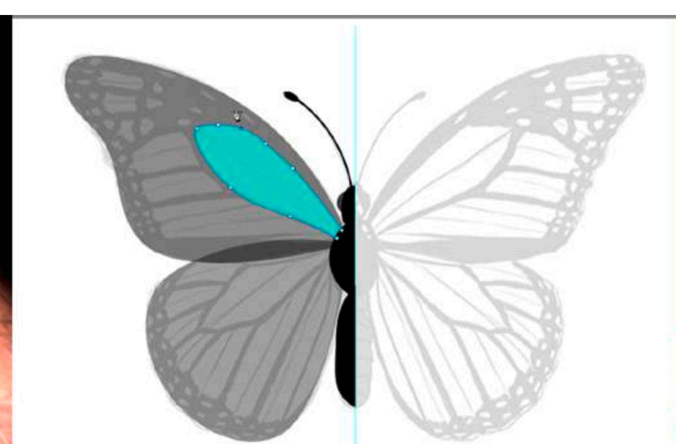
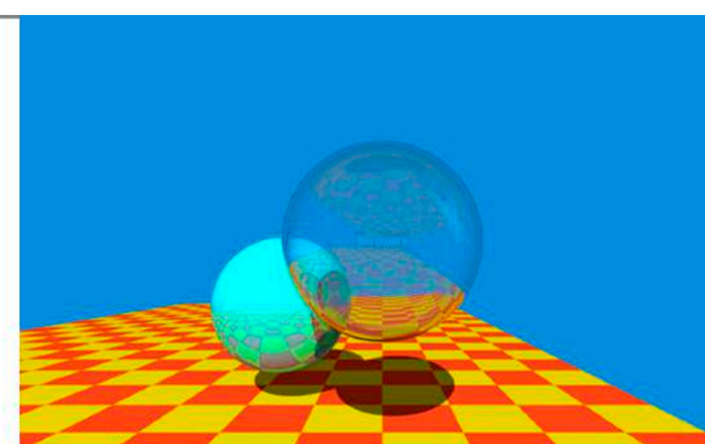
Introduction to Computer Graphics

GAMES101, Lingqi Yan, UC Santa Barbara

Lecture 2: Review of Linear Algebra


http://www.cs.ucsb.edu/~lingqi/teaching/cs180.html

Announcements

- Slides and recordings of Lecture 1 now available
- (Pre)-reading materials will be out before lectures

	日期	主题
第 1 周	Feb 11	**计算机图形学概述** [课件][录像]
	Feb 14	**向量与线性代数** 阅读材料：第 2 章（Miscellaneous Math），第 5 章（Linear Algebra）

- Happy Valentine's Day!

Last Lecture

- What is Computer Graphics?
- Why study Computer Graphics?
- Course Topics
- Course Logistics

A Swift and Brutal Introduction to Linear Algebra!

(in fact it's relatively easy…)

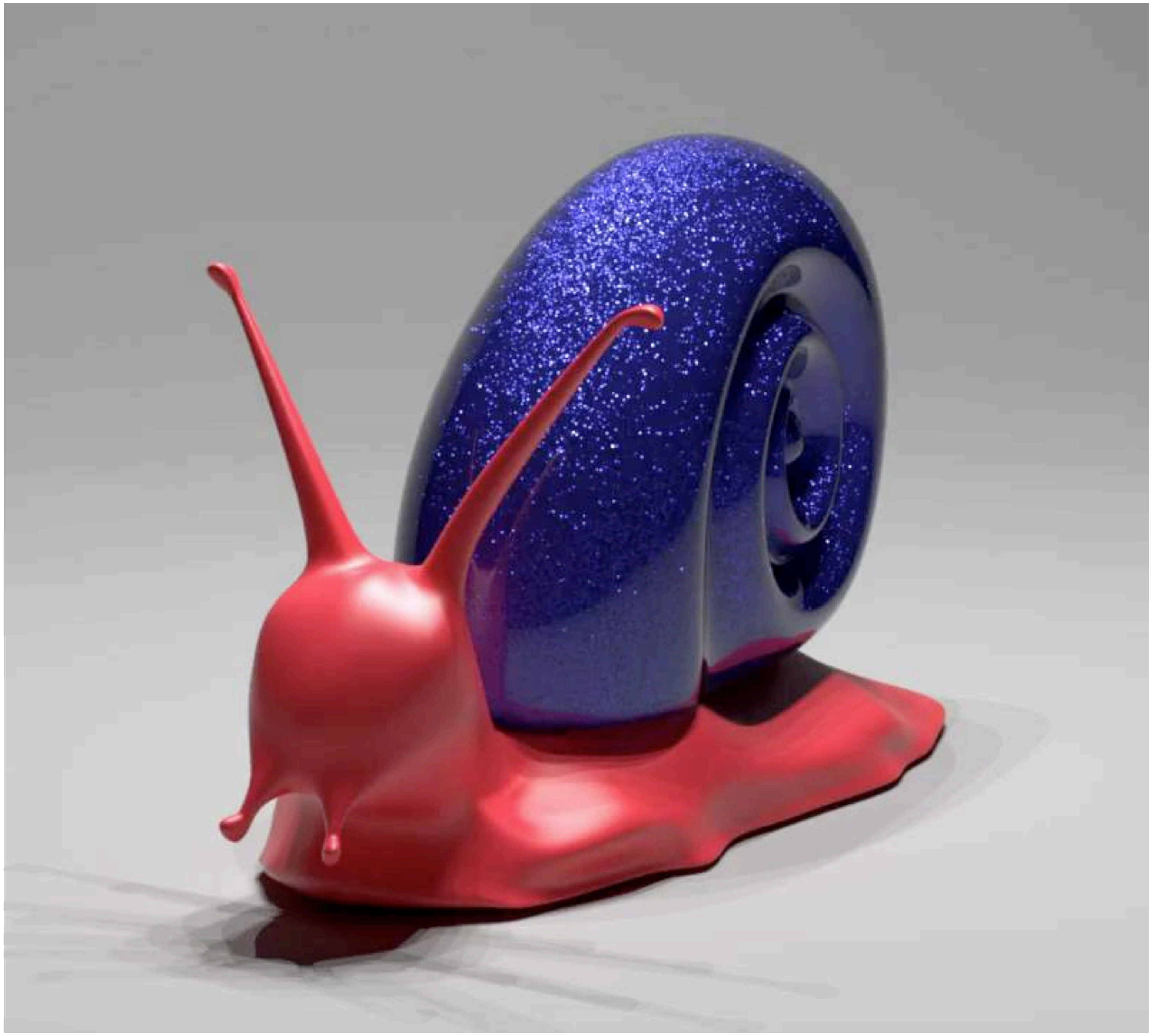
Graphics' Dependencies

- Basic mathematics
  - Linear algebra, calculus, statistics
- Basic physics
  - Optics, Mechanics
- Misc
  - Signal processing
  - Numerical analysis
- And a bit of aesthetics

This Course

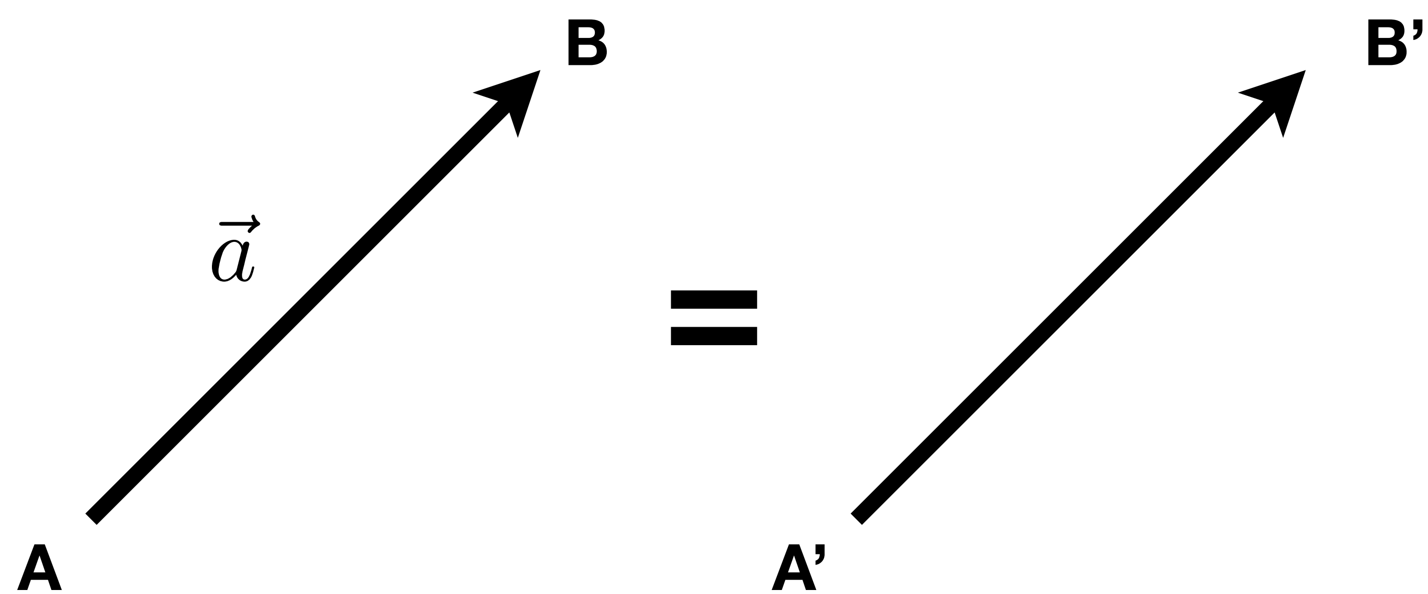
- More dependent on Linear Algebra
  - Vectors (dot products, cross products, …)
  - Matrices (matrix-matrix, matrix-vector mult., …)
- For example,
  - A point is a vector (?)
  - An operation like translating or rotating objects can be matrix-vector multiplication

An Example of Rotation



Rendering Glints on High-Resolution Normal-Mapped Specular Surfaces, Lingqi Yan, 2014

Vectors

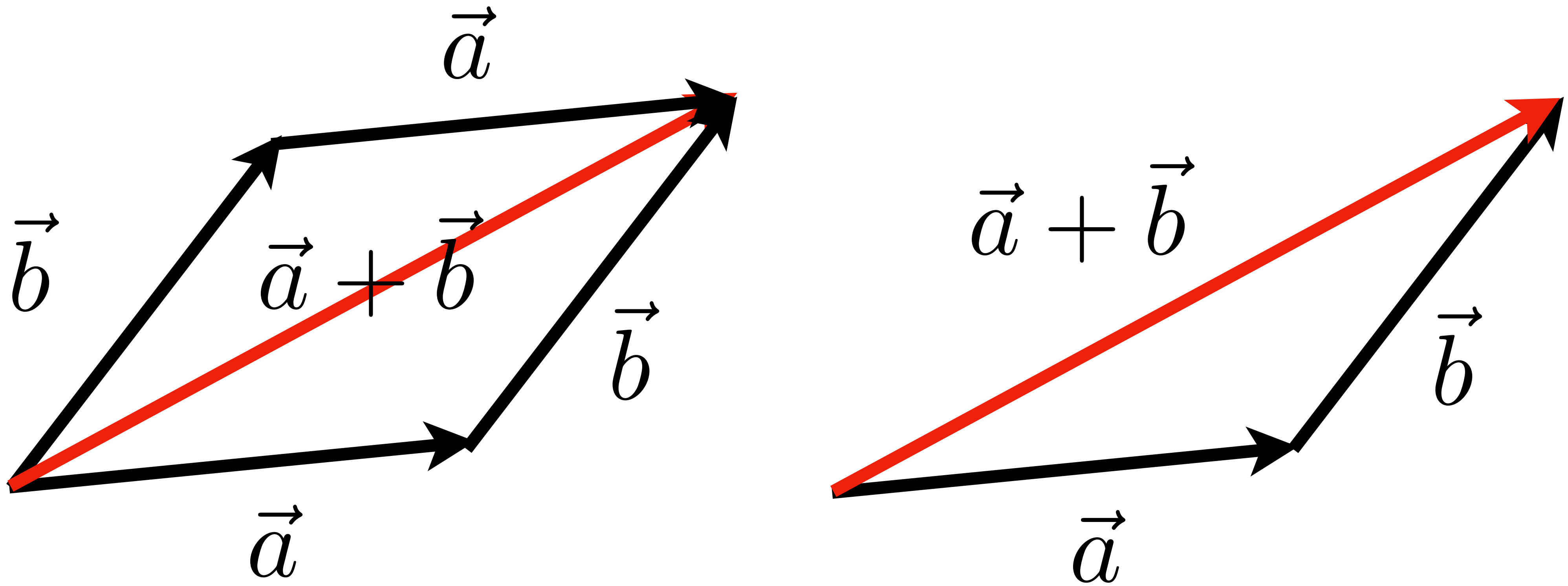


- Usually written as $\vec{a}$ or in bold $\boldsymbol{a}$
- Or using start and end points $\overrightarrow{AB} = B - A$
- Direction and length
- No absolute starting position

Vector Normalization

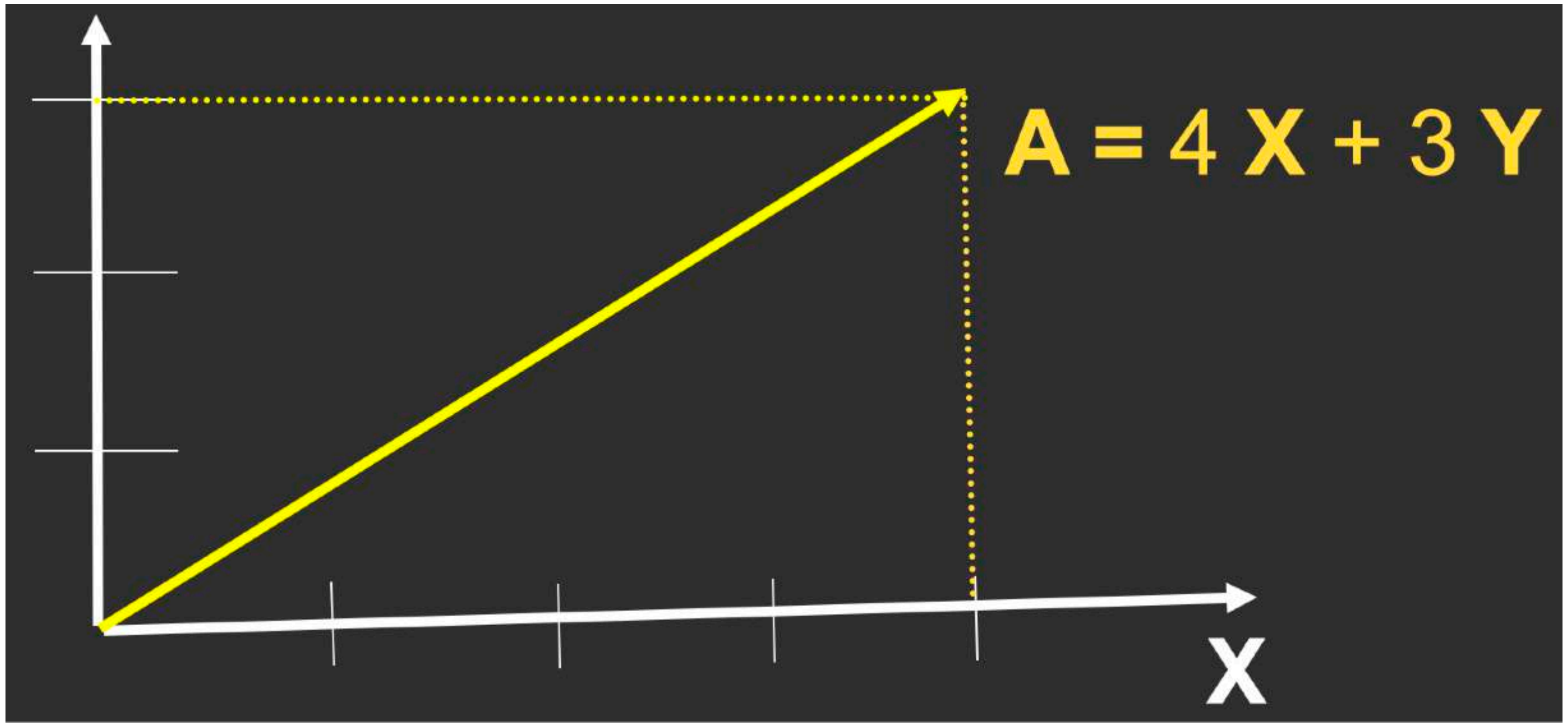
- Magnitude (length) of a vector written as $\|\vec{a}\|$
- Unit vector
  - A vector with magnitude of 1
  - Finding the unit vector of a vector (normalization): $\hat{a} = \vec{a}/\|\vec{a}\|$
  - Used to represent directions

Vector Addition



- Geometrically: Parallelogram law & Triangle law
- Algebraically: Simply add coordinates

Cartesian Coordinates



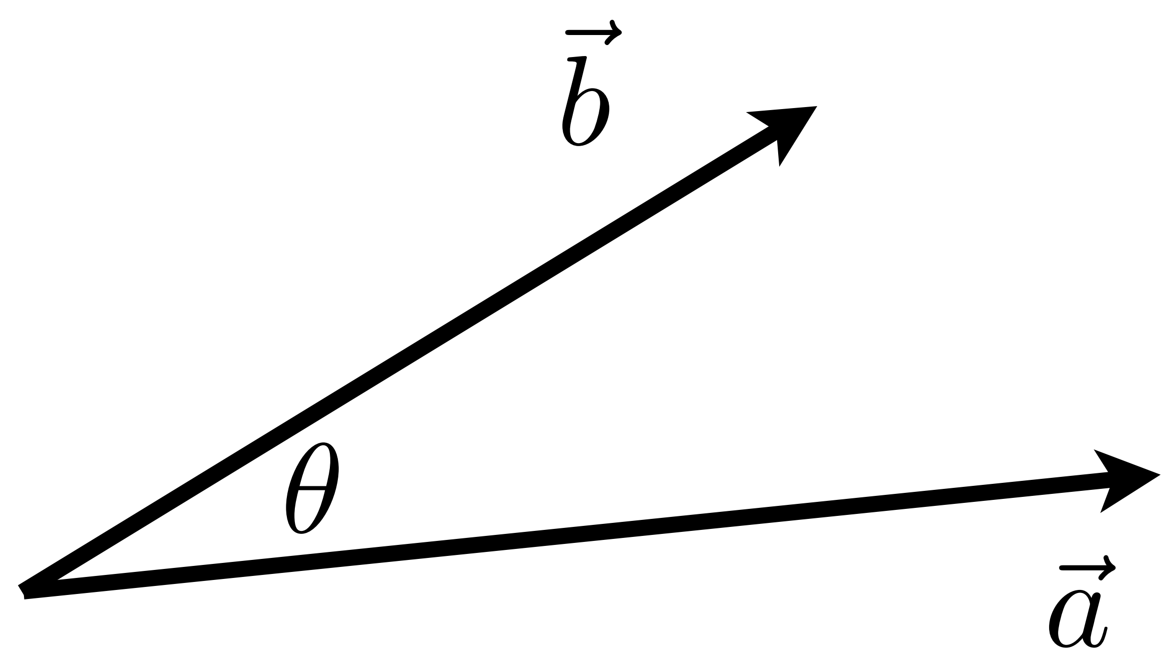
- X and Y can be any (usually orthogonal unit) vectors

$$\mathbf{A} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \mathbf{A}^T = (x, y) \qquad \|\mathbf{A}\| = \sqrt{x^2 + y^2}$$

Vector Multiplication

- Dot product
- Cross product
- Orthonormal bases and coordinate frames

Dot (scalar) Product



$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|}$$

- For unit vectors

$$\cos\theta = \hat{a} \cdot \hat{b}$$

Dot (scalar) Product

- Properties

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

$$(k\vec{a}) \cdot \vec{b} = \vec{a} \cdot (k\vec{b}) = k(\vec{a} \cdot \vec{b})$$

Dot Product in Cartesian Coordinates

- Component-wise multiplication, then adding up
  - In 2D

$$\vec{a} \cdot \vec{b} = \begin{pmatrix} x_a \\ y_a \end{pmatrix} \cdot \begin{pmatrix} x_b \\ y_b \end{pmatrix} = x_a x_b + y_a y_b$$

  - In 3D

$$\vec{a} \cdot \vec{b} = \begin{pmatrix} x_a \\ y_a \\ z_a \end{pmatrix} \cdot \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix} = x_a x_b + y_a y_b + z_a z_b$$

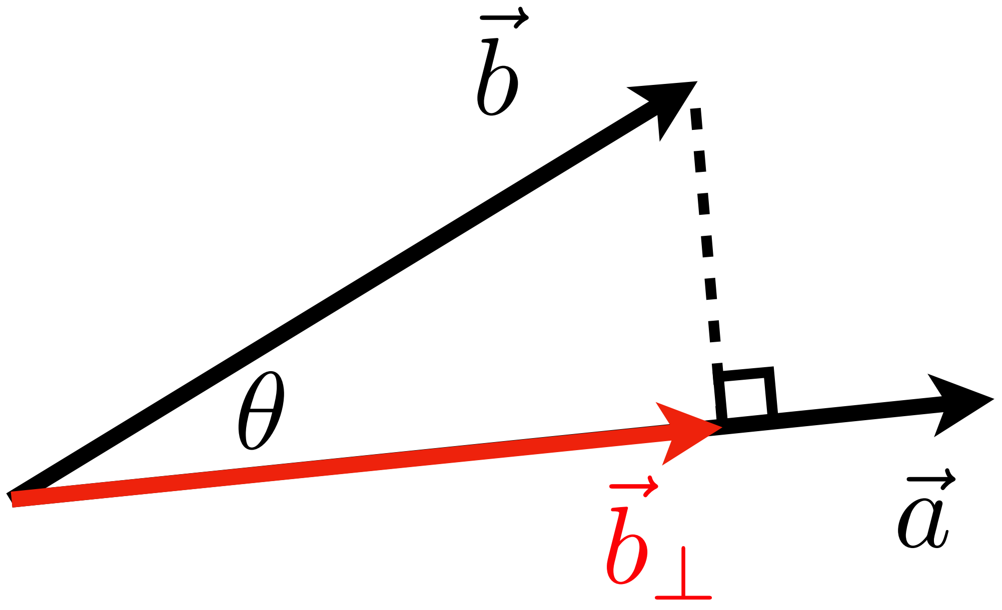
Dot Product in Graphics

- Find angle between two vectors
  (e.g. cosine of angle between light source and surface)

- Finding projection of one vector on another

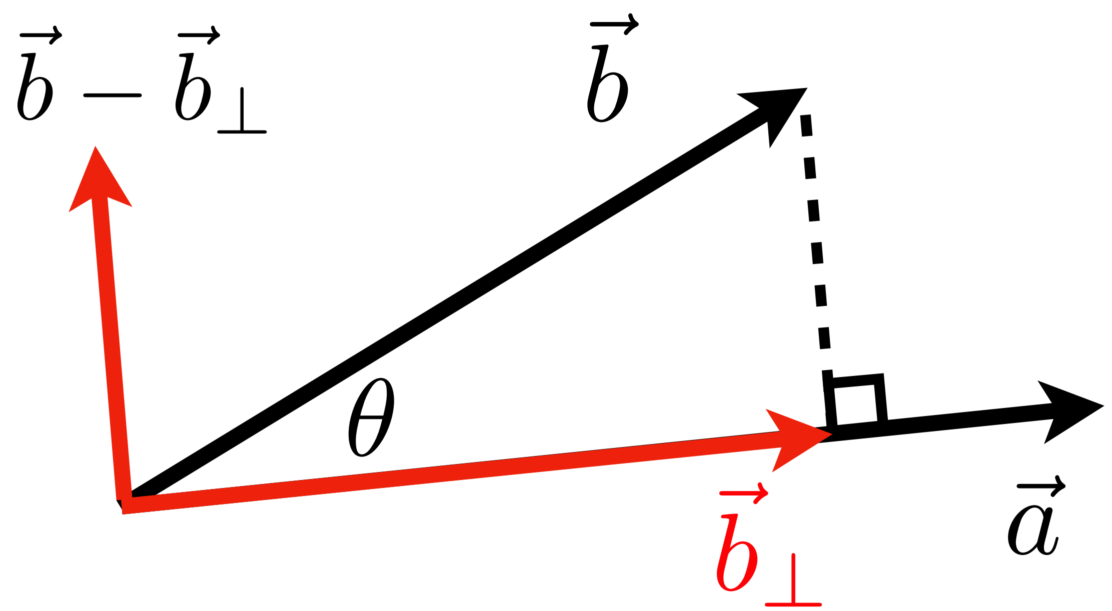
Dot Product for Projection

- $\vec{b}_\perp$: projection of $\vec{b}$ onto $\vec{a}$
  - $\vec{b}_\perp$ must be along $\vec{a}$ (or along $\hat{a}$)
    - $\vec{b}_\perp = k\hat{a}$
  - What's its magnitude k?
    - $k = \|\vec{b}_\perp\| = \|\vec{b}\| \cos\theta$



Dot Product in Graphics

- Measure how close two directions are
- Decompose a vector
- Determine forward / backward



Dot Product in Graphics

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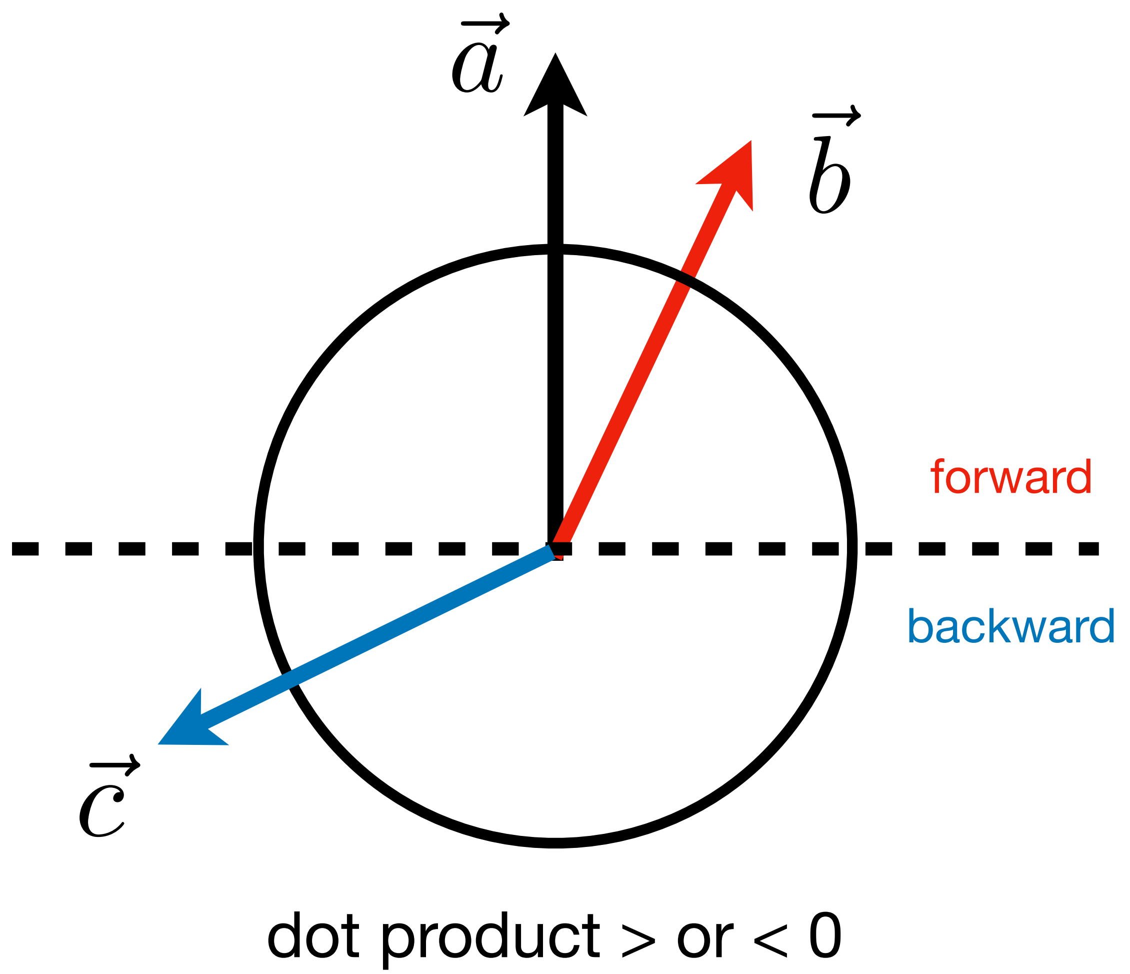
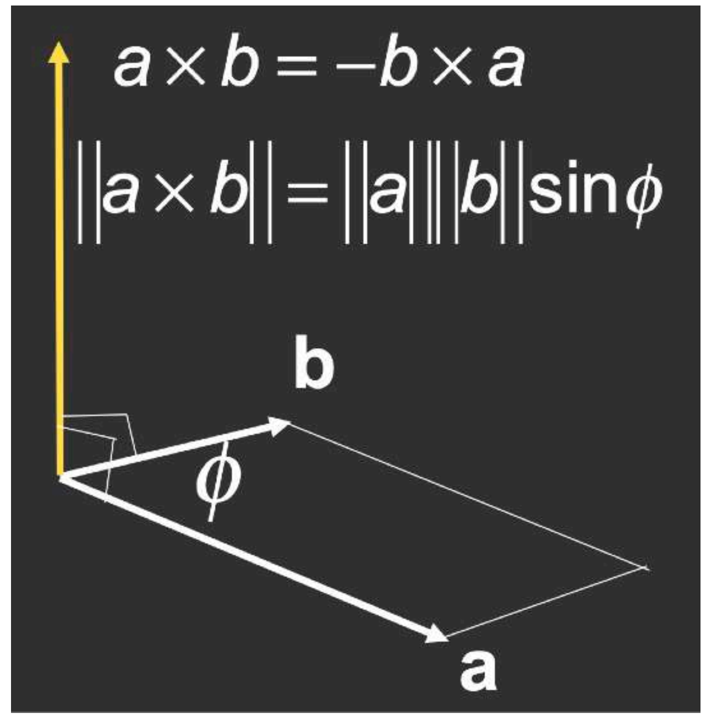


Vector Multiplication

- Dot product
- Cross product
- Orthonormal bases and coordinate frames

Cross (vector) Product



- Cross product is orthogonal to two initial vectors
- Direction determined by right-hand rule
- Useful in constructing coordinate systems (later)

Cross product: Properties

$$\vec{x} \times \vec{y} = +\vec{z}$$

$$\vec{y} \times \vec{x} = -\vec{z}$$

$$\vec{y} \times \vec{z} = +\vec{x}$$

$$\vec{z} \times \vec{y} = -\vec{x}$$

$$\vec{z} \times \vec{x} = +\vec{y}$$

$$\vec{x} \times \vec{z} = -\vec{y}$$

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

$$\vec{a} \times \vec{a} = \vec{0}$$

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

$$\vec{a} \times (k\vec{b}) = k(\vec{a} \times \vec{b})$$

Cross Product: Cartesian Formula?

$$\vec{a} \times \vec{b} = \begin{pmatrix} y_a z_b - y_b z_a \\ z_a x_b - x_a z_b \\ x_a y_b - y_a x_b \end{pmatrix}$$

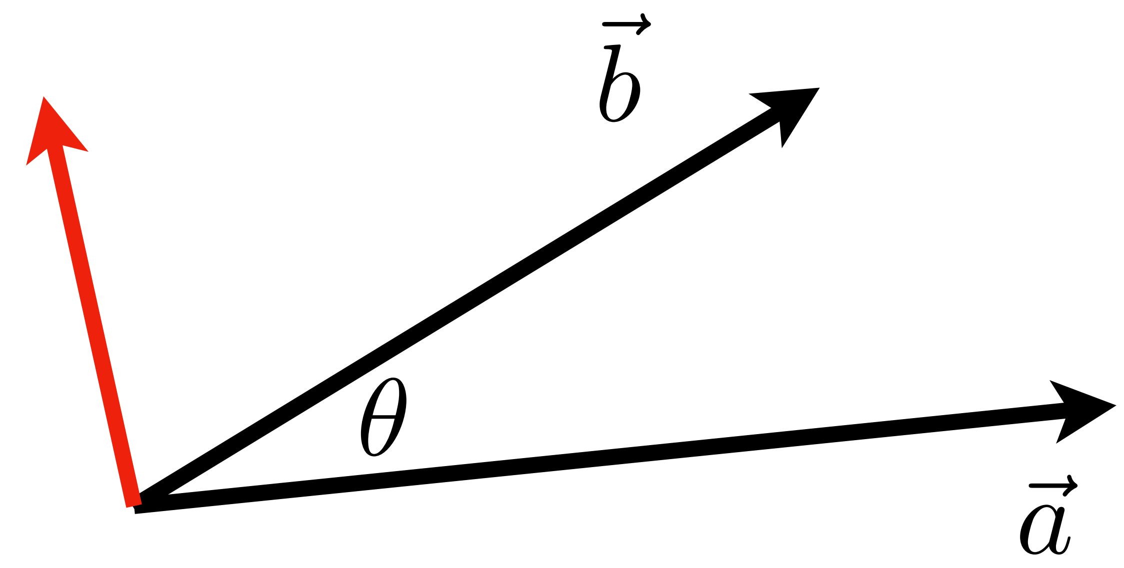
- Later in this lecture

$$\vec{a} \times \vec{b} = A^* b = \begin{pmatrix} 0 & -z_a & y_a \\ z_a & 0 & -x_a \\ -y_a & x_a & 0 \end{pmatrix} \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix}$$

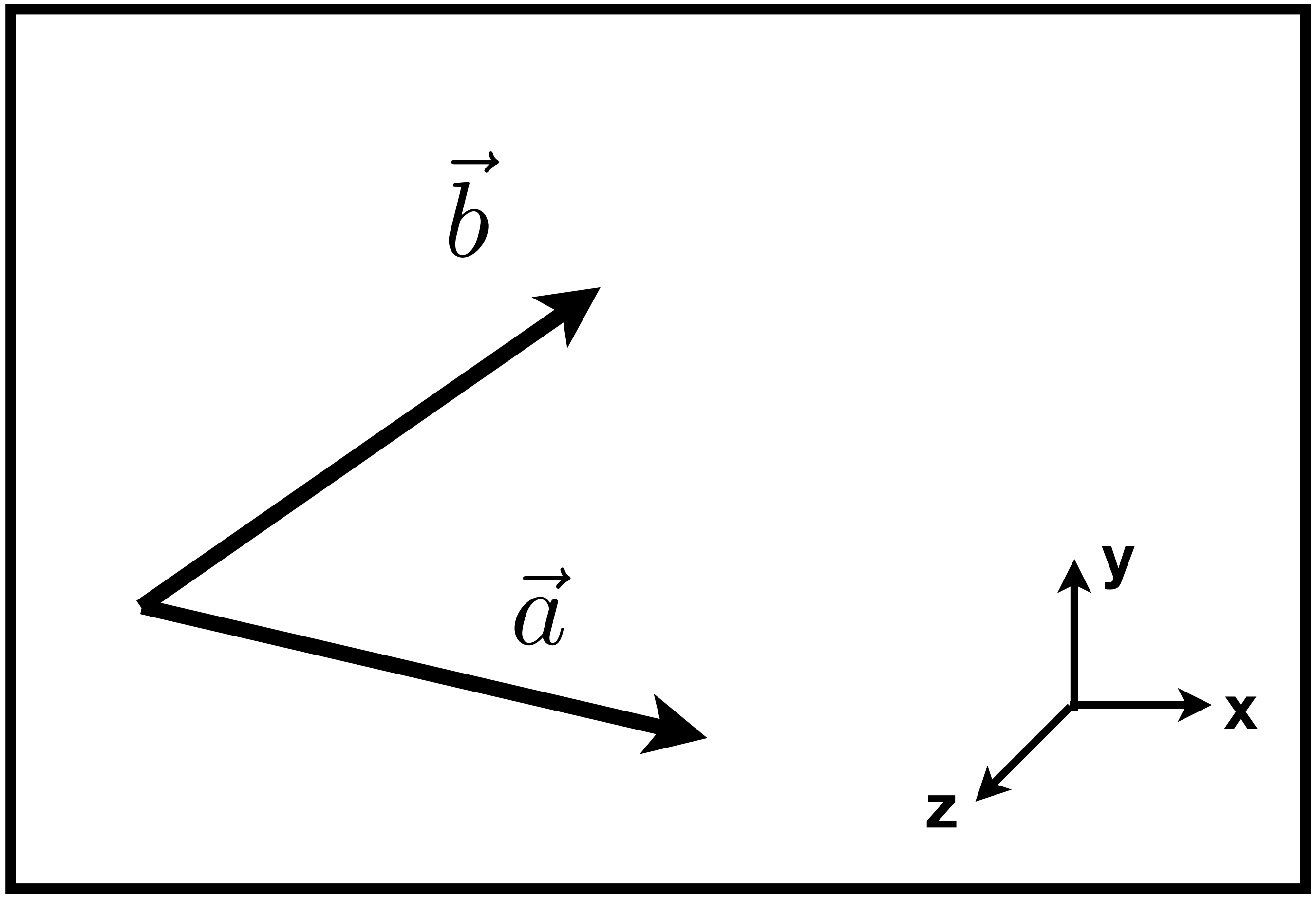
dual matrix of vector a

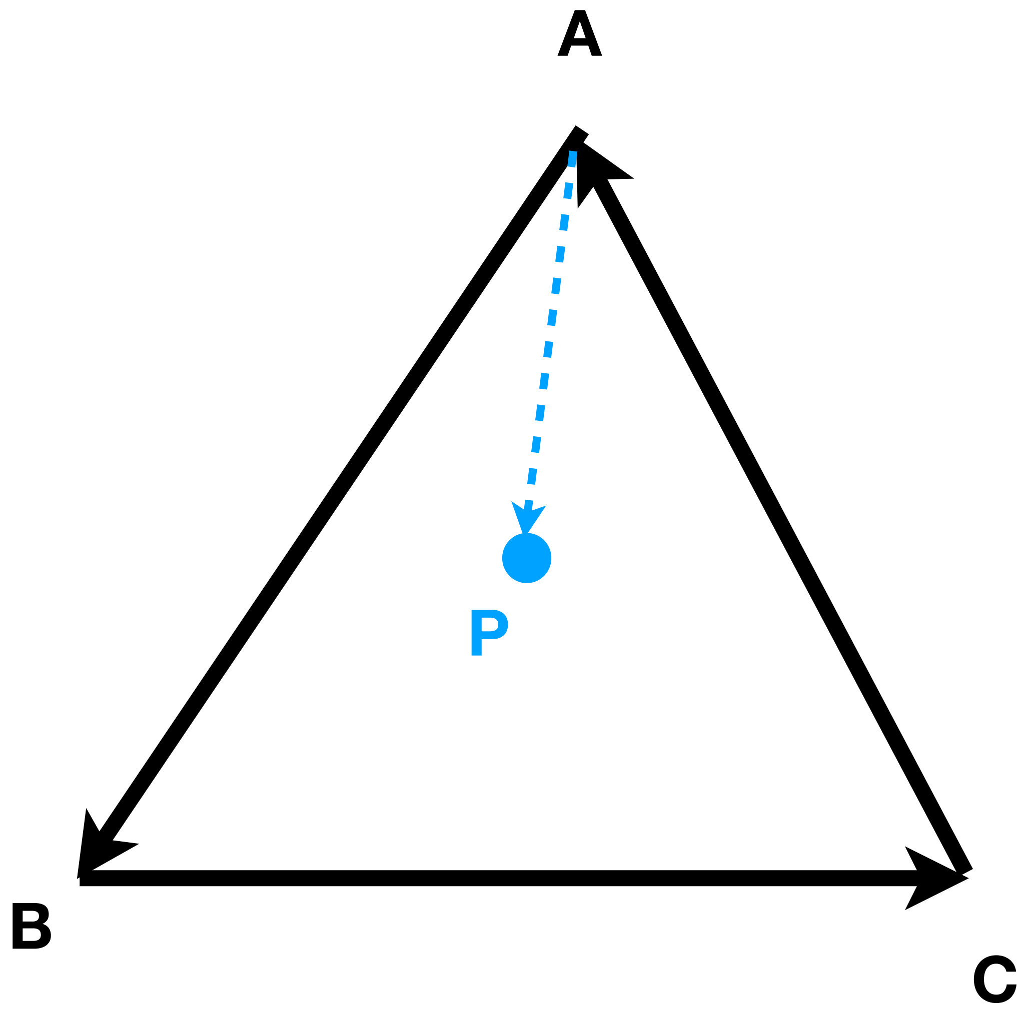
Cross Product in Graphics

- Determine left / right
- Determine **inside / outside**



Cross Product in Graphics





Vector Multiplication

- Dot product
- Cross product
- Orthonormal bases and coordinate frames

Orthonormal Bases / Coordinate Frames

- Important for representing points, positions, locations
- Often, many sets of coordinate systems
  - Global, local, world, model, parts of model (head, hands, …)
- Critical issue is transforming between these systems/ bases
  - A topic for next week

Orthonormal Coordinate Frames

- Any set of 3 vectors (in 3D) that

$$\|\vec{u}\| = \|\vec{v}\| = \|\vec{w}\| = 1$$

$$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{w} = \vec{u} \cdot \vec{w} = 0$$

$$\vec{w} = \vec{u} \times \vec{v}$$ (right-handed)

$$\vec{p} = (\vec{p} \cdot \vec{u})\vec{u} + (\vec{p} \cdot \vec{v})\vec{v} + (\vec{p} \cdot \vec{w})\vec{w}$$

(projection)

Questions?

Matrices

- Magical 2D arrays that haunt in every CS course
- In Graphics, pervasively used to represent transformations
    - Translation, rotation, shear, scale
      (more details in the next lecture)

What is a matrix

- Array of numbers ($m \times n$ = m rows, n columns)

$$\begin{pmatrix} 1 & 3 \\ 5 & 2 \\ 0 & 4 \end{pmatrix}$$

- Addition and multiplication by a scalar are trivial: element by element

Matrix-Matrix Multiplication

- # (number of) columns in A must = # rows in B
  (M x N) (N x P) = (M x P)

$$\begin{pmatrix} 1 & 3 \\ 5 & 2 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 3 & 6 & 9 & 4 \\ 2 & 7 & 8 & 3 \end{pmatrix}$$

Matrix-Matrix Multiplication

- \# (number of) columns in A must = # rows in B
  (M x N) (N x P) = (M x P)

$$\begin{pmatrix} 1 & 3 \\ 5 & 2 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 3 & 6 & 9 & 4 \\ 2 & 7 & 8 & 3 \end{pmatrix} = \begin{pmatrix} 9 & ? & 33 & 13 \\ 19 & 44 & 61 & 26 \\ 8 & 28 & 32 & ? \end{pmatrix}$$

- Element (i, j) in the product is
  the dot product of row i from A and column j from B

Matrix-Matrix Multiplication

- Properties
  - **Non-commutative**
    (AB and BA are different in general)
  - Associative and distributive
    - (AB)C=A(BC)
    - A(B+C) = AB + AC
    - (A+B)C = AC + BC

Matrix-Vector Multiplication

- Treat vector as a column matrix (m×1)
- Key for transforming points (next lecture)
- Official spoiler: 2D reflection about y-axis

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$$

Transpose of a Matrix

- Switch rows and columns (ij -> ji)

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$

- Property

$$(AB)^T = B^T A^T$$

Identity Matrix and Inverses

$$I_{3\times 3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$AA^{-1} = A^{-1}A = I$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

Vector multiplication in Matrix form

- Dot product?

$$\vec{a} \cdot \vec{b} = \vec{a}^T \vec{b}$$

$$= \begin{pmatrix} x_a & y_a & z_a \end{pmatrix} \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix} = \begin{pmatrix} x_a x_b + y_a y_b + z_a z_b \end{pmatrix}$$

- Cross product?

$$\vec{a} \times \vec{b} = A^* b = \begin{pmatrix} 0 & -z_a & y_a \\ z_a & 0 & -x_a \\ -y_a & x_a & 0 \end{pmatrix} \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix}$$

dual matrix of vector a

An Example of General Transformation

The Sponza Scene, rendered by Lingqi Yan using Real-time Ray Tracing (RTRT)

Questions?

Next

- Transform!

Transformers: The Last Knight, 2017 movie

Thank you!

(And thank Prof. Ravi Ramamoorthi and Prof. Ren Ng for many of the slides!)