# 混合有限元方法

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# Contents

# 第一章　混合变分问题

## 1.1　混合变分问题举例

### Poisson 方程

对 $f \in L^2(\Omega)$, 考虑 Poisson 方程齐次 Dirichlet 边值问题

$$
\begin{aligned}
-\Delta u &= f, && \text{in} \quad \Omega, \\
u &= 0, && \text{on} \quad \partial\Omega.
\end{aligned}
$$

其变分形式为, 找 $u \in V = H_0^1(\Omega)$, 使得

$$
(\nabla u, \nabla v) = (f, v), \quad \forall v \in V. \tag{1.1}
$$

对应的有限元方法为, 找 $u \in V_h$, 使得

$$
(\nabla_h u, \nabla_h v) = (f, v), \quad \forall v \in V_h. \tag{1.2}
$$

其中有限元空间 $V_h$ 是 $V$ 的某种分片多项式近似. 另一方面, 许多偏微分方程都是由物理过程决定的, 对上述泊松方程, 通过引入一个新的变量 $\boldsymbol{\sigma}$ 可以分解为如下形式

$$
\boldsymbol{\sigma} = \nabla u, \qquad \text{in} \quad \Omega \tag{1.3a}
$$

$$
-\operatorname{div} \boldsymbol{\sigma} = f, \qquad \text{in} \quad \Omega \tag{1.3b}
$$

$$
u = 0, \qquad \text{on} \quad \partial\Omega. \tag{1.3c}
$$

其中(1.3a)称为本构方程, (1.3b)为守恒方程. 本例中可以理解标量势函数 $u$(温度, 电势等) 决定了通量 $\boldsymbol{\sigma}$; 通量的散度 (通过分布积分, 对应于净流出量) 对应于源项 $f$, 即

$$
-\int_K \operatorname{div} \boldsymbol{\sigma} \mathrm{d}x = -\int_{\partial K} \boldsymbol{\sigma} \cdot \boldsymbol{n} \, \mathrm{d}s = \int_K f \, \mathrm{d}x,
$$

其中 $K$ 是 $\Omega$ 的任意子区域. 对于标准有限元方法(1.2), 上述守恒相当于测试函数空间 $V_h$ 包含分片间断函数, 显然不成立. 而混合形式(1.3a)-(1.3c)对应的混合有限元方法则可以满足守恒律.

混合形式(1.3a)-(1.3c)可以定义多种变分问题:

**Formula 1.1.** 找 $\boldsymbol{\sigma} \in H(\operatorname{div}, \Omega), u \in L^2(\Omega)$, 使得

$$
\begin{aligned}
(\boldsymbol{\sigma}, v) + (\operatorname{div} \boldsymbol{\tau}, u) &= 0, && \forall \boldsymbol{\tau} \in H(\operatorname{div}, \Omega), \\
-(\operatorname{div} \boldsymbol{\sigma}, v) &= (f, v), && \forall v \in L^2(\Omega)
\end{aligned} \tag{1.4}
$$

变分问题(1.4)称为 Possion 方程的混合对偶格式. 通过分部积分可以将微分算子转移到 $u, v$ 上, 得到与 Primal 变分形式(1.1)等价的混合变分问题(1.5).

**Formula 1.2.** 找 $\boldsymbol{\sigma} \in L_0^2(\Omega; \mathbb{R}^d), u \in H_0^1(\Omega)$, 使得

$$\begin{aligned} (\boldsymbol{\sigma}, \boldsymbol{\tau}) - (\boldsymbol{\tau}, \nabla u) &= 0, && \forall \boldsymbol{\tau} \in L_0^2(\Omega; \mathbb{R}^d), \\ (\boldsymbol{\sigma}, \nabla v) &= (f, v), && \forall v \in H_0^1(\Omega). \end{aligned} \tag{1.5}$$

若将(1.4)中的内积理解为 $H^{-1}(\Omega)$ 与 $H_0^1(\Omega)$ 上的对偶对, 则可得到

**Formula 1.3.** 找 $\boldsymbol{\sigma} \in L_0^2(\Omega; \mathbb{R}^d), u \in H_0^1(\Omega)$, 使得

$$\begin{aligned} (\boldsymbol{\sigma}, \boldsymbol{\tau}) + \langle \operatorname{div} \boldsymbol{\tau}, u \rangle &= 0, && \forall \boldsymbol{\tau} \in L_0^2(\Omega; \mathbb{R}^d), \\ -\langle \operatorname{div} \boldsymbol{\sigma}, v \rangle &= (f, v), && \forall v \in H_0^1(\Omega). \end{aligned} \tag{1.6}$$

## Stokes 方程

考虑 Stokes 方程:

$$-\Delta \boldsymbol{u} - \operatorname{grad} p = \boldsymbol{f}, \qquad \text{in} \quad \Omega, \tag{1.7a}$$

$$\operatorname{div} \boldsymbol{u} = 0, \qquad \text{in} \quad \Omega, \tag{1.7b}$$

$$\boldsymbol{u} = \boldsymbol{0}, \qquad \text{on} \quad \partial\Omega. \tag{1.7c}$$

其中 $\boldsymbol{u}$ 表示流体速度，$p$ 表示压力，(1.7a)为本构方程, (1.7b)为守恒方程. 通过引入伪应力 $\boldsymbol{\sigma}$, 可以将(1.7a)-(1.7c)改写为

$$\boldsymbol{\sigma} = \nabla \boldsymbol{u}, \qquad \text{in} \quad \Omega \tag{1.8a}$$

$$-\operatorname{div} \boldsymbol{\sigma} - \nabla p = f \qquad \text{in} \quad \Omega, \tag{1.8b}$$

$$\operatorname{div} \boldsymbol{u} = 0, \qquad \text{in} \quad \Omega \tag{1.8c}$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on} \quad \partial\Omega. \tag{1.8d}$$

注意到 $\operatorname{tr} \boldsymbol{\sigma} = \operatorname{div} \boldsymbol{u} = 0$，即 $\boldsymbol{\sigma}$ 属于无迹张量空间 $\mathbb{T}$.

(1.7a)-(1.7c)对应的变分问题为

**Formula 1.4.** 找 $\boldsymbol{u} \in H_0^1(\Omega; \mathbb{R}^d), p \in L_0^2(\Omega)$, 使得

$$\begin{aligned} (\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + (\operatorname{div} \boldsymbol{v}, p) &= (\boldsymbol{f}, \boldsymbol{v}), && \forall \boldsymbol{v} \in H_0^1(\Omega; \mathbb{R}^d), \\ (\operatorname{div} \boldsymbol{u}, q) &= 0, && \forall q \in L_0^2(\Omega). \end{aligned} \tag{1.9}$$

变分问题(1.9)说明, 与椭圆型偏微分方程 (eg. Possion 方程的 primal 形式) 不同, Stokes 方程对应的变分问题本身即为鞍点问题. (1.8a)-(1.8d)对应的变分问题为

**Formula 1.5.** 找 $\boldsymbol{\sigma} \in H(\operatorname{div}, \Omega; \mathbb{T}), u \in H_0(\operatorname{div}, \Omega)), p \in L_0^2(\Omega)$, 使得

$$\begin{aligned} (\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\operatorname{div} \boldsymbol{\tau}, \boldsymbol{u}) + (\operatorname{div} \boldsymbol{u}, q) &= 0, && \forall \boldsymbol{\tau} \in H(\operatorname{div}, \Omega; \mathbb{T}), q \in L_0^2(\Omega) \\ -(\operatorname{div} \boldsymbol{\sigma}, v) + (\operatorname{div} \boldsymbol{v}, p) &= (\boldsymbol{f}, \boldsymbol{v}), && \forall \boldsymbol{v} \in H_0(\operatorname{div}, \Omega). \end{aligned}$$

实际上, $\boldsymbol{\sigma}$ 的光滑性要求可以降低, 对应的分部式混合变分问题为

**Formula 1.6.** 找 $\boldsymbol{\sigma} \in H^{-1}(\operatorname{curl}\operatorname{div}, \Omega; \mathbb{T}), u \in H_0(\operatorname{div}, \Omega)), p \in L_0^2(\Omega)$, 使得

$$\begin{aligned}
(\boldsymbol{\sigma}, \boldsymbol{\tau}) + \langle \operatorname{div} \boldsymbol{\tau}, \boldsymbol{u} \rangle + (\operatorname{div} \boldsymbol{u}, q) &= 0, && \forall \boldsymbol{\tau} \in H^{-1}(\operatorname{curl}\operatorname{div}, \Omega; \mathbb{T}), q \in L_0^2(\Omega)\\
-\langle \operatorname{div} \boldsymbol{\sigma}, \boldsymbol{v} \rangle + (\operatorname{div} \boldsymbol{v}, p) &= (\boldsymbol{f}, \boldsymbol{v}), && \forall \boldsymbol{v} \in H_0(\operatorname{div}, \Omega),
\end{aligned}$$

其中 $H^{-1}(\operatorname{curl}\operatorname{div}, \Omega; \mathbb{T}) = \{\boldsymbol{\tau} \in L^2(\Omega; \mathbb{T}) : \operatorname{div} \boldsymbol{\tau} \in (H_0(\operatorname{div}, \Omega))'\}$, $\langle \cdot, \cdot \rangle$ 表示 $(H_0(\operatorname{div}, \Omega))'$ 与 $H_0(\operatorname{div}, \Omega)$ 间的对偶对.

在 Stokes 方程的基础上, 有更贴合物理性质的应变 Stokes 方程:

$$\begin{aligned}
-\operatorname{div}(\varepsilon(\boldsymbol{u})) - \operatorname{grad} p = \boldsymbol{f}, &\quad \text{in} \quad \Omega,\\
\operatorname{div} \boldsymbol{u} = 0, &\quad \text{in} \quad \Omega,\\
\boldsymbol{u} = \boldsymbol{0} &\quad \text{on} \quad \partial\Omega,
\end{aligned} \tag{1.10}$$

其中 $\varepsilon(\boldsymbol{u}) = \frac{1}{2}(\operatorname{grad} \boldsymbol{u} + \operatorname{grad} \boldsymbol{u}^\intercal)$ 为梯度的对称部分. 进一步地, 引入 $\boldsymbol{\sigma} = \varepsilon(\boldsymbol{u})$, 对应的方程为

$$\begin{aligned}
\boldsymbol{\sigma} = \varepsilon(\boldsymbol{u}), &\quad \text{in} \quad \Omega,\\
-\operatorname{div} \boldsymbol{\sigma} - \operatorname{grad} p = \boldsymbol{f}, &\quad \text{in} \quad \Omega,\\
\operatorname{div} \boldsymbol{u} = 0, &\quad \text{in} \quad \Omega,\\
\boldsymbol{u} = \boldsymbol{0} &\quad \text{on} \quad \partial\Omega.
\end{aligned} \tag{1.11}$$

方程(1.10), (1.11)对应的变分问题分别为

**Formula 1.7.** 找 $\boldsymbol{u} \in H_0^1(\Omega; \mathbb{R}^d), p \in L_0^2(\Omega)$, 使得

$$\begin{aligned}
(\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{u})) + (\operatorname{div} \boldsymbol{v}, p) &= (\boldsymbol{f}, \boldsymbol{v}), && \forall \boldsymbol{v} \in H_0^1(\Omega; \mathbb{R}^d),\\
(\operatorname{div} \boldsymbol{u}, q) &= 0, && \forall q \in L_0^2(\Omega).
\end{aligned}$$

**Formula 1.8.** 找 $\boldsymbol{\sigma} \in H(\operatorname{div}, \Omega; \mathbb{S} \cap \mathbb{T}), u \in H_0(\operatorname{div}, \Omega)), p \in L_0^2(\Omega)$, 使得

$$\begin{aligned}
(\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\operatorname{div} \boldsymbol{\tau}, \boldsymbol{u}) - (\operatorname{div} \boldsymbol{u}, q) &= 0, && \forall \boldsymbol{\tau} \in H(\operatorname{div}, \Omega; \mathbb{S} \cap \mathbb{T}), q \in L_0^2(\Omega)\\
(\operatorname{div} \boldsymbol{\sigma}, v) - (\operatorname{div} \boldsymbol{v}, p) &= -(\boldsymbol{f}, \boldsymbol{v}), && \forall \boldsymbol{v} \in H_0(\operatorname{div}, \Omega).
\end{aligned}$$

## 线弹性方程

线弹性方程为:

$$-\operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{f}, \qquad \text{in } \Omega, \tag{1.12a}$$

$$\boldsymbol{\sigma}(\boldsymbol{u}) = 2\mu\, \varepsilon(\boldsymbol{u}) + \lambda \operatorname{div} \boldsymbol{u} I \qquad \text{in } \Omega, \tag{1.12b}$$

$$\boldsymbol{u} = 0, \qquad \text{on } \partial\Omega. \tag{1.12c}$$

其中 (1.12a)为守恒方程, (1.12b)为本构方程, $\boldsymbol{u}$ 是位移向量，$\varepsilon(\boldsymbol{u})$ 是应变, $\boldsymbol{\sigma}(\boldsymbol{u})$ 是应力张量，$\mu$ 是正常数，$\lambda$ 称为 Lamé 系数. 若引入 $p = \lambda \operatorname{div} \boldsymbol{u}$，则线弹性方程可以改写为

$$\begin{aligned}
-\operatorname{div}(2\mu\varepsilon(\boldsymbol{u}) + pI) = \boldsymbol{f}, &\qquad \text{in } \Omega,\\
p = \lambda \operatorname{div} \boldsymbol{u} &\qquad \text{in } \Omega,\\
\boldsymbol{u} = 0, &\qquad \text{on } \partial\Omega.
\end{aligned} \tag{1.13}$$

(1.13)对应的混合变分问题为

**Formula 1.9.** 找 $\boldsymbol{u}\in H_0^1(\Omega;\mathbb{R}^d), p\in L_0^2(\Omega)$, 使得

$$2\mu(\varepsilon(\boldsymbol{u}),\varepsilon(\boldsymbol{v}))+(\operatorname{div}\boldsymbol{v},p)=(f,\boldsymbol{v}),\qquad \forall \boldsymbol{v}\in H_0^1(\Omega;\mathbb{R}^d), \tag{1.14a}$$

$$(\operatorname{div}\boldsymbol{u},q)-\frac{1}{\lambda}(p,q)=0,\qquad \forall q\in L_0^2(\Omega). \tag{1.14b}$$

(1.14a)-(1.14b)通常被称为 Lamé 系统. 当 $\lambda\to\infty$ 时, 其数值方法常会出现闭锁 (locking) 现象. 注意到当没有 $\frac{1}{\lambda}(p,q)$ 这一项时，即为应变 Stokes 方程, 即应变 Stokes 方程是线弹性方程的极限问题, 使用应变 Stokes 方程的有限元方法可以解决闭锁现象.

进一步地, 对线弹性方程(1.12a)-(1.12c)改写. 由 $\operatorname{tr}\boldsymbol{\sigma}=2\mu\operatorname{div}\boldsymbol{u}+d\lambda\operatorname{div}\boldsymbol{u}$，得 $\operatorname{div}\boldsymbol{u}=\frac{1}{d\lambda+2\mu}\operatorname{tr}\boldsymbol{\sigma}$, 即 $\boldsymbol{\sigma}=2\mu\varepsilon(\boldsymbol{u})+\frac{\lambda}{d\lambda+2\mu}\operatorname{tr}\boldsymbol{\sigma}I$, 故有

$$\varepsilon(u)=\frac{1}{2\mu}\Big(\boldsymbol{\sigma}-\frac{\lambda}{d\lambda+2\mu}\operatorname{tr}\boldsymbol{\sigma}I\Big).$$

为了简化记号，引入算子 $A$:

$$A\boldsymbol{\sigma}=\frac{1}{2\mu}\Big(\boldsymbol{\sigma}-\frac{\lambda}{d\lambda+2\mu}\operatorname{tr}\boldsymbol{\sigma}I\Big)=\frac{1}{2\mu}\operatorname{dev}\boldsymbol{\sigma}+\frac{1}{d(2\mu+d\lambda)}\operatorname{tr}\boldsymbol{\sigma}I,$$

其中 $\operatorname{dev}\boldsymbol{\sigma}=\boldsymbol{\sigma}-\frac{1}{d}\operatorname{tr}\boldsymbol{\sigma}I$，则对应的方程为

$$\begin{aligned} A\boldsymbol{\sigma}&=\varepsilon(\boldsymbol{u}), && \text{in }\Omega,\\ -\operatorname{div}\boldsymbol{\sigma}&=\boldsymbol{f}, && \text{in }\Omega,\\ \boldsymbol{u}&=0, && \text{on }\partial\Omega. \end{aligned}$$

其对应的混合变分问题为

**Formula 1.10.** 找 $\boldsymbol{\sigma}\in H(\operatorname{div},\Omega;\mathbb{S}), \boldsymbol{u}\in L^2(\Omega;\mathbb{R}^d)$, 使得

$$\begin{aligned} (A\boldsymbol{\sigma},\boldsymbol{\tau})+(\operatorname{div}\boldsymbol{\tau},\boldsymbol{u})&=0, && \forall\boldsymbol{\tau}\in H(\operatorname{div},\Omega;\mathbb{S}),\\ -(\operatorname{div}\boldsymbol{\sigma},\boldsymbol{v})&=(\boldsymbol{f},\boldsymbol{v}), && \forall\boldsymbol{v}\in L^2(\Omega;\mathbb{R}^d). \end{aligned}$$

**Formula 1.11.** 找 $\boldsymbol{\sigma}\in L^2(\Omega;\mathbb{S}), \boldsymbol{u}\in H_0^1(L^2(\Omega;\mathbb{R}^d)$, 使得

$$\begin{aligned} (A\boldsymbol{\sigma},\boldsymbol{\tau})-(\boldsymbol{\tau},\nabla\boldsymbol{u})&=0, && \forall\boldsymbol{\tau}\in L^2(\Omega;\mathbb{S}),\\ (\boldsymbol{\sigma},\nabla\boldsymbol{v})&=(\boldsymbol{f},\boldsymbol{v}), && \forall\boldsymbol{v}\in H_0^1(\Omega;\mathbb{R}^d). \end{aligned}$$

**Formula 1.12.** 找 $\boldsymbol{\sigma}\in H^{-1}(\operatorname{curl}\operatorname{div},\Omega;\mathbb{S}), \boldsymbol{u}\in H_0(\operatorname{div},\Omega)$, 使得

$$\begin{aligned} (A\boldsymbol{\sigma},\boldsymbol{\tau})+\langle\operatorname{div}\boldsymbol{\tau},\boldsymbol{u}\rangle&=0, && \forall\boldsymbol{\tau}\in H^{-1}(\operatorname{curl}\operatorname{div},\Omega;\mathbb{S}),\\ -\langle\operatorname{div}\boldsymbol{\sigma},\boldsymbol{v}\rangle&=(\boldsymbol{f},\boldsymbol{v}), && \forall\boldsymbol{v}\in H_0(\operatorname{div},\Omega), \end{aligned}$$

其中 $H^{-1}(\operatorname{curl}\operatorname{div},\Omega;\mathbb{S}):=\{\boldsymbol{\tau}\in L^2(\Omega;\mathbb{S}):\operatorname{div}\boldsymbol{\tau}\in(H_0(\operatorname{div},\Omega))'\}$, $\langle\cdot,\cdot\rangle$ 为 $(H_0(\operatorname{div},\Omega))'$ 与 $H_0(\operatorname{div},\Omega)$ 的对偶对.

**Formula 1.13.** 找 $\boldsymbol{\sigma}\in H^{-1}(\operatorname{div}\operatorname{div},\Omega;\mathbb{S}), \boldsymbol{u}\in H_0(\operatorname{curl},\Omega)$, 使得

$$\begin{aligned} (A\boldsymbol{\sigma},\boldsymbol{\tau})+\langle\operatorname{div}\boldsymbol{\tau},\boldsymbol{u}\rangle&=0, && \forall\boldsymbol{\tau}\in H^{-1}(\operatorname{div}\operatorname{div},\Omega;\mathbb{S}),\\ -\langle\operatorname{div}\boldsymbol{\sigma},\boldsymbol{v}\rangle&=(\boldsymbol{f},\boldsymbol{v}), && \forall\boldsymbol{v}\in H_0(\operatorname{curl},\Omega), \end{aligned}$$

其中 $H^{-1}(\operatorname{div}\operatorname{div},\Omega;\mathbb{S}):=\{\boldsymbol{\tau}\in L^2(\Omega;\mathbb{S}):\operatorname{div}\boldsymbol{\tau}\in(H_0(\operatorname{curl},\Omega))'\}$, $\langle\cdot,\cdot\rangle$ 为 $(H_0(\operatorname{curl},\Omega))'$ 与 $H_0(\operatorname{curl},\Omega)$ 的对偶对.

## 双旋度方程

双旋度方程为

$$\operatorname{curl}^2 \boldsymbol{u} = \boldsymbol{f}, \qquad \text{in } \Omega \tag{1.15a}$$
$$\operatorname{div} \boldsymbol{u} = 0, \qquad \text{in } \Omega \tag{1.15b}$$
$$\boldsymbol{u} \times \boldsymbol{n} = 0, \qquad \text{on } \partial\Omega, \tag{1.15c}$$

约束(1.15b)可以理解为 $\boldsymbol{u}$ 与 $H_0^1(\Omega)$ 的函数在 $L^2$ 意义下正交: $(u, \nabla v) = 0, \forall v \in H_0^1(\Omega)$. 其变分问题为

**Formula 1.14.** 找 $\boldsymbol{u} \in H_0(\operatorname{curl}, \Omega), p \in H_0^1(\Omega)$, 使得

$$\begin{aligned} (\operatorname{curl} \boldsymbol{u}, \operatorname{curl} \boldsymbol{v}) + (\boldsymbol{v}, \nabla p) &= (\boldsymbol{f}, \boldsymbol{v}), && \forall \boldsymbol{v} \in H_0(\operatorname{curl}, \Omega), \\ (\boldsymbol{u}, \nabla q) &= 0, && \forall q \in H_0^1(\Omega). \end{aligned}$$

## 重调和方程

齐次 Dirchlet 边界的重调和方程为

$$\begin{aligned} \Delta^2 u &= f, && \text{in } \Omega, \\ u = \partial_n u &= 0, && \text{on } \partial\Omega. \end{aligned} \tag{1.16}$$

四阶算子 $\Delta^2 = \operatorname{div}\operatorname{div} \nabla\nabla$, 通过引入变量 $\boldsymbol{\sigma}$, 得

$$\begin{aligned} \boldsymbol{\sigma} &= \nabla^2 u, && \text{in } \Omega, \\ \operatorname{div}\operatorname{div} \boldsymbol{\sigma} &= f, && \text{in } \Omega, \\ u = \partial_n u &= 0, && \text{on } \partial\Omega. \end{aligned} \tag{1.17}$$

其对应的变分问题为:

**Formula 1.15.** 找 $\boldsymbol{\sigma} \in L^2(\Omega; \mathbb{S}), u \in H_0^2(\Omega)$, 使得

$$\begin{aligned} (\boldsymbol{\sigma}, \boldsymbol{\tau}) - (\nabla^2 u, \boldsymbol{\tau}) &= 0, && \forall \boldsymbol{\tau} \in L^2(\Omega; \mathbb{S}), \\ (\boldsymbol{\sigma}, \nabla^2 v) &= (f, v), && \forall v \in H_0^2(\Omega). \end{aligned} \tag{1.18}$$

**Formula 1.16.** 找 $\boldsymbol{\sigma} \in H(\operatorname{div}\operatorname{div}, \Omega; \mathbb{S}), u \in L^2(\Omega)$, 使得

$$\begin{aligned} (\boldsymbol{\sigma}, \boldsymbol{\tau}) - (\operatorname{div}\operatorname{div} \boldsymbol{\tau}, u) &= 0, && \forall \boldsymbol{\tau} \in H(\operatorname{div}\operatorname{div}, \Omega; \mathbb{S}), \\ (\operatorname{div}\operatorname{div} \boldsymbol{\sigma}, v) &= (f, v), && \forall v \in L^2(\Omega). \end{aligned} \tag{1.19}$$

**Formula 1.17.** 找 $\boldsymbol{\sigma} \in H(\operatorname{div}, \Omega; \mathbb{S}), u \in H_0^1(\Omega)$, 使得

$$\begin{aligned} (\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\nabla u, \operatorname{div} \boldsymbol{\tau}) &= 0, && \forall \boldsymbol{\tau} \in H(\operatorname{div}, \Omega; \mathbb{S}), \\ -(\operatorname{div} \boldsymbol{\sigma}, \nabla v) &= (f, v), && \forall v \in H_0^1(\Omega). \end{aligned} \tag{1.20}$$

变分问题(1.18)称为双调和方程的 Primal 形式; (1.19)为对偶形式; (1.20)对前两者间空间的光滑性做了平衡, 但该变分问题适定性会出现问题, 解决的方法是将内积使用 $\langle \operatorname{div} \boldsymbol{\sigma}, \nabla v \rangle_{(H_0(\operatorname{curl}))' \times H_0(\operatorname{curl})}$ 代替. 另一种分部式变分问题为

**Formula 1.18.** 找 $\boldsymbol{\sigma} \in H^{-1}(\operatorname{div}\operatorname{div}, \Omega; \mathbb{S}), u \in H_0^1(\Omega)$, 使得

$$\begin{aligned} (\boldsymbol{\sigma}, \boldsymbol{\tau}) - \langle \operatorname{div}\operatorname{div} \boldsymbol{\tau}, u\rangle &= 0, && \forall \boldsymbol{\tau} \in H^{-1}(\operatorname{div}\operatorname{div}, \Omega; \mathbb{S}), \\ \langle \operatorname{div}\operatorname{div} \boldsymbol{\sigma}, v\rangle &= (f, v), && \forall v \in H_0^1(\Omega). \end{aligned}$$

双调和方程(1.16)也可以改写为

$$\begin{aligned} \sigma &= -\Delta u, && \text{in } \Omega, \\ -\Delta \sigma &= f, && \text{in } \Omega \\ u = \partial_n u &= 0, && \text{on } \partial\Omega \end{aligned} \tag{1.21}$$

该格式对应的变分问题(1.22)在 $\Omega$ 为凸区域时与原形式等价.

**Formula 1.19.** 找 $\sigma \in H^1(\Omega), u \in H_0^1(\Omega)$, 使得

$$\begin{aligned} (\sigma, \tau) - (\nabla \tau, \nabla u) &= 0, && \forall \tau \in H^1(\Omega) \\ (\nabla \sigma, \nabla v) &= (f, v), && \forall v \in H_0^1(\Omega). \end{aligned} \tag{1.22}$$

这种格式在适定性上存在一些问题，但可以理解为：

**Formula 1.20.** 找 $\sigma \in H^{-1}(\Delta, \Omega), u \in H_0^1(\Omega)$, 使得

$$\begin{aligned} (\sigma, \tau) + \langle \Delta \tau, u\rangle &= 0, && \forall \sigma \in H^{-1}(\Delta, \Omega) \\ -\langle \Delta \sigma, v\rangle &= (f, v), && \forall v \in H_0^1(\Omega). \end{aligned}$$

另一方面，我们也可以将原方程重写为一阶系统：

$$\begin{aligned} \boldsymbol{\phi} &= \nabla u, && \text{in } \Omega, \\ \boldsymbol{\sigma} &= \nabla \boldsymbol{\phi}, && \text{in } \Omega, \\ \boldsymbol{q} &= \operatorname{div} \boldsymbol{\sigma}, && \text{in } \Omega, \\ \operatorname{div} \boldsymbol{q} &= f && \text{in } \Omega, \\ u = \boldsymbol{\phi} \cdot \boldsymbol{n} &= 0, && \text{on } \partial\Omega, \end{aligned} \tag{1.23}$$

对应的混合变分问题为

**Formula 1.21.** 找 $\boldsymbol{\phi} \in L^2(\Omega; \mathbb{R}^d), \boldsymbol{\sigma} \in H(\operatorname{div}, \Omega; \mathbb{M}), \boldsymbol{q} \in H(\operatorname{div}, \Omega), u \in L^2(\Omega)$, 使得

$$\begin{aligned} (\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\operatorname{div} \boldsymbol{\tau}, \boldsymbol{\phi}) - (\boldsymbol{\phi}, \boldsymbol{p}) - (\operatorname{div} \boldsymbol{p}, u) &= 0 && \forall \boldsymbol{\tau} \in H(\operatorname{div}, \Omega; \mathbb{M}), \boldsymbol{p} \in H(\operatorname{div}, \Omega), \\ (\operatorname{div} \boldsymbol{\sigma}, \boldsymbol{\psi}) - (\boldsymbol{q}, \boldsymbol{\psi}) - (\operatorname{div} \boldsymbol{q}, v) &= -(f, v) && \forall \boldsymbol{\psi} \in L^2(\Omega; \mathbb{R}^d), v \in L^2(\Omega). \end{aligned} \tag{1.24}$$

## $m$ 重调和方程

一般地, 对 $m$ 重调和方程

$$(-\Delta)^m u = f,$$

通过引入 $\boldsymbol{\sigma}$ 可以改写为

$$\begin{aligned} \boldsymbol{\sigma} &= \nabla^m u, \\ (-\operatorname{div})^m \boldsymbol{\sigma} &= f. \end{aligned}$$

对应的混合变分问题为

**Formula 1.22.** 找 $\boldsymbol{\sigma} \in H(\operatorname{div}^m, \Omega; \mathbb{S}), u \in L^2(\Omega)$, 使得

$$\begin{aligned}(\boldsymbol{\sigma}, \boldsymbol{\tau}) - ((-\operatorname{div})^m \boldsymbol{\tau}, u) &= 0, \qquad \forall \boldsymbol{\tau} \in H(\operatorname{div}^m, \Omega; \mathbb{S}),\\ ((-\operatorname{div})^m \boldsymbol{\sigma}, v) &= (f, v), \quad \forall v \in L^2(\Omega).\end{aligned}$$

## 四阶奇异摄动方程

奇异摄动问题为

$$\begin{aligned}\varepsilon^2 \Delta^2 u - \Delta u &= f, \quad \text{in } \Omega,\\ u = \partial_n u &= 0, \quad \text{on } \partial\Omega.\end{aligned}$$

当 $\varepsilon$ 趋向 0 时, 该问题趋于 Possion 方程, 此时边界条件有多余, 会导致边界层现象. 引入 $\boldsymbol{\sigma}$, 上述问题可以改写为

$$\begin{aligned}\boldsymbol{\sigma} &= \varepsilon^2 \nabla^2 u, && \text{in } \Omega,\\ \operatorname{div}\operatorname{div} \boldsymbol{\sigma} - \Delta u &= f, && \text{in } \Omega,\\ u = \partial_n u &= 0, && \text{on } \partial\Omega.\end{aligned}$$

对应的混合变分问题为(1.25), 该方法可以避免边界层现象.

**Formula 1.23.** 找 $\boldsymbol{\sigma} \in H^{-1}(\operatorname{div}\operatorname{div}, \Omega; \mathbb{S}), u \in H_0^1(\Omega)$, 使得

$$\begin{aligned}\varepsilon^{-2}(\boldsymbol{\sigma}, \boldsymbol{\tau}) - \langle \operatorname{div}\operatorname{div} \boldsymbol{\tau}, u\rangle &= 0, && \forall \tau \in H^{-1}(\operatorname{div}\operatorname{div}, \Omega, \mathbb{S})\\ \langle \operatorname{div}\operatorname{div} \boldsymbol{\sigma}, v\rangle + (\nabla u, \nabla v) &= (f, v), && \forall v \in H_0^1(\Omega).\end{aligned} \tag{1.25}$$

## 应变梯度问题

应变梯度问题为

$$\begin{aligned}\operatorname{div}(\iota^2 \Delta - I)\boldsymbol{\sigma}(\boldsymbol{u}) &= \boldsymbol{f},\\ \boldsymbol{\sigma}(\boldsymbol{u}) &= 2\mu\varepsilon(\boldsymbol{u}) + \lambda \operatorname{div} \boldsymbol{u} I.\end{aligned}$$

引入新变量 $p$, 上述方程改写为

$$\begin{aligned}p &= \lambda \operatorname{div} \boldsymbol{u},\\ \operatorname{div}(\iota^2 \Delta - I)(2\mu\varepsilon(\boldsymbol{u}) + pI) &= \boldsymbol{f}.\end{aligned} \tag{1.26}$$

问题(1.26)对应的混合问题为

**Formula 1.24.** 找 $\boldsymbol{u} \in H_0^2(\Omega; \mathbb{R}^d), p \in H_0^1(\Omega) \cap L_0^2(\Omega)$, 使得

$$\begin{aligned}2\mu(\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v}))_\iota + (\operatorname{div} \boldsymbol{v}, p)_\iota &= (\boldsymbol{f}, \boldsymbol{v}), && \forall \boldsymbol{v} \in H_0^2(\Omega; \mathbb{R}^d),\\ (\operatorname{div} \boldsymbol{u}, q)_\iota - \tfrac{1}{\lambda}(p, q)_\iota &= 0, && \forall q \in H_0^1(\Omega) \cap L_0^2(\Omega).\end{aligned}$$

其中内积 $(p, q)_\iota := \iota^2(\nabla p, \nabla q) + (p, q)$.

或者，类似于线弹性方程的做法:

$$\iota^{-2} \bar{A}\bar{\boldsymbol{\sigma}} = \nabla\varepsilon(\boldsymbol{u}), \tag{1.27a}$$

$$\operatorname{div}\operatorname{div} \bar{\boldsymbol{\sigma}} - \operatorname{div} \varepsilon(\boldsymbol{u}) = f. \tag{1.27b}$$

得到混合格式:

$$(\iota^{-2} \bar{A}\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\tau}}) - \langle \operatorname{div}\operatorname{div} \bar{\boldsymbol{\tau}}, \boldsymbol{u}\rangle = 0 \quad \bar{\boldsymbol{\tau}} \in H^{-1}(\operatorname{div}\operatorname{div}, \Omega; \mathbb{S} \otimes \mathbb{R}^d), \tag{1.28a}$$

$$\langle \operatorname{div}\operatorname{div} \bar{\boldsymbol{\sigma}}, \boldsymbol{v}\rangle + (\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v})) = (f, \boldsymbol{v}) \quad \boldsymbol{u} \in H_0^1(\Omega; \mathbb{R}^d). \tag{1.28b}$$

## 1.2 Babuška 理论

算子方程 $Au = f$ 的适定性包括存在性、唯一性和稳定性三个方面.

### 1.2.1 抽象变分问题

给定变分问题：找 $u \in U$ 使得

$$a(u, v) = \langle f, v \rangle \quad \forall\, v \in V. \tag{1.29}$$

这里, $a(\cdot,\cdot) : U \times V \to \mathbb{R}$ 是 Banach 空间 $U$ 和 $V$ 上的双线性形式, 即关于每个变量都是线性的. 基于双线性形式 $a(\cdot,\cdot)$, 如下定义两个线性算子 $A : U \to V'$ 和 $A' : V \to U'$:

$$\langle Au, v \rangle = \langle u, A'v \rangle = a(u, v).$$

变分问题 (1.29) 等价于如下算子方程: 给定 $f \in V'$, 找 $u \in U$, 使得在对偶空间 $V'$ 里满足

$$Au = f. \tag{1.30}$$

假定双线性形式 $a(\cdot,\cdot)$ 是连续的, 即

$$a(u, v) \le C\|u\|_U\|v\|_V \quad \forall\, u \in U, v \in V. \tag{1.31}$$

将使得上述不等式成立的最小的常数记作 $\|a\|$, 即

$$\|a\| = \sup_{u\in U, v\in V} \frac{a(u,v)}{\|u\|_U\|v\|_V}.$$

在连续性条件 (1.31) 下, 易知 $A$ 和 $A'$ 都是有界算子, 且 $\|A\| = \|A'\| = \|a\|$. 为了讨论算子方程 (1.30) 的存在性和唯一性, 引入以下两个 inf-sup 条件:

$$\inf_{v\in V}\sup_{u\in U} \frac{a(u,v)}{\|u\|_U\|v\|_V} = \alpha_1 > 0, \tag{1.32}$$

$$\inf_{u\in U}\sup_{v\in V} \frac{a(u,v)}{\|u\|_U\|v\|_V} = \alpha_2 > 0. \tag{1.33}$$

**定理 1.25** ([?]). 假设双线性形式 $a(\cdot,\cdot)$ 是连续的, 即 (1.31) 成立. 算子方程 (1.30) 是适定的, 当且仅当 (1.32) 和 (1.33) 成立. 在 (1.32) 和 (1.33) 成立的前提下, 有

$$\|A^{-1}\| = \|(A')^{-1}\| = \alpha_1^{-1} = \alpha_2^{-1} = \alpha^{-1},$$

故算子方程 (1.30) 的解满足

$$\|u\|_U \le \frac{1}{\alpha}\|f\|_{V'}.$$

Proof. Inf-sup 条件 (1.32) 的算子形式为

$$\|A'v\|_{U'} \ge \alpha_1\|v\|_V \quad \forall\, v \in V,$$

这等价于算子 $A$ 是满射. 类似地, inf-sup 条件 (1.33) 的算子形式为

$$\|Au\|_{V'} \ge \alpha_2\|u\|_U \quad \forall\, u \in U,$$

这等价于算子 $A$ 是单射. 因此, inf-sup 条件 (1.32) 和 (1.33) 等价于 $A : U \to V$ 是同构算子. 进一步, 由开映射定理知, 算子 $A^{-1}$ 是有界的, 且易知 $A^{-1}$ 的范数等于 $\alpha_2^{-1}$. 关于 $A'$ 的结论类似可证. □

下面以 inf-sup 条件 (1.32) 为例来说明如何验证 inf-sup 条件.

**定理 1.26.** Inf-sup 条件 (1.32) 等价于：对任意的 $v \in V$ 存在 $u \in U$, 使得

$$a(u,v) \geq C_1 \|v\|_V^2, \qquad \|u\|_U \leq C_2 \|v\|_V. \tag{1.34}$$

Proof. 显然 (1.34) 意味着 (1.32), 其中 $\alpha_1 \geq C_1/C_2$. 现在来证明 (1.32) 意味着 (1.34). 对任意的 $v \in V$, 由 Hahn-Banach 定理的推论知, 存在 $f \in V'$ 满足 $f(v) = \|v\|_V^2$, 以及 $\|f\|_{V'} = \|v\|_V$. 由于 $A$ 是满射, 由开映射定理知, 总可以找到 $\tilde{u} \in U$ 满足 $A\tilde{u} = f$, 以及 $\inf\limits_{w \in \ker(A)} \|\tilde{u} + w\|_U \leq \alpha_E^{-1} \|f\|_{V'} = \alpha_E^{-1} \|v\|_V$. 于是, 存在 $u \in U$ 和 $C \geq 1$ 满足

$$\|u\|_U \leq C\alpha_E^{-1} \|v\|_V, \quad Au = f.$$

因此, $a(u,v) = \langle Au, v \rangle = f(v) = \|v\|_V^2$. 如果 Banach 空间 $U$ 是自反的, 还可以要求 $C = 1$. □

这样要验证 inf-sup 条件 (1.32) 只需构造一个合适的函数.

由定理 1.26 可知, inf-sup 条件 (1.32) 也等价于

$$\text{对任意的 } v \in V, \text{ 总存在非零 } u \in U, \text{ s.t. } a(u,v) \geq \alpha \|u\|_U \|v\|_V.$$

**注 1.27** ([61]). 设 $K$ 是 (实)Banach 空间 $X$ 的一个非空子集. 对任意的 $x \in X$, 称 $y \in K$ 是 $x$ 在 $K$ 上的**最佳逼近**, 如果

$$\|x - y\| = \inf\{\|x - z\| : z \in K\}.$$

如果任何一点 $x \in X$ 在 $K$ 上都有 (唯一) 最佳逼近, 则称集合 $K$ 是**可逼近的 (Chebyshev 集)**. 易知, 自反空间 $X$ 中的闭凸集合 $K$ 都是可逼近的. 进一步, 如果范数是严格凸的, 则 $K$ 是 Chebyshev 集. 但是, 如果 $X$ 不是自反的或 $K$ 不是凸的, 那么以上结论通常不成立.

满足 (1.34) 的 $u$ 依赖于 $v$. 一种特殊情形是当 $U = V$ 时 $u = v$, 相应的结果就是 Lax-Milgram 引理.

**引理 1.28** (Lax-Milgram 引理, [?]). 设 $V \times V$ 上的双线性形式 $a(\cdot,\cdot)$ 满足

1. 连续性: $a(u,v) \leq \beta \|u\|_V \|v\|_V$;

2. 强制性: $a(u,u) \geq \alpha \|u\|_V^2$,

则对任意的 $f \in V'$, 存在唯一的 $u \in V$ 满足

$$a(u,v) = \langle f, v \rangle,$$

且

$$\|u\|_V \leq \frac{1}{\alpha} \|f\|_{V'}.$$

最简单的情形是双线性形式 $a(\cdot,\cdot)$ 在 $V$ 上对称正定的. 此时, $a(\cdot,\cdot)$ 定义了一个新的内积, Lax-Milgram 引理就是 Riesz 表示定理.

### 1.2.2 变分问题的离散：Ritz 方法

假设 $a(\cdot,\cdot)$ 是对称正定的, 且 $U=V$. 抽象 Galerkin 变分问题 (1.29) 的 Ritz 变分问题为：找 $u\in U$, 使得

$$J(u)=\min_{w\in U} J(w), \tag{1.35}$$

其中 $J(u)=\dfrac{1}{2}a(u,u)-\langle f,u\rangle$.

因为 $U$ 通常是无穷维的函数空间, 直接求解 Ritz 变分问题 (1.35) 非常困难, 为此需要近似求解 (1.35). 设有限维子空间 $U_h\subset U$ 是 $U$ 的协调离散, 称 $U_h$ 为**试探函数空间**. Ritz 变分问题 (1.35) 的近似问题是：找 $u_h\in U_h$, 使得

$$J(u_h)=\min_{w_h\in U_h} J(w_h). \tag{1.36}$$

离散问题 (1.36) 称为 **Ritz 方法**.

设 $U_h$ 的一组基函数为 $\varphi_1,\dots,\varphi_N$, 即 $U_h=\mathrm{span}\{\varphi_1,\dots,\varphi_N\}$. 对任意的 $w_h\in U_h$, 有线性表示

$$w_h=\sum_{i=1}^N c_i\varphi_i,$$

其中系数 $c_1,\dots,c_n$ 是常数. 于是

$$J(w_h)=\frac{1}{2}\sum_{i,j=1}^N a(\varphi_i,\varphi_j)c_ic_j-\sum_{i=1}^N\langle f,\varphi_i\rangle c_i.$$

令 $A=(a_{ij})_{N\times N}$, $b=(b_i)_{N\times 1}$, $c=(c_i)_{N\times 1}$, 其中 $a_{ij}=a(\varphi_i,\varphi_j)$, $b_i=\langle f,\varphi_i\rangle$, 则

$$J(w_h)=\frac{1}{2}c^{\mathsf T}Ac-c^{\mathsf T}b.$$

由此, Ritz 方法 (1.36) 等价于以 $c$ 为自变量的二次函数的极值问题

$$\min_{c\in\mathbb{R}^N}\frac{1}{2}c^{\mathsf T}Ac-c^{\mathsf T}b.$$

设 $u_h=(\varphi_1,\dots,\varphi_N)c^*$, 其中 $c^*\in\mathbb{R}^N$. 于是向量 $c^*$ 满足线性代数方程组

$$Ac^*=b.$$

### 1.2.3 变分问题的离散：Galerkin 方法

考虑变分问题 (1.29) 在有限维子空间 $U_h\subset U$ 和 $V_h\subset V$ 上的协调离散, 称 $V_h$ 为**检验函数空间**. 变分问题 (1.29) 的离散方法为：找 $u_h\in U_h$ 使得

$$a(u_h,v_h)=\langle f,v_h\rangle\quad\forall\, v_h\in V_h. \tag{1.37}$$

若 $U_h\neq V_h$, 称离散方法 (1.37) 为 **Petrov-Galerkin 方法**；若 $U_h=V_h$, 称离散方法 (1.37) 为 **Galerkin 方法**. 离散问题 (1.37) 的存在唯一性等价于以下离散 inf-sup 条件:

$$\inf_{v_h\in V_h}\sup_{u_h\in U_h}\frac{a(u_h,v_h)}{\|u_h\|_U\|v_h\|_V}=\inf_{u_h\in U_h}\sup_{v_h\in V_h}\frac{a(u_h,v_h)}{\|u_h\|_U\|v_h\|_V}=\alpha_h>0. \tag{1.38}$$

选取有限维空间 $U_h$ 和 $V_h$ 的基函数, 设 $U_h = \text{span}\{\varphi_1, \dots, \varphi_N\}$, $V_h = \text{span}\{\psi_1, \dots, \psi_N\}$. 设 $u_h = \sum_{j=1}^{N} c_j^* \varphi_j$, 其中 $c^* = (c_1^*, \dots, c_N^*)^\intercal \in \mathbb{R}^N$. 代入离散方法 (1.37), 并取 $v_h = \psi_i$ $(i = 1, \dots, N)$, 得

$$\sum_{j=1}^{N} a(\varphi_j, \psi_i) c_j^* = \langle f, \psi_i \rangle, \quad i = 1, \dots, N.$$

令 $A = (a_{ij})_{N\times N}$, $b = (b_i)_{N\times 1}$, 其中 $a_{ij} = a(\varphi_j, \psi_i)$, $b_i = \langle f, \psi_i \rangle$. 由此, 离散方法 (1.37) 转化成线性代数方程组

$$Ac^* = b.$$

考虑到适定性, 要求系数矩阵 $A$ 是可逆的. 为确保一致稳定性, 要求常数 $\alpha_h$ 关于 $h$ 一致有下界, 且下界为正.

在双线性形式 $a(\cdot, \cdot)$ 是对称正定及 $U_h = V_h$ 的情形下, Ritz 方法 (1.36) 和 Galerkin 方法 (1.37) 是等价的. 如果双线性形式 $a(\cdot, \cdot)$ 不是对称的, 离散方法 (1.37) 仍可以用.

利用 inf-sup 条件可以建立离散方法 (1.37) 的误差估计. 协调离散的关键性质是 Garlerkin 正交性

$$a(u - u_h, v_h) = 0 \quad \forall\, v_h \in V_h.$$

引入投影算子 $P_h : U \to U_h$,

$$a(P_h u, v_h) = a(u, u_h) \qquad \forall\, u \in U, v_h \in V_h.$$

显然, $u_h = P_h u$, 并且 $P_h$ 限制在 $U_h$ 上是恒等算子.

**引理 1.29.** 成立

$$A_h P_h u = I' A u \quad \forall\, u \in U,$$

其中 $I' : V' \to V_h'$ 是嵌入算子 $I : V_h \to V$ 的对偶, 算子 $A_h : U_h \to V_h'$ 定义为

$$\langle A_h u_h, v_h \rangle = a(u_h, v_h) \quad \forall\, u_h \in U_h, v_h \in V_h.$$

Proof. 由 $P_h$ 和 $I$ 的定义, 有

$$a(P_h u, v_h) = a(u, v_h) = a(u, I v_h) = \langle Au, I v_h \rangle_{V' \times V} = \langle I' A u, v_h \rangle_{V_h' \times V_h}.$$

结合 $A_h$ 的定义, 即得证. □

**引理 1.30.** 投影算子 $P_h : U \to U_h$ 是有界的:

$$\|P_h u\|_U \le \frac{\|a\|}{\alpha_h} \|u\|_U \quad \forall\, u \in U. \tag{1.39}$$

Proof. 由离散 inf-sup 条件 (1.38) 和双线性形式 $a(\cdot, \cdot)$ 的有界性 (1.31), 有

$$\|P_h u\|_U \le \frac{1}{\alpha_h} \sup_{v_h \in V_h} \frac{a(P_h u, v_h)}{\|v_h\|_V} = \frac{1}{\alpha_h} \sup_{v_h \in V_h} \frac{a(u, v_h)}{\|v_h\|_V} \le \frac{1}{\alpha_h} \sup_{v \in V} \frac{a(u, v)}{\|v\|_V} \le \frac{\|a\|}{\alpha_h} \|u\|_U.$$

证毕. □

**引理 1.31.** 设 $U_h$ 是 $U$ 的非零子空间. 成立等式

$$\|I - P_h\| = \|P_h\|. \tag{1.40}$$

Proof. 由假设 $U_h$ 是 $U$ 的非零子空间可知, $P_h \neq 0$ 且 $P_h \neq I$. 又 $P_h^2 = P_h$, 应用文献 [77] 中的引理 5 可得等式 (1.40). □

**定理 1.32** (Céa 引理). *设双线性形式 $a(\cdot,\cdot)$ 满足 (1.31), (1.32), (1.33) 和 (1.38). 变分问题 (1.29) 存在唯一解 $u \in U$, 离散问题 (1.37) 存在唯一解 $u_h \in U_h$, 并成立*

$$\|u - u_h\|_U \leq \frac{\|a\|}{\alpha_h} \inf_{v_h \in U_h} \|u - v_h\|_U.$$

Proof. 在定理假设下, 对于 $f \in V'$, 变分问题 (1.29) 和离散问题 (1.37) 都是适定的. 对任意的 $w_h \in U_h$, 注意到 $P_h w_h = w_h$, 有

$$\|u - u_h\|_U = \|(I - P_h)(u - w_h)\|_U \leq \|I - P_h\| \|u - w_h\|_U.$$

应用等式 (1.40) 和 (1.39), 即可完成证明. □

## 1.3　鞍点系统：Brezzi 理论

### 1.3.1　混合变分问题

首先考虑一个抽象混合变分问题。设 $V$ 和 $P$ 为两个 Banach 空间。对于给定的 $(f, g) \in V' \times P'$，求 $(u, p) \in V \times P$ 使得

$$a(u, v) + b(v, p) = \langle f, v \rangle \quad \forall\, v \in V, \tag{1.41a}$$

$$b(u, q) = \langle g, q \rangle \quad \forall\, q \in P. \tag{1.41b}$$

现引入线性算子

$$A : V \to V', \qquad \text{定义为 } \langle Au, v \rangle = a(u, v)$$

与

$$B : V \to P', \quad B' : P \to V', \qquad \text{满足 } \langle Bv, q \rangle = \langle v, B'q \rangle = b(v, q).$$

抽象混合变分问题 (1.41) 对应的算子方程为

$$Au + B'p = f, \tag{1.42}$$

$$Bu = g, \tag{1.43}$$

或简记为

$$\begin{pmatrix} A & B' \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}.$$

### 1.3.2　Inf-sup 条件

我们将研究该抽象混合变分问题的适定性。

首先假设所有双线性形式均连续，即算子 $A, B, B'$ 连续：双线性形式 $a(\cdot,\cdot)$ 与 $b(\cdot,\cdot)$ 满足连续性条件

$$a(u, v) \leq C\|u\|_V \|v\|_V \quad \forall\, u, v \in V, \tag{1.44}$$

$$b(v, q) \leq C\|v\|_V \|q\|_P \quad \forall\, v \in V, q \in P. \tag{1.45}$$

第二个方程 (1.43) 的可解性等价于算子 $B$ 为满射（或算子 $B'$ 为单射且值域 $R(B')$ 闭），这又等价于如下 inf-sup 条件

$$\inf_{q\in P}\sup_{v\in V}\frac{b(v,q)}{\|v\|_V\|q\|_P}=\beta>0. \tag{1.46}$$

在条件 (1.46) 下，$B:V/\ker(B)\to P'$ 为同构映射。因此对任意 $g\in P'$，存在 $u_1\in V/\ker(B)$ 使得 $Bu_1=g$ 且 $\|u_1\|_V\le\beta^{-1}\|g\|_{P'}$。

确定唯一的 $u_1$ 后，我们将测试函数 $v$ 限制在 $\ker(B)$ 上考虑 (1.41a)。由于对 $v\in\ker(B)$ 有 $\langle v,B'q\rangle=\langle Bv,q\rangle=0$，得到如下变分问题：求 $u_0\in\ker(B)$ 使得

$$a(u_0,v)=\langle f,v\rangle-a(u_1,v)\quad\forall\, v\in\ker(B). \tag{1.47}$$

变分问题 (1.47) 的适定性等价于双线性形式 $a(u,v)$ 在空间 $Z=\ker(B)$ 上满足两个 inf-sup 条件：

$$\inf_{u\in Z}\sup_{v\in Z}\frac{a(u,v)}{\|u\|_V\|v\|_V}=\inf_{v\in Z}\sup_{u\in Z}\frac{a(u,v)}{\|u\|_V\|v\|_V}=\alpha>0. \tag{1.48}$$

如此确定唯一的 $u=u_0+u_1$ 后，通过求解

$$B'p=f-Au$$

得到 $p$。由于 $u_0$ 是 (1.47) 的解，右端项 $f-Au\in\ker(B)^\perp$。因此我们要求 $B':P\to\ker(B)^\perp$ 为同构映射，这也等价于条件 (1.46)。

**定理 1.33.** 设双线性形式 $a(\cdot,\cdot)$ 和 $b(\cdot,\cdot)$ 连续（即 (1.44)-(1.45) 成立），则混合变分问题 (1.41a)-(1.41b) 适定的充要条件是 (1.46) 与 (1.48) 同时成立。当这两个条件满足时，我们有稳定性估计

$$\|u\|_V+\|p\|_P\lesssim\|f\|_{V'}+\|g\|_{P'}.$$

以下关于算子 $B$ 的 inf-sup 条件的刻画非常实用。其验证过程可转化为构造一个合适的函数。该证明与定理 1.26 的证明类似，故在此从略。

**定理 1.34.** Inf-sup 条件 (1.46) 等价于：对任意 $q\in P$，存在 $v\in V$ 使得

$$b(v,q)\ge C_1\|q\|_P^2,\quad \text{且}\quad \|v\|_V\le C_2\|q\|_P.$$

需要指出的是，一般而言，构造符合要求的 $v=v(q)$（尤其是对其范数 $\|v\|_V$ 的控制）可能并非易事。

### 1.3.3 协调离散

我们考虑混合问题的有限元逼近：求 $u_h\in V_h$ 和 $p_h\in P_h$ 使得

$$a(u_h,v_h)+b(v_h,p_h)=\langle f,v_h\rangle\quad\forall\, v_h\in V_h, \tag{1.49}$$

$$b(u_h,q_h)=\langle g,q_h\rangle\quad\forall\, q_h\in P_h. \tag{1.50}$$

考虑协调有限元情形 $V_h\subset V$ 和 $P_h\subset P$。记 $B_h:V_h\to P_h'$，并令 $Z_h=\ker(B_h)$。回顾 $Z=\ker(B)$，在应用于 Stokes 方程时 $B=-\operatorname{div}$，因此 $Z$ 称为无散空间，$Z_h$ 称为离散无散空间。

**注 1.35.** 一般而言 $Z_h \not\subset Z$，即离散无散函数未必严格无散。只需比较 $B_h u_h = 0$（在 $(P_h)'$ 中）

$$\langle B_h u_h, q_h \rangle = 0 \quad \forall\, q_h \in P_h,$$

与 $Bu_h = 0$（在 $P'$ 中）

$$\langle Bu_h, q \rangle = 0 \quad \forall\, q \in P.$$

有限元逼近的离散 inf-sup 条件为:

$$\inf_{u_h \in Z_h} \sup_{v_h \in Z_h} \frac{a(u_h, v_h)}{\|u_h\|_V \|v_h\|_V} = \inf_{v_h \in Z_h} \sup_{u_h \in Z_h} \frac{a(u_h, v_h)}{\|u_h\|_V \|v_h\|_V} = \alpha_h > 0, \tag{1.51}$$

$$\inf_{q_h \in P_h} \sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|v_h\|_V \|q_h\|_P} = \beta_h > 0. \tag{1.52}$$

**定理 1.36.** 若 (1.44)-(1.45)、(1.46)、(1.48)、(1.51) 及 (1.52) 成立，则离散问题适定，且成立误差估计

$$\|u - u_h\|_V + \|p - p_h\|_P \le C \inf_{v_h \in V_h, q_h \in P_h} (\|u - v_h\|_V + \|p - q_h\|_P).$$

**Exercise 1.1.** 设 $U = V \times P$，并用定义在 $U$ 上的单个双线性形式重写混合变分形式。随后应用 Babuška 理论证明上述定理，并明确写出常数 $C$ 如何依赖于各 inf-sup 条件中的常数。

### 1.3.4 Fortin 算子

注意到连续层面的 inf-sup 条件 (1.46) 蕴含：对任意 $q_h \in P_h$，存在 $v \in V$ 使得

$$b(v, q_h) \ge \beta \|v\|_V \|q_h\|_P, \quad \|v\|_V \le C\|q_h\|_P.$$

为建立离散 inf-sup 条件，我们需要构造满足相应性质的 $v_h \in V_h$。一种方法是利用所谓的 Fortin 算子 [45]，通过 $v$ 构造出所需的 $v_h$。

**Definition 1.37** (Fortin 算子). 线性算子 $\Pi_h : V \to V_h$ 称为 Fortin 算子，若其满足:

1. 对任意 $q_h \in P_h$，$b(\Pi_h v, q_h) = b(v, q_h)$;
2. $|\Pi_h v|_V \le C|v|_V$。

**定理 1.38.** 若 inf-sup 条件 (1.46) 成立且存在 Fortin 算子 $\Pi_h$，则离散 inf-sup 条件 (1.52) 成立。

*Proof.* 由 inf-sup 条件 (1.46) 可得

$$\beta \le \inf_{q_h \in P_h} \sup_{v \in V} \frac{b(v, q_h)}{\|v\|_V \|q_h\|_P} \le C \inf_{q_h \in P_h} \sup_{v \in V} \frac{b(\Pi_h v, q_h)}{\|\Pi_h v\|_V \|q_h\|_P} \le C \inf_{q_h \in P_h} \sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|v_h\|_V \|q_h\|_P}$$

因此离散 inf-sup 条件 (1.52) 成立，且可取 $\beta_h = \beta / C$。 □

## 1.4 鞍点系统：Zulehner 理论

We refer to [79] for details in this section.

We shall consider an abstract mixed variational problem first. Let $V$ and $P$ be two Hilbert spaces. For given $(f, g) \in V' \times P'$, find $(u, p) \in V \times P$ such that

$$a(u, v) + b(v, p) = f(v) \quad \forall\, v \in V, \tag{1.53}$$

$$b(u, q) - c(p, q) = g(q) \quad \forall\, q \in P, \tag{1.54}$$

where $a$, $b$, and $c$ are bounded bilinear forms on $V \times V$ , $V \times P$, and $P \times P$, respectively. Assume in this section that $a$ and $c$ are symmetric, i.e.,

$$a(w, v) = a(v, w) \quad \forall\, w, v \in V, \quad c(p, q) = c(q, p) \quad \forall\, p, q \in P,$$

and $a$ and $c$ are nonnegative, i.e.,

$$a(v, v) \geq 0 \quad \forall\, v \in V, \quad c(q, q) \geq 0 \quad \forall\, q \in P.$$

Thus (1.53)-(1.54) is a symmetric and indefinite problem.

The mixed variational problem (1.53)-(1.54) in $V$ and $P$ can also be written as a variational problem on the product space $X = V \times P$. Find $x = (u, p) \in X$ such that

$$\mathcal{B}(x, y) = \mathcal{F}(y) \quad \forall\, y \in X \tag{1.55}$$

with

$$\mathcal{B}(x, y) = a(u, v) + b(v, p) + b(u, q) - c(p, q), \quad \mathcal{F}(y) = f(v) + g(q)$$

for $y = (v, q)$. Since we have assumed that the bilinear forms $a$, $b$, and $c$ are bounded, there is a constant $\overline{C}$ such that

$$\sup_{y \in X} \sup_{x \in X} \frac{\mathcal{B}(x, y)}{\|x\|_X \|y\|_X} \leq \overline{C}. \tag{1.56}$$

By the Babuška theory, problem (1.55) is well-posed if and only if there is a constant $\underline{C} > 0$ such that

$$\inf_{y \in X} \sup_{x \in X} \frac{\mathcal{B}(x, y)}{\|x\|_X \|y\|_X} \geq \underline{C}. \tag{1.57}$$

Then the following estimate holds

$$\frac{1}{\overline{C}} \|\mathcal{F}\|_{X'} \leq \|x\|_X \leq \frac{1}{\underline{C}} \|\mathcal{F}\|_{X'}.$$

Let us introduce linear operators

$$A : V \to V', \quad \text{as } \langle Au, v\rangle = a(u, v),$$

$$B : V \to P', \quad B' : P \to V', \quad \text{as } \langle Bv, q\rangle = \langle v, B'q\rangle = b(v, q).$$

and

$$C : P \to P', \quad \text{as } \langle Cp, q\rangle = c(p, q).$$

Written in the operator form, the problem becomes

$$Au + B'p = f, \tag{1.58}$$

$$Bu - Cp = g, \tag{1.59}$$

or in short

$$\begin{pmatrix} A & B' \\ B & -C \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}.$$

In a similar way, we associate a linear operator $\mathcal{A} \in L(X, X')$ to the bilinear form $\mathcal{B}$, given by

$$\langle \mathcal{A}x, y \rangle = \mathcal{B}(x, y).$$

Then problem (1.55), which is equivalent to (1.53)-(1.54), reads

$$\mathcal{A}x = \mathcal{F}. \tag{1.60}$$

In operator notation the conditions (1.56) and (1.57) can be written in the following form

$$\underline{C}\|y\|_X \le \|\mathcal{A}y\|_{X'} \le \overline{C}\|y\|_X \quad \forall\, y \in X. \tag{1.61}$$

**引理 1.39.** Let $f \in V'$ and $g \in P'$. Let $\mathcal{F} \in X'$ be given by $\mathcal{F}(v, q) = f(v) + g(q)$. Then

$$\|\mathcal{F}\|_{X'}^2 = \|f\|_{V'}^2 + \|g\|_{P'}^2.$$

Proof. It holds

$$\begin{aligned}
\|\mathcal{F}\|_{X'}^2 &= \sup_{(v,q)\in X} \frac{(f(v)+g(q))^2}{\|(v,q)\|_X^2} = \sup_{(v,q)\in X} \frac{(f(v)+g(q))^2}{\|v\|_V^2 + \|q\|_P^2} \\
&\le \sup_{(v,q)\in X} \frac{(\|f\|_{V'}\|v\|_V + \|g\|_{P'}\|q\|_P)^2}{\|v\|_V^2 + \|q\|_P^2} \le \|f\|_{V'}^2 + \|g\|_{P'}^2.
\end{aligned}$$

Since both $V$ and $P$ are Hilbert spaces, by the Riesz-isomorphism, there exist $w \in V$ and $p \in P$ such that

$$f(v) = (w, v)_V \quad \forall\, v \in V, \quad \|f\|_{V'} = \|w\|_V,$$

$$g(q) = (p, q)_P \quad \forall\, q \in P, \quad \|g\|_{P'} = \|p\|_P.$$

Hence

$$\frac{(f(w)+g(p))^2}{\|w\|_V^2 + \|p\|_P^2} = \|w\|_V^2 + \|p\|_P^2 = \|f\|_{V'}^2 + \|g\|_{P'}^2.$$

This end the proof. □

As a consequence of an estimate of the form (1.61) in $X$ we obtain two simple estimates, one in $V$ and one in $P$.

**定理 1.40.** If (1.61) holds for constants $\underline{C}, \overline{C} > 0$, then

$$\underline{C}^2\|v\|_V^2 \le \|Av\|_{V'}^2 + \|Bv\|_{P'}^2 \le \overline{C}^2\|v\|_V^2 \quad \forall\, v \in V, \tag{1.62}$$

$$\underline{C}^2\|q\|_P^2 \le \|Cq\|_{P'}^2 + \|B'q\|_{V'}^2 \le \overline{C}^2\|q\|_P^2 \quad \forall\, q \in P. \tag{1.63}$$

Proof. For $y = (v, q)$, we have

$$\|\mathcal{A}y\|_{X'} = \sup_{(w,r)\in X} \frac{\mathcal{B}((v,q),(w,r))}{\|(w,r)\|_X} = \sup_{(w,r)\in X} \frac{a(v,w) + b(w,q) + b(v,r) - c(q,r)}{\|(w,r)\|_X}.$$

Since

$$a(v,w) + b(w,q) = \langle Av + B'q, w \rangle, \quad b(v,r) - c(q,r) = \langle Bv - Cq, r \rangle,$$

we obtain from Lemma 1.39 that

$$\|\mathcal{A}y\|_{X'}^2 = \|Av + B'q\|_{V'}^2 + \|Bv - Cq\|_{P'}^2.$$

Then the estimates (1.62) and (1.63) immediately follow from (1.61) for $q = 0$ and for $v = 0$, respectively. $\square$

So, (1.62) and (1.63) are necessary conditions for (1.61). Next we will show that (1.62) and (1.63), not necessarily with the same constants, are also sufficient.

**定理 1.41.** If there are constants $\underline{C}_v, \overline{C}_v, \underline{C}_p, \overline{C}_p > 0$ such that

$$\underline{C}_v^2 \|v\|_V^2 \le \|Av\|_{V'}^2 + \|Bv\|_{P'}^2 \le \overline{C}_v^2 \|v\|_V^2 \quad \forall\, v \in V,$$

$$\underline{C}_p^2 \|q\|_P^2 \le \|Cq\|_{P'}^2 + \|B'q\|_{V'}^2 \le \overline{C}_p^2 \|q\|_P^2 \quad \forall\, q \in P,$$

then there are constants $\underline{C}, \overline{C} > 0$ such that

$$\underline{C}\|y\|_X \le \|\mathcal{A}y\|_{X'} \le \overline{C}\|y\|_X \quad \forall\, y \in X,$$

where $\underline{C}$ and $\overline{C} > 0$ depend only on $\underline{C}_v$, $\overline{C}_v$, $\underline{C}_p$ and $\overline{C}_p$.

Proof. For $y = (v, q)$, we have

$$\begin{aligned}\|\mathcal{A}y\|_{X'}^2 &= \|Av + B'q\|_{V'}^2 + \|Bv - Cq\|_{P'}^2 \le 2(\|Av\|_{V'}^2 + \|B'q\|_{V'}^2 + \|Bv\|_{P'}^2 + \|Cq\|_{P'}^2) \\ &\le 2(\overline{C}_v^2 \|v\|_V^2 + \overline{C}_p^2 \|q\|_P^2) \le 2\max\{\overline{C}_v^2, \overline{C}_p^2\}\|y\|_X^2,\end{aligned}$$

which proves the upper bound in (1.61) with $\overline{C}^2 = 2\max\{\overline{C}_v^2, \overline{C}_p^2\}$.

For showing a lower bound, we start with the following estimate based on the triangle inequality in $X'$

$$\begin{aligned}\|\mathcal{A}y\|_{X'} &= \left(\|Av + B'q\|_{V'}^2 + \|Bv - Cq\|_{P'}^2\right)^{1/2} \ge \left(\|B'q\|_{V'}^2 + \|Bv\|_{P'}^2\right)^{1/2} - \left(\|Av\|_{V'}^2 + \|Cq\|_{P'}^2\right)^{1/2} \\ &= \frac{\left(\|B'q\|_{V'}^2 + \|Bv\|_{P'}^2\right) - \left(\|Av\|_{V'}^2 + \|Cq\|_{P'}^2\right)}{\left(\|B'q\|_{V'}^2 + \|Bv\|_{P'}^2\right)^{1/2} + \left(\|Av\|_{V'}^2 + \|Cq\|_{P'}^2\right)^{1/2}}\end{aligned}$$

for $y = (v, q)$. A second lower bound follows from

$$\|\mathcal{A}y\|_{X'} = \sup_{(w,r)\in X} \frac{\mathcal{B}((v,q),(w,r))}{\|(w,r)\|_X} \ge \frac{\mathcal{B}((v,q),(v,-q))}{\|(v,-q)\|_X} = \frac{a(v,v) + c(q,q)}{\|y\|_X}.$$

Since

$$a(v,w)^2 \le a(v,v)a(w,w) \le a(v,v)\|Aw\|_{V'}\|w\|_V \le \overline{C}_v a(v,v)\|w\|_V^2,$$

we have

$$\|Av\|_{V'}^2 = \sup_{w\in V} \frac{a(v,w)^2}{\|w\|_V^2} \le \overline{C}_v a(v,v).$$

Analogously, we obtain

$$\|Cq\|_{P'}^2 \le \overline{C}_p c(q,q).$$

Hence

$$a(v,v) + c(q,q) \ge \frac{1}{\overline{C}_v}\|Av\|_{V'}^2 + \frac{1}{\overline{C}_p}\|Cq\|_{P'}^2 \ge \frac{1}{\max\{\overline{C}_v, \overline{C}_p\}}(\|Av\|_{V'}^2 + \|Cq\|_{P'}^2).$$

With this estimate and

$$\begin{aligned}\|y\|_X &\le \frac{1}{\min\{\underline{C}_v,\underline{C}_p\}}\left(\|B'q\|_{V'}^2+\|Bv\|_{P'}^2+\|Av\|_{V'}^2+\|Cq\|_{P'}^2\right)^{1/2}\\ &\le \frac{1}{\min\{\underline{C}_v,\underline{C}_p\}}\left(\|B'q\|_{V'}^2+\|Bv\|_{P'}^2\right)^{1/2}+\frac{1}{\min\{\underline{C}_v,\underline{C}_p\}}\left(\|Av\|_{V'}^2+\|Cq\|_{P'}^2\right)^{1/2},\end{aligned}$$

we obtain for the second lower bound

$$\|\mathcal{A}y\|_{X'} \ge \frac{\min\{\underline{C}_v,\underline{C}_p\}}{\max\{\overline{C}_v,\overline{C}_p\}}\frac{\|Av\|_{V'}^2+\|Cq\|_{P'}^2}{\left(\|B'q\|_{V'}^2+\|Bv\|_{P'}^2\right)^{1/2}+\left(\|Av\|_{V'}^2+\|Cq\|_{P'}^2\right)^{1/2}}.$$

Taking a weighted sum of the two lower bounds we immediately obtain

$$\begin{aligned}\|\mathcal{A}y\|_{X'} &= \frac{\min\{\underline{C}_v,\underline{C}_p\}}{\min\{\underline{C}_v,\underline{C}_p\}+2\max\{\overline{C}_v,\overline{C}_p\}}\|\mathcal{A}y\|_{X'}+\frac{2\max\{\overline{C}_v,\overline{C}_p\}}{\min\{\underline{C}_v,\underline{C}_p\}+2\max\{\overline{C}_v,\overline{C}_p\}}\|\mathcal{A}y\|_{X'}\\ &\ge \frac{\min\{\underline{C}_v,\underline{C}_p\}}{\min\{\underline{C}_v,\underline{C}_p\}+2\max\{\overline{C}_v,\overline{C}_p\}}\frac{\|B'q\|_{V'}^2+\|Bv\|_{P'}^2+\|Av\|_{V'}^2+\|Cq\|_{P'}^2}{\left(\|B'q\|_{V'}^2+\|Bv\|_{P'}^2\right)^{1/2}+\left(\|Av\|_{V'}^2+\|Cq\|_{P'}^2\right)^{1/2}}\\ &\ge \frac{\min\{\underline{C}_v,\underline{C}_p\}}{\sqrt{2}\min\{\underline{C}_v,\underline{C}_p\}+2\sqrt{2}\max\{\overline{C}_v,\overline{C}_p\}}\left(\|B'q\|_{V'}^2+\|Bv\|_{P'}^2+\|Av\|_{V'}^2+\|Cq\|_{P'}^2\right)^{1/2}.\end{aligned}$$

Hence

$$\begin{aligned}\|\mathcal{A}y\|_{X'} &\ge \frac{\min\{\underline{C}_v,\underline{C}_p\}}{\sqrt{2}\min\{\underline{C}_v,\underline{C}_p\}+2\sqrt{2}\max\{\overline{C}_v,\overline{C}_p\}}\left(\underline{C}_v^2\|v\|_V^2+\underline{C}_p^2\|q\|_P^2\right)^{1/2}\\ &\ge \frac{\left(\min\{\underline{C}_v,\underline{C}_p\}\right)^2}{\sqrt{2}\min\{\underline{C}_v,\underline{C}_p\}+2\sqrt{2}\max\{\overline{C}_v,\overline{C}_p\}}\|y\|_X,\end{aligned}$$

which concludes the proof by choosing $\underline{C}=\dfrac{\left(\min\{\underline{C}_v,\underline{C}_p\}\right)^2}{\sqrt{2}\min\{\underline{C}_v,\underline{C}_p\}+2\sqrt{2}\max\{\overline{C}_v,\overline{C}_p\}}$. □

In the following two lemmas it will be shown that conditions (1.62) and (1.63) of the last theorem can be replaced by two other conditions which will turn out to be easier to work with.

**引理 1.42.** If there are constants $\underline{\gamma}_v,\overline{\gamma}_v>0$ such that

$$\underline{\gamma}_v\|v\|_V^2 \le a(v,v)+\|Bv\|_{P'}^2 \le \overline{\gamma}_v\|v\|_V^2 \quad \forall\, v\in V, \tag{1.64}$$

then (1.62) is satisfied with constants $\underline{C}_v,\overline{C}_v>0$ that depend only on $\underline{\gamma}_v,\overline{\gamma}_v$.

And, vice versa, if there are constants $\underline{C}_v,\overline{C}_v>0$ such that (1.62) is satisfied, then (1.64) is satisfied with constants $\underline{\gamma}_v,\overline{\gamma}_v>0$ that depend only on $\underline{C}_v,\overline{C}_v$.

Proof. Assume that (1.64) is satisfied. Then we have

$$a(v,w)^2 \le a(v,v)a(w,w) \le \overline{\gamma}_v a(v,v)\|w\|_V^2,$$

which imples

$$\|Av\|_{V'}^2 \le \overline{\gamma}_v a(v,v).$$

Therefore

$$\|Av\|_{V'}^2+\|Bv\|_{P'}^2 \le \overline{\gamma}_v a(v,v)+\|Bv\|_{P'}^2 \le \max\{\overline{\gamma}_v,1\}(a(v,v)+\|Bv\|_{P'}^2) \le \max\{\overline{\gamma}_v,1\}\overline{\gamma}_v\|v\|_V^2.$$

This shows the upper bound in (1.62) for $\overline{C}_v^2 = \max\{\overline{\gamma}_v, 1\}\overline{\gamma}_v$.

For the lower bound observe that, for all $\varepsilon > 0$,

$$a(v,v) \le \|Av\|_{V'}\|v\|_V \le \frac{1}{2\varepsilon}\|Av\|_{V'}^2 + \frac{\varepsilon}{2}\|v\|_V^2, \tag{1.65}$$

which imples

$$\underline{\gamma}_v\|v\|_V^2 \le a(v,v) + \|Bv\|_{P'}^2 \le \frac{1}{2\varepsilon}\|Av\|_{V'}^2 + \frac{\varepsilon}{2}\|v\|_V^2 + \|Bv\|_{P'}^2,$$

and, therefore,

$$\left(\underline{\gamma}_v - \frac{\varepsilon}{2}\right)\|v\|_V^2 \le \frac{1}{2\varepsilon}\|Av\|_{V'}^2 + \|Bv\|_{P'}^2 \le \max\{\frac{1}{2\varepsilon}, 1\}(\|Av\|_{V'}^2 + \|Bv\|_{P'}^2).$$

For $\varepsilon = \underline{\gamma}_v$, we obtain the lower bound in (1.62) with $\underline{C}_v = \min\{\underline{\gamma}_v, 1/2\}\underline{\gamma}_v$.

Now assume that (1.62) is satisfied. Then we have (see the proof of the last theorem)

$$\|Av\|_{V'}^2 \le \overline{C}_v a(v,v),$$

and, therefore,

$$a(v,v) + \|Bv\|_{P'}^2 \ge \overline{C}_v^{-1}\|Av\|_{V'}^2 + \|Bv\|_{P'}^2 \ge \min\{1, \overline{C}_v^{-1}\}(\|Av\|_{V'}^2 + \|Bv\|_{P'}^2) \ge \min\{1, \overline{C}_v^{-1}\}\underline{C}_v^2\|v\|_V^2,$$

showing the lower bound in (1.64) for $\underline{\gamma}_v = \min\{1, \overline{C}_v^{-1}\}\underline{C}_v^2$.

For the upper bound we use (1.65) for $\varepsilon = 1/2$ and obtain

$$a(v,v) + \|Bv\|_{P'}^2 \le \|Av\|_{V'}^2 + \frac{1}{4}\|v\|_V^2 + \|Bv\|_{P'}^2 \le \left(\overline{C}_v^2 + \frac{1}{4}\right)\|v\|_V^2.$$

So, the upper bound in (1.65) is satisfied for $\overline{\gamma}_v = \overline{C}_v^2 + 1/4$. □

Completely analogously, we have the following lemma.

**引理 1.43.** If there are constants $\underline{\gamma}_p, \overline{\gamma}_p > 0$ such that

$$\underline{\gamma}_p\|q\|_P^2 \le c(q,q) + \|B'q\|_{V'}^2 \le \overline{\gamma}_p\|q\|_P^2 \quad \forall\, q \in P, \tag{1.66}$$

then (1.63) is satisfied with constants $\underline{C}_p, \overline{C}_p > 0$ that depend only on $\underline{\gamma}_p, \overline{\gamma}_p$.

And, vice versa, if there are constants $\underline{C}_p, \overline{C}_p > 0$ such that (1.63) is satisfied, then (1.66) is satisfied with constants $\underline{\gamma}_p, \overline{\gamma}_p > 0$ that depend only on $\underline{C}_p, \overline{C}_p$.

By summarizing the results of the last two theorems and lemmas we finally obtain the following theorem.

**定理 1.44.** If there are constants $\underline{\gamma}_v, \overline{\gamma}_v, \underline{\gamma}_p, \overline{\gamma}_p > 0$ such that

$$\underline{\gamma}_v\|v\|_V^2 \le a(v,v) + \|Bv\|_{P'}^2 \le \overline{\gamma}_v\|v\|_V^2 \quad \forall\, v \in V, \tag{1.67}$$

$$\underline{\gamma}_p\|q\|_P^2 \le c(q,q) + \|B'q\|_{V'}^2 \le \overline{\gamma}_p\|q\|_P^2 \quad \forall\, q \in P, \tag{1.68}$$

then

$$\underline{C}\|y\|_X \le \|\mathcal{A}y\|_{X'} \le \overline{C}\|y\|_X^2 \quad \forall\, y \in X \tag{1.69}$$

is satisfied with constants $\underline{C}, \overline{C} > 0$ that depend only on $\underline{\gamma}_v, \overline{\gamma}_v, \underline{\gamma}_p, \overline{\gamma}_p$. And, vice versa, if the estimates (1.69) are satisfied with constants $\underline{C}, \overline{C} > 0$, then the estimates (1.67) and (1.68) are satisfied with constants $\underline{\gamma}_v, \overline{\gamma}_v, \underline{\gamma}_p, \overline{\gamma}_p > 0$ that depend only on $\underline{C}, \overline{C}$.

**注 1.45.** In the case $C = 0$ (i.e., $c(q,q) \equiv 0$), the lower estimate in condition (1.68) has the special form

$$\underline{\gamma}_p \|q\|_P^2 \leq \|B'q\|_{V'}^2 \quad \forall\, q \in P. \tag{1.70}$$

From the lower estimate in (1.67) it immediately follows that

$$\underline{\gamma}_v \|v\|_V^2 \leq a(v,v) \quad \forall\, v \in \ker B = \{v \in V : Bv = 0\}. \tag{1.71}$$

On the other hand, from (1.70) and (1.71) the lower estimate in (1.67) easily follows, using the fact that (1.70) implies

$$\underline{\gamma}_p \|v\|_V^2 \leq \|Bv\|_{P'}^2 \quad \forall\, v \in (\ker B)^\perp,$$

where $(\ker B)^\perp$ denotes the orthogonal complement of $\ker B$. So we have recovered a classical result by Brezzi [21, 16]. Let $a$ and $b$ be bounded bilinear forms and $c \equiv 0$. Then problem (1.55) is well posed if and only if $a$ is coercive on $\ker B$ (see (1.71)) and the inf-sup condition for $b$ is satisfied (see (1.70)).

# 第二章　De Rham 复形及有限元离散

这一章讨论 de Rham 复形及其有限元离散. 为此，先引入一些记号. 设 $\Omega \subset \mathbb{R}^d$ 是 $d$ 维连通区域，取 $\Omega$ 内的任一点 $a$. 记 $\mathbb{M}, \mathbb{S}$ 和 $\mathbb{K}$ 分别为 $d$ 阶矩阵空间，对称矩阵空间和反对称矩阵空间. 对于矩阵 $\tau$, 定义对称部分 $\operatorname{sym}\tau = (\tau + \tau^{\intercal})/2$ 和反对称部分 $\operatorname{skw}\tau = (\tau - \tau^{\intercal})/2$. 显然有 $\tau = \operatorname{sym}\tau + \operatorname{skw}\tau$. 将两维向量 $v = (v_1, v_2)^{\intercal}$ 按顺时针旋转 $90^\circ$ 得到的向量记为 $v^{\perp}$，即

$$v^{\perp} = \begin{pmatrix} v_2 \\ -v_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

对于函数 $v$, 定义梯度算子 $\nabla v = \operatorname{grad} v = \left(\frac{\partial v}{\partial x_1}, \ldots, \frac{\partial v}{\partial x_d}\right)^{\intercal}$. 对于向量函数 $v = (v_1, \ldots, v_d)^{\intercal}$, 分别按行和按列定义 $\operatorname{grad} v$ 和 $\nabla v$，即

$$\operatorname{grad} v = \left(\frac{\partial v_i}{\partial x_j}\right)_{d\times d} = \begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \cdots & \frac{\partial v_1}{\partial x_d} \\ \vdots & & \vdots \\ \frac{\partial v_d}{\partial x_1} & \cdots & \frac{\partial v_d}{\partial x_d} \end{pmatrix}, \quad \nabla v = \left(\frac{\partial v_j}{\partial x_i}\right)_{d\times d} = \begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \cdots & \frac{\partial v_d}{\partial x_1} \\ \vdots & & \vdots \\ \frac{\partial v_1}{\partial x_d} & \cdots & \frac{\partial v_d}{\partial x_d} \end{pmatrix}.$$

显然有 $\operatorname{grad} v = (\nabla v)^{\intercal}$. 定义散度

$$\operatorname{div} v = \nabla \cdot v = \frac{\partial v_1}{\partial x_1} + \cdots + \frac{\partial v_d}{\partial x_d}.$$

易知 $\operatorname{div} v = \operatorname{tr}(\operatorname{grad} v) = \operatorname{tr}(\nabla v)$. 对于二元函数 $v$，定义旋度 $\operatorname{curl} v = (\nabla v)^{\intercal} = \left(\frac{\partial v}{\partial x_2}, -\frac{\partial v}{\partial x_1}\right)^{\intercal}$. 对于二元向量函数 $v = (v_1, v_2)^{\intercal}$，定义旋度

$$\operatorname{rot} v = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} = \operatorname{div} v^{\perp}, \quad \operatorname{curl} v = \begin{pmatrix} \frac{\partial v_1}{\partial x_2} & -\frac{\partial v_1}{\partial x_1} \\ \frac{\partial v_2}{\partial x_2} & -\frac{\partial v_2}{\partial x_1} \end{pmatrix}.$$

对于三元向量函数 $v = (v_1, v_2, v_3)^{\intercal}$，定义旋度

$$\operatorname{curl} v = \nabla \times v = \begin{pmatrix} \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} \\ \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \end{pmatrix}.$$

对于一般的 $d$ 元向量函数 $v = (v_1, \ldots, v_d)^{\intercal}$，定义旋度 $\operatorname{curl} v = \operatorname{skw} \operatorname{grad} v$.

定义 Sobolev 空间

$$\begin{aligned} H^1(\Omega) &= \{v \in L^2(\Omega) : \operatorname{grad} v \in L^2(\Omega; \mathbb{R}^d)\}, \\ H(\operatorname{div}, \Omega) &= \{v \in L^2(\Omega; \mathbb{R}^d) : \operatorname{div} v \in L^2(\Omega)\}, \\ H(\operatorname{rot}, \Omega) &= \{v \in L^2(\Omega; \mathbb{R}^2) : \operatorname{rot} v \in L^2(\Omega)\}, \\ H(\operatorname{curl}, \Omega) &= \{v \in L^2(\Omega; \mathbb{R}^d) : \operatorname{curl} v \in L^2(\Omega; \mathbb{K})\}, \quad d \geq 3. \end{aligned}$$

## 2.1 三维 de Rham 复形

设 $\Omega$ 是三维连通区域，取 $\Omega$ 内的任一点 $a$. 给出三维 de Rham 复形

$$\mathbb{R} \xrightarrow{\hookrightarrow} H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl}, \Omega) \xrightarrow{\text{curl}} H(\text{div}, \Omega) \xrightarrow{\text{div}} L^2(\Omega) \to 0. \tag{2.1}$$

之所以称为复形，是因为

$$\begin{aligned} \text{grad}\, H^1(\Omega) &\subseteq H(\text{curl}, \Omega) \cap \ker(\text{curl}), \\ \text{curl}\, H(\text{curl}, \Omega) &\subseteq H(\text{div}, \Omega) \cap \ker(\text{div}), \\ \text{div}\, H(\text{div}, \Omega) &\subseteq L^2(\Omega), \end{aligned}$$

其中核空间 $\mathbb{B} \cap \ker(A) = \{v \in \mathbb{B} : Av = 0\}$.

当上式中的 $\subseteq$ 都改成 $=$ 后，则称复形是正合的. 为证明三维 de Rham 复形在可缩区域上是正合的，引入 Poincaré 算子 [48, 54, 35]

$$\mathcal{P}_1 u = \int_0^1 u(a + t(x-a)) \cdot (x-a)\, \mathrm{d}t, \qquad u \in C^\infty(\Omega; \mathbb{R}^3), \tag{2.2}$$

$$\mathcal{P}_2 v = \int_0^1 t v(a + t(x-a)) \times (x-a)\, \mathrm{d}t, \qquad v \in C^\infty(\Omega; \mathbb{R}^3), \tag{2.3}$$

$$\mathcal{P}_3 w = (x-a) \int_0^1 t^2 w(a + t(x-a))\, \mathrm{d}t, \qquad w \in C^\infty(\Omega). \tag{2.4}$$

显然有复形

$$\mathbb{R} \xleftarrow{\mathcal{P}_0} C^\infty(\Omega) \xleftarrow{\mathcal{P}_1} C^\infty(\Omega; \mathbb{R}^3) \xleftarrow{\mathcal{P}_2} C^\infty(\Omega; \mathbb{R}^3) \xleftarrow{\mathcal{P}_3} C^\infty(\Omega) \leftarrow 0, \tag{2.5}$$

其中 $\mathcal{P}_0 : C^\infty(\Omega) \to \mathbb{R}$ 定义为 $\mathcal{P}_0 w = w - \mathcal{P}_1 \,\text{grad}\, w$. 直接计算可得

$$\mathcal{P}_0 w = w(x) - \int_0^1 (\text{grad}\, w)(a + t(x-a)) \cdot (x-a)\, \mathrm{d}t = w(x) - \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} w(a + t(x-a))\, \mathrm{d}t = w(a).$$

**引理 2.1.** 成立

$$\text{div}\, \mathcal{P}_3 w = w \quad \forall\, w \in C^\infty(\Omega), \tag{2.6}$$

$$\text{curl}\, \mathcal{P}_2 v + \mathcal{P}_3 \,\text{div}\, v = v \quad \forall\, v \in C^\infty(\Omega; \mathbb{R}^3), \tag{2.7}$$

$$\text{grad}\, \mathcal{P}_1 u + \mathcal{P}_2 \,\text{curl}\, u = u \quad \forall\, u \in C^\infty(\Omega; \mathbb{R}^3). \tag{2.8}$$

*Proof.* 容易验证

$$(t\frac{\mathrm{d}}{\mathrm{d}t}) w(a + t(x-a)) = ((x-a) \cdot \nabla) w(a + t(x-a)).$$

于是

$$\begin{aligned} \text{div}\, \mathcal{P}_3 w &= 3 \int_0^1 t^2 w(a + t(x-a))\, \mathrm{d}t + \int_0^1 t^2 ((x-a) \cdot \nabla) w(a + t(x-a))\, \mathrm{d}t \\ &= 3 \int_0^1 t^2 w(a + t(x-a))\, \mathrm{d}t + \int_0^1 t^3 \frac{\mathrm{d}}{\mathrm{d}t} w(a + t(x-a))\, \mathrm{d}t \\ &= t^3 w(a + t(x-a))|_{t=0}^1 = w(x). \end{aligned}$$

由 $\operatorname{curl}(v\times(x-a)) = -(x-a)\operatorname{div} v + ((x-a)\cdot\nabla)v + 2v$ 知，

$$
\begin{aligned}
\operatorname{curl}\mathcal{P}_2 v &= -\int_0^1 t(x-a)\operatorname{div}(v(a+t(x-a)))\,\mathrm{d}t + \int_0^1 t((x-a)\cdot\nabla)v(a+t(x-a))\,\mathrm{d}t \\
&\quad + \int_0^1 2tv(a+t(x-a))\,\mathrm{d}t \\
&= -\int_0^1 t^2(x-a)(\operatorname{div} v)(a+t(x-a))\,\mathrm{d}t + \int_0^1 t^2\frac{\mathrm{d}}{\mathrm{d}t}v(a+t(x-a))\,\mathrm{d}t \\
&\quad + \int_0^1 2tv(a+t(x-a))\,\mathrm{d}t \\
&= -\mathcal{P}_3\operatorname{div} v + t^2 v(a+t(x-a))|_{t=0}^1 = -\mathcal{P}_3\operatorname{div} v + v.
\end{aligned}
$$

对任意的向量 $y$ 成立 $(\nabla u)y = -(\operatorname{curl} u)\times y + (y\cdot\nabla)u$，故

$$
\begin{aligned}
\operatorname{grad}\mathcal{P}_1 u &= \int_0^1 t(\nabla u)|_{a+t(x-a)}(x-a)\,\mathrm{d}t + \int_0^1 u(a+t(x-a))\,\mathrm{d}t \\
&= -\int_0^1 t(\operatorname{curl} u)|_{a+t(x-a)}\times(x-a)\,\mathrm{d}t + \int_0^1 t\frac{\mathrm{d}}{\mathrm{d}t}u(a+t(x-a))\,\mathrm{d}t \\
&\quad + \int_0^1 u(a+t(x-a))\,\mathrm{d}t \\
&= -\mathcal{P}_2\operatorname{curl} u + tu(a+t(x-a))|_{t=0}^1 = -\mathcal{P}_2\operatorname{curl} u + u.
\end{aligned}
$$

□

**引理 2.2.** 对于 $u\in C^\infty(\Omega;\mathbb{R}^3), v\in C^\infty(\Omega;\mathbb{R}^3)$ 和 $w\in C^\infty(\Omega)$，有

$$\|\mathcal{P}_3 w\|_0 \le \frac{2}{3}h_\Omega\|w\|_0, \tag{2.9}$$

$$\|\mathcal{P}_2 v\|_0 \le 2h_\Omega\|v\|_0, \tag{2.10}$$

$$|\mathcal{P}_1 u|_1 \le \|u\|_0 + 2h_\Omega\|\operatorname{curl} u\|_0, \tag{2.11}$$

$$\|\mathcal{P}_1 u\|_{L^p(\Omega)} \le \frac{p}{(p-3)}h_\Omega\|u\|_{L^p(\Omega)}, \tag{2.12}$$

其中 $p>3$.

Proof. 设 $s=t^{1/\alpha}$ $(\alpha>1/3)$, 则

$$\mathcal{P}_3 w = \alpha(x-a)\int_0^1 s^{3\alpha-1}w(a+s^\alpha(x-a))\,\mathrm{d}s.$$

令 $y_s = a+s^\alpha(x-a), \Omega_s=\{a+s^\alpha(x-a): x\in\Omega\}\subseteq\Omega$. 从而

$$
\begin{aligned}
\|\mathcal{P}_3 w\|_0^2 &\le \alpha^2 h_\Omega^2\int_\Omega \mathrm{d}x\int_0^1 s^{6\alpha-2}w^2(a+s^\alpha(x-a))\,\mathrm{d}s = \alpha^2 h_\Omega^2\int_0^1 s^{6\alpha-2}\,\mathrm{d}s\int_\Omega w^2(a+s^\alpha(x-a))\,\mathrm{d}x \\
&= \alpha^2 h_\Omega^2\int_0^1 s^{6\alpha-2}\,\mathrm{d}s\int_\Omega w^2(y_s)\,\mathrm{d}x = \alpha^2 h_\Omega^2\int_0^1 s^{3\alpha-2}\,\mathrm{d}s\int_{\Omega_s} w^2(y_s)\,\mathrm{d}y_s \\
&\le \alpha^2 h_\Omega^2\int_0^1 s^{3\alpha-2}\,\mathrm{d}s\int_\Omega w^2(y_s)\,\mathrm{d}y_s = \frac{\alpha^2}{3\alpha-1}h_\Omega^2\|w\|_0^2.
\end{aligned}
$$

当 $\alpha=\frac{2}{3}$ 时，$\frac{\alpha^2}{3\alpha-1}$ 取到最小值 $\frac{4}{9}$，故有

$$\|\mathcal{P}_3 w\|_0 \le \frac{2}{3}h_\Omega\|w\|_0.$$

设 $s = t^{1/\alpha}$ $(\alpha > 1)$, 则

$$\mathcal{P}_2 v = \int_0^1 t v(a + t(x-a)) \times (x-a)\,\mathrm{d}t = \alpha \int_0^1 s^{2\alpha-1} v(a + s^\alpha(x-a)) \times (x-a)\,\mathrm{d}s.$$

令 $y_s = a + s^\alpha(x-a), \Omega_s = \{a + s^\alpha(x-a) : x \in \Omega\} \subseteq \Omega$. 从而

$$\begin{aligned}
\|\mathcal{P}_2 v\|_0^2 &\le \alpha^2 h_\Omega^2 \int_\Omega \mathrm{d}x \int_0^1 s^{4\alpha-2} |v(a + s^\alpha(x-a))|^2\,\mathrm{d}s = \alpha^2 h_\Omega^2 \int_0^1 s^{4\alpha-2}\,\mathrm{d}s \int_\Omega |v(a + s^\alpha(x-a))|^2\,\mathrm{d}x \\
&= \alpha^2 h_\Omega^2 \int_0^1 s^{4\alpha-2}\,\mathrm{d}s \int_\Omega |v(y_s)|^2\,\mathrm{d}x = \alpha^2 h_\Omega^2 \int_0^1 s^{\alpha-2}\,\mathrm{d}s \int_{\Omega_s} |v(y_s)|^2\,\mathrm{d}y_s \\
&\le \alpha^2 h_\Omega^2 \int_0^1 s^{\alpha-2}\,\mathrm{d}s \int_\Omega |v(y_s)|^2\,\mathrm{d}y_s = \frac{\alpha^2}{\alpha-1} h_\Omega^2 \|v\|_0^2.
\end{aligned}$$

当 $\alpha = 2$ 时，$\frac{\alpha^2}{\alpha-1}$ 取到最小值 4，故有

$$\|\mathcal{P}_2 v\|_0 \le 2h_\Omega \|v\|_0.$$

由(2.8)得

$$\|\operatorname{grad}\mathcal{P}_1 u\|_0 = \|u - \mathcal{P}_2 \operatorname{curl} u\|_0 \le \|u\|_0 + \|\mathcal{P}_2 \operatorname{curl} u\|_0 \le \|u\|_0 + 2h_\Omega \|\operatorname{curl} u\|_0.$$

设 $s = t^{1/\alpha}$ $(\alpha > (p-1)/(p-3))$, 则

$$\mathcal{P}_1 u = \int_0^1 u(a + t(x-a)) \cdot (x-a)\,\mathrm{d}t = \alpha \int_0^1 s^{\alpha-1} u(a + s^\alpha(x-a)) \cdot (x-a)\,\mathrm{d}s.$$

令 $y_s = a + s^\alpha(x-a), \Omega_s = \{a + s^\alpha(x-a) : x \in \Omega\} \subseteq \Omega$. 从而

$$\begin{aligned}
\|\mathcal{P}_1 u\|_{L^p(\Omega)}^p &\le \alpha^p h_\Omega^p \int_\Omega \mathrm{d}x \int_0^1 s^{p\alpha-p} |u(a + s^\alpha(x-a))|^p\,\mathrm{d}s \\
&= \alpha^p h_\Omega^p \int_0^1 s^{p\alpha-p}\,\mathrm{d}s \int_\Omega |u(a + s^\alpha(x-a))|^p\,\mathrm{d}x \\
&= \alpha^p h_\Omega^p \int_0^1 s^{p\alpha-p}\,\mathrm{d}s \int_\Omega |u(y_s)|^p\,\mathrm{d}x = \alpha^p h_\Omega^p \int_0^1 s^{(p-3)\alpha-p}\,\mathrm{d}s \int_{\Omega_s} |u(y_s)|^p\,\mathrm{d}y_s \\
&\le \alpha^p h_\Omega^p \int_0^1 s^{(p-3)\alpha-p}\,\mathrm{d}s \int_\Omega |u(y_s)|^p\,\mathrm{d}y_s = \frac{\alpha^p}{(p-3)\alpha-(p-1)} h_\Omega^p \|u\|_{L^p(\Omega)}^p.
\end{aligned}$$

当 $\alpha = \frac{p}{p-3}$ 时，$\frac{\alpha^p}{(p-3)\alpha-(p-1)}$ 取到最小值 $\alpha^p$，故有

$$\|\mathcal{P}_1 u\|_{L^p(\Omega)} \le \frac{p}{(p-3)} h_\Omega \|u\|_{L^p(\Omega)}.$$

□

**注 2.3.** 算子 $\mathcal{P}_1$ 不是 $L^2$ 有界的. 事实上，设 $\Omega \subseteq \mathbb{R}^3$ 包含坐标原点，取 $q(x) = x/|x|^2$, 则 $q(x) \in \mathbb{L}^2(\Omega;\mathbb{R}^3)$, 但是 $\mathcal{P}_1 q$ 没定义.

**定理 2.4.** 由式(2.2)-(2.4)定义的 Poincaré 算子 $\mathcal{P}_1, \mathcal{P}_2$ 和 $\mathcal{P}_3$ 可唯一的延拓成 Sobolev 空间上的有界线性算子，即

$$\|\mathcal{P}_3 w\|_{H(\mathrm{div})} \lesssim \|w\|_0 \qquad \forall\, w \in L^2(\Omega), \tag{2.13}$$

$$\|\mathcal{P}_2 v\|_{H(\mathrm{curl})} \lesssim \|v\|_{H(\mathrm{div})} \qquad \forall\, v \in H(\mathrm{div},\Omega), \tag{2.14}$$

$$|\mathcal{P}_1 u|_1 \lesssim \|u\|_{H(\mathrm{curl})} \qquad \forall\, u \in H(\mathrm{curl},\Omega). \tag{2.15}$$

同时成立复形

$$\mathbb{R} \xleftarrow{\mathcal{P}_0} H^1(\Omega) \xleftarrow{\mathcal{P}_1} H(\mathrm{curl},\Omega) \xleftarrow{\mathcal{P}_2} H(\mathrm{div},\Omega) \xleftarrow{\mathcal{P}_3} L^2(\Omega) \leftarrow 0. \tag{2.16}$$

且有恒等式

$$\operatorname{div}\mathcal{P}_3 w = w \quad \forall\, w \in L^2(\Omega), \tag{2.17}$$

$$\operatorname{curl}\mathcal{P}_2 v + \mathcal{P}_3 \operatorname{div} v = v \quad \forall\, v \in H(\mathrm{div},\Omega), \tag{2.18}$$

$$\operatorname{grad}\mathcal{P}_1 u + \mathcal{P}_2 \operatorname{curl} u = u \quad \forall\, u \in H(\mathrm{curl},\Omega), \tag{2.19}$$

$$\mathcal{P}_0 w + \mathcal{P}_1 \operatorname{grad} w = w \quad \forall\, w \in H^1(\Omega). \tag{2.20}$$

Proof. 利用(2.6)-(2.8)和(2.9)-(2.11)可得

$$\|\mathcal{P}_3 w\|_{H(\mathrm{div})}^2 = \|\mathcal{P}_3 w\|_0^2 + \|w\|_0^2 \le (1+\frac{4}{9}h_\Omega^2)\|w\|_0^2 \quad \forall\, w \in C^\infty(\Omega),$$

$$\|\mathcal{P}_2 v\|_{H(\mathrm{curl})}^2 = \|\mathcal{P}_2 v\|_0^2 + \|v - \mathcal{P}_3 \operatorname{div} v\|_0^2 \le (2+4h_\Omega^2)\|v\|_0^2 + \frac{8}{9}h_\Omega^2\|\operatorname{div} v\|_0^2 \quad \forall\, v \in C^\infty(\Omega;\mathbb{R}^3).$$

从而由稠密性可得(2.13)-(2.15).

同样利用稠密性，由(2.6)-(2.8)和(2.13)-(2.15)可得(2.17)-(2.19). □

**定理** 2.5. 设三维区域 $\Omega$ 是可缩的，则三维 de Rham 复形(2.1)

$$\mathbb{R} \xrightarrow{\hookrightarrow} H^1(\Omega) \xrightarrow{\mathrm{grad}} H(\mathrm{curl},\Omega) \xrightarrow{\mathrm{curl}} H(\mathrm{div},\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \to 0$$

是正合的.

Proof. 由恒等式(2.17)-(2.20)直接可得. □

## 2.2 多项式复形

在本节中，我们将讨论定义在可缩区域 $D \subset \mathbb{R}^d$ 上的多项式空间，其中 $d = 2, 3$. 不失一般性，我们假设 $D$ 包含坐标原点. 如果区域 $D$ 不包含坐标原点，可以平移区域 $D$ 使其包含坐标原点.

设整数 $k \ge 0$，记 $\mathbb{P}_k(D)$ 为区域 $D$ 上所有总次数不超过 $k$ 的多项式构成的空间。为方便表述，当 $k < 0$ 时，约定 $\mathbb{P}_k(D) := \{0\}$。记 $\mathbb{P}_k(D;\mathbb{X})$ 为 $\mathbb{P}_k(D)$ 的张量或向量版本，其中 $\mathbb{X}$ 可取 $\mathbb{M}$、$\mathbb{S}$、$\mathbb{K}$ 或 $\mathbb{R}^d$。定义 $k$ 次齐次多项式空间 $\mathbb{H}_k(D) := \mathbb{P}_k(D)\backslash\mathbb{P}_{k-1}(D)$. 可以直接验证

$$\boldsymbol{x}\cdot\nabla q = kq \quad \forall\, q \in \mathbb{H}_k(D), \tag{2.21}$$

$$\operatorname{div}(\boldsymbol{x}q) = (k+d)q \quad \forall\, q \in \mathbb{H}_k(D). \tag{2.22}$$

由等式 (2.21) 可直接推得 [32, (35)]

$$\mathbb{P}_k(T) \cap \ker(\boldsymbol{x}\cdot\nabla) = \mathbb{P}_0(T),$$

$$\mathbb{P}_k(T) \cap \ker(c + \boldsymbol{x}\cdot\nabla) = \{0\}, \tag{2.23}$$

其中 $c$ 为任意正数。

定义算子 $\pi_0 : \mathcal{C}^0(D) \to \mathbb{R}$ 为

$$\pi_0 v := v(\mathbf{0}).$$

### 2.2.1 二维多项式复形

**引理 2.6.** 设整数 $k \geq -1$，且 $D$ 为二维可缩区域. 多项式复形

$$\mathbb{R} \xrightarrow{\subseteq} \mathbb{P}_{k+1}(D) \xrightarrow{\text{curl}} \mathbb{P}_k(D;\mathbb{R}^2) \xrightarrow{\text{div}} \mathbb{P}_{k-1}(D) \to 0 \tag{2.24}$$

是正合的.

Proof. 由式 (2.22) 可知 $\text{div}(\boldsymbol{x}\mathbb{P}_{k-1}(D)) = \mathbb{P}_{k-1}(D)$，因此有

$$\text{div}\, \mathbb{P}_k(D;\mathbb{R}^2) = \mathbb{P}_{k-1}(D).$$

进一步通过直接计算可得

$$\dim \mathbb{P}_k(D;\mathbb{R}^2) = \dim \text{curl}\ \mathbb{P}_{k+1}(D) + \dim \mathbb{P}_{k-1}(D),$$

故复形 (2.24) 是正合的. □

通过旋转复形 (2.24)，我们得到以下正合的多项式复形：

$$\mathbb{R} \xrightarrow{\subseteq} \mathbb{P}_{k+1}(D) \xrightarrow{\text{grad}} \mathbb{P}_k(D;\mathbb{R}^2) \xrightarrow{\text{rot}} \mathbb{P}_{k-1}(D) \to 0.$$

**引理 2.7.** 设整数 $k \geq -1$，且 $D$ 为二维可缩区域. 多项式复形

$$0 \xrightarrow{\subseteq} \mathbb{P}_{k-1}(D) \xrightarrow{\boldsymbol{x}} \mathbb{P}_k(D;\mathbb{R}^2) \xrightarrow{\boldsymbol{x}^{\perp}\cdot} \mathbb{P}_{k+1}(D) \xrightarrow{\pi_0} \mathbb{R} \to 0 \tag{2.25}$$

是正合的.

Proof. 显然有 $\mathbb{P}_{k-1}(D) \cap \ker(\boldsymbol{x}) = 0$, 以及 $\boldsymbol{x}^{\perp} \cdot \mathbb{P}_k(D;\mathbb{R}^2) = \mathbb{P}_{k+1}(D) \cap \ker(\pi_0)$. 接下来证明

$$\mathbb{P}_{k-1}(D)\boldsymbol{x} = \mathbb{P}_k(D;\mathbb{R}^2) \cap \ker(\boldsymbol{x}^{\perp}).$$

易知, $\mathbb{P}_{k-1}(D)\boldsymbol{x} \subseteq \mathbb{P}_k(D;\mathbb{R}^2) \cap \ker(\boldsymbol{x}^{\perp})$. 另一方面, 设 $\boldsymbol{v} \in \mathbb{P}_k(D;\mathbb{R}^2) \cap \ker(\boldsymbol{x}^{\perp})$, 即 $\boldsymbol{v} \in \mathbb{P}_k(D;\mathbb{R}^2)$ 满足 $\boldsymbol{v} \cdot \boldsymbol{x}^{\perp} = 0$. 这表示 $\boldsymbol{v}$ 与 $\boldsymbol{x}^{\perp}$ 垂直，从而 $\boldsymbol{v}$ 平行于 $\boldsymbol{x}$，即 $\boldsymbol{v} \in \mathbb{P}_{k-1}(D)\boldsymbol{x}$. 证毕. □

复形 (2.24) 和 (2.25) 通过下列方式相连：

$$\mathbb{R} \underset{\pi_0}{\overset{\subset}{\rightleftarrows}} \mathbb{P}_{k+1}(D) \underset{\boldsymbol{x}^{\perp}\cdot}{\overset{\text{curl}}{\rightleftarrows}} \mathbb{P}_k(D;\mathbb{R}^2) \underset{\boldsymbol{x}}{\overset{\text{div}}{\rightleftarrows}} \mathbb{P}_{k-1}(D) \underset{\supset}{\rightleftarrows} 0\,. \tag{2.26}$$

与向量函数的 Koszul 复形不同，这里我们无法将恒等性质直接应用于齐次多项式。不过，利用 Koszul 算子与微分算子对多项式空间进行分解的方法仍然成立。

首先，我们有如下分解：

$$\mathbb{P}_{k+1}(D) = \boldsymbol{x}^{\perp} \cdot \mathbb{P}_k(D;\mathbb{R}^2) \oplus \mathbb{R}.$$

**引理 2.8.** 设整数 $k \geq -1$ 且 $D$ 为二维可缩区域，则有以下空间直和分解：

$$\mathbb{P}_k(D;\mathbb{R}^2) = \text{curl}\, \mathbb{P}_{k+1}(D) \oplus \mathbb{P}_{k-1}(D)\boldsymbol{x}. \tag{2.27}$$

Proof. 假设 $q \in \mathbb{P}_{k-1}(D)$ 且满足 $\boldsymbol{x}q \in \text{curl}\, \mathbb{P}_{k+1}(D)$，即

$$\text{div}(\boldsymbol{x}q) = 0.$$

由式 (2.22) 可知，$q = 0$. 因此有 $\text{curl}\, \mathbb{P}_{k+1}(D) \cap \mathbb{P}_{k-1}(D)\boldsymbol{x} = 0$. 由此结合维数关系

$$\dim \mathbb{P}_k(D;\mathbb{R}^2) = \dim \text{curl}\, \mathbb{P}_{k+1}(D) + \dim(\mathbb{P}_{k-1}(D)\boldsymbol{x}),$$

即可得到相应的分解。 □

类似地，我们有双向多项式复形

$$\mathbb{R} \underset{\pi_0}{\overset{\subset}{\rightleftarrows}} \mathbb{P}_{k+1}(D) \underset{\boldsymbol{x}\cdot}{\overset{\text{grad}}{\rightleftarrows}} \mathbb{P}_k(D;\mathbb{R}^2) \underset{\boldsymbol{x}^{\perp}}{\overset{\text{rot}}{\rightleftarrows}} \mathbb{P}_{k-1}(D) \underset{\supset}{\rightleftarrows} 0\,,$$

和多项式空间分解

$$\mathbb{P}_{k+1}(D) = \boldsymbol{x}\cdot\mathbb{P}_k(D;\mathbb{R}^2)\oplus\mathbb{R},$$
$$\mathbb{P}_k(D;\mathbb{R}^2) = \nabla\mathbb{P}_{k+1}(D)\oplus\mathbb{P}_{k-1}(D)\boldsymbol{x}^{\perp}.$$

### 2.2.2 三维多项式复形

**引理 2.9.** 设整数 $k\geq -1$，且 $D$ 为三维可缩区域. 多项式复形

$$\mathbb{R}\xrightarrow{\subset}\mathbb{P}_{k+1}(D)\xrightarrow{\text{grad}}\mathbb{P}_k(D;\mathbb{R}^3)\xrightarrow{\text{curl}}\mathbb{P}_{k-1}(D;\mathbb{R}^3)\xrightarrow{\text{div}}\mathbb{P}_{k-2}(D)\to 0 \tag{2.28}$$

是正合的.

Proof. 应用 (2.22) 可知 $\operatorname{div}\mathbb{P}_k(D;\mathbb{R}^2)=\mathbb{P}_{k-1}(D)$, 因此

$$\begin{aligned}\dim\mathbb{P}_k(D;\mathbb{R}^3)\cap\ker(\operatorname{div}) &= \dim\mathbb{P}_k(D;\mathbb{R}^3)-\dim\mathbb{P}_{k-1}(D)\\ &=\frac{1}{2}(k+1)(k+2)(k+3)-\frac{1}{6}k(k+1)(k+2)=\frac{1}{6}(k+1)(k+2)(2k+9).\end{aligned}$$

对于任意 $\boldsymbol{v}\in\mathbb{P}_{k+1}(D;\mathbb{R}^3)\cap\ker(\operatorname{curl})$, 线积分

$$q=\int_{\mathbf{0}}^{\boldsymbol{x}}\boldsymbol{v}\cdot\,\mathrm{d}\boldsymbol{x}\in\mathbb{P}_{k+2}(D)$$

与路径无关，因此它是适定的，并且成立 $\boldsymbol{v}=\operatorname{grad}q$. 从而，$\operatorname{grad}\mathbb{P}_{k+2}(D)=\mathbb{P}_{k+1}(D;\mathbb{R}^3)\cap\ker(\operatorname{curl})$, 且有

$$\begin{aligned}\dim\operatorname{curl}\mathbb{P}_{k+1}(D;\mathbb{R}^3)&=\dim\mathbb{P}_{k+1}(D;\mathbb{R}^3)-\dim\mathbb{P}_{k+1}(D;\mathbb{R}^3)\cap\ker(\operatorname{curl})\\ &=\dim\mathbb{P}_{k+1}(D;\mathbb{R}^3)-\dim\operatorname{grad}\mathbb{P}_{k+2}(D)\\ &=\frac{1}{2}(k+2)(k+3)(k+4)-\frac{1}{6}(k+3)(k+4)(k+5)+1\\ &=\frac{1}{6}(k+1)(k+2)(2k+9).\end{aligned}$$

注意到 $\dim\operatorname{curl}\mathbb{P}_{k+1}(D;\mathbb{R}^3)=\dim\mathbb{P}_k(D;\mathbb{R}^3)\cap\ker(\operatorname{div})$, 以及 $\operatorname{curl}\mathbb{P}_{k+1}(D;\mathbb{R}^3)\subseteq\mathbb{P}_k(D;\mathbb{R}^3)\cap\ker(\operatorname{div})$, 故有 $\operatorname{curl}\mathbb{P}_{k+1}(D;\mathbb{R}^3)=\mathbb{P}_k(D;\mathbb{R}^3)\cap\ker(\operatorname{div})$. 由此可知复形 (2.28) 是正合的。 □

**引理 2.10.** 设整数 $k\geq -1$，且 $D$ 为三维可缩区域. 多项式复形

$$0\xrightarrow{\subset}\mathbb{P}_{k-1}(D)\xrightarrow{\boldsymbol{x}}\mathbb{P}_k(D;\mathbb{R}^3)\xrightarrow{\boldsymbol{x}\times}\mathbb{P}_{k+1}(D;\mathbb{R}^3)\xrightarrow{\boldsymbol{x}\cdot}\mathbb{P}_{k+2}(D)\xrightarrow{\pi_0}\mathbb{R}\to 0 \tag{2.29}$$

是正合的.

Proof. 易知 $\pi_0\mathbb{P}_{k+2}(D)=\mathbb{R}$，以及 $\boldsymbol{x}\cdot\mathbb{P}_{k+1}(D;\mathbb{R}^3)=\mathbb{P}_{k+2}(D)\cap\ker(\pi_0)$. 因此，

$$\begin{aligned}\dim\mathbb{P}_{k+1}(D;\mathbb{R}^3)\cap\ker(\boldsymbol{x}\cdot)&=\dim\mathbb{P}_{k+1}(D;\mathbb{R}^3)-\dim\boldsymbol{x}\cdot\mathbb{P}_{k+1}(D;\mathbb{R}^3)\\ &=\dim\mathbb{P}_{k+1}(D;\mathbb{R}^3)-\dim\mathbb{P}_{k+2}(D)\cap\ker(\pi_0)\\ &=\frac{1}{2}(k+2)(k+3)(k+4)-\frac{1}{6}(k+3)(k+4)(k+5)+1\\ &=\frac{1}{6}(k+1)(k+2)(2k+9).\end{aligned}$$

对任意 $\boldsymbol{v} \in \mathbb{P}_k(D;\mathbb{R}^3) \cap \ker(\boldsymbol{x}\times)$，有 $\boldsymbol{x} \times \boldsymbol{v} = \boldsymbol{0}$，故 $\boldsymbol{v}$ 与 $\boldsymbol{x}$ 平行，即 $\boldsymbol{v} \in \boldsymbol{x}\mathbb{P}_{k-1}(D)$。因此 $\mathbb{P}_k(D;\mathbb{R}^3) \cap \ker(\boldsymbol{x}\times) = \boldsymbol{x}\mathbb{P}_{k-1}(D)$。于是，

$$
\begin{aligned}
\dim \boldsymbol{x} \times \mathbb{P}_k(D;\mathbb{R}^3) &= \dim \mathbb{P}_k(D;\mathbb{R}^3) - \dim \mathbb{P}_k(D;\mathbb{R}^3) \cap \ker(\boldsymbol{x}\times) \\
&= \dim \mathbb{P}_k(D;\mathbb{R}^3) - \dim \boldsymbol{x}\mathbb{P}_{k-1}(D) \\
&= \frac{1}{2}(k+1)(k+2)(k+3) - \frac{1}{6}k(k+1)(k+2) \\
&= \frac{1}{6}(k+1)(k+2)(2k+9).
\end{aligned}
$$

注意到 $\dim \boldsymbol{x}\times\mathbb{P}_k(D;\mathbb{R}^3) = \dim \mathbb{P}_{k+1}(D;\mathbb{R}^3)\cap\ker(\boldsymbol{x}\cdot)$，且 $\boldsymbol{x}\times\mathbb{P}_k(D;\mathbb{R}^3) \subseteq \mathbb{P}_{k+1}(D;\mathbb{R}^3)\cap\ker(\boldsymbol{x}\cdot)$，因此 $\boldsymbol{x} \times \mathbb{P}_k(D;\mathbb{R}^3) = \mathbb{P}_{k+1}(D;\mathbb{R}^3) \cap \ker(\boldsymbol{x}\cdot)$. 由此可得复形 (2.29) 的正合性。 □

复形 (2.28) 和 (2.29) 通过以下方式相连：

$$
\mathbb{R} \underset{\pi_0}{\overset{\subset}{\rightleftarrows}} \mathbb{P}_{k+2}(D) \underset{\boldsymbol{x}\cdot}{\overset{\mathrm{grad}}{\rightleftarrows}} \mathbb{P}_{k+1}(D;\mathbb{R}^3) \underset{\boldsymbol{x}\times}{\overset{\mathrm{curl}}{\rightleftarrows}} \mathbb{P}_k(D;\mathbb{R}^3) \underset{\boldsymbol{x}}{\overset{\mathrm{div}}{\rightleftarrows}} \mathbb{P}_{k-1}(D) \underset{\supset}{\rightleftarrows} 0\,. \tag{2.30}
$$

容易看出

$$
\mathbb{P}_{k+2}(D) = \boldsymbol{x} \cdot \mathbb{P}_{k+1}(D;\mathbb{R}^3) \oplus \mathbb{R}.
$$

接下来我们转向空间 $\mathbb{P}_{k+1}(D;\mathbb{R}^3)$。

**引理 2.11.** 设整数 $k \geq -1$ 且 $D$ 为三维可缩区域，则有以下空间直和分解：

$$
\mathbb{P}_{k+1}(D;\mathbb{R}^3) = \mathrm{grad}\,\mathbb{P}_{k+2}(D) \oplus \boldsymbol{x} \times \mathbb{P}_k(D;\mathbb{R}^3). \tag{2.31}
$$

*Proof.* 由于等式 (2.31) 左侧空间的维数等于右侧两空间维数之和，我们只需证明 (2.31) 中的和是直和。

对任意向量场 $\boldsymbol{v} \in \mathrm{grad}\,\mathbb{P}_{k+2}(D) \cap (\boldsymbol{x} \times \mathbb{P}_k(D;\mathbb{R}^3))$，设 $\boldsymbol{v} = \mathrm{grad}\,q$，其中 $q \in \mathbb{P}_{k+2}(D)$。由此可得 $(\boldsymbol{x}\cdot\nabla)q = 0$，故由 (2.21) 知 $q$ 为常值函数，从而 $\boldsymbol{v} = \boldsymbol{0}$。 □

**引理 2.12.** 设整数 $k \geq -1$ 且 $D$ 为三维可缩区域，则有以下空间直和分解：

$$
\mathbb{P}_k(D;\mathbb{R}^3) = \mathrm{curl}\,\mathbb{P}_{k+1}(D;\mathbb{R}^3) \oplus \boldsymbol{x}\mathbb{P}_{k-1}(D). \tag{2.32}
$$

*Proof.* 注意到 (2.32) 式左侧空间的维数等于右侧两个空间维数之和，我们只需证明该分解是直和。

为此，考虑交集 $\mathrm{curl}\,\mathbb{P}_{k+1}(D;\mathbb{R}^3)\cap\boldsymbol{x}\mathbb{P}_{k-1}(D)$ 中的任意向量场 $\boldsymbol{v}$。设 $\boldsymbol{v} = \boldsymbol{x}q$，其中 $q \in \mathbb{P}_{k-1}(D)$。由旋量场的性质，有 $\mathrm{div}(\boldsymbol{x}q) = 0$，应用 (2.22) 可得 $q = 0$，因此 $\boldsymbol{v} = \boldsymbol{0}$。 □

### 2.2.3 任意维多项式复形

借助外微分，可统一描述任意维空间中的多项式复形（包括多项式微分复形与 Koszul 复形），相关理论见有限元外微积分 [3, 6, 8]。这里采用反对称矩阵来表述部分相关结果。

**引理 2.13.** 设整数 $k \geq 1$ 且 $D$ 为 $d$ 维可缩区域，则成立以下空间直和分解：

$$
\mathbb{P}_{k-1}(D;\mathbb{R}^d) = \mathrm{grad}\,\mathbb{P}_k(D) \oplus (\ker(\cdot\boldsymbol{x}) \cap \mathbb{P}_{k-1}(D;\mathbb{R}^d)). \tag{2.33}
$$

Proof. 显然成立以下包含关系：

$$\operatorname{grad}\mathbb{P}_k(D)+(\ker(\cdot\boldsymbol{x})\cap\mathbb{P}_{k-1}(D;\mathbb{R}^d))\subseteq\mathbb{P}_{k-1}(D;\mathbb{R}^d).$$

根据公式 (2.21)，该和是直和。

映射 $\cdot\boldsymbol{x}:\mathbb{P}_{k-1}(D;\mathbb{R}^d)\to\mathbb{P}_k(D)\backslash\mathbb{R}$ 是满射，因此

$$\dim\mathbb{P}_{k-1}(D;\mathbb{R}^d)=\dim(\mathbb{P}_k(D)\backslash\mathbb{R})+\dim(\ker(\cdot\boldsymbol{x})\cap\mathbb{P}_{k-1}(D;\mathbb{R}^d)).$$

由于 $\dim(\mathbb{P}_k(D)\backslash\mathbb{R})=\dim\operatorname{grad}\mathbb{P}_k(D)$，通过维数计数即得分解 (2.33)。 □

文献 [64, Proposition 1] 中给出了 $\ker(\cdot\boldsymbol{x})\cap\mathbb{P}_{k-1}(D;\mathbb{R}^d)$ 的一种显式刻画，即：$\boldsymbol{v}\in\ker(\cdot\boldsymbol{x})\cap\mathbb{H}_{k-1}(D;\mathbb{R}^d)$ 当且仅当 $\boldsymbol{v}\in\mathbb{H}_{k-1}(D;\mathbb{R}^d)$ 且其 $(k-1)$ 阶导数 $\nabla^{k-1}\boldsymbol{v}$ 的对称部分为零。

我们将给出 $\ker(\cdot\boldsymbol{x})\cap\mathbb{P}_{k-1}(D;\mathbb{R}^d)$ 的另一种刻画。

**引理 2.14.** 设整数 $k\ge 1$ 且 $D$ 为 $d$ 维可缩区域。成立

$$\ker(\cdot\boldsymbol{x})\cap\mathbb{P}_{k-1}(D;\mathbb{R}^d)=\mathbb{P}_{k-2}(D;\mathbb{K})\boldsymbol{x},\tag{2.34}$$

其中 $\mathbb{K}$ 是 $\mathbb{R}^{d\times d}$ 中反对称矩阵子空间，以及以下空间直和分解：

$$\mathbb{P}_{k-1}(D;\mathbb{R}^d)=\operatorname{grad}\mathbb{P}_k(D)\oplus\mathbb{P}_{k-2}(D;\mathbb{K})\boldsymbol{x}.\tag{2.35}$$

Proof. 显然我们有 $\mathbb{P}_{k-2}(D;\mathbb{K})\boldsymbol{x}\subseteq\ker(\cdot\boldsymbol{x})\cap\mathbb{P}_{k-1}(D;\mathbb{R}^d)$。根据 (2.33)，只需证明 (2.35) 成立。

取 $\boldsymbol{q}\in\mathbb{P}_{k-1}(D;\mathbb{R}^d)$。不失一般性，由线性性质，只需考虑 $\boldsymbol{q}=\boldsymbol{x}^\alpha\boldsymbol{e}_\ell$ 的情形，其中 $|\alpha|=k-1$ 且 $1\le\ell\le d$。令

$$p=\frac{1}{k}\boldsymbol{x}\cdot\boldsymbol{q}=\frac{1}{k}\boldsymbol{x}^{\alpha+\boldsymbol{e}_\ell}\in\mathbb{P}_k(D),\quad \boldsymbol{\tau}=\frac{1}{k}\sum_{i=1}^d\alpha_i\boldsymbol{x}^{\alpha-\boldsymbol{e}_i}(\boldsymbol{e}_\ell\boldsymbol{e}_i^\intercal-\boldsymbol{e}_i\boldsymbol{e}_\ell^\intercal)\in\mathbb{P}_{k-2}(D;\mathbb{K}).$$

则有

$$\begin{aligned}\operatorname{grad}p+\boldsymbol{\tau}\boldsymbol{x}&=\frac{1}{k}\sum_{i=1}^d(\alpha_i+\delta_{i\ell})\boldsymbol{x}^{\alpha+\boldsymbol{e}_\ell-\boldsymbol{e}_i}\boldsymbol{e}_i+\frac{1}{k}\sum_{i=1}^d\alpha_i\boldsymbol{x}^{\alpha-\boldsymbol{e}_i}(\boldsymbol{e}_\ell x_i-\boldsymbol{e}_ix_\ell)\\&=\frac{1}{k}\sum_{i=1}^d(\alpha_i+\delta_{i\ell})\boldsymbol{x}^{\alpha+\boldsymbol{e}_\ell-\boldsymbol{e}_i}\boldsymbol{e}_i+\frac{1}{k}\sum_{i=1}^d\alpha_i\boldsymbol{x}^{\alpha}\boldsymbol{e}_\ell-\frac{1}{k}\sum_{i=1}^d\alpha_i\boldsymbol{x}^{\alpha+\boldsymbol{e}_\ell-\boldsymbol{e}_i}\boldsymbol{e}_i\\&=\frac{1}{k}\boldsymbol{x}^\alpha\boldsymbol{e}_\ell+\frac{|\alpha|}{k}\boldsymbol{x}^\alpha\boldsymbol{e}_\ell=\boldsymbol{x}^\alpha\boldsymbol{e}_\ell,\end{aligned}$$

即 $\boldsymbol{q}=\operatorname{grad}p+\boldsymbol{\tau}\boldsymbol{x}$。因此，成立

$$\mathbb{P}_{k-1}(D;\mathbb{R}^d)\subseteq\operatorname{grad}\mathbb{P}_k(D)\oplus\mathbb{P}_{k-2}(D;\mathbb{K})\boldsymbol{x}.$$

结合已知关系 $\operatorname{grad}\mathbb{P}_k(D)\oplus\mathbb{P}_{k-2}(D;\mathbb{K})\boldsymbol{x}\subseteq\mathbb{P}_{k-1}(D;\mathbb{R}^d)$，即得 (2.35)。 □

**引理 2.15.** 令整数 $k\ge 1$。设 $\boldsymbol{w}\in\mathbb{P}_{k-2}(D;\mathbb{K})\boldsymbol{x}$ 满足 $(\operatorname{skw}\operatorname{grad}\boldsymbol{w})\boldsymbol{x}=\boldsymbol{0}$，其中反对称算子 $\operatorname{skw}:\mathbb{R}^{d\times d}\to\mathbb{K}$ 定义为 $\operatorname{skw}\boldsymbol{\tau}:=\frac{1}{2}(\boldsymbol{\tau}-\boldsymbol{\tau}^\intercal)$。则有 $\boldsymbol{w}=\boldsymbol{0}$。

Proof. 由于

$$(\operatorname{skw}\operatorname{grad}\boldsymbol{w})\boldsymbol{x}=\frac{1}{2}(\operatorname{grad}\boldsymbol{w})\boldsymbol{x}-\frac{1}{2}(\operatorname{grad}\boldsymbol{w})^\intercal\boldsymbol{x}=\frac{1}{2}(I+\boldsymbol{x}\cdot\operatorname{grad})\boldsymbol{w}-\frac{1}{2}\operatorname{grad}(\boldsymbol{w}\cdot\boldsymbol{x}),$$

由 $\boldsymbol{w}\cdot\boldsymbol{x}=0$ 可得 $(I+\boldsymbol{x}\cdot\operatorname{grad})\boldsymbol{w}=\boldsymbol{0}$，结合(2.23)式即得 $\boldsymbol{w}=\boldsymbol{0}$。 □

**引理 2.16.** 设整数 $k \geq 1$ 且 $D$ 为 $d$ 维可缩区域。多项式复形

$$\mathbb{R} \to \mathbb{P}_k(D) \xrightarrow{\text{grad}} \mathbb{P}_{k-1}(D;\mathbb{R}^d) \xrightarrow{\text{skw grad}} \mathbb{P}_{k-2}(D;\mathbb{K}) \tag{2.36}$$

是正合的。

*Proof.* 显然(2.36)是一个复形。只需证明 $\mathbb{P}_{k-1}(D;\mathbb{R}^d) \cap \ker(\text{skw grad}) \subseteq \text{grad}\,\mathbb{P}_k(D)$。

对于任意 $\boldsymbol{v} \in \mathbb{P}_{k-1}(D;\mathbb{R}^d) \cap \ker(\text{skw grad})$，根据空间分解 (2.35)，存在 $q \in \mathbb{P}_k(D)$ 和 $\boldsymbol{w} \in \mathbb{P}_{k-2}(D;\mathbb{K})\boldsymbol{x}$ 使得 $\boldsymbol{v} = \text{grad}\,q + \boldsymbol{w}$。由 $\text{skw grad}\,\boldsymbol{v} = \mathbf{0}$ 可得 $\text{skw grad}\,\boldsymbol{w} = \mathbf{0}$。应用引理 2.15 即得 $\boldsymbol{w} = \mathbf{0}$。因此 $\boldsymbol{v} = \text{grad}\,q \in \text{grad}\,\mathbb{P}_k(D)$。 □

空间分解 (2.33) 与刻画 (2.34) 可总结为以下双向复形:

$$\mathbb{R} \underset{\pi_0}{\overset{\subseteq}{\rightleftarrows}} \mathbb{P}_k(D) \underset{\boldsymbol{v}\cdot\boldsymbol{x}}{\overset{\text{grad}}{\rightleftarrows}} \mathbb{P}_{k-1}(D;\mathbb{R}^d) \underset{\boldsymbol{\tau}\boldsymbol{x}}{\overset{\text{skw grad}}{\rightleftarrows}} \mathbb{P}_{k-2}(D;\mathbb{K}).$$

下面是反对称矩阵值多项式空间的一个直和分解。

**引理 2.17.** 设整数 $k \geq 1$ 且 $D$ 为 $d$ 维可缩区域。成立空间直和分解

$$\mathbb{P}_{k-2}(D;\mathbb{K}) = \text{skw grad}\,\mathbb{P}_{k-1}(D;\mathbb{R}^d) \oplus \big(\mathbb{P}_{k-2}(D;\mathbb{K}) \cap \ker(\boldsymbol{x})\big), \tag{2.37}$$

其中 $\mathbb{P}_{k-2}(D;\mathbb{K}) \cap \ker(\boldsymbol{x}) := \{\boldsymbol{\tau} \in \mathbb{P}_{k-2}(D;\mathbb{K}) : \boldsymbol{\tau}\boldsymbol{x} = \mathbf{0}\}$.

*Proof.* 根据空间直和分解 (2.35)，我们有

$$\text{skw grad}\,\mathbb{P}_{k-1}(D;\mathbb{R}^d) = \text{skw grad}(\mathbb{P}_{k-2}(D;\mathbb{K})\boldsymbol{x}).$$

由引理 2.15，$\text{skw grad}\,\mathbb{P}_{k-1}(D;\mathbb{R}^d) \cap \big(\mathbb{P}_{k-2}(D;\mathbb{K}) \cap \ker(\boldsymbol{x})\big) = \mathbf{0}$。因此只需验证维数关系。根据复形 (2.36)，

$$\dim \text{skw grad}\,\mathbb{P}_{k-1}(D;\mathbb{R}^d) = \dim \mathbb{P}_{k-1}(D;\mathbb{R}^d) - \dim \text{grad}\,\mathbb{P}_k(D)。 \tag{2.38}$$

另一方面，由空间分解 (2.35) 可得

$$\dim \mathbb{P}_{k-2}(D;\mathbb{K})\boldsymbol{x} = \dim \mathbb{P}_{k-1}(D;\mathbb{R}^d) - \dim \text{grad}\,\mathbb{P}_k(D).$$

因此，$\dim \text{skw grad}\,\mathbb{P}_{k-1}(D;\mathbb{R}^d) = \dim \mathbb{P}_{k-2}(D;\mathbb{K})\boldsymbol{x}$，由此即得 (2.37)。 □

由 (2.37) 和 (2.38) 可得

$$\dim \mathbb{P}_{k-2}(D;\mathbb{K}) \cap \ker(\boldsymbol{x}) = \dim \mathbb{P}_{k-2}(D;\mathbb{K}) + \dim \text{grad}\,\mathbb{P}_k(D) - \dim \mathbb{P}_{k-1}(D;\mathbb{R}^d). \tag{2.39}$$

接下来，给出空间 $\mathbb{P}_{k-1}(D;\mathbb{R}^d)$ 与 (2.35) 对偶的直和分解。

**引理 2.18.** 设整数 $k \geq 1$ 且 $D$ 为 $d$ 维可缩区域，则成立以下空间直和分解:

$$\mathbb{P}_{k-1}(D;\mathbb{R}^d) = (\mathbb{P}_{k-1}(D;\mathbb{R}^d) \cap \ker(\text{div})) \oplus (\mathbb{P}_{k-2}(D)\boldsymbol{x}). \tag{2.40}$$

*Proof.* 显然成立以下包含关系:

$$(\mathbb{P}_{k-1}(D;\mathbb{R}^d) \cap \ker(\text{div})) \oplus (\mathbb{P}_{k-2}(D)\boldsymbol{x}) \subseteq \mathbb{P}_{k-1}(D;\mathbb{R}^d).$$

根据公式 (2.22)，该和是直和。

又

$$\begin{aligned}\dim(\mathbb{P}_{k-1}(D;\mathbb{R}^d)\cap\ker(\mathrm{div})) &= \dim\mathbb{P}_{k-1}(D;\mathbb{R}^d) - \dim\mathrm{div}\,\mathbb{P}_{k-1}(D;\mathbb{R}^d)\\ &= \dim\mathbb{P}_{k-1}(D;\mathbb{R}^d) - \dim\mathbb{P}_{k-2}(D),\end{aligned}$$

通过比较可得分解 (2.40)。 □

**引理 2.19.** *设整数 $k\geq 1$ 且 $D$ 为 $d$ 维可缩区域。成立*

$$\mathbb{P}_{k-1}(D;\mathbb{R}^d)\cap\ker(\mathrm{div}) = \mathrm{div}\,\mathbb{P}_k(D;\mathbb{K}), \tag{2.41}$$

*以及以下空间直和分解:*

$$\mathbb{P}_{k-1}(D;\mathbb{R}^d) = (\mathrm{div}\,\mathbb{P}_k(D;\mathbb{K}))\oplus(\mathbb{P}_{k-2}(D)\boldsymbol{x}). \tag{2.42}$$

*Proof.* 根据空间分解 (2.40)，只需证明 (2.41) 即可。

对任意 $\boldsymbol{q}\in\mathbb{P}_{k-1}(D;\mathbb{R}^d)\cap\ker(\mathrm{div})$,

$$\mathrm{div}(2\,\mathrm{skw}(\boldsymbol{x}\otimes\boldsymbol{q})) = \mathrm{div}(\boldsymbol{x}\otimes\boldsymbol{q}) - \mathrm{div}(\boldsymbol{q}\otimes\boldsymbol{x}) = -(d-1+\boldsymbol{x}\cdot\mathrm{grad})\boldsymbol{q}.$$

这结合 (2.23) 表明 $\mathrm{div}(2\,\mathrm{skw}(\boldsymbol{x}\otimes\cdot))$ 在空间 $\mathbb{P}_{k-1}(D;\mathbb{R}^d)\cap\ker(\mathrm{div})$ 上是单射算子。因此,

$$\dim\mathrm{div}\,\mathbb{P}_k(D;\mathbb{K}) \geq \dim(\mathbb{P}_{k-1}(D;\mathbb{R}^d)\cap\ker(\mathrm{div})).$$

又显然有 $\mathrm{div}\,\mathbb{P}_k(D;\mathbb{K})\subseteq\mathbb{P}_{k-1}(D;\mathbb{R}^d)\cap\ker(\mathrm{div})$，故等式 (2.41) 成立。 □

**引理 2.20.** *设整数 $k\geq 1$ 且 $D$ 为 $d$ 维可缩区域。多项式复形*

$$\mathbb{P}_k(D;\mathbb{K}) \xrightarrow{\mathrm{div}} \mathbb{P}_{k-1}(D;\mathbb{R}^d) \xrightarrow{\mathrm{div}} \mathbb{P}_{k-2}(D) \to 0 \tag{2.43}$$

*是正合的。*

*Proof.* 显然 (2.43) 构成复形。根据 (2.22) 有 $\mathrm{div}\,\mathbb{P}_{k-1}(D;\mathbb{R}^d)=\mathbb{P}_{k-2}(D)$，再结合 (2.41) 即可得证。 □

在复形 (2.43) 的基础上，进一步有以下双向复形:

$$\mathbb{P}_k(D;\mathbb{K}) \underset{\mathrm{skw}(\boldsymbol{v}\otimes\boldsymbol{x})}{\overset{\mathrm{div}}{\rightleftarrows}} \mathbb{P}_{k-1}(D;\mathbb{R}^d) \underset{q\boldsymbol{x}}{\overset{\mathrm{div}}{\rightleftarrows}} \mathbb{P}_{k-2}(D) \underset{\supseteq}{\rightleftarrows} 0.$$

## 2.3 $t$-$n$ bases

For an $\ell$-dimensional sub-simplex $f\in\Delta_\ell(T)$, choose $\ell$ linearly independent tangential vectors $\{\boldsymbol{t}_1^f,\ldots,\boldsymbol{t}_\ell^f\}$ of $f$ and $d-\ell$ linearly independent normal vectors $\{\boldsymbol{n}_1^f,\ldots,\boldsymbol{n}_{d-\ell}^f\}$ of $f$. The set of $d$ vectors $\{\boldsymbol{t}_1^f,\ldots,\boldsymbol{t}_\ell^f,\boldsymbol{n}_1^f,\ldots,\boldsymbol{n}_{d-\ell}^f\}$ forms a basis of $\mathbb{R}^d$. Notice that for $\ell=0$, i.e., at vertices, there are no tangential vectors, and for $\ell=d$, there are no normal vectors. Define the tangent plane and normal plane of $f$ as

$$\mathscr{T}^f := \mathrm{span}\{\boldsymbol{t}_i^f, i=1,\ldots,\ell\}, \quad \mathscr{N}^f := \mathrm{span}\{\boldsymbol{n}_i^f, i=1,\ldots,d-\ell\}.$$

All vectors are normalized but $\{\boldsymbol{t}_i^f\}$ or $\{\boldsymbol{n}_i^f\}$ may not form an orthonormal basis.

Inside the subspace $\mathscr{T}^f$, we can find a basis $\{\hat{\boldsymbol{t}}_1^f,\dots,\hat{\boldsymbol{t}}_\ell^f\}$ dual to $\{\boldsymbol{t}_1^f,\dots,\boldsymbol{t}_\ell^f\}$, i.e., $\hat{\boldsymbol{t}}_i \in \mathscr{T}^f$ and $(\hat{\boldsymbol{t}}_i,\boldsymbol{t}_j)=\delta_{i,j}$ for $i,j=1,\dots,\ell$. Similarly we have a basis $\{\hat{\boldsymbol{n}}_1^f,\dots,\hat{\boldsymbol{n}}_{d-\ell}^f\}$ of $\mathscr{N}^f$ and $(\hat{\boldsymbol{n}}_i,\boldsymbol{n}_j)=\delta_{i,j}$ for $i,j=1,\dots,d-\ell$. As $\mathscr{T}^f\perp\mathscr{N}^f$, the basis $\{\hat{\boldsymbol{t}}_1^f,\dots,\hat{\boldsymbol{t}}_\ell^f,\hat{\boldsymbol{n}}_1^f,\dots,\hat{\boldsymbol{n}}_{d-\ell}^f\}$ is also dual to $\{\boldsymbol{t}_1^f,\dots,\boldsymbol{t}_\ell^f,\boldsymbol{n}_1^f,\dots,\boldsymbol{n}_{d-\ell}^f\}$.

Given a sub-simplex $f\in\Delta_\ell(T)$, we now present two bases for its normal plane $\mathscr{N}^f$ constructed in [?].

Recall that we label $F_i$ as the $(d-1)$-dimensional face opposite to the $i$-th vertex. Then $f\subseteq F_i$ for $i\in f^*$. One basis is composed by unit normal vectors of all such $(d-1)$-dimensional faces:

$$\{\boldsymbol{n}_{F_i}, i\in f^*\},$$

and will be called the face normal basis.

We now give its dual basis in $\mathscr{N}^f$. For $f\in\Delta_\ell(T),\ell=0,1,\dots,n-1$ and $i\in f^*$, let $f\cup\{i\}$ denotes the $(\ell+1)$-dimensional face in $\Delta_{\ell+1}(T)$ with vertices $\{i,f(0),\dots,f(\ell)\}$. Let $\boldsymbol{n}_{f\cup\{i\}}^f$ be a unit normal vector of $f$ but tangential to $f\cup\{i\}$. The basis

$$\{\boldsymbol{n}_{f\cup\{i\}}^f, i\in f^*\}$$

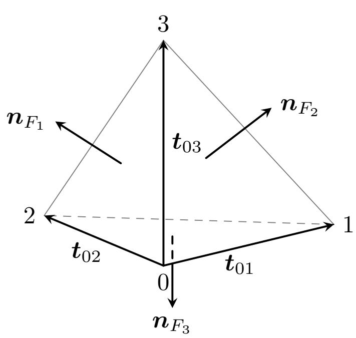
will be called the tangential normal basis.

**引理 2.21.** For $f\in\Delta_\ell(T)$, the rescaled tangential normal basis $\{\boldsymbol{n}_{f\cup\{i\}}^f/(\boldsymbol{n}_{f\cup\{i\}}^f\cdot\boldsymbol{n}_{F_i}), i\in f^*\}$ of $\mathscr{N}^f$ is dual to the face normal basis $\{\boldsymbol{n}_{F_i}, i\in f^*\}$.

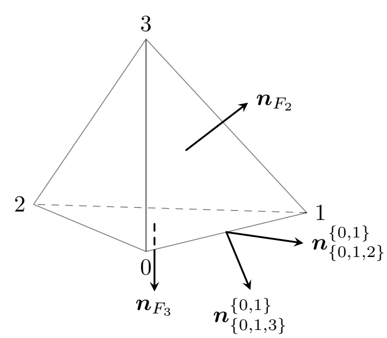
*Proof.* Clearly $\boldsymbol{n}_{f\cup\{i\}}^f,\boldsymbol{n}_{F_i}\in\mathscr{N}^f$ for $i\in f^*$. It suffices to prove

$$\boldsymbol{n}_{f\cup\{i\}}^f\cdot\boldsymbol{n}_{F_j}=0 \quad \text{for } i,j\in f^*, i\neq j,$$

which follows from the fact $f\cup\{i\}\subseteq F_j$ and $\boldsymbol{n}_{f\cup\{i\}}^f\in\mathscr{T}^{f\cup\{i\}}$. □




(a) Basis $\{\boldsymbol{t}_{0,1},\boldsymbol{t}_{0,2},\boldsymbol{t}_{0,3}\}$ and $\{\nabla\lambda_1,\nabla\lambda_2,\nabla\lambda_3\}$.




(b) Basis $\{\boldsymbol{n}_{F_2},\boldsymbol{n}_{F_3}\}$ and $\{\boldsymbol{n}_{\{0,1,2\}}^{\{0,1\}},\boldsymbol{n}_{\{0,1,3\}}^{\{0,1\}}\}$.

Figure 2.1: Face normal basis and tangential normal basis of a vertex and an edge in a tetrahedron.

**Example 2.1.** An important example is $f\in\Delta_0(T)$, i.e., $f$ is a vertex. Without loss of generality, let $f=\{0\}$. Then $\boldsymbol{n}_{f\cup\{i\}}^f$ is a unit normal vector of edge $\{0,i\}$: $\boldsymbol{t}_{0i}$ or $\boldsymbol{t}_{i0}$ depending on the orientation. Its dual basis is $\{\boldsymbol{n}_{F_i}/(\boldsymbol{n}_{F_i}\cdot\boldsymbol{t}_{0i}), i=1,\dots,n\}$. See Fig. 2.1 (a).

**Example 2.2.** Let $f=\{0,1\}$ be an edge of a tetrahedron. Then we have two bases of the normal plane $\mathscr{N}^f$: $\{\boldsymbol{n}_{F_2},\boldsymbol{n}_{F_3}\}$ and $\{\boldsymbol{n}_{\{0,1,2\}}^{\{0,1\}},\boldsymbol{n}_{\{0,1,3\}}^{\{0,1\}}\}$. They are dual to each other with an appropriate rescaling. See Fig. 2.1 (b).

## 2.4 $H(\mathrm{div})$ 有限元

在本节中，我们考虑单纯形上 $H(\mathrm{div})$-协调有限元的构造。

### 2.4.1 $H(\mathrm{div})$ 空间和迹算子

回顾 Sobolev 空间

$$H(\mathrm{div},\Omega) := \{\boldsymbol{v} \in L^2(\Omega;\mathbb{R}^d) : \mathrm{div}\,\boldsymbol{v} \in L^2(\Omega)\}.$$

其范数平方为 $\|\boldsymbol{v}\|_{\mathrm{div},\Omega}^2 := \|\boldsymbol{v}\|_{0,\Omega}^2 + \|\mathrm{div}\,\boldsymbol{v}\|_{0,\Omega}^2$.

**引理 2.22.** 设 $\boldsymbol{v} \in H(\mathrm{div},\Omega)$，则 $\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial\Omega} \in H^{-1/2}(\partial\Omega)$ 且格林公式成立：

$$(\mathrm{div}\,\boldsymbol{v}, w)_\Omega + (\boldsymbol{v}, \mathrm{grad}\,w)_\Omega = \langle \boldsymbol{v}\cdot\boldsymbol{n}, w\rangle_{H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)} \quad \forall\, w \in H^1(\Omega).$$

**引理 2.23.** 迹算子 $\boldsymbol{v} \in H(\mathrm{div},\Omega) \to \boldsymbol{v}\cdot\boldsymbol{n}|_\Gamma \in H^{-1/2}(\Gamma)$ 是满射。

### 2.4.2 Raviart-Thomas 元

设 $T \subset \mathbb{R}^d$ 是一个 $d$ 维单纯形，其中整数 $d \geq 2$. Raviart-Thomas (RT) 元 [67, 64, 31] 的形函数空间为: $k \geq 0$,

$$V_k^{\mathrm{RT}}(T) := \mathbb{P}_k(T;\mathbb{R}^d) + \boldsymbol{x}\mathbb{P}_k(T) = \mathbb{P}_k(T;\mathbb{R}^d) \oplus \boldsymbol{x}\mathbb{H}_k(T).$$

根据空间直和分解 (2.42)，成立直和分解

$$V_k^{\mathrm{RT}}(T) = (\mathrm{div}\,\mathbb{P}_{k+1}(T;\mathbb{K})) \oplus (\boldsymbol{x}\mathbb{P}_k(T)).$$

从而，由 (2.22) 可得

$$V_k^{\mathrm{RT}}(T) \cap \ker(\mathrm{div}) = \mathbb{P}_k(T;\mathbb{R}^d) \cap \ker(\mathrm{div}) = \mathrm{div}\,\mathbb{P}_{k+1}(T;\mathbb{K}).$$

自由度取为

$$(\boldsymbol{v}\cdot\boldsymbol{n}, q)_F, \quad q \in \mathbb{P}_k(F), F \in \Delta_{d-1}(T), \tag{2.44a}$$

$$(\boldsymbol{v}, \boldsymbol{q})_T, \quad \boldsymbol{q} \in \mathbb{P}_{k-1}(T;\mathbb{R}^d). \tag{2.44b}$$

**引理 2.24.** 设 $\boldsymbol{v} \in \mathbb{P}_k(T;\mathbb{R}^d)$ 满足

$$(\boldsymbol{v}, \boldsymbol{q})_T = 0 \quad \forall\, \boldsymbol{q} \in \mathbb{P}_{k-1}(T;\mathbb{R}^d), \tag{2.45}$$

则 $\boldsymbol{v} = \boldsymbol{0}$。

*Proof.* 为表述简便，记 $\Delta_{d-1}(T)$ 中的 $d+1$ 个面为 $F_i$，其中 $F_i$ 与 $T$ 的第 $i$ 个顶点相对，$\boldsymbol{n}_i$ 为 $F_i$ 的单位外法向量，$i = 0, 1, \cdots, d$。令 $\boldsymbol{t}_i$ $(i = 1, \ldots, d)$ 表示从顶点 0 指向顶点 $i$ 的单位切向量，则 $d$ 个向量 $\boldsymbol{t}_1, \ldots, \boldsymbol{t}_d$ 构成 $\mathbb{R}^d$ 的一组基（尽管它们一般不正交）。于是有

$$\boldsymbol{v} = \sum_{i=1}^d \frac{\boldsymbol{v}\cdot\boldsymbol{n}_i}{\boldsymbol{t}_i\cdot\boldsymbol{n}_i}\boldsymbol{t}_i.$$

由于 $\boldsymbol{v}\cdot\boldsymbol{n}_i|_{F_i} = 0$，存在 $p_i \in \mathbb{P}_{k-1}(T)$ 使得 $\boldsymbol{v}\cdot\boldsymbol{n}_i = \lambda_i p_i$。在 (2.45) 中取 $\boldsymbol{q} = \boldsymbol{n}_i p_i$ 可得

$$(\lambda_i p_i, p_i)_T = 0.$$

因此 $p_i = 0$，进而 $\boldsymbol{v}\cdot\boldsymbol{n}_i = 0$，最终得到 $\boldsymbol{v} = 0$。 □

**引理 2.25.** 自由度 (2.44) 对于形函数空间 $V_k^{\mathrm{RT}}(T)$ 是唯一可解的。

Proof. 根据 $\dim\mathbb{P}_k(F)+\dim\mathbb{P}_{k-1}(T)=\dim\mathbb{P}_k(T)$ 和 $\dim\mathbb{P}_k(F)=\dim\mathbb{H}_k(T)$ 可知，自由度 (2.44) 的总数为

$$(d+1)\dim\mathbb{P}_k(F)+d\dim\mathbb{P}_{k-1}(T)=\dim\mathbb{P}_k(F)+d\dim\mathbb{P}_k(T)=\dim V_k^{\mathrm{RT}}(T).$$

设 $\boldsymbol{v}\in V_k^{\mathrm{RT}}(T)$ 满足所有自由度 (2.44) 取值为零。利用分部积分及自由度 (2.44) 的零值条件，可得 $(\boldsymbol{v}\cdot\boldsymbol{n})|_{\partial T}=0$，且

$$(\operatorname{div}\boldsymbol{v},q)_T=(\boldsymbol{v}\cdot\boldsymbol{n},q)_{\partial T}-(\boldsymbol{v},\operatorname{grad}q)_T=0\quad\forall\, q\in\mathbb{P}_k(T).$$

因此 $\operatorname{div}\boldsymbol{v}=0$，进而 $\boldsymbol{v}\in\mathbb{P}_k(T;\mathbb{R}^d)$。再应用引理 2.24，由自由度 (2.44b) 为零即得 $\boldsymbol{v}=0$。 □

根据空间分解 (2.35)，自由度 (2.44) 可等价地改写为

$$(\boldsymbol{v}\cdot\boldsymbol{n},q)_F,\quad q\in\mathbb{P}_k(F),F\in\Delta_{d-1}(T),\tag{2.46a}$$
$$(\operatorname{div}\boldsymbol{v},q)_T,\quad q\in\mathbb{P}_k(T)/\mathbb{R},\tag{2.46b}$$
$$(\boldsymbol{v},\boldsymbol{q})_T,\quad \boldsymbol{q}\in\mathbb{P}_{k-2}(T;\mathbb{K})\boldsymbol{x}.\tag{2.46c}$$

自由度 (2.46) 阐明了有限元的一般构造思路：边界自由度 (2.46a) 由 Sobolev 空间的迹决定，确保 $H(\operatorname{div})$-协调性；内部自由度 (2.46b) 对应散度算子的像空间，体现微分复形结构；内部自由度 (2.46c) 对应泡函数空间或多项式空间的直和分解。基于微分复形与 Sobolev 空间迹的这一构造方法，同样可推广至其他 Sobolev 空间的有限元设计，详见文献 [31]。

最低次 RT 元 ($k=0$) 的基函数为 [16, Section 2.6]

$$\phi_i=(\boldsymbol{n}_{F_i}\cdot\nabla\lambda_i)(\mathtt{v}_i-\boldsymbol{x}),\quad i=0,1,\ldots,d,$$

其中 $\boldsymbol{n}_{F_i}$ 是 $(d-1)$ 维面 $F_i$ 的单位法向量，仅依赖于面 $F_i$，不依赖于所属的单纯形。

任意次 RT 元的基函数可参见文献 [7, Section 9, Tables 9.1-9.2]。事实上，任意维、任意次 RT 元具有如下几何分解（参见 [7]）

$$\mathbb{P}_k(T;\mathbb{R}^d)+\mathbb{P}_k(T)\boldsymbol{x}=\textstyle\bigoplus_{i=0}^d\mathbb{P}_k(F_i)(\mathtt{v}_i-\boldsymbol{x})\oplus\bigoplus_{i=1}^d\lambda_i\mathbb{P}_{k-1}(T)(\mathtt{v}_i-\boldsymbol{x}).$$

据此几何分解，可构造出 RT 元的基函数。

### 2.4.3 Brezzi-Douglas-Marini 元

Brezzi-Douglas-Marini (BDM) 元 [23, 22, 65, 31, 29, 33] 的形函数空间为 $V_k^{\mathrm{BDM}}(T):=\mathbb{P}_k(T;\mathbb{R}^d)$，其中 $k\geq 1$。自由度取为

$$(\boldsymbol{v}\cdot\boldsymbol{n},q)_F,\quad q\in\mathbb{P}_k(F),F\in\Delta_{d-1}(T),\tag{2.47a}$$
$$(\boldsymbol{v},\boldsymbol{q})_T,\quad \boldsymbol{q}\in\operatorname{grad}\mathbb{P}_{k-1}(T)\oplus\mathbb{P}_{k-2}(T;\mathbb{K})\boldsymbol{x}.\tag{2.47b}$$

**引理 2.26.** 自由度 (2.47) 对于形函数空间 $V_k^{\mathrm{BDM}}(T)$ 是唯一可解的。

Proof. 根据空间分解 (2.35)、关系式 $\dim\mathbb{P}_k(F)=\dim\mathbb{H}_k(T)$ 及 $\dim\mathbb{P}_k(F)+\dim\mathbb{P}_{k-1}(T)=\dim\mathbb{P}_k(T)$ 可知，自由度 (2.47) 的总数为

$$(d+1)\dim\mathbb{P}_k(F)+d\dim\mathbb{P}_{k-1}(T)-\dim\mathbb{H}_k(T)=d\dim\mathbb{P}_k(F)+d\dim\mathbb{P}_{k-1}(T)=\dim V_k^{\mathrm{BDM}}(T).$$

设 $\boldsymbol{v}\in V_k^{\mathrm{BDM}}(T)$ 满足所有自由度 (2.47) 取值为零。利用分部积分及自由度 (2.47) 的零值条件，可得 $(\boldsymbol{v}\cdot\boldsymbol{n})|_{\partial T}=0$，且

$$(\mathrm{div}\,\boldsymbol{v},q)_T=(\boldsymbol{v}\cdot\boldsymbol{n},q)_{\partial T}-(\boldsymbol{v},\mathrm{grad}\,q)_T=0\quad\forall\, q\in\mathbb{P}_{k-1}(T).$$

因此 $\mathrm{div}\,\boldsymbol{v}=0$，进而由分部积分得

$$(\boldsymbol{v},\mathrm{grad}\,q)_T=0\quad\forall\, q\in\mathbb{P}_k(T).$$

结合自由度 (2.47b) 的零值条件，可得

$$(\boldsymbol{v},\boldsymbol{q})_T=0\quad\forall\,\boldsymbol{q}\in\mathbb{P}_{k-1}(T;\mathbb{R}^d).$$

再应用引理 2.24 即得 $\boldsymbol{v}=0$。 □

接下来我们给出当 $k\geq 2$ 时 $H(\mathrm{div})$ 泡函数空间的显式刻画。定义多项式泡函数空间

$$\mathbb{B}_k(\mathrm{div},T):=\{\boldsymbol{v}\in\mathbb{P}_k(T;\mathbb{R}^d):\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial K}=0\}.$$

由 BDM 元的唯一可解性知，

$$\dim\mathbb{B}_k(\mathrm{div},T)=d\binom{k+d}{d}-(d+1)\binom{k+d-1}{d-1}=(k-1)\binom{k+d-1}{d-1}.$$

**引理** 2.27. *对于 $k\geq 2$ 成立*

$$\mathbb{B}_k(\mathrm{div},T)=\sum_{0\leq i<j\leq d}\lambda_i\lambda_j\mathbb{P}_{k-2}(T)\boldsymbol{t}_{i,j},\tag{2.48}$$

*其中 $\boldsymbol{t}_{i,j}:=\mathrm{v}_j-\mathrm{v}_i$，$\mathrm{v}_0,\mathrm{v}_1,\cdots,\mathrm{v}_d$ 是单纯形 $T$ 的顶点。*

Proof. 验证 $\lambda_i\lambda_j\mathbb{P}_{k-2}(T)\boldsymbol{t}_{i,j}\subseteq\mathbb{B}_k(\mathrm{div},T)$ 的依据是以下事实：

$$\lambda_i\lambda_j\boldsymbol{t}_{i,j}\cdot\boldsymbol{n}_\ell|_{F_\ell}=0,\quad \ell=0,1,\cdots,d.$$

事实上，若 $\ell=i$ 或 $\ell=j$，则 $\lambda_i\lambda_j|_{F_\ell}=0$；否则 $\boldsymbol{t}_{i,j}\cdot\boldsymbol{n}_\ell=0$。接下来证明 $\mathbb{B}_k(\mathrm{div},T)$ 中的每个函数均可表示为 $\lambda_i\lambda_j\boldsymbol{t}_{i,j}$ 的线性组合。

任取 $\boldsymbol{v}\in\mathbb{B}_k(\mathrm{div},T)$。由于 $\{\boldsymbol{t}_{0,1},\ldots,\boldsymbol{t}_{0,d}\}$ 构成 $\mathbb{R}^d$ 的一组基，故 $\boldsymbol{v}$ 可表示为

$$\boldsymbol{v}=p_1\boldsymbol{t}_{0,1}+\cdots+p_d\boldsymbol{t}_{0,d},$$

其中 $p_1,\ldots,p_d\in\mathbb{P}_k(T)$。注意到 $\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial T}=0$，因此

$$p_i\boldsymbol{t}_{0,i}\cdot\boldsymbol{n}_i|_{F_i}=0\quad(i=1,\ldots,d).$$

由此可知 $p_i|_{F_i}=0$ 对 $i=1,\ldots,d$ 成立，于是可以令

$$\boldsymbol{v}=\sum_{i=1}^d\lambda_iq_i\boldsymbol{t}_{0,i},\quad q_i\in\mathbb{P}_{k-1}(T).$$

进一步将 $q_i$ 展开为

$$q_i = \lambda_0 q_0 + \sum_{j=1}^{d} \lambda_j q_{ij},$$

其中

$$q_0 \in \mathbb{P}_{k-2}(T), \quad q_{ij} \in \sum_{\substack{\alpha_j+\cdots+\alpha_d=k-2\\ \alpha_j,\cdots,\alpha_d\in\mathbb{N}}} \mathrm{span}\{\lambda_j^{\alpha_j}\cdots\lambda_d^{\alpha_d}\},$$

并且 $q_{ij}$ 的表达中不包含 $\lambda_0,\lambda_1,\ldots,\lambda_{j-1}$。

由 $\boldsymbol{n}\cdot\boldsymbol{v}|_{F_0}=0$ 且 $\boldsymbol{t}_{0,i}\cdot\boldsymbol{n}_0\neq 0$，可得

$$\sum_{i=1}^{d}\lambda_i q_i|_{F_0}=0, \quad 即 \quad \sum_{i=1}^{d}\sum_{j=1}^{d}(\lambda_i\lambda_j q_{ij})|_{F_0}=0 \quad \Leftrightarrow \quad \sum_{i=1}^{d}\sum_{j=1}^{d}\lambda_i\lambda_j q_{ij}=0.$$

由于 $q_{ij}$ 的表达中不含 $\lambda_0,\lambda_1,\ldots,\lambda_{j-1}$，利用数学归纳法可证

$$\sum_{i=\ell}^{d}\sum_{j=\ell}^{d}\lambda_i\lambda_j q_{ij}=0, \quad \ell=1,\ldots,d.$$

可进一步得到

$$\sum_{j=i}^{d}\lambda_j q_{ij}+\sum_{j=i+1}^{d}\lambda_j q_{ji}=0, \quad i=1,\ldots,d.$$

由此可得

$$\sum_{i=1}^{d}\sum_{j=i}^{d}\lambda_i\lambda_j q_{ij}\boldsymbol{t}_{0,i}=-\sum_{i=1}^{d-1}\sum_{j=i+1}^{d}\lambda_i\lambda_j q_{ji}\boldsymbol{t}_{0,i}=-\sum_{j=1}^{d-1}\sum_{i=j+1}^{d}\lambda_i\lambda_j q_{ij}\boldsymbol{t}_{0,j}=-\sum_{i=2}^{d}\sum_{j=1}^{i-1}\lambda_i\lambda_j q_{ij}\boldsymbol{t}_{0,j}.$$

因此

$$\begin{aligned}
\boldsymbol{v} &= \sum_{i=1}^{d}\lambda_i q_i\boldsymbol{t}_{0,i}=\sum_{i=1}^{d}\lambda_i\lambda_0 q_0\boldsymbol{t}_{0,i}+\sum_{i=1}^{d}\sum_{j=1}^{d}\lambda_i\lambda_j q_{ij}\boldsymbol{t}_{0,i}\\
&=\sum_{i=1}^{d}\lambda_i\lambda_0 q_0\boldsymbol{t}_{0,i}+\sum_{i=2}^{d}\sum_{j=1}^{i-1}\lambda_i\lambda_j q_{ij}\boldsymbol{t}_{0,i}+\sum_{i=1}^{d}\sum_{j=i}^{d}\lambda_i\lambda_j q_{ij}\boldsymbol{t}_{0,i}\\
&=\sum_{i=1}^{d}\lambda_i\lambda_0 q_0\boldsymbol{t}_{0,i}+\sum_{i=2}^{d}\sum_{j=1}^{i-1}\lambda_i\lambda_j q_{ij}\boldsymbol{t}_{0,i}-\sum_{i=2}^{d}\sum_{j=1}^{i-1}\lambda_i\lambda_j q_{ij}\boldsymbol{t}_{0,j}\\
&=\sum_{i=1}^{d}\lambda_i\lambda_0 q_0\boldsymbol{t}_{0,i}+\sum_{i=2}^{d}\sum_{j=1}^{i-1}\lambda_i\lambda_j q_{ij}\boldsymbol{t}_{j,i},
\end{aligned}$$

得证。 □

最低次 BDM 元 ($k=1$) 的基函数为 [16, Section 2.6]

$$\phi_{\ell,i}=(\boldsymbol{n}_{F_\ell}\cdot\nabla\lambda_\ell)\lambda_i\boldsymbol{t}_{i,\ell}, \quad 0\le\ell\le d,\ 0\le i\neq\ell\le d.$$

任意次 BDM 元的基函数可参见文献 [7, Section 9, Table 9.1, Table 9.4]。

文献 [7] 中 BDM 元基函数还比较复杂，但它们都可以通过切向法向分解来构造。

引出切向法向分解 BDM 元的构造基函数。

## 2.5 $H(\mathrm{curl})$ 有限元

Assume $d=3$. Define

$$H(\mathrm{curl},\Omega):=\{\boldsymbol{v}\in L^2(\Omega;\mathbb{R}^3):\mathrm{curl}\,\boldsymbol{v}\in L^2(\Omega;\mathbb{R}^3)\}.$$

with squared norm $\|\boldsymbol{v}\|^2_{\mathrm{curl},\Omega}:=\|\boldsymbol{v}\|^2_{0,\Omega}+\|\mathrm{curl}\,\boldsymbol{v}\|^2_{0,\Omega}$.

The results on the trace of $H(\mathrm{curl},\Omega)$ can be found in [27, 25, 26]. For any regular vector field $\boldsymbol{v}$ in $\Omega$, we define the tangential trace $\gamma_t(\boldsymbol{v}):=\boldsymbol{n}\times\boldsymbol{v}|_\Gamma$, and the projection on the tangential plane $\pi_t(\boldsymbol{v}):=\boldsymbol{n}\times\boldsymbol{v}\times\boldsymbol{n}|_\Gamma$. Let

$$\boldsymbol{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma):=\{\boldsymbol{\lambda}\in V_\pi':\mathrm{div}_\Gamma\,\boldsymbol{\lambda}\in H^{-1/2}(\Gamma)\},$$
$$\boldsymbol{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma):=\{\boldsymbol{\lambda}\in V_\gamma':\mathrm{curl}_\Gamma\,\boldsymbol{\lambda}\in H^{-1/2}(\Gamma)\},$$

where $V_\gamma:=\gamma_t(\boldsymbol{H}^{1/2}(\Gamma))$ and $V_\pi:=\pi_t(\boldsymbol{H}^{1/2}(\Gamma))$. Spaces $\boldsymbol{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$ and $\boldsymbol{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)$ are dual to each other with the pivot space $L^2_t(\Gamma):=\{\boldsymbol{\lambda}\in L^2(\Gamma;\mathbb{R}^3):\boldsymbol{\lambda}\cdot\boldsymbol{n}=0\}$.

If the surface $\Gamma$ was regular, then

$$V_\gamma=V_\pi=\{\boldsymbol{\lambda}\in\boldsymbol{H}^{1/2}(\Gamma,\mathbb{R}^3):\boldsymbol{\lambda}\cdot\boldsymbol{n}=0\}.$$

In the case of piecewise regular surfaces, the spaces $V_\gamma$ and $V_\pi$ are different (see [25]).

**引理 2.28.** The trace operators $\gamma_t:H(\mathrm{curl},\Omega)\to\boldsymbol{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$ and $\pi_t:H(\mathrm{curl},\Omega)\to\boldsymbol{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)$ are linear, continuous and surjective. And we have the Green's formula

$$(\mathrm{curl}\,\boldsymbol{v},\boldsymbol{\phi})_\Omega-(\boldsymbol{v},\mathrm{curl}\,\boldsymbol{\phi})_\Omega=\langle\gamma_t\boldsymbol{v},\pi_t\boldsymbol{\phi}\rangle\quad\forall\,\boldsymbol{v},\boldsymbol{\phi}\in H(\mathrm{curl},\Omega).$$

Now introduce $H(\mathrm{curl})$-conforming finite elements on tetrahedrons. Assume $K\subset\mathbb{R}^3$ is a tetrahedron. Thanks to (2.31), define the shape function space

$$V^c_{k,\ell}(K):=\mathrm{grad}\,\mathbb{P}_{k+1}(K)\oplus\boldsymbol{x}\times\mathbb{P}_{\ell-1}(K;\mathbb{R}^3)=\mathbb{P}_k(K;\mathbb{R}^3)+\boldsymbol{x}\times\mathbb{P}_{\ell-1}(K;\mathbb{R}^3),$$

where $\ell=k,k+1$ for $k\geq1$, and $\ell=1$ for $k=0$. The degrees of freedom $\mathcal{N}^c_{k,\ell}(K)$ are given by

$$(\boldsymbol{v}\cdot\boldsymbol{t},q)_e\quad\forall\,q\in\mathbb{P}_k(e),e\in\mathcal{E}(K),\tag{2.49}$$
$$(\boldsymbol{n}\times\boldsymbol{v}\times\boldsymbol{n},\boldsymbol{q})_F\quad\forall\,q\in\mathrm{curl}_F\,\mathbb{P}_{\ell-1}(F)\oplus\boldsymbol{x}\mathbb{P}_{k-2}(F),F\in\mathcal{F}(K),\tag{2.50}$$
$$(\boldsymbol{v},\boldsymbol{q})_K\quad\forall\,\boldsymbol{q}\in\mathrm{curl}\,\mathbb{P}_{\ell-2}(K;\mathbb{R}^3)\oplus\boldsymbol{x}\mathbb{P}_{k-3}(K).\tag{2.51}$$

Since $\ell=k,k+1$, we have $\mathrm{curl}_F\,\mathbb{P}_{\ell-1}(F)\oplus\boldsymbol{x}\mathbb{P}_{k-2}(F)=\mathbb{P}_{\ell-2}(F;\mathbb{R}^2)+\boldsymbol{x}\mathbb{P}_{k-2}(F)$ and $\mathrm{curl}\,\mathbb{P}_{\ell-2}(K;\mathbb{R}^3)\oplus\boldsymbol{x}\mathbb{P}_{k-3}(K)=\mathbb{P}_{\ell-3}(F;\mathbb{R}^3)+\boldsymbol{x}\mathbb{P}_{k-3}(K)$.

Apparently $\boldsymbol{v}\cdot\boldsymbol{t}|_e\in\mathbb{P}_k(e)$ for any $\boldsymbol{v}\in V^c_{k,\ell}(K)$ and $e\in\mathcal{E}(K)$. And $\mathrm{curl}\,V^c_{k,\ell}(K)=\mathrm{curl}\,\mathbb{P}_\ell(K;\mathbb{R}^3)$.

**引理 2.29.** The finite element triple $(K,V^c_{k,\ell}(K),\mathcal{N}^c_{k,\ell}(K))$ is well-defined.

*Proof.* It is easy to check that the number of the degrees of freedom (2.49)-(2.51) is equal to $\dim V^c_{k,\ell}(K)$.

Take any $\boldsymbol{v}\in V^c_{k,\ell}(K)$ and suppose all the degrees of freedom (2.49)-(2.51) vanish. By the vanishing degrees of freedom (2.49)-(2.50), we get $\boldsymbol{v}\in H_0(\mathrm{curl},K)$. Now consider $\mathrm{curl}\,\boldsymbol{v}\in H_0(\mathrm{div},K)\cap\cap\mathbb{P}_{\ell-1}(K;\mathbb{R}^3)$. By the vanishing degrees of freedom (2.51), we get

$$(\mathrm{curl}\,\boldsymbol{v},\boldsymbol{q})_K=\quad\forall\,\boldsymbol{q}\in\mathbb{P}_{\ell-2}(K;\mathbb{R}^3).$$

Then by the well-posedness of the finite element triple $(K, V^d_{\ell-1,\ell-1}(K), \mathcal{N}^d_{\ell-1,\ell-1}(K))$, we obtain $\operatorname{curl}\boldsymbol{v} = 0$. Hence $\boldsymbol{v} \in \operatorname{grad}\mathbb{P}_{k+1}(K) \cap H_0(\operatorname{curl}, K)$. Then there exists $w \in \mathbb{P}_{k-3}(K)$ such that $\boldsymbol{v} = \nabla(b_K w)$, where $b_K := \lambda_1\lambda_2\lambda_3\lambda_4$ is the bubble function. It follows from the vanishing degrees of freedom (2.51) that

$$(b_K w, \operatorname{div}\boldsymbol{q})_K = \quad \forall\, \boldsymbol{q} \in \boldsymbol{x}\mathbb{P}_{k-3}(K).$$

Due to (2.32), it holds $\operatorname{div}(\boldsymbol{x}\mathbb{P}_{k-3}(K)) = \mathbb{P}_{k-3}(K)$. As a result, $w = 0$ and then $\boldsymbol{v} = \boldsymbol{0}$. □

We can see that the finite element triple $(K, V^c_{k,k+1}(K), \mathcal{N}^c_{k,k+1}(K))$ is the first kind Nédélec element [64], and $(K, V^c_{k,k}(K), \mathcal{N}^c_{k,k}(K))$ is the second kind Nédélec element [65].

Next we present the explicit expression of the $H(\operatorname{curl})$ bubble functions for $k \geq 3$. Let

$$\mathring{V}^c_k(K) := \{\boldsymbol{v} \in \mathbb{P}_k(K;\mathbb{R}^3) : \boldsymbol{v} \times \boldsymbol{n}|_{\partial K} = 0\}, \quad V^c_{k,b}(K) := \sum_{i=1}^4 b_{F_i}\mathbb{P}_{k-3}(K)\nabla\lambda_i.$$

**引理 2.30.** It holds for $k \geq 3$ that

$$\mathring{V}^c_k(K) = V^c_{k,b}(K). \tag{2.52}$$

Proof. It is easy to see that $V^c_{k,b}(K) \subseteq \mathring{V}^c_k(K)$. Next let's prove $\mathring{V}^c_k(K) \subseteq V^c_{k,b}(K)$. Take any $\boldsymbol{v} \in \mathring{V}^c_k(K)$. Since $\{\nabla\lambda_1, \nabla\lambda_2, \nabla\lambda_3\}$ forms a basis of $\mathbb{R}^3$, we can express

$$\boldsymbol{v} = q_1\nabla\lambda_1 + q_2\nabla\lambda_2 + q_3\nabla\lambda_3,$$

where $q_1, q_2, q_3 \in \mathbb{P}_k(K)$. Noting that $\boldsymbol{v} \times \boldsymbol{n}|_{\partial K} = 0$, we get

$$(q_2\nabla\lambda_2\times\boldsymbol{n}_1 + q_3\nabla\lambda_3\times\boldsymbol{n}_1)|_{F_1} = (q_1\nabla\lambda_1\times\boldsymbol{n}_2 + q_3\nabla\lambda_3\times\boldsymbol{n}_2)|_{F_2} = (q_1\nabla\lambda_1\times\boldsymbol{n}_3 + q_2\nabla\lambda_2\times\boldsymbol{n}_3)|_{F_3} = 0,$$

$$(q_1\nabla\lambda_1\times\boldsymbol{n}_4 + q_2\nabla\lambda_2\times\boldsymbol{n}_4 + q_3\nabla\lambda_3\times\boldsymbol{n}_4)|_{F_4} = 0. \tag{2.53}$$

Hence $\lambda_2\lambda_3|q_1$, $\lambda_1\lambda_3|q_2$ and $\lambda_1\lambda_2|q_3$. This means there exist $\tilde{q}_1, \tilde{q}_2, \tilde{q}_3 \in \mathbb{P}_{k-2}(K)$ such that $q_1 = \tilde{q}_1\lambda_2\lambda_3$, $q_2 = \tilde{q}_2\lambda_1\lambda_3$ and $q_3 = \tilde{q}_3\lambda_1\lambda_2$. Since $\nabla\lambda_1\times\boldsymbol{n}_4 + \nabla\lambda_2\times\boldsymbol{n}_4 + \nabla\lambda_3\times\boldsymbol{n}_4 = \boldsymbol{0}$, it follows from (2.53) that

$$((q_1 - q_3)\nabla\lambda_1\times\boldsymbol{n}_4 + (q_2 - q_3)\nabla\lambda_2\times\boldsymbol{n}_4)|_{F_4} = 0.$$

Consequently $\lambda_4|(\tilde{q}_1\lambda_3 - \tilde{q}_3\lambda_1)$ and $\lambda_4|(\tilde{q}_2\lambda_3 - \tilde{q}_3\lambda_2)$. Thus there exist $p_{ij} \in \mathbb{P}_{k-3}(K)$ $(i = 1,2,3, j = 1,2)$ such that

$$\tilde{q}_1 = \lambda_1 p_{11} + \lambda_4 p_{12}, \quad \tilde{q}_2 = \lambda_2 p_{21} + \lambda_4 p_{22}, \quad \tilde{q}_3 = \lambda_3 p_{31} + \lambda_4 p_{32}.$$

Moreover we have $\lambda_4|(p_{11} - p_{31})$ and $\lambda_4|(p_{21} - p_{31})$. Therefore we acquire from $\nabla\lambda_3 = -\nabla\lambda_1 - \nabla\lambda_2 - \nabla\lambda_4$ that

$$\begin{aligned}
\boldsymbol{v} &= \tilde{q}_1\lambda_2\lambda_3\nabla\lambda_1 + \tilde{q}_2\lambda_1\lambda_3\nabla\lambda_2 + \tilde{q}_3\lambda_1\lambda_2\nabla\lambda_3 \\
&= b_{F_4}(p_{11}\nabla\lambda_1 + p_{21}\nabla\lambda_2 + p_{31}\nabla\lambda_3) + p_{12}b_{F_1}\nabla\lambda_1 + p_{22}b_{F_2}\nabla\lambda_2 + p_{32}b_{F_3}\nabla\lambda_3 \\
&= b_{F_4}((p_{11} - p_{31})\nabla\lambda_1 + (p_{21} - p_{31})\nabla\lambda_2) - p_{31}b_{F_4}\nabla\lambda_4 + p_{12}b_{F_1}\nabla\lambda_1 + p_{22}b_{F_2}\nabla\lambda_2 + p_{32}b_{F_3}\nabla\lambda_3,
\end{aligned}$$

which ends the proof. □

We have from (2.52) that

$$\dim\mathring{V}^c_k(K) = 4\dim\mathbb{P}_{k-3}(K) - \dim\mathbb{P}_{k-4}(K) = \frac{1}{2}(k^2 - 1)(k - 2).$$

We can also acquire $\dim\mathring{V}^c_k(K)$ from the unisolvence of the second kind Nédélec element.

## 2.6 有限元 de Rham 复形

Recall that a Hilbert complex is a sequence of Hilbert spaces connected by a sequence of linear operators satisfying the property: the composition of two consecutive operators is vanished. A Hilbert complex is exact means the range of each map is the kernel of the succeeding map. As $\Omega$ is topologically trivial, the following de Rham Complex of $\Omega$ is exact

$$0 \to H^1(\Omega) \xrightarrow{\text{grad}} \boldsymbol{H}(\text{curl};\Omega) \xrightarrow{\text{curl}} \boldsymbol{H}(\text{div};\Omega) \xrightarrow{\text{div}} L^2(\Omega) \to 0. \tag{2.54}$$

给出泡函数复形。

# 第三章　Poisson 方程的混合有限元方法

## 3.1　Poisson 方程

考虑具有齐次 Dirichlet 边界条件的 Poisson 方程

$$\begin{cases} -\Delta u = f(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{3.1}$$

其中 $\Omega \subset \mathbb{R}^3$ 是有界多边形区域. 令 $\boldsymbol{\sigma} = \nabla u$ , 则

$$\begin{cases} \boldsymbol{\sigma} = \nabla u, & x \in \Omega, \\ -\operatorname{div} \boldsymbol{\sigma} = f(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{3.2}$$

引入函数空间

$$H(\operatorname{div}, \Omega) = \{\boldsymbol{\tau} : \boldsymbol{\tau} \in L^2(\Omega; \mathbb{R}^3), \operatorname{div} \boldsymbol{\tau} \in L^2(\Omega)\}, \tag{3.3}$$

并赋予以下函数

$$\|\boldsymbol{\tau}\|_{H(\operatorname{div},\Omega)}^2 = \|\boldsymbol{\tau}\|_0^2 + \|\operatorname{div} \boldsymbol{\tau}\|_0^2.$$

通过分部积分, 我们有以下混合变分问题: 求 $(\boldsymbol{\sigma}, u) \in H(\operatorname{div}, \Omega) \times L^2(\Omega)$, 使得

$$\begin{cases} (\boldsymbol{\sigma}, \boldsymbol{\tau}) + & (\operatorname{div} \boldsymbol{\tau}, u) = 0, \quad \forall\, \boldsymbol{\tau} \in H(\operatorname{div}, \Omega), \\ & (\operatorname{div} \boldsymbol{\sigma}, v) = -(f, v), \quad \forall\, v \in L^2(\Omega). \end{cases} \tag{3.4}$$

下面说明混合变分问题 (3.4) 是适定的. 首先, 有界性显然. 其次, 对任意 $\boldsymbol{\tau} \in H(\operatorname{div}, \Omega) \cap \ker(\operatorname{div})$ 成立

$$\|\boldsymbol{\tau}\|_{H(\operatorname{div},\Omega)}^2 \le (\boldsymbol{\tau}, \boldsymbol{\tau}).$$

最后, 我们需要证明以下 Inf-sup 条件.

**引理 3.1.** 对于任意 $v \in L^2(\Omega)$, 有以下结果成立

$$\|v\|_0 \lesssim \sup_{\boldsymbol{\tau} \in H(\operatorname{div},\Omega)} \frac{(\operatorname{div} \boldsymbol{\tau}, v)}{\|\boldsymbol{\tau}\|_{H(\operatorname{div},\Omega)}}. \tag{3.5}$$

Proof. 由 De Rham 复形可知, 对于任意 $v \in L^2(\Omega)$, 存在 $\boldsymbol{\tau} = \mathcal{P}v \in H(\operatorname{div}, \Omega)$ 使得

$$\operatorname{div} \boldsymbol{\tau} = \operatorname{div} \mathcal{P}v = v, \qquad \|\boldsymbol{\tau}\|_{H(\operatorname{div},\Omega)} \lesssim \|v\|_0,$$

其中 $\mathcal{P}$ 为 Poincaré 算子. 于是有

$$(\operatorname{div} \boldsymbol{\tau}, v) = \|v\|_0^2 \gtrsim \|v\|_0 \|\boldsymbol{\tau}\|_{H(\operatorname{div},\Omega)}.$$

由以上不等式可得 (3.5). □

综上可得混合变分问题 (3.4) 解的适定性, 于是有以下稳定性结果.

**定理 3.2.** 混合变分问题 (3.4) 存在唯一的解 $(\boldsymbol{\sigma}, u) \in H(\mathrm{div}, \Omega) \times L^2(\Omega)$, 且

$$\|\boldsymbol{\sigma}\|_{H(\mathrm{div},\Omega)} + \|u\|_0 \lesssim \sup_{\boldsymbol{\tau} \in H(\mathrm{div},\Omega), v \in L^2(\Omega)} \frac{(\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\mathrm{div}\, \boldsymbol{\tau}, u) + (\mathrm{div}\, \boldsymbol{\sigma}, v)}{\|\boldsymbol{\tau}\|_{H(\mathrm{div},\Omega)} + \|v\|_0}. \tag{3.6}$$

## 3.2 Poisson 方程的混合有限元格式

引入以下有限元空间:

$$\Sigma_h = \{\boldsymbol{\tau} \in H(\mathrm{div}, \Omega) : \boldsymbol{\tau}|_K \in \mathbb{P}_k(K; \mathbb{R}^3), K \in \mathcal{T}_h\},$$

$$V_h = \{v \in L^2(\Omega) : v|_K \in \mathbb{P}_{k-1}(K), K \in \mathcal{T}_h\}.$$

Poisson 方程的混合有限元格式如下: 求 $\boldsymbol{\sigma}_h \in \Sigma_h$ 和 $u_h \in V_h$, 使得

$$\begin{cases} (\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + & (\mathrm{div}\, \boldsymbol{\tau}_h, u_h) = 0, \quad \forall\, \boldsymbol{\tau}_h \in \Sigma_h, \\ & (\mathrm{div}\, \boldsymbol{\sigma}_h, v_h) = -(f, v_h), \quad \forall\, v_h \in V_h. \end{cases} \tag{3.7}$$

我们需要说明离散格式 (3.7) 是适定的. 首先, 有界性显然. 其次, 对于任意 $\boldsymbol{\tau}_h \in \Sigma_h \cap \ker(\mathrm{div})$, 强制性结果也显然. 下面证明离散的 Inf-sup 条件.

**引理 3.3.** 对于任意 $v_h \in V_h$, 有以下结果成立

$$\|v_h\|_0 \lesssim \sup_{\boldsymbol{\tau}_h \in \Sigma_h} \frac{(\mathrm{div}\, \boldsymbol{\tau}_h, v_h)}{\|\boldsymbol{\tau}_h\|_{H(\mathrm{div},\Omega)}}. \tag{3.8}$$

*Proof.* 由 Stokes 复形可知, 对于任意 $v_h \in L^2(\Omega)$, 存在 $\boldsymbol{\tau} \in H^1(\Omega; \mathbb{R}^3)$ 使得 $\mathrm{div}\, \boldsymbol{\tau} = v_h$, 且有

$$\|\boldsymbol{\tau}\|_1 \lesssim \|v_h\|_0. \tag{3.9}$$

定义插值算子 $I_h : H^1(\Omega; \mathbb{R}^3) \to \Sigma_h$ 如下: 对于任意 $\boldsymbol{\tau} \in H^1(\Omega; \mathbb{R}^3)$,

$$(I_h \boldsymbol{\tau} \cdot \mathbf{n}, q)_F = (\boldsymbol{\tau} \cdot \mathbf{n}, q)_F \quad \forall\, q \in \mathbb{P}_k(F),\ F \in \mathcal{F}(K),$$
$$(I_h \boldsymbol{\tau}, q)_K = (\boldsymbol{\tau}, q)_K \quad \forall\, q \in \nabla \mathbb{P}_{k-1}(K) \oplus \boldsymbol{x} \times \mathbb{P}_{k-2}(K; \mathbb{R}^3).$$

且有以下估计式:

$$\|\boldsymbol{\tau} - I_h \boldsymbol{\tau}\|_{0,K} \lesssim h_K |\boldsymbol{\tau}|_{1,K}, \tag{3.10}$$

$$|\boldsymbol{\tau} - I_h \boldsymbol{\tau}|_{1,K} \lesssim |\boldsymbol{\tau}|_{1,K}. \tag{3.11}$$

由 $I_h$ 的定义, 对于任意 $q \in V_h$, 我们有

$$\begin{aligned} (\mathrm{div}(\boldsymbol{\tau} - I_h \boldsymbol{\tau}), q) &= \sum_{K \in \mathcal{T}_h} (\mathrm{div}(\boldsymbol{\tau} - I_h \boldsymbol{\tau}), q)_K \\ &= \sum_{K \in \mathcal{T}_h} ((\boldsymbol{\tau} - I_h \boldsymbol{\tau}) \cdot \mathbf{n}, q)_{\partial K} - \sum_{K \in \mathcal{T}_h} (\boldsymbol{\tau} - I_h \boldsymbol{\tau}, \nabla q)_K = 0. \end{aligned}$$

由此可得 $\mathrm{div}\, I_h \boldsymbol{\tau} = \mathrm{div}\, \boldsymbol{\tau} = v_h$, 同时记 $I_h \boldsymbol{\tau} = \boldsymbol{\tau}_h$. 根据 (3.9)-(3.11) 可知

$$\begin{aligned} \|\boldsymbol{\tau}_h\|_{H(\mathrm{div},\Omega)}^2 &= \|\boldsymbol{\tau}_h\|_0^2 + \|\,\mathrm{div}\, \boldsymbol{\tau}_h\|^2 \\ &\lesssim \|\boldsymbol{\tau}\|_0^2 + \|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|_0^2 + \|\,\mathrm{div}\, \boldsymbol{\tau}_h\|_0^2 \\ &\lesssim \|\boldsymbol{\tau}\|_1^2 + \|v_h\|_0^2 \lesssim \|v_h\|_0^2. \end{aligned}$$

进一步, 我们有

$$(\operatorname{div}\boldsymbol{\tau}_h, v_h) = \|v_h\|_0^2 \gtrsim \|v_h\|_0\|\boldsymbol{\tau}_h\|_{H(\operatorname{div},\Omega)}.$$

(3.8) 由上式得证. □

综上可得混合有限元格式 (3.7) 解的适定性, 于是有以下稳定性结果.

**定理 3.4.** 混合有限元格式 (3.7) 存在唯一的解 $(\boldsymbol{\sigma}_h, u_h) \in \Sigma_h \times V_h$, 且

$$\|\boldsymbol{\sigma}_h\|_{H(\operatorname{div},\Omega)} + \|u_h\|_0 \lesssim \sup_{\boldsymbol{\tau}_h\in\Sigma_h, v_h\in V_h} \frac{(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + (\operatorname{div}\boldsymbol{\tau}_h, u_h) + (\operatorname{div}\boldsymbol{\sigma}_h, v_h)}{\|\boldsymbol{\tau}_h\|_{H(\operatorname{div},\Omega)} + \|v_h\|_0}. \tag{3.12}$$

## 3.3 误差分析

**定理 3.5.** 令 $(\boldsymbol{\sigma}, u)$ 为 (3.4) 的解, $(\boldsymbol{\sigma}_h, u_h)$ 为 (3.7) 的解, 假设 $\boldsymbol{\sigma} \in H^{k+1}(\Omega; \mathbb{R}^3)$ 且 $u \in H^k(\Omega)$, 则

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{H(\operatorname{div},\Omega)} + \|u - u_h\|_0 \lesssim h^k(\|\boldsymbol{\sigma}\|_{k+1} + \|u\|_k). \tag{3.13}$$

Proof. 通过将 (3.4) 与 (3.7) 作差, 得到误差方程:

$$\begin{cases} (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + & (\operatorname{div}\boldsymbol{\tau}_h, u - u_h) = 0, \quad \forall\, \boldsymbol{\tau}_h \in \Sigma_h, \\ & (\operatorname{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h), v_h) = 0, \quad \forall\, v_h \in V_h. \end{cases} \tag{3.14}$$

显然有

$$\begin{aligned} &\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{H(\operatorname{div},\Omega)} + \|u - u_h\|_0 \\ \le& \|\boldsymbol{\sigma} - I_h\boldsymbol{\sigma}\|_{H(\operatorname{div},\Omega)} + \|u - Q_h u\|_0 + \|I_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{H(\operatorname{div},\Omega)} + \|Q_h u - u_h\|_0, \end{aligned} \tag{3.15}$$

其中 $I_h$ 为引理 2.1 的证明中提及的插值算子, $Q_h$ 为 $k-1$ 次 $L^2$ 正交投影算子. 由 (3.12) 和 (3.14), 可得

$$\begin{aligned} &\|I_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{H(\operatorname{div},\Omega)} + \|Q_h u - u_h\|_0 \\ \lesssim& \sup_{\boldsymbol{\tau}_h\in\Sigma_h, v_h\in V_h} \frac{(I_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + (\operatorname{div}\boldsymbol{\tau}_h, Q_h u - u_h) + (\operatorname{div}(I_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h), v_h)}{\|\boldsymbol{\tau}_h\|_{H(\operatorname{div},\Omega)} + \|v_h\|_0} \\ \lesssim& \sup_{\boldsymbol{\tau}_h\in\Sigma_h, v_h\in V_h} \frac{(I_h\boldsymbol{\sigma} - \boldsymbol{\sigma}, \boldsymbol{\tau}_h) + (\operatorname{div}\boldsymbol{\tau}_h, Q_h u - u) + (\operatorname{div}(I_h\boldsymbol{\sigma} - \boldsymbol{\sigma}), v_h)}{\|\boldsymbol{\tau}_h\|_{H(\operatorname{div},\Omega)} + \|v_h\|_0} \lesssim \|I_h\boldsymbol{\sigma} - \boldsymbol{\sigma}\|_0. \end{aligned} \tag{3.16}$$

因此, 由 $I_h$ 和 $Q_h$ 的误差估计可得

$$\begin{aligned} &\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{H(\operatorname{div},\Omega)} + \|u - u_h\|_0 \\ \le& \|\boldsymbol{\sigma} - I_h\boldsymbol{\sigma}\|_{H(\operatorname{div},\Omega)} + \|u - Q_h u\|_0 + \|I_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{H(\operatorname{div},\Omega)} + \|Q_h u - u_h\|_0 \\ \lesssim& \|\boldsymbol{\sigma} - I_h\boldsymbol{\sigma}\|_1 + \|u - Q_h u\|_0 \lesssim h^k(\|\boldsymbol{\sigma}\|_{k+1} + \|u\|_k). \end{aligned} \tag{3.17}$$

□

此外, 在上述定理的证明过程中, 还能得到以下超收敛结果:

$$\|Q_h u - u_h\|_0 \lesssim h^{k+1}\|\boldsymbol{\sigma}\|_{k+1}. \tag{3.18}$$

下面我们说明通过对偶论证技巧, (3.18) 可以达到超 2 阶收敛的结果.

引入以下网格依赖范数:

$$
\begin{aligned}
\|\boldsymbol{\tau}\|_{0,h}^2 &:= \|\boldsymbol{\tau}\|_0^2 + \sum_{F\in\mathcal{F}_h} h_F\|\boldsymbol{\tau}\cdot\mathbf{n}\|_{0,F}^2,\\
|v|_{1,h}^2 &:= \sum_{K\in\mathcal{T}_h}\|\nabla v\|_{0,K}^2 + \sum_{F\in\mathcal{F}_h} h_F^{-1}\|[v]\|_{0,F}^2.
\end{aligned}
$$

需要说明在以上的范数下, 离散格式 (3.7) 同样是适定的. 首先, 有界性显然. 其次, 对于任意 $\boldsymbol{\tau}_h\in\Sigma_h\cap\ker(\mathrm{div})$, 强制性结果也显然. 下面证明离散的 Inf-sup 条件.

**引理 3.6.** 对于任意 $v_h\in V_h$, 有以下结果成立

$$
|v_h|_{1,h}\lesssim \sup_{\boldsymbol{\tau}_h\in\Sigma_h}\frac{(\mathrm{div}\,\boldsymbol{\tau}_h, v_h)}{\|\boldsymbol{\tau}_h\|_{0,h}}. \tag{3.19}
$$

Proof. 令 $\boldsymbol{\tau}_h\in\Sigma_h$ 使得对于任意 $F\in\mathcal{F}_h$ 和 $K\in\mathcal{T}_h$ 有

$$
\begin{aligned}
(\boldsymbol{\tau}_h\cdot\mathbf{n}, q)_F &= h_F^{-1}([v_h], q)_F \quad \forall\, q\in\mathbb{P}_k(F),\\
(\boldsymbol{\tau}_h, q)_K &= -(\nabla v_h, q)_K \quad \forall\, q\in\nabla\mathbb{P}_{k-1}(K)\oplus\boldsymbol{x}\times\mathbb{P}_{k-2}(K).
\end{aligned}
$$

使用尺度论证技巧, 可得

$$
\|\boldsymbol{\tau}_h\|_{0,h}^2\lesssim\sum_{K\in\mathcal{T}_h}\|\boldsymbol{\tau}_h\|_{0,K}^2\eqsim\sum_{K\in\mathcal{T}_h}\|\nabla v_h\|_{0,K}^2+\sum_{F\in\mathcal{F}_h}h_F^{-1}\|[v_h]\|_{0,F}^2\lesssim|v_h|_{1,h}^2. \tag{3.20}
$$

同时有

$$
\begin{aligned}
(\mathrm{div}\,\boldsymbol{\tau}_h, v_h) &= \sum_{F\in\mathcal{F}_h}(\boldsymbol{\tau}_h\cdot\mathbf{n}, [v_h])_F - \sum_{K\in\mathcal{T}_h}(\boldsymbol{\tau}_h, \nabla v_h)_K\\
&= \sum_{F\in\mathcal{F}_h}h_F^{-1}\|[v_h]\|_{0,F}^2 + \sum_{K\in\mathcal{T}_h}\|\nabla v_h\|_{0,K}^2 = |v_h|_{1,h}^2,
\end{aligned}
$$

将其与 (3.20) 联立即可证明 (3.19). □

同样对任意 $\boldsymbol{\sigma}_h\in\Sigma_h$ 和 $u_h\in V_h$ 有以下稳定性结果:

$$
\|\boldsymbol{\sigma}_h\|_{0,h}+\|u_h\|_{1,h}\lesssim\sup_{\boldsymbol{\tau}_h\in\Sigma_h, v_h\in V_h}\frac{(\boldsymbol{\sigma}_h,\boldsymbol{\tau}_h)+(\mathrm{div}\,\boldsymbol{\tau}_h, u_h)+(\mathrm{div}\,\boldsymbol{\sigma}_h, v_h)}{\|\boldsymbol{\tau}_h\|_{0,h}+\|v_h\|_{1,h}}. \tag{3.21}
$$

类似地, 在与定理 3.5 相同的假设下, 可得

$$
\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,h}\lesssim h^{k+1}\|\boldsymbol{\sigma}\|_{k+1}, \tag{3.22}
$$

$$
|Q_h u-u_h|_{1,h}\lesssim h^{k+1}\|\boldsymbol{\sigma}\|_{k+1}. \tag{3.23}
$$

接下来使用对偶论证来得到 $u_h$ 与 $Q_h u$ 在 $L^2$ 范数下的超 2 阶收敛结果. 令 $(\tilde{\boldsymbol{\sigma}},\tilde{u})$ 为以下辅助问题的解:

$$
\left\{
\begin{aligned}
&\tilde{\boldsymbol{\sigma}}=\nabla\tilde{u}, \quad x\in\Omega\\
&-\mathrm{div}\,\tilde{\boldsymbol{\sigma}}=Q_h u-u_h \quad x\in\Omega.
\end{aligned}
\right. \tag{3.24}
$$

假设 $\Omega$ 是凸区域, 那么 $\tilde{u}\in H^2(\Omega)$ 且

$$
\|\tilde{\boldsymbol{\sigma}}\|_1+\|\tilde{u}\|_2\lesssim\|Q_h u-u_h\|_0. \tag{3.25}
$$

**定理** 3.7. 令 $(\boldsymbol{\sigma}, u)$ 为 (3.4) 的解, $(\boldsymbol{\sigma}_h, u_h)$ 为 (3.7) 的解, 假设正则性结果 (3.25) 成立, 同时 $\boldsymbol{\sigma} \in H^{k+1}(\Omega; \mathbb{R}^3)$ 且 $u \in H^k(\Omega)$, 则

$$\|Q_h u - u_h\|_0 \lesssim h^{k+2} \|\boldsymbol{\sigma}\|_{k+1}. \tag{3.26}$$

Proof. 根据 $I_h$ 和 $Q_h$ 的定义, 有以下交换图性质:

$$\operatorname{div} I_h \boldsymbol{\sigma} = Q_h \operatorname{div} \boldsymbol{\sigma} \quad \forall\, \boldsymbol{\sigma} \in H^1(\Omega; \mathbb{R}^3). \tag{3.27}$$

由 (3.14), (3.24) 和 (3.27) 可得

$$\begin{aligned}
\|Q_h u - u_h\|_0^2 &= -(\operatorname{div} \widetilde{\boldsymbol{\sigma}}, Q_h u - u_h) \\
&= -(\operatorname{div}(\widetilde{\boldsymbol{\sigma}} - I_h \widetilde{\boldsymbol{\sigma}}), Q_h u - u_h) - (\operatorname{div}(I_h \widetilde{\boldsymbol{\sigma}}), Q_h u - u_h) \\
&= -(\operatorname{div}(I_h \widetilde{\boldsymbol{\sigma}}), Q_h u - u_h) = -(\operatorname{div}(I_h \widetilde{\boldsymbol{\sigma}}), u - u_h) \\
&= (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, I_h \widetilde{\boldsymbol{\sigma}}) = (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, I_h \widetilde{\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\sigma}}) + (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \widetilde{\boldsymbol{\sigma}}).
\end{aligned}$$

根据 $Q_h$ 的定义以及 (3.24), 再使用分部积分, 我们有

$$\begin{aligned}
(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \widetilde{\boldsymbol{\sigma}}) &= (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \nabla \widetilde{u}) = -(\operatorname{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h), \widetilde{u}) \\
&= (\operatorname{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h), Q_h \widetilde{u} - \widetilde{u}) = (\operatorname{div} \boldsymbol{\sigma}, Q_h \widetilde{u} - \widetilde{u}).
\end{aligned}$$

再由 $I_h$ 和 $Q_h$ 的误差估计, (3.22), (3.25) 以及 (3.27) 可得

$$\begin{aligned}
\|Q_h u - u_h\|_0^2 &= (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, I_h \widetilde{\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\sigma}}) + (\operatorname{div} \boldsymbol{\sigma}, Q_h \widetilde{u} - \widetilde{u}) \\
&= (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, I_h \widetilde{\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\sigma}}) + (\operatorname{div} \boldsymbol{\sigma} - \operatorname{div} I_h \boldsymbol{\sigma}, Q_h \widetilde{u} - \widetilde{u}) \\
&\lesssim \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 \|I_h \widetilde{\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\sigma}}\|_0 + \|\operatorname{div} \boldsymbol{\sigma} - Q_h \operatorname{div} \boldsymbol{\sigma}\|_0 \|Q_h \widetilde{u} - \widetilde{u}\|_0 \\
&\lesssim h \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 |\widetilde{\boldsymbol{\sigma}}|_1 + h^2 \|\operatorname{div} \boldsymbol{\sigma} - Q_h \operatorname{div} \boldsymbol{\sigma}\|_0 |\widetilde{u}|_2 \\
&\lesssim h^{k+2} \|\boldsymbol{\sigma}\|_{k+1} \|Q_h u - u_h\|_0.
\end{aligned}$$

□

## 3.4 后处理

在本节中, 我们将利用最优收敛阶结果 (3.13) 和超收敛结果 (3.26) 来构造关于 $u$ 的具有超收敛的逼近解.

定义有限元空间:

$$W_h = \{v \in L^2(\Omega) : v|_K \in \mathbb{P}_{k+1}(K), K \in \mathcal{T}_h\}.$$

定义 $u_h^* \in W_h$ 为以下问题的解: 对任意 $K \in \mathcal{T}_h$,

$$\begin{cases} (\nabla u_h^*, \nabla v)_K = (\boldsymbol{\sigma}_h, \nabla v)_K, & \forall\, v \in \mathbb{P}_{k+1}(K) \\ \int_K u_h^* \,\mathrm{d}x = \int_K u_h \,\mathrm{d}x. \end{cases} \tag{3.28}$$

**定理** 3.8. 假设正则性结果 (3.25) 成立, 同时 $\boldsymbol{\sigma} \in H^{k+1}(\Omega; \mathbb{R}^3)$ 且 $u \in H^{k+2}(\Omega)$, 则

$$\|u - u_h^*\|_0 \lesssim h^{k+2} (\|\boldsymbol{\sigma}\|_{k+1} + \|u\|_{k+2}). \tag{3.29}$$

Proof. 对于任意 $K \in \mathcal{T}_h$, 有

$$
\begin{aligned}
\|\nabla(Q_K^{k+1}u - u_h^*)\|_{0,K}^2 &= (\nabla(Q_K^{k+1}u - u_h^*), \nabla(Q_K^{k+1}u - u_h^*))_K \\
&= (\nabla(Q_K^{k+1}u - u), \nabla(Q_K^{k+1}u - u_h^*)) + (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \nabla(Q_K^{k+1}u - u_h^*)),
\end{aligned}
$$

其中 $Q_K^{k+1}$ 为 $K$ 上的 $k+1$ 次 $L^2$ 正交投影. 由上述等式可得

$$
\begin{aligned}
\|\nabla(u - u_h^*)\|_{0,K} &\le \|\nabla(u - Q_K^{k+1}u)\|_{0,K} + \|\nabla(Q_K^{k+1}u - u_h^*)\|_{0,K} \\
&\le 2\|\nabla(u - Q_K^{k+1}u)\|_{0,K} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,K}.
\end{aligned}
$$

记 $Q_h^{k+1}$ 为全局的 $k+1$ 次 $L^2$ 正交投影, 通过 $Q_h^{k+1}$ 的误差估计以及 (3.22), 我们有

$$
|u - u_h^*|_{1,h} \lesssim |u - Q_h^{k+1}u|_{1,h} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 \lesssim h^{k+1}(\|\boldsymbol{\sigma}\|_{k+1} + \|u\|_{k+2}). \tag{3.30}
$$

下面考虑 $L^2$ 误差. 容易看出

$$
\begin{aligned}
Q_K^0(Q_K^{k+1}u - u_h^*) &= Q_K^0 Q_K^{k+1}u - Q_K^0 u_h^* = Q_K^0 u - Q_K^0 u_h \\
&= Q_K^0 Q_h u - Q_K^0 u_h = Q_K^0(Q_h u - u_h),
\end{aligned}
$$

其中 $Q_K^0$ 为 $K$ 上的 0 次 $L^2$ 正交投影. 由 $Q_K^0$ 的误差估计可得

$$
\begin{aligned}
\|u - u_h^*\|_{0,K} &\le \|u - Q_K^{k+1}u\|_{0,K} + \|Q_K^{k+1}u - u_h^* - Q_K^0(Q_K^{k+1}u - u_h^*)\|_{0,K} + \|Q_K^0(Q_h u - u_h)\|_{0,K} \\
&\lesssim \|u - Q_K^{k+1}u\|_{0,K} + h_K|Q_K^{k+1}u - u_h^*|_{1,K} + \|Q_h u - u_h\|_{0,K}.
\end{aligned} \tag{3.31}
$$

通过 $Q_h^{k+1}$ 的误差估计, (3.26) 和 (3.30), 可以得出

$$
\|u - u_h^*\|_0 \lesssim \|u - Q_h^{k+1}u\|_0 + h|Q_h^{k+1}u - u_h^*|_{1,h} + \|Q_h u - u_h\|_0 \lesssim h^{k+2}(\|\boldsymbol{\sigma}\|_{k+1} + \|u\|_{k+2}).
$$

□

## 3.5 杂交化

考虑 $k=0$ 的情形，没有自由度

# 第四章 Stokes 方程的混合有限元方法

## 4.1 Stokes 方程

设 $\Omega \subset \mathbb{R}^d$ $(d \geq 2)$ 是有界区域. 考虑具有齐次 Dirichlet 边界条件下的 Stokes 方程

$$\begin{cases} -\Delta \boldsymbol{u} - \nabla p = \boldsymbol{f}(x), & \boldsymbol{x} \in \Omega, \\ \operatorname{div} \boldsymbol{u} = 0, & \boldsymbol{x} \in \Omega, \\ \boldsymbol{u} = 0, & \boldsymbol{x} \in \partial\Omega, \end{cases} \tag{4.1}$$

其中 $\boldsymbol{u}$ 表示流体速度, $p$ 表示压力. 问题 (4.1) 的混合变分问题为: 找 $\boldsymbol{u} \in H_0^1(\Omega; \mathbb{R}^d), p \in L_0^2(\Omega)$，使得

$$(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + (\operatorname{div} \boldsymbol{v}, p) = (\boldsymbol{f}, \boldsymbol{v}), \quad \boldsymbol{v} \in H_0^1(\Omega; \mathbb{R}^d), \tag{4.2}$$

$$(\operatorname{div} \boldsymbol{u}, q) = 0, \qquad q \in L_0^2(\Omega). \tag{4.3}$$

对于上述混合变分问题 (4.2)-(4.3), 根据 Brezzi 定理可以得到下面的适定性结果.

**定理 4.1.** 混合变分问题 (4.2)-(4.3) 存在唯一的解 $\boldsymbol{u} \in H_0^1(\Omega; \mathbb{R}^d), p \in L_0^2(\Omega)$, 且

$$\|\boldsymbol{u}\|_1 + \|p\|_0 \lesssim \|\boldsymbol{f}\|_{-1}.$$

**证明**: 为了得到混合变分问题 (4.2)-(4.3) 解的适定性, 我们来验证 Brezzi 定理的两个条件.

1. 强制性: 由 Poincáre 不等式可得强制性

$$\|\boldsymbol{v}\|_1^2 \lesssim (\nabla \boldsymbol{v}, \nabla \boldsymbol{v}), \quad \boldsymbol{v} \in H_0^1(\Omega; \mathbb{R}^d). \tag{4.4}$$

2. Inf-sup 条件

$$\|q\|_0 \lesssim \sup_{\boldsymbol{v} \in H_0^1(\Omega; \mathbb{R}^d)} \frac{(\operatorname{div} \boldsymbol{v}, q)}{\|\boldsymbol{v}\|_1}, \quad q \in L_0^2(\Omega). \tag{4.5}$$

由 $\operatorname{div} H_0^1(\Omega; \mathbb{R}^d) = L_0^2(\Omega)$ 可知, 存在 $\boldsymbol{u} \in H_0^1(\Omega; \mathbb{R}^d)$ 满足

$$\operatorname{div} \boldsymbol{u} = q \in L_0^2(\Omega), \quad 且 \quad \|\boldsymbol{u}\|_1 \lesssim \|q\|_0.$$

于是 $\|q\|_0 \|\boldsymbol{u}\|_1 \lesssim \|q\|_0^2 = (\operatorname{div} \boldsymbol{u}, q)$. 进而有

$$\|q\|_0 \lesssim \frac{(\operatorname{div} \boldsymbol{u}, q)}{\|\boldsymbol{u}\|_1} \lesssim \sup_{\boldsymbol{v} \in H_0^1(\Omega; \mathbb{R}^d)} \frac{(\operatorname{div} \boldsymbol{v}, q)}{\|\boldsymbol{v}\|_1}, \quad q \in L_0^2(\Omega). \tag{4.6}$$

即 inf-sup 条件成立.

事实上，inf-sup 条件(4.5)等价于 $\operatorname{div} H_0^1(\Omega; \mathbb{R}^d) = L_0^2(\Omega)$.

## 4.2 Stokes 方程的协调混合有限元方法

设 $V_h \subset H_0^1(\Omega;\mathbb{R}^d)$ 和 $P_h \subset L_0^2(\Omega)$ 为两个有限元空间，则混合变分问题 (4.2)-(4.3) 的有限元离散为: 找 $\boldsymbol{u}_h \in V_h$ 和 $p_h \in P_h$ 满足

$$(\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}) + (\operatorname{div} \boldsymbol{v}, p_h) = (\boldsymbol{f}, \boldsymbol{v}), \quad \boldsymbol{v} \in V_h, \tag{4.7}$$

$$(\operatorname{div} \boldsymbol{u}_h, q) = 0, \qquad q \in P_h. \tag{4.8}$$

为了得到混合元方法 (4.7)-(4.8) 解的适定性, 我们需要验证 Brezzi 定理的两个条件.

(a) 强制性:

$$\|\boldsymbol{v}\|_1^2 \lesssim (\nabla \boldsymbol{v}, \nabla \boldsymbol{v}), \quad \boldsymbol{v} \in V_h. \tag{4.9}$$

(b) 离散 inf-sup 条件

$$\|q\|_0 \lesssim \sup_{\boldsymbol{v} \in V_h} \frac{(\operatorname{div} \boldsymbol{v}, q)}{\|\boldsymbol{v}\|_1}, \quad q \in P_h. \tag{4.10}$$

由于 $V_h \subset H_0^1(\Omega;\mathbb{R}^d)$，连续情形强制性条件(4.4)意味着离散强制性条件(4.9).

下面考虑离散 inf-sup 条件(4.10). 连续情形，inf-sup 条件(4.5)等价于 $\operatorname{div} H_0^1(\Omega;\mathbb{R}^d) = L_0^2(\Omega)$. 但是，离散 inf-sup 条件(4.10)并不等价于 $\operatorname{div} V_h = P_h$.

**引理 4.2.** 记 $Q_h : L_0^2(\Omega) \to P_h$ 是 $L^2$ 正交投影算子. 假设 $\|\boldsymbol{v}\|_1 \eqsim \|Q_h \operatorname{div} \boldsymbol{v}\|_0$ 对所有的 $\boldsymbol{v} \in V_h/\ker(\operatorname{div})$ 成立, 则离散 inf-sup 条件(4.10)等价于 $Q_h \operatorname{div} V_h = P_h$.

**证明**: 先来证明离散 inf-sup 条件(4.10)意味着 $Q_h \operatorname{div} V_h = P_h$. 显然 $Q_h \operatorname{div} V_h \subseteq P_h$. 若存在 $q \in P_h$ 满足 $q$ 与 $Q_h \operatorname{div} V_h$ 正交，则与离散 inf-sup 条件(4.10)矛盾，故有 $Q_h \operatorname{div} V_h = P_h$.

再证 $Q_h \operatorname{div} V_h = P_h$ 意味着离散 inf-sup 条件(4.10). 对任意的 $q \in P_h$, 存在 $\boldsymbol{v} \in V_h/\ker(\operatorname{div})$ 使得 $Q_h \operatorname{div} \boldsymbol{v} = q$. 由假设条件可知 $\|\boldsymbol{v}\|_1 \eqsim \|Q_h \operatorname{div} \boldsymbol{v}\|_0 = \|q\|_0$. 于是

$$\|q\|_0 \|\boldsymbol{v}\|_1 \lesssim \|q\|_0^2 = (Q_h \operatorname{div} \boldsymbol{v}, q) = (\operatorname{div} \boldsymbol{v}, q).$$

故离散 inf-sup 条件(4.10)成立.

因此，$\operatorname{div} V_h = P_h$ 意味着离散 inf-sup 条件(4.10)成立，但是离散 inf-sup 条件(4.10)成立并不推出 $\operatorname{div} V_h = P_h$. 如果有限元空间 $V_h$ 和 $P_h$ 满足 $\operatorname{div} V_h = P_h$，由方程(4.8)可得 $\operatorname{div} \boldsymbol{u}_h = 0$，此时称混合元方法 (4.7)-(4.8) 是 **divergence-free** 的，或**质量守恒**的.

构建的有限元空间 $V_h$ 和 $P_h$ 必须满足离散 inf-sup 条件(4.10). Fortin[45]在 1977 年提出了一个简单而实用的判别准则，称之为 Fortin 准则.

**引理 4.3** (Fortin 准则). 离散 inf-sup 条件 (4.10) 等价于存在一个有界线性算子 $\Pi_h : H_0^1(\Omega;\mathbb{R}^d) \to V_h$ 满足

$$(\operatorname{div}(\boldsymbol{v} - \Pi_h \boldsymbol{v}), q) = 0, \qquad \boldsymbol{v} \in H_0^1(\Omega;\mathbb{R}^d), q \in P_h, \tag{4.11}$$

$$\|\Pi_h \boldsymbol{v}\|_1 \lesssim \|\boldsymbol{v}\|_1, \quad \boldsymbol{v} \in H_0^1(\Omega;\mathbb{R}^d). \tag{4.12}$$

算子 $\Pi_h$ 称为 Fortin 算子.

**证明**: 先证明存在 Fortin 算子意味着离散 inf-sup 条件 (4.10) 成立. 对于 $q \in P_h$, 由 inf-sup 条件 (4.5) 和(4.11)-(4.12) 可得

$$\|q\|_0 \lesssim \sup_{\boldsymbol{v}\in H_0^1(\Omega;\mathbb{R}^d)} \frac{(\operatorname{div}\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_1} = \sup_{\boldsymbol{v}\in H_0^1(\Omega;\mathbb{R}^d)} \frac{(\operatorname{div}(\Pi_h\boldsymbol{v}), q)}{\|\boldsymbol{v}\|_1} \lesssim \sup_{\boldsymbol{v}\in H_0^1(\Omega;\mathbb{R}^d)} \frac{(\operatorname{div}(\Pi_h\boldsymbol{v}), q)}{\|\Pi_h\boldsymbol{v}\|_1} \le \sup_{\boldsymbol{v}\in V_h} \frac{(\operatorname{div}\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_1}.$$

故离散 inf-sup 条件 (4.10) 成立.

再证明离散 inf-sup 条件 (4.10) 成立意味着存在 Fortin 算子. 对于 $\boldsymbol{v} \in H_0^1(\Omega;\mathbb{R}^d)$, $Q_h(\operatorname{div}\boldsymbol{v}) \in P_h$. 由离散 inf-sup 条件 (4.10) 知，存在 $\boldsymbol{v}_h \in V_h$ 满足

$$Q_h(\operatorname{div}\boldsymbol{v}_h) = Q_h(\operatorname{div}\boldsymbol{v}), \quad \|\boldsymbol{v}_h\|_1 \lesssim \|Q_h(\operatorname{div}\boldsymbol{v})\|_0 \lesssim \|\boldsymbol{v}\|_1.$$

记 $\Pi_h\boldsymbol{v} = \boldsymbol{v}_h$, 则 $\Pi_h$ 满足(4.11)-(4.12).

一般情况下分两步来构造 Fortin 算子 $\Pi_h$. 先构造两个有界线性算子 $\Pi_1, \Pi_2 : H_0^1(\Omega;\mathbb{R}^d) \to V_h$, 使其满足:

$$|\Pi_1\boldsymbol{v}|_1^2 + \sum_{T\in\mathcal{T}_h} h_T^{-2}\|\boldsymbol{v} - \Pi_1\boldsymbol{v}\|_{0,T}^2 \lesssim |\boldsymbol{v}|_1^2, \qquad \boldsymbol{v} \in H_0^1(\Omega;\mathbb{R}^d), \tag{4.13}$$

$$\|\Pi_2\boldsymbol{v}\|_{0,T} \lesssim \|\boldsymbol{v}\|_{0,\omega_T} + h_T|\boldsymbol{v}|_{1,\omega_T}, \qquad \boldsymbol{v} \in H_0^1(\Omega;\mathbb{R}^d), \tag{4.14}$$

$$(\operatorname{div}(\boldsymbol{v} - \Pi_2\boldsymbol{v}), q) = 0, \qquad q \in P_h, \boldsymbol{v} \in H_0^1(\Omega;\mathbb{R}^d), \tag{4.15}$$

其中 $\omega_T$ 是 $\mathcal{T}_h$ 中所有与 $T$ 相交非空的单形的并集. 再如下定义有界线性算子 $\Pi_h : H_0^1(\Omega;\mathbb{R}^d) \to V_h$:

$$\Pi_h\boldsymbol{v} = \Pi_1\boldsymbol{v} + \Pi_2(\boldsymbol{v} - \Pi_1\boldsymbol{v}), \qquad \boldsymbol{v} \in H_0^1(\Omega;\mathbb{R}^d). \tag{4.16}$$

这里有界线性算子 $\Pi_1$ 通常用于保证收敛阶, 有界线性算子 $\Pi_2$ 用于保证离散 inf-sup 条件.

**引理 4.4.** 假设(4.13)-(4.15)成立，则由式(4.16)定义的 $\Pi_h$ 是 Fortin 算子，即满足 (4.11) 和 (4.12).

**证明**: 利用逆不等式、(4.14)和(4.13),

$$|\Pi_2(\boldsymbol{v} - \Pi_1\boldsymbol{v})|_1^2 \lesssim \sum_{T\in\mathcal{T}_h} (h_T^{-2}\|\boldsymbol{v} - \Pi_1\boldsymbol{v}\|_{0,T}^2 + |\boldsymbol{v} - \Pi_1\boldsymbol{v}|_{1,T}^2) \lesssim |\boldsymbol{v}|_1^2, \quad \boldsymbol{v} \in H_0^1(\Omega;\mathbb{R}^d).$$

再由(4.13)可得

$$\|\Pi_h\boldsymbol{v}\|_1 \le \|\Pi_1\boldsymbol{v}\|_1 + \|\Pi_2(\boldsymbol{v} - \Pi_1\boldsymbol{v})\|_1 \lesssim \|\Pi_1\boldsymbol{v}\|_1 + |\boldsymbol{v}|_1 \lesssim \|\boldsymbol{v}\|_1, \quad \boldsymbol{v} \in H_0^1(\Omega;\mathbb{R}^d).$$

故 (4.12) 成立. 对于 $\boldsymbol{v} \in H_0^1(\Omega;\mathbb{R}^d)$ 和 $q \in P_h$, 利用(4.15)可得

$$(\operatorname{div}(\boldsymbol{v} - \Pi_h\boldsymbol{v}), q) = (\operatorname{div}(\boldsymbol{v} - \Pi_1\boldsymbol{v} - \Pi_2(\boldsymbol{v} - \Pi_1\boldsymbol{v})), q) = 0.$$

从而 (4.11) 成立.

**定理 4.5.** 设 $(\boldsymbol{u}, p) \in H_0^1(\Omega;\mathbb{R}^d) \times L_0^2(\Omega)$ 是 Stokes 方程(4.1)的解, $(\boldsymbol{u}_h, p_h) \in V_h \times P_h$ 是混合元方法(4.7)-(4.8) 的解，则有误差估计

$$|\boldsymbol{u} - \boldsymbol{u}_h|_1 \lesssim |\boldsymbol{u} - \Pi_h\boldsymbol{u}|_1 + \inf_{q\in P_h} \sup_{\boldsymbol{v}\in V_h} \frac{(\operatorname{div}\boldsymbol{v}, p - q)}{|\boldsymbol{v}|_1}, \tag{4.17}$$

$$\|p - p_h\|_0 \lesssim |\boldsymbol{u} - \Pi_h\boldsymbol{u}|_1 + \inf_{q\in P_h} \|p - q\|_0. \tag{4.18}$$

进一步，若有 $\operatorname{div} V_h = P_h$，则

$$|\boldsymbol{u} - \boldsymbol{u}_h|_1 \lesssim |\boldsymbol{u} - \Pi_h\boldsymbol{u}|_1. \tag{4.19}$$

**证明**: 将 (4.7)-(4.8) 减去 (4.2)-(4.3) 可得误差方程

$$(\nabla(\boldsymbol{u}-\boldsymbol{u}_h), \nabla \boldsymbol{v}) + (\operatorname{div} \boldsymbol{v}, p - p_h) = 0, \quad \boldsymbol{v} \in V_h, \tag{4.20}$$

$$(\operatorname{div}(\boldsymbol{u}-\boldsymbol{u}_h), q) = 0, \quad q \in P_h. \tag{4.21}$$

由(4.11)和误差方程(4.21)可知,

$$(\operatorname{div}(\Pi_h \boldsymbol{u}-\boldsymbol{u}_h), q) = 0, \quad q \in P_h. \tag{4.22}$$

将误差方程(4.20)中的 $\boldsymbol{v}$ 用 $\Pi_h \boldsymbol{u}-\boldsymbol{u}_h$ 代入，有

$$|\Pi_h \boldsymbol{u}-\boldsymbol{u}_h|_1^2 = -(\nabla(\boldsymbol{u}-\Pi_h \boldsymbol{u}), \nabla(\Pi_h \boldsymbol{u}-\boldsymbol{u}_h)) - (\operatorname{div}(\Pi_h \boldsymbol{u}-\boldsymbol{u}_h), p - p_h).$$

利用(4.22)可得

$$|\Pi_h \boldsymbol{u}-\boldsymbol{u}_h|_1^2 = -(\nabla(\boldsymbol{u}-\Pi_h \boldsymbol{u}), \nabla(\Pi_h \boldsymbol{u}-\boldsymbol{u}_h)) - (\operatorname{div}(\Pi_h \boldsymbol{u}-\boldsymbol{u}_h), p - q).$$

从而

$$|\Pi_h \boldsymbol{u}-\boldsymbol{u}_h|_1 \lesssim |\boldsymbol{u}-\Pi_h \boldsymbol{u}|_1 + \inf_{q\in P_h} \sup_{\boldsymbol{v}\in V_h} \frac{(\operatorname{div} \boldsymbol{v}, p-q)}{|\boldsymbol{v}|_1}$$

进一步利用三角不等式可得(4.17).

由离散 inf-sup 条件(4.10),

$$\|q-p_h\|_0 \lesssim \sup_{\boldsymbol{v}\in V_h} \frac{(\operatorname{div} \boldsymbol{v}, q-p_h)}{\|\boldsymbol{v}\|_1} \lesssim \|p-q\|_0 + \sup_{\boldsymbol{v}\in V_h} \frac{(\operatorname{div} \boldsymbol{v}, p-p_h)}{\|\boldsymbol{v}\|_1}.$$

利用误差方程(4.20),

$$\|q-p_h\|_0 \lesssim \|p-q\|_0 + \sup_{\boldsymbol{v}\in V_h} \frac{(\nabla(\boldsymbol{u}_h-\boldsymbol{u}), \nabla \boldsymbol{v})}{\|\boldsymbol{v}\|_1} \lesssim \|p-q\|_0 + |\boldsymbol{u}-\boldsymbol{u}_h|_1.$$

然后，借助三角不等式和(4.17)可得(4.18).

进一步，若有 $\operatorname{div} V_h = P_h$，可取 $q = Q_h p$, 则(4.17)式右端的第二项为零，故(4.19)成立.

如果 $\boldsymbol{u}-\boldsymbol{u}_h$ 的误差估计只依赖于速度 $\boldsymbol{u}$，不依赖于压力 $p$，则称混合元方法 (4.7)-(4.8) 是**压力鲁棒**的. 显然，divergence-free 的混合元方法 (4.7)-(4.8) 是压力鲁棒的. 关于 Stokes 方程数值格式的压力鲁棒性详见 [58].

### 4.2.1 压力间断的协调元方法

令 $\mathcal{T}_h$ 是 $\Omega$ 的单形网格剖分，假设 $\mathcal{T}_h$ 是形状正则的.

**两维 $P_2$-$P_0$ 元**

用向量值二次 Lagrange 元和分片常数分别离散速度和压力，即令

$$V_h := \{\boldsymbol{v} \in H_0^1(\Omega, \mathbb{R}^2) : \boldsymbol{v}|_T \in \mathbb{P}_2(T; \mathbb{R}^2), T \in \mathcal{T}_h\},$$

$$P_h := \{q \in L_0^2(\Omega) : q|_T \in \mathbb{P}_0(T), T \in \mathcal{T}_h\}.$$

二次 Lagrange 元的自由度为

$$\boldsymbol{v}(\mathrm{v}), \quad \mathrm{v} \in \Delta_0(T), \tag{4.23a}$$

$$\int_e \boldsymbol{v}\,\mathrm{d}s, \quad e \in \Delta_1(T). \tag{4.23b}$$

**引理 4.6.** $P_2$-$P_0$ 元满足离散 inf-sup 条件(4.10).

**证明**: 记 $I_h^{\mathrm{SZ}}: H_0^1(\Omega,\mathbb{R}^2)\to V_h$ 为 Scott-Zhang 插值算子 [70]，$I_h^{\mathrm{SZ}}$ 显然满足(4.13). 引入插值算子 $\Pi_2: H_0^1(\Omega,\mathbb{R}^2)\to V_h$, 定义如下

$$
\begin{aligned}
(\Pi_2\boldsymbol{v})(\mathrm{v}) &= 0, &\quad \mathrm{v}\in\Delta_0(\mathcal{T}_h),\\
\int_e \Pi_2\boldsymbol{v}\,\mathrm{d}s &= \int_e \boldsymbol{v}\,\mathrm{d}s, &\quad e\in\Delta_1(\mathcal{T}_h).
\end{aligned}
$$

由仿射等价性和迹不等式可得，

$$
\|\Pi_2\boldsymbol{v}\|_{0,T}^2 \eqsim h_T\sum_{e\in\partial T}\|Q_{0,e}\boldsymbol{v}\|_{0,e}^2 \lesssim h_T\|\boldsymbol{v}\|_{0,\partial T}^2 \lesssim \|\boldsymbol{v}\|_{0,T}^2 + h_T^2|\boldsymbol{v}|_{1,T}^2.
$$

故 $\Pi_2$ 满足(4.14). 利用分部积分可得

$$
(\operatorname{div}(\boldsymbol{v}-\Pi_2\boldsymbol{v}), q) = -(\boldsymbol{v}-\Pi_2\boldsymbol{v}, \nabla q) = 0, \qquad q\in P_h, \boldsymbol{v}\in H_0^1(\Omega;\mathbb{R}^d).
$$

即 $\Pi_2$ 满足(4.15).

根据(4.16)定义插值算子 $\Pi_h: H_0^1(\Omega;\mathbb{R}^d)\to V_h$，即 $\Pi_h\boldsymbol{v} = I_h^{\mathrm{SZ}}\boldsymbol{v} + \Pi_2(\boldsymbol{v}-I_h^{\mathrm{SZ}}\boldsymbol{v})$. 由引理4.4可知，$\Pi_h$ 是 Fortin 算子，故由 Fortin 准则可得离散 inf-sup 条件(4.10)成立.

**注 4.7.** 这里之所以选择二次 Lagrange 元不是线性 Lagrange 元来离散速度, 是因为线性 Lagrange 元只有顶点处函数值的自由度, 没有边上函数值积分平均的自由度, 这样就无法得到 $\int_T \operatorname{div}(\boldsymbol{v}-\Pi_2\boldsymbol{v})\,\mathrm{d}x = \int_{\partial T}(\boldsymbol{v}-\Pi_2\boldsymbol{v})\cdot\boldsymbol{n}\,\mathrm{d}s = 0$. 边上函数值积分平均的自由度保证了 $\operatorname{div} V_h$ 能映满分片常数.

当 $\boldsymbol{u}\in H_0^1(\Omega;\mathbb{R}^2)\cap H^3(\Omega;\mathbb{R}^2)$ 和 $p\in L_0^2(\Omega)\cap H^1(\Omega)$ 时，由定理 4.5可知 $P_2$-$P_0$ 元方法的误差估计

$$
\|\boldsymbol{u}-\boldsymbol{u}_h\|_1 + \|p-p_h\|_0 \lesssim h^2\|\boldsymbol{u}\|_3 + h\|p\|_1 \lesssim h(\|\boldsymbol{u}\|_3 + \|p\|_1).
$$

该误差估计的收敛阶对于压力 $p$ 是最优的，但对于速度 $\boldsymbol{u}$ 不是最优的.

将 $P_2$-$P_0$ 元推广到任意 $d$ 维，即 $P_d$-$P_0$ 元，此时离散速度的收敛阶的丢阶现象会更加严重. 下面考虑改进 $P_2$-$P_0$ 元方法.

### SMALL 元

为了证明离散 inf-sup 条件, 关键是要用到自由度(4.23b)的法向部分, 切向部分并不需要. 为此, 只要在向量值一次多项式空间的基础上增加边上的法向泡函数，即可避免丢阶现象. 由此得到 SMALL 元，参见 [16, Remark 8.4.2]和 [12, 46]. 在 SMALL 元方法中，压力仍然用分片常数逼近.

速度的形函数空间取为 $\mathbb{P}_1(T;\mathbb{R}^2)\oplus\operatorname{span}\{\lambda_i\lambda_j\nabla\lambda_k : i\neq j\neq k, 0\le i,j,k\le 2\}$. 自由度为

$$
\boldsymbol{v}(\mathrm{v}), \quad \mathrm{v}\in\Delta_0(T), \tag{4.24a}
$$

$$
\int_e \boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}s, \quad e\in\Delta_1(T). \tag{4.24b}
$$

**引理 4.8.** 形函数空间 $\mathbb{P}_1(T;\mathbb{R}^2)\oplus\operatorname{span}\{\lambda_i\lambda_j\nabla\lambda_k : i\neq j\neq k, 0\le i,j,k\le 2\}$ 由自由度(4.24)所唯一确定.

**证明**: 形函数空间的维数和自由度的个数均为 9. 设 $\boldsymbol{v} = \boldsymbol{q} + c_0\lambda_1\lambda_2\nabla\lambda_0 + c_1\lambda_2\lambda_0\nabla\lambda_1 + c_2\lambda_0\lambda_1\nabla\lambda_2$ 满足(4.24)中所有自由度为零，其中 $\boldsymbol{q} \in \mathbb{P}_1(T;\mathbb{R}^2), c_0, c_1, c_2 \in \mathbb{R}$. 由自由度(4.24a)可知 $\boldsymbol{q}$ 在所有顶点处取值为零, 故 $\boldsymbol{q} = 0$. 从而 $\boldsymbol{v} = c_0\lambda_1\lambda_2\nabla\lambda_0 + c_1\lambda_2\lambda_0\nabla\lambda_1 + c_2\lambda_0\lambda_1\nabla\lambda_2$. 在边 $e_0$ 上, $\boldsymbol{v}|_{e_0} = (c_0\lambda_1\lambda_2\nabla\lambda_0)|_{e_0}$, 故由自由度(4.24b)可知 $c_0 = 0$. 类似可得 $c_1 = c_2 = 0$. 于是 $\boldsymbol{v} = 0$.

类似 $P_2$-$P_0$ 元，我们可以定义 SMALL 元整体有限元空间，并证明离散 inf-sup 条件. Stokes 方程 SMALL 元方法的误差估计为

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_1 + \|p - p_h\|_0 \lesssim h(\|\boldsymbol{u}\|_2 + \|p\|_1).$$

任意 $d$ 维 SMALL 元的速度形函数空间为 $\mathbb{P}_1(T;\mathbb{R}^d) \oplus \operatorname{span}\{b_F\boldsymbol{n}_F, F \in \partial T\}$, 其中 $b_F$ 和 $\boldsymbol{n}_F$ 分别为面 $F$ 的泡函数和法向量. 自由度为

$$\begin{aligned} &\boldsymbol{v}(\mathrm{v}), \quad \mathrm{v} \in \Delta_0(T), \\ &\int_F \boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}s, \quad F \in \partial T. \end{aligned}$$

**Crouzeix-Raviart 元**

同样为了克服 $P_2$-$P_0$ 元的丢阶现象，Crouzeix-Raviart 元 [38]考虑将压力空间的多项式次数提高，同时对速度空间增补泡函数.

设 $k \geq 2$, Crouzeix-Raviart 元的速度形函数空间为 $\mathbb{P}_k(T;\mathbb{R}^2) + b_T\mathbb{P}_{k-2}(T;\mathbb{R}^2)$, 自由度为

$$\boldsymbol{v}(\mathrm{v}), \quad \mathrm{v} \in \Delta_0(T), \tag{4.25a}$$

$$(\boldsymbol{v}, \boldsymbol{q})_e, \quad \boldsymbol{q} \in \mathbb{P}_{k-2}(e;\mathbb{R}^2), e \in \Delta_1(T), \tag{4.25b}$$

$$(\boldsymbol{v}, \boldsymbol{q})_T, \quad \boldsymbol{q} \in \mathbb{P}_{k-2}(T;\mathbb{R}^2). \tag{4.25c}$$

**引理 4.9.** *设 $k \geq 2$, 形函数空间 $\mathbb{P}_k(T;\mathbb{R}^2) + b_T\mathbb{P}_{k-2}(T;\mathbb{R}^2)$ 由自由度(4.25)所唯一确定.*

**证明**: 形函数空间的维数为 $2(\dim\mathbb{P}_k(T) + \dim\mathbb{P}_{k-2}(T) - \dim\mathbb{P}_{k-3}(T))$ 和自由度的个数为

$$6 + 6(k-1) + 2\dim\mathbb{P}_{k-2}(T) = 2(\dim\mathbb{P}_k(T) + \dim\mathbb{P}_{k-2}(T) - \dim\mathbb{P}_{k-3}(T)).$$

设 $\boldsymbol{v} \in \mathbb{P}_k(T;\mathbb{R}^2) + b_T\mathbb{P}_{k-2}(T;\mathbb{R}^2)$ 满足(4.25)中所有自由度为零. 由自由度(4.25a)-(4.25b)可知 $\boldsymbol{v}|_{\partial T} = 0$，故 $\boldsymbol{v} \in b_T\mathbb{P}_{k-2}(T;\mathbb{R}^2)$. 再由自由度(4.25c)可得 $\boldsymbol{v} = 0$.

分别定义离散速度和压力的整体有限元空间

$$\begin{aligned} V_h &:= \{\boldsymbol{v} \in H_0^1(\Omega,\mathbb{R}^2) : \boldsymbol{v}|_T \in \mathbb{P}_k(T;\mathbb{R}^2) + b_T\mathbb{P}_{k-2}(T;\mathbb{R}^2), T \in \mathcal{T}_h\}, \\ P_h &:= \{q \in L_0^2(\Omega) : q|_T \in \mathbb{P}_{k-1}(T), T \in \mathcal{T}_h\}. \end{aligned}$$

**引理 4.10.** *设 $k \geq 2$, Crouzeix-Raviart 元满足离散 inf-sup 条件(4.10).*

**证明**: 令 $V_h^L$ 为 $k$ 次 Lagrange 元空间 $\{\boldsymbol{v} \in H_0^1(\Omega,\mathbb{R}^2) : \boldsymbol{v}|_T \in \mathbb{P}_k(T;\mathbb{R}^2), T \in \mathcal{T}_h\}$. 记 $I_h^{\mathrm{SZ}} : H_0^1(\Omega,\mathbb{R}^2) \to V_h^L$ 为 Scott-Zhang 插值算子 [70]，满足

$$(I_h^{\mathrm{SZ}}\boldsymbol{v}, \boldsymbol{q})_e = (\boldsymbol{v}, \boldsymbol{q})_e \quad \forall\, \boldsymbol{q} \in \mathbb{P}_{k-2}(e;\mathbb{R}^2), e \in \Delta_1(\mathcal{T}_h).$$

引入插值算子 $\Pi_2 : H_0^1(\Omega;\mathbb{R}^2) \to V_h$, 定义如下: 对任意的 $T \in \mathcal{T}_h$，$(\Pi_2\boldsymbol{v})|_T \in b_T\nabla\mathbb{P}_{k-1}(T)$ 满足

$$(\operatorname{div}(\Pi_2\boldsymbol{v}), q)_T = (\operatorname{div}\boldsymbol{v}, q)_T, \quad q \in \mathbb{P}_{k-1}(T) \cap L_0^2(T).$$

由仿射等价性可得，

$$\|\Pi_2 \boldsymbol{v}\|_{0,T} \lesssim h_T \|Q_{k-1,T} \operatorname{div} \boldsymbol{v}\|_{0,T} \leq h_T \|\operatorname{div} \boldsymbol{v}\|_{0,T}.$$

故 $\Pi_2$ 满足(4.14). 由 $\Pi_2 \boldsymbol{v}$ 的定义显然有

$$(\operatorname{div}(\boldsymbol{v} - \Pi_2 \boldsymbol{v}), q)_T = 0, \qquad q \in \mathbb{P}_{k-1}(T) \cap L_0^2(T), T \in \mathcal{T}_h, \boldsymbol{v} \in H_0^1(\Omega; \mathbb{R}^d).$$

根据(4.16)定义插值算子 $\Pi_h : H_0^1(\Omega; \mathbb{R}^d) \to V_h$，即 $\Pi_h \boldsymbol{v} = I_h^{\mathrm{SZ}} \boldsymbol{v} + \Pi_2(\boldsymbol{v} - I_h^{\mathrm{SZ}} \boldsymbol{v})$. 类似引理 4.4的证明可得

$$\|\Pi_h \boldsymbol{v}\|_1 \leq \|I_h^{\mathrm{SZ}} \boldsymbol{v}\|_1 + \|\Pi_2(\boldsymbol{v} - I_h^{\mathrm{SZ}} \boldsymbol{v})\|_1 \lesssim \|\boldsymbol{v}\|_1, \quad \forall\, \boldsymbol{v} \in H_0^1(\Omega; \mathbb{R}^d),$$

以及对任意的 $q \in P_h$ 成立

$$\begin{aligned}
(\operatorname{div}(\boldsymbol{v} - \Pi_h \boldsymbol{v}), q) &= \sum_{T \in \mathcal{T}_h} (\operatorname{div}(\boldsymbol{v} - I_h^{\mathrm{SZ}} \boldsymbol{v} - \Pi_2(\boldsymbol{v} - I_h^{\mathrm{SZ}} \boldsymbol{v})), q)_T \\
&= \sum_{T \in \mathcal{T}_h} (\operatorname{div}(\boldsymbol{v} - I_h^{\mathrm{SZ}} \boldsymbol{v} - \Pi_2(\boldsymbol{v} - I_h^{\mathrm{SZ}} \boldsymbol{v})), Q_T^0 q)_T \\
&= \sum_{T \in \mathcal{T}_h} (\operatorname{div}(\boldsymbol{v} - I_h^{\mathrm{SZ}} \boldsymbol{v}), Q_T^0 q)_T = 0.
\end{aligned}$$

这表明 $\Pi_h$ 是 Fortin 算子，故由 Fortin 准则可得离散 inf-sup 条件(4.10)成立.

当 $\boldsymbol{u} \in H_0^1(\Omega; \mathbb{R}^2) \cap H^{k+1}(\Omega; \mathbb{R}^2)$ 和 $p \in L_0^2(\Omega) \cap H^k(\Omega)$ 时，由定理 4.5可知 Crouzeix-Raviart 元方法的误差估计

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_1 + \|p - p_h\|_0 \lesssim h^k (\|\boldsymbol{u}\|_{k+1} + \|p\|_k).$$

设 $k \geq 2$, 三维情形 Crouzeix-Raviart 元 [16, Example 8.7.2]的压力空间仍为分片 $k-1$ 次多项式空间，速度形函数空间为

$$\begin{cases}
\mathbb{P}_2(T; \mathbb{R}^3) + b_T \mathbb{P}_0(T; \mathbb{R}^3) + \operatorname{span}\{b_F \boldsymbol{n}_F, F \in \partial T\}, & k = 2, \\
\mathbb{P}_k(T; \mathbb{R}^3) + b_T \mathbb{P}_{k-2}(T; \mathbb{R}^3), & k \geq 2.
\end{cases}$$

自由度为

$$\begin{aligned}
\boldsymbol{v}(\mathrm{v}), &\quad \mathrm{v} \in \Delta_0(T), \\
(\boldsymbol{v}, \boldsymbol{q})_e, &\quad \boldsymbol{q} \in \mathbb{P}_{k-2}(e; \mathbb{R}^3), e \in \Delta_1(T), \\
(\boldsymbol{v} \cdot \boldsymbol{n}, q)_F, &\quad q \in \mathbb{P}_0(F), F \in \Delta_2(T),\ \text{当 } k = 2 \text{ 时}, \\
(\boldsymbol{v}, \boldsymbol{q})_F, &\quad \boldsymbol{q} \in \mathbb{P}_{k-3}(F; \mathbb{R}^3), F \in \Delta_2(T),\ \text{当 } k \geq 3 \text{ 时}, \\
(\boldsymbol{v}, \boldsymbol{q})_T, &\quad \boldsymbol{q} \in \mathbb{P}_{k-2}(T; \mathbb{R}^3).
\end{aligned}$$

当 $k \geq 3$ 时, 自由度的个数为

$$\begin{aligned}
12 + 18(k-1) + 12\binom{k-1}{2} + 3 \dim \mathbb{P}_{k-2}(T) &= 6k^2 + 6 + 3 \dim \mathbb{P}_{k-2}(T) \\
&= 3 \dim \mathbb{P}_k(T) + 3 \dim \mathbb{P}_{k-2}(T) - 3 \dim \mathbb{P}_{k-4}(T).
\end{aligned}$$

**Scott-Vogelius 元**

两维 Scott-Vogelius 元 [69]离散速度和压力的有限元空间为

$$\begin{aligned} V_h &:= \{\boldsymbol{v} \in H_0^1(\Omega, \mathbb{R}^2) : \boldsymbol{v}|_T \in \mathbb{P}_k(T; \mathbb{R}^2), T \in \mathcal{T}_h\}, \\ P_h &:= \{q \in L_0^2(\Omega) : q|_T \in \mathbb{P}_{k-1}(T), T \in \mathcal{T}_h\}. \end{aligned}$$

对于三角剖分 $\mathcal{T}_h$ 中的顶点 $V$，记 $\theta_1, \ldots, \theta_n$ 为以 $V$ 顶点的角的角度，假设这些角按逆时针排列. 若 $V$ 为内部顶点，令 $\theta_{n+1} := \theta_1$. 定义

$$S(V) = \begin{cases} 0, & n = 1, \\ \max\{\pi - \theta_1 - \theta_2, \pi - \theta_1 - \theta_n\}, & n > 1, V \in \partial\Omega, \\ \max\{\pi - \theta_1 - \theta_2, \pi - \theta_n - \theta_{n+1}\}, & V \notin \partial\Omega. \end{cases}$$

显然，$S(V) = 0$ 当且仅当 $\mathcal{T}_h$ 中所有以 $V$ 为顶点的边落在两条直线上. 此时，称 $V$ 是奇异的. 若 $S(V)$ 是一个很小的正数，则 $V$ 接近奇异的. 因此，$S(V)$ 度量了 $V$ 的奇异程度.

**引理 4.11** ([69]). 设 $\mathcal{T}_h$ 是拟一致的三角剖分. 假设存在常数 $\delta > 0$ 使得

$$S(V) \geq \delta, \quad v \in \Delta_0(\mathcal{T}_h).$$

则当 $k \geq 4$ 时，Scott-Vogelius 元满足离散 inf-sup 条件，其中常数依赖于 $\delta$.

Scott-Vogelius 元是 divergence-free 的.

**注 4.12.** 在任意维单形剖分的 Alfeld 加密下，Scott-Vogelius 元对任意的 $k \geq d$ 均满足离散 inf-sup 条件 [51, 78, 9]. 当 $1 \leq k < d$ 时，速度形函数空间需要增加修正的 Bernardi-Raugel 面泡函数.

**注 4.13.** 文献 [33]详细研究了 Stokes 方程 divergence-free 的协调元,包括压力间断和压力连续的 divergence-free 协调元. Divergence-free 协调元一般含有超光滑自由度.

### 4.2.2 压力连续的协调元方法

这一节考虑压力连续的协调元方法，此时由分部积分公式知 $(\operatorname{div} \boldsymbol{v}, q) = -(\boldsymbol{v}, \nabla q)$，因此选取的空间 $V_h$ 的维数相对于 $\nabla P_h$ 的维数足够大即可.

**MINI 元**

MINI 元 [5]用 Lagrange 元离散压力，在 Lagrange 元的基础上增补泡函数离散速度，即

$$\begin{aligned} V_h &:= \{\boldsymbol{v} \in H_0^1(\Omega, \mathbb{R}^d) : \boldsymbol{v}|_T \in \mathbb{P}_k(T; \mathbb{R}^d) + b_T \nabla \mathbb{P}_k(T), T \in \mathcal{T}_h\}, \\ P_h &:= \{q \in H^1(\Omega) \cap L_0^2(\Omega) : q|_T \in \mathbb{P}_k(T), T \in \mathcal{T}_h\}, \end{aligned}$$

其中 $k \geq 1$. 当 $k = 1$ 时,

$$V_h = \{\boldsymbol{v} \in H_0^1(\Omega, \mathbb{R}^d) : \boldsymbol{v}|_T \in \mathbb{P}_1(T; \mathbb{R}^d) \oplus b_T \mathbb{P}_0(T; \mathbb{R}^d), T \in \mathcal{T}_h\}.$$

MINI 元速度空间 $V_h$ 在边界上的自由度同 Lagrange 元在边界上的自由度是一样的，其内部自由度为

$$(\boldsymbol{v}, \boldsymbol{q})_T, \quad \boldsymbol{q} \in \mathbb{P}_{k-d-1}(T; \mathbb{R}^d) + \nabla \mathbb{P}_k(T).$$

特别地，当 $k=1$ 时，其内部自由度为

$$(\boldsymbol{v},\boldsymbol{q})_T,\quad \boldsymbol{q}\in\mathbb{P}_0(T;\mathbb{R}^d).$$

**引理 4.14.** 设 $k\geq 1$, MINI 元满足离散 inf-sup 条件(4.10).

**证明**: 证明过程类似于 Crouzeix-Raviart 元. 记 $I_h^{\mathrm{SZ}}: H_0^1(\Omega,\mathbb{R}^d)\to V_h^L$ 为 Scott-Zhang 插值算子 [70], 其中 $V_h^L$ 为 $k$ 次 Lagrange 元空间 $\{\boldsymbol{v}\in H_0^1(\Omega,\mathbb{R}^d):\boldsymbol{v}|_T\in\mathbb{P}_k(T;\mathbb{R}^d),T\in\mathcal{T}_h\}$. 引入插值算子 $\Pi_2: H_0^1(\Omega,\mathbb{R}^d)\to V_h$, 定义如下: 对任意的 $T\in\mathcal{T}_h$，$(\Pi_2\boldsymbol{v})|_T\in b_T\nabla\mathbb{P}_k(T)$ 满足

$$(\Pi_2\boldsymbol{v},\boldsymbol{q})_T=(\boldsymbol{v},\boldsymbol{q})_T,\quad \boldsymbol{q}\in\nabla\mathbb{P}_k(T).$$

由仿射等价性可得，

$$\|\Pi_2\boldsymbol{v}\|_{0,T}\lesssim\|\boldsymbol{v}\|_{0,T}.$$

故 $\Pi_2$ 满足(4.14). 利用分部积分可得

$$(\operatorname{div}(\boldsymbol{v}-\Pi_2\boldsymbol{v}),q)=-(\boldsymbol{v}-\Pi_2\boldsymbol{v},\nabla q)=0,\qquad q\in P_h,\boldsymbol{v}\in H_0^1(\Omega;\mathbb{R}^d).$$

即 $\Pi_2$ 满足(4.15).

根据(4.16)定义插值算子 $\Pi_h: H_0^1(\Omega;\mathbb{R}^d)\to V_h$，即 $\Pi_h\boldsymbol{v}=I_h^{\mathrm{SZ}}\boldsymbol{v}+\Pi_2(\boldsymbol{v}-I_h^{\mathrm{SZ}}\boldsymbol{v})$. 由引理4.4可知，$\Pi_h$ 是 Fortin 算子，故由 Fortin 准则可得离散 inf-sup 条件(4.10)成立.

当 $\boldsymbol{u}\in H_0^1(\Omega;\mathbb{R}^d)\cap H^{k+1}(\Omega;\mathbb{R}^d)$ 和 $p\in L_0^2(\Omega)\cap H^{k+1}(\Omega)$ 时，由定理 4.5可知 Crouzeix-Raviart 元方法的误差估计

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_1+\|p-p_h\|_0\lesssim h^k(\|\boldsymbol{u}\|_{k+1}+h\|p\|_{k+1}).$$

该误差估计的收敛阶对于速度 $\boldsymbol{u}$ 是最优的，但对于压力 $p$ 不是最优的.

最低次 MINI 元因为简单常被用于离散 Stokes 方程.

#### Taylor-Hood 元

Taylor-Hood 元 [74, 14, 15]对速度 $\boldsymbol{u}$ 和压力 $p$ 均采用 Lagrange 元离散，其误差估计对速度 $\boldsymbol{u}$ 和压力 $p$ 均是最优的. 具体来说，Taylor-Hood 元离散速度 $\boldsymbol{u}$ 和压力 $p$ 的有限元空间分别为

$$\begin{aligned}V_h &:= \{\boldsymbol{v}\in H_0^1(\Omega,\mathbb{R}^d):\boldsymbol{v}|_T\in\mathbb{P}_k(T;\mathbb{R}^d),T\in\mathcal{T}_h\},\\ P_h &:= \{q\in H^1(\Omega)\cap L_0^2(\Omega): q|_T\in\mathbb{P}_{k-1}(T),T\in\mathcal{T}_h\},\end{aligned}$$

其中 $k\geq 2$.

对于 Taylor-Hood 元，对单形剖分 $\mathcal{T}_h$ 作如下假设:

$$\text{每一个 } d\text{-维单形 } T\in\mathcal{T}_h \text{ 至少有一个顶点位于开区域 } \Omega \text{ 内部.}$$

在此网格假设下，Taylor-Hood 元满足离散 inf-sup 条件(4.10)，其证明相对复杂，参考 [16, 14, 15, 47, 30, 41]. 这里给出文献 [41]中任意维最低次 Taylor-Hood 元离散 inf-sup 条件(4.10)的证明, 关键之处是 $\nabla V_h$ 为最低次棱元的子空间.

**引理 4.15.** 设 $k=2$, Taylor-Hood 元满足离散 inf-sup 条件(4.10).

**证明**: 记 $I_h^{\mathrm{SZ}}: H_0^1(\Omega,\mathbb{R}^d)\to V_h$ 为 Scott-Zhang 插值算子 [70]. 下面考虑插值算子 $\Pi_2: H_0^1(\Omega,\mathbb{R}^d)\to V_h$ 的构造.

记 $\mathcal{E}_h, \mathring{\mathcal{E}}_h$ 和 $\mathcal{E}_h^{\partial}$ 分别为单形剖分 $\mathcal{T}_h$ 所有一维边、内边和边界边的集合. 对端点为 $\mathtt{v}_i$ 和 $\mathtt{v}_j$ 的边 $e_{ij}\in\mathcal{E}_h$, 令 $\boldsymbol{b}_{ij}=\lambda_i\lambda_j\boldsymbol{t}_{ij}/\int_\Omega\lambda_i\lambda_j\,\mathrm{d}x$, 则有

$$(\operatorname{div}\boldsymbol{b}_{ij},\lambda_k)=-(\boldsymbol{b}_{ij},\nabla\lambda_k)=-\frac{1}{\int_\Omega\lambda_i\lambda_j\,\mathrm{d}x}\int_\Omega\lambda_i\lambda_j\boldsymbol{t}_{ij}\cdot\nabla\lambda_k\,\mathrm{d}x=-\boldsymbol{t}_{ij}\cdot\nabla\lambda_k=\delta_{ik}-\delta_{jk}.$$

易知, 当 $e_{ij}\in\mathring{\mathcal{E}}_h$ 时 $\boldsymbol{b}_{ij}\in V_h$, 当 $e_{ij}\in\mathcal{E}_h^{\partial}$ 时 $\boldsymbol{b}_{ij}\notin V_h$. 为此, 对 $\boldsymbol{b}_{ij}$ 进行修正. 对于边界边 $e_{ij}\in\mathcal{E}_h^{\partial}$, 由网格假设知, 存在一个 $\Omega$ 内的顶点 $\mathtt{v}_m$ 使得 $e_{im},e_{jm}\in\mathring{\mathcal{E}}_h$. 令

$$\boldsymbol{\psi}_{ij}=\begin{cases}\boldsymbol{b}_{ij}, & e_{ij}\in\mathring{\mathcal{E}}_h,\\ \boldsymbol{b}_{im}+\boldsymbol{b}_{mj}, & e_{ij}\in\mathcal{E}_h^{\partial},\end{cases}$$

则所有的 $\boldsymbol{\psi}_{ij}$ 均属于 $V_h$, 且成立

$$(\operatorname{div}\boldsymbol{\psi}_{ij},\lambda_k)=\delta_{ik}-\delta_{jk}.$$

现在如下定义插值算子 $\Pi_2: H_0^1(\Omega,\mathbb{R}^d)\to V_h$,

$$\Pi_2\boldsymbol{v}:=\sum_{e_{ij}\in\mathcal{E}_h,i<j}(\boldsymbol{v},\lambda_i\nabla\lambda_j-\lambda_j\nabla\lambda_i)\boldsymbol{\psi}_{ij}.$$

可知

$$\|\Pi_2\boldsymbol{v}\|_{0,T}\lesssim\|\boldsymbol{v}\|_{0,\omega_T}.$$

故 $\Pi_2$ 满足(4.14). 由 $\Pi_2\boldsymbol{v}$ 的定义可得,

$$\begin{aligned}(\operatorname{div}\Pi_2\boldsymbol{v},\lambda_k)&=\sum_{e_{ij}\in\mathcal{E}_h,\,i<j}(\boldsymbol{v},\lambda_i\nabla\lambda_j-\lambda_j\nabla\lambda_i)(\operatorname{div}\boldsymbol{\psi}_{ij},\lambda_k)\\&=\sum_{e_{ij}\in\mathcal{E}_h,\,i<j}(\boldsymbol{v},\lambda_i\nabla\lambda_j-\lambda_j\nabla\lambda_i)(\delta_{ik}-\delta_{jk})\\&=\sum_{e_{kj}\in\mathcal{E}_h,\,k<j}(\boldsymbol{v},\lambda_k\nabla\lambda_j-\lambda_j\nabla\lambda_k)-\sum_{e_{ik}\in\mathcal{E}_h,\,i<k}(\boldsymbol{v},\lambda_i\nabla\lambda_k-\lambda_k\nabla\lambda_i)\\&=\sum_{e_{ki}\in\mathcal{E}_h,\,i>k}(\boldsymbol{v},\lambda_k\nabla\lambda_i-\lambda_i\nabla\lambda_k)+\sum_{e_{ik}\in\mathcal{E}_h,\,i<k}(\boldsymbol{v},\lambda_k\nabla\lambda_i-\lambda_i\nabla\lambda_k)\\&=\sum_{e_{ki}\in\mathcal{E}_h}(\boldsymbol{v},\lambda_k\nabla\lambda_i-\lambda_i\nabla\lambda_k)\\&=(\boldsymbol{v},\lambda_k\nabla 1-\nabla\lambda_k)=-(\boldsymbol{v},\nabla\lambda_k)=(\operatorname{div}\boldsymbol{v},\lambda_k).\end{aligned}$$

即可推得 $(\operatorname{div}(\Pi_2\boldsymbol{v}-\boldsymbol{v}),\lambda_k)=0$. 故 $\Pi_2$ 满足(4.15).

根据(4.16)定义插值算子 $\Pi_h: H_0^1(\Omega;\mathbb{R}^d)\to V_h$, 即 $\Pi_h\boldsymbol{v}=I_h^{\mathrm{SZ}}\boldsymbol{v}+\Pi_2(\boldsymbol{v}-I_h^{\mathrm{SZ}}\boldsymbol{v})$. 由引理4.4可知, $\Pi_h$ 是 Fortin 算子, 故由 Fortin 准则可得离散 inf-sup 条件(4.10)成立.

当 $\boldsymbol{u}\in H_0^1(\Omega;\mathbb{R}^d)\cap H^{k+1}(\Omega;\mathbb{R}^d)$ 和 $p\in L_0^2(\Omega)\cap H^k(\Omega)$ 时, 由定理 4.5可知 Taylor-Hood 元方法的误差估计

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_1+\|p-p_h\|_0\lesssim h^k(\|\boldsymbol{u}\|_{k+1}+h\|p\|_k).$$

该误差估计的收敛阶对于速度 $\boldsymbol{u}$ 和压力 $p$ 均是最优的.

## 4.3 Stokes 方程的非协调混合元方法

这一节考虑 Stokes 方程的非协调混合元方法，即 $V_h \not\subset H_0^1(\Omega;\mathbb{R}^d)$. 此时，混合变分问题 (4.2)-(4.3) 的有限元离散为: 找 $\boldsymbol{u}_h \in V_h$ 和 $p_h \in P_h$ 满足

$$(\nabla_h \boldsymbol{u}_h, \nabla_h \boldsymbol{v}) + (\mathrm{div}_h\, \boldsymbol{v}, p_h) = (\boldsymbol{f}, \boldsymbol{v}), \quad \boldsymbol{v} \in V_h, \tag{4.26}$$

$$(\mathrm{div}_h\, \boldsymbol{u}_h, q) = 0, \qquad q \in P_h, \tag{4.27}$$

其中 $\nabla_h$ 和 $\mathrm{div}_h$ 关于网格剖分 $\mathcal{T}_h$ 分片定义的梯度算子和散度算子. 为了得到非协调元方法 (4.26)-(4.27) 解的适定性, 我们需要验证 Brezzi 定理的两个条件.

(a) 强制性: 即离散 Poincaré 不等式

$$\|\boldsymbol{v}\|_0^2 \lesssim (\nabla_h \boldsymbol{v}, \nabla_h \boldsymbol{v}), \quad \boldsymbol{v} \in V_h. \tag{4.28}$$

(b) 离散 inf-sup 条件

$$\|q\|_0 \lesssim \sup_{\boldsymbol{v}\in V_h} \frac{(\mathrm{div}_h\, \boldsymbol{v}, q)}{\|\nabla_h \boldsymbol{v}\|_0}, \quad q \in P_h. \tag{4.29}$$

类似于引理 4.3，离散 inf-sup 条件 (4.29) 等价于存在 Fortin 算子，即存在一个有界线性算子 $\Pi_h : H_0^1(\Omega;\mathbb{R}^d) \to V_h$ 满足

$$(\mathrm{div}_h(\boldsymbol{v} - \Pi_h \boldsymbol{v}), q) = 0, \qquad \boldsymbol{v} \in H_0^1(\Omega;\mathbb{R}^d), q \in P_h, \tag{4.30}$$

$$\|\nabla_h(\Pi_h \boldsymbol{v})\|_0 \lesssim \|\nabla_h \boldsymbol{v}\|_0, \quad \boldsymbol{v} \in H_0^1(\Omega;\mathbb{R}^d). \tag{4.31}$$

**定理 4.16.** 设 $(\boldsymbol{u}, p) \in H_0^1(\Omega;\mathbb{R}^d) \times L_0^2(\Omega)$ 是 Stokes 方程(4.1)的解, $(\boldsymbol{u}_h, p_h) \in V_h \times P_h$ 是混合元方法 (4.26)-(4.27) 的解，则有误差估计

$$\|\nabla_h(\boldsymbol{u} - \boldsymbol{u}_h)\|_0 \lesssim \|\nabla_h(\boldsymbol{u} - \Pi_h \boldsymbol{u})\|_0 + \sup_{\boldsymbol{v}\in V_h} \frac{E_h(\boldsymbol{u}, p; \boldsymbol{v})}{\|\nabla_h \boldsymbol{v}\|_0} + \inf_{q\in P_h} \sup_{\boldsymbol{v}\in V_h} \frac{(\mathrm{div}_h\, \boldsymbol{v}, p - q)}{\|\nabla_h \boldsymbol{v}\|_0}, \tag{4.32}$$

$$\|p - p_h\|_0 \lesssim \|\nabla_h(\boldsymbol{u} - \Pi_h \boldsymbol{u})\|_0 + \sup_{\boldsymbol{v}\in V_h} \frac{E_h(\boldsymbol{u}, p; \boldsymbol{v})}{\|\nabla_h \boldsymbol{v}\|_0} + \inf_{q\in P_h} \|p - q\|_0, \tag{4.33}$$

其中 $E_h(\boldsymbol{u}, p; \boldsymbol{v}) := (\nabla \boldsymbol{u}, \nabla_h \boldsymbol{v}) + (\mathrm{div}_h\, \boldsymbol{v}, p) - (\boldsymbol{f}, \boldsymbol{v})$. 进一步，若有 $\mathrm{div}_h\, V_h = P_h$，则

$$\|\nabla_h(\boldsymbol{u} - \boldsymbol{u}_h)\|_0 \lesssim \|\nabla_h(\boldsymbol{u} - \Pi_h \boldsymbol{u})\|_0 + \sup_{\boldsymbol{v}\in V_h} \frac{E_h(\boldsymbol{u}, p; \boldsymbol{v})}{\|\nabla_h \boldsymbol{v}\|_0}. \tag{4.34}$$

这里 $\sup_{\boldsymbol{v}\in V_h} \frac{E_h(\boldsymbol{u}, p; \boldsymbol{v})}{\|\nabla_h \boldsymbol{v}\|_0}$ 称为非协调元的相容性误差.

**证明**: 任取 $q \in P_h$, 记 $\boldsymbol{v} = \Pi_h \boldsymbol{u} - \boldsymbol{u}_h$, 则由(4.30)和(4.27)可得

$$(\mathrm{div}_h\, \boldsymbol{v}, q - p_h) = (\mathrm{div}_h(\Pi_h \boldsymbol{u} - \boldsymbol{u}_h), q - p_h) = 0.$$

接着利用(4.26)可得

$$\begin{aligned}
\|\nabla_h(\Pi_h \boldsymbol{u} - \boldsymbol{u}_h)\|_0^2 &= (\nabla_h(\Pi_h \boldsymbol{u} - \boldsymbol{u}_h), \nabla_h \boldsymbol{v}) = (\nabla_h(\Pi_h \boldsymbol{u} - \boldsymbol{u}_h), \nabla_h \boldsymbol{v}) + (\mathrm{div}_h\, \boldsymbol{v}, q - p_h) \\
&= (\nabla_h(\Pi_h \boldsymbol{u}), \nabla_h \boldsymbol{v}) + (\mathrm{div}_h\, \boldsymbol{v}, q) - (\boldsymbol{f}, \boldsymbol{v}) \\
&= (\nabla_h(\Pi_h \boldsymbol{u} - \boldsymbol{u}), \nabla_h \boldsymbol{v}) + (\mathrm{div}_h\, \boldsymbol{v}, q - p) + (\nabla \boldsymbol{u}, \nabla_h \boldsymbol{v}) + (\mathrm{div}_h\, \boldsymbol{v}, p) - (\boldsymbol{f}, \boldsymbol{v}).
\end{aligned}$$

故由 Cauchy-Schwarz 可得

$$\|\nabla_h(\Pi_h \boldsymbol{u} - \boldsymbol{u}_h)\|_0 \lesssim \|\nabla_h(\boldsymbol{u} - \Pi_h \boldsymbol{u})\|_0 + \sup_{\boldsymbol{v}\in V_h} \frac{E_h(\boldsymbol{u}, p; \boldsymbol{v})}{\|\nabla_h \boldsymbol{v}\|_0} + \inf_{q\in P_h} \sup_{\boldsymbol{v}\in V_h} \frac{(\operatorname{div}_h \boldsymbol{v}, p-q)}{\|\nabla_h \boldsymbol{v}\|_0}.$$

再结合三角不等式即有(4.32).

对任意的 $\boldsymbol{v} \in V_h$，利用(4.26)可推得

$$(\operatorname{div}_h \boldsymbol{v}, p - p_h) = (\operatorname{div}_h \boldsymbol{v}, p) + (\nabla_h \boldsymbol{u}_h, \nabla_h \boldsymbol{v}) - (\boldsymbol{f}, \boldsymbol{v}) = (\nabla_h(\boldsymbol{u}_h - \boldsymbol{u}), \nabla_h \boldsymbol{v}) + E_h(\boldsymbol{u}, p; \boldsymbol{v}).$$

由离散 inf-sup 条件(4.10)，

$$\begin{aligned}
\|q - p_h\|_0 &\lesssim \sup_{\boldsymbol{v}\in V_h} \frac{(\operatorname{div}_h \boldsymbol{v}, q - p_h)}{\|\nabla_h \boldsymbol{v}\|_0} \lesssim \|p - q\|_0 + \sup_{\boldsymbol{v}\in V_h} \frac{(\operatorname{div}_h \boldsymbol{v}, p - p_h)}{\|\nabla_h \boldsymbol{v}\|_0} \\
&\lesssim \|p - q\|_0 + \|\nabla_h(\boldsymbol{u} - \boldsymbol{u}_h)\|_0 + \sup_{\boldsymbol{v}\in V_h} \frac{E_h(\boldsymbol{u}, p; \boldsymbol{v})}{\|\nabla_h \boldsymbol{v}\|_0}.
\end{aligned}$$

进而借助三角不等式和(4.32)可推得(4.33).

进一步，若有 $\operatorname{div}_h V_h = P_h$，可取 $q = Q_h p$, 则(4.32)式右端的第三项为零，故(4.34)成立.

### 4.3.1 非协调 $P_1$-$P_0$ 元

向量值非协调线性元的形函数空间为 $\mathbb{P}_1(T; \mathbb{R}^d)$，自由度为

$$\int_F \boldsymbol{v} \,\mathrm{d}s, \quad F \in \Delta_{d-1}(T).$$

在文献 [38]中，分别用非协调线性元和分片常数离散速度和压力，即令

$$\begin{aligned}
V_h &:= \Big\{\boldsymbol{v} \in L^2(\Omega, \mathbb{R}^d) : \boldsymbol{v}|_T \in \mathbb{P}_1(T, \mathbb{R}^d), T \in \mathcal{T}_h, \int_F [\boldsymbol{v}] \,\mathrm{d}s = 0,\ F \in \triangle_{d-1}(\mathcal{T}_h)\Big\}, \\
P_h &:= \{q \in L_0^2(\Omega) : q|_T \in \mathbb{P}_0(T), T \in \mathcal{T}_h\},
\end{aligned}$$

其中 $[\boldsymbol{v}]|_F$ 为 $\boldsymbol{v}$ 跨过面 $F$ 的跳量; 当 $F \subset \partial\Omega$ 时，$[\boldsymbol{v}]|_F = \boldsymbol{v}$. 显然 $V_h \not\subset H_0^1(\Omega; \mathbb{R}^d)$. 对于非协调元空间 $V_h$，成立离散 Poincaré 不等式 [18]

$$\|\boldsymbol{v}\|_0 \lesssim \|\nabla_h \boldsymbol{v}\|_0, \quad \boldsymbol{v} \in V_h.$$

**引理 4.17.** 非协调 $P_1$-$P_0$ 元满足 $\operatorname{div}_h V_h = P_h$ 和离散 inf-sup 条件(4.29).

**证明**: 如下定义插值算子 $\Pi_h : H_0^1(\Omega; \mathbb{R}^d) \to V_h$:

$$\int_F \Pi_h \boldsymbol{v} \,\mathrm{d}s = \int_F \boldsymbol{v} \,\mathrm{d}s, \quad \boldsymbol{v} \in H_0^1(\Omega; \mathbb{R}^d),\ F \in \triangle_{d-1}(\mathcal{T}_h). \tag{4.35}$$

由分部积分可得

$$(\operatorname{div}_h(\boldsymbol{v} - \Pi_h \boldsymbol{v}), q) = 0, \quad \boldsymbol{v} \in H_0^1(\Omega; \mathbb{R}^d),\ q \in P_h.$$

故 $\operatorname{div}_h(\Pi_h \boldsymbol{v}) = Q_h(\operatorname{div} \boldsymbol{v})$ 对 $\boldsymbol{v} \in H_0^1(\Omega; \mathbb{R}^d)$ 成立. 于是, $\operatorname{div}_h V_h = P_h$. 由尺度论证技巧可证

$$\|\nabla_h(\Pi_h \boldsymbol{v})\|_0 \lesssim \|\nabla_h \boldsymbol{v}\|_0, \quad \boldsymbol{v} \in H_0^1(\Omega; \mathbb{R}^d).$$

因此，$\Pi_h$ 是 Fortin 算子，从而由 Fortin 准则知离散 inf-sup 条件(4.29)成立.

非协调 $P_1$-$P_0$ 元方法是分片 divergence-free 的，但不是 divergence-free 的.

**引理 4.18.** 设 $(\boldsymbol{u},p)\in H_0^1(\Omega;\mathbb{R}^d)\times L_0^2(\Omega)$ 是 Stokes 方程(4.1)的解, $(\boldsymbol{u}_h,p_h)\in V_h\times P_h$ 是非协调 $P_1$-$P_0$ 元方法 (4.26)-(4.27) 的解. 假设 $\boldsymbol{u}\in H^2(\Omega;\mathbb{R}^d)$, $p\in H^1(\Omega)$, 则有误差估计

$$
\|\nabla_h(\boldsymbol{u}-\boldsymbol{u}_h)\|_0+\|p-p_h\|_0\lesssim h(\|\boldsymbol{u}\|_2+\|p\|_1). \tag{4.36}
$$

**证明**: 由定理 4.16和 $\Pi_h$ 的插值误差估计，我们只要估计相容性误差即可.

由分部积分和弱连续性可推得，

$$
\begin{aligned}
E_h(\boldsymbol{u},p;\boldsymbol{v})&=(\nabla\boldsymbol{u},\nabla_h\boldsymbol{v})+(\operatorname{div}_h\boldsymbol{v},p)-(\boldsymbol{f},\boldsymbol{v})=(\nabla\boldsymbol{u},\nabla_h\boldsymbol{v})+(\operatorname{div}_h\boldsymbol{v},p)+(\Delta\boldsymbol{u}+\nabla p,\boldsymbol{v})\\
&=\sum_{T\in\mathcal{T}_h}\big((\partial_n\boldsymbol{u},\boldsymbol{v})_{\partial T}+(p,\boldsymbol{v}\cdot\boldsymbol{n})_{\partial T}\big)=\sum_{F\in\mathcal{F}_h}\big((\partial_n\boldsymbol{u},[\boldsymbol{v}])_F+(p,[\boldsymbol{v}\cdot\boldsymbol{n}])_F\big)\\
&=\sum_{F\in\mathcal{F}_h}\big((\partial_n\boldsymbol{u}-Q_{0,F}(\partial_n\boldsymbol{u}),[\boldsymbol{v}-Q_{0,F}\boldsymbol{v}])_F+(p-Q_{0,F}p,[(\boldsymbol{v}-Q_{0,F}\boldsymbol{v})\cdot\boldsymbol{n}])_F\big).
\end{aligned}
$$

再利用投影算子 $Q_{0,F}$ 的误差估计可得

$$
E_h(\boldsymbol{u},p;\boldsymbol{v})\lesssim h(\|\boldsymbol{u}\|_2+\|p\|_1).
$$

得证.

由误差估计(4.36)知，非协调 $P_1$-$P_0$ 元方法不是压力鲁棒的. 通过对(4.26)右端项中的 $\boldsymbol{v}$ 进行最低次 Raviart–Thomas 重构 [62, 17]，得到的非协调 $P_1$-$P_0$ 元方法是压力鲁棒的. 更多 Stokes 方程数值方法压力鲁棒性的介绍参见 [58].

### 4.3.2 一个新的非协调元

记二次非协调泡函数

$$
b_T^{\mathrm{NC}}=2-(d+1)(\lambda_0^2+\lambda_1^2+\cdots+\lambda_d^2).
$$

泡函数 $b_T^{\mathrm{NC}}$ 在单形重心处取值为 1.

**引理 4.19.** 对任意的 $F\in\Delta_{d-1}(T)$, 成立

$$
(b_T^{\mathrm{NC}},q)_F=0,\quad q\in\mathbb{P}_1(F).
$$

**证明**: 不妨设面 $F$ 对应顶点 $\mathrm{v}_0$，故等价于证明

$$
(b_T^{\mathrm{NC}},\lambda_i)_F=0,\quad i=1,2,\ldots,d.
$$

通过直接计算

$$
\frac{1}{|F|}(b_T^{\mathrm{NC}},\lambda_i)_F=\frac{2}{d}-(d+1)\sum_{j=1}^{d}\frac{1}{|F|}\int_F\lambda_j^2\lambda_i\,\mathrm{d}s=\frac{2}{d}-(d+1)\frac{(d-1)!}{(d+2)!}(2d+4)=0.
$$

得证.

速度的形函数空间取为 $\mathbb{P}_1(T;\mathbb{R}^d)+b_T^{\mathrm{NC}}\mathbb{P}_0(T;\mathbb{R}^d)$，自由度为

$$
\begin{aligned}
&\int_F\boldsymbol{v}\,\mathrm{d}s,\quad F\in\Delta_{d-1}(T),\\
&\int_T\boldsymbol{v}\,\mathrm{d}x.
\end{aligned}
$$

压力的形函数空间取为 $\mathbb{P}_1(T)$

- 压力用分片线性元离散，由此给出 Stokes 方程的一种非协调元方法，进行误差分析和数值试验
- 压力用非协调线性元离散，由此给出 Stokes 方程的一种非协调元方法，进行误差分析和数值试验
- 压力用线性元 Lagrange 元离散，由此给出 Stokes 方程的一种非协调元方法，进行误差分析和数值试验

### 4.3.3 Divergence-free 非协调元

文献 [76]在 $H(\mathrm{div})$ 协调元的基础上，通过泡函数增加切向连续性来构造 divergence-free 非协调元. 文献 [76]给出了两维、三维低阶 divergence-free 非协调元，这里统一给出任意维 divergence-free 非协调元.

记 $\mathbb{K}$ 为所有 $d$ 阶反对称矩阵所组成的线性空间. 设 $d$ 维单形 $T$ 的顶点为 $\mathtt{v}_0,\mathtt{v}_1,\dots,\mathtt{v}_d$，相对应的 $(d-1)$ 维面为 $F_i$ $(i=0,1,\dots,d)$, 面 $F_i$ 的法向量记为 $\boldsymbol{n}_{F_i}$, 不在引起混淆的情况下简记为 $\boldsymbol{n}_i$.

**引理 4.20.** 设 $\boldsymbol{\tau}_0,\boldsymbol{\tau}_1,\dots,\boldsymbol{\tau}_d\in\mathbb{K}$ 满足

$$\boldsymbol{n}_i^{\intercal}\boldsymbol{\tau}_j\boldsymbol{n}_k=\boldsymbol{n}_j^{\intercal}\boldsymbol{\tau}_i\boldsymbol{n}_k,\quad 0\le i,j,k\le d, i\ne j, i\ne k, j\ne k,$$

则 $\boldsymbol{\tau}_i=0$, 其中 $i=0,1,\dots,d$.

**证明**: 由 $\boldsymbol{\tau}_i$ 的反对称性知 $\boldsymbol{n}_k^{\intercal}\boldsymbol{\tau}_j\boldsymbol{n}_i=-\boldsymbol{n}_i^{\intercal}\boldsymbol{\tau}_j\boldsymbol{n}_k$, $\boldsymbol{n}_k^{\intercal}\boldsymbol{\tau}_i\boldsymbol{n}_j=-\boldsymbol{n}_j^{\intercal}\boldsymbol{\tau}_i\boldsymbol{n}_k$, 因此

$$\boldsymbol{n}_k^{\intercal}\boldsymbol{\tau}_j\boldsymbol{n}_i=\boldsymbol{n}_k^{\intercal}\boldsymbol{\tau}_i\boldsymbol{n}_j,\quad 0\le i,j,k\le d, i\ne j, i\ne k, j\ne k.$$

从而，$\boldsymbol{n}_j^{\intercal}\boldsymbol{\tau}_i\boldsymbol{n}_k$ 关于前两个下标和后两个下标均是对称的. 由对称性可得

$$\boldsymbol{n}_k^{\intercal}\boldsymbol{\tau}_i\boldsymbol{n}_j=\boldsymbol{n}_i^{\intercal}\boldsymbol{\tau}_k\boldsymbol{n}_j=\boldsymbol{n}_i^{\intercal}\boldsymbol{\tau}_j\boldsymbol{n}_k=\boldsymbol{n}_j^{\intercal}\boldsymbol{\tau}_i\boldsymbol{n}_k.$$

又由 $\boldsymbol{\tau}_i$ 的反对称性知 $\boldsymbol{n}_k^{\intercal}\boldsymbol{\tau}_i\boldsymbol{n}_j=-\boldsymbol{n}_j^{\intercal}\boldsymbol{\tau}_i\boldsymbol{n}_k$. 故

$$\boldsymbol{n}_j^{\intercal}\boldsymbol{\tau}_i\boldsymbol{n}_k=0,\quad 0\le i,j,k\le d, j\ne i, k\ne i.$$

注意到 $\{\boldsymbol{n}_j\boldsymbol{n}_k^{\intercal}:0\le j,k\le d, j\ne i, k\ne i\}$ 构成了 $d$ 阶矩阵的一组基. 于是 $\boldsymbol{\tau}_i=0$，得证.

引入低次 $H(\mathrm{div})$ 有限元形函数空间 $V^{\mathrm{div}}(T)=\mathbb{P}_1(T;\mathbb{R}^d)+\boldsymbol{x}\mathbb{P}_k(T)$, $k=0,1$. 当 $k=0$ 时，$V^{\mathrm{div}}(T)=\mathbb{P}_1(T;\mathbb{R}^d)$ 为最低次 Brezzi-Douglas-Marini (BDM) 元 [23, 22, 65]的形函数空间；当 $k=1$ 时，$V^{\mathrm{div}}(T)$ 为一次 Raviart-Thomas (RT) 元 [67, 64, 31]的形函数空间. 定义刚体运动空间 $\mathrm{RM}(T):=\mathbb{P}_0(T;\mathbb{R}^d)+\mathbb{K}\boldsymbol{x}$. $\mathrm{RM}(T)$ 也是第一类最低次 Nedelec 元的形函数空间 [64]，且有

$$\mathrm{RM}(T)=\mathrm{span}\{\lambda_i\nabla\lambda_j-\lambda_j\nabla\lambda_i:0\le i<j\le d\},\quad \dim\mathrm{RM}(T)=\frac{1}{2}d(d+1).$$

当 $d=1$ 时，$\mathrm{RM}(T)=\mathbb{P}_0(T)$.

现在在低次 $H(\mathrm{div})$ 有限元的基础上定义 divergence-free 非协调元. 形函数空间取为 $V(T):=V^{\mathrm{div}}(T)\oplus\mathrm{div}(b_T\mathbb{P}_1(T;\mathbb{K}))$, 其中 $b_T:=\lambda_0\lambda_1\dots\lambda_d$ 为泡函数. 自由度为

$$(\boldsymbol{v}\cdot\boldsymbol{n},q)_F,\quad q\in\mathbb{P}_1(F), F\in\Delta_{d-1}(T),\tag{4.37a}$$

$$(\boldsymbol{n}\times\boldsymbol{v}\times\boldsymbol{n},\boldsymbol{q})_F,\quad \boldsymbol{q}\in\mathrm{RM}(F), F\in\Delta_{d-1}(T),\tag{4.37b}$$

$$(\boldsymbol{v},\boldsymbol{q})_T,\quad \boldsymbol{q}\in\mathbb{P}_0(T;\mathbb{R}^d)\quad \text{if } k=1.\tag{4.37c}$$

**引理 4.21.** 设 $T$ 是一个 $d$ 维单形，$\boldsymbol{\tau} \in \mathbb{P}_1(T;\mathbb{K})$，则 $\boldsymbol{v} = \operatorname{div}(b_T\boldsymbol{\tau})$ 满足 $\operatorname{div}\boldsymbol{v} = 0$，以及 $(\boldsymbol{v}\cdot\boldsymbol{n})|_{\partial T} = 0$.

**证明**: 由 $\boldsymbol{\tau}$ 的反对称性可得 $\operatorname{div}\boldsymbol{v} = 0$. 对于 $F \in \Delta_{d-1}(T)$，由 $\boldsymbol{\tau}$ 的反对称性和 $b_T|_F = 0$ 可得

$$(\boldsymbol{v}\cdot\boldsymbol{n})|_F = (\operatorname{div}(b_T\boldsymbol{n}^\intercal\boldsymbol{\tau}))|_F = (\operatorname{div}(b_T\boldsymbol{n}^\intercal\boldsymbol{\tau}\Pi_F))|_F = (\operatorname{div}_F(b_T\boldsymbol{n}^\intercal\boldsymbol{\tau}\Pi_F))|_F = 0.$$

**引理 4.22.** 形函数空间 $V(T) = V^{\operatorname{div}}(T) \oplus \operatorname{div}(b_T\mathbb{P}_1(T;\mathbb{K}))$ 由自由度(4.37)所唯一确定.

**证明**: 先证明 $V^{\operatorname{div}}(T) \cap \operatorname{div}(b_T\mathbb{P}_1(T;\mathbb{K})) = 0$. 对于 $\boldsymbol{v} \in V^{\operatorname{div}}(T) \cap \operatorname{div}(b_T\mathbb{P}_1(T;\mathbb{K}))$, 由 $\operatorname{div}\boldsymbol{v} = 0$ 得 $\boldsymbol{v} \in \mathbb{P}_1(T;\mathbb{R}^d)$. 注意到 $\boldsymbol{v} \in \operatorname{div}(b_T\mathbb{P}_1(T;\mathbb{K}))$ 意味着 $(\boldsymbol{v}\cdot\boldsymbol{n})|_{\partial T} = 0$，故由 BDM 元的唯一可解性知 $\boldsymbol{v} = 0$. 于是，形函数空间 $V(T)$ 的维数为

$$\dim V^{\operatorname{div}}(T) + \dim\mathbb{P}_1(T;\mathbb{K}) = d(d+1) + dk + \frac{1}{2}d(d^2-1),$$

恰好等于自由度(4.37)的个数.

设 $\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2 \in V(T)$ 满足(4.37)中所有自由度为零，其中 $\boldsymbol{v}_1 \in V^{\operatorname{div}}(T)$，$\boldsymbol{v}_2 \in \operatorname{div}(b_T\mathbb{P}_1(T;\mathbb{K}))$. 易知 $\operatorname{div}\boldsymbol{v}_2 = 0$, 以及 $(\boldsymbol{v}_2\cdot\boldsymbol{n})|_{\partial T} = 0$. 因此 $\operatorname{div}\boldsymbol{v} = \operatorname{div}\boldsymbol{v}_1$, 以及 $(\boldsymbol{v}\cdot\boldsymbol{n})|_{\partial T} = (\boldsymbol{v}_1\cdot\boldsymbol{n})|_{\partial T}$. 借助分部积分，由自由度(4.37a)和(4.37c)可推

$$(\operatorname{div}\boldsymbol{v}, q)_T = (\boldsymbol{v}\cdot\boldsymbol{v}, q)_{\partial T} - (\boldsymbol{v}, \nabla q)_T = 0, \quad q \in \mathbb{P}_1(T).$$

于是 $\boldsymbol{v}_1 \in \mathbb{P}_1(T;\mathbb{R}^d)$ 且满足 $(\boldsymbol{v}_1\cdot\boldsymbol{n})|_{\partial T} = 0$. 由 BDM 元的唯一可解性知 $\boldsymbol{v}_1 = 0$. 故 $\boldsymbol{v} \in \operatorname{div}(b_T\mathbb{P}_1(T;\mathbb{K}))$.

设 $\boldsymbol{v} = \sum_{i=0}^d \operatorname{div}(b_T\lambda_i\boldsymbol{\tau}_i)$, 其中 $\boldsymbol{\tau}_0, \boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_d \in \mathbb{K}$. 由自由度(4.37b)可得

$$\sum_{m=0}^{d}\big(\operatorname{div}(b_T\lambda_m\boldsymbol{\tau}_m), \lambda_i\nabla_F\lambda_j - \lambda_j\nabla_F\lambda_i\big)_{F_\ell} = \big(\boldsymbol{v}, \lambda_i\nabla_F\lambda_j - \lambda_j\nabla_F\lambda_i\big)_{F_\ell} = 0,$$

其中 $0 \le i,j,\ell \le d, i \ne \ell, j \ne \ell$. 注意到 $(\operatorname{div} b_T)|_{F_\ell} = b_{F_\ell}\nabla\lambda_\ell$,

$$\sum_{m=0}^{d}\big(b_{F_\ell}\lambda_m\boldsymbol{\tau}_m\boldsymbol{n}_\ell, \lambda_i\nabla_F\lambda_j - \lambda_j\nabla_F\lambda_i\big)_{F_\ell} = 0.$$

直接计算可得

$$2\sum_{m\ne\ell}(\nabla\lambda_j)^\intercal\boldsymbol{\tau}_m\boldsymbol{n}_\ell + (\nabla\lambda_j)^\intercal\boldsymbol{\tau}_i\boldsymbol{n}_\ell - 2\sum_{m\ne\ell}(\nabla\lambda_i)^\intercal\boldsymbol{\tau}_m\boldsymbol{n}_\ell - (\nabla\lambda_i)^\intercal\boldsymbol{\tau}_j\boldsymbol{n}_\ell = 0,$$

其中 $0 \le i,j,\ell \le d, i \ne \ell, j \ne \ell$. 将上式关于下标 $i$ 从 0 到 $d$ 除了 $j$ 和 $\ell$ 进行求和可得

$$(2d+1)\sum_{m\ne\ell}(\nabla\lambda_j)^\intercal\boldsymbol{\tau}_m\boldsymbol{n}_\ell = 0, \quad 0 \le j,\ell \le d, j \ne \ell.$$

因此 $(\nabla\lambda_j)^\intercal\boldsymbol{\tau}_i\boldsymbol{n}_\ell = (\nabla\lambda_i)^\intercal\boldsymbol{\tau}_j\boldsymbol{n}_\ell$, 也即

$$\boldsymbol{n}_j^\intercal\boldsymbol{\tau}_i\boldsymbol{n}_\ell = \boldsymbol{n}_i^\intercal\boldsymbol{\tau}_j\boldsymbol{n}_\ell, \quad 0 \le i,j,\ell \le d, i \ne \ell, j \ne \ell.$$

由引理 4.20知, $\boldsymbol{\tau}_i = 0, i = 0,1,\ldots,d$. 故 $\boldsymbol{v} = 0$.

分别定义离散速度和压力的整体有限元空间

$$\begin{aligned} V_h := \{\boldsymbol{v} \in L^2(\Omega;\mathbb{R}^d) : \boldsymbol{v}|_T \in V^{\operatorname{div}}(T) \oplus \operatorname{div}(b_T\mathbb{P}_1(T;\mathbb{K})), T \in \mathcal{T}_h, \\ \text{所有自由度(4.37)跨过内部边界是连续的}, \\ \text{自由度(4.37a)-(4.37b)在边界上为零}\}, \end{aligned}$$

$$P_h := \{q \in L_0^2(\Omega) : q|_T \in \mathbb{P}_k(T), T \in \mathcal{T}_h\}.$$

显然有 $V_h \subset H_0(\mathrm{div}, \Omega)$ 但 $V_h \not\subseteq H^1(\Omega; \mathbb{R}^d)$, 且有弱连续性

$$\int_F [\boldsymbol{v}] \,\mathrm{d}s = 0, \quad F \in \mathcal{F}_h. \tag{4.38}$$

**引理 4.23.** Divergence-free 非协调元满足 $\mathrm{div}\, V_h = P_h$ 和离散 inf-sup 条件(4.29).

**证明**: 令 $\Pi_h : H_0^1(\Omega; \mathbb{R}^d) \to V_h$ 为基于自由度(4.37)的节点插值算子，则利用分部积分和尺度论证技巧可推得 $\Pi_h$ 是 Fortin 算子，且有 $\mathrm{div}\, V_h = P_h$.

Divergence-free 非协调元方法是 divergence-free 的.

**引理 4.24.** 设 $(\boldsymbol{u}, p) \in H_0^1(\Omega; \mathbb{R}^d) \times L_0^2(\Omega)$ 是 Stokes 方程(4.1)的解, $(\boldsymbol{u}_h, p_h) \in V_h \times P_h$ 是 divergence-free 非协调元方法 (4.26)-(4.27) 的解. 假设 $\boldsymbol{u} \in H^2(\Omega; \mathbb{R}^d)$，$p \in H^1(\Omega)$，则有误差估计

$$\|\nabla_h(\boldsymbol{u} - \boldsymbol{u}_h)\|_0 \lesssim h\|\boldsymbol{u}\|_2, \tag{4.39}$$

$$\|p - p_h\|_0 \lesssim h(\|\boldsymbol{u}\|_2 + h^k\|p\|_1). \tag{4.40}$$

**证明**: 由定理 4.16和 $\Pi_h$ 的插值误差估计，我们只要估计相容性误差即可.

由分部积分和弱连续性可推得，

$$\begin{aligned} E_h(\boldsymbol{u}, p; \boldsymbol{v}) &= (\nabla \boldsymbol{u}, \nabla_h \boldsymbol{v}) + (\mathrm{div}\, \boldsymbol{v}, p) - (\boldsymbol{f}, \boldsymbol{v}) = (\nabla \boldsymbol{u}, \nabla_h \boldsymbol{v}) + (\mathrm{div}\, \boldsymbol{v}, p) + (\Delta \boldsymbol{u} + \nabla p, \boldsymbol{v}) \\ &= \sum_{T \in \mathcal{T}_h} (\partial_n \boldsymbol{u}, \boldsymbol{v})_{\partial T} = \sum_{F \in \mathcal{F}_h} (\partial_n \boldsymbol{u}, [\boldsymbol{v}])_F \\ &= \sum_{F \in \mathcal{F}_h} (\partial_n \boldsymbol{u} - Q_{0,F}(\partial_n \boldsymbol{u}), [\boldsymbol{v} - Q_{0,F} \boldsymbol{v}])_F. \end{aligned}$$

再利用投影算子 $Q_{0,F}$ 的误差估计可得

$$E_h(\boldsymbol{u}, p; \boldsymbol{v}) \lesssim h\|\boldsymbol{u}\|_2.$$

得证.

当 $d = 2, 3$ 时，我们得到了文献 [76]中的两维、三维低阶 divergence-free 非协调元.

只要有弱连续性(4.38), 即可得到误差估计(4.39)和(4.40). 因此，可以减少自由度(4.37b)中的矩量. 令反对称矩阵线性函数约化空间

$$\mathbb{P}_1^r(T; \mathbb{K}) = \mathbb{P}_1(T; \mathbb{K}) / \oplus_{i=0}^d (d\lambda_i - 1)\mathbb{K}_i,$$

其中 $\mathbb{K}_i$ 是面 $F_i$ 上全部切向反对称矩阵所形成的线性空间.

**引理 4.25.** $\dim \mathbb{P}_1^r(T; \mathbb{K}) = d^2 - 1$.

**证明**: 由于

$$\sum_{i=0}^d \dim \mathbb{K}_i = \frac{1}{2}(d-2)(d^2-1),$$

故只需证明 $\{(d\lambda_i - 1)\mathbb{K}_i, i = 0, 1, \ldots, d\}$ 线性无关. 设

$$\sum_{i=0}^d (d\lambda_i - 1)\boldsymbol{\tau}_i = 0, \quad \boldsymbol{\tau}_i \in \mathbb{K}_i.$$

将重心的坐标代入，即 $\lambda_0=\lambda_1=\ldots=\lambda_d=\frac{1}{d+1}$，可推得

$$\sum_{i=0}^{d}\boldsymbol{\tau}_i=0,\quad \sum_{i=0}^{d}\lambda_i\boldsymbol{\tau}_i=0.$$

由 $\lambda_0,\lambda_1,\ldots,\lambda_d$ 的线性无关性可得 $\boldsymbol{\tau}_i=0, i=0,1,\ldots,d$.

约化后的形函数空间取为 $V^r(T):=V^{\mathrm{div}}(T)\oplus\mathrm{div}(b_T\mathbb{P}_1^r(T;\mathbb{K}))$. 自由度为

$$(\boldsymbol{v}\cdot\boldsymbol{n},q)_F,\quad q\in\mathbb{P}_1(F), F\in\Delta_{d-1}(T), \tag{4.41a}$$

$$(\boldsymbol{n}\times\boldsymbol{v}\times\boldsymbol{n},\boldsymbol{q})_F,\quad \boldsymbol{q}\in\mathbb{P}_0(F;\mathbb{R}^{d-1}), F\in\Delta_{d-1}(T), \tag{4.41b}$$

$$(\boldsymbol{v},\boldsymbol{q})_T,\quad \boldsymbol{q}\in\mathbb{P}_0(T;\mathbb{R}^d)\quad \text{if } k=1. \tag{4.41c}$$

**引理 4.26.** *形函数空间 $V^r(T)=V^{\mathrm{div}}(T)\oplus\mathrm{div}(b_T\mathbb{P}_1^r(T;\mathbb{K}))$ 由自由度(4.41)所唯一确定.*

**证明**: 形函数空间 $V^r(T)$ 的维数为

$$\dim V^r(T)=d(d+1)+dk+d^2-1,$$

等于自由度(4.37)的个数为 $d(d+1)+d^2-1+dk$.

设 $\boldsymbol{v}\in V^r(T)$ 满足(4.41)中所有自由度为零. 类似于引理 4.22的证明可得 $\boldsymbol{v}\in\mathrm{div}(b_T\mathbb{P}_1^r(T;\mathbb{K}))$.

设 $\boldsymbol{v}=\sum_{i=0}^d\mathrm{div}(b_T\lambda_i\boldsymbol{\tau}_i)$, 其中 $\boldsymbol{\tau}_0,\boldsymbol{\tau}_1,\ldots,\boldsymbol{\tau}_d\in\mathbb{K}$. 由自由度(4.41b)可得

$$\sum_{i=0}^{d}\int_{F_j}b_{F_j}\lambda_i\boldsymbol{\tau}_i\nabla\lambda_j\,\mathrm{d}s=\int_{F_j}\boldsymbol{n}\times\boldsymbol{v}\times\boldsymbol{n}\,\mathrm{d}s=0,\quad j=0,1,\ldots,d.$$

由此可得

$$\sum_{i=0}^{d}\boldsymbol{\tau}_i\boldsymbol{n}_j=\boldsymbol{\tau}_j\boldsymbol{n}_j,\quad j=0,1,\ldots,d.$$

由 $\boldsymbol{\tau}_i$ 的反对称性可得

$$\boldsymbol{\tau}_j=\boldsymbol{\sigma}_j+\sum_{i=0}^{d}\boldsymbol{\tau}_i,\quad j=0,1,\ldots,d,$$

其中 $\boldsymbol{\sigma}_j\in\mathbb{K}_j$. 对上式关于 $j$ 进行求和有 $d\sum_{i=0}^d\boldsymbol{\tau}_i=-\sum_{i=0}^d\boldsymbol{\sigma}_i$, 故

$$\boldsymbol{\tau}_j=\boldsymbol{\sigma}_j-\frac{1}{d}\sum_{i=0}^{d}\boldsymbol{\sigma}_i,\quad j=0,1,\ldots,d.$$

从而

$$\sum_{i=0}^{d}\lambda_i\boldsymbol{\tau}_i=\sum_{i=0}^{d}\lambda_i\boldsymbol{\sigma}_i-\frac{1}{d}\sum_{i=0}^{d}\boldsymbol{\sigma}_i=\frac{1}{d}\sum_{i=0}^{d}(d\lambda_i-1)\boldsymbol{\sigma}_i\in\oplus_{i=0}^d(d\lambda_i-1)\mathbb{K}_i.$$

另一方面，$\sum_{i=0}^d\lambda_i\boldsymbol{\tau}_i\in\mathbb{P}_1^r(T;\mathbb{K})=\mathbb{P}_1(T;\mathbb{K})/\oplus_{i=0}^d(d\lambda_i-1)\mathbb{K}_i$. 所以，$\sum_{i=0}^d\lambda_i\boldsymbol{\tau}_i=0$, 从而 $\boldsymbol{v}=0$.

约化后的 divergence-free 非协调元方法具有与(4.39)和(4.40)一样的误差估计.

三维情形 $d=3$，约化向量有限元形函数空间为 $V^{\mathrm{div}}(T)\oplus\mathrm{curl}(b_T\mathbb{P}_1^r(T;\mathbb{R}^3))$, 其中 $\mathbb{P}_1^r(T;\mathbb{R}^3):=\mathbb{P}_1(T;\mathbb{R}^3)/\mathrm{span}\{(3\lambda_i-1)\nabla\lambda_i, i=0,1,2,3\}$, 与文献 [76]中的相同.

## 4.4 Stokes 复形

### 4.4.1 有限元 Stokes 复形

回顾二维 de Rham 复形

$$0 \to H_0^1(\Omega) \xrightarrow{\text{curl}} H_0(\text{div}, \Omega) \xrightarrow{\text{div}} L_0^2(\Omega) \to 0,$$

$$\mathbb{R} \to H^1(\Omega) \xrightarrow{\text{curl}} H(\text{div}, \Omega) \xrightarrow{\text{div}} L^2(\Omega) \to 0,$$

$$0 \to H_0^{s+2}(\Omega) \xrightarrow{\text{curl}} H_0^{s+1}(\Omega; \mathbb{R}^2) \xrightarrow{\text{div}} H_0^s(\Omega) \cap L_0^2(\Omega) \to 0,$$

$$\mathbb{R} \to H^{s+2}(\Omega) \xrightarrow{\text{curl}} H^{s+1}(\Omega; \mathbb{R}^2) \xrightarrow{\text{div}} H^s(\Omega) \to 0.$$

回顾三维 de Rham 复形

$$0 \to H_0^1(\Omega) \xrightarrow{\text{grad}} H_0(\text{curl}, \Omega) \xrightarrow{\text{curl}} H_0(\text{div}, \Omega) \xrightarrow{\text{div}} L_0^2(\Omega) \to 0,$$

$$\mathbb{R} \to H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl}, \Omega) \xrightarrow{\text{curl}} H(\text{div}, \Omega) \xrightarrow{\text{div}} L^2(\Omega) \to 0,$$

$$0 \to H_0^{s+3}(\Omega) \xrightarrow{\text{grad}} H_0^{s+2}(\Omega; \mathbb{R}^3) \xrightarrow{\text{curl}} H_0^{s+1}(\Omega; \mathbb{R}^3) \xrightarrow{\text{div}} H_0^s(\Omega) \cap L_0^2(\Omega) \to 0,$$

$$\mathbb{R} \to H^{s+3}(\Omega) \xrightarrow{\text{grad}} H^{s+2}(\Omega; \mathbb{R}^3) \xrightarrow{\text{curl}} H^{s+1}(\Omega; \mathbb{R}^3) \xrightarrow{\text{div}} H^s(\Omega) \to 0.$$

# 第五章　重调和方程混合元方法

The biharmonic equation with homogenous Dirichlet boundary condition is given by

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u|_{\partial\Omega} = \partial_n u|_{\partial\Omega} = 0. \end{cases} \tag{5.1}$$

The biharmonic equation (5.1) arises in areas of continuum mechanics, including linear elasticity theory and the solution of Stokes flows. In two dimensionas, the above problem is the Kirchhoff model describing the deflection of an elastic thin plate subject to a vertical load $f$ [44, 68].

Let $f \in H^{-2}(\Omega)$. The variational problem for the biharmonic equation (5.1) is to find $u \in H_0^2(\Omega)$ satisfying

$$a(u, v) = \langle f, v\rangle \quad \forall\, v \in H_0^2(\Omega), \tag{5.2}$$

where $a(u, v) := (\nabla^2 u, \nabla^2 v)$. The variational problem (5.2) is wellposed since

$$\|v\|_2 \lesssim |v|_2 \quad \forall\, v \in H_0^2(\Omega).$$

By the elliptic regularity theory for non-smooth domains (cf. [13, 49, 39, 50]), there exists $\alpha \in (\frac{1}{2}, 1]$ such that for $f \in H^{-2+\alpha}(\Omega)$, the solution $u$ of problem (5.2) belongs to $H^{2+\alpha}(\Omega)$ and

$$\|u\|_{H^{2+\alpha}(\Omega)} \lesssim \|f\|_{H^{-2+\alpha}(\Omega)}.$$

When $\Omega$ is convex, we can take $\alpha = 1$, i.e.,

$$\|u\|_{H^3(\Omega)} \lesssim \|f\|_{H^{-1}(\Omega)}.$$

## 5.1 Hellan-Herrmann-Johnson Mixed Method

In this section, assume $f \in H^{-1}(\Omega)$. Introducing $\boldsymbol{\sigma} := -\nabla^2 u$, rewrite the biharmonic equation (5.1)

$$\begin{cases} \boldsymbol{\sigma} = -\nabla^2 u & \text{in } \Omega, \\ \operatorname{div}\operatorname{div}\boldsymbol{\sigma} = -f & \text{in } \Omega, \\ u|_{\partial\Omega} = \partial_n u|_{\partial\Omega} = 0. \end{cases} \tag{5.3}$$

Define the Hilbert space (cf. [66])

$$\boldsymbol{H}^{-1}(\operatorname{div}\operatorname{div}, \Omega; \mathbb{S}) := \{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega; \mathbb{S}) : \operatorname{div}\operatorname{div}\boldsymbol{\tau} \in H^{-1}(\Omega)\}$$

with squared norm $\|\boldsymbol{\tau}\|^2_{\boldsymbol{H}^{-1}(\operatorname{div}\operatorname{div})} := \|\boldsymbol{\tau}\|_0^2 + \|\operatorname{div}\operatorname{div}\boldsymbol{\tau}\|_{-1}^2$.

The Hellan-Herrmann-Johnson (HHJ) mixed formulation [60, 52, 53, 59] of the biharmonic equation (5.3) in two dimensions is to find $(\boldsymbol{\sigma}, u) \in \boldsymbol{H}^{-1}(\operatorname{div}\operatorname{div}, \Omega; \mathbb{S}) \times H_0^1(\Omega)$ such that

$$a(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, u) = 0 \qquad \forall\, \boldsymbol{\tau} \in \boldsymbol{H}^{-1}(\operatorname{div}\operatorname{div}, \Omega; \mathbb{S}), \tag{5.4}$$

$$b(\boldsymbol{\sigma}, v) = -\langle f, v\rangle \quad \forall\, v \in H_0^1(\Omega). \tag{5.5}$$

where

$$a(\boldsymbol{\sigma}, \boldsymbol{\tau}) := (\boldsymbol{\sigma}, \boldsymbol{\tau}), \quad b(\boldsymbol{\tau}, v) := \langle \operatorname{div}\operatorname{div} \boldsymbol{\tau}, v\rangle.$$

Obviously

$$a(\boldsymbol{\sigma}, \boldsymbol{\tau}) \leq \|\boldsymbol{\sigma}\|_0 \|\boldsymbol{\tau}\|_0 \leq \|\boldsymbol{\sigma}\|_{\boldsymbol{H}^{-1}(\operatorname{div}\operatorname{div})} \|\boldsymbol{\tau}\|_{\boldsymbol{H}^{-1}(\operatorname{div}\operatorname{div})} \quad \forall\, \boldsymbol{\sigma}, \boldsymbol{\tau} \in \boldsymbol{H}^{-1}(\operatorname{div}\operatorname{div}, \Omega; \mathbb{S}),$$

$$b(\boldsymbol{\tau}, v) \leq \|\operatorname{div}\operatorname{div} \boldsymbol{\tau}\|_{-1} |v|_1 \leq \|\boldsymbol{\tau}\|_{\boldsymbol{H}^{-1}(\operatorname{div}\operatorname{div})} |v|_1 \quad \forall\, \boldsymbol{\tau} \in \boldsymbol{H}^{-1}(\operatorname{div}\operatorname{div}, \Omega; \mathbb{S}), v \in H_0^1(\Omega).$$

Define $B : \boldsymbol{H}^{-1}(\operatorname{div}\operatorname{div}, \Omega; \mathbb{S}) \to H^{-1}(\Omega)$ by

$$\langle B\boldsymbol{\tau}, v\rangle = b(\boldsymbol{\tau}, v) = \langle \operatorname{div}\operatorname{div} \boldsymbol{\tau}, v\rangle.$$

Since $\ker B = \{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega; \mathbb{S}) : \operatorname{div}\operatorname{div} \boldsymbol{\tau} = 0\}$, we have the coercivity on the kernel $\ker B$

$$a(\boldsymbol{\tau}, \boldsymbol{\tau}) = \|\boldsymbol{\tau}\|_0^2 = \|\boldsymbol{\tau}\|^2_{\boldsymbol{H}^{-1}(\operatorname{div}\operatorname{div})} \quad \forall\, \boldsymbol{\tau} \in \ker B.$$

Given a scalar function $v$, we can embed it into the symmetric tensor space as $\iota(v) = v\boldsymbol{I}_{2\times 2}$. Following the proofs in [16, 24], we can see that for $v \in H_0^1(\Omega)$,

$$b(-\iota(v), v) = -\langle \operatorname{div}\operatorname{div}(\iota(v)), v\rangle = -\langle \Delta v, v\rangle = |v|_1^2,$$

$$\|-\iota(v)\|^2_{\boldsymbol{H}^{-1}(\operatorname{div}\operatorname{div})} = \|\iota(v)\|_0^2 + |v|_1^2 \lesssim |v|_1^2.$$

Therefore we obtain the following inf-sup condition

$$\sup_{\boldsymbol{\tau} \in \boldsymbol{H}^{-1}(\operatorname{div}\operatorname{div}, \Omega; \mathbb{S})} \frac{b(\boldsymbol{\tau}, v)}{\|\boldsymbol{\tau}\|_{\boldsymbol{H}^{-1}(\operatorname{div}\operatorname{div})}} \geq \frac{b(-\iota(v), v)}{\|-\iota(v)\|_{\boldsymbol{H}^{-1}(\operatorname{div}\operatorname{div})}} \gtrsim |v|_1.$$

Then by the Brezzi Theory, the HHJ mixed formulation (5.4)-(5.5) is well-posed.

**引理 5.1** (Corollary 2.3 in [60]). The problem (5.2) and the mixed formulation (5.4)-(5.5) are fully equivalent, i.e., if $u \in H_0^2(\Omega)$ solves problem (5.2), then $\boldsymbol{\sigma} = -\nabla^2 u \in \boldsymbol{H}^{-1}(\operatorname{div}\operatorname{div}, \Omega; \mathbb{S})$ and $(\boldsymbol{\sigma}, u)$ solves the mixed formulation (5.4)-(5.5). And, vice versa, if $(\boldsymbol{\sigma}, u) \in \boldsymbol{H}^{-1}(\operatorname{div}\operatorname{div}, \Omega; \mathbb{S}) \times H_0^1(\Omega)$ solves the mixed formulation (5.4)-(5.5), then $u \in H_0^2(\Omega)$ and $u$ solves problem (5.2).

Proof. Both problems are uniquely solvable. Therefore, it suffices to show that $(\boldsymbol{\sigma}, u)$ with $\boldsymbol{\sigma} = -\nabla^2 u$ solves the mixed formulation (5.4)-(5.5), if $u$ solves problem (5.2). Assume $u \in H_0^2(\Omega)$ is the solution of problem (5.2). Then, obviously, $\boldsymbol{\sigma} \in \boldsymbol{L}^2(\Omega; \mathbb{S})$ and

$$(\boldsymbol{\sigma}, \nabla^2 v) = -\langle f, v\rangle \quad \forall\, v \in H_0^2(\Omega),$$

which implies that $\operatorname{div}\operatorname{div} \boldsymbol{\sigma} = -f \in H^{-1}(\Omega)$ in the distributional sense. Therefore, $\boldsymbol{\sigma} \in \boldsymbol{H}^{-1}(\operatorname{div}\operatorname{div}, \Omega; \mathbb{S})$ and (5.5) immediately follows.

By the definition of $\operatorname{div}\operatorname{div}\boldsymbol{\tau}$ in the distributional sense we have

$$\langle \operatorname{div}\operatorname{div}\boldsymbol{\tau}, v\rangle = (\boldsymbol{\tau}, \nabla^2 v) \quad \forall\, v \in C_0^\infty(\Omega).$$

Since $C_0^\infty(\Omega)$ is dense in $H_0^2(\Omega)$, it follows for $v = u$ that

$$\langle \operatorname{div}\operatorname{div}\boldsymbol{\tau}, u\rangle = (\boldsymbol{\tau}, \nabla^2 u) = -(\boldsymbol{\tau}, \boldsymbol{\sigma}),$$

which shows (5.4). □

Let

$$H_{n,0}^{1/2}(\Gamma) := \{\partial_n \varphi : \varphi \in H^2(\Omega) \cap H_0^1(\Omega)\},$$

which is equipped with norm

$$\|q\|_{H_{n,0}^{1/2}(\Gamma)} := \inf_{\varphi \in H^2(\Omega)\cap H_0^1(\Omega), \partial_n\varphi = q} \|\varphi\|_{H^2(\Omega)}.$$

Define

$$H_n^{-1/2}(\Gamma) := \left(H_{n,0}^{1/2}(\Gamma)\right)'.$$

**引理** 5.2 (Theorem 3.34 in [71]). Let $\Omega \subset \mathbb{R}^n$ with $n = 2, 3$ be a bounded domain with smooth or Lipschitz boundary $\Gamma = \partial\Omega$.

(a) The trace operator $\boldsymbol{\tau} \in \boldsymbol{H}^{-1}(\operatorname{div}\operatorname{div}, \Omega; \mathbb{S}) \to (\boldsymbol{n}^\intercal \boldsymbol{\tau}\boldsymbol{n})|_\Gamma \in H_n^{-1/2}(\Gamma)$ is surjective. It holds

$$\|\boldsymbol{n}^\intercal \boldsymbol{\tau}\boldsymbol{n}\|_{H_n^{-1/2}(\Gamma)} \lesssim \|\boldsymbol{\tau}\|_{\boldsymbol{H}^{-1}(\operatorname{div}\operatorname{div})} \quad \forall\, \boldsymbol{\tau} \in \boldsymbol{H}^{-1}(\operatorname{div}\operatorname{div}).$$

(b) For any $g \in H_n^{-1/2}(\Gamma)$, there exists some $\boldsymbol{\tau} \in \boldsymbol{H}^{-1}(\operatorname{div}\operatorname{div})$ such that

$$(\boldsymbol{n}^\intercal \boldsymbol{\tau}\boldsymbol{n})|_\Gamma = g \quad \text{and} \quad \|\boldsymbol{\tau}\|_{\boldsymbol{H}^{-1}(\operatorname{div}\operatorname{div})} \lesssim \|g\|_{H_n^{-1/2}(\Gamma)}.$$

**引理** 5.3 (Theorem 2.1 in [66] and Theorem 3.36 in [71]). Let $\Omega$, $\Omega_1$ and $\Omega_2$ are three open and bounded domains in $\mathbb{R}^n$. Suppose $\bar\Omega = \bar\Omega_1 \cup \bar\Omega_2$ and $\Omega_1 \cap \Omega_2 = \varnothing$. Let $\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega; \mathbb{S})$ satisfy $\boldsymbol{\tau}|_{\Omega_1} \in \boldsymbol{C}^0(\bar\Omega_1; \mathbb{S}) \cap \boldsymbol{H}^1(\Omega_1; \mathbb{S})$, $\boldsymbol{\tau}|_{\Omega_2} \in \boldsymbol{C}^0(\bar\Omega_2; \mathbb{S}) \cap \boldsymbol{H}^1(\Omega_2; \mathbb{S})$, $\boldsymbol{t}^\intercal(\boldsymbol{\tau}|_{\Omega_1})\boldsymbol{n} \in H^{1/2}(\partial\Omega_1)$ and $\boldsymbol{t}^\intercal(\boldsymbol{\tau}|_{\Omega_2})\boldsymbol{n} \in H^{1/2}(\partial\Omega_2)$. If the normal-normal component $\boldsymbol{n}^\intercal\boldsymbol{\tau}\boldsymbol{n}$ is continuous across the interface $\partial\Omega_1 \cap \partial\Omega_2$, then $\boldsymbol{\tau} \in \boldsymbol{H}^{-1}(\operatorname{div}\operatorname{div}, \Omega; \mathbb{S})$.

**注** 5.4. The condition $\boldsymbol{t}^\intercal(\boldsymbol{\tau}|_{\Omega_1})\boldsymbol{n} \in H^{1/2}(\partial\Omega_1)$ is an inner-elemental continuity constraint for the tangential stress vector $\boldsymbol{t}^\intercal(\boldsymbol{\tau}|_{\Omega_1})\boldsymbol{n}$ at vertices in 2D and at edges in 3D. If we slightly weaken the regularity, the constraint disappears, and much simpler elements can be used.

### 5.1.1 HHJ method

By Lemma 5.3, it is natural to find the approximation of the moment $\boldsymbol{\sigma}$ in the finite-dimensional subspace of

$$\Sigma^{\operatorname{div}\operatorname{div}} := \{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega; \mathbb{S}) : \boldsymbol{\tau} \in \boldsymbol{H}^1(K; \mathbb{S}) \quad \forall\, K \in \mathcal{T}_h \text{ and } [\![\boldsymbol{n}^\intercal\boldsymbol{\tau}\boldsymbol{n}]\!]|_e = 0 \text{ for each } e \in \mathcal{E}_h^i\}.$$

For any $\boldsymbol{\tau} \in \Sigma^{\operatorname{div}\operatorname{div}}$ and $v \in C_0^\infty(\Omega)$, we have

$$
\begin{aligned}
\langle \operatorname{div}\operatorname{div}\boldsymbol{\tau}, v\rangle = (\boldsymbol{\tau}, \nabla^2 v) &= -\sum_{K\in\mathcal{T}_h}(\operatorname{div}\boldsymbol{\tau}, \nabla v)_K + \sum_{K\in\mathcal{T}_h}(\boldsymbol{\tau}\boldsymbol{n}, \nabla v)_{\partial K}\\
&= -\sum_{K\in\mathcal{T}_h}(\operatorname{div}\boldsymbol{\tau}, \nabla v)_K + \sum_{K\in\mathcal{T}_h}(\boldsymbol{t}^\intercal\boldsymbol{\tau}\boldsymbol{n}, \partial_t v)_{\partial K} + \sum_{K\in\mathcal{T}_h}(\boldsymbol{n}^\intercal\boldsymbol{\tau}\boldsymbol{n}, \partial_n v)_{\partial K}\\
&= -\sum_{K\in\mathcal{T}_h}(\operatorname{div}\boldsymbol{\tau}, \nabla v)_K + \sum_{K\in\mathcal{T}_h}(\boldsymbol{t}^\intercal\boldsymbol{\tau}\boldsymbol{n}, \partial_t v)_{\partial K}.
\end{aligned}
$$

Define finite element spaces

$$
\begin{aligned}
&\Sigma_h^{\operatorname{div}\operatorname{div}} := \{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega;\mathbb{S}) : \boldsymbol{\tau} \in \mathbb{P}_{k-1}(K;\mathbb{S}) \quad \forall\, K \in \mathcal{T}_h \text{ and } [\![\boldsymbol{n}^\intercal\boldsymbol{\tau}\boldsymbol{n}]\!]|_e = 0 \text{ for each } e \in \mathcal{E}_h^i\},\\
&V_h := \left\{v \in H_0^1(\Omega) : v|_K \in \mathbb{P}_k(K) \quad \forall\, K \in \mathcal{T}_h\right\}
\end{aligned}
$$

with $k \geq 1$. Given $K \in \mathcal{T}_h$, the local degrees of freedom of the space $\boldsymbol{\Sigma}_h$ are

- $\int_e (\boldsymbol{n}^\intercal\boldsymbol{\tau}\boldsymbol{n}) v \,\mathrm{d}s \quad \forall\, v \in \mathbb{P}_{k-1}(e), e \in \mathcal{E}_h(K)$;
- $\int_K \boldsymbol{\tau} : \boldsymbol{\varsigma} \,\mathrm{d}x \quad \forall\, \boldsymbol{\varsigma} \in \mathbb{P}_{k-2}(K, \mathbb{S})$.

The local degrees of freedom of the space $V_h$ are

- $w(p)$ for each vertex $p$ of $K$;
- $\int_e wv \,\mathrm{d}s \quad \forall\, v \in \mathbb{P}_{k-2}(e), e \in \mathcal{E}_h(K)$;
- $\int_K wv \,\mathrm{d}x \quad \forall\, v \in \mathbb{P}_{k-3}(K)$.

The basis functions for symmetric tensors are

- Corresponding to the degrees of freedom on edge with vertices $p_i$ and $p_j$:
$$\lambda_i^\ell \lambda_j^{k-1-\ell}\operatorname{sym}(\boldsymbol{t}_i \otimes \boldsymbol{t}_j) \text{ for } \ell = 0, 1, \cdots, k-1;$$
- Corresponding to the interior degrees of freedom:
$$\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\lambda_3^{\alpha_3}\lambda_\ell \operatorname{sym}(\boldsymbol{t}_i \otimes \boldsymbol{t}_j) \text{ with } \alpha_1 + \alpha_2 + \alpha_3 = k-2,$$
where $(ij\ell)$ is a circular permutation of $(123)$.

The Hellan-Herrmann-Johnson mixed method for the mixed formulation (5.4)-(5.5) is to find $(\boldsymbol{\sigma}_h, u_h) \in \Sigma_h^{\operatorname{div}\operatorname{div}} \times V_h$ such that

$$
\begin{aligned}
a(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b_h(\boldsymbol{\tau}_h, u_h) &= 0 && \forall\, \boldsymbol{\tau}_h \in \Sigma_h^{\operatorname{div}\operatorname{div}}, && (5.6)\\
b_h(\boldsymbol{\sigma}_h, v_h) &= -\langle f, v_h\rangle && \forall\, v_h \in V_h, && (5.7)
\end{aligned}
$$

where

$$b_h(\boldsymbol{\sigma}_h, v_h) := -\sum_{K\in\mathcal{T}_h}(\operatorname{div}\boldsymbol{\sigma}_h, \nabla v_h)_K + \sum_{K\in\mathcal{T}_h}(\boldsymbol{t}^\intercal\boldsymbol{\sigma}_h\boldsymbol{n}, \partial_t v_h)_{\partial K}.$$

Applying integration by parts, it holds

$$b_h(\boldsymbol{\sigma}_h, v_h) = \sum_{K\in\mathcal{T}_h}(\boldsymbol{\sigma}_h, \nabla^2 v_h)_K - \sum_{K\in\mathcal{T}_h}(\boldsymbol{n}^\intercal\boldsymbol{\sigma}_h\boldsymbol{n}, \partial_n v_h)_{\partial K}.$$

Introduce mesh dependent norms

$$\|\boldsymbol{\tau}\|_{0,h}^2 := \|\boldsymbol{\tau}\|_0^2 + \sum_{e\in\mathcal{E}_h} h_e\|\boldsymbol{n}^\intercal\boldsymbol{\tau}\boldsymbol{n}\|_{0,e}^2 \quad \forall\,\boldsymbol{\tau}\in\Sigma^{\operatorname{div}\operatorname{div}},$$

$$\|v\|_{2,h}^2 := \sum_{K\in\mathcal{T}_h}|v|_{2,K}^2 + \sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\![\partial_{n_e}v]\!]\|_{0,e}^2 \quad \forall\, v\in V := H^2(\mathcal{T}_h)\cap H_0^1(\Omega).$$

To derive the error estimates of the HHJ mixed method (5.6)-(5.7), we need some interpolation operators. First, define $I_h^{\operatorname{div}\operatorname{div}} : \Sigma^{\operatorname{div}\operatorname{div}} \to \Sigma_h^{\operatorname{div}\operatorname{div}}$ in the following way (cf. [10, 43, 37, 16]): given $\boldsymbol{\tau}\in\Sigma^{\operatorname{div}\operatorname{div}}$, for any element $K\in\mathcal{T}_h$ and any edge $e$ of $K$,

$$\int_e \boldsymbol{n}^\intercal(\boldsymbol{\tau} - I_h^{\operatorname{div}\operatorname{div}}\boldsymbol{\tau})\boldsymbol{n}\, q\,\mathrm{d}s = 0 \quad \forall\, q\in\mathbb{P}_{k-1}(e),$$

$$\int_K (\boldsymbol{\tau} - I_h^{\operatorname{div}\operatorname{div}}\boldsymbol{\tau}) : \boldsymbol{\varsigma}\,\mathrm{d}x = 0 \quad \forall\,\boldsymbol{\varsigma}\in\mathbb{P}_{k-2}(K,\mathbb{S}).$$

It admits (cf. [10, 43, 37, 16])

$$b_h(\boldsymbol{\tau} - I_h^{\operatorname{div}\operatorname{div}}\boldsymbol{\tau}, v_h) = 0 \quad \forall\,\boldsymbol{\tau}\in\Sigma^{\operatorname{div}\operatorname{div}}, v_h\in V_h. \tag{5.8}$$

Next, define $P_h : V \to V_h$ in the following way (cf. [10, 43, 37, 73]) : given $w\in V$, for any element $K\in\mathcal{T}_h$, any vertex $p$ of $K$ and any edge $e$ of $K$,

$$P_h w(p) = w(p),$$

$$\int_e (w - P_h w)v\,\mathrm{d}s = 0 \quad \forall\, v\in\mathbb{P}_{k-2}(e),$$

$$\int_K (w - P_h w)v\,\mathrm{d}x = 0 \quad \forall\, v\in\mathbb{P}_{k-3}(K).$$

According to the definition of $P_h$, we have (cf. [10, 43, 37, 16])

$$b_h(\boldsymbol{\tau}_h, v - P_h v) = 0 \quad \forall\,\boldsymbol{\tau}_h\in\Sigma_h^{\operatorname{div}\operatorname{div}}, v\in V. \tag{5.9}$$

The error estimates for interpolation operators $I_h^{\operatorname{div}\operatorname{div}}$ and $P_h$ are summarized in the following lemma(cf. [10, 43, 37, 73]).

**引理 5.5.** For all $v\in H^{m+3}(K)$, $\boldsymbol{\tau}\in\boldsymbol{H}^{m+1}(\Omega,\mathbb{S})$ with $m$ a non-negative integer and all $K\in\mathcal{T}_h$, we have the estimates

$$\left\|\boldsymbol{\tau} - I_h^{\operatorname{div}\operatorname{div}}\boldsymbol{\tau}\right\|_{0,K} + h_K^{1/2}\left|\boldsymbol{\tau} - I_h^{\operatorname{div}\operatorname{div}}\boldsymbol{\tau}\right|_{0,\partial K} \lesssim h_K^{\min\{m+1,k\}}|\boldsymbol{\tau}|_{m+1,K},$$

$$\begin{aligned}\|v - P_h v\|_{0,K} + h_K\,|v - P_h v|_{1,K} + h_K^2\,|v - P_h v|_{2,K}&\\ +h_K^{3/2}\,\|\boldsymbol{\nabla}(v - P_h v)\|_{0,\partial K} &\lesssim h_K^{\min\{m+2,k\}+1}|v|_{m+3,K}.\end{aligned}$$

**引理 5.6** (Lemma 4.2 in [56]). There holds the following inf-sup condition

$$\|v_h\|_{2,h} \lesssim \sup_{\boldsymbol{\tau}_h\in\Sigma_h^{\operatorname{div}\operatorname{div}}} \frac{b_h(\boldsymbol{\tau}_h, v_h)}{\|\boldsymbol{\tau}_h\|_{0,h}} \quad \forall\, v_h\in V_h. \tag{5.10}$$

*Proof.* Let $\boldsymbol{\tau}_h\in\Sigma_h^{\operatorname{div}\operatorname{div}}$ such that for any $K\in\mathcal{T}_h$ and $e\in\mathcal{E}_h$,

$$\boldsymbol{n}^\intercal\boldsymbol{\tau}\boldsymbol{n} = \frac{-1}{h_e}[\![\partial_{n_e}v_h]\!] \quad \text{on } e,$$

$$\int_K \boldsymbol{\tau}_h : \boldsymbol{\varsigma}\,\mathrm{d}x = \int_K \nabla^2 v_h : \boldsymbol{\varsigma}\,\mathrm{d}x \quad \forall\,\boldsymbol{\varsigma}\in\mathbb{P}_{k-2}(K;\mathbb{S}).$$

It follows from the scaling argument

$$\|\boldsymbol{\tau}_h\|_{0,h} \lesssim \|v_h\|_{2,h}. \tag{5.11}$$

And we also have

$$b_h(\boldsymbol{\tau}_h, v_h) = \sum_{K\in\mathcal{T}_h} (\boldsymbol{\tau}_h, \nabla^2 v_h)_K + \sum_{K\in\mathcal{T}_h} (\boldsymbol{n}^\intercal \boldsymbol{\tau}_h \boldsymbol{n}, \partial_n v_h)_{\partial K} = \|v_h\|_{2,h}^2,$$

which together with (5.11) implies (5.10). □

Apparently both the bilinear forms $a(\cdot,\cdot)$ and $b_h(\cdot,\cdot)$ are continuous with respect to the mesh dependent norms $\|\cdot\|_{0,h}$ and $\|\cdot\|_{2,h}$. And by the inverse inequality, we have

$$\|\boldsymbol{\tau}_h\|_{0,h}^2 \lesssim \|\boldsymbol{\tau}_h\|_0^2 = a(\boldsymbol{\tau}_h, \boldsymbol{\tau}_h) \quad \forall\, \boldsymbol{\tau}_h \in \Sigma_h^{\operatorname{div}\operatorname{div}}.$$

Then by Brezzi's theory, we have the following inf-sup condition

$$\|\widetilde{\boldsymbol{\sigma}}_h\|_{0,h} + \|\widetilde{u}_h\|_{2,h} \lesssim \sup_{\boldsymbol{\tau}_h\in\Sigma_h^{\operatorname{div}\operatorname{div}}, v_h\in V_h} \frac{a(\widetilde{\boldsymbol{\sigma}}_h, \boldsymbol{\tau}_h) + b_h(\boldsymbol{\tau}_h, \widetilde{u}_h) + b_h(\widetilde{\boldsymbol{\sigma}}_h, v_h)}{\|\boldsymbol{\tau}_h\|_{0,h} + \|v_h\|_{2,h}} \tag{5.12}$$

for any $\widetilde{\boldsymbol{\sigma}}_h \in \Sigma_h^{\operatorname{div}\operatorname{div}}$ and $\widetilde{u}_h \in V_h$.

The well-posedness of the HHJ mixed method (5.6)-(5.7) follows from the inf-sup condition (5.12).

**定理 5.7.** Let $(\boldsymbol{\sigma}, u) \in \boldsymbol{H}^{-1}(\operatorname{div}\operatorname{div}, \Omega; \mathbb{S}) \times H_0^1(\Omega)$ be the solution of the HHJ mixed formulation (5.4)-(5.5), and $(\boldsymbol{\sigma}_h, u_h) \in \Sigma_h^{\operatorname{div}\operatorname{div}} \times V_h$ be the HHJ mixed finite element method (5.6)-(5.7). Assume $\boldsymbol{\sigma} \in \boldsymbol{H}^{m+1}(\Omega, \mathbb{S})$ and $u \in H^{m+3}(\Omega)$ for some non-negative integer $m$. Then

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,h} + \|P_h u - u_h\|_{2,h} \lesssim h^{\min\{m+1,k\}} \|\boldsymbol{\sigma}\|_{m+1}, \tag{5.13}$$

$$\|u - u_h\|_{2,h} \lesssim h^{\min\{m+1,k-1\}} \|u\|_{m+3}.$$

Proof. Using integration by parts, we get from problem (5.3)

$$\begin{aligned} a(\boldsymbol{\sigma}, \boldsymbol{\tau}_h) + b_h(\boldsymbol{\tau}_h, u) &= 0 \qquad\quad \forall\, \boldsymbol{\tau}_h \in \Sigma_h^{\operatorname{div}\operatorname{div}}, \\ b_h(\boldsymbol{\sigma}, v_h) &= -\langle f, v_h\rangle \quad \forall\, v_h \in V_h. \end{aligned}$$

Subtracting (5.6) and (5.7) from the last two inequalities, we achieve the following orthogonality

$$a(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b_h(\boldsymbol{\tau}_h, u - u_h) = 0 \qquad \forall\, \boldsymbol{\tau}_h \in \Sigma_h^{\operatorname{div}\operatorname{div}}, \tag{5.14}$$

$$b_h(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, v_h) = 0 \qquad \forall\, v_h \in V_h. \tag{5.15}$$

Then from (5.8) and (5.9), it holds for any $\boldsymbol{\tau}_h \in \Sigma_h^{\operatorname{div}\operatorname{div}}$ and $v_h \in V_h$

$$\begin{aligned} & a(I_h^{\operatorname{div}\operatorname{div}}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b_h(\boldsymbol{\tau}_h, P_h u - u_h) + b_h(I_h^{\operatorname{div}\operatorname{div}}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, v_h) \\ =\,& a(I_h^{\operatorname{div}\operatorname{div}}\boldsymbol{\sigma} - \boldsymbol{\sigma}, \boldsymbol{\tau}_h) + b_h(\boldsymbol{\tau}_h, P_h u - u) + b_h(I_h^{\operatorname{div}\operatorname{div}}\boldsymbol{\sigma} - \boldsymbol{\sigma}, v_h) \\ =\,& a(I_h^{\operatorname{div}\operatorname{div}}\boldsymbol{\sigma} - \boldsymbol{\sigma}, \boldsymbol{\tau}_h). \end{aligned}$$

Taking $\widetilde{\boldsymbol{\sigma}}_h = I_h^{\operatorname{div}\operatorname{div}}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h$ and $\widetilde{u}_h = P_h u - u_h$ in (5.12), we have

$$\begin{aligned} & \|I_h^{\operatorname{div}\operatorname{div}}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,h} + \|P_h u - u_h\|_{2,h} \\ \lesssim & \sup_{\boldsymbol{\tau}_h\in\Sigma_h^{\operatorname{div}\operatorname{div}}, v_h\in V_h} \frac{a(I_h^{\operatorname{div}\operatorname{div}}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b_h(\boldsymbol{\tau}_h, P_h u - u_h) + b_h(I_h^{\operatorname{div}\operatorname{div}}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, v_h)}{\|\boldsymbol{\tau}_h\|_{0,h} + \|v_h\|_{2,h}} \\ \lesssim & \sup_{\boldsymbol{\tau}_h\in\Sigma_h^{\operatorname{div}\operatorname{div}}, v_h\in V_h} \frac{a(I_h^{\operatorname{div}\operatorname{div}}\boldsymbol{\sigma} - \boldsymbol{\sigma}, \boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{0,h} + \|v_h\|_{2,h}} \lesssim \|\boldsymbol{\sigma} - I_h^{\operatorname{div}\operatorname{div}}\boldsymbol{\sigma}\|_0, \end{aligned}$$

which combined with the triangle inequality gives

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,h}+\|P_h u-u_h\|_{2,h}\lesssim\|\boldsymbol{\sigma}-I_h^{\operatorname{div}\operatorname{div}}\boldsymbol{\sigma}\|_{0,h}.$$

Finally we ends the proof by applying Lemma 5.5. □

To derive the error estimate of $|u-u_h|_1$, we present the second discrete inf-sup condition. Define $(\operatorname{div}\operatorname{div})_h:\Sigma_h^{\operatorname{div}\operatorname{div}}\to V_h$ by

$$((\operatorname{div}\operatorname{div})_h\boldsymbol{\tau}_h,v_h)=b_h(\boldsymbol{\tau}_h,v_h).$$

Then equip space $\Sigma_h^{\operatorname{div}\operatorname{div}}$ with the squared mesh-dependent norm

$$\|\boldsymbol{\tau}_h\|^2_{H_h^{-1}(\operatorname{div}\operatorname{div})}:=\|\boldsymbol{\tau}_h\|_0^2+\|(\operatorname{div}\operatorname{div})_h\boldsymbol{\tau}_h\|^2_{-1,h}\quad\text{ with }\|w\|_{-1,h}:=\sup_{v_h\in V_h}\frac{(w,v_h)}{|v_h|_1}.$$

By (5.8) and (5.15), we have

$$(\operatorname{div}\operatorname{div})_h(I_h^{\operatorname{div}\operatorname{div}}\sigma)=(\operatorname{div}\operatorname{div})_h\sigma=(\operatorname{div}\operatorname{div})_h\sigma_h.$$

**引理 5.8.** We have $(\operatorname{div}\operatorname{div})_h\Sigma_h^{\operatorname{div}\operatorname{div}}=V_h$, and the discrete inf-sup condition

$$\|v_h\|_1\lesssim\sup_{\boldsymbol{\tau}_h\in\Sigma_h^{\operatorname{div}\operatorname{div}}}\frac{b_h(\boldsymbol{\tau}_h,v_h)}{\|\boldsymbol{\tau}_h\|_{H_h^{-1}(\operatorname{div}\operatorname{div})}}\quad\forall\, v_h\in V_h.\tag{5.16}$$

*Proof.* For $v_h\in V_h$, set $\boldsymbol{\tau}_h=-I_h^{\operatorname{div}\operatorname{div}}(v_hI)$. By (5.8),

$$b_h(\boldsymbol{\tau}_h,v_h)=-b_h(v_hI,v_h)=|v_h|_1^2.$$

And

$$\|\boldsymbol{\tau}_h\|_{H_h^{-1}(\operatorname{div}\operatorname{div})}\lesssim\|\boldsymbol{\tau}_h\|_0+\sup_{v_h\in V_h}\frac{b_h(\boldsymbol{\tau}_h,v_h)}{|v_h|_1}\lesssim|v_h|_1.$$

Thus the discrete inf-sup condition (5.16) holds. Combine the discrete inf-sup condition (5.16) and the fact $(\operatorname{div}\operatorname{div})_h\Sigma_h^{\operatorname{div}\operatorname{div}}\subseteq V_h$ to get $(\operatorname{div}\operatorname{div})_h\Sigma_h^{\operatorname{div}\operatorname{div}}=V_h$. □

Both the bilinear forms $a(\cdot,\cdot)$ and $b_h(\cdot,\cdot)$ are continuous with respect to norms $\|\cdot\|_{H_h^{-1}(\operatorname{div}\operatorname{div})}$ and $|\cdot|_1$. And by the inverse inequality, we have

$$\|\boldsymbol{\tau}_h\|^2_{H_h^{-1}(\operatorname{div}\operatorname{div})}=\|\boldsymbol{\tau}_h\|_0^2=a(\boldsymbol{\tau}_h,\boldsymbol{\tau}_h)\quad\forall\,\boldsymbol{\tau}_h\in\Sigma_h^{\operatorname{div}\operatorname{div}}\ker((\operatorname{div}\operatorname{div})_h).$$

Then by Brezzi's theory, we have the following inf-sup condition

$$\|\widetilde{\boldsymbol{\sigma}}_h\|_{H_h^{-1}(\operatorname{div}\operatorname{div})}+|\widetilde{u}_h|_1\lesssim\sup_{\boldsymbol{\tau}_h\in\Sigma_h^{\operatorname{div}\operatorname{div}},v_h\in V_h}\frac{a(\widetilde{\boldsymbol{\sigma}}_h,\boldsymbol{\tau}_h)+b_h(\boldsymbol{\tau}_h,\widetilde{u}_h)+b_h(\widetilde{\boldsymbol{\sigma}}_h,v_h)}{\|\boldsymbol{\tau}_h\|_{H_h^{-1}(\operatorname{div}\operatorname{div})}+|v_h|_1}\tag{5.17}$$

for any $\widetilde{\boldsymbol{\sigma}}_h\in\Sigma_h^{\operatorname{div}\operatorname{div}}$ and $\widetilde{u}_h\in V_h$.

**定理 5.9.** Let $(\boldsymbol{\sigma},u)\in\boldsymbol{H}^{-1}(\operatorname{div}\operatorname{div},\Omega;\mathbb{S})\times H_0^1(\Omega)$ be the solution of the HHJ mixed formulation (5.4)-(5.5), and $(\boldsymbol{\sigma}_h,u_h)\in\Sigma_h^{\operatorname{div}\operatorname{div}}\times V_h$ be the HHJ mixed finite element method (5.6)-(5.7). Assume $\boldsymbol{\sigma}\in\boldsymbol{H}^{m+1}(\Omega,\mathbb{S})$ and $u\in H^{m+3}(\Omega)$ for some non-negative integer $m$. Then

$$|u-u_h|_1\lesssim h^{\min\{m+1,k\}}\|u\|_{m+3}.$$

Proof. Taking $\widetilde{\boldsymbol{\sigma}}_h = \boldsymbol{\Pi}_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h$ and $\widetilde{u}_h = P_h u - u_h$ in (5.17), we have

$$\begin{aligned}
&\|\boldsymbol{\Pi}_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{H_h^{-1}(\operatorname{div}\operatorname{div})} + |P_h u - u_h|_1 \\
\lesssim &\sup_{\boldsymbol{\tau}_h\in\Sigma_h^{\operatorname{div}\operatorname{div}}, v_h\in V_h} \frac{a(\boldsymbol{\Pi}_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b_h(\boldsymbol{\tau}_h, P_h u - u_h) + b_h(\boldsymbol{\Pi}_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, v_h)}{\|\boldsymbol{\tau}_h\|_{H_h^{-1}(\operatorname{div}\operatorname{div})} + |v_h|_1} \\
\lesssim &\sup_{\boldsymbol{\tau}_h\in\Sigma_h^{\operatorname{div}\operatorname{div}}, v_h\in V_h} \frac{a(\boldsymbol{\Pi}_h\boldsymbol{\sigma} - \boldsymbol{\sigma}, \boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{H_h^{-1}(\operatorname{div}\operatorname{div})} + |v_h|_1} \lesssim \|\boldsymbol{\sigma} - \boldsymbol{\Pi}_h\boldsymbol{\sigma}\|_0,
\end{aligned}$$

which combined with the triangle inequality gives

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 + |P_h u - u_h|_1 \lesssim \|\boldsymbol{\sigma} - \boldsymbol{\Pi}_h\boldsymbol{\sigma}\|_{0,h}.$$

Finally we end the proof by applying Lemma 5.5. □

Using the usual duality argument, we can additionally derive a superconvergent error estimate between $u_h$ and $P_h u$ in $H^1$ norm. Let $(\widetilde{\boldsymbol{\sigma}}, \widetilde{u})$ be the solution of the auxiliary problem:

$$\begin{cases} \widetilde{\boldsymbol{\sigma}} = -\nabla^2\widetilde{u} & \text{in } \Omega, \\ \operatorname{div}\operatorname{div}\widetilde{\boldsymbol{\sigma}} = \Delta(P_h u - u_h) & \text{in } \Omega, \\ \widetilde{u} = \partial_n\widetilde{u} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.18}$$

Since $\triangle(P_h u - u_h) \notin L^2(\Omega)$, the second equation of (5.18) is interpreted by the following relation

$$\int_\Omega (\operatorname{div}\widetilde{\boldsymbol{\sigma}})\cdot\nabla v\,\mathrm{d}x = \int_\Omega \nabla(P_h u - u_h)\cdot\nabla v\,\mathrm{d}x \quad \forall\, v\in H_0^1(\Omega). \tag{5.19}$$

We assume that $\widetilde{u}\in H^3(\Omega)\cap H_0^2(\Omega)$ with the bound

$$\|\widetilde{\boldsymbol{\sigma}}\|_1 + \|\widetilde{u}\|_3 \lesssim |P_h u - u_h|_1. \tag{5.20}$$

When $\Omega$ is a convex bounded polygonal domain, the regularity result (5.20) has been obtained in [39, 50].

**定理 5.10.** Let $(\boldsymbol{\sigma}, u)\in \boldsymbol{H}^{-1}(\operatorname{div}\operatorname{div},\Omega;\mathbb{S})\times H_0^1(\Omega)$ be the solution of the HHJ mixed formulation (5.4)-(5.5), and $(\boldsymbol{\sigma}_h, u_h)\in\Sigma_h^{\operatorname{div}\operatorname{div}}\times V_h$ be the HHJ mixed finite element method (5.6)-(5.7). Assume the regularity condition (5.20) holds, $\boldsymbol{\sigma}\in\boldsymbol{H}^{m+1}(\Omega,\mathbb{S})$ and $u\in H^{m+3}(\Omega)$ for some non-negative integer $m$. Then

$$|P_h u - u_h|_1 \lesssim h^{\min\{m+1,k\}+1}(\|\boldsymbol{\sigma}\|_{m+1} + \delta_{k1}\|f\|_0). \tag{5.21}$$

Proof. Taking $v = P_h u - u_h$ in (5.19), we get from (5.8)-(5.9) and (5.14)

$$\begin{aligned}
|P_h u - u_h|_1^2 &= (\operatorname{div}\widetilde{\boldsymbol{\sigma}}, \nabla(P_h u - u_h)) = -b_h(\widetilde{\boldsymbol{\sigma}}, P_h u - u_h) \\
&= -b_h(I_h^{\operatorname{div}\operatorname{div}}\widetilde{\boldsymbol{\sigma}}, P_h u - u_h) = -b_h(I_h^{\operatorname{div}\operatorname{div}}\widetilde{\boldsymbol{\sigma}}, u - u_h) \\
&= a(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, I_h^{\operatorname{div}\operatorname{div}}\widetilde{\boldsymbol{\sigma}}).
\end{aligned}$$

Using (5.18), (5.15) and (5.9), it holds

$$\begin{aligned}
a(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \widetilde{\boldsymbol{\sigma}}) &= -a(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \nabla^2\widetilde{u}) = -(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \nabla^2\widetilde{u}) \\
&= -b_h(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \widetilde{u}) = -b_h(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \widetilde{u} - P_h\widetilde{u}) \\
&= -b_h(\boldsymbol{\sigma}, \widetilde{u} - P_h\widetilde{u}).
\end{aligned}$$

Hence

$$
\begin{aligned}
|P_h u - u_h|_1^2 &= a(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, I_h^{\operatorname{div}\operatorname{div}}\widetilde{\boldsymbol{\sigma}}) = a(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, I_h^{\operatorname{div}\operatorname{div}}\widetilde{\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\sigma}}) + a(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \widetilde{\boldsymbol{\sigma}}) \\
&= a(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, I_h^{\operatorname{div}\operatorname{div}}\widetilde{\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\sigma}}) - b_h(\boldsymbol{\sigma}, \widetilde{u} - P_h\widetilde{u}).
\end{aligned} \tag{5.22}
$$

If $k = 1$, we acquire from (5.22), (5.5), Lemma 5.5 and (5.13)

$$
\begin{aligned}
|P_h u - u_h|_1^2 &= a(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, I_h^{\operatorname{div}\operatorname{div}}\widetilde{\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\sigma}}) + (f, \widetilde{u} - P_h\widetilde{u}) \\
&\lesssim \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 \|I_h^{\operatorname{div}\operatorname{div}}\widetilde{\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\sigma}}\|_0 + \|f\|_0 \|\widetilde{u} - P_h\widetilde{u}\|_0 \\
&\lesssim h\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 \|\widetilde{\boldsymbol{\sigma}}\|_1 + h^2\|f\|_0\|\widetilde{u}\|_2 \lesssim (h\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 + h^2\|f\|_0)(\|\widetilde{\boldsymbol{\sigma}}\|_1 + \|\widetilde{u}\|_2) \\
&\lesssim h^2(\|\boldsymbol{\sigma}\|_1 + \|f\|_0)(\|\widetilde{\boldsymbol{\sigma}}\|_1 + \|\widetilde{u}\|_2)
\end{aligned}
$$

Then it follows from (5.20)

$$
|P_h u - u_h|_1 \lesssim h^2(\|\boldsymbol{\sigma}\|_1 + \|f\|_0).
$$

If $k \geq 2$, we get from (5.22), (5.9), Lemma 5.5,

$$
\begin{aligned}
|P_h u - u_h|_1^2 &= a(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, I_h^{\operatorname{div}\operatorname{div}}\widetilde{\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\sigma}}) - b_h(\boldsymbol{\sigma}, \widetilde{u} - P_h\widetilde{u}) \\
&= a(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, I_h^{\operatorname{div}\operatorname{div}}\widetilde{\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\sigma}}) - b_h(\boldsymbol{\sigma} - I_h^{\operatorname{div}\operatorname{div}}\boldsymbol{\sigma}, \widetilde{u} - P_h\widetilde{u}) \\
&\lesssim \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 \|I_h^{\operatorname{div}\operatorname{div}}\widetilde{\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\sigma}}\|_0 + \|\boldsymbol{\sigma} - I_h^{\operatorname{div}\operatorname{div}}\boldsymbol{\sigma}\|_{0,h} \|\widetilde{u} - P_h\widetilde{u}\|_{2,h} \\
&\lesssim h\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 \|\widetilde{\boldsymbol{\sigma}}\|_1 + h\|\boldsymbol{\sigma} - I_h^{\operatorname{div}\operatorname{div}}\boldsymbol{\sigma}\|_{0,h}\|\widetilde{u}\|_3 \\
&\lesssim h(\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 + \|\boldsymbol{\sigma} - I_h^{\operatorname{div}\operatorname{div}}\boldsymbol{\sigma}\|_{0,h})(\|\widetilde{\boldsymbol{\sigma}}\|_1 + \|\widetilde{u}\|_3).
\end{aligned}
$$

Then it follows from (5.20) and (5.13)

$$
|P_h u - u_h|_1 \lesssim h(\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 + \|\boldsymbol{\sigma} - I_h^{\operatorname{div}\operatorname{div}}\boldsymbol{\sigma}\|_{0,h}) \lesssim h^{\min\{m+1,k\}+1}\|\boldsymbol{\sigma}\|_{m+1}.
$$

□

### 5.1.2 Postprocessing

We will construct a new superconvergent approximation to deflection $u$ in virtue of the optimal result of moment in (5.13) and the superconvergent result (5.21) in this section. For any integer $l \geq 1$, let $I_h^l$ be the Lagrange interpolation operator onto the element-wise $l$-th Lagrange element space (cf. [36, 19]) with respect to $\mathcal{T}_h$. Denote

$$
V_h^* := \left\{ v \in L^2(\Omega) : v|_K \in \mathbb{P}_{k+1}(K) \quad \forall K \in \mathcal{T}_h \right\}.
$$

With this space, define a new approximation $u_h^* \in V_h^*$ to $u$ piecewisely as a solution of the following problem: for any $K \in \mathcal{T}_h$,

$$
u_h^*(\mathtt{v}_i) = u_h(\mathtt{v}_i) \text{ for } i = 1, 2, 3, \tag{5.23}
$$

$$
\int_K \nabla^2 u_h^* : \nabla^2 v \,\mathrm{d}x = -\int_K \boldsymbol{\sigma}_h : \nabla^2 v \,\mathrm{d}x \tag{5.24}
$$

for any $v \in \mathbb{P}_{k+1}(K)$ with $v(\mathtt{v}_i) = 0$ $(i = 1, 2, 3)$, where $\{\mathtt{v}_i\}_{i=1}^3$ are three vertices of $K$. By (5.23) and error estimate of $I_h^1$, we have

$$
|I_h^1(I_h^{k+1}u - u_h^*)|_1 = |I_h^1(P_h u - u_h)|_1 \lesssim |P_h u - u_h|_1. \tag{5.25}
$$

Denote $z := (I - I_h^1)(I_h^{k+1}u - u_h^*)$. It is easy to see that $I_h^1 z = 0$, $z \in V_h^*$, and $z(\mathtt{v}) = 0$ for each vertex $\mathtt{v}$ of triangulation $\mathcal{T}_h$. Thus we have from error estimate of $I_h^1$ that

$$\|z\|_{0,K} + h_K|z|_{1,K} = \|z - I_h^1 z\|_{0,K} + h_K|z - I_h^1 z|_{1,K} \lesssim h_K^2|z|_{2,K}. \tag{5.26}$$

**定理 5.11.** Assume the regularity condition (5.20) holds, $\boldsymbol{\sigma} \in \boldsymbol{H}^{m+1}(\Omega, \mathbb{S})$ and $u \in H^{m+3}(\Omega)$ for some non-negative integer $m$. Then

$$|u - u_h^*|_{1,h} \lesssim h^{\min\{m+1,k\}+1}(\|\boldsymbol{\sigma}\|_{m+1} + \|u\|_{m+3} + \delta_{k1}\|f\|_0).$$

Proof. From (5.24) with $v = z$ and the first equation of problem (5.3), it follows that

$$\int_K \nabla^2(u - u_h^*) : \nabla^2 z \, \mathrm{d}x = -\int_K (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) : \nabla^2 z \, \mathrm{d}x.$$

Noting the definition of $z$, we have

$$\begin{aligned}
&\int_K \nabla^2 z : \nabla^2 z \, \mathrm{d}x \\
=&\int_K \nabla^2(I_h^{k+1}u - u_h^*) : \nabla^2 z \, \mathrm{d}x \\
=&\int_K \nabla^2(I_h^{k+1}u - u) : \nabla^2 z \, \mathrm{d}x + \int_K \nabla^2(u - u_h^*) : \nabla^2 z \, \mathrm{d}x \\
=&\int_K \nabla^2(I_h^{k+1}u - u) : \nabla^2 z \, \mathrm{d}x - \int_K (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) : \nabla^2 z \, \mathrm{d}x.
\end{aligned}$$

On the other hand, by the Cauchy-Schwarz inequality,

$$|z|_{2,K}^2 \lesssim |I_h^{k+1}u - u|_{2,K}|z|_{2,K} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,K}|z|_{2,K},$$

which together with (5.26) gives

$$|z|_{1,K} \lesssim h_K|z|_{2,K} \lesssim h_K|I_h^{k+1}u - u|_{2,K} + h_K\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,K}.$$

Then we have from the triangle inequality and (5.25) that

$$\begin{aligned}
|u - u_h^*|_{1,h} \le& |u - I_h^{k+1}u|_{1,h} + |I_h^1(I_h^{k+1}u - u_h^*)|_1 + |z|_{1,h} \\
\lesssim& |u - I_h^{k+1}u|_{1,h} + |P_h u - u_h|_1 + h(|u - I_h^{k+1}u|_{2,h} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0).
\end{aligned}$$

Finally, the required result follows readily from the error estimate of $I_h^{k+1}$, (5.21) and (5.13). □

### 5.1.3 Hybridization

We will hybridize the HHJ mixed formulation (5.4)-(5.5) in this subsection [4, 57, 55]. Introduce two finite element spaces

$$\begin{aligned}
\Sigma_h &:= \{\boldsymbol{\tau}_h \in L^2(\Omega; \mathbb{S}) : \boldsymbol{\tau}_h|_T \in \mathbb{P}_{k-1}(T; \mathbb{S}) \text{ for each } T \in \mathcal{T}_h\}, \\
M_h &:= \{\mu_h \in L^2(\mathcal{E}_h) : \mu_h|_e \in \mathbb{P}_{k-1}(e) \text{ for each } e \in \mathring{\mathcal{E}}_h, \text{ and } \mu_h = 0 \text{ on } \mathcal{E}_h \backslash \mathring{\mathcal{E}}_h\}.
\end{aligned}$$

Equip the multiplier space $M_h$ with squared norm

$$\|\mu_h\|_{\alpha,h}^2 := \sum_{T \in \mathcal{T}_h} \sum_{e \in \mathcal{E}(T)} h_e^{-2\alpha}\|\mu_h\|_{0,e}^2, \quad \alpha = \pm 1/2.$$

The hybridization of the HHJ mixed formulation (5.4)-(5.5) is to find $(\boldsymbol{\sigma}_h, u_h, \lambda_h) \in \Sigma_h \times V_h \times M_h$ such that

$$a(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b_h(\boldsymbol{\tau}_h; u_h, \lambda_h) = 0 \qquad \forall\, \boldsymbol{\tau}_h \in \Sigma_h, \tag{5.27}$$

$$b_h(\boldsymbol{\sigma}_h; v_h, \mu_h) = -(f, v_h) \quad \forall\, v_h \in V_h, \mu_h \in M_h. \tag{5.28}$$

where

$$b_h(\boldsymbol{\tau}_h; u_h, \lambda_h) := (\boldsymbol{\tau}_h, \nabla_h^2 u_h) + \sum_{T\in\mathcal{T}_h} (\boldsymbol{n}^\intercal \boldsymbol{\tau}_h \boldsymbol{n}_e, \lambda_h - \partial_{n_e} u_h)_{\partial T},$$

where $\nabla_h$ is the piecewise counterpart of $\nabla$ with respect to $\mathcal{T}_h$. Here $\boldsymbol{n}$ is the unit outward normal to $\partial T$, and $\boldsymbol{n}_e$ is a fixed unit normal vector of edge $e$.

**引理 5.12.** There holds the following inf-sup condition

$$\|\nabla_h^2 v_h\|_0 + \|\mu_h - \partial_{n_e} v_h\|_{1/2,h} \lesssim \sup_{\boldsymbol{\tau}_h\in\Sigma_h} \frac{b_h(\boldsymbol{\tau}_h; v_h, \mu_h)}{\|\boldsymbol{\tau}_h\|_{0,h}} \quad \forall\, v_h \in V_h, \mu_h \in M_h. \tag{5.29}$$

*Proof.* Let $\boldsymbol{\tau}_h \in \Sigma_h$ such that

$$(\boldsymbol{n}^\intercal \boldsymbol{\tau}_h \boldsymbol{n}_e)|_e = \frac{1}{h_e}(\mu_h - \partial_{n_e} v_h), \qquad e \in \mathcal{E}(T), T \in \mathcal{T}_h,$$

$$\int_T \boldsymbol{\tau}_h : \boldsymbol{\varsigma}\,\mathrm{d}x = \int_T \nabla^2 v_h : \boldsymbol{\varsigma}\,\mathrm{d}x \quad \forall\, \boldsymbol{\varsigma} \in \mathbb{P}_{k-2}(T;\mathbb{S}), T \in \mathcal{T}_h.$$

Then

$$\|\boldsymbol{\tau}_h\|_{0,h} \lesssim \|\nabla_h^2 v_h\|_0 + \|\mu_h - \partial_{n_e} v_h\|_{1/2,h},$$

$$b_h(\boldsymbol{\tau}_h; v_h, \mu_h) = \|\nabla_h^2 v_h\|_0^2 + \|\mu_h - \partial_{n_e} v_h\|_{1/2,h}^2.$$

Therefore the inf-sup (5.29) holds. □

Thanks to the discrete inf-sup conditions (5.29) and the fact $\|\boldsymbol{\tau}_h\|_{0,h} \lesssim \|\boldsymbol{\tau}_h\|_0$ for $\boldsymbol{\tau}_h \in \Sigma_h$, the well-posedness of the mixed finite element method (5.27)-(5.28) follows from the Babuška-Brezzi theory [16].

**定理 5.13.** The hybridized mixed finite element method (5.27)-(5.28) is well-posed. We have the discrete stability

$$\begin{aligned}&\|\widetilde{\boldsymbol{\sigma}}_h\|_{0,h} + \|\nabla_h^2 \widetilde{u}_h\|_0 + \|\widetilde{\lambda}_h - \partial_{n_e}\widetilde{u}_h\|_{1/2,h} \\ \lesssim &\sup_{\boldsymbol{\tau}_h\in\Sigma_h, v_h\in V_h, \mu\in M_h} \frac{A_h(\widetilde{\boldsymbol{\sigma}}_h, \widetilde{u}_h, \widetilde{\lambda}_h; \boldsymbol{\tau}_h, v_h, \mu_h)}{\|\boldsymbol{\tau}_h\|_{0,h} + \|\nabla_h^2 v_h\|_0 + \|\mu_h - \partial_{n_e} v_h\|_{1/2,h}}\end{aligned} \tag{5.30}$$

for any $\widetilde{\boldsymbol{\sigma}}_h \in \Sigma_h$, $\widetilde{u}_h \in V_h$ and $\widetilde{\lambda}_h \in M_h$, where

$$A_h(\widetilde{\boldsymbol{\sigma}}_h, \widetilde{u}_h, \widetilde{\lambda}_h; \boldsymbol{\tau}_h, v_h, \mu_h) := (\widetilde{\boldsymbol{\sigma}}_h, \boldsymbol{\tau}_h) + b_h(\boldsymbol{\tau}_h; \widetilde{u}_h, \widetilde{\lambda}_h) + b_h(\widetilde{\boldsymbol{\sigma}}_h; v_h, \mu_h).$$

**定理 5.14.** Let $(\boldsymbol{\sigma}, u) \in \boldsymbol{H}^{-1}(\operatorname{div}\operatorname{div}, \Omega; \mathbb{S}) \times H_0^1(\Omega)$ be the solution of the HHJ mixed formulation (5.4)-(5.5), and $(\boldsymbol{\sigma}_h, u_h, \lambda_h)$ be the solution of the mixed finite element method (5.27)-(5.28). Assume $\boldsymbol{\sigma} \in \boldsymbol{H}^{m+1}(\Omega, \mathbb{S})$ and $u \in H^{m+3}(\Omega)$ for some non-negative integer $m$. Then

$$\|\partial_{n_e}(P_h u - u_h) - (Q_{\mathcal{E}_h}^{k-1}(\partial_{n_e} u) - \lambda_h)\|_{1/2,h} \lesssim h^{\min\{m+1,k\}} \|\boldsymbol{\sigma}\|_{m+1}, \tag{5.31}$$

$$\|\partial_{n_e} u_h - \lambda_h\|_{-1/2,h} \lesssim h^{\min\{m+2,k\}} \|u\|_{m+3}. \tag{5.32}$$

When $\Omega$ is convex, we have

$$\|Q_{\mathcal{E}_h}^{k-1}(\partial_{n_e} u) - \lambda_h\|_{-1/2,h} \lesssim h^{\min\{m+1,k\}+1} (\|\boldsymbol{\sigma}\|_{m+1} + \delta_{k1}\|f\|_0). \tag{5.33}$$

Proof. Apply (5.8), we can show that

$$A_h(I_h^{\mathrm{div}\,\mathrm{div}}\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, P_h u-u_h, Q_{\mathcal{E}_h}^{k-1}(\partial_{n_e}u)-\lambda_h;\boldsymbol{\tau}_h, v_h, \mu_h) = (I_h^{\mathrm{div}\,\mathrm{div}}\boldsymbol{\sigma}-\boldsymbol{\sigma},\boldsymbol{\tau}_h)$$

holds for any $\boldsymbol{\tau}_h\in\boldsymbol{\Sigma}_h$, $v_h\in V_h$ and $\mu\in M_h$.

It follows from the stability results (5.30) that

$$\|\partial_{n_e}(P_h u-u_h)-(Q_{\mathcal{E}_h}^{k-1}(\partial_{n_e}u)-\lambda_h)\|_{1/2,h}\lesssim\|\boldsymbol{\sigma}-I_h^{\mathrm{div}\,\mathrm{div}}\boldsymbol{\sigma}\|_0,$$

which together with the interpolation error estimate of $I_h^{\mathrm{div}\,\mathrm{div}}$ indicates (5.31). And (5.32) follows from the triangle inequality, (5.31) and the interpolation error estimates of $P_h$ and $Q_{\mathcal{E}_h}^{k-1}$. Finally (5.33) follows from (5.31), (5.21) and the inverse inequality. □

Define $\nabla_w^2: V_h\times M_h\to\Sigma_h$ by

$$(\nabla_w^2(v_h,\mu_h),\boldsymbol{\tau}_h)=b_h(\boldsymbol{\tau}_h;v_h,\mu_h)\quad\forall\,\boldsymbol{\tau}_h\in\Sigma_h.$$

Then the hybridized mixed finite element method (5.27)-(5.28) can be recast as: find $(u_h,\lambda_h)\in V_h\times M_h$ such that

$$(\nabla_w^2(u_h,\lambda_h),\nabla_w^2(v_h,\mu_h))=(f,v_h)\quad\forall\,v_h\in V_h,\mu_h\in M_h.$$

### 5.1.4 Equivalence between HHJ method and modified Morley element method

Recall the Morley element space

$$\begin{aligned}V_h^M:=\Big\{&v_h\in L^2(\Omega):v_h|_K\in\mathbb{P}_2(K),\ \forall\,K\in\mathcal{T}_h;\int_e[\![\partial_n v_h]\!]\,\mathrm{d}s=0,\ \forall\,e\in\mathcal{E}_h;\\ &v_h\text{ is continous at each vertex }\mathtt{v}\in\mathcal{N}_h;v_h(\mathtt{v})=0,\ \forall\,\mathtt{v}\in\mathcal{N}_h\cap\partial\Omega\Big\}.\end{aligned}$$

A modified Morley element method is to find $w_h\in V_h^M$ such that

$$(\nabla_h^2 w_h,\nabla_h^2 v_h)=(f,I_h^1 v_h)\quad\forall\,v_h\in V_h^M.\tag{5.34}$$

**定理** 5.15. Let $w_h\in V_h^M$ be the solution of Morley element method (5.34). Then $(-\nabla_h^2 w_h, I_h^1 w_h, Q_{\mathcal{E}_h}^0(\partial_{n_e}w_h))\in\Sigma_h\times V_h\times M_h$ is the solution of the hybridized mixed finite element method (5.27)-(5.28) with $k=1$.

Proof. Choose $v_h\in V_h^M$ such that $v_h$ vanishes at all vertices of $\mathcal{T}_h$, then $I_h^1 v_h=0$. Applying integration by parts on the left hand side of (5.34), we get

$$\sum_{e\in\mathcal{E}_h^i}([n^\intercal(\nabla_h^2 w_h)n],\partial_n v_h)_e=\sum_{T\in\mathcal{T}_h}(n^\intercal(\nabla_h^2 w_h)n,\partial_n v_h)_{\partial T}=0.$$

As a result, $[n^\intercal(\nabla_h^2 w_h)n]_e=0$ for all $e\in\mathcal{E}_h^i$, that is $\nabla_h^2 w_h\in\Sigma_h^{\mathrm{div}\,\mathrm{div}}$.

For $v_h\in V_h^M$, by (5.9) and (5.34),

$$\begin{aligned}b_h(\nabla_h^2 w_h;I_h^1 v_h,\mu_h)&=b_h(\nabla_h^2 w_h,I_h^1 v_h)=b_h(\nabla_h^2 w_h,v_h)\\&=(\nabla_h^2 w_h,\nabla_h^2 v_h)-\sum_{T\in\mathcal{T}_h}(n^\intercal(\nabla_h^2 w_h)n,\partial_n u_h)_{\partial T}\\&=(\nabla_h^2 w_h,\nabla_h^2 v_h)=-(f,I_h^1 v_h).\end{aligned}$$

Then $(-\nabla_h^2 w_h, I_h^1 w_h, Q_{\mathcal{E}_h}^0(\partial_{n_e} w_h))$ satisfies (5.28)

It follows from (5.9) that

$$(-\nabla_h^2 w_h, \boldsymbol{\tau}_h) + b_h(\boldsymbol{\tau}_h; I_h^1 w_h, Q_{\mathcal{E}_h}^0(\partial_{n_e} w_h)) = b_h(\boldsymbol{\tau}_h, w_h - I_h^1 w_h) = 0,$$

which ends the proof. □

# Appendices

# 第一章 Bernstein 多项式和 Lagrange 元几何分解

## A.1 单纯形和子单纯形

设 $T \subset \mathbb{R}^d$ 是一个 $d$ 维单纯形，其顶点为 $\mathtt{v}_0, \mathtt{v}_1, \ldots, \mathtt{v}_d$。依照 [7] 的记法，我们用 $\Delta(T)$ 表示 $T$ 的所有子单纯形所构成的集合，而 $\Delta_\ell(T)$ 表示维数为 $\ell$ 的子单纯形集合，其中 $0 \leq \ell \leq d$。

对于子单纯形 $f \in \Delta_\ell(T)$，我们将重载符号 $f$ 的含义，使其同时表示几何单纯形和代数指标集合。即一方面 $f = \{f(0), \ldots, f(\ell)\} \subseteq \{0, 1, \ldots, d\}$，另一方面

$$f = \mathrm{Convex}(\mathtt{v}_{f(0)}, \ldots, \mathtt{v}_{f(\ell)}) \in \Delta_\ell(T)$$

是由顶点 $\mathtt{v}_{f(0)}, \ldots, \mathtt{v}_{f(\ell)}$ 张成的 $\ell$ 维单纯形。若 $f \in \Delta_\ell(T)$，其中 $\ell = 0, \ldots, d-1$，则 $f^* \in \Delta_{d-\ell-1}(T)$ 表示 $T$ 中与 $f$ 相对的子单纯形。在代数上，若将 $f$ 视为 $\{0, 1, \ldots, d\}$ 的子集，则 $f^* \subseteq \{0, 1, \ldots, d\}$ 满足 $f \cup f^* = \{0, 1, \ldots, d\}$，即 $f^*$ 是集合 $f$ 的补集。在几何上，

$$f^* = \mathrm{Convex}(\mathtt{v}_{f^*(1)}, \ldots, \mathtt{v}_{f^*(d-\ell)}) \in \Delta_{d-\ell-1}(T)$$

是由不包含在 $f$ 中的顶点张成的 $(d-\ell-1)$ 维单纯形。关于 $f$ 和 $f^*$ 的图示，请参阅 [?, Fig. 2]。

记 $F_i$ 为与顶点 $\mathtt{v}_i$ 相对的 $(d-1)$ 维面，即 $F_i = \{i\}^*$。此处大写字母 $F$ 专用于表示 $T$ 的 $(d-1)$ 维面。对于较低维的子单纯形，我们有时采用更常规的记法。例如，顶点将记为 $\mathtt{v}_i$，由 $\mathtt{v}_i$ 和 $\mathtt{v}_j$ 生成的边将记为 $\mathtt{e}_{ij}$。

## A.2 重心坐标和 Bernstein 多项式

对于区域 $D \subseteq \mathbb{R}^d$ 及整数 $r \geqslant 0$，记 $\mathbb{P}_r(D)$ 为定义在 $D$ 上的 $r$ 次实值多项式空间。为简化表述，令 $\mathbb{P}_r = \mathbb{P}_r\left(\mathbb{R}^d\right)$。于是当 $d$ 维区域 $D$ 具有非空内部时，有 $\dim \mathbb{P}_r(D) = \dim \mathbb{P}_r = \binom{r+d}{d}$。当 $D = \mathtt{v}$ 为单点集时，对任意 $r \geqslant 0$ 均有 $\mathbb{P}_r(\mathtt{v}) = \mathbb{R}$；而当 $r < 0$ 时，规定 $\mathbb{P}_r(D) = 0$。记 $\mathbb{H}_r(D)$ 为定义在 $D$ 上的 $r$ 次齐次实值多项式空间。

对 $d$ 维单纯形 $T$，设 $\lambda_0, \lambda_1, \ldots, \lambda_d$ 为其重心坐标函数，即满足 $\lambda_i \in \mathbb{P}_1(T)$ 且 $\lambda_i(\mathtt{v}_j) = \delta_{i,j}$ $(0 \leqslant i, j \leqslant d)$，其中 $\delta_{i,j}$ 为 Kronecker delta 函数。函数集 $\{\lambda_i, i = 0, 1, \ldots, d\}$ 构成 $\mathbb{P}_1(T)$ 的一组基，满足 $\sum_{i=0}^d \lambda_i(x) = 1$，且对任意 $x \in T$ 有 $0 \leq \lambda_i(x) \leq 1$（$i = 0, 1, \ldots, d$）。$T$ 的子单纯形与重心坐标的零点集一一对应：具体地，对任意 $f \in \Delta_\ell(T)$，有 $f = \{x \in T \mid \lambda_i(x) = 0, i \in f^*\}$。

我们将使用多重指标记号 $\alpha \in \mathbb{N}^d$，即 $\alpha = (\alpha_1, \ldots, \alpha_d)$，其中各整数 $\alpha_i \geq 0$。定义

$$x^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d}, \qquad |\alpha| := \sum_{i=1}^d \alpha_i。$$

同时，我们还将使用多重指标集合 $\mathbb{N}^{0:d}$，其元素为 $\alpha = (\alpha_0, \ldots, \alpha_d)$，并定义

$$\lambda^\alpha := \lambda_0^{\alpha_0} \cdots \lambda_d^{\alpha_d}, \quad \alpha \in \mathbb{N}^{0:d}。$$

接下来引入单纯形格点集 [?, ?]（亦称主格 [?]）：$d$ 维 $r$ 阶单纯形格点集是一个指标和为 $r$ 的 $(d+1)$ 维非负整数指标集，即

$$\mathbb{T}_r^d = \left\{\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_d) \in \mathbb{N}^{0:d} \mid \alpha_0 + \alpha_1 + \ldots + \alpha_d = r\right\}.$$

集合中的元素 $\alpha \in \mathbb{T}_r^d$ 称为格点。

在单纯形 $T$ 上，$r$ 次多项式空间的 Bernstein 表示为

$$\mathbb{P}_r(T) := \mathrm{span}\{\lambda^\alpha = \lambda_0^{\alpha_0}\lambda_1^{\alpha_1} \ldots \lambda_d^{\alpha_d}, \alpha \in \mathbb{T}_r^d\}.$$

在 Bernstein 形式下，对于任意 $f \in \Delta_\ell(T)$，有

$$\mathbb{P}_r(f) = \mathrm{span}\{\lambda_f^\alpha = \lambda_{f(0)}^{\alpha_0}\lambda_{f(1)}^{\alpha_1} \ldots \lambda_{f(\ell)}^{\alpha_\ell}, \alpha \in \mathbb{T}_r^\ell\}.$$

通过重心坐标自然延拓，可得到 $\mathbb{P}_r(f) \subseteq \mathbb{P}_r(T)$。

定义 $(\ell+1)$ 次泡多项式

$$b_f := \lambda_f = \lambda_{f(0)}\lambda_{f(1)} \cdots \lambda_{f(\ell)} \in \mathbb{P}_{\ell+1}(f).$$

泡多项式 $b_f$ 满足以下性质.

**引理 A.1.** 设 $f, e \in \Delta(T)$。若 $f \not\subseteq e$，则 $b_f|_e = 0$。

*Proof.* 由分解 $f = (f \cap e^*) \cup (f \cap e)$ 且 $f \not\subseteq e$，可得 $f \cap e^* \neq \varnothing$。因此 $b_f$ 包含某个 $i \in e^*$ 对应的 $\lambda_i$，故 $b_f|_e = 0$。 □

特别地，$b_f$ 在除 $f$ 以外的所有维度不大于 $\dim f$ 的子单纯形上，以及不包含 $f$ 的高维子单纯形上均为零。

## A.3 Lagrange 元几何分解

这一节内容取材于文献 [34, 29]. 下述 Lagrange 元的几何分解最初在 [7, (2.6)] 中给出。有关该几何分解的示意，可参见 [7, Fig. 2.1]。对于 0 维面（即顶点 $\mathtt{v}$），约定

$$\int_{\mathtt{v}} u \,\mathrm{d}s = u(\mathtt{v}), \quad 1 \in \mathbb{P}_r(\mathtt{v}) = \mathbb{R}.$$

**定理 A.2.** 设 $T$ 为 $d$ 维单纯形，$\mathbb{P}_r(T)$ 为 $r \geq 1$ 次多项式空间，则有如下分解：

$$\mathbb{P}_r(T) = \bigoplus_{\ell=0}^d \bigoplus_{f \in \Delta_\ell(T)} b_f \mathbb{P}_{r-(\ell+1)}(f). \tag{A.1}$$

且任意 $u \in \mathbb{P}_r(T)$ 由以下自由度唯一确定：

$$\int_f u\, p \,\mathrm{d}s, \qquad p \in \mathbb{P}_{r-(\ell+1)}(f), f \in \Delta_\ell(T), \ell = 0, 1, \ldots, d. \tag{A.2}$$

Proof. 我们首先证明分解 (A.1)。每个部分 $b_f\mathbb{P}_{r-(\ell+1)}(f)\subseteq\mathbb{P}_r(T)$，并且由于 $b_f$ 的性质（参见引理 A.1），这些子空间的直和成立。通过观察 $(1+x)^{d+1}(1+x)^{r-1}=(1+x)^{d+r}$ 中 $x^r$ 的系数可知组合恒等式

$$\sum_{\ell=0}^{d}\binom{d+1}{\ell+1}\binom{r-1}{r-\ell-1}=\binom{d+r}{r}$$

成立。利用此恒等式计算维数可得 (A.1)。

为了证明自由度的唯一可解性，我们按照分解 (A.1) 选择 $\mathbb{P}_r(T)$ 的基 $\phi_i$，并将自由度 (A.2) 记为 $N_i$。由构造可知，基的数量与自由度的数量一致。对应的自由度-基矩阵 $(N_i(\phi_j))$ 是方阵，并且是块下三角形矩阵。具体地，对于 $\phi_f\in b_f\mathbb{P}_{r-(\ell+1)}(f)$，由引理 A.1 中 $b_f$ 的性质有

$$\int_e \phi_f p\,\mathrm{d}s=0,\qquad e\in\Delta_m(T),m\leq\ell,e\neq f,p\in\mathbb{P}_{r-\dim e-1}(e).$$

每个对角块矩阵是测度 $b_f\,\mathrm{d}x_f$ 下的 Gram 矩阵

$$\int_f pqb_f\,\mathrm{d}x_f,\quad p,q\in\mathbb{P}_{r-(\ell+1)}(f),$$

因此对称且正定，故可逆。由此，下三角矩阵的可逆性保证了自由度的唯一可解性；下图给出示意:

$$
\begin{array}{cc}
N_f\backslash\,\phi_f & \begin{array}{ccccc} 0 & 1 & \cdots & d-1 & d \end{array}\\
\begin{array}{c} 0\\ 1\\ \vdots\\ d-1\\ d \end{array} &
\left(\begin{array}{c|c|c|c|c}
\square & 0 & \cdots & 0 & 0\\ \hline
\square & \square & \cdots & 0 & 0\\ \hline
\vdots & \vdots & \ddots & \vdots & \vdots\\ \hline
\square & \square & \cdots & \square & 0\\ \hline
\square & \square & \cdots & \square & \square
\end{array}\right)
\end{array}
\tag{A.3}
$$

□

**注 A.3.** 需要注意的是，当 $r<\ell+1$ 时，有 $\mathbb{P}_{r-(\ell+1)}(f)=\{0\}$。因此，在几何分解 (A.1) 中，最后一个非零项对应于 $\ell\leq\min\{r-1,,d\}$。这意味着多项式的次数决定了分解 (A.1) 中所涉及子单纯形的维数。例如，对于二次多项式，求和仅包含边泡函数，而不包括面泡及更高维泡函数。尽管如此，为了书写简便，仍保留完整的求和符号 $\bigoplus_{\ell=0}^d$，默认理解为其中的零子空间会自动截断求和范围。

设 $\mathcal{T}_h$ 为区域 $\Omega$ 的一族协调单纯形剖分。记 $\Delta_\ell(\mathcal{T}_h)$ 为剖分 $\mathcal{T}_h$ 中所有 $\ell$ 维子单纯形的集合，其中 $\ell=0,1,\ldots,d$。Lagrange 有限元空间定义为

$$S_h^r:=\{v\in C(\Omega):v|_T\in\mathbb{P}_r(T),\forall\,T\in\mathcal{T}_h,\ \text{自由度 (A.2) 是单值的}\},$$

其具有如下几何分解形式:

$$S_h^r=\bigoplus_{\ell=0}^d\bigoplus_{f\in\Delta_\ell(\mathcal{T}_h)}b_f\mathbb{P}_{r-(\ell+1)}(f).$$

这里，多项式空间 $b_f\,\mathbb{P}_{r-(\ell+1)}(f)$ 通过重心坐标的 Bernstein 形式自然扩展到包含 $f$ 的各单元 $T$ 上，从而得到在整个区域 $\Omega$ 上连续的分片多项式函数。因此，$S_h^r\subset H^1(\Omega)$，并且其维数为

$$\dim S_h^r=\sum_{\ell=0}^d|\Delta_\ell(\mathcal{T}_h)|\binom{r-1}{\ell},$$

其中 $|\Delta_\ell(\mathcal{T}_h)|$ 表示剖分 $\mathcal{T}_h$ 中 $\ell$ 维单纯形的数量。

向量型 Lagrange 元的几何分解是上述结果的直接推广：

$$\mathbb{P}_r(T;\mathbb{R}^d) = \bigoplus_{\ell=0}^{d}\bigoplus_{f\in\Delta_\ell(T)}\left[b_f\mathbb{P}_{r-(\ell+1)}(f)\otimes\mathbb{R}^d\right]. \tag{A.4}$$

在式 (A.4) 中，默认使用 $\mathbb{R}^d$ 的一组固定正交基展开向量分量，通常取描述区域 $\Omega$ 的笛卡尔坐标系。

# 第二章　算子理论

## B.1 Preliminary From Functional Analysis

Let $X$ and $Y$ be two Banach spaces in this section. See [20, 75] for details in this section.

### B.1.1 Closed operators

**Definition B.1** (Unbounded linear operator). Let $X$ and $Y$ be two Banach spaces. An **unbounded linear operator** from $X$ into $Y$ is linear map $A : D(A) \subset X \to Y$ defined on a linear subspace $D(A) \subset X$ with values in $Y$. The set $D(A)$ is called the domain of $A$.

**Definition B.2** (Graph). Let $X$ and $Y$ be two Banach spaces. Let $A : D(A) \subset X \to Y$ be a linear operator. Define the **graph** of $A$ as the following linear subspace

$$\{(x, Ax) \in X \times Y : X \in D(A)\}.$$

The linear operator $A$ is said to be **closed** if the graph of $A$ is closed in $X \times Y$.

**引理 B.3.** The linear operator $A : D(A) \subset X \to Y$ is closed if and only if its domain $D(A)$ is a complete space with respect to the graph norm $\sqrt{\|x\|_X^2 + \|Ax\|_Y^2}$.

**引理 B.4.** If the linear operator $A : D(A) \subset X \to Y$ is closed, then $N(A)$ is closed; however, $R(A)$ need not be closed.

**定理 B.5** (Closed graph theorem). Let $X$ and $Y$ be two Banach spaces. If $A : D(A) \subset X \to Y$ is a closed linear operator and $D(A)$ is closed, then $A$ is continuous (bounded).

### B.1.2 Adjoints in Banach spaces

**Definition B.6** (Adjoint $A'$). Suppose $A : D(A) \subset X \to Y$ be a linear operator that is densely defined. Let

$$D(A') = \{y' \in Y' : \exists x' \in X' \text{ such that } \langle y', Ax\rangle = \langle x', x\rangle \quad \forall x \in D(A)\},$$

where $X'$ and $Y'$ are the dual spaces of $X$ and $Y$ respectively. Then we define $A'y' := x'$. The linear operator

$$A' : D(A') \subset Y' \to X'$$

is called the **adjoint** of $A$. The **domain** $D(A')$ can be rewritten as

$$D(A') = \{y' \in Y' : \exists c \geq 0 \text{ such that } |\langle y', Ax\rangle| \leq c\|x\|_X \quad \forall x \in D(A)\}.$$

The fundamental relation between $A$ and $A'$ is given by

$$\langle y', Ax\rangle_{Y'\times Y} = \langle A'y', x\rangle_{X'\times X} \qquad \forall x \in D(A), \quad y' \in D(A').$$

**引理 B.7.** If $A : X \to Y$ is a linear bounded operator then $A' : Y' \to X'$ is also a linear bounded operator and, moreover,

$$\|A'\|_{\mathcal{L}(Y',X')} = \|A\|_{\mathcal{L}(X,Y)}.$$

**引理 B.8.** Let $A : D(A) \subset X \to Y$ be a densely defined linear operator. Then the adjoint $A' : D(A') \subset Y' \to X'$ is closed.

**定理 B.9.** Let $X$ and $Y$ be two reflexive Banach spaces. Let $A : D(A) \subset X \to Y$ be an unbounded linear operator that is densely defined and closed. Then $D(A')$ is dense in $Y'$. Thus $A''$ is well defined ($A'' : D(A'') \subset X'' \to Y''$) and it may also be viewed as an unbounded operator from $X$ into $Y$. Then we have

$$A'' = A.$$

**Definition B.10.** If $M \subset X$ is a linear subspace, we define the annihilator of $M$ in $X'$

$$M^\perp = \{f \in X' : \langle f, x\rangle = 0 \quad \forall x \in M\}.$$

If $N \subset X'$ is a linear subspace we set

$$N^\perp = \{x \in X : \langle f, x\rangle = 0 \quad \forall f \in N\}.$$

**引理 B.11.** Let $M \subset X$ be a linear subspace. Then

$$(M^\perp)^\perp = \overline{M}.$$

Let $N \subset X'$ be a linear subspace. Then

$$(N^\perp)^\perp \supset \overline{N}.$$

Moreover, if $X$ is reflexive, then

$$(N^\perp)^\perp = \overline{N}.$$

**定理 B.12.** Let $A : D(A) \subset X \to Y$ be an unbounded linear operator that is densely defined and closed. Then

$$N(A) = R(A')^\perp, \quad N(A') = R(A)^\perp,$$

$$N(A)^\perp \supset \overline{R(A')}, \quad N(A')^\perp = \overline{R(A)}.$$

Moreover, if $X$ is reflexive, then

$$N(A)^\perp = \overline{R(A')}.$$

**定理 B.13** (Closed range theorem). Let $A : D(A) \subset X \to Y$ be an unbounded linear operator that is densely defined and closed. The following properties are equivalent:

(i) $R(A)$ is closed.

(ii) $R(A')$ is closed.

(iii) $R(A) = N(A')^{\perp}$.

(iv) $R(A') = N(A)^{\perp}$.

**定理 B.14.** Let $A : D(A) \subset X \to Y$ be a closed unbounded linear operator. Then $R(A)$ is closed if and only if there exists a constant $C$ such that

$$\mathrm{dist}(x, N(A)) \leq C\|Ax\|_Y \quad \forall x \in D(A).$$

**定理 B.15.** Let $A : D(A) \subset X \to Y$ be an unbounded linear operator that is densely defined and closed. The following properties are equivalent:

(a) $A$ is surjective, i.e. $R(A) = Y$,

(b) there is a constant $C$ such that

$$\|y\|_{Y'} \leq C\|A'y\|_{X'} \quad \forall y \in D(A'),$$

(c) $N(A') = \{0\}$ and $R(A')$ is closed.

**定理 B.16.** Let $A : D(A) \subset X \to Y$ be an unbounded linear operator that is densely defined and closed. The following properties are equivalent:

(a) $A'$ is surjective, i.e. $R(A') = X'$,

(b) there is a constant $C$ such that

$$\|x\|_X \leq C\|Ax\|_Y \quad \forall x \in D(A),$$

(c) $N(A) = \{0\}$ and $R(A)$ is closed.

# 第三章 Sobolev Spaces and Elliptic Equations

Sobolev spaces are fundamental in the study of partial differential equations and their numerical approximations. In this chapter, we shall give brief discussions on the Sobolev spaces and the regularity theory for elliptic boundary value problems.

## C.1 Sobolev Spaces

We shall state and explain main results (without proofs) on Sobolev spaces. We refer to [1, 20] for comprehensive treatment.

### C.1.1 Preliminaries

We first set up the environment of our discussion: Lipschitz domains, multi-index notation for differentiation, and some basic functional spaces.

Given two metric spaces $(X, d_X)$ and $(Y, d_Y)$, where $d_X$ denotes the metric on the set $X$ and $d_Y$ is the metric on set $Y$, a function $f : X \to Y$ is called **Lipschitz continuous** if there exists a real constant $K \geq 0$ such that, for all $x_1$ and $x_2$ in $X$,

$$d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2).$$

In particular, a real-valued function $f : R \to R$ is called Lipschitz continuous if there exists a positive real constant $K$ such that, for all real $x_1$ and $x_2$,

$$|f(x_1) - f(x_2)| \leq K|x_1 - x_2|.$$

A domain (a connected open set) $\Omega \subset \mathbb{R}^n$ is called a **Lipschitz domain** if its boundary $\partial\Omega$ can be locally represented by Lipschitz continuous function; namely for any $x \in \partial\Omega$, there exists a neighborhood of $x$, $G \subset \mathbb{R}^n$, such that $G \cap \partial\Omega$ is the graph of a Lipschitz continuous function under a proper local coordinate system.

**Example C.1.**

- All the smooth domains are Lipschitz.
- A non-smooth example is that every polygonal domain in $\mathbb{R}^2$ or polyhedron in $\mathbb{R}^3$ is Lipschitz.
- Every convex domain in $\mathbb{R}^n$ is Lipschitz.
- A simple example of non-Lipschitz domain is two polygons touching at one vertex only.
- A more interesting non-Lipschitz domain is a domain with cusp points on the boundary.

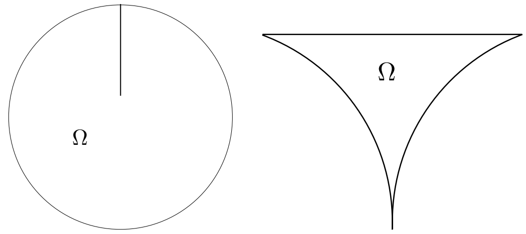



Figure C.1: Domains that are not Lipschitz: slit domain (left) and cusp domain (right)

Let $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}_+^n$, where $\mathbb{Z}_+^n$ is the set of non-negative integers, be a vector of nonnegative integers. Denote by $|\alpha| = \sum_{i=1}^n \alpha_i$. For a smooth function $v$ and $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, denote

$$D^\alpha v = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}, \quad \text{and } x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

Several basic Banach spaces will often be used in this book. $C(\bar{\Omega})$ is the space of continuous functions on $\bar{\Omega}$ with the usual maximum norm

$$\|v\|_{C(\bar{\Omega})} = \max_{x \in \bar{\Omega}} |v(x)|.$$

$C_0^\infty(\Omega)$ denotes the space of infinitely differential functions in $\Omega$ that vanish in some neighborhood of $\partial\Omega$; namely $\text{supp}(v) \subset \Omega$, where

$$\text{supp}(v) = \text{closure of } \{x \in \Omega : v(x) \neq 0\}.$$

A function $u$ that is measurable on $\Omega$ is said to be **essentially bounded** on $\Omega$ if there is a constant $K$ such that $|u(x)| \leq K$ a.e. on $\Omega$. The greatest lower bound of such constants $K$ is called the essential supremum of $|u|$ on $\Omega$, and is denoted by $\text{ess sup}_{x \in \Omega} |u(x)|$.

Given $1 \leq p < \infty$, denote by $L^p(\Omega)$ the class of all measurable functions $u$ defined on $\Omega$ for which $\int_\Omega |u(x)|^p \, \mathrm{d}x < \infty$, and by $L^\infty(\Omega)$ the vector space of all functions $u$ that are essentially bounded on $\Omega$. The norm is defined by

$$\|u\|_{p,\Omega} = \left( \int_\Omega |u|^p \, \mathrm{d}x \right)^{1/p} \quad \text{for } 1 \leq p < \infty,$$

and

$$\|u\|_{\infty,\Omega} = \operatorname*{ess\,sup}_{x \in \Omega} |u(x)|.$$

We identify in $L^p(\Omega)$ with $1 \leq p \leq \infty$ functions that are equal almost everywhere in $\Omega$; the elements of $L^p(\Omega)$ are thus equivalence classes of measurable functions satisfying $\|u\|_{p,\Omega} < \infty$, two functions being equivalent if they are equal a.e. in $\Omega$.

**Exercise C.1.** Prove that $L^p(\Omega)$, $1 \leq p \leq \infty$ defined above are Banach spaces.

### C.1.2 Definition of Sobolev spaces

The conventional way to understand a function is through its point-wise values which is not adequate. It is better to understand a function as a functional through its action on a space of unproblematic test functions (convential and well-behaved functions). In this way, the concept of functions can be generalized to distributions. "Integration by parts" is used to extend the differentiation operators from classic differentiable functions to distributions by shifting the operators to the test functions. Sobolev spaces will be first defined here for integer orders using the concept of distributions and their weak derivatives. The fractional order Sobolev spaces will be introduced by looking at the $p$th power integrable of quotient of difference. Definitions will also be given to Sobolev spaces satisfying certain zero boundary conditions.

Distributions and weak derivatives. We begin with the nice function space $C_0^\infty(\Omega)$. Obviously $C_0^\infty(\Omega)$ is a real vector space and can be turned into a topological vector space by a proper topology. The space $C_0^\infty(\Omega)$ equipped with the following topology is denoted by $\mathcal{D}(\Omega)$: a sequence of functions $\{\phi_k\} \subset C_0^\infty(\Omega)$ is said to be convergent to a function $\phi \in C_0^\infty(\Omega)$ in the space $\mathcal{D}(\Omega)$ if

1. there exists a compact set $K \subset \Omega$ such that for all $k$, $\operatorname{supp}(\phi_k - \phi) \subset K$, and
2. for every $\alpha \in \mathbb{Z}_+^n$, we have $\lim_{k\to\infty} D^\alpha \phi_k(x) = D^\alpha \phi(x)$ uniformly on $K$.

The space, denoted by $\mathcal{D}'(\Omega)$, of all continuous linear functionals on $\mathcal{D}(\Omega)$ is called the (Schwarz) distribution space. The space $\mathcal{D}'(\Omega)$ will be equipped with the weak star topology. Namely, in $\mathcal{D}'(\Omega)$, a sequence $T_n$ converge to $T$ in the distribution sense if and only if $\langle T_n, \phi\rangle \to \langle T, \phi\rangle$ for all $\phi \in \mathcal{D}(\Omega)$, where $\langle \cdot, \cdot\rangle : \mathcal{D}'(\Omega) \times \mathcal{D}(\Omega) \to \mathbb{R}$ is the duality pair. A function $\phi$ belonging to $\mathcal{D}(\Omega)$ is called a test function since the action of a distribution on $\phi$ can be thought as a test.

**Example C.2.** By the definition, an element in $\mathcal{D}'(\Omega)$ is uniquely determined by its action. The action could be very general and abstract as long as it is linear and continuous. As an example, let us introduce the Dirac delta distribution $\delta \in \mathcal{D}'(\Omega)$ with $0 \in \Omega \subset \mathbb{R}^n$ defined as

$$\langle \delta, \phi\rangle = \phi(0) \quad \text{for all } \phi \in \mathcal{D}(\Omega).$$

One important class of distributions is to use the integration as the action. A function is called locally integrable if it is Lebesgue integrable over every compact subset of $\Omega$. We define the space $L^1_{\mathrm{loc}}(\Omega)$ as the space containing all locally integrable functions. We can embed $L^1_{\mathrm{loc}}(\Omega)$ into $\mathcal{D}'(\Omega)$ using the integration as the duality action. For a function $u \in L^1_{\mathrm{loc}}(\Omega)$, define $T_u \in \mathcal{D}'(\Omega)$ as

$$\langle T_u, \phi\rangle := \int_\Omega u\phi \,\mathrm{d}x \quad \text{for all } \phi \in \mathcal{D}(\Omega).$$

We shall still denoted $T_u$ by $u$. The correspondence $u \mapsto T_u$ is often used to identify an "ordinary" function as a distribution.

A distribution is often also known as a generalized function as the concept of distribution is a more general than the concept of the classic function. One of the basic distribution which is not an "ordinary" function is the Dirac $\delta$-distribution introduced in Example C.2. Indeed, one motivation of the invention of distribution space is to include Dirac delta "function".

**Exercise C.2.** Prove that it is not possible to represent delta distribution by a locally integrable function.

If $u$ is a smooth function, it follows from integration by parts that, for any $\alpha \in \mathbb{Z}_+^n$

$$\int_\Omega D^\alpha u(x)\phi(x)\,\mathrm{d}x = (-1)^{|\alpha|}\int_\Omega u(x) D^\alpha \phi(x)\,\mathrm{d}x \quad \text{for all } \phi \in \mathcal{D}(\Omega).$$

There are no boundary terms, since $\phi$ has compact support in $\Omega$ and thus $\phi$, together with its derivatives, vanishes near $\partial\Omega$. The above identity is the basis for defining derivatives for a distribution. If $T \in \mathcal{D}'(\Omega)$, then for any $\alpha \in \mathbb{Z}_+^n$, we define weak derivative $D^\alpha T$ as the distribution given by

$$\langle D^\alpha T, \phi\rangle = (-1)^{|\alpha|}\langle T, D^\alpha \phi\rangle \quad \text{for all } \phi \in \mathcal{D}(\Omega).$$

It is easy to see for a differentiable function, its weak derivative coincides with its classical derivative. But in general, the weak derivative is much weaker than the classical one such that the differential operator can be extended from differential functions to a much larger space: the space of distributions. For example, we can even talk about the derivative of a discontinuous function.

**Example C.3.** The Heaviside step function is defined as $S(x) = 1$ for $x > 0$ and $S(x) = 0$ for $x < 0$. By the definition

$$\int_{\mathbb{R}} S'\phi \,\mathrm{d}x = -\int_{\mathbb{R}} S\phi' \,\mathrm{d}x = -\int_0^\infty \phi' \,\mathrm{d}x = \phi(0).$$

Therefore $S' = \delta$ in the distribution sense but $\delta$ is not a function in $L^1_{\mathrm{loc}}(\Omega)$.

Integer order Sobolev spaces. The Sobolev space of index $(k,p)$, where $k$ is a nonnegative integer and $p \geq 1$, is defined by

$$W^{k,p}(\Omega) := \{v \in L^p(\Omega) : D^\alpha v \in L^p(\Omega) \text{ for all } |\alpha| \leq k\}$$

with a norm $\|\cdot\|_{k,p,\Omega}$ given by

$$\|v\|_{k,p,\Omega}^p := \sum_{|\alpha|\leq k} \|D^\alpha v\|_{0,p,\Omega}^p.$$

We will have occasions to use the seminorm $|\cdot|_{k,p,\Omega}$ given by

$$|v|_{k,p,\Omega}^p := \sum_{|\alpha|= k} \|D^\alpha v\|_{0,p,\Omega}^p.$$

For $p = 2$, it is customary to write $H^k(\Omega) := W^{k,2}(\Omega)$, which is a Hilbert space together with an inner product as follows

$$(u,v)_{H^k(\Omega)} = \sum_{|\alpha|\leq k} (D^\alpha u, D^\alpha v)_{L^2(\Omega)} = \sum_{|\alpha|\leq k} \int_\Omega D^\alpha u D^\alpha v \,\mathrm{d}x$$

and the corresponding norm is denoted by $\|v\|_{k,\Omega} = \|v\|_{k,2,\Omega}$.

**Exercise C.3.** Prove that $W^{k,p}(\Omega)$ is a Banach space.

For a function in $L^p(\Omega)$, treating it as a distribution, its weak derivatives always exists as distributions. But the weak derivative may not be in the space $L^p(\Omega)$. Therefore an element in $W^{k,p}(\Omega)$ possesses certain smoothness.

**Example C.4.** We consider the Heaviside function restricted to $(-1,1)$ and still denote by $S$. The weak derivative of $S$ is Delta distribution which is not integrable. Therefore $S \notin H^1(-1,1)$.

**Example C.5.** Let $u(x) = |x|$ for $x \in (-1,1)$ be an anti-derivative of $2(S - 1/2)$. Obviously $u \in L^2(-1,1)$ and $u' \in L^2(-1,1)$. Therefore $u \in H^1(-1,1)$.

Fractional order Sobolev spaces. In the definition of classic derivatives, it takes the pointwise limit of the quotient of difference. For functions in Sobolev space, we shall use the $p$th power integrability of the quotient difference to characterize the differentiability.

For $0 < \theta < 1$ and $1 \le p < \infty$, we define

$$W^{\theta,p}(\Omega) = \left\{ v \in L^p(\Omega) : \int_\Omega \int_\Omega \frac{|v(x) - v(y)|^p}{|x-y|^{n+\theta p}} \,\mathrm{d}x\,\mathrm{d}y < \infty \right\}$$

and

$$H^\theta = W^{\theta,2}(\Omega).$$

In $W^{\theta,p}(\Omega)$, we define the following semi-norm

$$|v|_{\theta,p,\Omega} := \left( \int_\Omega \int_\Omega \frac{|v(x) - v(y)|^p}{|x-y|^{n+\theta p}} \,\mathrm{d}x\,\mathrm{d}y \right)^{1/p}, \tag{C.1}$$

and norm

$$\|v\|_{\theta,p,\Omega} := \left( \|v\|_{0,p,\Omega}^p + |v|_{\theta,p,\Omega}^p \right)^{1/p}.$$

**Example C.6.** For the Heaviside function S restricted to $(-1, 1)$, we look at the integral (C.1). To be in the space $W^{\theta,p}(\Omega)$, it is essential to have the integral

$$\int_0^1 \frac{1}{x^{\theta p}} \,\mathrm{d}x < \infty.$$

That is to require $\theta < 1/p$. In particular, we conclude $S \in H^{1/2-\epsilon}(-1, 1)$ for any $0 < \epsilon \le 1/2$ but $S \notin H^{1/2}(-1, 1)$.

Given $s = k + \theta$ with a real number $\theta \in (0, 1)$ and an integer $k \ge 0$, define

$$W^{s,p}(\Omega) := \{v \in W^{k,p}(\Omega) : D^\alpha v \in W^{\theta,p}(\Omega), |\alpha| = k\}.$$

In $W^{s,p}(\Omega)$, we define the following semi-norm and norm

$$|v|_{s,p,\Omega} = \left( \sum_{|\alpha|=k} |D^\alpha v|_{\theta,p,\Omega}^p \right)^{1/p}, \quad \|v\|_{s,p,\Omega} = \left( \|v\|_{k,p,\Omega}^p + |v|_{s,p,\Omega}^p \right)^{1/p}.$$

Negative order Sobolev spaces. $W^{k,p}(\Omega)$ is a Banach space, i.e., it is complete in the topology induced by the norm $\|\cdot\|_{k,p,\Omega}$. Indeed $\{\phi \in C^\infty(\Omega) : \|\phi\|_{k,p,\Omega} < \infty\}$ is dense in $W^{k,p}(\Omega)$. When $\Omega$ satisfies the segment condition, $C^\infty(\overline{\Omega})$ is dense in $W^{k,p}(\Omega)$ [2]. The closure of $C_0^\infty(\Omega)$ with respect to the same topology is denoted by $W_0^{k,p}(\Omega)$. For $p = 2$, we usually write $H_0^k(\Omega) = W_0^{k,2}(\Omega)$. Roughly speaking for $u \in H_0^1(\Omega)$, $u|_{\partial\Omega} = 0$ in an appropriate sense. Except $k = 0$ or $\Omega = \mathbb{R}^n$, $W_0^{k,p}(\Omega)$ is a proper subspace of $W^{k,p}(\Omega)$.

For $k \in \mathbb{N}$, $W^{-k,p}(\Omega)$ is defined as the dual space of $W_0^{k,p'}(\Omega)$, where $p'$ is the conjugate of $p$, i.e., $1/p + 1/p' = 1$. In particular $H^{-k}(\Omega) = (H_0^k(\Omega))'$, and for $f \in H^{-k}(\Omega)$

$$\|f\|_{-k,\Omega} = \sup_{v \in H_0^k(\Omega)} \frac{\langle f, v\rangle}{\|v\|_{k,\Omega}}.$$

### C.1.3 Extension theorems

The extension theorem presented below is a fundamental result for Sobolev spaces. Fourier transform is a powerful tool. But unfortunately it only works for functions defined in the entire space $\mathbb{R}^n$. To generalize results proved on the whole $\mathbb{R}^n$ to a bounded domain $\Omega$, we can try to extend the function defined in $W^{k,p}(\Omega)$ to $W^{k,p}(\mathbb{R}^n)$. The extension of a function $u \in L^p(\Omega)$ is trivial. For example, we can simply set $u(x) = 0$ when $x \notin \Omega$ which is called zero extension. But such extension will create a bad discontinuity along the boundary and thus cannot control the norm of derivatives especially the boundary is non-smooth. The extension of Sobolev space $W^{k,p}(\Omega)$ is subtle. We only present the result here.

**定理 C.1.** For any bounded Lipschitz domain $\Omega$, for any $s \geq 0$ and $1 \leq p \leq \infty$, there exits a linear operator $E : W^{s,p}(\Omega) \to W^{s,p}(\mathbb{R}^n)$ such that

1. $Eu|_\Omega = u$, and
2. $E$ is continuous. Namely there exists a constant $C(s,\Omega)$ which is increasing with respect to $s \geq 0$ such that, for all $1 \leq p \leq \infty$,

$$\|Ev\|_{s,p,\mathbb{R}^n} \leq C(s,\Omega)\|v\|_{s,p,\Omega} \quad \text{for all } v \in W^{s,p}(\Omega).$$

Theorem C.1 is well-known for integer order Sobolev spaces defined on smooth domains and the corresponding proof can be found in most text books on Sobolev spaces. But for Lipschitz domains and especially for fractional order spaces Theorem C.1 is less well-known and the the proof of the theorem for these cases is quite complicated. For integer order Sobolev spaces on Lipschitz domains, we refer to Calderón [28] or Stein [72]. For factional order Sobolev spaces on Lipschitz domains, we refer to the book by McLean [63].

### C.1.4 Embedding theorems

Embedding theorems of Sobolev spaces are what make the Sobolev spaces interesting and important. The Sobolev spaces $W^{k,p}(\Omega)$ are defined using weak derivatives. The smoothness using weak derivatives is weaker than that using classic derivatives.

**Example C.7.** In two dimensions, consider the function $u(x) = \ln|\ln|x||$ when $|x| < 1/e$ and $u(x) = 0$ when $|x| \geq 1/e$. It is easy to verify that $u \in H^1(\mathbb{R}^2)$. But $u$ is unbounded, i.e., $u \notin C(\mathbb{R}^2)$.

Sobolev embedding theorem connects ideas of smoothness using “weak” and “classic” derivatives. Roughly speaking, it says that if a function is weakly smooth enough, then it implies certain classic smoothness with less order of smoothness.

**Exercise C.4.** Prove that if $u \in W^{1,1}(0,1)$, then $u \in L^\infty(0,1)$.

We now present the general embedding theorem. For two Banach spaces $B_1$ and $B_0$, we say $B_1$ is continuously embedded into $B_0$, denoted by $B_1 \hookrightarrow B_0$, if for any $u \in B_1$, it is also in $B_0$ and the embedding map is continuous, i.e., for all $u \in B_1$

$$\|u\|_{B_0} \lesssim \|u\|_{B_1}.$$

**定理 C.2** (General Sobolev embedding). Let $1 \leq p \leq \infty$, $k \in \mathbb{Z}_+$ and $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$.

Case 1. $kp > n$

$$W^{k,p}(\Omega) \hookrightarrow C(\bar{\Omega}).$$

Case 2. $kp = n$

$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega) \quad \text{for all } q \in [1, \infty).$$

Furthermore

$$W^{n,1}(\Omega) \hookrightarrow C(\bar{\Omega}).$$

Case 3. $kp < n$

$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega) \quad \text{with } \frac{1}{q} = \frac{1}{p} - \frac{k}{n}.$$

We take the following visualization of Sobolev spaces from DeVore [40, page 93]. This will give us a simple way to keep track of various results. We shall do this by using points in the upper right quadrant of the plane. The $x$-axis will correspond to the $L^p$ spaces except that $L^p$ is identified with $x = 1/p$ not with $x = p$. The $y$-axis will correspond to the order of smoothness. For example, the point $(1/p, k)$ represents the Sobolve space $W^{k,p}(\Omega)$. The line with slope $n$ (the dimension of Euclidean spaces) passing through $(1/p, 0)$ is the demarcation line for embeddings of Sobolev spaces into $L^p(\Omega)$ (see Figure C.2). Any Sobolev space with indices corresponding to a point above that line is embedded into $L^p(\Omega)$ and space below that line cannot be embedded into $L^p(\Omega)$.

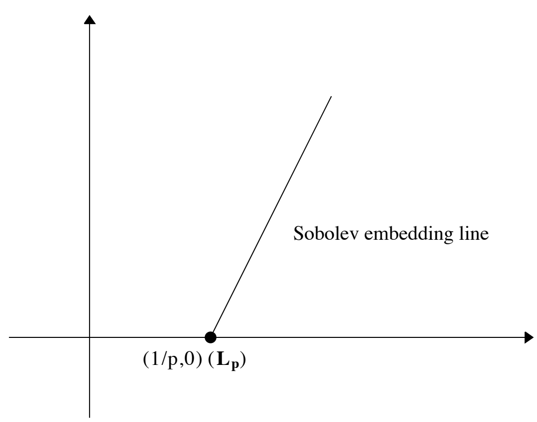



Figure C.2: Sobolev embedding line

To be quickly determine if a point lies above the demarcation line or not, we introduce the Sobolev number

$$\mathrm{sob}_n(k, p) = k - \frac{n}{p}.$$

If $\mathrm{sob}_n(k, p) > 0$, functions from $W^{k,p}(\Omega)$ are continuous (or more precisely can find a continuous representative in its equivalent class). In general

$$W^{k,p}(\Omega) \hookrightarrow W^{l,q}(\Omega) \quad \text{if } k > l \text{ and } \mathrm{sob}_n(k, p) > \mathrm{sob}_n(l, q).$$

Sobolev spaces corresponding to points on the demarcation line may or may not be embedded in $L^p(\Omega)$. For example, for $n \geq 1$

$$W^{n,1}(\Omega) \hookrightarrow L^\infty(\Omega) \quad \text{but } W^{1,n}(\Omega) \not\hookrightarrow L^\infty(\Omega).$$

Note that $W^{1,n}(\Omega) \hookrightarrow L^p(\Omega)$ for all $1 \leq p < \infty$. Furthermore there exists a constant $C(n,\Omega)$ depending on the dimension of Euclidean space and the Lipschitz domain such that

$$\|v\|_{0,p,\Omega} \leq C(n,\Omega)p^{1-1/n}\|v\|_{1,n,\Omega} \quad \text{for all } 1 \leq p < \infty.$$

**Example C.8.** By the emebedding theorem, we have

1. $H^1(\Omega) \hookrightarrow C(\bar{\Omega})$ in one dimension (Exercise C.4).

2. $H^1(\Omega) \not\hookrightarrow C(\bar{\Omega})$ for $n \geq 2$ (Example C.7). Namely there is no continuous representation of a function in $H^1(\Omega)$ for $n \geq 2$ and thus its point-wise value is not well defined.

3. In 2-D, $H^1(\Omega) \hookrightarrow L^p(\Omega)$ for any $1 \leq p < \infty$; in 3-D, $H^1(\Omega) \hookrightarrow L^p(\Omega)$ for any $1 \leq p \leq 6$.

### C.1.5 Trace Theorem

The trace theorem is to define function values on the boundary. If $u \in C(\bar{\Omega})$, then $u(x)$ is well defined for $x \in \partial\Omega$. But for $u \in W^{k,p}(\Omega)$, the function is indeed defined as an equivalent class of Lebesgue integrable functions, i.e., $u \sim v$ if and only if $u = v$ almost everywhere. The boundary $\partial\Omega$ is a measure zero set (in $n$th dimensional Lebesgue measure) and thus the pointwise value of $u|_{\partial\Omega}$ is not well defined for functions in Sobolev spaces.

**定理 C.3.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth or Lipschitz boundary $\Gamma = \partial\Omega$. Then the trace operator $\gamma : C^1(\bar{\Omega}) \to C(\Gamma)$ can be continuously extended to $\gamma : H^1(\Omega) \to H^{1/2}(\Gamma)$, i.e., the trace inequality holds:

$$\|\gamma(u)\|_{1/2,\Gamma} \lesssim \|u\|_{1,\Omega} \quad \text{for all } u \in H^1(\Omega).$$

Furthermore the trace operator is surjective and has a continuous right inverse.

More general, if $\Omega$ is a Lipschitz domain, then the trace operator $\gamma : H^s(\Omega) \to H^{s-1/2}(\Gamma)$ is bounded for $1/2 < s < 3/2$. See McLean [63, pages 100-106].

**定理 C.4.** Suppose that $\Omega \subset \mathbb{R}^n$ has a Lipschitz boundary, and that $p$ is a real number in the range $1 \leq p \leq \infty$. Then there is a constant $C$ such that

$$\|v\|_{0,p,\partial\Omega} \leq C\|v\|_{0,p,\Omega}^{1-1/p}\|v\|_{1,p,\Omega}^{1/p} \quad \forall\, v \in W^{1,p}(\Omega).$$

### C.1.6 Norm Equivalence Theorem

The definition of norm $\|\cdot\|_{k+1,p}$ involves all the $i$th derivatives for $i \leq k+1$. But the highest one is the critical one. That is we can prove [2, Theorem 5.2 in page 135]

$$\|u\|_{k+1,p} \eqsim \|u\|_{0,p} + |u|_{k+1,p} \quad \text{for all } u \in W^{k+1,p}(\Omega).$$

**定理 C.5** (Sobolev norm equivalence). Let $P_k(\Omega)(k \geq 0)$ be the set of all polynomials in $\Omega$ with the total degree no more than $k$. Let $N = \dim P_k(\Omega)$. Assume $f_i \in (W^{k+1,p}(\Omega))'$, $i = 1, 2, \cdots, N$, $p \in [1, \infty]$ such that for $q \in P_k(\Omega)$, $q = 0$ if $f_i(q) = 0$ for all $1 \leq i \leq N$. Then

$$\|v\|_{k+1,p,\Omega} \lesssim |v|_{k+1,p,\Omega} + \sum_{i=1}^{N} |f_i(v)| \quad \text{for all } v \in W^{k+1,p}(\Omega). \tag{C.2}$$

As a special case of the Sobolev norm equivalence theorem, we present the following variants of Poincaré or Friedrichs inequalities. Assume that $\Gamma$ is a measurable subset of $\Omega$ with positive measure (in $n-1$ dimensional Lebesgue measure). Choosing $f_0(v) = \int_\Gamma v \,\mathrm{d}s$ in (C.2), we obtain the Friedrichs inequality

$$\|v\|_{1,p,\Omega} \lesssim |v|_{1,p,\Omega} + \left| \int_\Gamma v \,\mathrm{d}s \right| \quad \text{for all } v \in W^{1,p}(\Omega).$$

Consequently, we have the Poincaré inequality

$$\|v\|_{1,p,\Omega} \lesssim |v|_{1,p,\Omega} \quad \text{for all } v \in W_0^{1,p}(\Omega). \tag{C.3}$$

Choosing $f_0(v) = \int_\Omega v \,\mathrm{d}x$, we obtain the Poincaré-Friedrichs inequality

$$\|v\|_{1,p,\Omega} \lesssim |v|_{1,p,\Omega} + \left| \int_\Omega v \,\mathrm{d}x \right| \quad \text{for all } v \in W^{1,p}(\Omega).$$

Consequently, let $\bar{v} = \int_\Omega v \,\mathrm{d}x / |\Omega|$ denote the average of $v$ over $\Omega$, we get the averaged Poincaré inequality

$$\|v - \bar{v}\|_{1,p,\Omega} \lesssim |v|_{1,p,\Omega} \quad \text{for all } v \in W^{1,p}(\Omega).$$

**定理 C.6.** Let $k \geq 0$ and $p \in [1, \infty]$. Then

$$\inf_{q \in P_k(\Omega)} \|v + q\|_{k+1,p,\Omega} \lesssim |v|_{k+1,p,\Omega} \quad \text{for all } v \in W^{k+1,p}(\Omega),$$

$$\|\dot{v}\|_{k+1,p,\Omega} \lesssim |\dot{v}|_{k+1,p,\Omega} \quad \text{for all } v \in W^{k+1,p}(\Omega)/P_k(\Omega).$$

All constants hidden in the notation $\lesssim$ depends on the size and shape of the domain $\Omega$. This is important when apply them to one simplex with diameter $h$ in finite element methods.

### C.1.7 Green's formulas

Let $\nu = (\nu_1, \nu_2, \cdots, \nu_n)$ be the unit outward normal to $\partial\Omega$. The (outer) normal derivative operator

$$\partial_\nu = \sum_{i=1}^{n} \nu_i \partial_i$$

is defined almost everywhere along $\partial\Omega$ for smooth functions. Given two functions $u, v \in H^1(\Omega)$, the following fundamental Green's formula

$$\int_\Omega u \partial_i v \,\mathrm{d}x = -\int_\Omega \partial_i u v \,\mathrm{d}x + \int_{\partial\Omega} u v \nu_i \,\mathrm{d}s$$

holds for any $i = 1, \cdots, n$. From this formula, other Green's formulas may be easily deduced. For example, replacing $u$ by $\partial_i u$ and taking the sum from 1 to $n$, we get

$$\int_\Omega \sum_{i=1}^{n} \partial_i u \partial_i v \,\mathrm{d}x = -\int_\Omega \Delta u v \,\mathrm{d}x + \int_{\partial\Omega} \partial_\nu u v \,\mathrm{d}s$$

for all $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$. As a consequence, we obtain by subtraction

$$\int_\Omega (u\Delta v - \Delta u v)\,\mathrm{d}x = \int_{\partial\Omega} (u\partial_\nu v - \partial_\nu u v)\,\mathrm{d}s$$

for all $u \in H^2(\Omega)$ and $v \in H^2(\Omega)$.

**定理 C.7.** Let $m$ be a positive integer. Let $\Omega, \Omega_1$ and $\Omega_2$ be three open and bounded domains in $\mathbb{R}^n$. Suppose $\bar\Omega = \bar\Omega_1 \cup \bar\Omega_2$ and $\Omega_1 \cap \Omega_2 = \varnothing$. Let $u$ be a function defined in $\Omega$ such that $u|_{\Omega_1} \in C^{m-1}(\bar\Omega_1) \cap H^m(\Omega_1)$ and $u|_{\Omega_2} \in C^{m-1}(\bar\Omega_2) \cap H^m(\Omega_2)$. Then $u \in H^m(\Omega)$ if and only if $u \in C^{m-1}(\bar\Omega)$.

For a piecewise smooth function which is also in $H^1(\Omega)$, then it is globally continuous.

**Exercise C.5.** Let $\Omega$, $\Omega_1$ and $\Omega_2$ are three open and bounded domains in $\mathbb{R}^n$. Suppose $\bar\Omega = \bar\Omega_1 \cup \bar\Omega_2$ and $\Omega_1 \cap \Omega_2 = \varnothing$. Let $u$ be a function defined in $\Omega$ such that $u|_{\Omega_1} \in C(\bar\Omega_1) \cap H^1(\Omega_1)$ and $u|_{\Omega_2} \in C(\bar\Omega_2) \cap H^1(\Omega_2)$. Prove that $u \in H^1(\Omega)$ if and only if $u \in C(\bar\Omega)$.

## C.2 Elliptic Boundary Value Problems and Regularity

### C.2.1 Weak formulation

Let us take the Poisson equation with homogenous Dirichlet boundary condition

$$-\Delta u = f \quad \text{in } \Omega, \qquad \text{and } u|_{\partial\Omega} = 0, \tag{C.4}$$

as an example to illustrate the main idea. If there exists a function $u \in C_0^2(\Omega)$ satisfying the Poisson equation, we call $u$ a classic solution. The smoothness of $u$ excludes many interesting solutions for physical problems. We need to seek a weak solution in more broader spaces: Sobolev spaces. Here "weak" means the smoothness is imposed by weak derivatives.

Recall the basic idea of Sobolev space is to treat function as functional. Let us try to understand the equation (C.4) in the distribution sense. We seek a solution $u \in \mathcal{D}'(\Omega)$ such that for any $\phi \in C_0^\infty(\Omega)$,

$$\langle -\nabla\cdot\nabla u, \phi\rangle := \langle \nabla u, \nabla\phi\rangle = \langle f, \phi\rangle,$$

which suggests a weak formulation: find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v\,\mathrm{d}x = \int_\Omega f v\,\mathrm{d}x \quad \text{for all } v \in H_0^1(\Omega). \tag{C.5}$$

And we can take $f \in H^{-1}(\Omega) := (H_0^1(\Omega))'$.

A generalization of weak formulation (C.5) is: given an $f \in V'$ and a continuous bilinear form $a(\cdot,\cdot) : U \times V \to \mathbb{R}$, find a solution $u \in U$ such that

$$a(u, v) = \langle f, v\rangle \quad \text{for all } v \in V.$$

Here the space $U$ is the one we seek a solution and thus called trial space, and $V$ is still called test space.

### C.2.2 Regularity theory

The weak solution $u$ of (C.5) can be proved to be a solution of (C.4) in a more classic sense if $u$ is smooth enough such that we can integration by parts back. The theory for proving the smoothness of the weak solution is called regularity theory, which is the bridge to connect classical and weak solutions.

First we assume right hand side $f \in L^2(\Omega)$ which will imply $\Delta u \in L^2(\Omega)$.

**定理 C.8.** Let $u$ be the solution of (C.5). If $f \in L^2(\Omega)$, then $-\Delta u = f$ in $L^2(\Omega)$, i.e.

$$\int_\Omega -\Delta u \, v \, \mathrm{d}x = \int_\Omega f v \, \mathrm{d}x \quad \forall\, v \in L^2(\Omega). \tag{C.6}$$

Furthermore $u = 0$ on $\partial\Omega$ in the trace sense.

*Proof.* When $f \in L^2(\Omega)$, $\langle f, v\rangle = (f, v)$. Choosing $v \in C_0^\infty(\Omega) \subset H_0^1(\Omega)$ in (C.5) implies that

$$\langle -\Delta u, v\rangle = (f, v) \quad \forall\, v \in C_0^\infty(\Omega).$$

That is $-\Delta u$ as a distribution is equal to $f \in L^2(\Omega)$. Since $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$ and both sides are continuous in $L^2$ topology, we conclude (C.6). □

**定理 C.9** ([42]). Let $\Omega$ be a smooth and bounded domain of $\mathbb{R}^n$. Then for each $f \in L^2(\Omega)$, there exists a unique $u \in H^2(\Omega)$, the solution of (C.5), that satisfies

$$\|u\|_{2,\Omega} \le C\|f\|_{0,\Omega}$$

where $C$ is a positive constant depending on $\Omega$.

Theorem C.9, however, does not hold on general Lipschitz domains. The requirement of $\Omega$ is necessary. When $\partial\Omega$ is only Lipschitz continuous, we may not be able to glue boundary pieces on the corner points. This is not only a technical difficulty. We now give an example in which the regularity of $u$ depends on the shape of $\Omega$.

**Example C.9.** Let us give a simple counter example. Given $\beta \in (0, 1)$, consider the following nonconvex domain

$$\Omega = \{(r, \theta) : 0 < r < 1, 0 < \theta < \pi/\beta\}.$$

Let $v = r^\beta \sin(\beta\theta)$. Being the imaginary part of the complex analytic function $z^\beta$, $v$ is harmonic in $\Omega$. Define $u = (1 - r^2)v$. A direct calculation shows that

$$-\Delta u = 4(1+\beta)v \quad \text{in } \Omega, \quad \text{and } u|_{\partial\Omega} = 0.$$

Note that $4(1+\beta)v \in L^\infty(\Omega) \subset L^2(\Omega)$, but $u \notin H^2(\Omega)$.

Nevertheless, a slightly weaker result does hold for general Lipschitz domains.

**定理 C.10** ([39, 11]). Assume that $\Omega$ is a bounded Lipschitz domain. Then there exists a constant $\alpha \in (0, 1]$ such that

$$\|u\|_{1+\alpha,\Omega} \le C\|f\|_{\alpha-1,\Omega}$$

for the solution $u$ of (C.5), where $C$ is a constant depending on the domain $\Omega$.

A remarkable fact is that we can take $\alpha = 1$ in the above theorem for convex domains. This means that Theorem C.9 can be extended to convex domains.

**定理 C.11.** Let $\Omega$ be a convex, bounded domain of $\mathbb{R}^n$. Then for each $f \in L^2(\Omega)$, there exists a unique $u \in H^2(\Omega)$, the solution of (C.5), that satisfies

$$\|u\|_{2,\Omega} \le C\|f\|_{0,\Omega}$$

where $C$ is a positive constant depending on $\Omega$.

When $\Omega$ is a concave polygon in $\mathbb{R}^2$, we do not have full regularity, i.e., $\alpha < 1$. There are at least two ways to obtain analogous (but weaker) results for concave domain. The first one (see [39, 11]) is to use fractional Sobolev spaces: namely $-\Delta$: $H_0^1(\Omega) \cap H^{1+s}(\Omega) \to H^{-1+s}(\Omega)$ which holds for any $s \in [0, s_0)$ for some $s_0 \in (0, 1]$ depending on $\Omega$.

Another approach is to use $L^p$ space, instead of $L^2$. It can be shown that (cf. [49])

$$-\Delta : H_0^1(\Omega) \cap W^{2,p}(\Omega) \to L^p(\Omega)$$

is an isomorphism, which holds for any $p \in (1, p_0)$ for some $p_0 > 1$ that depends on the domain $\Omega$.

# 第四章　微分复形在偏微分方程中的应用

## D.1　Stokes Complex 及应用

Complex 1.

$$\mathbb{R} \longrightarrow H_0^1(\Omega) \xrightarrow{\text{grad}} \boldsymbol{H}_0(\text{curl},\Omega) \xrightarrow{\text{curl}} \boldsymbol{H}_0(\text{div},\Omega) \xrightarrow{\text{div}} L_0^2(\Omega) \longrightarrow 0, \tag{D.1}$$

Example 1：The Possion equation

$$\begin{cases} -\Delta u = f, & \text{in}\,\Omega \\ u = 0, & \text{on}\,\partial\Omega. \end{cases} \tag{D.2}$$

引入新变量

$$\begin{cases} \boldsymbol{\sigma} = \nabla u \\ \operatorname{div}\boldsymbol{\sigma} = -f. \end{cases} \tag{D.3}$$

则方程 (D.2) 变为

$$\begin{cases} \operatorname{div}\boldsymbol{\sigma} + f = 0, & \text{in}\,\Omega \\ \boldsymbol{\sigma} - \nabla u = 0, & \text{in}\,\Omega \\ u = 0, & \text{on}\,\partial\Omega. \end{cases} \tag{D.4}$$

在结合复形 (D.1) 的后半部分, 则方程 (D.4) 的变分形式为: $\boldsymbol{\tau} \in \boldsymbol{H}_0(\text{div},\Omega)$, $u \in L_0^2(\Omega)$, 使得

$$\begin{cases} (\boldsymbol{\sigma},\boldsymbol{\tau}) + (\operatorname{div}\boldsymbol{\tau}, u) = 0 & \forall\ \boldsymbol{\tau} \in \boldsymbol{H}_0(\text{div},\Omega), \\ (\operatorname{div}\boldsymbol{\sigma}, v) = 0 & \forall\ v \in L_0^2(\Omega), \end{cases} \tag{D.5}$$

下面利用复形 (D.1) 来证明 inf-sup 条件和强制性.

(1) inf-sup 条件: 对 $v \in L_0^2(\Omega)$, 根据复形 (D.1) 可知 $\exists\boldsymbol{\tau} \in \boldsymbol{H}_0(\text{div},\Omega)$, 使得 $\operatorname{div}\boldsymbol{\tau} = v$, 则有

$$\|v\|_0 = \frac{(\operatorname{div}\boldsymbol{\tau}, v)}{\|\operatorname{div}\boldsymbol{\tau}\|_0} \lesssim \sup_{\boldsymbol{\tau}\in\boldsymbol{H}_0(\text{div},\Omega)} \frac{(\operatorname{div}\boldsymbol{\tau}, u)}{\|\boldsymbol{\tau}\|_{\boldsymbol{H}(\text{div})}}$$

(2) 强制性

$$\begin{aligned} \ker\boldsymbol{B} &= \{\boldsymbol{\tau} \in \boldsymbol{H}_0(\text{div},\Omega),\ (\operatorname{div}\boldsymbol{\tau}, v) = 0, \quad \forall v \in L_0^2(\Omega)\} \\ &= \{\boldsymbol{\tau} \in \boldsymbol{H}(\text{div},\Omega),\ \operatorname{div}\boldsymbol{\tau} = 0\} \end{aligned}$$

所以

$$\|\boldsymbol{\tau}\|_{\boldsymbol{H}(\text{div})}^2 = \|\boldsymbol{\tau}\|_0^2 + \|\operatorname{div}\boldsymbol{\tau}\|_0^2 \lesssim \|\boldsymbol{\tau}\|_0^2 = (\boldsymbol{\tau},\boldsymbol{\tau}), \quad \forall\boldsymbol{\tau} \in \ker\boldsymbol{B}.$$

Example 2: The Maxwell equation

$$\begin{cases} \operatorname{curl}\operatorname{curl}\boldsymbol{u} = \boldsymbol{f}, & \text{in } \Omega \\ \operatorname{div} u = 0, & \text{in } \Omega \\ \boldsymbol{u}\times\boldsymbol{n} = 0, & \text{on } \partial\Omega. \end{cases} \tag{D.6}$$

则方程 (D.6) 的变分形式为: $\boldsymbol{u} \in \boldsymbol{H}_0(\operatorname{curl},\Omega)$, $p \in H_0^1(\Omega)$, 使得

$$\begin{cases} (\operatorname{curl}\boldsymbol{u}, \operatorname{curl}\boldsymbol{v}) + (\boldsymbol{v}, \nabla\boldsymbol{p}) = (\boldsymbol{f},\boldsymbol{v}) & \forall\ \boldsymbol{v} \in \boldsymbol{H}_0(\operatorname{curl},\Omega), \\ (\boldsymbol{u}, \nabla\boldsymbol{q}) = 0 & \forall\ q \in H_0^1(\Omega), \end{cases} \tag{D.7}$$

下面利用复形 (D.1) 来证明 inf-sup 条件和强制性.
(1) inf-sup 条件: 对 $q \in H_0^1(\Omega)$, 根据复形 (D.1) 可知 $\exists\, \boldsymbol{v} = \nabla q \in \boldsymbol{H}_0(\operatorname{curl},\Omega)$, 又由 poincáre 不等式可得

$$\|q\|_{H_0^1(\Omega)} \lesssim |q|_{H_0^1(\Omega)} = \frac{(\nabla q, \nabla q)}{\|\nabla q\|_0} = \frac{(\boldsymbol{v}, \nabla q)}{\|\boldsymbol{v}\|_{\boldsymbol{H}(\operatorname{curl})}} \lesssim \sup_{\boldsymbol{v}\in \boldsymbol{H}_0(\operatorname{curl},\Omega)} \frac{(\boldsymbol{v}, \nabla q)}{\|\boldsymbol{v}\|_{\boldsymbol{H}(\operatorname{curl})}}$$

(2) 强制性

$$\begin{aligned} \ker \boldsymbol{B} &= \{\boldsymbol{v} \in \boldsymbol{H}_0(\operatorname{curl},\Omega),\ (\boldsymbol{v}, \nabla q) = 0, \quad \forall q \in H_0^1(\Omega)\} \\ &= \{\boldsymbol{v} \in \boldsymbol{H}(\operatorname{curl},\Omega),\ \operatorname{div}\boldsymbol{v} = 0\} \end{aligned}$$

又由 $\|\boldsymbol{v}\|_0 \lesssim \|\operatorname{curl}\boldsymbol{v}\|_0$, 所以

$$\|\boldsymbol{v}\|^2_{\boldsymbol{H}(\operatorname{curl})} = \|\boldsymbol{v}\|_0^2 + \|\operatorname{curl}\boldsymbol{v}\|_0^2 \lesssim \|\operatorname{curl}\boldsymbol{v}\|_0^2, \quad \forall \boldsymbol{v} \in \ker \boldsymbol{B}.$$

Complex 2.

$$\mathbb{R} \longrightarrow H^3(\Omega) \xrightarrow{\text{grad}} \boldsymbol{H}^2(\Omega;\mathbb{R}^3) \xrightarrow{\text{curl}} \boldsymbol{H}^1(\Omega;\mathbb{R}^3) \xrightarrow{\text{div}} L^2(\Omega) \longrightarrow 0. \tag{D.8}$$

Example 3: The Stokes equation

$$\begin{cases} -\Delta\boldsymbol{u} - \nabla p = \boldsymbol{f}, & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u} = 0, & \text{in } \Omega \\ \boldsymbol{u} = 0, & \text{on } \partial\Omega \end{cases} \tag{D.9}$$

则方程 (D.9) 的变分形式为: $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega,\mathbb{R}^3)$, $p \in L_0^2(\Omega)$, 使得

$$\begin{cases} (\nabla\boldsymbol{u}, \nabla\boldsymbol{v}) + (\operatorname{div}\boldsymbol{v}, \boldsymbol{p}) = (\boldsymbol{f},\boldsymbol{v}) & \forall\ \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega,\mathbb{R}^3), \\ (\operatorname{div}\boldsymbol{u}, q) = 0 & \forall\ q \in L_0^2(\Omega), \end{cases} \tag{D.10}$$

下面利用复形 (D.8) 来证明 inf-sup 条件和强制性.
(1) inf-sup 条件: 对 $q \in L_0^2(\Omega)$, 根据复形 (D.8) 可知 $\exists\, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega,\mathbb{R}^3)$, $\operatorname{div}\boldsymbol{v} = q$, 且由 poincáre 不等式可得 $\|\boldsymbol{v}\|_1 \lesssim \|\operatorname{div}\boldsymbol{v}\|_0 = \|q\|_0$, 则

$$(\operatorname{div}\boldsymbol{v}, q) = \|q\|_0^2 \gtrsim \|q\|_0\|\boldsymbol{v}\|_1,$$

则有

$$\|q\|_0 \lesssim \frac{(\operatorname{div}\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_1} \lesssim \sup_{\boldsymbol{v}\in \boldsymbol{H}_0(\Omega,\mathbb{R}^3)} \frac{(\operatorname{div}\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_1}$$

(2) 强制性又由 poincáre 不等式可得

$$\|\boldsymbol{v}\|_1^2 \lesssim (\nabla \boldsymbol{v}, \nabla \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega, \mathbb{R}^3).$$

Complex 3.

$$\mathbb{R} \longrightarrow H_0^1(\Omega) \xrightarrow{\text{grad}} \boldsymbol{H}_0(\text{grad}\,\text{curl}, \Omega) \xrightarrow{\text{curl}} \boldsymbol{H}_0^1(\Omega; \mathbb{R}^3) \xrightarrow{\text{div}} L_0^2(\Omega) \longrightarrow 0. \tag{D.11}$$

Example 4： The quad-curl problems

$$\begin{cases} -\operatorname{curl} \triangle \operatorname{curl} \boldsymbol{u} + \nabla p = \boldsymbol{f}, & \text{in}\, \Omega, \\ \operatorname{div} \boldsymbol{u} = 0, & \text{in}\, \Omega \\ \boldsymbol{u} \times \boldsymbol{n} = \operatorname{curl} \boldsymbol{u} = 0, & \text{on}\, \partial\Omega \end{cases} \tag{D.12}$$

则方程 (D.12) 的变分形式为: $\boldsymbol{u} \in \boldsymbol{H}_0(\text{grad}\,\text{curl}, \Omega)$, $p \in H_0^1(\Omega)$, 使得

$$\begin{cases} (\text{grad}\,\text{curl}\, \boldsymbol{u}, \text{grad}\,\text{curl}\, \boldsymbol{v}) + (\boldsymbol{v}, \nabla p) = (\boldsymbol{f}, \boldsymbol{v}) & \forall\, \boldsymbol{v} \in \boldsymbol{H}_0(\text{grad}\,\text{curl}, \Omega), \\ (\boldsymbol{u}, \nabla q) = 0 & \forall\, q \in H_0^1(\Omega), \end{cases} \tag{D.13}$$

下面利用复形 (D.11) 来证明 inf-sup 条件和强制性.
(1) inf-sup 条件: 对 $q \in H_0^1(\Omega)$, 根据复形 (D.11) 可知 $\exists\, \boldsymbol{v} = \nabla q \in \boldsymbol{H}_0(\text{grad}\,\text{curl}, \Omega)$, 又由 poincáre 不等式可得

$$\|q\|_{H_0^1(\Omega)} \lesssim |q|_{H_0^1(\Omega)} = \frac{(\nabla q, \nabla q)}{\|\nabla q\|_0} = \frac{(\boldsymbol{v}, \nabla q)}{\|\boldsymbol{v}\|_{\boldsymbol{H}(\text{grad}\,\text{curl})}} \lesssim \sup_{\boldsymbol{v} \in \boldsymbol{H}_0(\text{grad}\,\text{curl}, \Omega)} \frac{(\boldsymbol{v}, \nabla q)}{\|\boldsymbol{v}\|_{\boldsymbol{H}(\text{grad}\,\text{curl})}}$$

(2) 强制性

$$\begin{aligned} \ker \boldsymbol{B} &= \{\boldsymbol{v} \in \boldsymbol{H}_0(\text{grad}\,\text{curl}, \Omega),\ (\boldsymbol{v}, \nabla q) = 0, \quad \forall q \in H_0^1(\Omega)\} \\ &= \{\boldsymbol{v} \in \boldsymbol{H}(\text{grad}\,\text{curl}, \Omega),\ \operatorname{div} \boldsymbol{v} = 0\} \end{aligned}$$

又由 poincáre 不等式可得 $\|\boldsymbol{v}\|_0^2 + \|\operatorname{curl} \boldsymbol{v}\|_0^2 \lesssim \|\operatorname{curl} \boldsymbol{v}\|_0^2 \lesssim \|\text{grad}\,\text{curl}\, \boldsymbol{v}\|_0$, 所以

$$\|\boldsymbol{v}\|_{\boldsymbol{H}(\text{grad}\,\text{curl})}^2 = \|\boldsymbol{v}\|_0^2 + \|\operatorname{curl} \boldsymbol{v}\|_0^2 + + \|\text{grad}\,\text{curl}\, \boldsymbol{v}\|_0^2 \lesssim \|\text{grad}\,\text{curl}\, \boldsymbol{v}\|_0^2, \quad \forall \boldsymbol{v} \in \ker \boldsymbol{B}.$$

Complex 4.

$$\mathbb{R} \longrightarrow H^1(\Omega) \xrightarrow{\text{grad}} \boldsymbol{H}(\text{Hess}\,\text{curl}, \Omega) \xrightarrow{\text{curl}} \boldsymbol{H}^2(\Omega; \mathbb{R}^3) \xrightarrow{\text{div}} H^1(\Omega) \longrightarrow 0. \tag{D.14}$$

Example 5 :The strain gradient problems
这里引入应变梯度 $\varepsilon(\boldsymbol{u}) = \text{sym}\,\text{grad}\, \boldsymbol{u}$ 则应变梯度问题的变分形式为: $\boldsymbol{u} \in \boldsymbol{H}_0^2(\Omega; \mathbb{R}^3)$, $p \in H_0^1(\Omega)$, 使得

$$\begin{cases} (\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v})) + (\operatorname{div} \boldsymbol{v}, p) = (\boldsymbol{f}, \boldsymbol{v}) & \forall\, \boldsymbol{v} \in \boldsymbol{H}_0^2(\Omega; \mathbb{R}^3), \\ (\operatorname{div} \boldsymbol{u}, \boldsymbol{q}) = 0 & \forall\, q \in H_0^1(\Omega), \end{cases} \tag{D.15}$$

进一步改写成带参数 $l$ 的形式

$$\begin{cases} (\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v}))_l + (\operatorname{div} \boldsymbol{v}, p)_l = (\boldsymbol{f}, \boldsymbol{v}) & \forall\, \boldsymbol{v} \in \boldsymbol{H}_0^2(\Omega; \mathbb{R}^3), \\ (\operatorname{div} \boldsymbol{u}, \boldsymbol{q})_l = 0 & \forall\, q \in H_0^1(\Omega), \end{cases} \tag{D.16}$$

这里

$$(p,q)_l = l^2(\nabla p, \nabla q) + (p,q).$$

还是下面利用复形 (D.14) 来证明 inf-sup 条件和强制性, 证明方法和上面类似.

Complex 5.

$$\mathbb{R} \longrightarrow H^2(\Omega) \xrightarrow{\text{grad}} \boldsymbol{H}^1(\text{curl},\Omega) \xrightarrow{\text{curl}} \boldsymbol{H}^1(\Omega;\mathbb{R}^3) \xrightarrow{\text{div}} L^2(\Omega) \longrightarrow 0. \tag{D.17}$$

Example 5 :The biharmonic equation

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \partial_v u = 0 & \text{on } \partial\Omega \end{cases} \tag{D.18}$$

原始变分形式为找 $u \in H_0^2(\Omega)$ 使得

$$(\nabla^2 u, \nabla^2 v) = \langle f, v \rangle \quad \forall\ v \in H_0^2(\Omega) \tag{D.19}$$

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