Part I: PG methods as black box optimization
Part II: Variance reduction and baselines

Part III: Theoretical Foundations

Sham Kakade

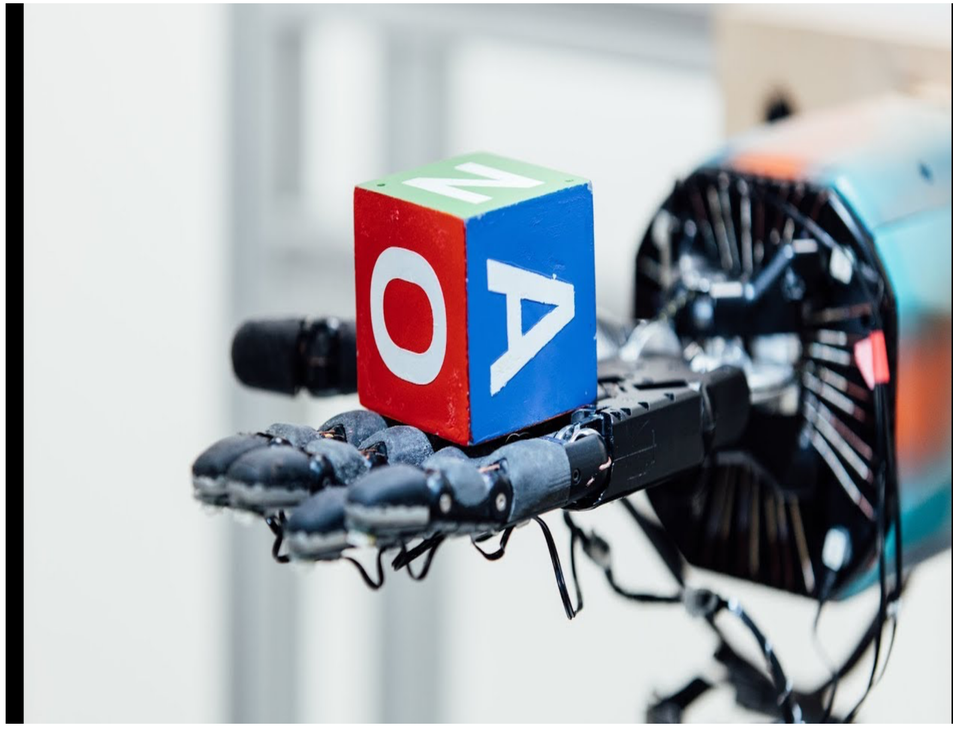
University of Washington & Microsoft Research

(with Nicolas Le Roux and Martha White)

Policy Optimization in RL

[AlphaZero, Silver et.al, 17]

[OpenAI Five, 18]



[OpenAI,19]

What should we be asking?

The “Tabular” Dynamic Programming approach

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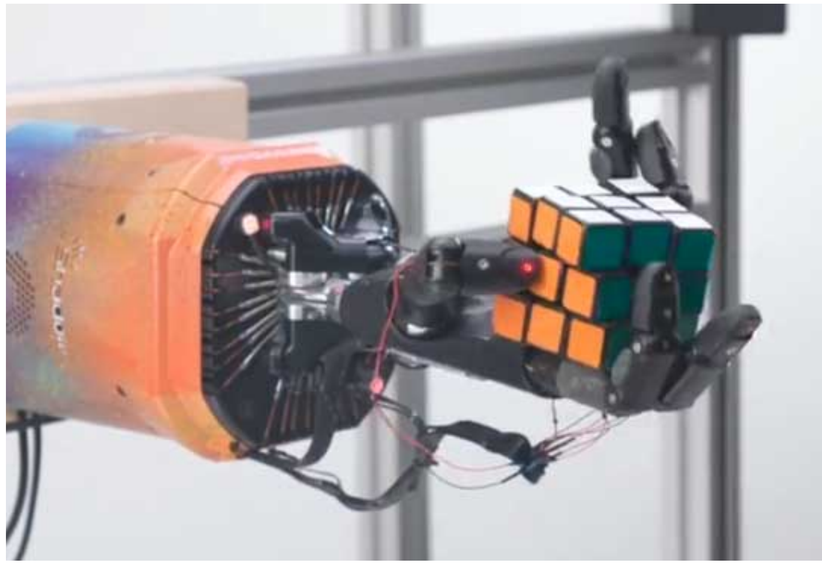
- Convergence:
  - Policy (& value) iteration geometrically converge to the global optima.
- Generalization:
  - There are worst case approximation guarantees.

In practice, policy gradient methods rule…

Why do we like them?



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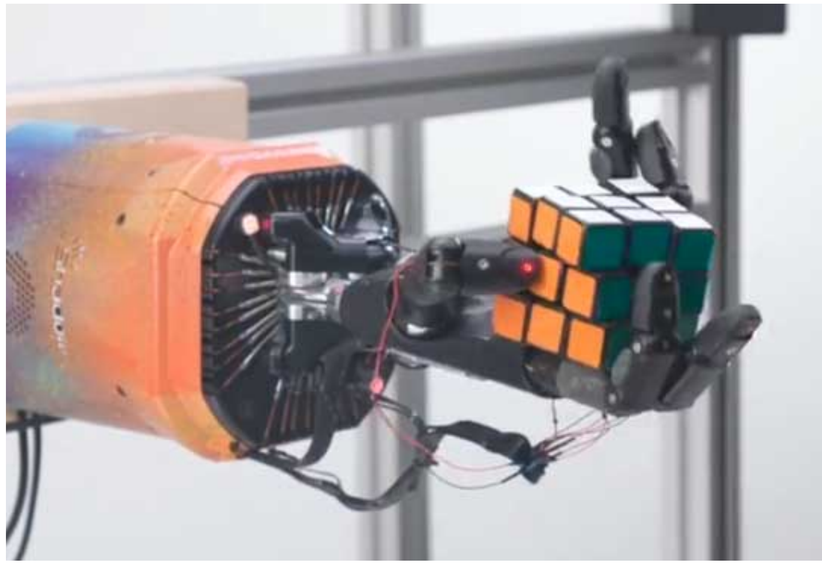

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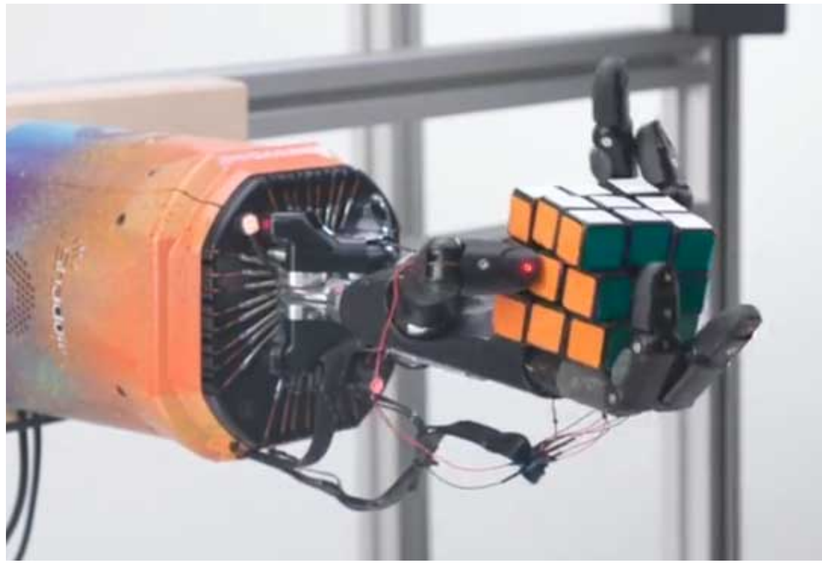

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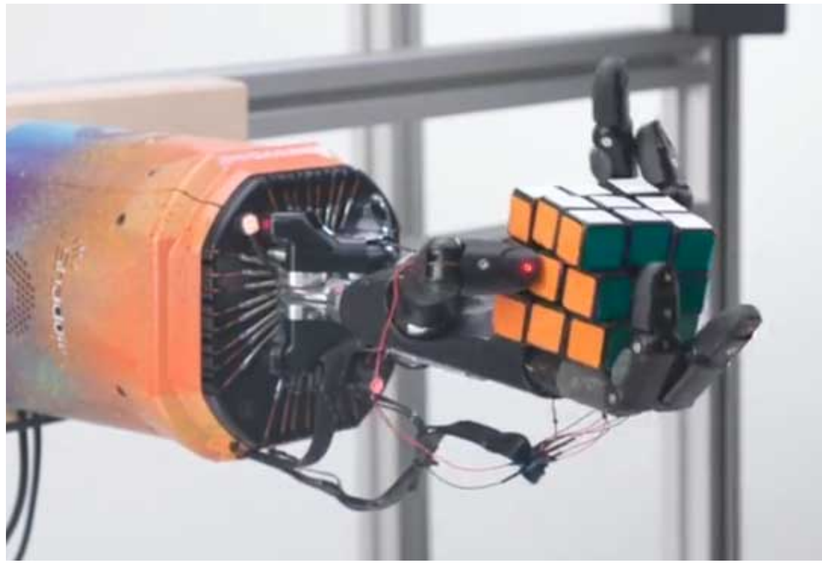

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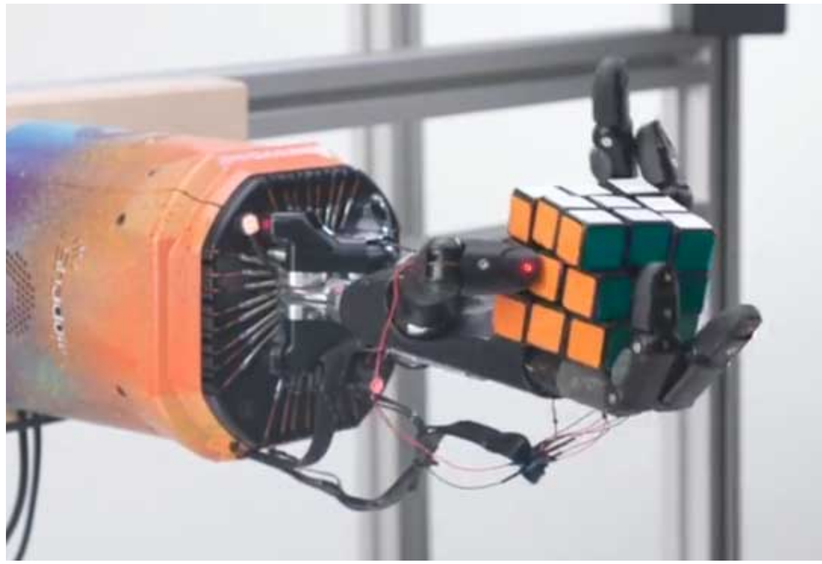

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Questions:

- Convergence: Do policy gradient methods globally converge?
- Generalization: How do they deal with infinite states/actions?

Part III: Theoretical Foundations

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Supplementary material:

- AJKS Book: Reinforcement Learning: Theory and Algorithms, Section 3
- Following along with the colab notebooks!

Part A: Small State Spaces

(and the softmax policy class)

Policy Optimization over the ”softmax” policy class
(let’s start simple!)

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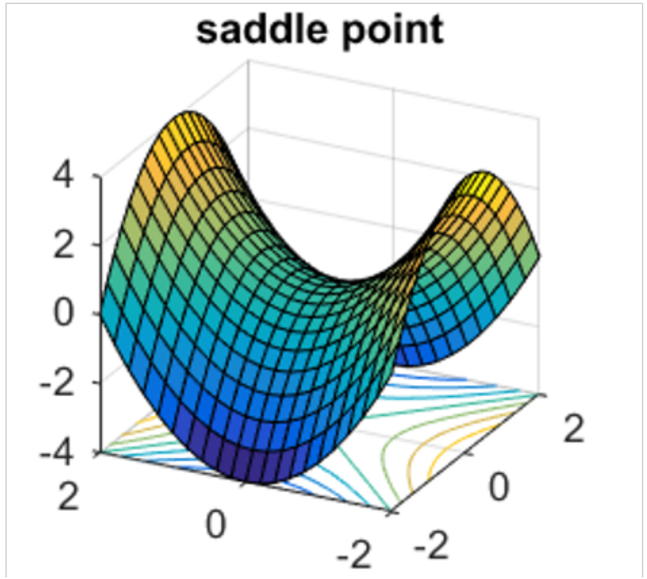
The policy optimization problem $\max_\theta V^{\pi_\theta}(s_0)$ is non-convex.

Do we have global convergence?

The Optimization Landscape



Supervised Learning:

- Gradient descent tends to ‘just work’ in practice (not sensitive to initialization)
- Saddle points not a problem…

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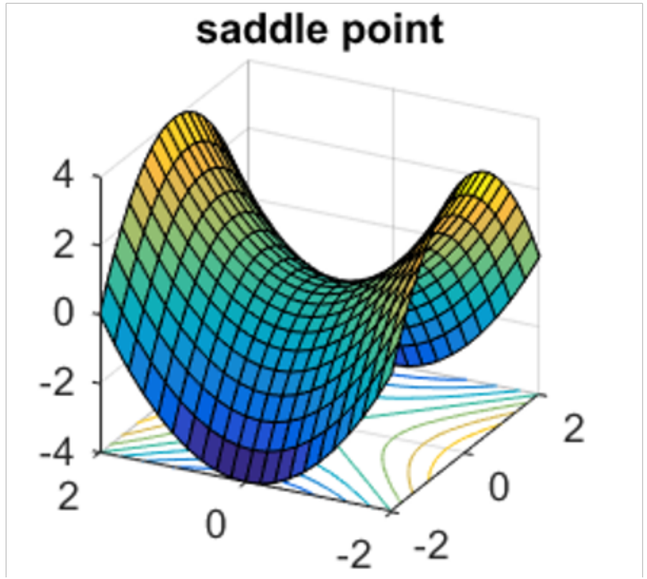

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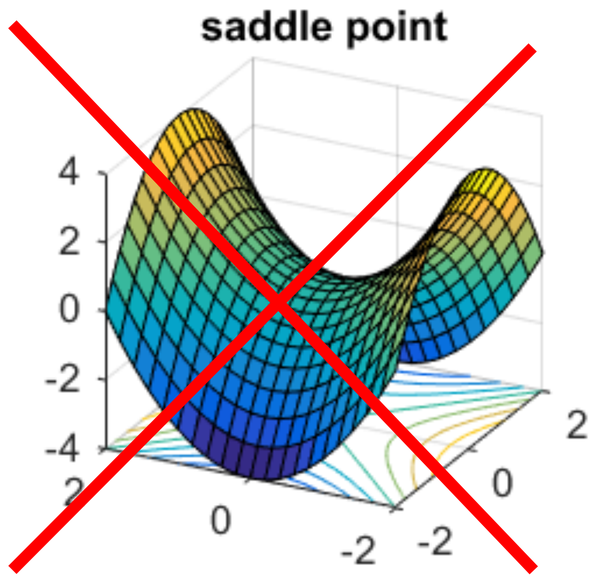

Reinforcement Learning:

- In many real RL problems, we have “very” flat regions.
- Gradients can be exponentially small in the “horizon” due to lack of exploration.

The Optimization Landscape

Lemma: [Higher order vanishing gradients]

Suppose there are $S \leq 1/(1-\gamma)$ states in the MDP. With random initialization, all k-th higher-order gradients, for $k < S/\log(S)$, the spectral norm of the gradients are bounded by $2^{-S/2}$.

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Thrun ‘92

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How might we address this?

Colab Notebook III-a: PG on $V^{\theta}(\mu)$

MDP:

- $S = 20, \;\; 1/(1-\gamma) = 20$
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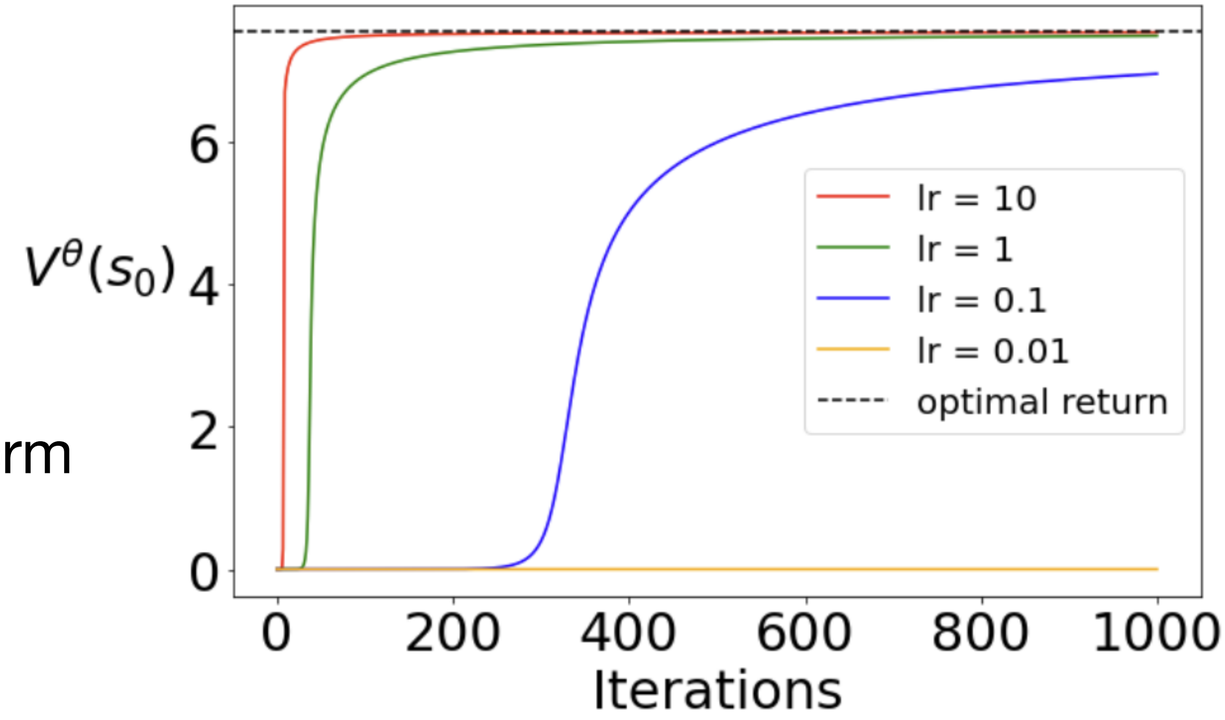

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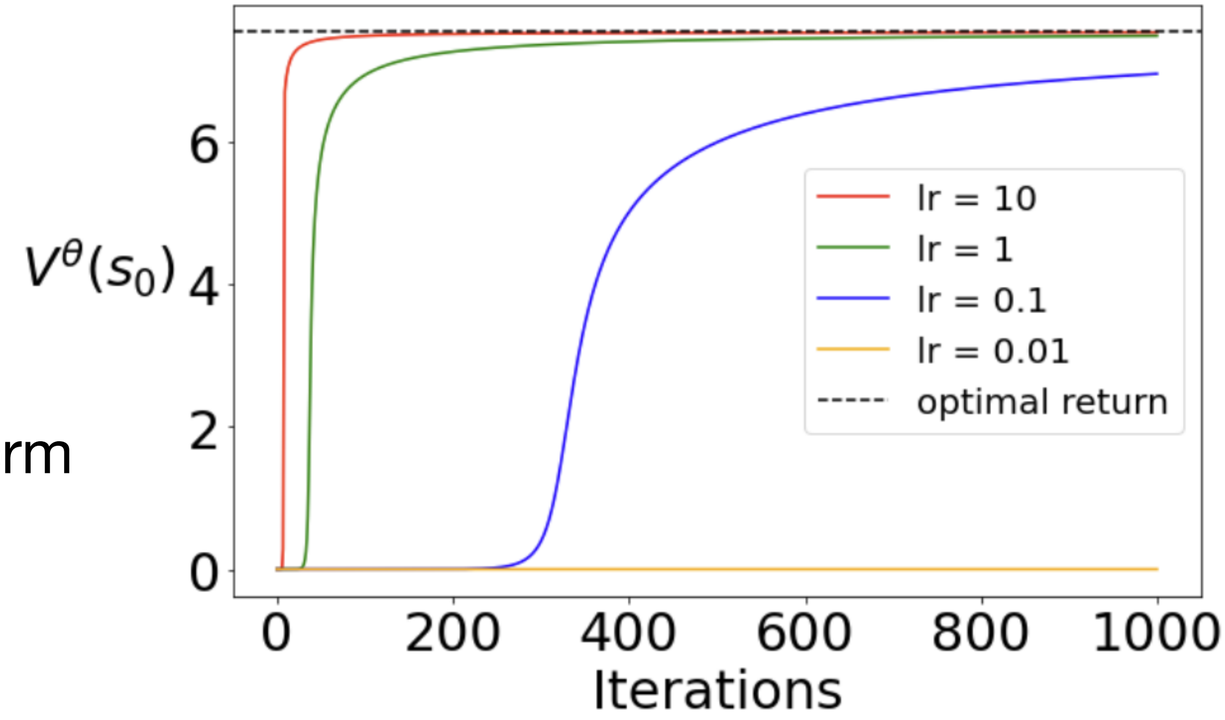
- Does this idea always work?



Global Convergence of PG for Softmax

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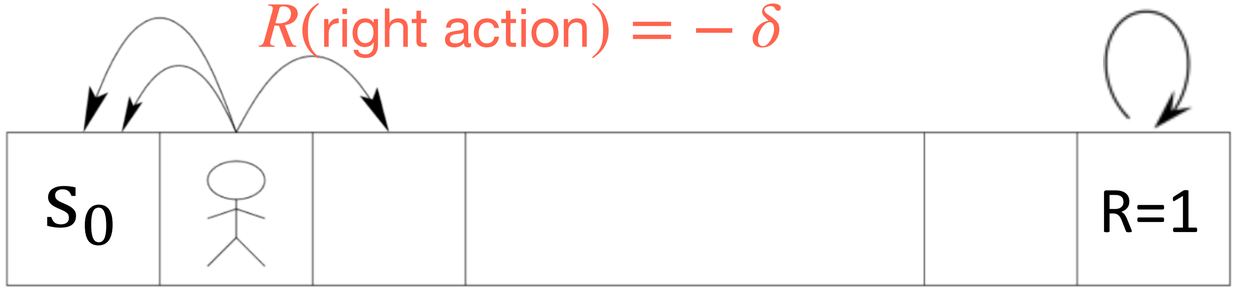
- Rate could be exponentially slow in terms of $\#\text{states}$
  - Let's take a closer look!

Colab Notebook III-b: the “anti-shaped” MDP

“Anti-shaped” modificiation:

- $-\delta = 0.5$ reward for moving right
  (otherwise the same MDP)
- $V^\star(s_0) \approx 0.05$



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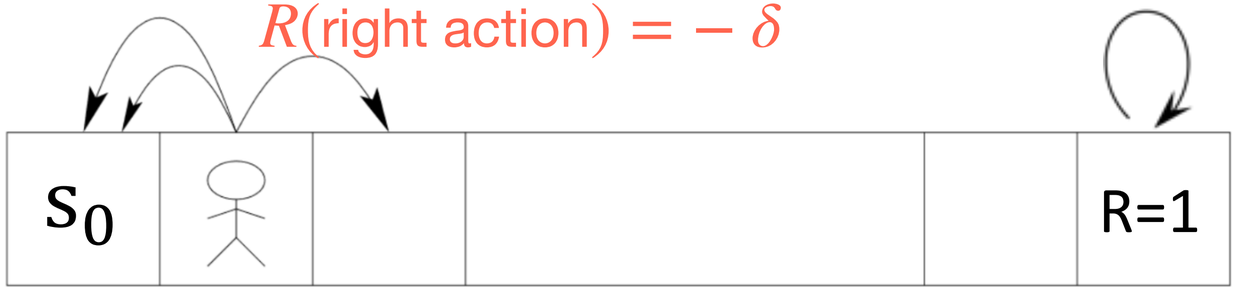

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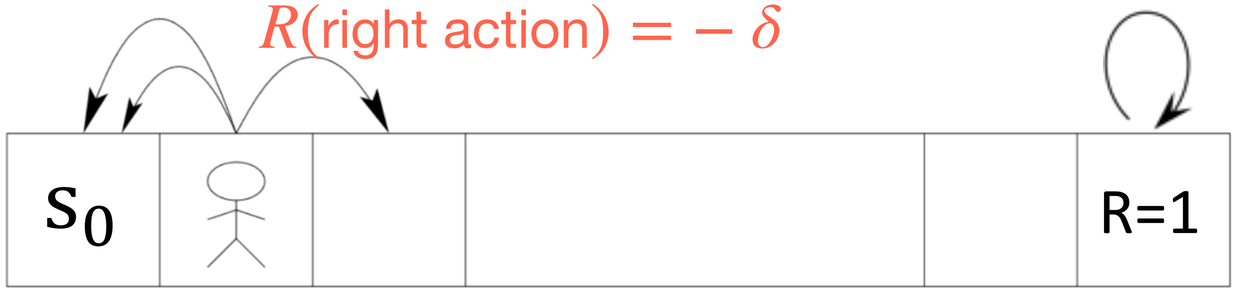

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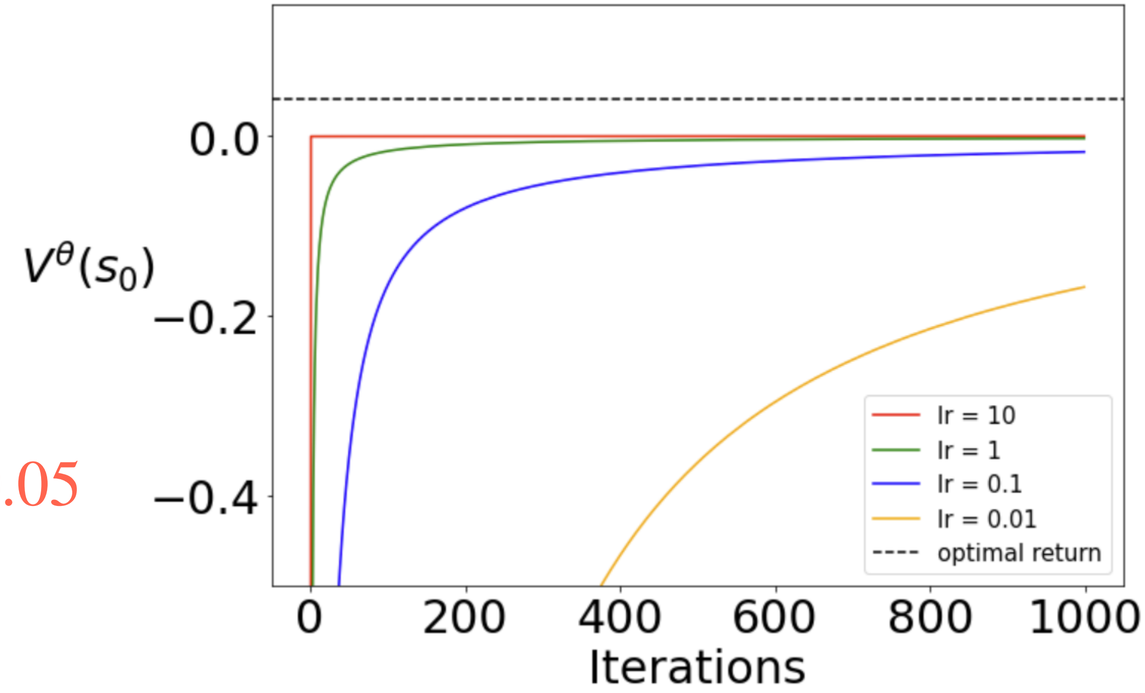
Practice:

- we never see convergence to opt, $V^\star(s_0) \approx 0.05$
  (stalls at $V(s_0) = 0$)
- rate could be exponentially slow



Colab Notebook III-c: Entropy Regularization

- (for softmax) we get flat gradients when π becomes deterministic
- heuristic: use entropy regularization to penalize against deterministic policies

$$L_{ent}(\theta) := V^{\theta}(\mu) + \lambda \sum_{s,a} \mu(s)\pi_{\theta}(a \mid s) \log \pi_{\theta}(a \mid s)$$

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  (see AJKS, Chapter 10: Further Reading)

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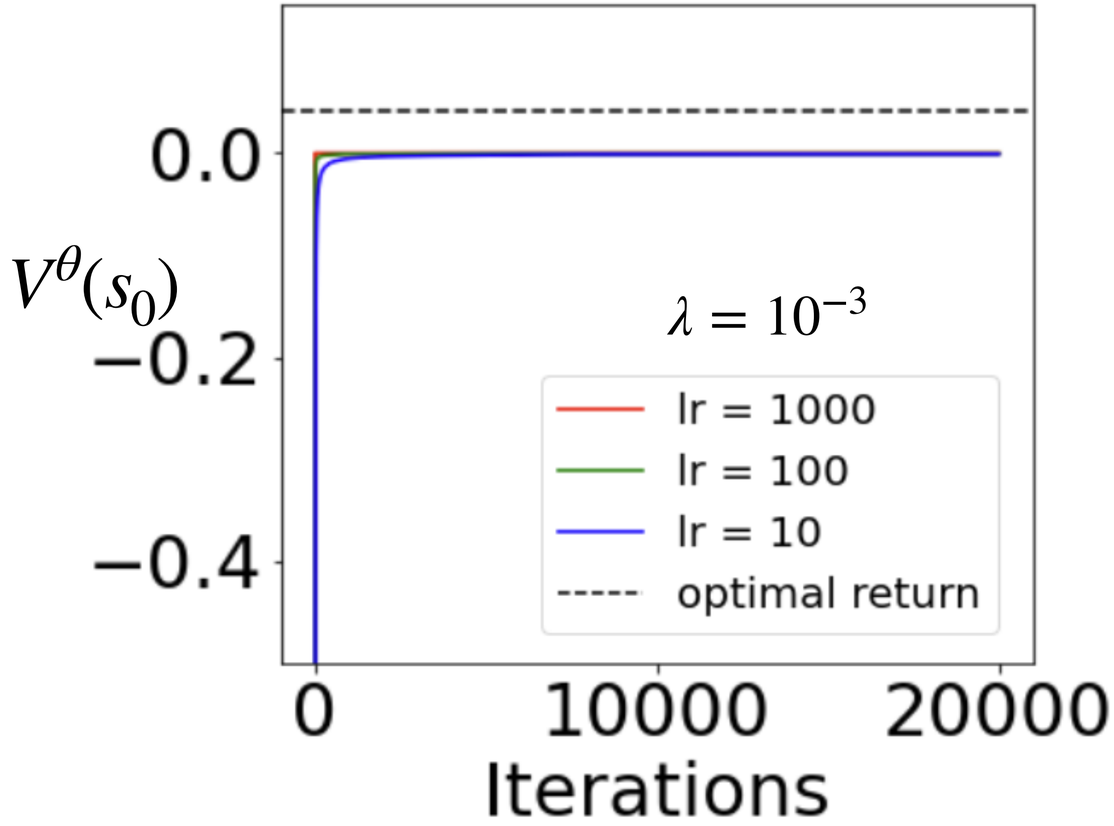
- rate could be exponentially slow



Global Convergence: Softmax + Log Barrier regularization

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Theorem [PG: Softmax+Log Barrier]

S : #states, A : #actions, H : Horizon $= 1/(1-\gamma)$

Suppose $\mu = \text{uniform}_S$ and with appropriate settings of λ and η

After $\frac{S^4A^2H^6}{\epsilon^2}$ iterations, we have for all s,

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- Even though problem is non-convex, we a have poly iteration complexity.
- Log barrier and uniform μ helps with conditioning problems.
  - succinct proof: shows that $\pi_\theta(a\,|\,s)$ doesn't become too small (see AJKS)
  - log barrier reg = KL-regularization ≠ entropy regularization

Colab Notebook III-d: Log Barrier Regularization

- Log barrier aggressively penalizes against deterministic policies
- polynomial upper bound is slow

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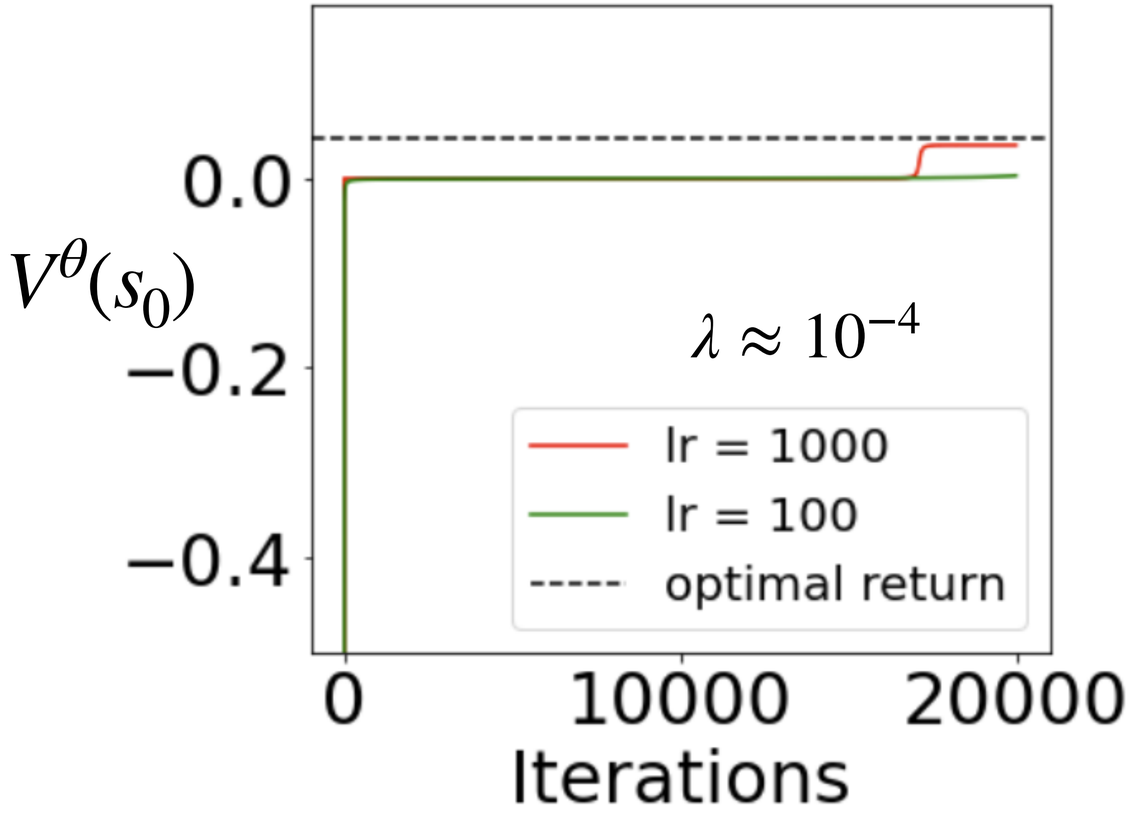
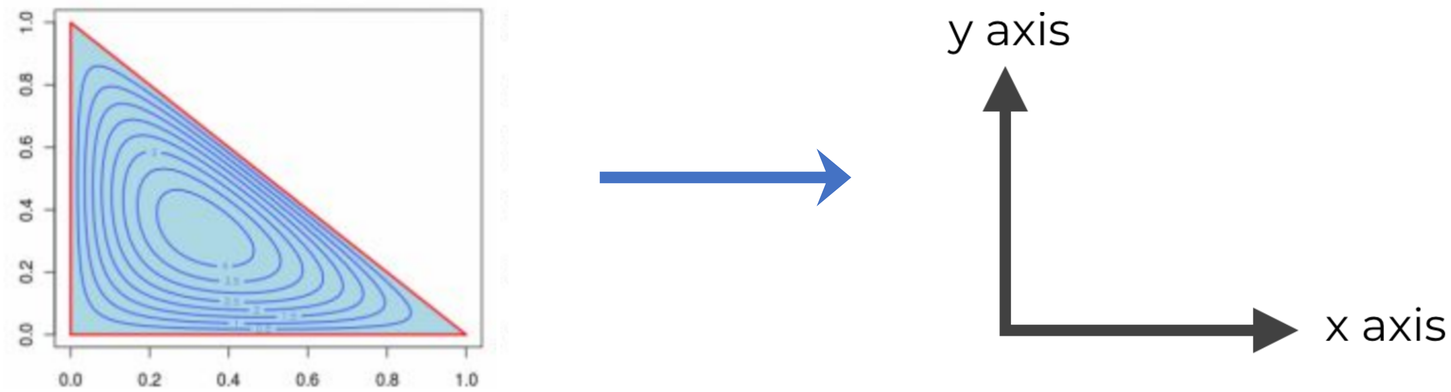
Practice: (on the anti-shaped example)

- for small λ & appropriate learn. rate, we convergence near to opt
- rate is polynomial, yet slow



Preconditioning: The Natural Policy Gradient (NPG)



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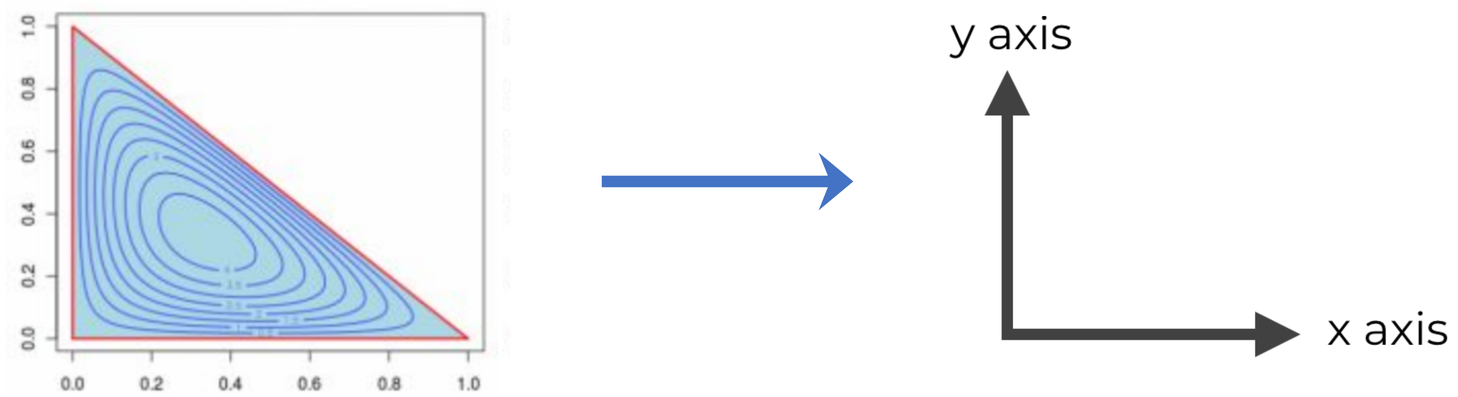

- Practice: most methods are gradient based, usually variants of:
  NPG [K. ‘01]; TRPO [Schulman ‘15]; PPO [Schulman ‘17]
- NPG warps the distance metric to stretch the corners out (using the Fisher information metric) to move ‘more’ near the boundaries. The update is:

$$F(\theta) = E_{s,a\sim\pi_\theta}\left[\nabla \log \pi_\theta(a \mid s)\, \nabla \log \pi_\theta(a \mid s)^\top\right]$$

$$\theta \leftarrow \theta + \eta F(\theta)^{-1} \nabla V^\theta(s_0)$$

NPG and “soft” policy iteration

- The softmax policy class: $\pi_\theta(a \mid s) \propto \exp(\theta_{s,a})$

NPG and “soft” policy iteration

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- At iteration t, the NPG update rule:

$$\theta \leftarrow \theta + \eta F(\theta)^{-1} \nabla V^\theta(s_0)$$

is equivalent to a “soft” policy iteration update rule:

$$\pi(a \mid s) \leftarrow \pi(a \mid s) \frac{\exp(\eta Q^\pi(s,a))}{Z}$$

NPG and “soft” policy iteration

- The softmax policy class: $\pi_\theta(a \mid s) \propto \exp(\theta_{s,a})$
- At iteration t, the NPG update rule:

$$\theta \leftarrow \theta + \eta F(\theta)^{-1} \nabla V^\theta(s_0)$$

is equivalent to a “soft” policy iteration update rule:

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What happens for this non-convex update rule?

Global Convergence of NPG

Theorem [NPG]

Set $\eta = (1-\gamma)^2 \log A$.

For the softmax policy class, we have after T iterations and for all states s_0

$$V^{(T)}(s_0) \geq V^{\star}(s_0) - \frac{2}{(1-\gamma)^2 T}$$

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- Dimension free iteration complexity: (No dependence on S, A, μ)
- Also a “fast rate”.
- Even though problem is non-convex, a mirror descent analysis applies. Analysis idea from [Even-Dar, K., Mansour 2009]

Colab Notebook III-e: NPG

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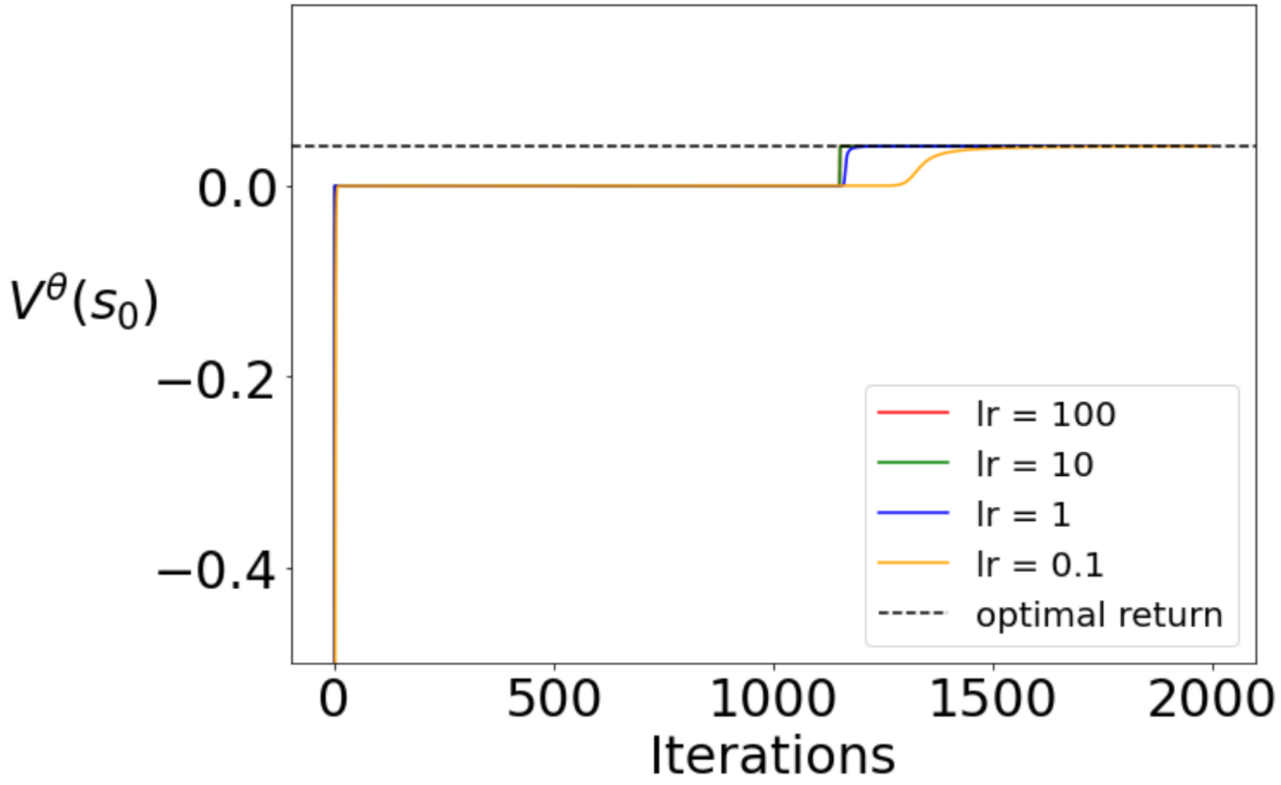
Practice: (on the anti-shaped example)
- convergence to opt, $V^{\star}(s_0) \approx 0.05$ is relatively fast



Part-B: Large State Spaces

What about generalization and sampling?

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- Neural policy class: $f_\theta(s, a)$ is a neural network

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2. policy update

$$\pi(a \mid s) \leftarrow \frac{\pi(a \mid s) \exp(\eta w_\star \cdot \phi_{s,a})}{Z_s}$$

(Z_s is the normalizing constant)

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- This is a distribution shift factor, due to misalignment between $d^{(t)}$ and $d^\star$.
- This guarantee is stronger than approximate dynamic programming.
- See AJKS!

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Thanks for making the notebooks!


Alan Chan


Shivam Garg


Dhawal Gupta


Abhishek Naik