Referee report on the manuscript SINUM-M150419 entitled

"Stabilization-Free Virtual Element Methods"

Authors: Chunyu Chen, Xuehai Huang and Huayi Wei

This article designs a stabilization-free nonconforming VEM, and a stabilization-free conforming VEM for a reaction diffusion problem. The authors make significant effort in designing a discrete bilinear form which induces a norm equivalent to the H^1 norm on the corresponding virtual spaces. Numerical tests are conducted in two dimensions. The work appears to be mostly new and of some interest to the mathematical community. However, I do not think the material is sufficiently novel, nor the writing of high enough quality to be published in SINUM. In particular, I do not believe the paper satisfies the SINUM editorial policy which states that "Articles for SINUM must be written in clear and idiomatic English and must be accessible and of interest to a range of numerical analysts. They must contain substantially new results and relate them to the extant literature in a scholarly fashion." Moreover, I worry about the validity of some of the results in the article. I therefore do not recommend publication.

I suggest that the authors submit to another journal after significant revision. Here are some issues I noticed during my review of the article:

- A significantly more thorough review of the literature is required. Indeed, very little mention of other stabilization free methods are given. It is difficult to gauge how this work compares to the existing literature and where the true novelty of the work lies. For example, in the reference [2] a stabilisation free method is designed by considering a HHO space of unknowns and a reconstruction in a VEM space. See [2, Remark 5.1]. How does the current article compare to this work? Moreover, the authors do not make it clear what the benefit of a "stabilization-free" VEM is.

- At equation (3.4) (and continuing below) the space $\mathbb{P}_{k-2}(T;\mathbb{K})\boldsymbol{x}$ is considered. Is this standard notation? At first I thought $\boldsymbol{x}$ was a predefined point (e.g. the centre of T) and the space $\mathbb{P}_{k-2}(T;\mathbb{K})\boldsymbol{x}$ was the set of functions that could be written as a polynomial valued anti-symmetric tensor times the predefined point $\boldsymbol{x}$. However, I think what is actually meant is $\mathbb{P}_{k-2}(T;\mathbb{K})\boldsymbol{x} = \{f : T \to \mathbb{R}^d : f(\boldsymbol{x}) = \boldsymbol{P}_{k-2}(\boldsymbol{x})\boldsymbol{x}, \quad \forall \boldsymbol{x} \in T, \boldsymbol{P}_{k-2} \in \mathbb{P}_{k-2}(T;\mathbb{K})\}$. The authors should clarify here.

- Line 149: "$(I + \boldsymbol{x}\cdot\nabla)\boldsymbol{w} = \boldsymbol{0}$, which implies $\boldsymbol{w} = \boldsymbol{0}$". I do not see how this follows. We have that

$$(\nabla \boldsymbol{w}(\boldsymbol{x}))\boldsymbol{x} = -\boldsymbol{w}(\boldsymbol{x}) \qquad \forall \boldsymbol{x} \in T.$$

It is not clear to me how the conclusion that $\boldsymbol{w} = \boldsymbol{0}$ follows. The authors should precise their reasoning.

- In the proof of Lemma 12, the authors apply a discrete trace inequality to the quantity
$$\sum_{F\in\mathcal{F}^{\partial}(\mathcal{T}_K)} h_F^{1/2}\|\boldsymbol{\phi}\cdot\boldsymbol{n}\|_{0,F}$$
and cite [11, (2.18)]. However, [11, (2.18)] is a continuous trace inequality and there is no justification given for why the discrete trace inequality holds in this particular case. Moreover, it is not explained how one concludes that $\|\operatorname{div}\boldsymbol{\phi}\|_{-1,K}\lesssim\|\boldsymbol{\phi}\|_{0,K}$.

- In the proof of Lemma 4.1, the authors claim that
$$h_K^{\frac{1}{2}}\|\partial_n v\|_{0,\partial K}\lesssim |v|_{1,K}.$$
This is a sort of discrete trace inequality and the authors justify it by citing (A.3)-(A.4) in [15]. However, it is not apparent to me how this follows from (A.3)-(A.4). Moreover, the authors conclude that "with the multiplicative trace inequality"
$$h_K^{-\frac{1}{2}}\|v\|_{0,\partial K}\lesssim h_K^{-1}\|v\|_{0,K}.$$
Again, this is a *discrete* trace inequality, and it is not specified why it is valid on the virtual space $V_k(K)$. The same goes for the arguments used in the proof of Lemma 4.3.

- When introducing the nonconforming virtual element method in Section 4, the authors should give some references for NCVEM. In particular, I don't think the space $V_k(K)$ is the classical NCVEM space (e.g. that defined in [1]). The authors should give reference to where this space is first introduced.

- It seems that the main novelty of this work is the construction of a space
$$\begin{aligned}\mathbb{V}_{k-1}^{\mathrm{div}}(K) := \{\boldsymbol{\phi}\in L^2(K) : \operatorname{div}\boldsymbol{\phi}\in\mathbb{P}_{k-2}(K),\\ \boldsymbol{\phi}\cdot\boldsymbol{n}_F\in\mathbb{P}^{k-1}(F)\ \forall F\in\mathcal{F}(K), \boldsymbol{\phi}|_T\in\mathbb{P}_{k-1}(T)^d\ \forall T\in\mathcal{T}_K\}\end{aligned}$$
and a computable L^2 projector $Q_{K,k-1}^{\mathrm{div}}$ onto $\mathbb{V}_{k-1}^{\mathrm{div}}(K)$ such that
$$\|Q_{K,k-1}^{\mathrm{div}}\nabla v\|_{0,K}\simeq\|\nabla v\|_{0,K}\qquad\forall v\in V_k(K),$$
where $V_k(K)$ is the non-conforming space defined at line 378 (or in Section 5 the authors consider the conforming space defined at line 544). This result is stated in Lemma 4.4 and its proof relies on the norm equivalence (3.32) and the set of uni-solvent DOFs (3.29)-(3.31). However, upon reading Section 3.2 for the first time, it is difficult to grasp what it is the authors are actually trying to achieve. Are all of the spaces, definitions and lemmas in Section 3.2 really necessary to prove Lemma 4.4? If so, the authors should state the main result of Lemma 4.4 in an earlier section, and leave much of the details of Section 3.2 and the proof of Lemma 4.4 to a later section so that readers can understand what the authors are trying to achieve.

- In Section 4, the authors design a stabilization free NCVEM for a reaction-diffusion problem, and in section 5 a stabilization free VEM. This seems an odd choice as one does not require a Poincaré inequality for the coercivity of the continuous blinear form. Indeed, if

$$\mathrm{a}(u,v) = (\nabla u, \nabla v)_\Omega + \alpha(u,v)_\Omega$$

with $\alpha > 0$ then it holds trivially that

$$\|u\|_{1,\Omega}^2 \leq \max(1, \alpha^{-1})\mathrm{a}(u,u).$$

While things mightn't be so simple at the discrete level, it would seem more appropriate to consider a Poisson problem.

- In the numerical section, the authors should compare their results to standard VEM and NCVEM to highlight any benefit their approach has.
- I suggest the authors conduct a thorough proof read of the manuscript as there are many typos, grammatical errors and poorly constructed sentences.

References

[1] B. Ayuso de Dios, K. Lipnikov, and G. Manzini. "The nonconforming virtual element method". In: *ESAIM: Mathematical Modelling and Numerical Analysis* 50.3 (2016), pp. 879–904.

[2] M. Cicuttin, A. Ern, and S. Lemaire. "A hybrid high-order method for highly oscillatory elliptic problems". In: *Computational Methods in Applied Mathematics* 19.4 (2019), pp. 723–748.