ell}{4\pi r}\sin\theta\cos(kr-\omega t) \tag{1.4.12}$$

It is seen that both $\overline{H}$ and $\overline{E}$ are tangent to the surface of a large sphere with radius r. The field vectors $\overline{H}$ and $\overline{E}$ are perpendicular to each other. As a function of θ, the magnitudes of both the electric and magnetic fields are proportional to $\sin\theta$. We plot the radiation field pattern in Fig. 1.4.2. The length E is proportional to the magnitude of the electric field in the direction θ.

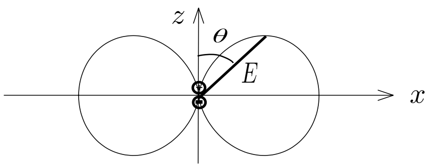


Figure 1.4.2 Radiation field pattern.

Case B) For static fields when $\omega = 0$, $k = \omega/c = 0$, we find

$$\overline{E} = \frac{q\ell}{4\pi\epsilon_o r^3}(\hat{r}2\cos\theta + \hat{\theta}\sin\theta), \qquad \overline{H} = 0 \tag{1.4.13}$$

There is only electric field for a static dipole.

Case C) In the immediate neighborhood of the dipole, $kr \to 0$. Keeping terms of the orders $1/r^2$, the magnetic field vector is

$$\overline{H} = -\hat{\phi}\frac{\omega q\ell}{4\pi r^2}\sin\theta\sin\omega t = \hat{\phi}\frac{d(q\cos\omega t)}{dt}\frac{\ell}{4\pi r^2}\sin\theta = \hat{\phi}\frac{I\ell}{4\pi r^2}\sin\theta \tag{1.4.14}$$

This corresponds to the field produced by an element of length ℓ carrying current I along the z axis, and is known as the Biot-Savart law.

Jean-Baptiste Biot (21 April 1774 – 3 February 1862), professor of mathematical physics at the Collège de France since 1800, reported experiments with his assistant **Felix Savart** (30 June 1791 – 16 March 1841) following Orsted's discovery in April 1820 to the Académie des Sciences in October 1820 which led to the Bior-Savart law. Savart became Professor at the Collège de France in 1836.

Example 1.4.2

Apply the Biot-Savart law (1.4.14) to determine the magnetic field of an infinitely long wire at a distance ρ from the wire. We place the observation point at (ρ, z), let $\ell = dz'$, and integrate (1.4.14) to obtain [Fig. E1.4.2.1]

$$\overline{H} = \hat{\phi}\frac{1}{4\pi}\int_{-\infty}^{+\infty} dz' \frac{I\sin\theta}{z'^2+\rho^2} \tag{E1.4.2.1}$$

The integration can be carried out with the substitution $z' = -\rho\cot\theta$. We find $dz' = \rho d\theta/\sin^2\theta$, $z'^2+\rho^2 = \rho^2/\sin^2\theta$, and (E1.4.2.1) becomes

$$\overline{H} = \hat{\phi}\frac{1}{4\pi}\int_0^{\pi} d\theta \frac{I\sin\theta}{\rho} = \hat{\phi}\frac{I}{2\pi\rho}$$

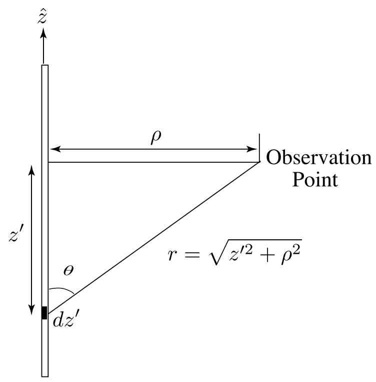


Figure E1.4.2.1 Integration of current elements in an infinitely long wire.

The above result can also be obtained by applying Stokes' theorem to Ampère's law $\nabla \times \overline{H} = \overline{J}$.

$$\oint_C d\bar{l}\cdot\overline{H} = \iint d\overline{S}\cdot\overline{J} = I$$

The integration path for the line integral is a circle of radius ρ around the line source whose area integral gives rise to the current I. The result is $2\pi\rho H_\phi = I$.

— End of Example 1.4.2 —

C. Electric Field Pattern

To study and sketch the electric and magnetic field lines, Hertz introduced a parameter Q in terms of a radial distance $\rho = r\sin\theta$ in the cylindrical coordinate system. We have

$$\begin{aligned}
Q &= \rho\frac{\partial\Pi}{\partial\rho} = \rho\frac{\partial r}{\partial\rho}\frac{\partial\Pi}{\partial r} \\
&= r\sin^2\theta\frac{\partial}{\partial r}\left[\frac{q\ell}{4\pi r}\cos(kr-\omega t)\right] \\
&= \frac{kq\ell}{4\pi}\sin^2\theta\left[-\sin(kr-\omega t)-\frac{1}{kr}\cos(kr-\omega t)\right] \\
\nabla Q &= \hat{r}\frac{\partial Q}{\partial r}+\hat{\theta}\frac{1}{r}\frac{\partial Q}{\partial\theta} \\
&= \hat{r}\frac{k^2q\ell}{4\pi}\sin^2\theta\left[\frac{1}{kr}\sin(kr-\omega t)+(\frac{1}{k^2r^2}-1)\cos(kr-\omega t)\right] \\
&\quad +\hat{\theta}\frac{kq\ell}{2\pi r}\sin\theta\cos\theta\left[-\sin(kr-\omega t)-\frac{1}{kr}\cos(kr-\omega t)\right]
\end{aligned}$$

which is the product of two factors, one depends only on θ, and the other on r and t. From (1.4.9) and (1.4.10), we find

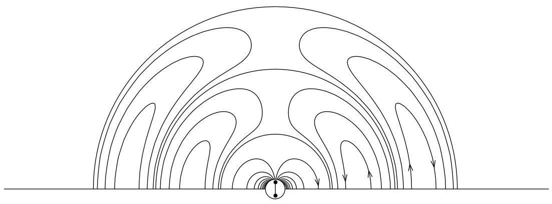
$$\overline{H} = \hat{\phi}\frac{\omega kq\ell}{4\pi r}\sin\theta\left[\frac{1}{kr}\sin(kr-\omega t)-\cos(kr-\omega t)\right] \tag{1.4.15}$$

$$\begin{aligned}
\overline{E} = \frac{k^2q\ell}{4\pi\epsilon_o r}\Bigg\{&\hat{r}2\cos\theta\left[\frac{1}{kr}\sin(kr-\omega t)+\frac{1}{k^2r^2}\cos(kr-\omega t)\right] \\
&+\hat{\theta}\sin\theta\left[\frac{1}{kr}\sin(kr-\omega t)+(\frac{1}{k^2r^2}-1)\cos(kr-\omega t)\right]\Bigg\}
\end{aligned} \tag{1.4.16}$$

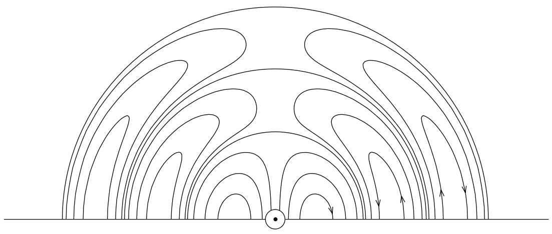
Thus in terms of Q,

$$\begin{aligned}
\overline{H} &= -\hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial t}Q \\
\overline{E} &= \frac{1}{\epsilon_o\rho}\hat{\phi}\times\nabla Q
\end{aligned}$$

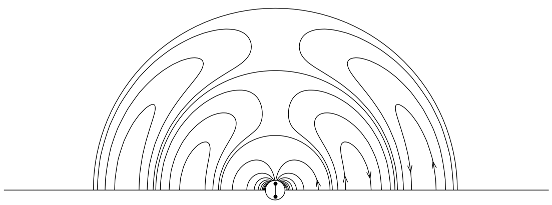
The electric field lines on any ρ-z plane are seen to follow the intersection of $Q=$ constant surfaces with the ρ-z plane. In Fig. 1.4.3, we plot the electric field lines at different times.



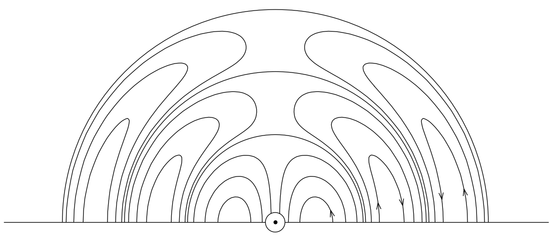
$\omega t = 0$



$\omega t = \pi/2$



$\omega t = \pi$



$\omega t = 3\pi/2$

Figure 1.4.3 Electric field patterns.

EXAMPLE 1.4.3

Consider the radiation field zone when $kr \gg 1$ and

$$Q = -\frac{kq\ell}{4\pi}\sin^2\theta\sin(kr-\omega t)$$

Construct three constant Q surfaces at $\omega t = -\pi/2$ (or $3\pi/2$) and indicate the electric field line directions.

ANSWER: Consider

$$\sin^2\theta\cos(kr) = c$$

We sketch the three cases of $c = 0,\ \frac{1}{2},\ 1$ in Fig. E1.4.3.1.

$$\begin{array}{lll}
\text{For} \quad c=0, & kr = 2m\pi \pm \frac{\pi}{2} & \\
\text{For} \quad c=\frac{1}{2}, & kr = 2m\pi & \text{for } \theta = \pm\frac{\pi}{4} \\
 & kr = 2m\pi \pm \frac{\pi}{3} & \text{for } \theta = \frac{\pi}{2} \\
\text{For} \quad c=1, & \theta = \frac{\pi}{2} \quad \text{and} \quad kr = 2m\pi. &
\end{array}$$

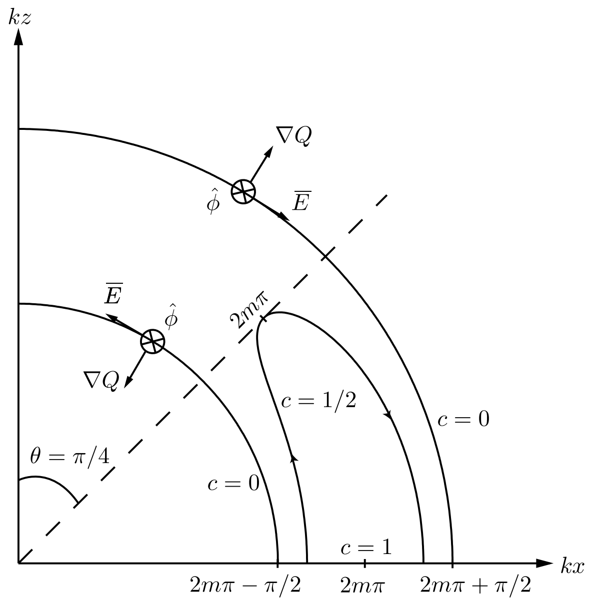


Figure E1.4.3.1 Radiation field plot.

— END OF EXAMPLE 1.4.3 —

Example 1.4.4

In the radiation far field of a Hertzian dipole with $kr \gg 1$, the electric and magnetic fields are

$$\overline{E} = -\hat{\theta}\eta\frac{\omega qk\ell}{4\pi r}\sin\theta\cos(kr-\omega t) \tag{E1.4.4.1}$$

$$\overline{H} = -\hat{\phi}\frac{\omega qk\ell}{4\pi r}\sin\theta\cos(kr-\omega t) \tag{E1.4.4.2}$$

It is seen that both $\overline{H}$ and $\overline{E}$ are tangent to the surface of a large sphere with radius r. The field vectors $\overline{H}$ and $\overline{E}$ are perpendicular to each other and their magnitudes are related by $\eta = (\mu_o/\epsilon_o)^{1/2}$.

To investigate the power and energy issues, Hertz invoked Poynting's theorem. Poynting's power density vector $\overline{S}$ for fields for $kr \gg 1$ is

$$\overline{S} = \overline{E}\times\overline{H} = \hat{r}\eta\left(\frac{\omega qk\ell}{4\pi r}\right)^2\sin^2\theta\cos^2(kr-\omega t)$$

which is seen to be pointing in the $\hat{r}$-direction away from the large sphere. We now calculate the time-average power density

$$<\overline{S}> = \frac{1}{2\pi}\int_0^{2\pi}d(\omega t)\;\overline{E}\times\overline{H} = \hat{r}\;\frac{\eta}{2}\left(\frac{\omega qk\ell}{4\pi r}\right)^2\sin^2\theta$$

The radiation pattern is shown in Fig. E1.4.4.1. The length P is proportional to the magnitude of the radiated power in the direction θ.

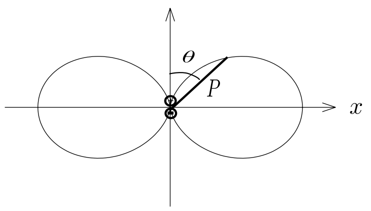


Figure E1.4.4.1 Radiation power pattern.

Integrating the $\hat{r}$ directed power over the surface of a sphere of radius r gives

$$P = \oint\!\!\oint dS\,\hat{r}\cdot<\overline{S}>= \frac{4\pi\eta}{3}\left(\frac{\omega qk\ell}{4\pi}\right)^2 = \frac{\eta}{12\pi}(\omega qk\ell)^2 = 10(\omega qk\ell)^2 \tag{E1.4.4.3}$$

Notice that the total time-average power leaving the dipole source can be calculated with a spherical surface of any radius r, and yields the same result.

— End of Example 1.4.4 —

Poynting's Power Vector for Hertzian Waves

For a Hertzian dipole, the magnetic field $\overline{H}$ and the electric field $\overline{E}$ are

$$\overline{H} = \hat{\phi}\frac{\omega kq\ell}{4\pi r}\sin\theta\left[\frac{1}{kr}\sin(kr-\omega t) - \cos(kr-\omega t)\right] \tag{1.4.17}$$

$$\begin{aligned}\overline{E} = \eta\frac{\omega kq\ell}{4\pi r}\Big\{&\hat{r}2\cos\theta\left[\frac{1}{kr}\sin(kr-\omega t) + \frac{1}{k^2r^2}\cos(kr-\omega t)\right]\\ &+\hat{\theta}\sin\theta\left[\frac{1}{kr}\sin(kr-\omega t) + (\frac{1}{k^2r^2}-1)\cos(kr-\omega t)\right]\Big\}\end{aligned} \tag{1.4.18}$$

The Poynting vector power density is

$$\begin{aligned}\overline{S} &= \overline{E}\times\overline{H}\\ &= \eta\left(\frac{\omega kq\ell}{4\pi\epsilon_o r}\right)^2\Big\{-\hat{\theta}\ \sin 2\theta\left[(\frac{1}{k^3r^3}-\frac{1}{kr})\frac{1}{2}\sin 2(kr-\omega t)\right.\\ &\qquad\qquad\left.-\frac{1}{k^2r^2}\cos 2(kr-\omega t)\right]\\ &\qquad +\hat{r}\ \sin^2\theta\left[(\frac{1}{k^3r^3}-\frac{2}{kr})\frac{1}{2}\sin 2(kr-\omega t)\right.\\ &\qquad\qquad\left.-\frac{1}{k^2r^2}\cos 2(kr-\omega t)+\cos^2(kr-\omega t)\right]\Big\}\end{aligned} \tag{1.4.19}$$

We now calculate the time-average of $\overline{S}$. Notice that the time average of $\sin 2(kr-\omega t)$ and $2\cos(kr-\omega t)$ is zero, and the time average of either $\sin^2(kr-\omega t)$ or $\cos^2(kr-\omega t)$ is $1/2$. The above expression, after integration, is equal to the time-average power density at any point $\overline{r}$.

$$<\overline{S}> = \frac{1}{2\pi}\int_0^{2\pi} d(\omega t)\overline{E}\times\overline{H} = \hat{r}\ \frac{\eta}{2}\left(\frac{\omega qk\ell}{4\pi r}\right)^2\sin^2\theta$$

Integrating the $\hat{r}$ directed power over the surface of a sphere of radius r gives [Fig. 1.4.4]

$$\begin{aligned}P &= \oiint dS\,\hat{r}\cdot <\overline{S}> = \int_0^{2\pi} d\phi\int_0^{\pi} d\theta\ r^2\sin\theta\left[\frac{\eta}{2}\left(\frac{\omega qk\ell}{4\pi r}\right)^2\sin^2\theta\right]\\ &= \int_0^{\pi} d\theta\ 2\pi r^2\sin^3\theta\left[\frac{\omega k^3}{2\epsilon_o}\left(\frac{q\ell}{4\pi r}\right)^2\right] = \frac{4\pi\eta}{3}\left(\frac{\omega qk\ell}{4\pi}\right)^2 = \frac{\eta}{12\pi}(\omega qk\ell)^2\end{aligned}$$

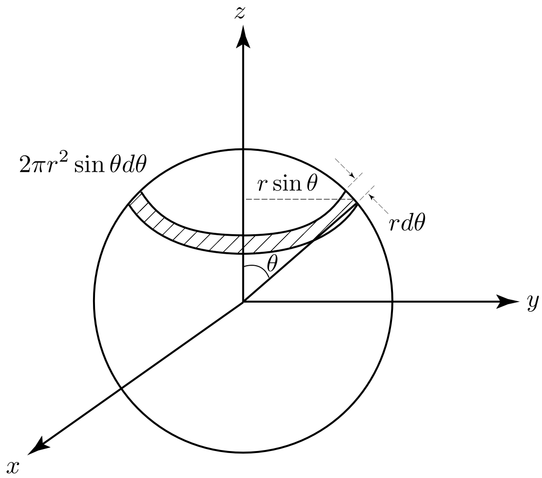


Figure 1.4.4 Integration geometry for time-average power density.

where $2\pi r^2 \sin\theta$ is the ribbon-like surface element to be integrated from $\theta = 0$ to $\theta = \pi$. Notice that the total time-average power leaving the dipole source obtained by calculating with a spherical surface with any radius r is the same.

EXAMPLE 1.4.5

For a Hertzian dipole, the time average of $\overline{E}\cdot\overline{J}$ is, with $\overline{J} = \hat{z}\omega q\ell \sin\omega t \delta(\overline{r})$, and as $r \to 0$,

$$
\begin{aligned}
<\overline{E}\cdot\overline{J}> = \frac{\eta k}{2}\frac{(\omega q\ell)^2}{4\pi r}\bigg\{ & 2\cos^2\theta\left[-\frac{1}{kr}\cos kr + \frac{1}{k^2r^2}\sin kr\right] \\
& - \sin^2\theta\left[-\frac{1}{kr}\cos kr + (\frac{1}{k^2r^2} - 1)\sin kr\right]\bigg\}\delta(\overline{r}) \\
= \frac{\eta k}{2}\frac{(\omega q\ell)^2}{4\pi r}\bigg\{ & \cos^2\theta\left[-\frac{3}{kr}(1-\frac{k^2r^2}{2}+...)+(\frac{3}{k^2r^2}-1)(kr-\frac{k^3r^3}{6}+...)\right] \\
& - \left[-\frac{1}{kr}(1-\frac{k^2r^2}{2}+...)+(\frac{1}{k^2r^2}-1)(kr-\frac{k^3r^3}{6}+...)\right]\bigg\}\delta(\overline{r}) \\
= \frac{\eta k}{2}\frac{(\omega q\ell)^2}{4\pi r} & \left\{\left[\frac{2kr}{3}\right]\right\}\delta(\overline{r}) = \frac{\eta}{12\pi}(\omega qk\ell)^2\delta(\overline{r})
\end{aligned}
$$

where we have Talor expanded $\sin kr$ and $\cos kr$ around $r = 0$.

— END OF EXAMPLE 1.4.5 —

Example 1.4.6

For a dipole moment $\overline{p} = q\overline{\ell}$, the magnetic and electric fields are

$$\overline{H} = \frac{\omega k}{4\pi r}(\overline{p} \times \hat{r})\left[\frac{1}{kr}\sin(kr - \omega t) - \cos(kr - \omega t)\right] \tag{E1.4.6.1}$$

$$\begin{aligned}\overline{E} = \frac{k^2}{4\pi\epsilon_o r}\Bigg\{&[(\overline{p} \times \hat{r}) \times \hat{r} + 2\hat{r}(\hat{r} \cdot \overline{p})]\left[\frac{1}{k^2r^2}\cos(kr - \omega t) + \frac{1}{kr}\sin(kr - \omega t)\right] \\ &- [(\overline{p} \times \hat{r}) \times \hat{r}]\cos(kr - \omega t)\Bigg\}\end{aligned} \tag{E1.4.6.2}$$

Applying the Biot-Savart law to derive the magnetic field of an infinitely long wire, we first make use of the first term in (E1.4.6.1) with the same approximation as for (1.4.14) to obtain

$$\overline{H} \approx \frac{\omega k}{4\pi r}(\overline{p} \times \hat{r})\frac{1}{kr}\sin(-\omega t) = \frac{1}{4\pi r^2}\left[\frac{d(q\cos\omega t)}{dt}\overline{\ell} \times \hat{r}\right] = \frac{1}{4\pi r^3}(I\overline{\ell} \times \overline{r})$$

where I is the current and $\overline{\ell}$ denotes the direction and length of the current element. The vector $\overline{r} = \hat{\rho}\rho + \hat{z}z'$ points from the source element to the observation point.

— End of Example 1.4.6 —

Example 1.4.7

Consider the scattering of electromagnetic waves by particles of size much smaller than a wavelength, such as sunlight by air molecules. Model the particle as a small sphere of radius a with an induced dipole moment proportional to a and the intensity of the illuminating electric field,

$$ql = 4\pi\epsilon_o a^3\left(\frac{\epsilon_a - \epsilon_o}{\epsilon_a + 2\epsilon_o}\right)E_0$$

where ϵ_a is the dielectric constant of the air molecule and E_0 is the incident electric field intensity. The total power P_s re-radiated by the particle acting as a Hertzian dipole is, by virtue of (E1.4.4.3)

$$P_s = \frac{\eta}{12\pi}(\omega qk\ell)^2 = \frac{4\pi}{3\eta}\left(\frac{\epsilon_a - \epsilon_o}{\epsilon_a + 2\epsilon_o}\right)^2 k^4a^6E_0^2$$

The scattering cross-section is defined as

$$\sigma_s = \frac{P_s}{E_0^2/2\eta} = \frac{8\pi}{3}\left(\frac{\epsilon_a - \epsilon_o}{\epsilon_a + 2\epsilon_o}\right)^2 k^4a^6$$

This is known as the result of Rayleigh scattering, which has been used to explain why the sky is blue.

— End of Example 1.4.7 —

John William Strutt (Lord Rayleigh) (12 November 1842 – 30 June 1919) entered Trinity College, Cambridge, in 1861, and graduated in 1865. His first paper in 1865 was on Maxwell's electromagnetic theory. His theory of scattering (1871) provided the explanation of why the sky is blue. From 1879–1884 he succeeded Maxwell as the second Cavendish professor of experimental physics at Cambridge.

Problems

P1.4.1

The magnetic field $\overline{H}$ and electric field $\overline{E}$ of a Hertzian dipole at very large distances $(kr \gg 1)$ are

$$\overline{H} = -\hat{\phi}\frac{\omega k q\ell}{4\pi r}\sin\theta\cos(kr - \omega t)$$
$$\overline{E} = -\hat{\theta}\frac{k^2 q\ell}{4\pi\epsilon_o r}\sin\theta\cos(kr - \omega t)$$

(a) Find the Poynting's power density vector $\overline{S}$ as a function of time. What is the time-averaged power density vector $\langle\overline{S}\rangle$?
(b) By integrating the Poynting vector over the surface of a sphere of radius r , find the time-averaged power P radiated by the Hertzian dipole.
(c) The amplitude of the current in the Hertzian dipole is $I_o = \omega q$. By using $P = \frac{1}{2}I_o^2 R_{rad}$, find the radiation resistance R_{rad} of the Hertzian dipole.
(d) A radio station is 15 km away from a city. The transmitting antenna tower may be modeled as a Hertzian dipole antenna of dipole moment $q\ell$. To maintain the FCC standard of 25 mV/m field strength in the city, how much radiation power P must be provided?

P1.4.2

Determine the static electric field for a Hertzian dipole oriented in a general direction $\overline{p} = \hat{x}p_x + \hat{y}p_y + \hat{z}p_z$, with dipole moment $p = q\ell$.

P1.4.3

Sun navigation was first observed in 1911. It was found that some species of ants, horseshoe crabs, honeybees, etc., are sensitive to polarized light. These creatures can navigate as long as there is a small patch of blue sky. The sky polarization depends upon the angle ϕ between the sun's rays to a particular point in the sky and an observer's line of sight to the same point. The sunlight, which is unpolarized, or randomly polarized, excites air molecules which behave like small dipole antennas when irradiated by the incident electric fields of the sunlight. The scattered electric field $\overline{E}_s$ for each excited dipole antenna is linearly polarized in planes perpendicular to the sunlight path; and looking along the sun ray path the scattered wave is unpolarized, or randomly polarized.

At sunset, if an ant looks directly at the sun $(\phi = 0)$, what is the polarization? What is the polarization if the ant looks at the zenith $(\phi = 90°)$ perpendicular to the sun ray path? Show that the sky light appears to be partially linearly polarized when it looks at other parts of the sky [*Scientific American*, July 1955].

P1.4.4

(a) For the electromagnetic field solution of a Hertzian dipole with dipole moment $p = ql$, let $k \to 0$ and show that $\overline{H} = 0$. Determine the electric field $\overline{E}$ of a static dipole with $k = 0$.

(b) Consider the Rayleigh scattering of electromagnetic waves by particles of size much smaller than a wavelength, such as sunlight by air molecules. Model the particle when illuminated with a light wave as an induced Hertzian dipole with dipole moment p, which is proportional to the incident field amplitude E_o, and can be expressed as $p = p_o E_o$. Find the total power P_s re-radiated by the particle. Find the scattering cross-section defined by $2\eta P/E_0^2$. The above result is usually used to explain why the sky is blue.

P1.4.5

Why is sky blue (but why isn't it purple?) ?

P1.4.6

(a) Consider an optical fiber with cross section area A. The electromagnetic wave guided inside the fiber is scattered by the atoms and the molecules making up the fiber. Since the sizes of the scattering particles are much smaller than the guided light wavelength, the process can again be described by Rayleigh scattering. Assume $\epsilon = 2\epsilon_o$, show that the scattered power from each particle is $\frac{\pi}{12\eta} k^4 a^6 E_o^2$.

(b) Assume the guided light has intensity E_0, wavelength 10^{-6} m, and particle radius $a = 10^{-10}$ m. Find the guided power flow in watts and the total scattered power of a fiber with a length of 1 km in terms of the density of the particles inside the fiber N. Calculate the ratio of the scattered power to the guided power.

(c) Assume the particle density is approximately $3/4\pi a^3$ per m^3, estimate, with the numbers given above, the fiber loss per kilometer (in dB/km) due to the Rayleigh scattering.

P1.4.7

Two Hertzian dipole antennas are located at $(0, 0, 0)$ and $(0, d, 0)$ with dipole moments $p_1 = q_1 l$ and $p_2 = q_2 l$ current densities:

$$\overline{J}_1 = \hat{z} I_1 \delta(x)\delta(y)\delta(z) \qquad \text{and} \qquad \overline{J}_2 = \hat{x} I_2 \delta(x)\delta(y-d)\delta(z)$$

as shown in Figure P1.4.7.1. The two in phase dipoles are oriented in z and x directions respectively.

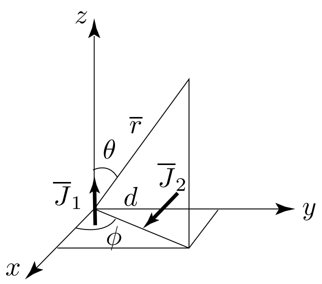


Figure P1.4.7.1

(a) For the x-oriented dipole, the far field ($r \gg 1$) expression of $\overline{E}$ on the yz-plane is:

$$\overline{E}_2 = \hat{x}\frac{k^2 q_2 \ell}{4\pi r\epsilon_0}\cos(k\sqrt{x^2+(y-d)^2+z^2}-\omega t)$$

Show that as $d \ll \sqrt{x^2+y^2+z^2} = r$

$$\overline{E}_2 = \hat{x}\frac{k^2 q_2 \ell}{4\pi r\epsilon_0}\cos(kr - kd\sin\theta - \omega t)$$

(b) Find the total far field $\overline{E}$ on the yz-plane.

(c) Let q_1 and q_2 be real and positive. On the yz-plane, if the far field $\overline{E}$ for $\theta = 45°$ is circularly polarized,

(i) Find the minimum d in terms of λ.

(ii) What is the ratio of q_1/q_2 ?

(iii) Specify the handness of the circularly polarized field.

P1.4.8

The Biot-Savart law states that the magnetic field at (r,θ,ϕ) produced by an element of length ℓ at the origin carrying current I along the z axis is

$$\overline{B} = \hat{\phi}\frac{\mu_o I\ell}{4\pi r^2}\sin\theta$$

Consider a wire with infinite length carrying current I in the direction of z. Use the Biot-Savart law to show that the magnetic field produced by the wire is

$$\overline{B} = \hat{\phi}\frac{\mu_o I}{2\pi\rho}$$

where ρ is the distance from the wire. Apply Stokes' theorem to Ampère's law without the displacement term, find $\overline{B}$ and confirm the above result.

For a high-voltage transmission line carrying current $I = 1\,\text{kA}$, find the magnetic field strength 10 meters away from the wire, and compare with the earth magnetic field strength which is approximately 5×10^{-5} Tesla.

1.5 Constitutive Relations

Maxwell's equations govern the behavior of electric field vectors $\overline{D}$ and $\overline{E}$, magnetic field vectors $\overline{B}$ and $\overline{H}$, and source fields $\overline{J}$ and ρ.

$$\nabla \times \overline{H} = \frac{\partial}{\partial t}\overline{D} + \overline{J} \tag{1.5.1}$$

$$\nabla \times \overline{E} = -\frac{\partial}{\partial t}\overline{B} \tag{1.5.2}$$

$$\nabla \cdot \overline{D} = \rho \tag{1.5.3}$$

$$\nabla \cdot \overline{B} = 0 \tag{1.5.4}$$

$$\nabla \cdot \overline{J} = -\frac{\partial}{\partial t}\rho \tag{1.5.5}$$

Equation (1.5.3) can be derived by taking the divergence of (1.5.1) and introducing (1.5.5). Similarly, Eq. (1.5.4) is derivable from divergence of (1.5.2). Giving sources $\overline{J}$ and ρ satisfying (1.5.5), we have a total of six independent scalar equations, three from (1.5.1) and three from (1.5.2), to determine 12 components of the field vectors $\overline{D}$, $\overline{E}$, $\overline{H}$, and $\overline{B}$. Thus we need six more scalar equations.These are the constitutive relations, which provide a mathematical description of the electromagnetic properties of all media.

I proposed that we call them *bianisotropic media* [Kong, 1968] when material media are characterized by the following constitutive relations:

$$\overline{D} = \overline{\overline{\epsilon}} \cdot \overline{E} + \overline{\overline{\xi}} \cdot \overline{H} \tag{1.5.6}$$

$$\overline{B} = \overline{\overline{\zeta}} \cdot \overline{E} + \overline{\overline{\mu}} \cdot \overline{H} \tag{1.5.7}$$

where $\overline{\overline{\epsilon}}$, $\overline{\overline{\mu}}$, $\overline{\overline{\xi}}$, and $\overline{\overline{\zeta}}$ are all 3×3 matrices. Their elements are called *constitutive parameters*. In its most general form, a constitutive parameter can be cast in the form of integro-differential operators. In this section, we discuss special cases of the constitutive relations.

The bianisotropic description of material has fundamental importance from the point of view of relativity. The principle of relativity requires that all physical laws of nature must be characterized by mathematical equations that are form-invariant from one observer to the other. Although the numerical values of the field quantities may vary from one observer to another, the forms of the Maxwell equations in (1.5.1) to (1.5.5) are invariant, and so are the bianisotropic form as expressed in (1.5.6) and (1.5.7) for the constitutive relations.

A. Isotropic Media

For isotropic media, $\overline{\overline{\xi}} = \overline{\overline{\zeta}} = 0$, and $\overline{\overline{\mu}} = \mu \overline{\overline{I}}$ with $\overline{\overline{I}}$ denoting the 3×3 identity matrix. The constitutive relations for an isotropic medium can be written simply as

$$\overline{D} = \epsilon \overline{E} \qquad \text{where } \epsilon = \text{permittivity} \tag{1.5.8}$$

$$\overline{B} = \mu \overline{H} \qquad \text{where } \mu = \text{permeability} \tag{1.5.9}$$

By isotropy we mean that the field vector $\overline{E}$ is parallel to $\overline{D}$ and the field vector $\overline{H}$ is parallel to $\overline{B}$. In free space void of any matter, $\mu = \mu_o$ and $\epsilon = \epsilon_o$,

$$\mu_o = 4\pi \times 10^{-7} \qquad \text{henry/meter}$$

$$\epsilon_o \approx 8.85 \times 10^{-12} \qquad \text{farad/meter}$$

Inside a material medium, the permittivity ϵ is determined by the electrical properties of the medium and the permeability μ by the magnetic properties of the medium.

EXAMPLE 1.5.1

A dielectric material can be described by a free-space part and a part that is due to the material alone. The material part can be characterized by a polarization vector $\overline{P}$ such that

$$\overline{D} = \epsilon \overline{E} = \epsilon_o \overline{E} + \overline{P} \tag{E1.5.1.1}$$

The polarization $\overline{P}$ symbolizes the electric dipole moment per unit volume of the dielectric material. In the presence of an external electric field, the polarization vector may be caused by induced dipole moments, alignment of the permanent dipole moments of the medium, or migration of ionic charges.

A magnetic material can also be described by a free-space part and a part characterized by a magnetization vector $\overline{M}$ such that

$$\overline{B} = \mu \overline{H} = \mu_o \overline{H} + \mu_o \overline{M} \tag{E1.5.1.2}$$

A medium is diamagnetic if $\mu < \mu_o$ and paramagnetic if $\mu > \mu_o$. Diamagnetism is caused by induced magnetic moments that tend to oppose the externally applied magnetic field. Paramagnetism is due to alignment of magnetic moments. When placed in an inhomogeneous magnetic field, a diamagnetic material tends to move toward regions of weaker magnetic field, and a paramagnetic material toward regions of stronger magnetic field. Ferromagnetism and antiferromagnetism are highly nonlinear effects.

— END OF EXAMPLE 1.5.1 —

B. Anisotropic Media

For anisotropic media, $\overline{\overline{\xi}} = \overline{\overline{\zeta}} = 0$, and the constitutive relations are usually written as

$$\overline{D} = \overline{\overline{\epsilon}} \cdot \overline{E} \qquad \text{where } \overline{\overline{\epsilon}} = \text{permittivity tensor} \tag{1.5.10}$$

$$\overline{B} = \overline{\overline{\mu}} \cdot \overline{H} \qquad \text{where } \overline{\overline{\mu}} = \text{permeability tensor} \tag{1.5.11}$$

The field vector $\overline{E}$ is no longer parallel to $\overline{D}$, and the field vector $\overline{H}$ is no longer parallel to $\overline{B}$. A medium is *electrically anisotropic* if it is described by the permittivity tensor $\overline{\overline{\epsilon}}$ and a scalar permeability μ, and *magnetically anisotropic* if it is described by the permeability tensor $\overline{\overline{\mu}}$ and a scalar permittivity ϵ. Note that a medium can be both electrically and magnetically anisotropic as described by both $\overline{\overline{\epsilon}}$ and $\overline{\overline{\mu}}$ in (1.5.10) and (1.5.11).

Crystals are described in general by symmetric permittivity tensors. There always exists a coordinate transformation that transforms a symmetric matrix into a diagonal matrix. In this coordinate system, called the *principal system,*

$$\overline{\overline{\epsilon}} = \begin{bmatrix} \epsilon_x & 0 & 0 \\ 0 & \epsilon_y & 0 \\ 0 & 0 & \epsilon_z \end{bmatrix} \tag{1.5.12}$$

The three coordinate axes are referred to as the principal axes of the crystal. For cubic crystals, $\epsilon_x = \epsilon_y = \epsilon_z$ and they are isotropic. In tetragonal, hexagonal, and rhombohedral crystals, two of the three parameters are equal. Such crystals are *uniaxial.* Here there is a two-dimensional degeneracy; the principal axis that exhibits this anisotropy is called the *optic axis.* For a uniaxial crystal with

$$\overline{\overline{\epsilon}} = \begin{bmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon_z \end{bmatrix} \tag{1.5.13}$$

the z axis is the optic axis. The crystal is *positive uniaxial* if $\epsilon_z > \epsilon$; it is *negative uniaxial* if $\epsilon_z < \epsilon$. In orthorhombic, monoclinic, and triclinic crystals, all three crystallographic axes are unequal. We have $\epsilon_x \neq \epsilon_y \neq \epsilon_z$, and the medium is *biaxial.*

C. Bianisotropic Media

For isotropic or anisotropic media, the constitutive relations relate the two electric field vectors and the two magnetic field vectors by either a scalar or a tensor. Such media become polarized when placed in an electric field and become magnetized when placed in a magnetic field. A bianisotropic medium provides the cross-coupling between the electric and magnetic fields. When placed in an electric or a magnetic field, a bianisotropic medium becomes both polarized and magnetized. The constitutive relations for a bianisotropic medium take the form

$$\overline{D} = \overline{\overline{\epsilon}} \cdot \overline{E} + \overline{\overline{\xi}} \cdot \overline{H} \tag{1.5.14a}$$

$$\overline{B} = \overline{\overline{\zeta}} \cdot \overline{E} + \overline{\overline{\mu}} \cdot \overline{H} \tag{1.5.14b}$$

where $\overline{D}$ depends on both $\overline{E}$ and $\overline{H}$, and so does $\overline{B}$.

Magnetoelectric Media

Magnetoelectric materials, theoretically predicted by Dzyaloshinskii, and Landau and Lifshitz [1960], were observed experimentally in 1960 by Astrov [1960] in antiferromagnetic chromium oxide. The constitutive relations that Dzyaloshinskii proposed for chromium oxide have the following form:

$$\overline{D} = \begin{bmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon_z \end{bmatrix} \cdot \overline{E} + \begin{bmatrix} \xi & 0 & 0 \\ 0 & \xi & 0 \\ 0 & 0 & \xi_z \end{bmatrix} \cdot \overline{H} \tag{1.5.15a}$$

$$\overline{B} = \begin{bmatrix} \xi & 0 & 0 \\ 0 & \xi & 0 \\ 0 & 0 & \xi_z \end{bmatrix} \cdot \overline{E} + \begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu_z \end{bmatrix} \cdot \overline{H} \tag{1.5.15b}$$

It was then shown by Indenbom [1960] and by Birss [1963] that 58 magnetic crystal classes can exhibit the magnetoelectric effect. Rado [1964] proved that the effect is not restricted to antiferromagnetics; ferromagnetic gallium iron oxide is also magnetoelectric.

Moving Media

Media in motion were the first bianisotropic media to receive attention in electromagnetic theory. In 1888, Wilhelm Röentgen (1845–1923) discovered that a moving dielectric becomes magnetized when it is placed in an electric field. In 1905, H. A. Wilson showed that a moving dielectric in a uniform magnetic field becomes electrically polarized. Almost any medium becomes bianisotropic when it is in motion.

D. Biisotropic Media

Tellegen Media

In 1948, the gyrator was introduced by B. D. H. Tellegen as a new element, in addition to the resistor, the capacitor, the inductor, and the ideal transformer, for describing a network. To realize his new network element, Tellegen conceived of a medium possessing constitutive relations of the form

$$\overline{D} = \epsilon \overline{E} + \tau \overline{H} \tag{1.5.16a}$$

$$\overline{B} = \tau \overline{E} + \mu \overline{H} \tag{1.5.16b}$$

where $\tau^2/\mu\epsilon$ is nearly equal to 1. Tellegen considered that the model of the medium had elements possessing permanent electric and magnetic dipoles parallel or antiparallel to each other, so that an applied electric field that aligns the electric dipoles simultaneously aligns the magnetic dipoles; and a magnetic field that aligns the magnetic dipoles simultaneously aligns the electric dipoles. Tellegen also wrote general constitutive relations (1.5.14) and examined the symmetry properties by energy conservation.

Chiral Media

Chiral media, which include many classes of sugar solutions, amino acids, DNA, and natural substances have the following constitutive relations

$$\overline{D} = \epsilon \overline{E} + \chi \frac{\partial \overline{H}}{\partial t} \tag{1.5.17a}$$

$$\overline{B} = \mu \overline{H} - \chi \frac{\partial \overline{E}}{\partial t} \tag{1.5.17b}$$

where χ is the chiral parameter. Media characterized by the constitutive relations (1.5.16) and (1.5.17) are biisotropic media.

E. Constitutive Matrices

Constitutive relations in the most general form can be written as

$$c\overline{D} = \overline{\overline{P}} \cdot \overline{E} + \overline{\overline{L}} \cdot c\overline{B} \tag{1.5.18a}$$

$$\overline{H} = \overline{\overline{M}} \cdot \overline{E} + \overline{\overline{Q}} \cdot c\overline{B} \tag{1.5.18b}$$

where $c = 3 \times 10^8$ m/s is the velocity of light in vacuum, and $\overline{\overline{P}}$, $\overline{\overline{Q}}$, $\overline{\overline{L}}$, and $\overline{\overline{M}}$ are all 3×3 matrices. Their elements are called *constitutive parameters.* In the definition of the constitutive relations, the constitutive matrices $\overline{\overline{L}}$ and $\overline{\overline{M}}$ relate electric and magnetic fields. When $\overline{\overline{L}}$ and $\overline{\overline{M}}$ are not identically zero, the medium is *bianisotropic.* When there is no coupling between electric and magnetic fields, $\overline{\overline{L}} = \overline{\overline{M}} = 0$ and the medium is *anisotropic.* For an anisotropic medium, if $\overline{\overline{P}} = c\epsilon\overline{\overline{I}}$ and $\overline{\overline{Q}} = (1/c\mu)\overline{\overline{I}}$ with $\overline{\overline{I}}$ denoting the 3×3 unit matrix, the medium is *isotropic.* The reason that we write constitutive relations in the present form is based on relativistic considerations. First, the fields $\overline{E}$ and $c\overline{B}$ form a single tensor in four-dimensional space, and so do $c\overline{D}$ and $\overline{H}$. Second, constitutive relations written in the form (1.5.18) are Lorentz-covariant. These aspects will be discussed in Chapter 8.

Equation (1.5.18) can be rewritten in the form

$$\begin{bmatrix} c\overline{D} \\ \overline{H} \end{bmatrix} = \overline{\overline{C}} \cdot \begin{bmatrix} \overline{E} \\ c\overline{B} \end{bmatrix} \tag{1.5.19a}$$

and $\overline{\overline{C}}$ is a 6×6 constitutive matrix:

$$\overline{\overline{C}} = \begin{bmatrix} \overline{\overline{P}} & \overline{\overline{L}} \\ \overline{\overline{M}} & \overline{\overline{Q}} \end{bmatrix} \tag{1.5.19b}$$

which has the dimension of admittance.

The constitutive matrix $\overline{\overline{C}}$ may be a function of space-time coordinates, thermodynamical and continuum-mechanical variables, or electromagnetic field strengths. According to the functional dependence of $\overline{\overline{C}}$, we can classify the various media as (i) inhomogeneous if $\overline{\overline{C}}$ is a function of space coordinates, (ii) nonstationary if $\overline{\overline{C}}$ is a function of time, (iii) time-dispersive if $\overline{\overline{C}}$ contains time derivatives, (iv) spatially dispersive if $\overline{\overline{C}}$ contains spatial derivatives, (v) nonlinear if $\overline{\overline{C}}$

depends on the electromagnetic field, and so forth. In the general case $\overline{\overline{C}}$ may be a function of integro-differential operators and coupled to fundamental equations of other physical disciplines.

We have defined constitutive relations by expressing $\overline{D}$ and $\overline{H}$ in terms of $\overline{E}$ and $\overline{B}$. We may also express constitutive relations in the form of $\overline{D}$ and $\overline{B}$ as a function of $\overline{E}$ and $\overline{H}$:

$$\begin{bmatrix} \overline{D} \\ \overline{B} \end{bmatrix} = \overline{\overline{C}}_{EH} \cdot \begin{bmatrix} \overline{E} \\ \overline{H} \end{bmatrix} \tag{1.5.20a}$$

where in view of (1.5.14) and (1.5.18),

$$\overline{\overline{C}}_{EH} = \begin{bmatrix} \overline{\overline{\epsilon}} & \overline{\overline{\xi}} \\ \overline{\overline{\zeta}} & \overline{\overline{\mu}} \end{bmatrix} = \frac{1}{c} \begin{bmatrix} \overline{\overline{P}} - \overline{\overline{L}} \cdot \overline{\overline{Q}}^{-1} \cdot \overline{\overline{M}} & \overline{\overline{L}} \cdot \overline{\overline{Q}}^{-1} \\ -\overline{\overline{Q}}^{-1} \cdot \overline{\overline{M}} & \overline{\overline{Q}}^{-1} \end{bmatrix} \tag{1.5.20b}$$

Here $\overline{\overline{C}}_{EH}$ is the constitutive matrix under $\overline{E}\,\overline{H}$ representation.

To express $\overline{E}$ and $\overline{H}$ in terms of $\overline{B}$ and $\overline{D}$, we write

$$\begin{bmatrix} \overline{E} \\ \overline{H} \end{bmatrix} = \overline{\overline{C}}_{DB} \cdot \begin{bmatrix} \overline{D} \\ \overline{B} \end{bmatrix} \tag{1.5.21a}$$

where

$$\overline{\overline{C}}_{DB} = \begin{bmatrix} \overline{\overline{\kappa}} & \overline{\overline{\chi}} \\ \overline{\overline{\gamma}} & \overline{\overline{\nu}} \end{bmatrix} = c \begin{bmatrix} \overline{\overline{P}}^{-1} & -\overline{\overline{P}}^{-1} \cdot \overline{\overline{L}} \\ \overline{\overline{M}} \cdot \overline{\overline{P}}^{-1} & \overline{\overline{Q}} - \overline{\overline{M}} \cdot \overline{\overline{P}}^{-1} \cdot \overline{\overline{L}} \end{bmatrix} \tag{1.5.21b}$$

In terms of parameters in $\overline{EH}$ representation, we find

$$\overline{\overline{\kappa}} = \left[\overline{\overline{\epsilon}} - \overline{\overline{\xi}} \cdot \overline{\overline{\mu}}^{-1} \cdot \overline{\overline{\zeta}}\right]^{-1}$$

$$\overline{\overline{\chi}} = -\overline{\overline{\kappa}} \cdot \overline{\overline{\xi}} \cdot \overline{\overline{\mu}}^{-1}$$

$$\overline{\overline{\nu}} = \left[\overline{\overline{\mu}} - \overline{\overline{\zeta}} \cdot \overline{\overline{\epsilon}}^{-1} \cdot \overline{\overline{\xi}}\right]^{-1}$$

$$\overline{\overline{\gamma}} = -\overline{\overline{\nu}} \cdot \overline{\overline{\zeta}} \cdot \overline{\overline{\epsilon}}^{-1}$$

Here $\overline{\overline{C}}_{DB}$ is the constitutive matrix under $\overline{D}\,\overline{B}$ representation. The other possible construction for expressing $\overline{E}$ and $\overline{B}$ in terms of $\overline{H}$ and $\overline{D}$ is not shown because it will not be needed in later developments.

Problems

P1.5.1

For each of the following constitutive relations, state whether the given medium is

(1) Isotropic/anisotropic/bianisotropic,
(2) Linear/nonlinear,
(3) Spatially/temporally dispersive,
(4) Homogeneous/inhomogeneous.

(a) Cholesteric liquid crystals can be modeled by a spiral structure with constitutive relations given by

$$\overline{D} = \begin{pmatrix} \epsilon(1+\delta\cos Kz) & \epsilon\delta\sin Kz & 0 \\ \epsilon\delta\sin Kz & \epsilon(1-\delta\cos Kz) & 0 \\ 0 & 0 & \epsilon_z \end{pmatrix} \cdot \overline{E}$$

where the spiral direction is along the z axis.

(b) In view of the optical activities in quartz crystals, the constitutive relation for a quartz crystal is phenomenologically described as

$$E_j = \kappa_{ij} D_i + \frac{1}{\mu_o \epsilon_o} G_{ij} \frac{\partial}{\partial t} B_i$$
$$H_j = \frac{1}{\mu_o} B_j - \frac{1}{\mu_o \epsilon_o} G_{ij} \frac{\partial}{\partial t} D_i$$

(c) When a magnetic field $\overline{B}_0$ is applied to a conductor carrying a current, an electric field $\overline{E}$ is developed. This is called the *Hall effect*, discovered by Edwin Herbert Hall in 1879 while he was a graduate student at the Johns Hopkins University. Assuming the conduction carrier drifts with a mean velocity $\overline{v}$ proportional to $R\sigma\overline{E}$, the constitutive relation that takes care of the Hall effect is given by

$$\overline{J} = \sigma\left(\overline{E} + R\sigma\overline{E} \times \overline{B}_0\right)$$

where σ is the conductivity and R is the Hall coefficient. For copper, $\sigma \approx 6.7 \times 10^7\,\text{mho/m}$ and $R \approx -5.5 \times 10^{-11}\,\text{m}^3/\text{coul}$.

(d) The phenomenon of natural optical activity can be explained with the use of the constitutive relation

$$D_i = \epsilon_{ij} E_j + \gamma_{ijk} \frac{\partial E_j}{\partial x_k}$$

where ϵ_{ij} and γ_{ijk} are functions of frequency and $\gamma_{ijk} = -\gamma_{jik}$.

(e) The phenomenon of pyroelectricity in a crystal is observed when it is heated. The constitutive relation for a *pyroelectric material* can be written as

$$\overline{D} = \overline{D}_0 + \overline{\overline{\epsilon}} \cdot \overline{E}$$

where a spontaneous term $\overline{D}_0$ exists even in the absence of an external field.

(f) The phenomenon in which dipole moments are induced in a crystal by mechanical stress is called *piezoelectricity.* A piezoelectric material is characterized by a piezoelectric tensor $\gamma_{i,kl} = \gamma_{i,lk}$ such that

$$D_i = D_{0i} + \epsilon_{ik}E_k + \gamma_{i,kl}s_{kl}$$

where s_{kl} is the stress tensor to second order in electric fields. All pyroelectric media are also piezoelectric.

(g) An isotropic dielectric can exhibit the Kerr effect when placed in an electric field. In this case the permittivity can be written as

$$\epsilon_{ij} = \epsilon\delta_{ij} + \sigma E_i E_j$$

where ϵ is the unperturbed permittivity. The principal axis of ϵ_{ij} coincides with the electric field.

(h) In an electrooptical material that exhibits Pockel's effect, the constitutive relation can be written as

$$D_i = \epsilon_{ij}E_j + \sigma_{ijk}E_j E_k$$

where $\sigma_{ijk} = \sigma_{jik}$ is a third-rank tensor symmetrical in i and j, and therefore has 18 independent elements.

P1.5.2

Similar to the expression of the constitutive relation $\overline{D} = \overline{\overline{\epsilon}} \cdot \overline{E} = \epsilon_o\overline{E} + \overline{P}$, the constitutive relation $\overline{B} = \overline{\overline{\mu}} \cdot \overline{H}$ can also be represented in terms of a "free-space" part $\mu_o\overline{H}$ and a magnetization vector $\overline{M}$ such that

$$\overline{B} = \mu_o\overline{H} + \mu_o\overline{M}$$

Notice that while $\overline{P}$ has the same dimension as $\overline{D}$, $\overline{M}$ has the same dimension as $\overline{H}$.

In the case of media possessing permanent moments, the polarization $\overline{P}$ and the magnetization $\overline{M}$ are given classically by the Langevin equation

$$L(x) = \coth x - \frac{1}{x}$$

For a paramagnetic material with magnetic moments Nm,

$$M = NmL\left(\frac{mH}{kT}\right)$$

where $k = 1.38 \times 10^{-23}$ joule/kelvin is Boltzmann's constant, and T is the absolute temperature in kelvins. Show that in the low-field limit, since $mH \ll kT$, the medium is linear.

1.6 Boundary Conditions

A. Continuity of Electric and Magnetic Field Components

Assume that there is a plane boundary surface at $z = 0$ separating Regions 1 and 2, we can derive the boundary condition for $\overline{H}$ by using a small pill-box [Fig. 1.6.1] and letting Δz go to zero. As across the boundary, field amplitudes may be discontinuous while on the x-y plane they are not varying much. We thus ignore partial derivatives with respect to x and y, and keep only partial derivatives with respect to z. We find that

$$\begin{aligned}\nabla \times \overline{H} &= \frac{\partial}{\partial z}\Big\{\hat{z} \times \overline{H}\Big\} \\ &= \lim_{\Delta z \to 0} \frac{1}{\Delta z}\Big\{\hat{z} \times \Big[\overline{H}(x_0, y_0, z_0 + \frac{\Delta z}{2}) - \overline{H}(x_0, y_0, z_0 - \frac{\Delta z}{2})\Big]\Big\} \\ &= \lim_{\Delta z \to 0} \frac{1}{\Delta z}\Big\{\hat{z} \times \Big[\overline{H}_1 - \overline{H}_2\Big]\Big\} \qquad (1.6.1)\end{aligned}$$

where $\overline{H}(x_0, y_0, z_0 + \frac{\Delta z}{2}) = \overline{H}_2$ is in region 2, and $\overline{H}(x_0, y_0, z_0 - \frac{\Delta z}{2}) = \overline{H}_1$ is in region 1.

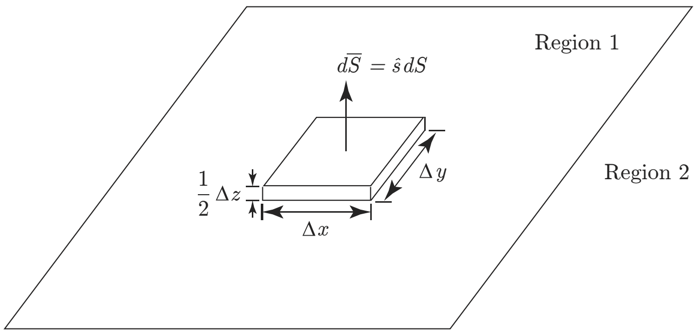


Figure 1.6.1 Small pill-box volume.

From Ampère's law, letting the surface normal $\hat{n} = \hat{z}$, we find

$$\hat{n} \times (\overline{H}_1 - \overline{H}_2) = \lim_{\Delta z \to 0} \Delta z \Big\{\frac{\partial \overline{D}}{\partial t} + \overline{J}\Big\} \qquad (1.6.2)$$

Assume that the time derivative of $\overline{D}$, $\frac{\partial \overline{D}}{\partial t}$ and the vector current density $\overline{J}$ are both finite, we obtain from (1.6.2) $H_{1y} = H_{2y}; H_{1x} = H_{2x}$ or that

$$\boxed{\hat{n} \times (\overline{H}_1 - \overline{H}_2) = 0} \tag{1.6.3}$$

Thus the tangential components of the magnetic field $\overline{H}$ are continuous across the boundary surface.

Similar derivations apply to the electric field components. From Faraday's law across the boundary, we conclude that

$$\boxed{\hat{n} \times (\overline{E}_1 - \overline{E}_2) = 0} \tag{1.6.4}$$

Thus the tangential components of the electric field $\overline{E}$ are continuous across the boundary surface.

Letting Δz go to zero by using the small pill-box in [Fig. 1.6.1], we find from Gauss' law

$$\begin{aligned}\nabla \cdot \overline{D} &= \lim_{\Delta z \to 0} \frac{1}{\Delta z} \left[D_z(x_0, y_0, z_0 + \frac{\Delta z}{2}) - D_z(x_0, y_0, z_0 - \frac{\Delta z}{2}) \right] \\ &= \lim_{\Delta z \to 0} \frac{1}{\Delta z} \left[\hat{z} \cdot (\overline{D}_1 - \overline{D}_2) \right] \end{aligned} \tag{1.6.5}$$

where $D_z(x_0, y_0, z_0 + \frac{\Delta z}{2}) = D_{1z}$ and $D_z(x_0, y_0, z_0 - \frac{\Delta z}{2}) = D_{2z}$. We find

$$\hat{n} \cdot (\overline{D}_1 - \overline{D}_2) = \lim_{\Delta z \to 0} \rho \, \Delta z \tag{1.6.6}$$

Assume that the charge density is finite across the boundary, we find

$$\boxed{\hat{n} \cdot (\overline{D}_1 - \overline{D}_2) = 0} \tag{1.6.7}$$

Thus the normal components of the electric field $\overline{D}$ are continuous across the boundary surface.

Similarly from Gauss' law $\nabla \cdot \overline{B} = 0$, we find

$$\boxed{\hat{n} \cdot (\overline{B}_1 - \overline{B}_2) = 0} \tag{1.6.8}$$

The normal component of the magnetic field $\overline{B}$ is continuous across the boundary surface. The magnetic field $\overline{H}$ is continuous.

B. Surface Charge and Current Densities

It is often convenient, in particular mathematically, to define regions where the electric and magnetic fields are zero. The media occupying such regions are called perfect conductors, which are idealizations of media where the fields inside are vanishingly small. We assume that all fields in Region 2 are zero, $\overline{E}_2 = \overline{H}_2 = \overline{B}_2 = \overline{D}_2 = 0$.

Electric charges and currents are located primarily in a very thin layer on the surface of perfect conductors. Thus on the surface of perfect conductors, we assume ρ is infinite contained in a zero thickness. We may define a surface charge density

$$\rho_s = \lim_{\Delta z \to 0} \rho \Delta z$$

which is finite and has dimension coulombs/m^2. The concept of surface charge density will have very practical usefulness. As $\overline{D}_2 = 0$, Equation (1.6.6) becomes

$$\boxed{\rho_s = \hat{n} \cdot \overline{D}_1} \tag{1.6.9}$$

Thus the difference between the $\overline{D}$ field components normal to the boundary surface is equal to the surface charge density at the boundary surface.

On the right hand side of (1.6.2), the time derivatives $\partial D_x/\partial t$ and $\partial D_y/\partial t$ are finite but we may assume J_x and J_y to be infinite to create a surface current density $\overline{J}_s$ when $\Delta z \to 0$:

$$\overline{J}_s = \lim_{\substack{\Delta z \to 0 \\ \overline{J} \to \infty}} \overline{J}\, \Delta z \tag{1.6.10}$$

We obtain from (1.6.1), as $\overline{H}_2 = 0$,

$$\boxed{\overline{J}_s = \hat{n} \times \overline{H}_1} \tag{1.6.11}$$

Thus the discontinuity in the tangential components of $\overline{H}$ is equal to the surface current at the boundary surface.

The boundary conditions (1.6.8) and (1.6.4) remain unchanged,

$$\begin{aligned} \hat{n} \times \overline{E}_1 &= 0 \\ \hat{n} \cdot \overline{B}_1 &= 0 \end{aligned}$$

the normal component of the magnetic field $\overline{B}$ and the tangential components of the electric field $\overline{E}$ are continuous.

C. Boundary Conditions

The Maxwell equations have been written in differential form. They must be supplemented with boundary conditions and initial conditions wherever derivatives do not exist. The boundary conditions can be derived from either the differential form or the integral form of the Maxwell equations. The field vectors $\overline{E}$, $\overline{B}$, $\overline{D}$, and $\overline{H}$ are assumed to be finite but may be discontinuous across the boundary. The volume current and charge densities $\overline{J}$ and $\overline{\rho}$, however, may be infinite, such as on the surface of a perfect conductor, where we can define the surface current density $\overline{J}_s = \delta\overline{J}$ in the limit as $\delta \rightarrow 0$ and $\overline{J} \rightarrow \infty$,

$$\overline{J}_s = \lim_{\substack{\delta\rightarrow 0 \\ \overline{J}\rightarrow\infty}} \overline{J}\,\delta \tag{1.6.12}$$

and the surface charge density $\rho_s = \delta\rho$ in the limit as $\delta \rightarrow 0$ and $\rho \rightarrow \infty$

$$\rho_s = \lim_{\substack{\delta\rightarrow 0 \\ \rho\rightarrow\infty}} \rho\,\delta \tag{1.6.13}$$

The surface current density has dimension amp/m and the surface charge density has dimension coul/m^2.

For a stationary boundary separating regions 1 and 2, we let the surface normal $\hat{n}$ point from region 2 to region 1. The boundary conditions are as follows:

$$\hat{n} \times (\overline{E}_1 - \overline{E}_2) = 0 \tag{1.6.14}$$

$$\hat{n} \times (\overline{H}_1 - \overline{H}_2) = \overline{J}_s \tag{1.6.15}$$

$$\hat{n} \cdot (\overline{B}_1 - \overline{B}_2) = 0 \tag{1.6.16}$$

$$\hat{n} \cdot (\overline{D}_1 - \overline{D}_2) = \rho_s \tag{1.6.17}$$

where subscripts 1 and 2 denote fields in regions 1 and 2, respectively. Essentially the boundary conditions state that the tangential components of $\overline{E}$ and the normal components of $\overline{B}$ are continuous across the boundary; the discontinuity of the tangential components of $\overline{H}$ is equal to the surface current density $\overline{J}_s$; and the discontinuity of the normal components of $\overline{D}$ is equal to the surface charge density ρ_s.

EXAMPLE 1.6.1 Derivation of boundary conditions.

We now derive the boundary conditions by using integral formulas. First we consider the integration of a vector field $\overline{A}$ over a volume V enclosed by a surface S with surface normal $\hat{s}$. The following formulas are useful:

$$\iiint dV\,\nabla\cdot\overline{A} = \oint\!\!\!\!\int dS\,\hat{s}\cdot\overline{A} \tag{E1.6.1.1a}$$

$$\iiint dV\,\nabla\times\overline{A} = \oint\!\!\!\!\int dS\,\hat{s}\times\overline{A} \tag{E1.6.1.1b}$$

where (E1.6.1.1a) is the familiar Gauss' theorem which relates integration of the divergence of the vector field $\overline{A}$ over the volume V to the integration of the field over the surface S enclosing V. Equation (E1.6.1.1b) is derived from (E1.6.1.1a) by noting that $\nabla\cdot(\overline{C}\times\overline{A}) = -\overline{C}\cdot\nabla\times\overline{A}$ where $\overline{C}$ is a constant vector independent of position. Applying the Gauss' theorem (E1.6.1.1a) to $\nabla\cdot(\overline{C}\times\overline{A})$, we obtain

$$-\overline{C}\cdot\iiint dV\,\nabla\times\overline{A} = \oint\!\!\!\!\int dS\,\hat{s}\cdot\overline{C}\times\overline{A} = -\overline{C}\cdot\oint\!\!\!\!\int dS\,\hat{s}\times\overline{A}$$

This is seen to be (E1.6.1.1b) dot-multiplied by $\overline{C}$ on both sides. Letting $\overline{C}$ be an arbitrary vector, the result is then (E1.6.1.1b).

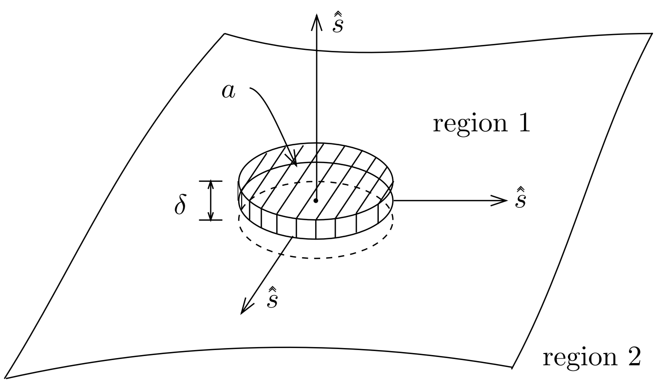


Figure E1.6.1.1 Pillbox for derivation of boundary conditions.

Now consider an interface separating regions 1 and 2 [Fig. E1.6.1.1]. Assume a small pillbox volume across the interface. Integrating Maxwell equations over the volume and applying (E1.6.1.1), we obtain

$$\oiint dS\, \hat{s} \times \overline{E} = -\iiint dV \frac{\partial}{\partial t} \overline{B} \tag{E1.6.1.2}$$

$$\oiint dS\, \hat{s} \times \overline{H} = \iiint dV \frac{\partial}{\partial t} \overline{D} + \iiint dV\, \overline{J} \tag{E1.6.1.3}$$

$$\oiint dS\, \hat{s} \cdot \overline{B} = 0 \tag{E1.6.1.4}$$

$$\oiint dS\, \hat{s} \cdot \overline{D} = \iiint dV\, \rho \tag{E1.6.1.5}$$

These are the Maxwell equations in integral form, which will be used to derive boundary conditions for both stationary and moving boundaries.

If we assume that the boundary surface is not in motion, then for the terms involving partial derivatives with time, $\partial/\partial t$ can be moved to the outside of the integral. Since the integration is over the volume, the result is a function of time only, and the partial derivatives become total derivatives. Therefore, for stationary boundary surfaces, the Maxwell equations in integral form become

$$\oiint dS\, \hat{s} \times \overline{E} = -\frac{d}{dt} \iiint dV\, \overline{B} \tag{E1.6.1.6}$$

$$\oiint dS\, \hat{s} \times \overline{H} = \frac{d}{dt} \iiint dV\, \overline{D} + \iiint dV\, \overline{J} \tag{E1.6.1.7}$$

$$\oiint dS\, \hat{s} \cdot \overline{B} = 0 \tag{E1.6.1.8}$$

$$\oiint dS\, \hat{s} \cdot \overline{D} = \iiint dV\, \rho \tag{E1.6.1.9}$$

Now we let the volume of the pillbox approach zero in such a manner that the thickness of the ribbon side, δ, goes to zero before the top and bottom areas a shrink to a point. We dispose of terms of the order of δ.

We see that the terms involving time derivatives in (E1.6.1.6) and (E1.6.1.7) drop out because they are proportional to δ. We then consider the right-hand sides of (E1.6.1.7) and (E1.6.1.9) which become $\delta a \overline{J}$ and $\delta a \rho$, respectively. If $\overline{J}$ and ρ are finite, both terms will be zero because they are proportional to δ. When there are surface charges and currents at the boundary, the right-hand sides of (E1.6.1.7) and (E1.6.1.9) become $a\overline{J}_s$ and $a\rho_s$. We then see that the surface integral terms involving cross and dot products will be dropped except when $\hat{s}$ is in the directions $\hat{n}$ or $-\hat{n}$. After canceling a on both sides of the equations, we obtain from (E1.6.1.6)–(E1.6.1.9) the boundary conditions (1.6.14)–(1.6.17).

— End of Example 1.6.1 —

Example 1.6.2

Consider an electromagnetic wave with

$$\overline{E}_i = \hat{x} E_0 \cos(kz - \omega t) \tag{E1.6.2.1a}$$
$$\overline{H}_i = \hat{y} H_0 \cos(kz - \omega t) \tag{E1.6.2.1b}$$

impinging upon the surface of a perfectly conducting surface [Fig. E1.6.2.1]. The boundary condition at the surface of the boundary requires that

$$\hat{n} \times (\overline{E}_1 - \overline{E}_2) = 0 \tag{E1.6.2.2a}$$
$$\hat{n} \times (\overline{H}_1 - \overline{H}_2) = \overline{J}_s \tag{E1.6.2.2b}$$

where $\hat{n} = -\hat{z}$ is the normal to the surface. A perfect conductor is defined to have fields zero inside, thus $\overline{E}_2 = \overline{H}_2 = 0$.

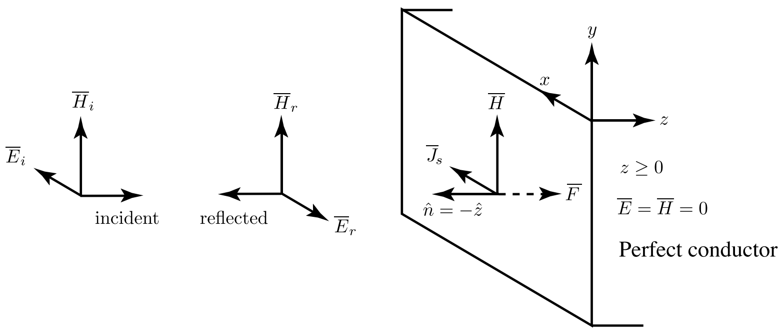


Figure E1.6.2.1 Reflection by a perfect conductor.

The reflected wave that satisfies the boundary conditions (E1.6.2.2) is

$$\overline{E}_r = -\hat{x} E_0 \cos(kz + \omega t) \tag{E1.6.2.3a}$$
$$\overline{H}_r = \hat{y} H_0 \cos(kz + \omega t) \tag{E1.6.2.3b}$$

which is propagating in the $-\hat{z}$ direction. The surface current $\overline{J}_s$ at $z = 0$ is found to be

$$\overline{J}_s = \hat{n} \times [(\overline{H}_i + \overline{H}_r) - 0]_{z=0} = \hat{x} 2 H_0 \cos \omega t$$

The magnetic field at $z = 0$ is $\overline{B} = \mu_o(\overline{H}_i + \overline{H}_r) = \hat{y} 2 \mu_o H_0 \cos \omega t$. From the Lorentz force law, the force density acting on $\overline{J}_s$ is

$$\overline{F} = \frac{1}{2} \overline{J}_s \times \overline{B} = \hat{z} 2 \mu_o H_0^2 \cos^2 \omega t$$

The factor $1/2$ is due to the fact that there is magnetic field only on one side of the current sheet. The time-average value is thus

$$F = \mu_o H_0^2$$

which is twice the value of the incident radiation pressure in Example 1.3.13. This is because the reflected wave is in the $-\hat{z}$ direction, and it exerts a recoil force on the conductor when it launches the reflected wave.

— END OF EXAMPLE 1.6.2 —

Problems

P1.6.1

Derive boundary conditions for $\overline{E}$ and $\overline{H}$ by applying Stokes' theorem to [P1.6.1.1].

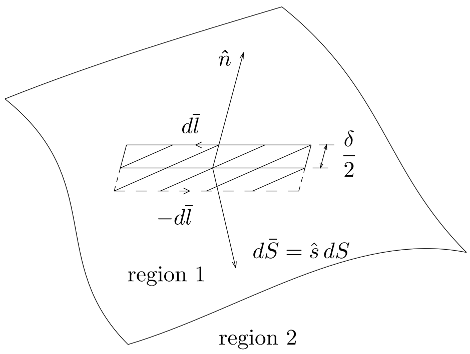


Figure P1.6.1.1 Derivation of boundary condition with Stokes' theorem.

P1.6.2

Derive the boundary conditions for $\overline{H}$ by applying the curl theorem to a small pill-box volume on the x-y plane which has an area A and an infinitesimal thickness Δz.

P1.6.3

Applying the divergence theorem (1.1.19) and integrating over the pillbox volume in Fig. E1.6.1.1 with area a and circumferential length l to find boundary condition for $\overline{D}$.

1.7 Reflection and Guidance

A. Wave Vector $\overline{k}$

The electric field $\overline{E}(\overline{r}, t)$ is governed by the Helmholtz wave equation.

$$\left(\nabla^2 - \mu\epsilon \frac{\partial^2}{\partial t^2}\right) \overline{E}(\overline{r}, t) = 0 \tag{1.7.1}$$

with

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{1.7.2}$$

as the Laplacian operator ∇^2 in rectangular coordinate system.

Consider the solution

$$\overline{E}(\overline{r}, t) = \overline{E} \cos\left(k_x x + k_y y + k_z z - \omega t\right) \tag{1.7.3}$$

where $\overline{E}$ is a constant vector. The electric field vector in (1.7.3) represents a linearly polarized wave. Since a general polarization can be expressed as a combination of two linear polarizations, the following analysis applies to all polarizations.

Substituting (1.7.3) into (1.7.1), we obtain the dispersion relation

$$\boxed{k_x^2 + k_y^2 + k_z^2 = \omega^2 \mu\epsilon = k^2} \tag{1.7.4}$$

We define a vector

$$\overline{k} = \hat{x} k_x + \hat{y} k_y + \hat{z} k_z \tag{1.7.5}$$

The vector $\overline{k}$ is called the wave vector, the propagation vector, or simply the $\overline{k}$ vector. By virtue of the dispersion relation (1.7.4), we see that the magnitude of the $\overline{k}$ vector is equal to $\omega(\mu\epsilon)^{1/2}$.

The scalar product of the wave vector $\overline{k} = \hat{x} k_x + \hat{y} k_y + \hat{z} k_z$ and the position vector $\overline{r} = \hat{x} x + \hat{y} y + \hat{z} z$ gives

$$\overline{k} \cdot \overline{r} = k_x x + k_y y + k_z z$$

A constant phase front is determined by $\overline{k} \cdot \overline{r} = \text{constant}$, which indicates that the front is a plane perpendicular to the $\overline{k}$ vector [Fig. 1.7.1]. The phase front is a plane and the amplitude of the electric field on the plane is a constant. We call the solution in (1.7.3) a uniform plane

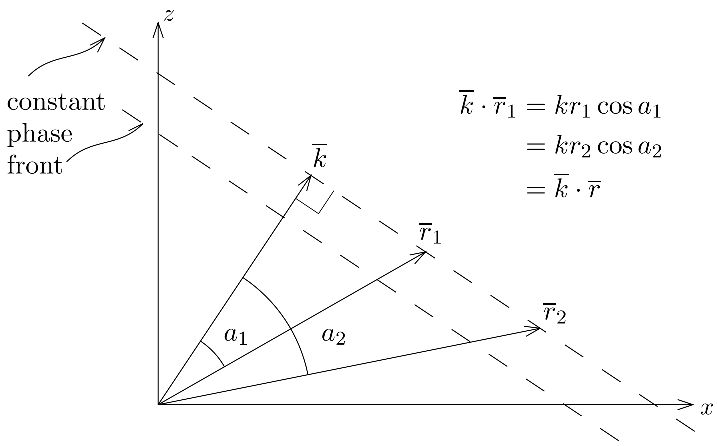


Figure 1.7.1 Constant phase fronts of a plane wave.

wave. A plane wave is non-uniform if its phase front is a plane but the amplitudes of the field are not constant. Since the constant phase front must be perpendicular to $\overline{k}$ at all times, we conclude that this phase front propagates in the direction of $\overline{k}$.

B. Reflection and Transmission of TE Waves

Consider a plane wave incident from a medium with permittivity ϵ_0 and permeability μ_0 upon a dielectric medium with permittivity ϵ_t and permeability μ_0. The boundary surface of the two media is situated at $x = 0$. Let the incident plane wave be linearly polarized with the electric field vector in the $\hat{y}$ direction [Fig. 1.7.2].

We call the x-z plane the plane of incidence, which is formally defined as the plane formed by the normal to the boundary surface and the incident wave vector $\overline{k}$. The incident electric field vector $\overline{E}_i$ is perpendicular to the plane of incidence and the magnetic field vector $\overline{H}_i$ is parallel to the plane of incidence. We call the incident wave a transverse electric (TE) wave. The TE wave is also called perpendicularly polarized, horizontally polarized, or simply the E wave or s wave.

An incident wave of general polarization can be decomposed into two linearly polarized waves; one with the electric field vector perpendicular to the plane of incidence which is the TE wave, and one with the electric field vector parallel to the plane of incidence which is called

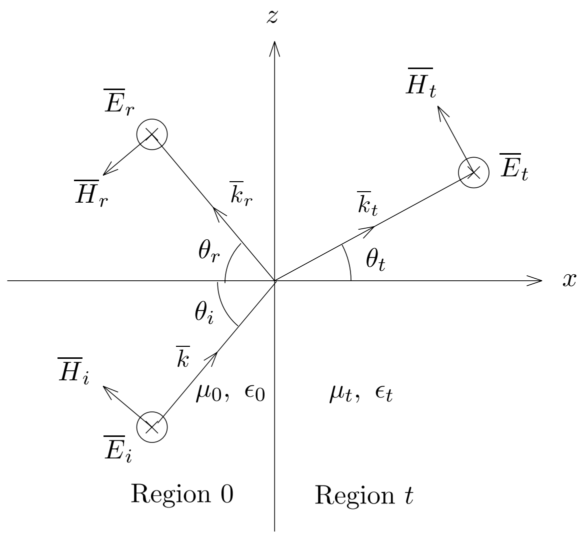


Figure 1.7.2 Reflection and transmission of TE waves at a plane boundary separating Regions 0 **and** t.

the transverse magnetic (TM) wave. The TM wave will have the magnetic field vector perpendicular to the plane of incidence and is also called parallelly polarized, vertically polarized, or simply the H wave or p wave. We shall first study the case of TE wave incidence.

The incident electric field vector is assumed to have unit amplitude and is written as

$$\begin{aligned}\overline{E}_i(\overline{r},t) &= \hat{y}\cos(\overline{k}\cdot\overline{r}-\omega t)\\ &= \hat{y}\cos(k_x x + k_z z - \omega t)\end{aligned} \tag{1.7.6a}$$

with the wave vector

$$\overline{k} = \hat{x}k_x + \hat{z}k_z$$

The magnetic field vector

$$\overline{H}_i(\overline{r},t) = \frac{1}{\omega\mu_0}(-\hat{x}k_z + \hat{z}k_x)\cos(k_x x + k_z z - \omega t) \tag{1.7.6b}$$

The Poynting vector power density for the incident plane wave is

$$\begin{aligned}\overline{S}_i(\overline{r},t) &= \overline{E}_i(\overline{r},t)\times\overline{H}_i(\overline{r},t)\\ &= \overline{k}\frac{1}{\omega\mu_0}\cos^2(k_x x + k_z z - \omega t)\end{aligned} \tag{1.7.6c}$$

which is in the direction of the wave vector $\overline{k}$.

The reflected fields for the incident TE wave are

$$\overline{k}_r = -\hat{x}k_{rx} + \hat{z}k_{rz} \tag{1.7.7a}$$

$$\overline{E}_r(\overline{r},t) = \hat{y}R\cos(-k_{rx}x + k_{rz}z - \omega t) \tag{1.7.7b}$$

$$\overline{H}_r(\overline{r},t) = -\frac{1}{\omega\mu_0}(\hat{x}k_{rz} + \hat{z}k_{rx})R\cos(k_x x + k_z z - \omega t) \tag{1.7.7c}$$

The Poynting vector power density for the reflected plane wave is

$$\overline{S}_r(\overline{r},t) = \overline{k}_r \frac{R^2}{\omega\mu_0}\cos^2(k_{rx}x + k_{rz}z - \omega t) \tag{1.7.7d}$$

where R is the reflection coefficient for the electric field component.

The incident wave vector $\overline{k} = \hat{x}k_x + \hat{z}k_z$ and the reflected wave vector $\overline{k}_r = -\hat{x}k_{rx} + \hat{z}k_{rz}$ are governed by the dispersion relations

$$k_x^2 + k_z^2 = \omega^2\mu_0\epsilon_0 = k^2 \tag{1.7.8}$$

$$k_{rx}^2 + k_{rz}^2 = \omega^2\mu_0\epsilon_0 = k_r^2 \tag{1.7.9}$$

This is seen by substituting (1.7.6a) and (1.7.7a) in the Helmholtz wave equations for E_{iy} and E_{ry}.

In Region t, we write the transmitted TE wave solution in the following form

$$\overline{k}_t = \hat{x}k_{tx} + \hat{z}k_{tz} \tag{1.7.10a}$$

$$\overline{E}_t(\overline{r},t) = \hat{y}T\cos(k_{tx}x + k_{tz}z - \omega t) \tag{1.7.10b}$$

$$\overline{H}_t(\overline{r},t) = \frac{T}{\omega\mu_t}(-\hat{x}k_{tz} + \hat{z}k_{tx})\cos(k_{tx}x + k_{tz}z - \omega t) \tag{1.7.7c}$$

$$\overline{S}_t(\overline{r},t) = \overline{k}_t \frac{T^2}{\omega\mu_t}\cos^2(k_x x + k_z z - \omega t) \tag{1.7.7d}$$

where T is the transmission coefficient, and the dispersion relation

$$k_{tx}^2 + k_{tz}^2 = \omega^2\mu_t\epsilon_t = k_t^2 \tag{1.7.11}$$

governs the magnitude k_t for the transmitted wave vector $\overline{k}_t = \hat{x}k_{tx} + \hat{z}k_{tz}$.

Let the boundary surface be at $x = 0$ where the tangential components of $\overline{E}$ and $\overline{H}$ are continuous for all z and t. We obtain

$$\cos(k_z z - \omega t) + R\cos(k_{rz} z - \omega t) = T\cos(k_{tz} z - \omega t) \quad (1.7.12)$$

$$\frac{k_x}{\mu_0}\cos(k_z z - \omega t) - \frac{k_{rx}}{\mu_0} R\cos(k_{rz} z - \omega t) = \frac{k_{tx}}{\mu_t} T\cos(k_{tz} z - \omega t) \quad (1.7.13)$$

Since (1.7.12) and (1.7.13) must hold for all z and t, it follows that

$$\boxed{k_z = k_{rz} = k_{tz}} \quad (1.7.14)$$

This is called the phase matching condition.

From the dispersion relations (1.7.8) and (1.7.9), we find $k_{rx} = k_x$. Equations (1.7.12) and (1.7.13) then reduce to

$$1 + R = T \quad (1.7.15)$$

$$1 - R = \frac{\mu_0 k_{tx}}{\mu_t k_x} T \quad (1.7.16)$$

Note that the boundary conditions of normal $\overline{D}$ and normal $\overline{B}$ components continuous at $x = 0$ are satisfied since the condition of continuous normal $\overline{B}$ yields the same equation as (1.7.15) and there is no normal $\overline{D}$ component.

The reflection and transmission coefficients R and T are determined from (1.7.15) and (1.7.16), giving

$$R = R_{0t}^{TE} = \frac{1 - p_{0t}^{TE}}{1 + p_{0t}^{TE}} \quad (1.7.17)$$

$$T = T_{0t}^{TE} = \frac{2}{1 + p_{0t}^{TE}} \quad (1.7.18)$$

where

$$p_{0t}^{TE} = \frac{\mu_0 k_{tx}}{\mu_t k_x} \quad (1.7.19)$$

With p_{0t}^{TE} for the TE waves defined in (1.7.19), R_{0t}^{TE} in (1.7.17) is called the Fresnel reflection coefficient for a TE wave incident from Region 0 and reflected at the boundary separating Regions 0 and t. In (1.7.18), T_{0t}^{TE} is the transmission coefficient from Region 0 to Region t.

Augustin Jean Fresnel (10 May 1788 – 14 July 1827)
Augustin Fresnel was educated at the Ecole Polytechnique and served as an engineer in various departments of France. With his mathematical analysis, he removed a number of objections to the wave theory, and used the wave theory to calculate diffraction patterns that agreed with experimental observations. He developed a system of lenses which has revolutionized lighthouse illumination throughout the world.

Equation (1.7.14), the phase matching condition, is a very important formula arising from the boundary conditions. In terms of the angle of incidence θ_i, the angle of reflection θ_r, and the angle of transmission θ_t, and the relation $k_r = k$ as seen from (1.7.8) and (1.7.9), the phase matching condition (1.7.14) gives

$$k \sin\theta_i = k_r \sin\theta_r = k_t \sin\theta_t$$

Thus the angle of reflection is equal to the angle of incidence $\theta_r = \theta_i$, and

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{k}{k_t} = \frac{\sqrt{\mu_0\epsilon_0}}{\sqrt{\mu_t\epsilon_t}} = \frac{n_0}{n_t} \tag{1.7.20}$$

where $n_0 = c\sqrt{\mu_0\epsilon_0}$ is called the refractive index for Region 0 and $n_t = c\sqrt{\mu_t\epsilon_t}$ is the refractive index for Region t. Equation (1.7.20) is known as Snell's law.

Willebrord van Roijen Snell (1580 – 1626) studied at the University of Leiden and received his degree in 1607. In 1613 he succeeded his father as professor of mathematics at the University of Leiden. Snell's law for the refraction of light between two media was experimentally discovered in 1621.

Power Conservation

The time-average Poynting vectors for the incident, the reflected, and the transmitted waves are calculated to be

$$<\overline{S}_i> = \frac{1}{2\omega\mu_0}\overline{k} = \frac{1}{2\omega\mu_0}(\hat{x}k_x + \hat{z}k_z) \tag{1.7.21}$$

$$<\overline{S}_r> = \frac{|R|^2}{2\omega\mu_0}\overline{k}_r = \frac{|R|^2}{2\omega\mu_0}(-\hat{x}k_x + \hat{z}k_z) \tag{1.7.22}$$

$$<\overline{S}_t> = \frac{|T|^2}{2\omega\mu_t}\overline{k}_t = \frac{|T|^2}{2\omega\mu_t}(\hat{x}k_{tx} + \hat{z}k_z) \tag{1.7.23}$$

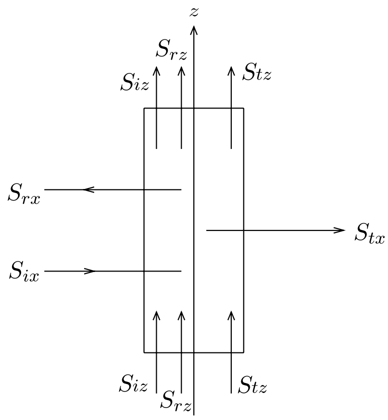


Figure 1.7.3 Power conservation at a plane boundary.

Power conservation is observed by considering a control volume across the boundary surface [Fig. 1.7.3]. We must prove that the x components of all the Poynting vectors entering and exiting the control volume are equal. We define the power reflection coefficient or the *reflectivity* to be

$$r = \frac{-\hat{x}\cdot <\overline{S}_r>}{\hat{x}\cdot <\overline{S}_i>} = |R|^2 \tag{1.7.24}$$

and the power transmission coefficient or the *transmissivity* to be

$$t = \frac{\hat{x}\cdot <\overline{S}_t>}{\hat{x}\cdot <\overline{S}_i>} = p_{0t}\,|T|^2 \tag{1.7.25}$$

By virtue of (1.7.17)–(1.7.18), we see that

$$r + t = 1$$

This demonstrates power conservation for reflection and transmission at a plane boundary.

Exercise 1.7.1 Notice that

$$<S_{ix}> - <S_{rx}> = <S_{tx}>$$
$$<S_{iz}> - <S_{rz}> \neq <S_{tz}>$$

— End of Exercise 1.7.1 —

C. Reflection and Transmission of TM Waves

The reflection and transmission of TM waves [Fig. 1.7.4] by a plane boundary can be carried out in a manner similar to the treatment of TE waves. The incident magnetic field vector $\overline{H}_i = \hat{y} H_{iy}$ is assumed to have unit amplitude and the magnetic and electric field components are written as

$$H_{iy} = \cos(k_x x + k_z z - \omega t) \tag{1.7.26a}$$

$$E_{ix} = \frac{k_z}{\omega \epsilon_0} \cos(k_x x + k_z z - \omega t) \tag{1.7.26b}$$

$$E_{iz} = -\frac{k_x}{\omega \epsilon_0} \cos(k_x x + k_z z - \omega t) \tag{1.7.26c}$$

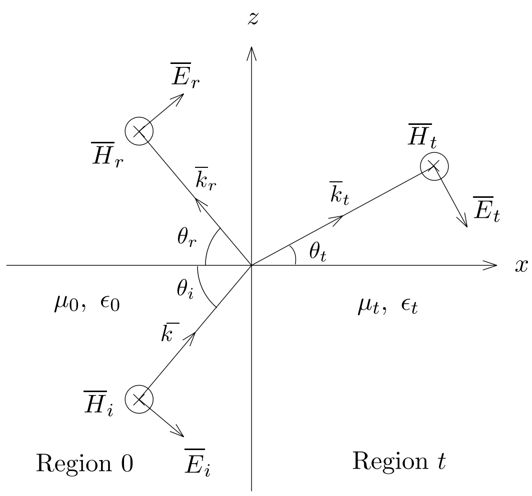


Figure 1.7.4 Reflection and transmission of TM waves.

The reflected field components for the incident TM wave are

$$H_{ry} = R^{TM} \cos(-k_{rx} x + k_{rz} z - \omega t) \tag{1.7.27a}$$

$$E_{rx} = \frac{k_{rz}}{\omega \epsilon_0} R^{TM} \cos(-k_{rx} x + k_{rz} z - \omega t) \tag{1.7.27b}$$

$$E_{rz} = \frac{k_{rx}}{\omega \epsilon_0} R^{TM} \cos(-k_{rx} x + k_{rz} z - \omega t) \tag{1.7.27c}$$

where R^{TM} is the reflection coefficient for the magnetic field component H_{iy}. In Region t, the transmitted TM field components are

$$H_{ty} = T^{TM}\cos(k_{tx}x + k_{tz}z - \omega t) \tag{1.7.28a}$$

$$E_{tx} = \frac{k_{tz}}{\omega\epsilon_t}T^{TM}\cos(k_{tx}x + k_{tz}z - \omega t) \tag{1.7.28b}$$

$$E_{tz} = -\frac{k_{tx}}{\omega\epsilon_t}T^{TM}\cos(k_{tx}x + k_{tz}z - \omega t) \tag{1.7.28c}$$

where T^{TM} is the transmission coefficient for the magnetic field component H_{iy}.

The incident wave vector $\overline{k} = \hat{x}k_x + \hat{z}k_z$, the reflected wave vector $\overline{k}_r = -\hat{x}k_{rx} + \hat{z}k_{rz}$, and the transmitted wave vector satisfy the same dispersion relations (1.7.8), (1.7.9), and (1.7.11) as for the TE wave case. Matching the boundary conditions of tangential components of $\overline{E}$ and $\overline{H}$ continuous at $x = 0$, we obtain the same phase matching condition (1.7.14) and the reflection and transmission coefficients R^{TM} and T^{TM}

$$R^{TM} = R_{0t}^{TM} = \frac{1 - p_{0t}^{TM}}{1 + p_{0t}^{TM}} \tag{1.7.29}$$

and

$$T^{TM} = T_{0t}^{TM} = \frac{2}{1 + p_{0t}^{TM}} \tag{1.7.30}$$

where

$$p_{0t}^{TM} = \frac{\epsilon_0 k_{tx}}{\epsilon_t k_x} \tag{1.7.31}$$

Note that the Fresnel reflection coefficient for TM waves is now representing the ratio of the reflected and incident *magnetic* fields.

EXERCISE 1.7.2 At the surface of a perfect conductor, we may calculate the reflection coefficients by letting $\epsilon_t \to \infty$. We find that for TE waves $p_{0t}^{TE} \to \infty$ and $R_{0t}^{TE} \to -1$ while for TM waves $p_{0t}^{TM} \to 0$ and $R_{0t}^{TM} \to 1$. Thus the tangential electric field vanishes at the boundary and the tangential magnetic field doubles its strength in order to support the induced surface currents.

— END OF EXERCISE 1.7.2 —

D. Brewster Angle and Zero Reflection

The Brewster angle θ_b is the incident angle $\theta_i = \theta_b$ at which there is no reflected power. Setting $R = 0$ or $p_{0t} = 1$ we find, from (1.7.19), for TE waves $k_{tx} = k_x$ or

$$k_t \cos\theta_t = k \cos\theta_i \tag{1.7.32}$$

To solve for the incident angle, we make use of Snell's law

$$k_t \sin\theta_t = k \sin\theta_i \tag{1.7.33}$$

It follows from (1.7.32) and (1.7.33) that $\theta_t = \theta_i$ and $\epsilon_t = \epsilon_0$. Thus there is zero reflection since there is no boundary.

For TM waves, we obtain from (1.7.31), $\epsilon_0 k_{tx} = \epsilon_t k_x$ or

$$\epsilon_0 k_t \cos\theta_t = \epsilon_t k \cos\theta_i \tag{1.7.34}$$

Since $k = \omega\sqrt{\mu_0\epsilon}$ and $k_t = \omega\sqrt{\mu_0\epsilon_t}$, we obtain from (1.7.34)

$$k \cos\theta_t = k_t \cos\theta_i \tag{1.7.35}$$

Multiplying (1.7.33) and (1.7.35), we obtain

$$\sin 2\theta_b = \sin 2\theta_t$$

In addition to the trivial solution $\theta_t = \theta_b$, we also obtain

$$\theta_b + \theta_t = \frac{\pi}{2} \tag{1.7.36}$$

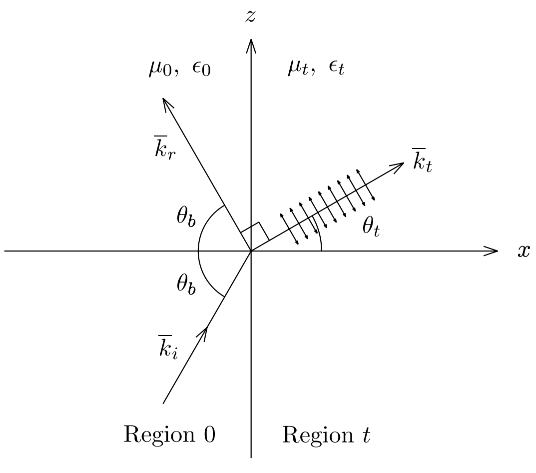


Figure 1.7.5 Incidence at the Brewster angle.

Since the reflected direction is perpendicular to the transmitted direction, the reflected wave vector $\overline{k}_r$ is perpendicular to the transmitted wave vector $\overline{k}_t$ [Fig. 1.7.5].

Physically we can explain this by visualizing the dielectric media as consisting of dipoles that are excited by the transmitted wave and radiating at the same frequency. Each individual dipole has a radiation pattern that is maximum in a direction perpendicular to the dipole axis and null along the dipole axis. For a TM wave excitation, all dipoles oscillate parallel to the plane of incidence along the $\overline{E}$-field lines. At the Brewster angle of incidence, the reflected $\overline{k}_r$ vector is in the same direction as the dipole oscillation in the transmitted medium. Thus, no TM wave is reflected.

Substituting (1.7.36) in (1.7.35), we obtain the Brewster angle

$$\theta_b = \tan^{-1}\frac{k_t}{k} = \tan^{-1}\sqrt{\frac{\epsilon_t}{\epsilon_0}} \tag{1.7.37}$$

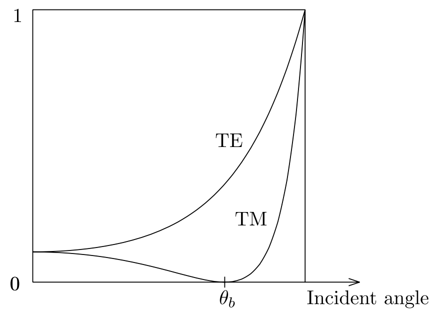


Figure 1.7.6 Reflectivity of TE and TM waves.

In Fig. 1.7.6, we plot the reflectivities as functions of the incident angle. In general, on a solid dielectric surface, the TE waves reflect more than the TM waves. For an unpolarized incident wave, the reflected wave becomes linearly polarized perpendicular to the plane of incidence. Thus the Brewster angle is also referred to as the polarization angle.

David Brewster (11 December 1781 – 10 February 1868)

David Brewster entered the University of Edinburgh at the age of 11. He was knighted in 1831, and his Treatise on Optics was also published in 1831. He taught at St. Andrews and in 1838 was promoted to principal. In 1859, he became principal of the University of Edinburgh.

The reflection and transmission of TM waves by a plane boundary has been carried out in a manner similar to the treatment of TE waves. We can also invoke the principle of duality and write down the answers directly. Making the replacements $\overline{E} \to \overline{H}$, $\overline{H} \to -\overline{E}$, $\mu_0 \rightleftarrows \epsilon_0$, and the boundary conditions of continuous tangential $\overline{H}$ and $\overline{E}$ at $x = 0$, we find the dual of the TE problem [Fig. 1.7.2] to be precisely the TM problem [Fig. 1.7.4]. We obtain the reflection and transmission coefficients as in (1.7.29)–(1.7.30) with p_{0t}^{TE} in (1.7.19) replaced by $p_{0t}^{TM} = \epsilon_0 k_{tx}/\epsilon_t k_x$.

EXAMPLE 1.7.1

Consider an electromagnetic wave impinging normally upon a dielectric half space (Region 2) with permittivity ϵ_2 from a medium (Region 1) with permittivity ϵ_1.

(a) Let $\epsilon_1 = \epsilon_o$ and $\epsilon_2 = 4\epsilon_o$. What are the reflection coefficient R_{12} and the transmission coefficient T_{12}?
(b) What is the sum of Poynting power of the wave on either side of the interface? Do they conserve?
(c) What is the sum of momentum density of the wave on either side of the interface? Do they conserve?
(d) Find the radiation pressure exerted on both sides of the boundary. Do they match?
(e) Will the half space move towards the incident wave or away from it?

SOLUTION:

(a)

$$p_{0t} = \frac{\mu_0 k_{tx}}{\mu_t k_x} = \frac{\mu_0 \omega \sqrt{\mu_0 4\epsilon_0}}{\mu_0 \omega \sqrt{\mu_0 \epsilon_0}} = 2$$

$$R_{12} = \frac{1 - p_{0t}}{1 + p_{0t}} = \frac{1-2}{1+2} = -\frac{1}{3}$$

$$T_{12} = 1 + R_{12} = \frac{2}{1 + p_{0t}} = \frac{2}{3}$$

(b) Computing the time averaged Poynting power of the incident, reflected, and transmitted waves, we find

$$< \overline{S}_i > = \hat{x}\frac{E_0^2}{2\eta_1} = \hat{x}\frac{E_0^2}{2\eta_0}$$

$$< \overline{S}_r > = -\hat{x}\frac{R_{12}^2 E_0^2}{2\eta_1} = -\hat{x}\frac{R_{12}^2 E_0^2}{2\eta_0} = -\hat{x}\left(\frac{1}{9}\right)\frac{E_0^2}{2\eta_0}$$

$$< \overline{S}_t > = \hat{x}\frac{T_{12}^2 E_0^2}{2\eta_2} = \hat{x}\frac{2T_{12}^2 E_0^2}{2\eta_0} = \hat{x}\left(\frac{8}{9}\right)\frac{E_0^2}{2\eta_0}$$

and since

$$\hat{x} \cdot < \overline{S}_i >= -\hat{x} \cdot < \overline{S}_r > + \hat{x} \cdot < \overline{S}_t >$$

we see that power is conserved.

(c) The momentum density of the field is given by $\overline{g} = \mu\epsilon\overline{S}$, so that

$$
\begin{aligned}
< \overline{g}_i > &= \hat{x}\mu_0\epsilon_0 \frac{E_0^2}{2\eta_0} = \hat{x}\frac{E_0^2}{2\eta_0 c^2} \\
< \overline{g}_r > &= -\hat{x}\left(\frac{1}{9}\right)\frac{E_0^2}{2\eta_0 c^2} = -\hat{x}\mu_0\epsilon_0\frac{E_0^2}{18\eta_0 c^2} \\
< \overline{g}_t > &= \hat{x}\mu_0(4\epsilon_0)\left(\frac{8}{9}\right)\frac{E_0^2}{2\eta_0} = \hat{x}\frac{16E_0^2}{9\eta_0 c^2}
\end{aligned}
$$

The total momentum density of the field is not conserved which implies there exists a mechanical momentum. Assuming that the plates are initially at rest, in order for total momentum to be conserved we need the mechanical momentum,

$$< \overline{g}_{\text{mech}} >=< \overline{g}_i > - < \overline{g}_r > - < \overline{g}_t >= -\hat{x}\frac{11E_0^2}{9\eta_0 c^2}$$

(d) The radiation pressure magnitude is given by $|\overline{F}| = \sqrt{\mu\epsilon}\ |\overline{S}|$. The direction in which the force is applied depends on whether the wave is an impinging wave (force acts in same direction as $\overline{S}$) or a launched wave (force acts in opposite direction as $\overline{S}$ due to recoil effect). For the incident, reflected, and transmitted fields we find,

$$
\begin{aligned}
< \overline{F}_i > &= \sqrt{\mu_0\epsilon_0} < \overline{S}_i >= \hat{x}\frac{E_0^2}{2\eta_0 c} \\
< \overline{F}_r > &= -\sqrt{\mu_0\epsilon_0} < \overline{S}_r >= \hat{x}\frac{E_0^2}{18\eta_0 c} \\
< \overline{F}_t > &= -\sqrt{\mu_0(4\epsilon_0)} < \overline{S}_t >= -\hat{x}\frac{8E_0^2}{9\eta_0 c}
\end{aligned}
$$

so that there is a net force of

$$< \overline{F}_{\text{tot}} >= -\hat{x}\frac{13E_0^2}{18\eta_0 c}$$

acting on the half space.

(e) Using the results of either part (c) or (d) we find that the half space will move towards the incident wave.

— End of Example 1.7.1 —

E. Guidance by Conducting Parallel Plates

Consider the guidance of electromagnetic waves by a pair of perfectly conducting plates at $x = 0$ and $x = d$ [Fig. 1.7.7]. For TM waves, the Maxwell equations are

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} - \mu\epsilon\frac{\partial^2}{\partial t^2}\right)H_y = 0 \tag{1.7.38a}$$

$$\epsilon\frac{\partial}{\partial t}E_x = -\frac{\partial}{\partial z}H_y \tag{1.7.38b}$$

$$\epsilon\frac{\partial}{\partial t}E_z = \frac{\partial}{\partial x}H_y \tag{1.7.38c}$$

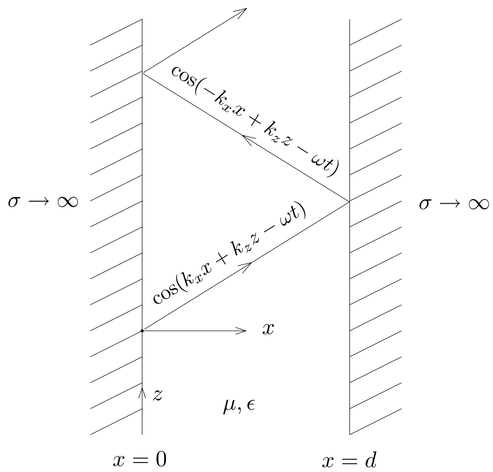


Figure 1.7.7 Parallel-plate waveguide.

In the parallel-plate waveguide, the wave is guided along the $\pm\hat{z}$ directions. The two wave solutions with wave vectors $\overline{k}$ and $\overline{k}_r$ in the guided region are

$$\overline{H}_i = \hat{y}\cos(k_x x + k_z z - \omega t) \tag{1.7.39}$$

$$\overline{E}_i = [\hat{x}k_z - \hat{z}k_x]\frac{1}{\omega\epsilon}\cos(k_x x + k_z z - \omega t) \tag{1.7.40}$$

$$\overline{H}_r = \hat{y}\ R\cos(-k_x x + k_z z - \omega t) \tag{1.7.41}$$

$$\overline{E}_r = [\hat{x}k_z + \hat{z}k_x]\frac{R}{\omega\epsilon}\cos(-k_x x + k_z z - \omega t) \tag{1.7.42}$$

The boundary conditions at the parallel plates require that the tangential electric field be zero at $x = 0$ and $x = d$.

$$-\cos(k_z z - \omega t) + R\cos(k_z z - \omega t) = 0 \qquad (1.7.43a)$$

$$-\cos(k_x d + k_z z - \omega t) + R\cos(-k_x d + k_z z - \omega t) = 0 \qquad (1.7.43b)$$

Solution to the above equations yields $R = 1$ and

$$\boxed{2k_x d = 2m\pi} \qquad (1.7.44)$$

which is known as the guidance condition. It states that in the $\hat{x}$ direction the bouncing waves must interfere constructively with $2k_x d = 2m\pi$ in order for the wave to be guided [Fig. 1.7.8].

The dispersion relation is $k_x^2 + k_y^2 = k^2$. The set of discrete k_x values admissible inside the guide is

$$\boxed{k_x = \frac{m\pi}{d}\ \ \mathrm{m}^{-1} = \frac{m}{2d}\ \ \mathrm{K}_o = k_{cm}} \qquad (1.7.45)$$

where m is any integer. We name the guided waves TM_m modes.

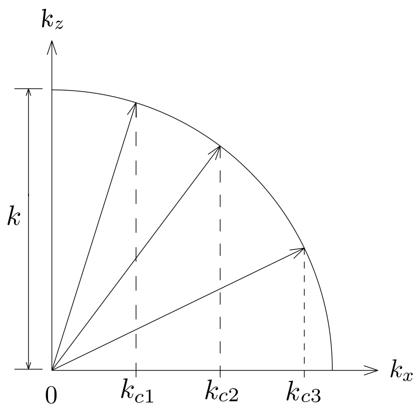


Figure 1.7.8 Interpretation of the guidance condition.

Thus as a result of the boundary condition at $x = 0$ and $x = d$, the spatial variation along the $\hat{x}$ direction of a guided wave must be an integer number in a distance of $2d$. The magnetic and electric vector fields are

$$\overline{H} = \hat{y}\cos k_x x\ \cos(k_z z - \omega t) \qquad (1.7.46)$$

$$\overline{E} = \hat{x}\frac{k_z}{\omega\epsilon}\cos k_x x\ \cos(k_z z - \omega t) + \hat{z}\frac{k_x}{\omega\epsilon}\sin k_x x\ \sin(k_z z - \omega t) \qquad (1.7.47)$$

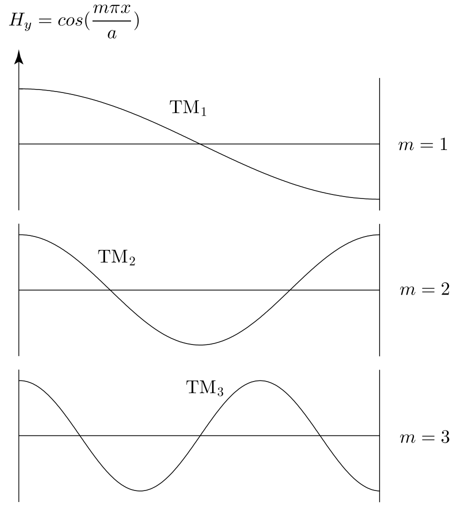


Figure 1.7.9 Field amplitudes for TM_1, TM_2, **and** TM_3 **modes.**

In Fig. 1.7.9, we plot H_y for $m = 1, 2, 3$. They are standing waves in the transversal x direction and propagate in the z direction. We see that there are more spatial variations in the waveguide with separation of d, when the x component of the spatial frequency, $k_x = (m/2d)\,\mathrm{K}_o$, is higher with larger m. The velocity of the TM_m mode in the z direction is determined from the dispersion relation

$$k_z^2 = k^2 - k_{cm}^2 \tag{1.7.48}$$

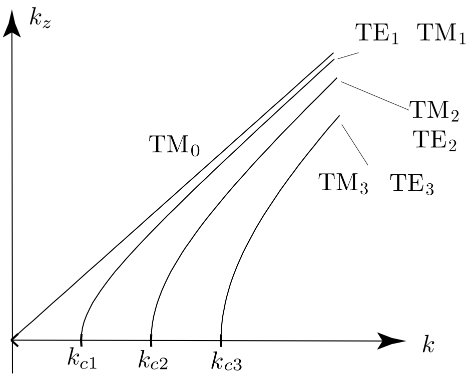


Figure 1.7.10 ω–k_z **diagram.**

The phase and group velocities are, as $\omega = ck$ and $k_z dk_z = kdk$,

$$v_p = \omega/k_z = ck/k_z \tag{1.7.49}$$

$$v_g = d\omega/dk_z = cdk/dk_z = ck_z/k \tag{1.7.50}$$

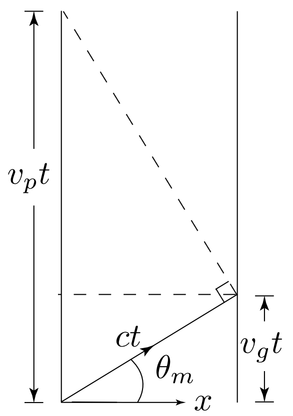


Figure 1.7.11 Distances traveled with phase and group velocities.

where $c = 1/\sqrt{\mu\epsilon}$. The phase velocity v_p is larger than c, as seen from Fig. 1.7.11. Let $\sin\theta_m = k_{cm}/k = m\pi/kd = m\lambda/2d$. We see that $v_p = c/\sin\theta_m$ and $v_g = c\sin\theta_m$, thus $v_p v_g = c^2$. In Figure 1.7.12 we show that for a propagating TM_m mode, as frequency increases, the angle θ_m increases, and the group velocity $v_g = c\Delta k/\Delta k_z$ increases.

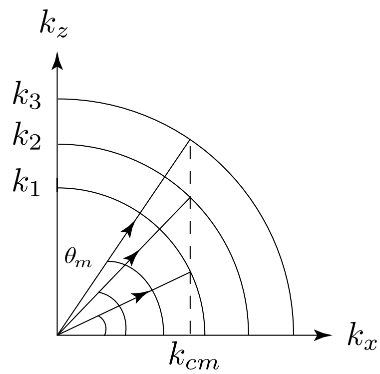


Figure 1.7.12 Guidance with increasing frequency.

It is seen from (1.7.48) that as $k < k_{cm}$, $k_z^2 = -(k_{cm}^2 - k^2) = -k_{zI}^2$, suggesting that the guided wave will attenuate in the $\hat{z}$ direction. The fields satisfying the Maxwell equations and the boundary conditions become

$$\overline{H} = \hat{y}\cos k_x x\, e^{-k_{zI}z}\cos\omega t \tag{1.7.51}$$

$$\overline{E} = \hat{x}\frac{k_{zI}}{\omega\epsilon}\cos k_x x\, e^{-k_{zI}z}\sin\omega t - \hat{z}\frac{k_x}{\omega\epsilon}\sin k_x x\, e^{-k_{zI}z}\sin\omega t \tag{1.7.52}$$

The time-average power in the $\hat{z}$ direction is zero, and the guided modes for $k < k_{cm}$ are evanescent.

The spatial frequency at which $k_z = 0$ is called the cutoff spatial frequency k_{cm}

$$k_{cm} = \frac{m}{2d}\ \mathrm{K}_o \tag{1.7.53}$$

corresponding to cutoff wavelength $\lambda_{cm} = 2d/m$. In order for the mth order TM mode to propagate, the spatial frequency k must be larger than k_{cm} or the wavelength must be smaller than λ_{cm} . Notice that if the TM_m mode is propagating, then all TM_l modes with $l < m$ can also propagate. Thus for a given spatial frequency k such that $k_{cm} < k < k_{c(m+1)}$, there will be $m+1$ TM modes admissible inside the waveguide. The lowest-order TM mode is TM_0 whose $k_{c0} = 0$.

The electric and magnetic fields for the TM_0 mode are, since $k_x = 0$ and $k_z = k$,

$$H_y = \cos(kz - \omega t) \tag{1.7.54a}$$

$$E_x = \frac{k}{\omega\epsilon}\cos(kz - \omega t) \tag{1.7.54b}$$

which is equivalent to a plane wave propagating in the $\hat{z}$ direction. The TM_0 mode is also called the fundamental mode or the TEM mode in the parallel-plate waveguide.

Example 1.7.2 TE modes.

We write the solution for TE waves as

$$E_y = (A\cos k_x x - B\sin k_x x)\sin(k_z z - \omega t) \tag{E1.7.2.1}$$

The boundary conditions at $x = 0, d$ require $E_y = 0$ which gives $A = 0$ and the same guidance condition (1.7.45). We thus obtain the electric and magnetic fields for TE modes

$$\overline{E} = -\hat{y}\; B \sin k_x x \;\; \sin(k_z z - \omega t) \tag{E1.7.2.2}$$

$$\begin{aligned}\overline{H} &= \hat{x}\; \frac{k_z}{\omega\mu} B \sin k_x x \;\; \sin(k_z z - \omega t) \\ &\quad + \hat{z}\; \frac{k_x}{\omega\mu} B \cos k_x x \;\; \cos(k_z z - \omega t)\end{aligned} \tag{E1.7.2.3}$$

where

$$k_x = \frac{m\pi}{d} \;\; \mathrm{m}^{-1} = \frac{m}{2d} \;\; \mathrm{K}_o = k_{cm} \tag{E1.7.2.4}$$

The above result can be interpreted in terms of plane waves reflecting from the conducting plates in the same way as for the TM waves. One important difference is that TE_0 does not exist and the lowest-order TE mode is TE_1 .

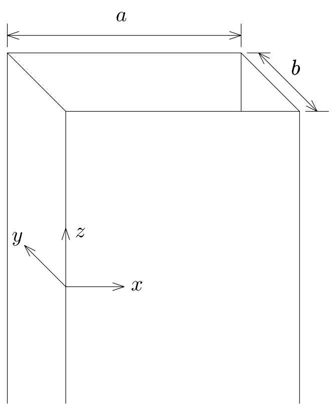


Figure E1.7.2.1 Metallic rectangular waveguide.

Consider a metallic rectangular waveguide having dimensions a along the x axis and b along the y axis [Fig. E1.7.2.1]. The TE wave fields inside the guided region can be written as (E1.7.2.2) and (E1.7.2.3). The boundary conditions at $x = 0, a$ require $E_y = E_z = 0$ and at $y = 0, b$ require $E_x = E_z = 0$ which give rise to the same guidance condition (E1.7.2.4) with d replaced by a .

Surface charges are $\rho_s = \mp B \sin k_x x \;\; \sin(k_z z - \omega t)$ at $y = 0, b$. Surface currents are $\overline{J}_s = \mp \hat{z} B \sin k_x x \sin(k_z z - \omega t)$ at $y = 0, b$. Since there is no variation in the $\hat{y}$ directions, the fields are for TE_{m0} modes. The fundamental mode is TE_{10} and the lowest cutoff spatial frequency is $k = k_{c1} = (1/2a)\; \mathrm{K}_o$ corresponding to a cutoff wavelength of $\lambda_{c1} = 2a$.

— End of Example 1.7.2 —

Problems

P1.7.1

Consider an electromagnetic wave propagating in an isotropic medium with permittivity ϵ and permeability μ. It has the following electric field vector

$$\overline{E} = (\hat{x}E_x + \hat{y}E_y + \hat{z}E_z)\cos(k_x x + k_z z - \omega t)$$

where E_x, E_y, and E_z are real constants.

(a) Determine the constraints on E_x, E_y, and E_z, in terms of k_x and k_z, such that the above electric field vector represents an electromagnetic wave.
(b) Let $k_x = \sqrt{3}k/2$, $k_z = k/2$ and $E_x = E_y = E_o$. What is the polarization of the wave?
(c) Add another plane wave component to the wave shown above, so that the total electric wave is left-hand circularly polarized.

P1.7.2

When the incident k vector is normal to a plane boundary, a TE wave becomes a TEM wave; a TM wave also becomes a TEM wave. Compare the reflection and transmission coefficients for TE and TM waves at normal incidence. Do both TE and TM results reduce to the same numbers? If not, why? Do the reflectivities and transmissivities for TE and TM waves at normal incidence reduce to the same result?

P1.7.3

The gas laser depicted in Fig. P1.7.3.1 uses "Brewster angle" quartz windows on the gas discharge tube in order to minimize reflection losses. Determine the angle θ if the index of refraction for quartz at the wavelength of interest is $n = 1.46$. Because of these windows, the laser output is almost completely linearly polarized. What is the direction of polarization, i.e., is $\overline{E}$ parallel or perpendicular to the paper? Why?

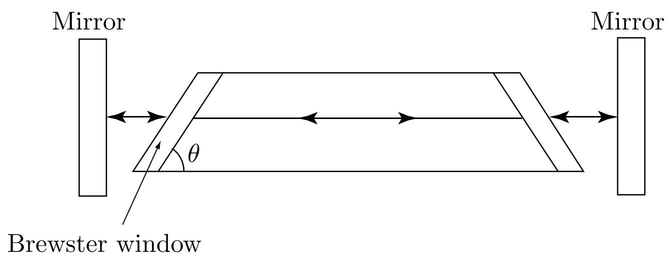


Figure P1.7.3.1 A gas laser with Brewster windows.

P1.7.4

Sun light glares caused by reflections from plane surfaces are partially linearly polarized.

(a) Determine the Brewster angle for $\epsilon_t = 9\epsilon_o$. The Brewster angle, θ_B, is also called the polarization angle because at θ_B the reflected wave is entirely TE polarized.

(b) Your polaroid glasses absorb one linear component of incident light. To minimize sun glare, what component, TE or TM, reaches your eyes after passing through the glasses? Explain why.

P1.7.5

Consider a plane wave incident on a planar boundary at $x = 0$ from a dielectric medium with $\epsilon = 9\epsilon_o$ upon another dielectric medium with μ_o and ϵ_t. The right-hand circularly polarized incident electric field is

$$\overline{E}_i = E_0(\sqrt{3}\hat{x} + \hat{z})\cos(k_x x - k_z z - \omega t) + 2\hat{y}\sin(k_x x - k_z z - \omega t)$$

where E_0 is a real constant. The reflected field is

$$\overline{E}_r = E_0\left[2R^{TE}\hat{y}\sin(k_x x + k_z z - \omega t) + R^{TM}(-\sqrt{3}\hat{x} + \hat{z})\cos(k_x x + k_z z - \omega t)\right]$$

(a) Show that the incident angle is 30°.

(b) For $k_x = 1\,\mathrm{K}_o$, find the frequency (Hz) and wavelength (m) in region 1.

(c) Find the value of ϵ_t $(0 < \epsilon_t/\epsilon_o < \infty)$ for which the reflected wave is linearly polarized.

P1.7.6

A laser beam in free space with the polarization of electric field parallel to the paper is incident normally upon a glass surface. There is 16% power of the incident wave being reflected and the rest being transmitted. Neglect the reflection on the bottom surface. The reflection coefficients of TE and TM incident waves are given by, respectively,

$$R^{TE} = \frac{\cos\theta_i - \sqrt{n^2 - \sin^2\theta_i}}{\cos\theta_i + \sqrt{n^2 - \sin^2\theta_i}}$$

$$R^{TM} = \frac{n^2\cos\theta_i - \sqrt{n^2 - \sin^2\theta_i}}{n^2\cos\theta_i + \sqrt{n^2 - \sin^2\theta_i}}$$

where $n = \sqrt{\epsilon/\epsilon_o}$ is the refraction index and θ_i is the incident angle.

(a) What is the amplitude of the reflected electric field $\overline{E}_r$ in terms of the amplitude of the incident electric field $\overline{E}_i$?

(b) What is the refraction index $(n = \sqrt{\epsilon/\epsilon_o})$ of the glass?

(c) Let the surface of the glass rotate by $\theta = \sin^{-1}(2/3)$ about an axis perpendicular to the paper. How much of the incident power is reflected?

(d) Let the surface of the glass rotate by θ about an axis perpendicular to the paper, so that the laser beam is totally transmitted without reflection. What is the rotation angle θ in radians?

P1.7.7

Find the cutoff wavelength λ_{cm} and the cutoff angular frequency ω_{cm} corresponding to the cutoff spatial frequency $k_{cm} = (m/2d)\ \mathrm{K}_o$.

P1.7.8

An AM radio in an automobile cannot receive any signal when the car is inside a tunnel. Consider, for example, the Lincoln Tunnel under the Hudson River, which was built in 1939. A cross-section of the tunnel is shown in Figure P1.7.8.1. Ignore the air ducts; assume that they are closed. Model the tunnel as a rectangular waveguide of dimension $6.55\mathrm{m} \times 4.19\mathrm{m}$.

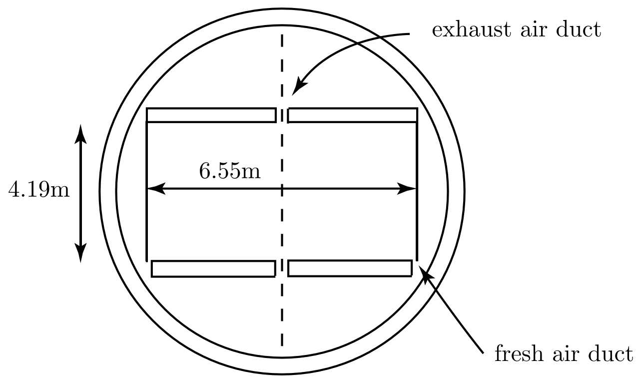


Figure P1.7.8.1 Tunnel modeled as rectangular waveguide.

(a) Give the range of frequencies for which only the dominant mode, TE_{10} , may propagate.
(b) Explain why AM signals cannot be received.
(c) Can FM signals be received? Above what frequencies?

Answers

P1.1.1

$\overline{A}+\overline{B}+\overline{C}=0$ and $\overline{A}+\overline{B}-\overline{C}=-2\overline{C}$.

P1.1.2

$|\overline{A}\times\overline{B}|^2=(\overline{A}\times\overline{B})\cdot(\overline{A}\times\overline{B})=\overline{A}\cdot(\overline{B}\times(\overline{A}\times\overline{B}))=\overline{A}\cdot(\overline{A}B^2-\overline{B}(\overline{A}\cdot\overline{B}))=A^2B^2-(\overline{A}\cdot\overline{B})^2$

P1.1.3

$r=\sqrt{8},\theta=\pi/4,\phi=\pi/4$; and $\rho=2,\phi=\pi/4,z=2$.

P1.1.4

$\hat{c}=\hat{x}\,0.6+\hat{z}\,0.8$.

P1.1.5

$\overline{A}\cdot\overline{B}=A_xB_x=AB_x=AB\cos\theta_{AB}$

P1.1.6

From $B^2\sin^2\theta_{AB}=|\overline{A}-\overline{B}|^2-(A-B\cos\theta_{AB})^2$, we find $|\overline{A}-\overline{B}|^2=A^2+B^2-2AB\cos\theta_{AB}$. It follows that $AB\cos\theta_{AB}=\frac{1}{2}[A^2+B^2-(\overline{A}-\overline{B})\cdot(\overline{A}-\overline{B})]=\overline{A}\cdot\overline{B}$

P1.1.7

$\overline{A}\times\overline{B}=\hat{z}(A_xB_y-A_yB_x)$ is in the $\hat{z}$ direction perpendicular to both $\overline{A}$ and $\overline{B}$.

P1.1.8

$$\begin{aligned}|\overline{A}\times\overline{B}|^2&=(A_yB_z-A_zB_y)^2+(A_zB_x-A_xB_z)^2+(A_xB_y-A_yB_x)^2\\&=A^2B^2-(A_xB_x+A_yB_y+A_zB_z)^2\\&=A^2B^2-(\overline{A}\cdot\overline{B})^2=A^2B^2(1-\cos^2\theta_{AB})=(AB\sin\theta_{AB})^2\end{aligned}$$

P1.1.9

For $\Phi(x)=x^2,\ \nabla\Phi(x)=\hat{x}2x$. For $\Phi(x)=-x^3,\ \nabla\Phi(x)=-\hat{x}3x^2$.

P1.1.10

Its gradient is $\nabla\Phi=\hat{x}2x+\hat{y}4y$.
For the ellipse with $\Phi=x^2+2y^2$ equals a constant,

$$d\Phi=2xdx+4ydy=(\hat{x}2x+\hat{y}4y)\cdot(\hat{x}dx+\hat{y}dy)=\nabla\Phi\cdot d\overline{r}=0$$

where $d\overline{r}$ is tangent to the ellipse. Thus the gradient $\nabla\Phi$ is normal to the ellipse and pointing in the directions of an expanding ellipse.

P1.1.11

The gradient of the function is $\nabla\Phi=\hat{x}+\hat{y}$. For $\Phi_2=x_2+y_2>\Phi_1=x_1+y_1$, $\nabla\Phi$ is pointing in the direction of increasing Φ .

P1.1.12

$$\begin{aligned}
&\nabla\cdot\left(\overline{E}\times\overline{H}\right)=\nabla\cdot\begin{bmatrix}\hat{x} & \hat{y} & \hat{z}\\ E_x & E_y & E_z\\ H_x & H_y & H_z\end{bmatrix}\\
&=\frac{\partial}{\partial x}\left(E_yH_z-E_zH_y\right)+\frac{\partial}{\partial y}\left(E_zH_x-E_xH_z\right)+\frac{\partial}{\partial z}\left(E_xH_y-E_yH_x\right)\\
&=H_x\left(\frac{\partial}{\partial y}E_z-\frac{\partial}{\partial z}E_y\right)+H_z\left(\frac{\partial}{\partial x}E_y-\frac{\partial}{\partial y}E_x\right)+H_y\left(\frac{\partial}{\partial z}E_x-\frac{\partial}{\partial x}E_z\right)\\
&-E_x\left(\frac{\partial}{\partial y}H_z-\frac{\partial}{\partial z}H_y\right)-E_y\left(\frac{\partial}{\partial z}H_x-\frac{\partial}{\partial x}H_z\right)-E_z\left(\frac{\partial}{\partial x}H_y-\frac{\partial}{\partial y}H_x\right)\\
&=\overline{H}\cdot\left(\nabla\times\overline{E}\right)-\overline{E}\cdot\left(\nabla\times\overline{H}\right)\\
&\nabla\cdot\left(\nabla\times\overline{A}\right)=\nabla\cdot\begin{bmatrix}\hat{x} & \hat{y} & \hat{z}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ A_x & A_y & A_z\end{bmatrix}=0\\
&\nabla\times\left(\nabla\phi\right)=\begin{bmatrix}\hat{x} & \hat{y} & \hat{z}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ \partial\phi/\partial x & \partial\phi/\partial y & \partial\phi/\partial z\end{bmatrix}=0\\
&\nabla\times\left(\nabla\times\overline{E}\right)=\nabla\times\begin{bmatrix}\hat{x} & \hat{y} & \hat{z}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ E_x & E_y & E_z\end{bmatrix}\\
&=\begin{bmatrix}\hat{x} & \hat{y} & \hat{z}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ \left(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}\right) & \left(\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}\right) & \left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)\end{bmatrix}\\
&=\left[\frac{\partial}{\partial x}\left(\frac{\partial E_y}{\partial y}+\frac{\partial E_x}{\partial x}+\frac{\partial E_z}{\partial z}\right)-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)E_x\right]\hat{x}\\
&+\left[\frac{\partial}{\partial y}\left(\frac{\partial E_y}{\partial y}+\frac{\partial E_x}{\partial y}+\frac{\partial E_z}{\partial z}\right)-\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)E_y\right]\hat{y}\\
&+\left[\frac{\partial}{\partial z}\left(\frac{\partial E_y}{\partial y}+\frac{\partial E_x}{\partial y}+\frac{\partial E_z}{\partial z}\right)-\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)E_z\right]\hat{z}\\
&=\nabla\left(\nabla\cdot\overline{E}\right)-\nabla^2\overline{E}
\end{aligned}$$

To prove (1.1.9), we may also write

$$\begin{aligned}
\nabla\cdot(\overline{E}\times\overline{H}) &= \begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ E_x & E_y & E_z\\ H_x & H_y & H_z\end{vmatrix} = \begin{vmatrix}H_x & H_y & H_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ E_x & E_y & E_z\end{vmatrix} - \begin{vmatrix}E_x & E_y & E_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ H_x & H_y & H_z\end{vmatrix}\\
&= \overline{H}\cdot(\nabla\times\overline{E})-\overline{E}\cdot(\nabla\times\overline{H})
\end{aligned}$$

P1.1.13

We write (1.1.21) as $\nabla \times \overline{H} = \lim_{\Delta V \to 0} \oiint dS\, \hat{s} \times \overline{H} / \Delta V$
Applying the above result to a large V containing an infinite number of such differential volumes, we find the curl theorem

$$\iiint_V dV\, \nabla \times \overline{H} = \oiint_S dS\, \hat{s} \times \overline{H}$$

This is the curl theorem similar to the divergence theorem except that now the result is in vector form.

P1.1.14

If the surface integral of $\nabla \times \overline{H}$ is carried out over a closed surface, there will be no external contour enclosing the surface and the result will be zero.

$$\oiint_S d\overline{S} \cdot (\nabla \times \overline{H}) = 0 \qquad \text{(A1.1.14.1)}$$

This scalar equation should not be confused with Stokes theorem which is obtained by integrating over an open surface or the curl theorem in P1.1.13 for which we integrated over a volume V enclosed by a surface S, which is a vector relation.

P1.1.15

$\nabla \cdot \overline{A} = 3\rho + 2$, $\iiint dV\, \nabla \cdot \overline{A} = 6\pi \int (3\rho^2 + 2\rho) = 6\pi(5^3 + 5^2) = 900\pi$

$$\oiint_S d\overline{S} \cdot \overline{A} = 10\pi \int_0^3 dz\ 5^2 + 6\pi 5^2 = 900\pi$$

P1.1.16

$[\overline{A} \times (\nabla \times \overline{B})]_i = \epsilon_{ijk}\epsilon_{klm} A_j \partial_l B_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) A_j \partial_l B_m = A_m \partial_i B_m - A_l \partial_l B_i = A_j \partial_i B_j - [(\overline{A} \cdot \nabla)\overline{B}]_i$

P1.1.17

$\partial_i(\overline{A} \cdot \overline{A}) = A_j \partial_i B_j + B_j \partial_i A_j = A_j \partial_j B_i + B_j \partial_j A_i + A_j \partial_i B_j - A_j \partial_j B_i + B_j \partial_i A_j - B_j \partial_j A_i = A_j \partial_j B_i + (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) A_j \partial_l B_m + A_j \partial_j B_i + (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) B_j \partial_l A_m = A_j \partial_j B_i + B_j \partial_j A_i + \epsilon_{ijk} A_j \epsilon_{klm} \partial_l A_m + \epsilon_{ijk} A_j \epsilon_{klm} \partial_l A_m = [(\overline{A} \cdot \nabla)\overline{B} + \overline{A} \times (\nabla \times \overline{B}) + (\overline{B} \cdot \nabla)\overline{A} + \overline{B} \times (\nabla \times \overline{A})]_i$

or $[(\overline{A} \cdot \nabla)\overline{B} + \overline{A} \times (\nabla \times \overline{B})]_i = [(\overline{A} \cdot \nabla)\overline{B}]_i + A_j \partial_i B_j - [(\overline{A} \cdot \nabla)\overline{B}]_i$.

P1.1.18

$\partial_i(\overline{A} \cdot \overline{A}) = 2A_j \partial_i A_j = 2A_j \partial_j A_i + 2A_j \partial_i A_j - 2A_j \partial_j A_i = 2A_j \partial_j A_i + 2(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) A_j \partial_l A_m = 2A_j \partial_j A_i + 2\epsilon_{ijk} A_j \epsilon_{klm} \partial_l A_m = 2[(\overline{A} \cdot \nabla)\overline{A} + \overline{A} \times (\nabla \times \overline{A})]_i$

or $[(\overline{A}\cdot\nabla)\overline{A}+\overline{A}\times(\nabla\times\overline{A})]_i=[(\overline{A}\cdot\nabla)\overline{A}]_i+A_j\partial_i A_j-[(\overline{A}\cdot\nabla)\overline{A}]_i\,.$

P1.1.19

$$\overline{E}=-\nabla\Phi=\frac{C}{(x^2+y^2+z^2)^{3/2}}\,[\hat{x}x+\hat{y}y+\hat{z}z]=\overline{r}\frac{C}{r^3}=\hat{r}\frac{C}{r^2}$$

in terms of the position vector $\overline{r}=\hat{x}x+\hat{y}y+\hat{z}z$, and the length of the position vector $r=\sqrt{x^2+y^2+z^2}$, and $\hat{r}$ is pointing in the direction of $\overline{r}$ with unit length. Assuming that the electric field is due to a charged particle q situated at the origin, we can integrate Gauss' law over a small spherical volume with radius $r=\delta$ surrounding the origin to obtain

$$q=\oiint_S d\overline{S}\cdot\overline{D}=\int_0^{\pi}\int_0^{2\pi}d\theta\;d\phi\;\delta^2\;\sin\theta\;\frac{\epsilon_o C}{\delta^2}=4\pi\epsilon_o C$$

Thus the constant $C=q/4\pi\epsilon_o$ and the static electric field

$$\overline{E}=\hat{r}\,\frac{q}{4\pi\epsilon_o r^2}$$

P1.2.1

$\overline{E}_1$ and $\overline{E}_3$ qualify as electromagnetic waves.
$\overline{E}_2$ and $\overline{E}_4$ violate Gauss' law $\nabla\cdot\overline{E}=0$.

P1.2.2

$\overline{E}=\hat{x}E_0\cos(kz+\omega t)$. As time t increases, z must decrease in order for $kz+\omega t=$ constant, thus the wave is propagating in the $-\hat{z}$ direction.

P1.2.3

Wavelength $\lambda=2\pi/k_0=0.01\,\text{m}$.
Frequency $f=c/\lambda=30\,\text{GHz}$.
For $\lambda=632.8\,\text{nm}$, $k=1/\lambda=1.58\times10^6\,\text{K}_o$.
For $f=2.4\,\text{GHz}$, $k=f/c=2.4\times10^9\,\text{Hz}/3\times10^8\,\text{m/s}=8\,\text{K}_o$.

P1.2.4

(a) (i) 60 Hz: $\lambda=c/f=5\times10^6\,(\text{m})$
(ii) AM radio (535–1605 kHz): $\lambda=186.9\sim560.8\,(\text{m})$
(iii) FM radio (88–108 MHz): $\lambda=2.778\sim3.409\,(\text{m})$
(iv) Visible light ($\sim10^{14}$ Hz): $\lambda=\sim3\times10^{-6}\,(\text{m})$
(v) X-rays ($\sim10^{18}$ Hz): $\lambda=\sim3\times10^{-10}\,(\text{m})$

(b) (i) 1 km: $f=c/\lambda=3\times10^5\,(\text{Hz})$
(ii) 1 m: $f=3\times10^8\,(\text{Hz})$
(iii) 1 mm: $f=3\times10^{11}\,(\text{Hz})$
(iv) 1 μm: $f=3\times10^{14}\,(\text{Hz})$
(v) 1 Å: $f=3\times10^{18}\,(\text{Hz})$

(c) (i) 1 km: $k=2\pi/\lambda=\text{K}_o/\lambda=10^{-3}\,\text{K}_o$

(ii) 1 m: $k = 1\,\mathrm{K}_o$
(iii) 1 mm: $k = 10^3\,\mathrm{K}_o$
(iv) $1\,\mu$m: $k = 10^6\,\mathrm{K}_o$
(v) 1 Å: $k = 10^{10}\,\mathrm{K}_o$

(d) (i) 1 km: $\hbar\omega = 1.24 \times 10^{-9}$ eV
(ii) 1 m: $\hbar\omega = 1.24 \times 10^{-6}$ eV
(iii) 1 mm: $\hbar\omega = 1.24 \times 10^{-3}$ eV
(iv) $1\,\mu$m: $\hbar\omega = 1.24$ eV
(v) 1 Å: $\hbar\omega = 1.24 \times 10^4$ eV

P1.2.5

$v_p = v_g = c$

P1.2.6

(a) At $z = z_0$, $E_x = -2\sin(kz_0 - \omega t)$, and $E_y = \frac{1}{\sqrt{2}}\cos(kz_0 - \omega t) - \frac{1}{\sqrt{2}}\sin(kz_0 - \omega t)$. Since $E_x^2/2 - \sqrt{2}E_xE_y + 2E_y^2 = 1$, the wave is elliptically polarized.
(b) $\overline{E} = \frac{1}{2}\left[\hat{x}\cos(kx - \omega t) + \hat{y}\sin(kz - \omega t)\right] + \frac{1}{2}\left[\hat{x}\cos(kx - \omega t) - \hat{y}\sin(kz - \omega t)\right]$
(c) $\overline{E} = \hat{x}\cos(kz - \omega t + \pi/4) + \hat{y}\cos(kz - \omega t - \pi/4)$. This is the superposition of two linearly polarized waves.

P1.2.7

The wave has wavelength 1 cm, and is right-hand circularly polarized, the helix is left-handed, and its pitch is 1 cm.

P1.2.8

For a right-handed circularly polarized wave $\alpha = \pi/4$, then

$$Q = I\cos(2\pi/4)\cos(2\beta) = 0$$
$$U = I\cos(2\pi/4)\sin(2\beta) = 0$$
$$V = I\sin(2\pi/4) = I$$

For a left-handed circularly polarized wave $\alpha = -\pi/4$, then

$$Q = I\cos(-2\pi/4)\cos(2\beta) = 0$$
$$U = I\cos(-2\pi/4)\sin(2\beta) = 0$$
$$V = I\sin(-2\pi/4) = -I$$

For linearly polarized wave $\alpha = 0$, then

$$V = I\sin 0 = 0$$

P1.3.1

$mv^2/R = Ze^2/4\pi\epsilon R^2 \Rightarrow R = 4\pi\epsilon n^2\hbar^2/Zme^2 \approx 0.52\,n^2 \times 10^{-10}\,\mathrm{m}$ for $Z = 1$.

P1.3.2

(a) $T = 150 \times 10^9/c = 500\,\text{sec} = 8.33\,\text{min}$
(b) $P_r = 1.5\,\text{kW/m}^2 \times \pi \times (6.4 \times 10^6)^2\,\text{m}^2 = 1.93 \times 10^{14}\,\text{kW}$
(c) $S = P$ (power density per Hz) $\times\, W$ (bandwidth) $= 10^{-11}\,\text{Wm}^{-2}$
$E = \sqrt{2\eta S} = 8.68 \times 10^{-5}\,\text{volt/m}$

P1.3.3

The power density is $P = 1.2 \times 10^{10}\,\text{W/m}^2$. The radiation pressure is $p = 40\,\text{N/m}^2$. The area required is $20\,\text{m}^2$.

P1.3.5

(a) $\overline{B} = \hat{x} I_0 \mu_o / 2\pi d$
(b) $\overline{F} = \hat{y} N q v B_0$

P1.3.4

$\overline{F} = \hat{x} m \frac{d^2x}{dt^2} + \hat{y} m \frac{d^2y}{dt^2} = -m\omega_c^2(\hat{x}x + \hat{y}y) = -m\hat{\rho}R\omega_c^2 = -\hat{\rho}mv^2/R$

P1.3.6

The Lorentz force acting on the particle is qvB_0 and the centrifugal force acting on the particle is mv^2/R, where R is the radius of the circle. We have $qvB_0 = mv^2/R$. The time it takes the particle to complete one revolution is $2\pi R/v = 2\pi m/qB_0$. The cyclotron frequency is thus $\omega_c = v/R = qB_0/m$, and the radius is $R = mv/qB_0$.

P1.3.7

(a) Because there is a magnetic field, the effective electric field that drives conduction current is approximately

$$\overline{E}_{eff} \cong \overline{E} + \overline{v} \times \overline{B}$$

Hence $\overline{J} \cong \sigma(\overline{E} + \overline{v} \times \overline{B})$. When $\sigma \to \infty$, $\overline{E}, \overline{B}$ still remains finite, then we have to impose $\overline{E} + \overline{v} \times \overline{B} = 0$ or $\overline{E} = -\overline{v} \times \overline{B}$. This is used in approximating solar wind fields.

(b) Let $\overline{v} = \overline{v}_n + \overline{v}_\parallel$, where $\overline{v}_n$ is normal to $\overline{B}$ and $\overline{v}_\parallel$ is parallel to $\overline{B}$. Then $\overline{E} = -\overline{v} \times \overline{B} = -\overline{v}_n \times \overline{B}$. The Poynting vector

$$\begin{aligned}\overline{S} = \overline{E} \times \overline{H} &= \frac{1}{\mu_0}\overline{E} \times \overline{B} = -\frac{1}{\mu_0}(\overline{v}_n \times \overline{B}) \times \overline{B} = \frac{B^2}{\mu_0}\overline{v}_n \\ &\approx \frac{(5 \times 10^{-9})^2}{4\pi \times 10^{-7}} \times 4 \times 10^5 \times \cos 45^\circ \approx 5.6\mu\text{W/m}^2\end{aligned}$$

(c) Kinetic energy density $W_k = \frac{1}{2}\rho_m v^2 \approx \frac{1}{2} \times (1800 \times 9.1 \times 10^{-31} \times 10^7) \times (4 \times 10^5)^2 = 1.31 \times 10^{-9}\,\text{joule/m}^3$.
Electric energy density $W_e = \frac{1}{2}\epsilon_0 E^2 \approx \frac{1}{2} \times 8.85 \times 10^{-12} \times (4 \times 10^5 \times 5 \times 10^{-9} \times \sin 45^\circ)^2 = 8.85 \times 10^{-18}\,\text{joule/m}^3$.

Magnetic energy density $W_m = \frac{1}{2\mu_0}B^2 \approx \frac{1}{2\times 4\pi\times 10^{-7}} \times (5\times 10^{-9})^2 = 9.9\times 10^{-12}\text{joule/m}^3$ Therefore $W_k \gg W_m \gg W_e$.
Kinetic energy density is the largest.

P1.3.8

a) For $\delta = 0$ our model becomes $\frac{\partial^2 x}{\partial t^2} + \omega_0^2 x + \frac{qE}{m} = 0$. Assuming that driving and driven quantities have sinusoidal time dependency ω , we may write $\left(\omega_0^2 - \omega^2\right)\overline{x} = -\frac{q\overline{E}}{m}$ or $x = \frac{qE}{m\left(\omega^2-\omega_0^2\right)}$. For $\omega > \omega_0$ the electrons are in phase with the E-field, but for $\omega < \omega_0$ the electrons are $180°$ out of phase. In terms of current (or radiation) the oscillation is $180°$ out of phase for $\omega > \omega_0$ (for electrons or ions) thus tending to cancel the exciting field (by radiating a competing field $180°$ out of phase). This cancellation becomes complete if there are many particles participating and if their amplitudes are large enough. Thus we want $\omega > \omega_0$ (for opacity) but not so large as to render $\overline{x}$ too small and we want a high density. This condition is in fact met by $0 < \omega^2 - \omega_0^2 < \omega_p^2$ as is the case in many metals with ω in optical regime and ω_0 much smaller and ω_p in the ultraviolet regime. Thus these metals appear opaque.

b) Poynting theorem $-\nabla\cdot(\overline{E}\times\overline{H}) - \frac{\mu_0}{2}\frac{\partial}{\partial t}H^2 - \frac{\epsilon_0}{2}\frac{\partial}{\partial t}E^2 = \overline{E}\cdot\overline{J}$.

$\overline{E} = -\frac{m}{q}\left(\frac{\partial^2\overline{x}}{\partial t^2} + \delta\frac{\partial\overline{x}}{\partial t} + \omega_0^2\overline{x}\right)$. Assume a particle density n and velocity v , $\overline{J} = qnv = qn\frac{\partial\overline{x}}{\partial t}$. Thus

$$\begin{aligned} E\cdot J &= -mn\frac{\partial\overline{x}}{\partial t}\cdot\left(\frac{\partial^2\overline{x}}{\partial t^2} + \delta\frac{\partial\overline{x}}{\partial t} + \omega_0^2\overline{x}\right) \\ &= -mn\left(\frac{\partial}{\partial t}\left[\frac{1}{2}\left(\frac{\partial\overline{x}}{\partial t}\right)^2\right] + \delta\left(\frac{\partial x}{\partial t}\right)^2 + \omega_0^2\frac{\partial}{\partial t}\left(\frac{1}{2}x^2\right)\right) \end{aligned}$$

$$W \equiv \frac{\mu_0 H^2}{2} + \frac{\epsilon_0 E^2}{2} + \frac{mn}{2}\left(\frac{\partial x}{\partial t}\right)^2 + \frac{mn\omega_0^2}{2}x^2 \text{ (Energy desity)}$$

$$P_D \equiv mn\delta\left(\frac{\partial x}{\partial t}\right)^2 \text{ (Power density dissipated through collision)}$$

$$\frac{\mu_0 H^2}{2} = \text{magnetic energy density}$$

$$\frac{\epsilon_0 E^2}{2} = \text{electric energy density}$$

$$\frac{mn}{2}\left(\frac{\partial x}{\partial t}\right)^2 = \text{particle kinetic energy density}$$

$$\frac{mn\omega_0^2 x^2}{2} = \text{particle potential energy density}$$

P1.3.9

(a) Using Gauss' law, the electric field between the two plates due to the charges is given by $\overline{E} = \hat{x}\ \sigma/\epsilon_0$. The Poynting vector and the momentum density vector are given by $\overline{S} = \overline{E} \times \overline{H} = \hat{x}\frac{\sigma}{\epsilon_0} \times \hat{y}\frac{B_0}{\mu_0}e^{-\gamma t} = \hat{z}\frac{B_0\sigma}{\epsilon_0\mu_0}e^{-\gamma t}$ and $\overline{g}_f = \epsilon_0\mu_0\overline{S} = \hat{z}B_0\sigma e^{-\gamma t}$.

(b) By Faraday's law, the induced electric field will exist along the surface of the plate. Accordingly,

$$\oint_C \overline{E} \cdot d\bar{l} = -\frac{\partial}{\partial t}\int_S \overline{B} \cdot d\overline{S} \Rightarrow 2E_0 l = \gamma l d B_0 e^{-\gamma t} \Rightarrow E_0 = \frac{\gamma d B_0}{2}e^{-\gamma t}$$

By symmetry, both the E-field at the top and bottom will be equal in magnitude, however opposite in direction. The total force density along the top and bottom of the plate will be $\overline{F} = \hat{z}\ 2\sigma E_0 = \hat{z}\ \gamma d B_0 \sigma e^{-\gamma t}$.

(c) The mechanical momentum density vector, $\overline{g}_m$, can then be found by integrating the force density vector.

$$\overline{g}_m = \hat{z}\int_0^t \gamma B_0 \sigma e^{-\gamma t'} dt' = \hat{z}\left[-B_0\sigma e^{-\gamma t'}\right]_0^t = \hat{z}B_0\sigma(1 - e^{-\gamma t})$$

(d) Adding the field and mechanical momentum terms, we see that the total momentum of the system is conserved, $\overline{g} = \overline{g}_f + \overline{g}_m = \hat{z}\ B_0\sigma$.

P1.3.10

For the plane current loop, we let the line charge density be ρ amp/m. The magnetic moment for the segment dl is

$$d\overline{M} = \frac{1}{2}(\rho dl)\overline{r} \times \overline{v} = \frac{1}{2}dl\overline{r} \times \overline{I} = \frac{1}{2}I\overline{r} \times d\bar{l}$$

The total magnetic moment of the loop is thus $\overline{M} = \oint d\overline{M} = \frac{1}{2}I\oint \overline{r} \times d\bar{l}$. For the plane loop $\oint \overline{r} \times d\bar{l} = 2\hat{m}A$. In the case of a circle with radius R , $\oint \overline{r} \times d\bar{l} = \hat{z}2\pi R^2$.

P1.3.11

(a) $\gamma = q/2m$. For a complicated structure of charged distributions, the gyromagnetic ratio is $\gamma = gq/2m$, where the g-factor g describes the magnetic structure.

(b) Let $\overline{M} = \hat{x}M_x + \hat{y}M_y + \hat{z}M_z$, $\frac{d\overline{M}}{dt} = \gamma\overline{M} \times \overline{B}$ gives

$\frac{dM_y}{dt} = -\gamma M_x B_0 \qquad \frac{dM_x}{dt} = \gamma M_y B_0 \qquad \frac{dM_z}{dt} = 0$ which yield

$M_x = M_0\cos(\gamma B_0 t + \phi_0) \qquad M_y = -M_0\sin(\gamma B_0 t + \phi_0) \qquad M_z = M_{0z}$

Thus the angular Larmor frequency of precession is $\omega = \gamma B_0$ . .

(c) $\nabla \cdot \overline{B} = 0$ gives $B_z = B_0 - z$

(d) The angular precession Larmor frequency is . $\omega = \gamma B_z = \gamma(B_0 - z)$.

(e) $\delta f = \delta\omega/2\pi = \gamma \times \delta_z/2\pi = 43\,\text{kHz}$

P1.3.12

The magnetic field, $\overline{H}$ at the position of the loop due to the straight wire carrying current I_0 is $\overline{H} = \hat{\phi}\frac{I_0}{2\pi d} = \hat{x}\frac{I_0}{2\pi d}$.

$$\overline{T} = \overline{M} \times \overline{B} = \hat{z}\frac{A_0 I_l I_0 \mu_0}{2\pi d}$$

which means that the current loop will move about the z-axis in a counter-clockwise direction.

P1.3.13

The dissipated power per unit volume is $P_d = \overline{f} \cdot \overline{v} = \rho\overline{v} \cdot \overline{E} = \overline{J} \cdot \overline{E}$.

P1.3.14

The Poynting vector is calculated to be

$$\overline{G} = \overline{D} \times \overline{B} = \hat{z}\sqrt{\frac{\epsilon_o}{\mu_o}}\, c^2 E_0^2 \cos^2(kz - \omega t)$$

The Maxwell stress tensor is

$$\overline{\overline{T}} = \frac{1}{2}\left(\mu_o H_0^2 + \epsilon_o E_0^2\right)\cos^2(kz - \omega t)\overline{\overline{I}} - \left(\hat{x}\hat{x}\epsilon_o E_0^2 + \hat{y}\hat{y}\mu_o H_0^2\right)\cos^2(kz - \omega t)$$

From (1.3.12) we find the force density

$$\overline{f} = -\nabla \cdot \overline{\overline{T}} - \frac{\partial \overline{G}}{\partial t} = \hat{z}k(\mu_o H_0^2 + \epsilon_o E_0^2)\sin(kz - \omega t)\cos(kz - \omega t)$$

P1.3.15

For $\partial_i = \frac{\partial}{\partial x_i}$, Maxwell's equations can be written in index notation as

$$\begin{aligned}
\nabla \times \overline{H} = \frac{\partial \overline{D}}{\partial t} \quad &\Longleftrightarrow \quad \frac{\partial D_i}{\partial t} = \varepsilon_{ijk}\partial_j H_k \\
\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} \quad &\Longleftrightarrow \quad \frac{\partial B_i}{\partial t} = -\varepsilon_{ijk}\partial_j E_k \\
\nabla \cdot \overline{D} = 0 \quad &\Longleftrightarrow \quad \partial_i D_i = 0 \\
\nabla \cdot \overline{B} = 0 \quad &\Longleftrightarrow \quad \partial_i B_i = 0
\end{aligned}$$

The i^{th} component of the time derivative of $\overline{D} \times \overline{B}$ is

$$\begin{aligned}
\frac{\partial}{\partial t}(\overline{D} \times \overline{B})_i &= \frac{\partial}{\partial t}(\varepsilon_{ijk} D_j B_k) = \varepsilon_{ijk} D_j \frac{\partial B_k}{\partial t} + \varepsilon_{ijk}\frac{\partial D_j}{\partial t} B_k \\
&= -\varepsilon_{ijk}\varepsilon_{pqk} D_j(\partial_p E_q) + \varepsilon_{kij}\varepsilon_{mnj} B_k \partial_m H_n \\
&= -(\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}) D_j(\partial_p E_q) + (\delta_{km}\delta_{in} - \delta_{kn}\delta_{im}) B_k(\partial_m H_n) \\
&= D_j\partial_j E_i - D_j\partial_i E_j + B_k\partial_k H_i - B_k\partial_i H_k \qquad \text{(A1.3.15.1)}
\end{aligned}$$

where we use the identity $\varepsilon_{ijk}\varepsilon_{rsk} = \delta_{ir}\delta_{js} - \delta_{is}\delta_{jr}$. Identify the terms in the right hand side of equation (A1.3.15.1):

$$\begin{aligned}
D_j \partial_i E_j &= \epsilon_0 E_j \partial_i E_j = \partial_i (\frac{1}{2}\epsilon_0 \overline{E}\cdot\overline{E}) = \partial_i(\frac{1}{2}\overline{D}\cdot\overline{E}) \\
B_k \partial_i H_k &= \mu_0 H_k \partial_i H_k = \partial_i(\frac{1}{2}\mu_0 \overline{H}\cdot\overline{H}) = \partial_i(\frac{1}{2}\overline{B}\cdot\overline{H}) \\
D_j \partial_j E_i &= D_j \partial_j E_i + E_i \partial_j D_j = \partial_j (D_j E_i) = \nabla\cdot(\overline{D}E_i) \\
B_k \partial_k H_i &= B_k \partial_k H_i + H_i \partial_k B_k = \partial_k (B_k H_i) = \nabla\cdot(\overline{B}H_i)
\end{aligned}$$

we find $\frac{\partial}{\partial t}(\overline{D}\times\overline{B})_i = -\partial_i(\frac{1}{2}\overline{D}\cdot\overline{E} + \frac{1}{2}\overline{B}\cdot\overline{H}) + \nabla\cdot(\overline{D}E_i + \overline{B}H_i)$, which is identical to

$$\frac{\partial}{\partial t}(\overline{D}\times\overline{B}) + \nabla\cdot\left(W\overline{\overline{I}} - \overline{DE} - \overline{BH}\right) = 0$$

P1.4.1

(a) $\left\langle \overline{S}\right\rangle = \hat{r}\frac{\omega k^3}{2\epsilon_o}\left(\frac{q\ell}{4\pi r}\right)^2 \sin^2\theta$

(b) $P = \displaystyle\int_0^\pi d\theta 2\pi r^2 \sin\theta \left[\frac{\omega k^3}{2\epsilon_o}\left(\frac{q\ell}{4\pi r}\right)^2 \sin^2\theta\right] = \frac{4\pi\omega k^3}{3\epsilon_o}\left(\frac{q\ell}{4\pi}\right)^2$

(c) $R_{rad} = \frac{2P}{I_o^2} = \frac{8\pi k^3}{3\epsilon_o\omega}\left(\frac{\ell}{4\pi}\right)^2$

(d) For $\theta = \pi/2$ $E_o = -\frac{k^2 q\ell}{4\pi\epsilon_o r}$, $q\ell = -\frac{4\pi\epsilon_o r}{k^2}E_o$. Notice that the radiation field is only in the upper half space for the radio antenna, therefore
$P = \frac{2\pi}{3\eta_o}\left(E_o r\right)^2 = \frac{1}{180}\left(25\times10^{-3}\times15\times10^3\right)^2 = 781.25(W)$

P1.4.2

For the p_z component, the electric field vector in the rectangular coordinate system is

$$\begin{aligned}
\overline{E}_{p_z} &= [\hat{r}2\cos\theta + \hat{\theta}\sin\theta]\frac{p_z}{4\pi\epsilon_o r^3} \\
&= \frac{p_z}{4\pi\epsilon_o}\left\{\hat{x}\frac{3}{r^3}\left(\frac{xz}{r^2}\right) + \hat{y}\frac{3}{r^3}\left(\frac{yz}{r^2}\right) + \hat{z}\frac{3}{r^3}\left(\frac{z}{r}\right)^2 - \hat{z}\frac{1}{r^3}\right\}
\end{aligned}$$

The total electric field due to all three components is therefore

$$\overline{E} = [3\hat{r}(\hat{r}\cdot\overline{p}) - \overline{p}]\frac{1}{4\pi\epsilon_o r^3} = [(\overline{p}\times\hat{r})\times\hat{r} + 2\hat{r}(\hat{r}\cdot\overline{p})]\frac{1}{4\pi\epsilon_o r^3}$$

P1.4.3

Looking at $\phi = 0$, the sky is unpolarized, looking at the zenith ($\phi = 90^\circ$) the sky is linearly polarized, looking at other parts of the sky, it is partially linearly polarized.

P1.4.4

(a) $\overline{E} = \frac{p}{4\pi\epsilon r^3}\left(\hat{\theta}\sin\theta + \hat{r}\,2\cos\theta\right)$

(b) The total power scattered by a Hertzian dipole with dipole moment $p_0 E_0$

$$P = \int_0^\pi d\theta \; 2\pi r^2 \sin\theta \left[\frac{\omega k^3}{2\epsilon_o} \left(\frac{p_0 E_0}{4\pi r} \right)^2 \sin^2\theta \right] = \frac{4\pi\omega k^3}{3\epsilon_o} \left(\frac{p_0 E_0}{4\pi} \right)^2 = \frac{k^4 p_0^2 E_0^2}{12\pi\eta\epsilon_0^2}$$

Scattering cross section $2\eta P_s / E_o^2 = \frac{k^4}{6\pi\epsilon_0^2}\, p_0^2$.

P1.4.5

$P_s = \frac{4\pi}{3\eta}\left(\frac{\epsilon_a - \epsilon_o}{\epsilon_a + 2\epsilon_o}\right)^2 k^4 a^6 E_0^2$. Sky is blue as blue light has a larger k and thus scatters more. It is not violet because there is less violet light reaching the lower atmosphere for scattering and the color receptors in our eyes are stimulated differently. The red and green cones are stimulated about equally and the blue cones are stimulated more strongly, resulting in perceiving a pale sky blue color.

P1.4.6

(a) $P_{\text{scatt}} = \frac{4\pi}{3\eta}\left[\frac{\epsilon_p - \epsilon_o}{\epsilon_p + 2\epsilon_o}\right]^2 k^4 a^6 E_o^2 = \frac{\pi}{12\eta} k^4 a^6 E_o^2$

(b) The total power loss of a control-volume with area A and length dl is

$$\begin{aligned} \frac{1}{P}\frac{dP}{d\ell} &= \frac{2\eta}{AE_0^2} \times \frac{\pi k^4 \times 10^{-60}}{12\eta} E_0^2 \times N \times A \\ &= \frac{\pi k^4 \times 10^{-60}}{6} N = \frac{\pi (2\pi \times 10^6)^4 \times 10^{-60}}{6} N \\ &= \frac{8N}{3}\pi^5 \times 10^{-36}\ \mathrm{m}^{-1} = \frac{8N}{3}\pi^5 \times 10^{-33}\ \mathrm{km}^{-1} \end{aligned}$$

which gives rise to a loss of $(300.88 - 10\log N)$ dB/km.

(c) $\frac{1}{P}\frac{dP}{d\ell} \approx 0.2\,\mathrm{km}^{-1}$ gives rise to a loss of 6.99 dB/km.

P1.4.7

(a) $\sqrt{x^2 + (y-d)^2 + z^2} = \sqrt{x^2 + y^2 + z^2 - 2yd + d^2} \approx \sqrt{r^2 - 2yd}$
$= r\sqrt{1 - 2\frac{y}{r^2}d} \approx r(1 - \frac{1}{2} \times 2\frac{y}{r^2}d) = r - \frac{y}{r}d = r - d\sin\theta$

(b) $\overline{E}_{tot} = -\frac{k^2\ell}{4\pi r\epsilon_0}\left[\hat{\theta} q_1 \cos(kr - \omega t)\sin\theta - \hat{x} q_2 \cos(kr - kd\sin\theta - \omega t)\right]$

(c)

(i) $d\sin\theta = \frac{\lambda}{4}$; $d = \sqrt{2}\frac{\lambda}{4}$

(ii) $\frac{\sqrt{2}}{2} q_1 = q_2$; $\quad q_1/q_2 = \sqrt{2}$

(iii) $\overline{E}_{tot} = -\frac{k^2 \ell q_2}{4\pi r\epsilon_0}\left[\hat{\theta}\cos(\omega t - kr) - \hat{\phi}\sin(\omega t - kr)\right] \Rightarrow L.H.C.P$

P1.4.8

Writing $\ell = dz$, $r = \sqrt{\rho^2 + z^2}$, $\sin\theta = \frac{\rho}{\sqrt{\rho^2 + z^2}}$, and $z = \rho\tan\alpha$, yields

$$\overline{B} = \hat{\phi}\int_{-\infty}^{\infty} dz \frac{\mu_o \rho I}{4\pi(\rho^2 + z^2)^{3/2}} = \hat{\phi}\int_{-\pi/2}^{\pi/2} \sec^2\alpha\, d\alpha \frac{\mu_o \rho^2 I}{4\pi \sec^3\alpha} = \hat{\phi}\frac{\mu_o I}{2\pi\rho}.$$

The magnetic field is $\overline{B} = \hat{\phi}\frac{\mu_o I}{2\pi\rho} = \hat{\phi}\frac{4\pi\times10^{-7}\times10^3}{2\pi\times10} = \hat{\phi}\,2\times10^{-5}$ Tesla.

P1.5.1

(a) This constitutive relation for cholesteric liquid crystals is
  (1) *Anisotropic*
  (2) *Linear*
  (4) *Inhomogeneous*: $\overline{\overline{\epsilon}}$ depends on position.

(b) This constitutive relation for the quartz crystals is
  (1) *Bianisotropic*
  (2) *Linear*
  (3) *Temporally dispersive*
  (4) *Homogeneous*

Another answer is

$$E_j = \kappa_{ij} D_i + c^2 G_{ij} \frac{\partial}{\partial t} B_i = \kappa_{ij} D_i - c^2 G_{ij} (\nabla \times \overline{E})_i$$
$$H_j = \frac{1}{\mu_0} B_j - c^2 G_{ij} \frac{\partial}{\partial t} D_i = \frac{1}{\mu_0} B_j - c^2 G_{ij} (\nabla \times \overline{H})_i$$

Express $\overline{D}$ and $\overline{B}$ in terms of $\overline{E}$ and $\overline{H}$.

$$D_j = \kappa_{ij}^{-1}\left[E_i + c^2 G_{ki} (\nabla \times \overline{E})_k\right]$$
$$B_j = \mu_0 \left[H_j + c^2 G_{ij} (\nabla \times \overline{H})_i\right]$$

Then the constitutive relation is
  (1) *Anisotropic*
  (2) *Linear*
  (3) *Spatial dispersive*
  (4) *Homogeneous*

(c) We can write $\overline{J} \simeq \sigma(\overline{E} + R\sigma\overline{E}\times\overline{B}_0)$, in matrix form

$$\begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix} = \begin{bmatrix} \sigma & R\sigma^2 B_{0z} & -R\sigma^2 B_{0y} \\ -R\sigma^2 B_{0z} & \sigma & R\sigma^2 B_{0x} \\ R\sigma^2 B_{0y} & -R\sigma^2 B_{0x} & \sigma \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

$$\begin{aligned} \nabla\times\overline{H} &= -i\omega\epsilon_0\overline{E} + \overline{J} \\ &= -i\omega \begin{bmatrix} \epsilon_0 + i\dfrac{\sigma}{\omega} & i\dfrac{R\sigma^2}{\omega}B_{0z} & -i\dfrac{R\sigma^2}{\omega}B_{0y} \\ -i\dfrac{R\sigma^2}{\omega}B_{0z} & \epsilon_0 + i\dfrac{\sigma}{\omega} & i\dfrac{R\sigma^2}{\omega}B_{0x} \\ i\dfrac{R\sigma^2}{\omega}B_{0y} & -i\dfrac{R\sigma^2}{\omega}B_{0x} & \epsilon_0 + i\dfrac{\sigma}{\omega} \end{bmatrix} \cdot \overline{E} = -i\omega\overline{\overline{\epsilon}}\cdot\overline{E} \end{aligned}$$

The constitutive relation is thus
  (1) *Anisotropic*

(2) *Linear*
(3) *Temporally dispersive*: Permittivity depends on ω.
(4) *Homogeneous*

(d) Consider the following dispersion relation:

$$D_i = \epsilon_{ij} E_j + \gamma_{ijk} \frac{\partial E_j}{\partial x_k}$$

A repeated index in a product implies summation over that index from 1 to 3 (e.g., $A_i B_i = A_1 B_1 + A_2 B_2 + A_3 B_3$). An equation or an inequality holds for each of the unrepeated indices.
The constitutive relation is
(1) *Anisotropic*: $\overline{D}$ and $\overline{E}$ are not related by a scalar factor.
(2) *Linear*
(3) *Spatially dispersive*: The constitutive relation involves space derivatives of $\overline{E}$.
(4) *Homogeneous*: ϵ_{ij} and γ_{ijk} do not depend on $\overline{r}$.

(e) The constitutive relation for pyroelectricity is
(1) *Anisotropic*
(2) *Linear*: Variations of $\overline{D}$ and $\overline{E}$ are linearly related. $\delta\overline{D} = \overline{\overline{\epsilon}} \cdot \delta E$.
(4) *Homogeneous*

(f) The constitutive relation for piezoelectricity is
(1) *Anisotropic*
(2) *Linear*: Variations of $\overline{D}$ and $\overline{E}$ are linearly related. $\delta\overline{D} = \overline{\overline{\epsilon}} \cdot \delta E$.
(4) *Homogeneous*

Note: S_{kl} is the mechanical stress tensor. The force acting on an imaginary surface S in a solid is

$$F_k = \int_S ds\, S_{kl} n_l$$

The dimensions of S_{kl} are Force/Area.

(g) For the Kerr effect, the constitutive relation is
(1) *Anisotropic*
(2) *Nonlinear*
(4) *Homogeneous*

(h) For the Pockel's effect, the constitutive relation is
(1) *Anisotropic*
(2) *Nonlinear*
(4) *Homogeneous*

P1.5.2

In the low-field limit, $L(x) \approx \frac{x}{3}$, and $M \approx \frac{Nm^2H}{3kT}$, and the medium is linear.

P1.6.1

Consider a ribbon-like surface as shown in Fig. P1.6.1.1. Integrating over the surface of the ribbon area, Faraday's law and Ampère's law become

$$\oint d\bar{l} \cdot \overline{E} = -\frac{d}{dt} \iint dS\, \hat{s} \cdot \overline{B}$$
$$\oint d\bar{l} \cdot \overline{H} = \frac{d}{dt} \iint dS\, \hat{s} \cdot \overline{D} + \iint dS\, \hat{s} \cdot \overline{J}$$

Let the ribbon area approach zero in such a manner that δ goes to zero first and the terms involving δ are discarded. To relate $\overline{E}_1, \overline{H}_1$ in region 1 to $\overline{E}_2, \overline{H}_2$ in region 2, we proceed as follows.

The integral forms of Faraday's law and Ampère's law as applied to the ribbon area in Fig. P1.6.1.1 yield, as $\delta \to 0$,

$$\frac{d}{dt} \iint dS\, \hat{s} \cdot \overline{B} = 0 = \frac{d}{dt} \iint dS\, \hat{s} \cdot \overline{D}$$

because $d(\hat{s} \cdot \overline{B})/dt$ and $d(\hat{s} \cdot \overline{D})/dt$ remain finite while the ribbon area approaches zero. Therefore

$$d\bar{l} \cdot (\overline{E}_1 - \overline{E}_2) = 0$$
$$d\bar{l} \cdot (\overline{H}_1 - \overline{H}_2) = \hat{s} \cdot \overline{J} \delta\, dl$$

The electric field $\overline{E}$ in the $d\bar{l}$ direction is tangential to the surface and can be written as $d\bar{l} \cdot \overline{E} = dl\hat{s} \cdot \hat{n} \times \overline{E} = dl\hat{s} \times \hat{n} \cdot \overline{E}$ for all $dl\hat{s}$ along the interface and similarly for $\overline{H}$. We thus have

$$\hat{n} \times (\overline{E}_1 - \overline{E}_2) = 0$$
$$\hat{n} \times (\overline{H}_1 - \overline{H}_2) = \lim_{\delta \to 0} \overline{J} \delta \equiv \overline{J}_s$$

P1.6.2

We apply the curl theorem to a small pill-box volume on the x-y plane [Fig. P1.7.8.1], which has an area A and an infinitesimal thickness Δz. We let $\Delta z \to 0$ faster than $A \to 0$, such that terms involving Δz can be neglected:

$$\iiint dV\, \nabla \times \overline{H} \approx A\hat{z} \times (\overline{H}_{z>0} - \overline{H}_{z<0})$$

Such results are useful in the derivation of boundary conditions for the Maxwell equations. Integrating Ampere's law $\nabla \times H = \partial \overline{D}/\partial t + \overline{J}$ over the pill-box volume, we have $A\hat{z} \times (\overline{H}_{z>0} - \overline{H}_{z<0}) = A\Delta z \partial \overline{D}/\partial t + A\Delta z \overline{J}$. The first term on the right-hand side is neglected because physically $\partial \overline{D}/\partial t$ is finite. However if $\overline{J}$ is infinite in the pill-box then $\Delta z \overline{J} = \overline{J}_s$ is finite, where $\overline{J}_s = \hat{z} \times (\overline{H}_{z>0} - \overline{H}_{z<0})$. We call $\overline{J}_s$ surface current.

P1.6.3

Using Gauss' law of $\nabla \cdot \overline{D} = \rho$, we find $a\delta\rho = \hat{s}(a + l\delta) \cdot \left(\overline{D}_1 - \overline{D}_2\right)$. In the limit of $\delta \to 0$, $\delta\rho = \rho_s$, the last term vanishes, and we obtain (1.6.9).

P1.7.1

(a) $\nabla \cdot \overline{E} = 0$ gives $\begin{cases} E_x k_x + E_z k_z = 0 \\ E_y = \text{arbitrary} \end{cases}$

(b) $\overline{E} = \left(\hat{x} + \hat{y} - \hat{z}\sqrt{3}\right) E_o \cos\left(k_x x + k_z z - \omega t\right)$ is linearly polarized.

(c) Let $\overline{E}_{add} = (\hat{x}E_1 + \hat{y}E_2 + \hat{z}E_3) \sin\left(k_x x + k_z z - \omega t\right)$ and we require

$$\begin{cases} E_3 = -\sqrt{3}E_1 \\ (\hat{x}E_x + \hat{y}E_y + \hat{z}E_z) \cdot (\hat{x}E_1 + \hat{y}E_2 + \hat{z}E_3) = 0 \\ |\hat{x}E_x + \hat{y}E_y + \hat{z}E_z| = |\hat{x}E_1 + \hat{y}E_2 + \hat{z}E_3| \end{cases}$$

Thus $\overline{E}_{add} = \left(-\frac{1}{2}\hat{x} + 2\hat{y} + \hat{z}\frac{\sqrt{3}}{2}\right) E_o \sin\left(k_x x + k_z z - \omega t\right)$.

P1.7.2

R^{TE} and T^{TE} are for electric field vectors while R^{TM} and T^{TM} are for magnetic field vectors. They do not reduce to the same numbers.

$$R^{TE} = \frac{1-n}{1+n} \qquad R^{TM} = \frac{n-1}{n+1}$$

As for reflectivity and transmissivity, the two cases yield identical results.

P1.7.3

$\theta_B = \tan^{-1} n = \tan^{-1} 1.46 = 55.59°$, $\theta = 90° - \theta_B = 34.41°$.

P1.7.4

(a) The Brewster angle for $\epsilon_t = 9$ is $\theta_B = \tan^{-1}\sqrt{\epsilon_t} = \tan^{-1}\sqrt{9} = 71.57°$.

(b) The dominant portion of the sun glares is TE polarized wave. The polaroid glasses absorb the TE component of the incident light, thus the TM component reaches the eyes after passing through the polaroid glasses.

P1.7.5

(a) $\overline{E}_i \cdot \overline{k}_i = 0 \Rightarrow \sqrt{3}k_x - k_z = 0 \Rightarrow \theta_i = \tan^{-1}(k_x/k_z) = \tan^{-1}(1/\sqrt{3}) = 30°$.

(b) For $k_x = 1\,(K_o)$, we get $k_z = \sqrt{3}k_x = \sqrt{3}\,(K_o)$. $\Rightarrow k = \sqrt{k_x^2 + k_z^2} = 2\,(K_o)$, and $k = \omega\sqrt{\mu_o 9\epsilon_o} = 3\omega/c$. So $f = \omega/2\pi = ck/(3 \cdot 2\pi) = 2 \times 10^8$ (Hz) and $\lambda = 2\pi/k = 0.5\,\text{m}$.

(c) If the totally reflected wave is linearly polarized, the incident angle is the Brewster angle, thus $\theta_i = 30° = \tan^{-1}\sqrt{\epsilon_t/9\epsilon_o} \Rightarrow \epsilon_t = 9\epsilon_o \tan^2 30° = 3\epsilon_o$.

P1.7.6

(a) $P_r = 0.16P_i$, so $\left|\overline{E}_r\right| = \sqrt{0.16}\left|\overline{E}_i\right| = 0.4\left|\overline{E}_i\right|$.

(b) $R = (n-1)/(n+1) = 0.4$ so $n = 7/3$.

(c) This problem is the TM wave case, so $P_r/P_i = \left|R^{TM}\right|^2 = (11/38)^2$.

(d) The tilted angle is the Brewster angle, $\theta_B = \tan^{-1} n = \tan^{-1}(7/3)$.

P1.7.7

$\lambda_{cm} = 2d/m$ and $\omega_{cm} = m\pi/d(\mu\epsilon)^{1/2}$.

P1.7.8

(a) $f_{c10} = \frac{\omega_{c10}}{2\pi} = \frac{c}{2\pi}\left(\frac{\pi}{a}\right) = \frac{3\times10^8}{2\pi} \times \frac{\pi}{6.55} = 22.9\,(\text{MHz}) < f < f_{c01} = \frac{c}{2\pi}\left(\frac{\pi}{b}\right) = \frac{3\times10^8}{2\pi} \times \frac{\pi}{4.19} = 35.8\,(\text{MHz})$

(b) An AM radio operates in the range of 500 to 1600 (KHz) is below the cutoff frequency of the fundamental mode TE_{10}. Therefore, AM signals can not be received in the tunnel.

(c) FM signals operating in the range of 88.1 to 107.9 (MHz) can be received in the tunnel.