Homework 4

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1. Prove that there are no positive integer solutions to the equation $x^2 - y^2 = 1$

   We can rewrite the statement as such: $\forall x, y \in Z^+, x^2 - y^2 \neq 1$.
   For the sake of contradiction let $\forall x, y \in Z^+, x^2 - y^2 = 1$. Then the equation can be rewritten as $(x-y)(x+y) = 1$. Because both x and y are positive integers, $(x-y)$ is an integer and $(x+y)$ is a positive integer. Therefore, for the multiple to be 1, both $(x+y) = (x-y) = 1$. However there is a contradiction. the value of the sum of two positive integer is at least 2. $x + y \geq 2$
   Therefore, the statement is true.

2. Use the Well-Ordering Principle to prove that any integer greater than or equal to 8 can be represented as the sum of integer multiples of 3 and 5.

   We can rewrite the statement as: $n \in \mathbb{Z}, n \geq 8, n = 3a + 5b$ $(a, b \in \mathbb{Z})$
   For the sake of contradiction the statement is false. Then there exists a set of counter examples, $X \neq \emptyset$, and the least element being, $x \in X$.
   $x \neq 8$ because when $n = 8$, $a = 1, b = 1$. Also, $x \neq 9$, because when $n = 9$, $a = 3, b = 0$. So, $x > 9$.
   $(x-1) \notin X$, and $(x-2) \notin X$.
   Let $(x-2) = 3a_1 + 5b_1$. If $x = 3a_1 + 5b_1 + 2$. This can be re written as $x = 3(a_1 - 1) + 5b_1 + 2 + 3 = 3(a_1 - 1) + 5(b_1 + 1)$
   But then there is a contradiction, $x \notin X$. X has no least element. Therefore, $X = \emptyset$.
   Then, statement is true $\forall n \geq 8, n \in \mathbb{N}$

3. Use the Well-Ordering Principle to prove that $n! > 2^n$ for all natural numbers $n \geq 4$

   For the sake of contradiction suppose that the statement is false. Then there exists a set of counter examples, $X \neq \emptyset$. Let the least element of the set be $x \in X$.
   $x \neq 4$. $4! = 24$, $2^4 = 16$. Then, $x > 4$.
   $(x-1)! > 2^(x-1)$. Then $(x-1)! > 2^x \times \frac{1}{2}$. So, $2(x-1)! > 2^x$.
   Since $x > 4$, $x! > 2(x-1)! > 2^x$. There is a contradiction.

$x \notin X$. X has no least element. Therefore, $X = \emptyset$. Then, statement is true $\forall (n \geq 4) \in \mathbb{N}$

4. Use the Well-Ordering Principle to prove that $n! \leq n^n$ for all positive integers n

   For the sake of contradiction suppose that the statement is false. Then there exists a set of counter examples, $X \neq \emptyset$. Let the least element of the set be $x \in X$.
   Then, $x \neq 1$. $1! = 1^1$. So, $x > 1$.
   Since $x - 1$ is not in X, $(x-1)! \leq (x-1)^{(x-1)}$.
   $(x-1)!(x) \leq (x-1)^{(x-1)}(x) \leq x^x$.
   Then there is a contradiction. $x \notin X$. X has no least element. Therefore, $X = \emptyset$.
   Then, statement is true $\forall n \in \mathbb{Z}^+$

5. Use the Well-Ordering Principle to prove that $\sum_{i=0}^{n} i^3 = (\frac{n(n+1)}{2})^2$

   For the sake of contradiction suppose that the statement is false. Then there exists a set of counter examples, $X \neq \emptyset$. Let the least element of the set be $x \in X$.
   $x \neq 0$. $0^3 = 0^2 = 0$. Then, $x > 0$.
   $x - 1$ is not in X. $\sum_{i=0}^{x-1} i^3 = (\frac{(x-1)x}{2})^2$.
   $\sum_{i=0}^{x-1} i^3 + x^3 = (\frac{(x-1)x}{2})^2 + x^3$
   $= \frac{((x-1)^2)(x^2)+4x^3}{4} = \frac{((x^2-2x+1+4x)(x^2)}{4} = \frac{((x+1)^2(x^2)}{4} = (\frac{((x+1)x}{2})^2$
   Then there is a contradiction. $x \notin X$. X has no least element.
   Therefore, $X = \emptyset$. Then, statement is true $\forall n \in \mathbb{N}$

6. Use induction to prove that $\sum_{i=0}^{n} i^3 = (\frac{n(n+1)}{2})^2$

   The least element of $n \in \mathbb{N}$ is $n = 0$. The statement is true for when $n = 0$. $\sum_{i=0}^{0} i^3 = 0$, and $(\frac{0(0+1)}{2})^2 = 0$
   Then assume that the statement is true for $k \in \mathbb{N}$. Then, $\sum_{i=0}^{k} i^3 = (\frac{k(k+1)}{2})^2$.
   $\sum_{i=0}^{k+1} i^3 = (\frac{k(k+1)}{2})^2 + (k+1)^3$
   $= \frac{k^2(k+1)^2+4(k+1)^3}{4} = \frac{(k^2+4(k+1))(k+1)^2}{4} = \frac{(k+2)^2(k+1)^2}{4}$
   Therefore by the Principle of Mathematical Induction, $\forall n \in \mathbb{N}$ the statement is true.

7. Using induction, prove that for all positive integers n, $\frac{5}{4}8^n + 3^{3n-1}$ is divisible by 19.

   Let $P(n) = \frac{5}{4}8^n + 3^{3n-1}$.
   Then the statement can be rewritten as; $\forall n \in \mathbb{Z}^+$, $19 | P(n)$ For all $n \in \mathbb{Z}^+$ the least element is 1. When $n = 1$, $\frac{5}{4}8^1 + 3^2 = 19$ Therefore the statement is true.
   Let for some $k \in \mathbb{Z}+$ be true of the statement. $19|(\frac{5}{4}8^k + 3^{3k-1})$.
   Then $\frac{5}{4}8^{k+1} + 3^{3(k+1)-1} = \frac{5}{4}8^k \times 8 + 3^{3k-1} \times 27$
   $= \frac{5}{4}8^k \times 8 + 3^{3k-1} \times (8 + 19) = (\frac{5}{4}8^k + 3^{3k-1})8 + 19 \times 3^{3k-1}$

Therefore when $19|P(k)$ then $19|P(k+1)$.
By the Principle of Mathematical Induction, $\forall n \in \mathbb{Z}^+$, $19|P(n)$.

8. Prove that every integer $n \geq 1$ can be expressed as the sum of distinct terms in the Fibonacci sequence.

   Fibonacci series is a series of numbers in which each number is the sum of the two preceding numbers. When $n = 1$, then n can be expressed as $n = 0 + 1$. Let some number $F_k \leq k \leq F_{k+1}$ be true to the statement, and the statement is true for all numbers less than F_k. $k - F_k \leq F_{k-1}$. But because any number less or equal to F_k can be expressed as the sum of distinct terms of the Fibonacci sequence, $k - F_k$ can also be expressed as the sum of Fibonacci sequence. Therefore the statement is true.

9. Use induction to show that for any natural number $n \geq 1$, given pairs, $(a_1, b_1), (a_2, b_2), \cdots (a_n, b_n)$ of integer numbers, there exist integer numbers c and d such that $((a_1)^2+(b_1)^2)((a_2)^2+(b_2)^2) \cdots ((a_n)^2 + (b_n)^2) = c^2 + d^2$.

   When n=1, for any $a, b \in \mathbb{N}$, let $(a_1, b_1) = (a, b)$, the statement is true. When n=2, let $a, b, e, f \in \mathbb{N}$, the statement is true.
   Then $(a^2+b^2)(e^2+f^2) = ae^2+af^2+be^2+bf^2 = (ae^2+bf^2)+(af^2+be^2)+2abef - 2abef = (ae+bf)^2 + (af - be)^2 = c^2 + d^2$
   For some $n = k$, $((a_1)^2 + (b_1)^2)((a_2)^2 + (b_2)^2) \cdots ((a_k)^2 + (b_k)^2) = c^2 + d^2$.
   Then when $n = k+1$, $(c^2+d^2)((a_{k+1})^2+(b_{k+1})^2) = (ca_{k+1}+db_{k+1})^2+(cb_{k+1}-da_{k+1})^2$.
   So it is also true.
   By the Principle of Mathematical Induction, the statement is true for all $n \geq 1, n \in \mathbb{N}$.

10. You are planning a dinner party with at least twelve guests. You have tables that sit 4-5 people, and you need to assign your guests to tables. Use induction to prove that $n \geq 12$ people can be split up into groups such that each group has exactly 4 or 5 members.
    We can rewrite the statement as such: when $n \geq 12, n \in \mathbb{N}, n = 4a + 5b$ for some a and b that is a natural number.
    When $n = 12$ then the statement is true, $12 = 4(3) + 5(0)$.
    Lets assume that for $n = k$ the statement is true.
    When $a \neq 0$; $k = 4a+5b$. Then $4a+5b+1 = 4(a-1)+5b+4+1 = 4(a-1)+5(b+1) = k+1$.
    When $a = 0$; $k = 5b$. Then $5b + 1 = 5(b-3) + 15 + 1 = 5(b-3) + 4(4) = k + 1$.
    Because $n \geq 12$ for when $a = 0$, $b \geq 3$ and so $(b-3) \in \mathbb{N}$.
    Therefore by the Principle of mathematical induction the statement is true.