Deep BSDE: General Form

We start from a general semilinear parabolic PDE:

$$\frac{\partial u}{\partial t}(t,x) + \frac{1}{2}\operatorname{Tr}\left[\sigma\sigma^\top(t,x)\cdot \operatorname{Hess}_x u(t,x)\right] + \nabla u(t,x)\cdot\mu(t,x) + f\left(t,x,u(t,x),\sigma^\top(t,x)\nabla u(t,x)\right) = 0$$

with terminal condition:

$$u(T,x) = g(x)$$

Define the forward SDE:

$$dX_t = \mu(t,X_t)\,dt + \sigma(t,X_t)\,dW_t, \quad X_0 = \xi$$

Apply Itô's formula to the composite process $Y_t = u(t,X_t)$:

$$\begin{aligned} du(t,X_t) &= \frac{\partial u}{\partial t}(t,X_t)\,dt + \nabla u(t,X_t)^\top\,dX_t + \frac{1}{2}\operatorname{Tr}\left[\sigma\sigma^\top(t,X_t)\cdot\operatorname{Hess}_x u(t,X_t)\right]dt \\ &= \left(\frac{\partial u}{\partial t} + \nabla u^\top\mu + \frac{1}{2}\operatorname{Tr}[\sigma\sigma^\top \operatorname{Hess} u]\right)dt + \nabla u^\top\sigma\,dW_t \end{aligned}$$

From the PDE:

$$\frac{\partial u}{\partial t} + \nabla u^\top\mu + \frac{1}{2}\operatorname{Tr}[\sigma\sigma^\top\operatorname{Hess} u] = -f(t,X_t,u(t,X_t),\sigma^\top\nabla u(t,X_t))$$

So:

$$du(t,X_t) = -f(t,X_t,u(t,X_t),\sigma^\top\nabla u(t,X_t))\,dt + \nabla u^\top\sigma\,dW_t$$

Let $Y_t = u(t,X_t)$, $Z_t = \sigma^\top(t,X_t)\nabla u(t,X_t)$. Then:

$$\boxed{dY_t = -f(t,X_t,Y_t,Z_t)\,dt + Z_t^\top\,dW_t}$$

Integral form of the BSDE

$$\begin{aligned} u(t,X_t) - u(0,X_0) = &-\int_0^t f(s,X_s,u(s,X_s),\sigma^\top\nabla u(s,X_s))\,ds \\ &+ \int_0^t \left[\nabla u(s,X_s)^\top\sigma(s,X_s)\right]\,dW_s \end{aligned}$$

To numerically solve the BSDE using the Deep BSDE method, we discretize time with a uniform grid:

$$0 = t_0 < t_1 < \cdots < t_N = T, \quad \Delta t = t_{n+1} - t_n$$

The forward SDE is simulated using Euler–Maruyama:

$$X_{n+1} = X_n + \mu(t_n,X_n)\,\Delta t + \sigma(t_n,X_n)\,\Delta W_n$$

We then approximate the backward BSDE dynamics as:

$$Y_{n+1} = Y_n - f(t_n,X_n,Y_n,Z_n)\,\Delta t + Z_n^\top\,\Delta W_n$$

Here:
- $X_n \approx X_{t_n}$ is the simulated forward process,
- $Y_n \approx u(t_n,X_n)$ is the estimated solution,
- $Z_n \approx \sigma^\top(t_n,X_n)\nabla u(t_n,X_n)$ is predicted via a neural network in Deep BSDE,
- $\Delta W_n \sim \mathcal{N}(0,\Delta t)$ is the Brownian increment.

The goal is to train the parameters of the neural network $u(t_0,X_{t_0})$ at each time stepss approximating Y_0 and $\nabla u(t_n,X_{t_n})$ approximating Z_n such that:

$$u(t_0,X_{t_0}) \approx Y_0, \nabla u(t_n,X_{t_n}) \approx Z_n$$

At the final time step $t = T$, we are given the terminal condition of the PDE:

$$Y_N \stackrel{?}{\approx} g(X_N)$$

To guide the training of the Deep BSDE network, we define the loss function based on the discrepancy at the terminal time:

$$\mathcal{L} = \mathbb{E}\left[|Y_N - g(X_N)|^2\right]$$

Deep BSDE for the BS Model

We consider the Black–Scholes PDE for a European call option:

$$\frac{\partial u}{\partial t}(t,S) + rS\frac{\partial u}{\partial S}(t,S) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 u}{\partial S^2}(t,S) = ru(t,S)$$

with terminal condition:

$$u(T,S) = (S-K)^+$$

Define the forward SDE under the risk-neutral measure $\mathbb{Q}$:

$$dS_t = rS_t\,dt + \sigma S_t\,d\widetilde{W}_t, \quad S_0 = s$$

Let the value function be:

$$u(t,S_t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[(S_T-K)^+ \mid \mathcal{F}_t]$$

Define the composite process:

$$Y_t = u(t,S_t)$$

Apply Itô's lemma to $Y_t = u(t,S_t)$:

$$\begin{aligned} du(t,S_t) &= \frac{\partial u}{\partial t}(t,S_t)\,dt + \frac{\partial u}{\partial S}(t,S_t)\,dS_t + \frac{1}{2}\frac{\partial^2 u}{\partial S^2}(t,S_t)\,d\langle S\rangle_t \\ &= \left(\frac{\partial u}{\partial t} + rS_t\frac{\partial u}{\partial S} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 u}{\partial S^2}\right)dt + \sigma S_t\frac{\partial u}{\partial S}\,d\widetilde{W}_t \end{aligned}$$

Since u satisfies the Black–Scholes PDE, the drift term equals ru. So:

$$du(t,S_t) = ru(t,S_t)\,dt + \sigma S_t\frac{\partial u}{\partial S}(t,S_t)\,d\widetilde{W}_t$$

This is a backward SDE of the form:

$$dY_t = rY_t\,dt + Z_t\,d\widetilde{W}_t, \quad \text{where } Z_t = \sigma S_t\frac{\partial u}{\partial S}(t,S_t)$$

To numerically solve this BSDE using the Deep BSDE method, we discretize time with a uniform grid:

$$0 = t_0 < t_1 < \cdots < t_N = T, \quad \Delta t = t_{n+1} - t_n$$

Simulate the forward SDE using Euler–Maruyama:

$$S_{t_{n+1}} = S_{t_n} + rS_{t_n}\,\Delta t + \sigma S_{t_n}\,\Delta W_n, \quad \Delta W_n \sim \mathcal{N}(0,\Delta t)$$

Discretize the BSDE for $Y_t = u(t,S_t)$:

$$\begin{aligned} Y_{t_{n+1}} &= Y_{t_n} + rY_{t_n}\Delta t + Z_{t_n}\Delta W_n \\ &= (1+r\Delta t)\,Y_{t_n} + Z_{t_n}\Delta W_n \end{aligned}$$

We approximate:

- $Y_{t_n} \approx u(t_n, S_{t_n})$
- $\delta(t_n, S_{t_n}) \approx \frac{\partial u}{\partial S}(t_n, S_{t_n})$ is predicted by a neural network
- $Z_{t_n} = \sigma S_{t_n}\cdot\delta(t_n, S_{t_n})$

At the final step $t_N = T$, the terminal condition becomes:

$$Y_T = u(T,S_T) \approx (S_T-K)^+$$

We define the loss function as the mismatch at terminal time:

$$\mathcal{L} = \mathbb{E}\left[\left(Y_T - (S_T-K)^+\right)^2\right]$$

Deep BSDE for the SABR Model

We consider the SABR stochastic volatility model:

$$\begin{cases} dF_t = \alpha_t F_t^{\beta}\, dW_t \\ d\alpha_t = \nu \alpha_t\, dZ_t \\ dW_t\, dZ_t = \rho\, dt \end{cases}$$

We define the value function:

$$u(t, F, \alpha) = \mathbb{E}[(F_T - K)^+ \mid F_t = F, \alpha_t = \alpha]$$

Under the risk-neutral measure, the function $u(t, F, \alpha)$ satisfies the following parabolic PDE:

$$\frac{\partial u}{\partial t} + \frac{1}{2}\alpha^2 F^{2\beta} \frac{\partial^2 u}{\partial F^2} + \rho\nu\alpha^2 F^{\beta} \frac{\partial^2 u}{\partial F\, \partial\alpha} + \frac{1}{2}\nu^2\alpha^2 \frac{\partial^2 u}{\partial \alpha^2} = 0$$

with terminal condition:

$$u(T, F, \alpha) = (F - K)^+$$

We apply Itô's lemma to the composite process $Y_t = u(t, F_t, \alpha_t)$:

$$\begin{aligned} du(t, F_t, \alpha_t) &= \frac{\partial u}{\partial t}\, dt + \frac{\partial u}{\partial F}\, dF_t + \frac{\partial u}{\partial \alpha}\, d\alpha_t \\ &\quad + \frac{1}{2}\frac{\partial^2 u}{\partial F^2}\, d\langle F\rangle_t + \frac{1}{2}\frac{\partial^2 u}{\partial \alpha^2}\, d\langle \alpha\rangle_t + \frac{\partial^2 u}{\partial F \partial \alpha}\, d\langle F, \alpha\rangle_t \end{aligned}$$

We compute the quadratic variations:

$$\begin{aligned} d\langle F\rangle_t &= \alpha_t^2 F_t^{2\beta}\, dt \\ d\langle \alpha\rangle_t &= \nu^2 \alpha_t^2\, dt \\ d\langle F, \alpha\rangle_t &= \rho\nu\alpha_t^2 F_t^{\beta}\, dt \end{aligned}$$

Substitute all components into the Itô expansion:

$$\begin{aligned} du = &\left[\frac{\partial u}{\partial t} + \frac{1}{2}\alpha_t^2 F_t^{2\beta}\frac{\partial^2 u}{\partial F^2} + \frac{1}{2}\nu^2\alpha_t^2 \frac{\partial^2 u}{\partial \alpha^2} + \rho\nu\alpha_t^2 F_t^{\beta} \frac{\partial^2 u}{\partial F \partial \alpha}\right] dt \\ &+ \alpha_t F_t^{\beta} \frac{\partial u}{\partial F}\, dW_t + \nu\alpha_t \frac{\partial u}{\partial \alpha}\, dZ_t \end{aligned}$$

Now, recall that $u(t, F, \alpha)$ satisfies the SABR PDE:

$$\frac{\partial u}{\partial t} + \frac{1}{2}\alpha^2 F^{2\beta} \frac{\partial^2 u}{\partial F^2} + \rho\nu\alpha^2 F^{\beta} \frac{\partial^2 u}{\partial F\, \partial\alpha} + \frac{1}{2}\nu^2\alpha^2 \frac{\partial^2 u}{\partial \alpha^2} = 0$$

Therefore, the entire drift term (the dt-part) cancels out.
We are left with only the stochastic (martingale) part:

$$du = \alpha_t F_t^{\beta}\, \frac{\partial u}{\partial F}\, dW_t + \nu\alpha_t\, \frac{\partial u}{\partial \alpha}\, dZ_t$$

Discretize time: $0 = t_0 < t_1 < \cdots < t_N = T$
Euler–Maruyama forward simulation:

$$\begin{cases} F_{t_{n+1}} = F_{t_n} + \alpha_{t_n} F_{t_n}^{\beta}\, \Delta W_n \\ \alpha_{t_{n+1}} = \alpha_{t_n} + \nu\alpha_{t_n}\, \Delta Z_n \end{cases}$$

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Discretized BSDE:

$$u(t_{n+1}, F_{t_{n+1}}, \alpha_{t_{n+1}}) \approx u(t_n, F_{t_n}, \alpha_{t_n}) \\ + \alpha_{t_n} F_{t_n}^{\beta}\, u_F(t_n, F_{t_n}, \alpha_{t_n})\, \Delta W_n + \nu \alpha_{t_n}\, u_\alpha(t_n, F_{t_n}, \alpha_{t_n})\, \Delta Z_n$$

We approximate:

- $Y_{t_n} \approx u(t_n, F_{t_n}, \alpha_{t_n})$
- $\delta_F(t_n) \approx u_F(t_n, F_{t_n}, \alpha_{t_n})$ is predicted by a neural network
- $\delta_\alpha(t_n) \approx u_\alpha(t_n, F_{t_n}, \alpha_{t_n})$ is predicted by a neural network

At each step t_n, a neural network takes $(t_n, F_{t_n}, \alpha_{t_n})$ as input and predicts δ_F and δ_α.

Terminal Condition and Loss

At $t = T$, we match the terminal payoff:

$$Y_T \approx u(T, F_T, \alpha_T) \approx (F_T - K)^+$$

Loss function:

$$\mathcal{L} = \mathbb{E}\left[\left(Y_T - (F_T - K)^+\right)^2\right]$$

The Deep BSDE method learns to approximate gradients u_F, u_α, and recursively predicts $u(t, F, \alpha)$ backward from maturity using Monte Carlo paths and neural networks.