STABILIZATION-FREE VIRTUAL ELEMENT METHODS*

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Abstract. Stabilization-free virtual element methods in arbitrary degree of polynomial are developed for second order elliptic problems, including a nonconforming virtual element method in arbitrary dimension and a conforming virtual element method in two dimensions. The key is to construct local $H(\mathrm{div})$-conforming macro finite element spaces such that the associated L^2 projection of the gradient of virtual element functions is computable, and the L^2 projector has a uniform lower bound on the gradient of virtual element function spaces in L^2 norm. Optimal error estimates are derived for these stabilization-free virtual element methods. Numerical results are provided to verify the convergence rates.

Key words. virtual element, stabilization-free, macro finite element, norm equivalence, error analysis

MSC codes. 65N12, 65N22, 65N30

1. Introduction.

Recently a stabilization-free linear virtual element method, based on a higher order polynomial projection of the gradient of virtual element functions, is devised for Poisson equation in two dimensions in [9, 10], where the degree of polynomial used in projection depends on the number of vertices of the polygon. We refer to [20] for a discussion on similar stabilization-free virtual element methods for plane elasticity problem. The idea in [9, 10] is not easy to extend to construct higher order stabilization-free virtual element methods, and the analysis is rather elaborate. This motivates us to construct stabilization-free virtual element methods in arbitrary degree of polynomial and arbitrary dimension in a unified way.

The key to construct stabilization-free virtual element methods is to find a finite-dimensional space $\mathbb{V}(K)$ and a projector Q_K onto space $\mathbb{V}(K)$ such that

(C1) It holds the norm equivalence

$$\|Q_K \nabla v\|_{0,K} \eqsim \|\nabla v\|_{0,K} \quad \forall\, v \in V_k(K) \tag{1.1}$$

on shape function space $V_k(K)$ of virtual elements;

(C2) The projection $Q_K \nabla v$ is computable based on the degrees of freedom (DoFs) of virtual elements for $v \in V_k(K)$.

We can choose Q_K as the L^2-orthogonal projector with respect to the inner product $(\cdot,\cdot)_K$. The norm equivalence (1.1) implies space $\mathbb{V}(K)$ should be sufficiently large compared with the virtual element space $V_k(K)$. In standard virtual element methods, $Q_{k-1}^K \nabla v$ [8] or $\nabla \Pi_k^K v$ [6, 7, 1, 4] are used, where Q_{k-1}^K is the L^2-orthogonal projector onto the $(k-1)$-th order polynomial space $\mathbb{P}_{k-1}(K;\mathbb{R}^d)$, and Π_k^K is the H^1 projection operator onto the k-th order polynomial space $\mathbb{P}_k(K)$. While only

$$\|Q_{k-1}^K \nabla v\|_{0,K} \lesssim \|\nabla v\|_{0,K}, \quad \|\nabla \Pi_k^K v\|_{0,K} \lesssim \|\nabla v\|_{0,K}$$

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hold rather than the norm equivalence (1.1), then the additional stabilization term is usually necessary to ensure the coercivity of the discrete bilinear form. To remove the additional stabilization term, $\mathbb{V}(K)$ is taken as $\mathbb{P}_l(K;\mathbb{R}^d)$ with large enough l even for the lowest order case $k=1$ in [9, 10, 20], and the virtual element space $V_k(K)$ has to be modified accordingly to make $Q_l^K\nabla v$ computable. Instead, we employ k-th order or $(k-1)$-th order $H(\mathrm{div})$-conforming macro finite elements as $\mathbb{V}(K)$ in this paper, and keep the virtual element space $V_k(K)$ as usual ones.

We first construct $H(\mathrm{div})$-conforming macro finite elements based on a simplicial partition $\mathcal{T}_K$ of polytope K in arbitrary dimension. The shape function space $\mathbb{V}_k^{\mathrm{div}}(K)$ is a subspace of the k-th order Brezzi-Douglas-Marini (BDM) element space on the simplicial partition $\mathcal{T}_K$ for $k\geq 1$ and the lowest order Raviart-Thomas (RT) element space for $k=0$, with some constraints. To ensure the L^2 projection $Q_{K,k}^{\mathrm{div}}\nabla v$ onto space $\mathbb{V}_k^{\mathrm{div}}(K)$ is computable for virtual element function $v\in V_k(K)$, we require $\operatorname{div}\boldsymbol{\phi}\in\mathbb{P}_{\max\{k-1,0\}}(K)$ and $\boldsymbol{\phi}\cdot\boldsymbol{n}$ on each $(d-1)$-dimensional face of K is a polynomial for $\boldsymbol{\phi}\in\mathbb{V}_k^{\mathrm{div}}(K)$. Based on these considerations and the direct decomposition of an $H(\mathrm{div})$-conforming macro finite element space related to $\mathbb{V}_k^{\mathrm{div}}(K)$, we propose the unisolvent DoFs for space $\mathbb{V}_k^{\mathrm{div}}(K)$, and establish the L^2 norm equivalence. By the way, we use the matrix-vector language to review a conforming finite element for differential $(d-2)$-form in [3, 2].

By the aid of projector $Q_{K,k}^{\mathrm{div}}$, we advance a stabilization-free nonconforming virtual element method in arbitrary dimension and a stabilization-free conforming virtual element method in two dimensions for second order elliptic problems. Indeed, these stabilization-free virtual element methods can be equivalently recast as primal mixed virtual element methods. We prove the norm equivalence (1.1) and the well-posedness of these stabilization-free virtual element methods, and derive the optimal error estimates.

The idea on constructing stabilization-free virtual element methods in this paper is simple, and can be extended to more virtual element methods and more partial differential equations.

The rest of this paper is organized as follows. Notation and mesh conditions are presented in Section 2. In Section 3, $H(\mathrm{div})$-conforming macro finite elements in arbitrary dimension are constructed. A stabilization-free nonconforming virtual element method in arbitrary dimension is developed in Section 4. And a stabilization-free conforming virtual element method in two dimensions is devised in Section 5. Some numerical results are shown in Section 6.

2. Preliminaries.

2.1. Notation. Let $\Omega\subset\mathbb{R}^d$ be a bounded polytope. Given a bounded domain $K\subset\mathbb{R}^d$ and a non-negative integer m, let $H^m(K)$ be the usual Sobolev space of functions on K. The corresponding norm and semi-norm are denoted respectively by $\|\cdot\|_{m,K}$ and $|\cdot|_{m,K}$. By convention, let $L^2(K)=H^0(K)$. Let $(\cdot,\cdot)_K$ be the standard inner product on $L^2(K)$. If K is Ω, we abbreviate $\|\cdot\|_{m,K}$, $|\cdot|_{m,K}$ and $(\cdot,\cdot)_K$ by $\|\cdot\|_m$, $|\cdot|_m$ and $(\cdot,\cdot)$, respectively. Let $H_0^m(K)$ be the closure of $\mathcal{C}_0^\infty(K)$ with respect to the norm $\|\cdot\|_{m,K}$, and $L_0^2(K)$ consist of all functions in $L^2(K)$ with zero mean value. For integer $k\geq 0$, notation $\mathbb{P}_k(K)$ stands for the set of all polynomials over K with the total degree no more than k. Set $\mathbb{P}_{-1}(K)=\{0\}$. For a banach space $B(K)$, let $B(K;\mathbb{X}):=B(K)\otimes\mathbb{X}$ with $\mathbb{X}=\mathbb{R}^d$ and $\mathbb{K}$, where $\mathbb{K}$ is the set of antisymmetric matrices. Denote by Q_k^K the L^2-orthogonal projector onto $\mathbb{P}_k(K)$ or $\mathbb{P}_k(K;\mathbb{X})$. For tensor $\boldsymbol{\tau}$, let $\operatorname{skw}\boldsymbol{\tau}:=(\boldsymbol{\tau}-\boldsymbol{\tau}^{\intercal})/2$ be the antisymmetric part of $\boldsymbol{\tau}$. Denote by $\#S$ the number of elements in a finite set S.

For d-dimensional polytope K, let $\mathcal{F}(K)$ and $\mathcal{E}(K)$ be the set of all $(d-1)$-dimensional faces and $(d-2)$-dimensional faces of K respectively. For $F \in \mathcal{F}(K)$, denote by $\boldsymbol{n}_{K,F}$ be the unit outward normal vector to ∂K, which will be abbreviate as $\boldsymbol{n}_F$ or $\boldsymbol{n}$ if not causing any confusion.

For d-dimensional simplex T, let $F_i \in \mathcal{F}(T)$ be the $(d-1)$-dimensional face opposite to vertex $\mathtt{v}_i$, $\boldsymbol{n}_i$ be the unit outward normal to the face F_i, and λ_i be the barycentric coordinate of $\boldsymbol{x}$ corresponding to vertex $\mathtt{v}_i$, for $i = 0, 1, \cdots, d$. Clearly $\{\boldsymbol{n}_1, \boldsymbol{n}_2, \cdots, \boldsymbol{n}_d\}$ spans $\mathbb{R}^d$, and $\{\mathrm{skw}(\boldsymbol{n}_i\boldsymbol{n}_j^{\intercal})\}_{1\leq i<j\leq d}$ spans the antisymmetric space $\mathbb{K}$. For $F \in \mathcal{F}(T)$, let $\mathcal{E}(F) := \{e \in \mathcal{E}(T) : e \subset \partial F\}$. For $e \in \mathcal{E}(F)$, denote by $\boldsymbol{n}_{F,e}$ be the unit vector outward normal to ∂F but parallel to F.

Let $\{\mathcal{T}_h\}$ denote a family of partitions of Ω into nonoverlapping simple polytopes with $h := \max_{K\in\mathcal{T}_h} h_K$ and $h_K := \mathrm{diam}(K)$. Denote by $\mathcal{F}_h^r$ the set of all $(d-r)$-dimensional faces of the partition $\mathcal{T}_h$ for $r = 1, \ldots, d$. Set $\mathcal{F}_h := \mathcal{F}_h^1$ for simplicity. Let $\mathcal{F}_h^{\partial}$ be the subset of $\mathcal{F}_h$ including all $(d-1)$-dimensional faces on $\partial\Omega$. For any $F \in \mathcal{F}_h$, let h_F be its diameter and fix a unit normal vector $\boldsymbol{n}_F$. For a piecewise smooth function v, define

$$\|v\|_{1,h}^2 := \sum_{K\in\mathcal{T}_h} \|v\|_{1,K}^2, \quad |v|_{1,h}^2 := \sum_{K\in\mathcal{T}_h} |v|_{1,K}^2.$$

For domain K, we use $\boldsymbol{H}(\mathrm{div}, K)$ and $\boldsymbol{H}_0(\mathrm{div}, K)$ to denote the standard divergence vector spaces. For smooth vector function $\boldsymbol{v}$, let $\nabla\boldsymbol{v} := (\partial_i v_j)_{1\leq i,j\leq d}$. On face $F \in \mathcal{F}_h$, define surface divergence

$$\mathrm{div}_F\, \boldsymbol{v} = \mathrm{div}(\boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{n}) = \mathrm{div}\, \boldsymbol{v} - \partial_n(\boldsymbol{v}\cdot\boldsymbol{n}).$$

For smooth function v, define surface gradient $\nabla_F v := \nabla v - (\partial_n v)\boldsymbol{n}$.

2.2. Mesh conditions. We impose the following conditions on the mesh $\mathcal{T}_h$ in this paper:

- (A1) Each element $K \in \mathcal{T}_h$ and each face $F \in \mathcal{F}_h^r$ for $1 \leq r \leq d-1$ is star-shaped with a uniformly bounded chunkiness parameter.
- (A2) There exists a quasi-uniform simplicial mesh $\mathcal{T}_h^*$ such that each $K \in \mathcal{T}_h$ is a union of some simplexes in $\mathcal{T}_h^*$.

For $K \in \mathcal{T}_h$, let $\boldsymbol{x}_K$ be the center of the largest ball contained in K. Throughout this paper, we use "$\lesssim \cdots$" to mean that "$\leq C\cdots$", where C is a generic positive constant independent of mesh size h, but may depend on the chunkiness parameter of the polytope, the degree of polynomials k, the dimension of space d, and the shape regularity and quasi-uniform constants of the virtual triangulation $\mathcal{T}_h^*$, which may take different values at different appearances. And $A \eqsim B$ means $A \lesssim B$ and $B \lesssim A$.

For polytope $K \in \mathcal{T}_h$, denote by $\mathcal{T}_K$ the simplicial partition of K, which is induced from $\mathcal{T}_h^*$. Let $\mathcal{F}(\mathcal{T}_K)$ and $\mathcal{E}(\mathcal{T}_K)$ be the set of all $(d-1)$-dimensional faces and $(d-2)$-dimensional faces of the simplicial partition $\mathcal{T}_K$ respectively. Set

$$\mathcal{F}^{\partial}(\mathcal{T}_K) := \{F \in \mathcal{F}(\mathcal{T}_K) : F \subset \partial K\}, \quad \mathcal{E}^{\partial}(\mathcal{T}_K) := \{e \in \mathcal{E}(\mathcal{T}_K) : e \subset \partial K\}.$$

Hereafter we use T to represent a simplex, and K to denote a general polytope.

3. $H(\mathrm{div})$-Conforming Macro Finite Elements. In this section we will construct $H(\mathrm{div})$-conforming macro finite elements in arbitrary dimension.

3.1. $H(\mathrm{div})$**-conforming finite elements.** For d-dimensional polytope $K \in \mathcal{T}_h$ and $k \geq 2$, let

$$\boldsymbol{V}_{k-1}^{BDM}(K) := \{\boldsymbol{\phi} \in \boldsymbol{H}(\mathrm{div}, K) : \boldsymbol{\phi}|_T \in \mathbb{P}_{k-1}(T; \mathbb{R}^d) \text{ for each } T \in \mathcal{T}_K\}$$

be the local Brezzi-Douglas-Marini (BDM) element space [13, 12, 24], whose degrees of freedom (DoFs) are given by [16]

$$(\boldsymbol{v} \cdot \boldsymbol{n}, q)_F, \quad q \in \mathbb{P}_{k-1}(F), F \in \mathcal{F}(T), \tag{3.1}$$

$$(\mathrm{div}\, \boldsymbol{v}, q)_T, \quad q \in \mathbb{P}_{k-2}(T)/\mathbb{R}, \tag{3.2}$$

$$(\boldsymbol{v}, \boldsymbol{q})_T, \quad \boldsymbol{q} \in \mathbb{P}_{k-3}(T; \mathbb{K})\boldsymbol{x}. \tag{3.3}$$

Define $\mathring{\boldsymbol{V}}_{k-1}^{BDM}(K) := \boldsymbol{V}_{k-1}^{BDM}(K) \cap \boldsymbol{H}_0(\mathrm{div}, K)$.

We also need the lowest order Raviart-Thomas (RT) element space [13, 12, 24]

$$\boldsymbol{V}^{RT}(K) := \{\boldsymbol{\phi} \in \boldsymbol{H}(\mathrm{div}, K) : \boldsymbol{\phi}|_T \in \mathbb{P}_0(T; \mathbb{R}^d) + \boldsymbol{x}\mathbb{P}_0(T) \text{ for each } T \in \mathcal{T}_K\}.$$

Define $\mathring{\boldsymbol{V}}^{RT}(K) := \boldsymbol{V}^{RT}(K) \cap \boldsymbol{H}_0(\mathrm{div}, K)$.

3.2. Finite element for differential $(d-2)$**-form.** Now recall the finite element for differential $(d-2)$-form, i.e. $H\Lambda^{d-2}$-conforming finite element in [3, 2]. We will present the finite element for differential $(d-2)$-form using the proxy of the differential form rather than the differential form itself as in [3, 2].

By Lemma 3.12 in [16], we have the decomposition

$$\mathbb{P}_{k-1}(T; \mathbb{R}^d) = \nabla P_k(T) \oplus \mathbb{P}_{k-2}(T; \mathbb{K})\boldsymbol{x}. \tag{3.4}$$

LEMMA 3.1. *For* $\boldsymbol{w} \in \mathbb{P}_{k-2}(T; \mathbb{K})\boldsymbol{x}$ *satisfying* $(\mathrm{skw}\, \nabla \boldsymbol{w})\boldsymbol{x} = \boldsymbol{0}$*, it holds* $\boldsymbol{w} = \boldsymbol{0}$.

Proof. Since

$$(\mathrm{skw}\, \nabla \boldsymbol{w})\boldsymbol{x} = \frac{1}{2}(\nabla \boldsymbol{w})\boldsymbol{x} - \frac{1}{2}(\nabla \boldsymbol{w})^{\intercal}\boldsymbol{x} = \frac{1}{2}\nabla(\boldsymbol{w} \cdot \boldsymbol{x}) - \frac{1}{2}(I + \boldsymbol{x} \cdot \nabla)\boldsymbol{w},$$

we acquire from $\boldsymbol{w} \cdot \boldsymbol{x} = 0$ that $(I + \boldsymbol{x} \cdot \nabla)\boldsymbol{w} = \boldsymbol{0}$, which implies $\boldsymbol{w} = \boldsymbol{0}$. ∎

LEMMA 3.2. *The polynomial complex*

$$\mathbb{R} \to \mathbb{P}_k(T) \xrightarrow{\nabla} \mathbb{P}_{k-1}(T; \mathbb{R}^d) \xrightarrow{\mathrm{skw}\, \nabla} \mathbb{P}_{k-2}(T; \mathbb{K}) \tag{3.5}$$

is exact.

Proof. Clearly (3.5) is a complex. It suffices to prove $\mathbb{P}_{k-1}(T; \mathbb{R}^d) \cap \ker(\mathrm{skw}\, \nabla) \subseteq \nabla \mathbb{P}_k(T)$.

For $\boldsymbol{v} \in \mathbb{P}_{k-1}(T; \mathbb{R}^d) \cap \ker(\mathrm{skw}\, \nabla)$, by decomposition (3.4), there exist $q \in \mathbb{P}_k(T)$ and $\boldsymbol{w} \in \mathbb{P}_{k-2}(T; \mathbb{K})\boldsymbol{x}$ such that $\boldsymbol{v} = \nabla q + \boldsymbol{w}$. By $\mathrm{skw}\, \nabla \boldsymbol{v} = \boldsymbol{0}$, we get $\mathrm{skw}\, \nabla \boldsymbol{w} = \boldsymbol{0}$. Apply Lemma 3.1 to derive $\boldsymbol{w} = \boldsymbol{0}$. Thus $\boldsymbol{v} = \nabla q \in \nabla \mathbb{P}_k(T)$. ∎

LEMMA 3.3. *It holds the decomposition*

$$\mathbb{P}_{k-2}(T; \mathbb{K}) = \mathrm{skw}\, \nabla \mathbb{P}_{k-1}(T; \mathbb{R}^d) \oplus \big(\mathbb{P}_{k-2}(T; \mathbb{K}) \cap \ker(\boldsymbol{x})\big), \tag{3.6}$$

where $\mathbb{P}_{k-2}(T; \mathbb{K}) \cap \ker(\boldsymbol{x})$ *means the kernel of operator* $\boldsymbol{x} : \mathbb{P}_{k-2}(T; \mathbb{K}) \to \mathbb{P}_{k-1}(T; \mathbb{R}^d)$.

Proof. Thanks to decomposition (3.4), we have

$$\operatorname{skw} \nabla \mathbb{P}_{k-1}(T;\mathbb{R}^d) = \operatorname{skw} \nabla(\mathbb{P}_{k-2}(T;\mathbb{K})\boldsymbol{x}).$$

By Lemma 3.1, $\operatorname{skw} \nabla \mathbb{P}_{k-1}(T;\mathbb{R}^d) \cap \big(\mathbb{P}_{k-2}(T;\mathbb{K}) \cap \ker(\boldsymbol{x})\big) = \{\boldsymbol{0}\}$. Then we only need to check dimensions. Due to complex (3.5),

$$\dim \operatorname{skw} \nabla \mathbb{P}_{k-1}(T;\mathbb{R}^d) = \dim \mathbb{P}_{k-1}(T;\mathbb{R}^d) - \dim \nabla \mathbb{P}_k(T). \tag{3.7}$$

On the other side, by space decomposition (3.4),

$$\dim \mathbb{P}_{k-2}(T;\mathbb{K})\boldsymbol{x} = \dim \mathbb{P}_{k-1}(T;\mathbb{R}^d) - \dim \nabla \mathbb{P}_k(T).$$

Hence $\dim \operatorname{skw} \nabla \mathbb{P}_{k-1}(T;\mathbb{R}^d) = \dim \mathbb{P}_{k-2}(T;\mathbb{K})\boldsymbol{x}$, which yields (3.6). ∎

By (3.6) and (3.7), it follows

$$\dim \mathbb{P}_{k-2}(T;\mathbb{K}) \cap \ker(\boldsymbol{x}) = \dim \mathbb{P}_{k-2}(T;\mathbb{K}) + \dim \nabla \mathbb{P}_k(T) - \dim \mathbb{P}_{k-1}(T;\mathbb{R}^d). \tag{3.8}$$

With the decomposition (3.6) and $\mathbb{P}_{k-1}(F;\mathbb{R}^{d-1}) = \nabla_F P_k(F) \oplus \mathbb{P}_{k-2}(F;\mathbb{K})\boldsymbol{x}$, we are ready to define the finite element for differential $(d-2)$-form. Take $\mathbb{P}_k(T;\mathbb{K})$ as the space of shape functions. The degrees of freedom are given by

$$((\boldsymbol{n}_1^e)^\intercal \boldsymbol{\tau} \boldsymbol{n}_2^e, q)_e, \quad q \in \mathbb{P}_k(e), e \in \mathcal{E}(T), \tag{3.9}$$

$$(\operatorname{div}_F(\boldsymbol{\tau}\boldsymbol{n}), q)_F, \quad q \in \mathbb{P}_{k-1}(F)/\mathbb{R}, F \in \mathcal{F}(T), \tag{3.10}$$

$$(\boldsymbol{\tau}\boldsymbol{n}, \boldsymbol{q})_F, \quad \boldsymbol{q} \in \mathbb{P}_{k-2}(F;\mathbb{K})\boldsymbol{x}, F \in \mathcal{F}(T), \tag{3.11}$$

$$(\operatorname{div} \boldsymbol{\tau}, \boldsymbol{q})_T, \quad \boldsymbol{q} \in \mathbb{P}_{k-3}(T;\mathbb{K})\boldsymbol{x}, \tag{3.12}$$

$$(\boldsymbol{\tau}, \boldsymbol{q})_T, \quad \boldsymbol{q} \in \mathbb{P}_{k-2}(T;\mathbb{K}) \cap \ker(\boldsymbol{x}). \tag{3.13}$$

In DoF (3.9), $\boldsymbol{n}_1^e$ and $\boldsymbol{n}_2^e$ are two unit normal vectors of e satisfying $\boldsymbol{n}_1^e \cdot \boldsymbol{n}_2^e = 0$.

LEMMA 3.4. *For $e \in \mathcal{E}(T)$, let $\tilde{\boldsymbol{n}}_1$ and $\tilde{\boldsymbol{n}}_2$ be another two unit normal vectors of e satisfying $\tilde{\boldsymbol{n}}_1 \cdot \tilde{\boldsymbol{n}}_2 = 0$. Then*

$$\operatorname{skw}(\tilde{\boldsymbol{n}}_1 \tilde{\boldsymbol{n}}_2^\intercal) = \pm \operatorname{skw}(\boldsymbol{n}_1^e (\boldsymbol{n}_2^e)^\intercal).$$

Proof. Notice that there exists an orthonormal matrix $H \in \mathbb{R}^{2\times 2}$ such that $(\tilde{\boldsymbol{n}}_1, \tilde{\boldsymbol{n}}_2) = (\boldsymbol{n}_1^e, \boldsymbol{n}_2^e)H$. Then

$$\begin{aligned} 2\operatorname{skw}(\tilde{\boldsymbol{n}}_1 \tilde{\boldsymbol{n}}_2^\intercal) = \tilde{\boldsymbol{n}}_1 \tilde{\boldsymbol{n}}_2^\intercal - \tilde{\boldsymbol{n}}_2 \tilde{\boldsymbol{n}}_1^\intercal &= (\tilde{\boldsymbol{n}}_1, \tilde{\boldsymbol{n}}_2) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{\boldsymbol{n}}_1^\intercal \\ \tilde{\boldsymbol{n}}_2^\intercal \end{pmatrix} \\ &= (\boldsymbol{n}_1^e, \boldsymbol{n}_2^e) H \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} H^\intercal (\boldsymbol{n}_1^e, \boldsymbol{n}_2^e)^\intercal. \end{aligned}$$

By direct computation, $H \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} H^\intercal = \det(H) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Hence

$$2\operatorname{skw}(\tilde{\boldsymbol{n}}_1 \tilde{\boldsymbol{n}}_2^\intercal) = 2\det(H)\operatorname{skw}(\boldsymbol{n}_1^e (\boldsymbol{n}_2^e)^\intercal),$$

which ends the proof. ∎

LEMMA 3.5. *Let $\boldsymbol{\tau} \in \mathbb{P}_k(T;\mathbb{K})$ and $F \in \mathcal{F}(T)$. Assume the degrees of freedom* (3.9)-(3.11) *on F vanish. Then $\boldsymbol{\tau}\boldsymbol{n}|_F = \boldsymbol{0}$.*

Proof. Due to (3.9), we get $(\boldsymbol{n}_1^e)^\intercal \boldsymbol{\tau} \boldsymbol{n}_2^e|_e = 0$ on each $e \in \mathcal{E}(F)$, which together with Lemma 3.4 indicates $\boldsymbol{n}_{F,e}^\intercal \boldsymbol{\tau} \boldsymbol{n}_F|_e = 0$. By the unisolvence of BDM element on face F, cf. DoFs (3.1)-(3.3), it follows from DoFs (3.10)-(3.11) that $\boldsymbol{\tau}\boldsymbol{n}|_F = \boldsymbol{0}$. ∎

LEMMA 3.6. *For $\boldsymbol{\tau} \in \mathbb{P}_k(T;\mathbb{K})$, $\boldsymbol{\tau}\boldsymbol{n}|_{F_i} = \boldsymbol{0}$ for $i = 1, \ldots, d$, if and only if*

$$\boldsymbol{\tau} = \sum_{1 \le i < j \le d} \lambda_i \lambda_j q_{ij} \boldsymbol{N}_{ij} \tag{3.14}$$

for some $q_{ij} \in \mathbb{P}_{k-2}(T)$. Here $\{\boldsymbol{N}_{ij}\}_{1\le i<j\le d}$ denotes the basis of $\mathbb{K}$ being dual to $\{\operatorname{skw}(\boldsymbol{n}_i\boldsymbol{n}_j^\intercal)\}_{1\le i<j\le d}$, i.e.,

$$\boldsymbol{N}_{ij} : \operatorname{skw}(\boldsymbol{n}_l \boldsymbol{n}_m^\intercal) = \delta_{il}\delta_{jm}, \quad 1 \le i < j \le d,\ 1 \le l < m \le d.$$

Proof. For $1 \le l \le d$ but $l \ne i, j$, by the definition of $\boldsymbol{N}_{ij}$, it holds $\boldsymbol{N}_{ij}\boldsymbol{n}_l = \boldsymbol{0}$. Hence for $\boldsymbol{\tau} = \sum\limits_{1\le i<j\le d} \lambda_i\lambda_j q_{ij}\boldsymbol{N}_{ij}$, obviously we have $\boldsymbol{\tau}\boldsymbol{n}|_{F_i} = \boldsymbol{0}$ for $i = 1, \ldots, d$.

On the other side, assume $\boldsymbol{\tau}\boldsymbol{n}|_{F_i} = \boldsymbol{0}$ for $i = 1, \ldots, d$. Express $\boldsymbol{\tau}$ as

$$\boldsymbol{\tau} = \sum_{1\le i<j\le d} p_{ij}\boldsymbol{N}_{ij},$$

where $p_{ij} = \boldsymbol{n}_i^\intercal \boldsymbol{\tau} \boldsymbol{n}_j \in \mathbb{P}_k(T)$. Therefore $p_{ij}|_{F_i} = p_{ij}|_{F_j} = 0$, which ends the proof. ∎

LEMMA 3.7. *The degrees of freedom* (3.9)-(3.13) *are uni-solvent for* $\mathbb{P}_k(T;\mathbb{K})$.

Proof. By $\mathbb{P}_{k-1}(F;\mathbb{R}^{d-1}) = \nabla_F P_k(F) \oplus \mathbb{P}_{k-2}(F;\mathbb{K})\boldsymbol{x}$, the number of degrees of freedom (3.10)-(3.11) is $(d^2+d)\binom{k+d-2}{k-1} - (d+1)\binom{k+d-1}{k}$. Using (3.4) and (3.8), the number of degrees of freedom (3.9)-(3.13) is

$$\begin{aligned}&\frac{1}{2}(d^2+d)\binom{k+d-2}{k} + (d^2+d)\binom{k+d-2}{k-1} - (d+1)\binom{k+d-1}{k}\\ &+ \frac{1}{2}(d^2+d)\binom{k+d-2}{k-2} + \binom{k+d}{k} - (d+1)\binom{k+d-1}{k-1} = \frac{1}{2}(d^2-d)\binom{k+d}{k},\end{aligned}$$

which equals to $\dim \mathbb{P}_k(T;\mathbb{K})$.

Assume $\boldsymbol{\tau} \in \mathbb{P}_k(T;\mathbb{K})$ and all the degrees of freedom (3.9)-(3.13) vanish. It holds from Lemma 3.5 that $\boldsymbol{\tau}\boldsymbol{n}|_{\partial T} = \boldsymbol{0}$. Noting that $\boldsymbol{\tau}$ is antisymmetric, we also have $\boldsymbol{n}^\intercal\boldsymbol{\tau}|_{\partial T} = \boldsymbol{0}$. On each $F \in \mathcal{F}(T)$, it holds

$$\boldsymbol{n}^\intercal \operatorname{div}\boldsymbol{\tau} = \operatorname{div}(\boldsymbol{n}^\intercal\boldsymbol{\tau}) = \operatorname{div}_F(\boldsymbol{n}^\intercal\boldsymbol{\tau}) + \partial_n(\boldsymbol{n}^\intercal\boldsymbol{\tau}\boldsymbol{n}) = \operatorname{div}_F(\boldsymbol{n}^\intercal\boldsymbol{\tau}). \tag{3.15}$$

Hence $\boldsymbol{n}^\intercal \operatorname{div}\boldsymbol{\tau}|_{\partial T} = 0$. Thanks to DoFs (3.1)-(3.3) for BDM element, we acquire from DoF (3.12) and $\operatorname{div}\operatorname{div}\boldsymbol{\tau} = 0$ that $\operatorname{div}\boldsymbol{\tau} = \boldsymbol{0}$, which together with DoF (3.13) and decomposition (3.6) gives

$$(\boldsymbol{\tau}, \boldsymbol{q})_T = 0 \quad \forall\, \boldsymbol{q} \in \mathbb{P}_{k-2}(T;\mathbb{K}).$$

Applying Lemma 3.6, $\boldsymbol{\tau}$ has the expression as in (3.14). Taking $\boldsymbol{q} = q_{ij}\operatorname{skw}(\boldsymbol{n}_i\boldsymbol{n}_j^\intercal)$ in the last equation for $1 \le i < j \le d$, we get $q_{ij} = 0$. Thus $\boldsymbol{\tau} = \boldsymbol{0}$. ∎

For polygon $K \in \mathcal{T}_h$, define the local finite element space for differential $(d-2)$-form

$$\begin{aligned}\boldsymbol{V}_k^{d-2}(K) := \{\boldsymbol{\tau} \in \boldsymbol{L}^2(K;\mathbb{K}) : \boldsymbol{\tau}|_T \in \mathbb{P}_k(T;\mathbb{K}) \text{ for each } T \in \mathcal{T}_K,\\ \text{all the DoFs (3.9)-(3.11) are single-valued}\}.\end{aligned}$$

Thanks to Lemma 3.5, space $\boldsymbol{V}_k^{d-2}(K)$ is $H\Lambda^{d-2}$-conforming. Define $\mathring{\boldsymbol{V}}_k^{d-2}(K) := \boldsymbol{V}_k^{d-2}(K) \cap \mathring{H}\Lambda^{d-2}(K)$, where $\mathring{H}\Lambda^{d-2}(K)$ is the subspace of $H\Lambda^{d-2}(K)$ with homogeneous boundary condition. Notice that $\boldsymbol{V}_k^{d-2}(K)$ is the Lagrange element space for $d=2$, and $\boldsymbol{V}_k^{d-2}(K)$ is the second kind Nédélec element space for $d=3$ [24].

Recall the local finite element de Rham complexes in [3, 2]. For completeness, we will prove the exactness of these complexes.

Lemma 3.8. *Let $k \geq 2$. Finite element complexes*

$$\boldsymbol{V}_k^{d-2}(K) \xrightarrow{\operatorname{div}\operatorname{skw}} \boldsymbol{V}_{k-1}^{BDM}(K) \xrightarrow{\operatorname{div}} V_{k-2}^{L^2}(K) \to 0, \tag{3.16}$$

$$\boldsymbol{V}_1^{d-2}(K) \xrightarrow{\operatorname{div}\operatorname{skw}} \boldsymbol{V}^{RT}(K) \xrightarrow{\operatorname{div}} V_0^{L^2}(K) \to 0, \tag{3.17}$$

$$\mathring{\boldsymbol{V}}_k^{d-2}(K) \xrightarrow{\operatorname{div}\operatorname{skw}} \mathring{\boldsymbol{V}}_{k-1}^{BDM}(K) \xrightarrow{\operatorname{div}} \mathring{V}_{k-2}^{L^2}(K) \to 0, \tag{3.18}$$

$$\mathring{\boldsymbol{V}}_1^{d-2}(K) \xrightarrow{\operatorname{div}\operatorname{skw}} \mathring{\boldsymbol{V}}^{RT}(K) \xrightarrow{\operatorname{div}} \mathring{V}_0^{L^2}(K) \to 0, \tag{3.19}$$

are exact, where $\mathring{V}_{k-2}^{L^2}(K) := V_{k-2}^{L^2}(K)/\mathbb{R}$, and

$$V_{k-2}^{L^2}(K) := \{v \in L^2(K) : v|_T \in \mathbb{P}_{k-2}(T) \text{ for each } T \in \mathcal{T}_K\}.$$

Proof. We only prove complex (3.16), since the argument for the rest complexes is similar. Clearly (3.16) is a complex. We refer to [17, Section 4] for the proof of $\operatorname{div} \boldsymbol{V}_{k-1}^{BDM}(K) = V_{k-2}^{L^2}(K)$.

Next prove $\boldsymbol{V}_{k-1}^{BDM}(K) \cap \ker(\operatorname{div}) = \operatorname{div}\operatorname{skw} \boldsymbol{V}_k^{d-2}(K)$. For $\boldsymbol{v} \in \boldsymbol{V}_{k-1}^{BDM}(K) \cap \ker(\operatorname{div})$, by Theorem 1.1 in [19], there exists $\boldsymbol{\tau} \in \boldsymbol{H}^1(K;\mathbb{K})$ satisfying $\operatorname{div}\boldsymbol{\tau} = \operatorname{div}\operatorname{skw}\boldsymbol{\tau} = \boldsymbol{v}$. Let $\boldsymbol{\sigma} \in \boldsymbol{V}_k^{d-2}(K)$ be the nodal interpolation of $\boldsymbol{\tau}$ based on DoFs (3.9)-(3.13). Thanks to DoF (3.9), it follows from the integration by parts that

$$(\operatorname{div}_F(\boldsymbol{\sigma}\boldsymbol{n}), 1)_F = (\boldsymbol{v}\cdot\boldsymbol{n}, 1)_F \quad \forall\, F \in \mathcal{F}(\mathcal{T}_K),$$

which together with (3.15) and DoF (3.10) that

$$(\boldsymbol{n}^{\intercal} \operatorname{div}\boldsymbol{\sigma}, q)_F = (\operatorname{div}_F(\boldsymbol{\sigma}\boldsymbol{n}), q)_F = (\boldsymbol{v}\cdot\boldsymbol{n}, q)_F \quad \forall\, q \in \mathbb{P}_{k-1}(F), F \in \mathcal{F}(\mathcal{T}_K).$$

Therefore, due to DoF (3.12) and the fact $\operatorname{div}\operatorname{div}\boldsymbol{\sigma} = \operatorname{div}\boldsymbol{v} = 0$, we acquire from the unisolvence of DoFs (3.1)-(3.3) for BDM element that $\boldsymbol{v} = \operatorname{div}\boldsymbol{\sigma} \in \operatorname{div}\operatorname{skw}\boldsymbol{V}_k^{d-2}(K)$. ∎

Note that $\operatorname{div}\operatorname{skw} = \operatorname{curl}$ for $d = 2, 3$. For $k \geq 1$, by finite element complexes (3.16)-(3.19), we have

$$\dim \operatorname{div}\operatorname{skw} \boldsymbol{V}_k^{d-2}(K) - \dim \operatorname{div}\operatorname{skw} \mathring{\boldsymbol{V}}_k^{d-2}(K) = \binom{k+d-2}{d-1} \#\mathcal{F}^{\partial}(\mathcal{T}_K) - 1. \tag{3.20}$$

3.3. $H(\operatorname{div})$-conforming macro finite element. For each polygon $K \in \mathcal{T}_h$, define shape function space

$$\boldsymbol{V}_{k-1}^{\operatorname{div}}(K) := \{\boldsymbol{\phi} \in \boldsymbol{V}_{k-1}^{BDM}(K) : \operatorname{div}\boldsymbol{\phi} \in \mathbb{P}_{k-2}(K)\},$$

for $k \geq 2$, and

$$\boldsymbol{V}_0^{\operatorname{div}}(K) := \{\boldsymbol{\phi} \in \boldsymbol{V}_0^{RT}(K) : \operatorname{div}\boldsymbol{\phi} \in \mathbb{P}_0(K)\}.$$

Apparently $\mathbb{P}_{k-1}(K;\mathbb{R}^d) \subseteq \boldsymbol{V}_{k-1}^{\operatorname{div}}(K)$, $\boldsymbol{V}_0^{\operatorname{div}}(K) \cap \ker(\operatorname{div}) = \boldsymbol{V}_0^{RT}(K) \cap \ker(\operatorname{div})$, and $\boldsymbol{V}_{k-1}^{\operatorname{div}}(K) \cap \ker(\operatorname{div}) = \boldsymbol{V}_{k-1}^{BDM}(K) \cap \ker(\operatorname{div})$ for $k \geq 2$.

In the following lemma we present a direct sum decomposition of space $\boldsymbol{V}_{k-1}^{\operatorname{div}}(K)$.

LEMMA 3.9. *For $k \geq 1$, it holds*

$$\boldsymbol{V}_{k-1}^{\mathrm{div}}(K) = \operatorname{div}\operatorname{skw}\boldsymbol{V}_k^{d-2}(K) \oplus (\boldsymbol{x}-\boldsymbol{x}_K)\mathbb{P}_{\max\{k-2,0\}}(K). \tag{3.21}$$

Then the complex

$$\boldsymbol{V}_k^{d-2}(K) \xrightarrow{\operatorname{div}\operatorname{skw}} \boldsymbol{V}_{k-1}^{\mathrm{div}}(K) \xrightarrow{\operatorname{div}} \mathbb{P}_{\max\{k-2,0\}}(K) \to 0$$

is exact.

Proof. We only prove the case $k \geq 2$, as the proof for case $k = 1$ is similar. Since $\operatorname{div} : (\boldsymbol{x}-\boldsymbol{x}_K)\mathbb{P}_{k-2}(K) \to \mathbb{P}_{k-2}(K)$ is bijective [16, Lemma 3.1], we have $\operatorname{div}\operatorname{skw}\boldsymbol{V}_k^{d-2}(K) \cap (\boldsymbol{x}-\boldsymbol{x}_K)\mathbb{P}_{k-2}(K) = \{\boldsymbol{0}\}$. Clearly $\operatorname{div}\operatorname{skw}\boldsymbol{V}_k^{d-2}(K) \oplus (\boldsymbol{x}-\boldsymbol{x}_K)\mathbb{P}_{k-2}(K) \subseteq \boldsymbol{V}_{k-1}^{\mathrm{div}}(K)$.

On the other side, for $\boldsymbol{\phi} \in \boldsymbol{V}_{k-1}^{\mathrm{div}}(K)$, by $\operatorname{div}\boldsymbol{\phi} \in \mathbb{P}_{k-2}(K)$, there exists a $q \in \mathbb{P}_{k-2}(K)$ such that $\operatorname{div}((\boldsymbol{x}-\boldsymbol{x}_K)q) = \operatorname{div}\boldsymbol{\phi}$, i.e. $\boldsymbol{\phi}-(\boldsymbol{x}-\boldsymbol{x}_K)q \in \boldsymbol{V}_{k-1}^{\mathrm{div}}(K)\cap\ker(\operatorname{div}) = \boldsymbol{V}_{k-1}^{BDM}(K) \cap \ker(\operatorname{div})$. Thanks to finite element complex (3.16), $\boldsymbol{\phi} - (\boldsymbol{x}-\boldsymbol{x}_K)q \in \operatorname{div}\operatorname{skw}\boldsymbol{V}_k^{d-2}(K)$. Thus (3.21) follows. ∎

Based on the space decomposition (3.21) and the degrees of freedom of BDM element, we propose the following DoFs for space $\boldsymbol{V}_{k-1}^{\mathrm{div}}(K)$

$$(\boldsymbol{\phi}\cdot\boldsymbol{n}, q)_F \quad \forall\, q \in \mathbb{P}_{k-1}(F) \text{ on each } F \in \mathcal{F}^{\partial}(\mathcal{T}_K), \tag{3.22}$$

$$(\operatorname{div}\boldsymbol{\phi}, q)_K \quad \forall\, q \in \mathbb{P}_{\max\{k-2,0\}}(K)/\mathbb{R}, \tag{3.23}$$

$$(\boldsymbol{\phi}, \boldsymbol{q})_K \quad \forall\, \boldsymbol{q} \in \operatorname{div}\operatorname{skw}\mathring{\boldsymbol{V}}_k^{d-2}(K) = \operatorname{div}\mathring{\boldsymbol{V}}_k^{d-2}(K). \tag{3.24}$$

LEMMA 3.10. *The set of DoFs* (3.22)-(3.24) *is uni-solvent for space* $\boldsymbol{V}_{k-1}^{\mathrm{div}}(K)$.

Proof. By (3.20) and (3.21), the number of DoFs (3.22)-(3.24) is

$$\begin{aligned}&\binom{k+d-2}{d-1}\#\mathcal{F}^{\partial}(\mathcal{T}_K) + \dim\mathbb{P}_{\max\{k-2,0\}}(K) - 1 + \dim\operatorname{div}\operatorname{skw}\mathring{\boldsymbol{V}}_k^{d-2}(K)\\ &= \dim\operatorname{div}\operatorname{skw}\boldsymbol{V}_k^{d-2}(K) + \dim\mathbb{P}_{\max\{k-2,0\}}(K) = \dim\boldsymbol{V}_{k-1}^{\mathrm{div}}(K).\end{aligned}$$

Assume $\boldsymbol{\phi} \in \boldsymbol{V}_{k-1}^{\mathrm{div}}(K)$ and all the DoFs (3.22)-(3.24) vanish. By the vanishing DoF (3.22), $\boldsymbol{\phi} \in \boldsymbol{H}_0(\operatorname{div}, K)$ and $\operatorname{div}\boldsymbol{\phi} \in L_0^2(K)$. Then it follows from the vanishing DoF (3.23) that $\operatorname{div}\boldsymbol{\phi} = 0$. Thanks to the exactness of complexes (3.18)-(3.19), $\boldsymbol{\phi} \in \operatorname{div}\operatorname{skw}\mathring{\boldsymbol{V}}_k^{d-2}(K)$. Therefore $\boldsymbol{\phi} = \boldsymbol{0}$ holds from the vanishing DoF (3.24). ∎

Remark 3.11. When K is a simplex and $\mathcal{T}_K = \{K\}$, thanks to DoF (3.3) for the BDM element, DoF (3.24) can be replaced by

$$(\boldsymbol{\phi}, \boldsymbol{q})_K \quad \forall\, \boldsymbol{q} \in \mathbb{P}_{k-3}(K;\mathbb{K})\boldsymbol{x}$$

for $k \geq 3$. And DoF (3.24) disappears for $k = 1$ and $k = 2$.

Next we consider the norm equivalence of space $\boldsymbol{V}_{k-1}^{\mathrm{div}}(K)$.

LEMMA 3.12. *For $\boldsymbol{\phi} \in \boldsymbol{V}_{k-1}^{\mathrm{div}}(K)$, it holds the norm equivalence*

$$\|\boldsymbol{\phi}\|_{0,K} \eqsim h_K\|\operatorname{div}\boldsymbol{\phi}\|_{0,K} + \sup_{\boldsymbol{\psi}\in\operatorname{div}\mathring{\boldsymbol{V}}_k^{d-2}(K)} \frac{(\boldsymbol{\phi},\boldsymbol{\psi})_K}{\|\boldsymbol{\psi}\|_{0,K}} + \sum_{F\in\mathcal{F}^{\partial}(\mathcal{T}_K)} h_F^{1/2}\|\boldsymbol{\phi}\cdot\boldsymbol{n}\|_{0,F}. \tag{3.25}$$

Proof. By the inverse inequality [21, Lemma 10] and the trace inequality [11, (2.18)], we have

$$h_K\|\operatorname{div}\boldsymbol{\phi}\|_{0,K} + \sup_{\boldsymbol{\psi}\in\operatorname{div}\mathring{\boldsymbol{V}}_k^{d-2}(K)}\frac{(\boldsymbol{\phi},\boldsymbol{\psi})_K}{\|\boldsymbol{\psi}\|_{0,K}} + \sum_{F\in\mathcal{F}^\partial(\mathcal{T}_K)} h_F^{1/2}\|\boldsymbol{\phi}\cdot\boldsymbol{n}\|_{0,F}$$
$$\lesssim \|\operatorname{div}\boldsymbol{\phi}\|_{-1,K} + \|\boldsymbol{\phi}\|_{0,K} \lesssim \|\boldsymbol{\phi}\|_{0,K}.$$

Next we focus on the proof of the other side. Again we only prove the case $k\geq 2$, whose argument can be applied to case $k=1$. Take $\boldsymbol{\phi}_1 \in \boldsymbol{V}_{k-1}^{BDM}(K)$ such that $\boldsymbol{\phi}_1\cdot\boldsymbol{n}|_{\partial K} = \boldsymbol{\phi}\cdot\boldsymbol{n}_{\partial K}$, and all the DoFs (3.1)-(3.3) of $\boldsymbol{\phi}_1$ in the interior of K vanish. We have

$$\|\boldsymbol{\phi}_1\|_{0,K} \eqsim \sum_{F\in\mathcal{F}^\partial(\mathcal{T}_K)} h_F^{1/2}\|\boldsymbol{\phi}\cdot\boldsymbol{n}\|_{0,F}, \tag{3.26}$$

$$\|\operatorname{div}\boldsymbol{\phi}_1\|_{0,T}^2 = \|Q_0^T(\operatorname{div}\boldsymbol{\phi}_1)\|_{0,T}^2 \leq \frac{1}{|T|}\sum_{F\in\mathcal{F}(T)\cap\mathcal{F}^\partial(\mathcal{T}_K)} |F|\|\boldsymbol{\phi}\cdot\boldsymbol{n}\|_{0,F}^2 \quad \forall\, T\in\mathcal{T}_K.$$

Then let $w\in H^1(K)/\mathbb{R}$ be the solution of

$$\begin{cases} -\Delta w = \operatorname{div}(\boldsymbol{\phi}-\boldsymbol{\phi}_1) & \text{in } K,\\ \partial_n w = 0 & \text{on } \partial K.\end{cases}$$

The weak formulation is

$$(\nabla w,\nabla v)_K = (\operatorname{div}(\boldsymbol{\phi}-\boldsymbol{\phi}_1), v)_K \quad \forall\, v\in H^1(K)/\mathbb{R}.$$

Obviously we have

$$\|\nabla w\|_{0,K} \lesssim h_K\|\operatorname{div}(\boldsymbol{\phi}-\boldsymbol{\phi}_1)\|_{0,K} \lesssim h_K\|\operatorname{div}\boldsymbol{\phi}\|_{0,K} + \sum_{F\in\mathcal{F}^\partial(\mathcal{T}_K)} h_F^{1/2}\|\boldsymbol{\phi}\cdot\boldsymbol{n}\|_{0,F}.$$

Let $I_K^{\operatorname{div}}: \boldsymbol{H}_0(\operatorname{div},K)\to\mathring{\boldsymbol{V}}_{k-1}^{BDM}(K)$ be the L^2-bounded commuting projection operator in [18]. Set $\boldsymbol{\phi}_2 = -I_K^{\operatorname{div}}(\nabla w)\in\mathring{\boldsymbol{V}}_{k-1}^{BDM}(K)$. We have

$$\operatorname{div}\boldsymbol{\phi}_2 = -\Delta w = \operatorname{div}(\boldsymbol{\phi}-\boldsymbol{\phi}_1), \tag{3.27}$$

$$\|\boldsymbol{\phi}_2\|_{0,K} \lesssim \|\nabla w\|_{0,K} \lesssim h_K\|\operatorname{div}\boldsymbol{\phi}\|_{0,K} + \sum_{F\in\mathcal{F}^\partial(\mathcal{T}_K)} h_F^{1/2}\|\boldsymbol{\phi}\cdot\boldsymbol{n}\|_{0,F}. \tag{3.28}$$

By (3.27), $\boldsymbol{\phi}-\boldsymbol{\phi}_1-\boldsymbol{\phi}_2\in\mathring{\boldsymbol{V}}_{k-1}^{BDM}(K)\cap\ker(\operatorname{div})$, which together the exactness of complex (3.18) indicates $\boldsymbol{\phi}-\boldsymbol{\phi}_1-\boldsymbol{\phi}_2\in\operatorname{div}\mathring{\boldsymbol{V}}_k^{d-2}(K)$. Hence

$$\begin{aligned}\|\boldsymbol{\phi}\|_{0,K} &\lesssim \|\boldsymbol{\phi}_1\|_{0,K} + \|\boldsymbol{\phi}_2\|_{0,K} + \|\boldsymbol{\phi}-\boldsymbol{\phi}_1-\boldsymbol{\phi}_2\|_{0,K}\\ &\lesssim \|\boldsymbol{\phi}_1\|_{0,K} + \|\boldsymbol{\phi}_2\|_{0,K} + \sup_{\boldsymbol{\psi}\in\operatorname{div}\mathring{\boldsymbol{V}}_k^{d-2}(K)}\frac{(\boldsymbol{\phi}-\boldsymbol{\phi}_1-\boldsymbol{\phi}_2,\boldsymbol{\psi})_K}{\|\boldsymbol{\psi}\|_{0,K}}\\ &\lesssim \|\boldsymbol{\phi}_1\|_{0,K} + \|\boldsymbol{\phi}_2\|_{0,K} + \sup_{\boldsymbol{\psi}\in\operatorname{div}\mathring{\boldsymbol{V}}_k^{d-2}(K)}\frac{(\boldsymbol{\phi},\boldsymbol{\psi})_K}{\|\boldsymbol{\psi}\|_{0,K}}.\end{aligned}$$

Finally (3.25) holds from (3.26) and (3.28). ∎

Let

$$\mathbb{V}_{k-1}^{\mathrm{div}}(K) := \{\boldsymbol{\phi} \in \boldsymbol{V}_{k-1}^{\mathrm{div}}(K) : \boldsymbol{\phi}\cdot\boldsymbol{n}|_F \in \mathbb{P}_{k-1}(F) \quad \forall\, F \in \mathcal{F}(K)\}.$$

Due to DoFs (3.22)-(3.24) for $\boldsymbol{V}_{k-1}^{\mathrm{div}}(K)$, a set of unisolvent DoFs for $\mathbb{V}_{k-1}^{\mathrm{div}}(K)$ is

$$(\boldsymbol{\phi}\cdot\boldsymbol{n}, q)_F \quad \forall\, q \in \mathbb{P}_{k-1}(F) \text{ on each } F \in \mathcal{F}(K), \tag{3.29}$$

$$(\operatorname{div}\boldsymbol{\phi}, q)_K \quad \forall\, q \in \mathbb{P}_{\max\{k-2,0\}}(K)/\mathbb{R}, \tag{3.30}$$

$$(\boldsymbol{\phi}, \boldsymbol{q})_K \quad \forall\, \boldsymbol{q} \in \operatorname{div}\mathring{\boldsymbol{V}}_k^{d-2}(K). \tag{3.31}$$

As an immediate result of Lemma 3.12, we get the following norm equivalence of space $\mathbb{V}_{k-1}^{\mathrm{div}}(K)$.

Corollary 3.13. *For $\boldsymbol{\phi} \in \mathbb{V}_{k-1}^{\mathrm{div}}(K)$, it holds the norm equivalence*

$$\|\boldsymbol{\phi}\|_{0,K} \eqsim h_K\|\operatorname{div}\boldsymbol{\phi}\|_{0,K} + \sup_{\boldsymbol{\psi}\in\operatorname{div}\mathring{\boldsymbol{V}}_k^{d-2}(K)} \frac{(\boldsymbol{\phi},\boldsymbol{\psi})_K}{\|\boldsymbol{\psi}\|_{0,K}} + \sum_{F\in\mathcal{F}(K)} h_F^{1/2}\|\boldsymbol{\phi}\cdot\boldsymbol{n}\|_{0,F}. \tag{3.32}$$

For later use, let $Q_{K,k-1}^{\mathrm{div}}$ be the L^2-orthogonal projection operator onto $\mathbb{V}_{k-1}^{\mathrm{div}}(K)$ with respect to the inner product $(\cdot,\cdot)_K$. Introduce discrete spaces

$$\mathbb{V}_{h,k-1}^{\mathrm{div}} := \{\boldsymbol{\phi}_h \in \boldsymbol{L}^2(\Omega;\mathbb{R}^d) : \boldsymbol{\phi}_h|_K \in \mathbb{V}_{k-1}^{\mathrm{div}}(K) \text{ for each } K \in \mathcal{T}_h\},$$
$$\mathbb{P}_l(\mathcal{T}_h) := \{q_h \in L^2(\Omega) : q_h|_K \in \mathbb{P}_l(K) \text{ for each } K \in \mathcal{T}_h\}$$

with non-negative integer l. For $\boldsymbol{\phi} \in \boldsymbol{L}^2(\Omega;\mathbb{R}^d)$, let $Q_{h,k-1}^{\mathrm{div}}\boldsymbol{\phi} \in \mathbb{V}_{h,k-1}^{\mathrm{div}}$ be determined by $(Q_{h,k-1}^{\mathrm{div}}\boldsymbol{\phi})|_K = Q_{K,k-1}^{\mathrm{div}}(\boldsymbol{\phi}|_K)$ for each $K \in \mathcal{T}_h$. For $v \in L^2(\Omega)$, let $Q_h^l v \in \mathbb{P}_l(\mathcal{T}_h)$ be determined by $(Q_h^l v)|_K = Q_l^K(v|_K)$ for each $K \in \mathcal{T}_h$. For simplicity, the vector version of Q_h^l is still denoted by Q_h^l. And we abbreviate Q_h^k as Q_h if $l = k$.

4. Stabilization-free nonconforming virtual element method.

In this section we will develop a stabilization-free nonconforming virtual element method for the second order elliptic problem in arbitrary dimension

$$\begin{cases} -\Delta u + \alpha u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $\Omega \subseteq \mathbb{R}^d$ is a bounded polygon, $f \in L^2(\Omega)$ and α is a nonnegative constant. The weak formulation of problem (4.1) is to find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = (f, v) \quad \forall\, v \in H_0^1(\Omega), \tag{4.2}$$

where the bilinear form $a(u, v) := (\nabla_h u, \nabla_h v) + \alpha(u, v)$ with ∇_h being the piecewise counterpart of ∇ with respect to $\mathcal{T}_h$.

4.1. H^1-nonconforming virtual element.

Recall the H^1-nonconforming virtual element in [15, 21, 4]. The degrees of freedom are given by

$$\frac{1}{|F|}(v, q)_F \quad \forall\, q \in \mathbb{P}_{k-1}(F), F \in \mathcal{F}(K), \tag{4.3}$$

$$\frac{1}{|K|}(v, q)_K \quad \forall\, q \in \mathbb{P}_{k-2}(K). \tag{4.4}$$

To define the space of shape functions, we need a local H^1 projection operator $\Pi_k^K : H^1(K) \to \mathbb{P}_k(K)$: given $v \in H^1(K)$, let $\Pi_k^K v \in \mathbb{P}_k(K)$ be the solution of the problem

$$(\nabla \Pi_k^K v, \nabla q)_K = (\nabla v, \nabla q)_K \quad \forall\, q \in \mathbb{P}_k(K), \tag{4.5}$$

$$\int_{\partial K} \Pi_k^K v \,\mathrm{d}s = \int_{\partial K} v \,\mathrm{d}s. \tag{4.6}$$

It holds

$$\Pi_k^K q = q \quad \forall\, q \in \mathbb{P}_k(K). \tag{4.7}$$

With the help of operator Π_k^K, the space of shape functions is defined as

$$\begin{aligned} V_k(K) := \{v \in H^1(K) : \Delta v \in \mathbb{P}_k(K),\ \partial_n v|_F \in \mathbb{P}_{k-1}(F) \text{ for each face } F \in \mathcal{F}(K), \\ \text{and } (v - \Pi_k^K v, q)_K = 0 \quad \forall\, q \in \mathbb{P}_k(K)/\mathbb{P}_{k-2}(K)\}, \end{aligned}$$

where $\mathbb{P}_k(K)/\mathbb{P}_{k-2}(K)$ means the orthogonal complement space of $\mathbb{P}_{k-2}(K)$ in $\mathbb{P}_k(K)$ with respect to the inner product $(\cdot,\cdot)_K$. Due to (4.7), it holds $\mathbb{P}_k(K) \subseteq V_k(K)$. DoFs (4.3)-(4.4) are uni-solvent for the shape function space $V_k(K)$.

For $v \in V_k(K)$, the H^1 projection $\Pi_k^K v$ is computable using only DoFs (4.3)-(4.4), and the L^2 projection

$$Q_k^K v = \Pi_k^K v + Q_{k-2}^K v - Q_{k-2}^K \Pi_k^K v \tag{4.8}$$

is also computable using only DoFs (4.3)-(4.4).

We will prove the inverse inequality and the norm equivalence for the virtual element space $V_k(K)$.

Lemma 4.1. *It holds the inverse inequality*

$$|v|_{1,K} \lesssim h_K^{-1} \|v\|_{0,K} \quad \forall\, v \in V_k(K). \tag{4.9}$$

Proof. Apply the integration by parts to get

$$|v|_{1,K}^2 = -(\Delta v, v)_K + (\partial_n v, v)_{\partial K} \le \|\Delta v\|_{0,K}\|v\|_{0,K} + \|\partial_n v\|_{0,\partial K}\|v\|_{0,\partial K}.$$

By $h_K\|\Delta v\|_{0,K} + h_K^{1/2}\|\partial_n v\|_{0,\partial K} \lesssim |v|_{1,K}$, i.e. (A.3)-(A.4) in [15], we obtain

$$|v|_{1,K} \lesssim h_K^{-1}\|v\|_{0,K} + h_K^{-1/2}\|v\|_{0,\partial K},$$

which together with the multiplicative trace inequality yields (4.9). ∎

Lemma 4.2. *For $v \in V_k(K)$, we have*

$$\|\Pi_k^K v\|_{0,K}^2 + h_K^2 |\Pi_k^K v|_{1,K}^2 \lesssim \|Q_{k-2}^K v\|_{0,K}^2 + \sum_{F \in \mathcal{F}(K)} h_F \|Q_{k-1}^F v\|_{0,F}^2, \tag{4.10}$$

$$\|Q_k^K v\|_{0,K}^2 \lesssim \|Q_{k-2}^K v\|_{0,K}^2 + \sum_{F \in \mathcal{F}(K)} h_F \|Q_{k-1}^F v\|_{0,F}^2. \tag{4.11}$$

Proof. We get from (4.5) and the integration by parts that

$$
\begin{aligned}
|\Pi_k^K v|_{1,K}^2 &= (\nabla v, \nabla \Pi_k^K v)_K = -(v, \Delta \Pi_k^K v)_K + (v, \partial_n(\Pi_k^K v))_{\partial K} \\
&= -(Q_{k-2}^K v, \Delta \Pi_k^K v)_K + \sum_{F\in\mathcal{F}(K)} (Q_{k-1}^F v, \partial_n(\Pi_k^K v))_F \\
&\leq \|Q_{k-2}^K v\|_{0,K} \|\Delta \Pi_k^K v\|_{0,K} + \sum_{F\in\mathcal{F}(K)} \|Q_{k-1}^F v\|_{0,F} \|\partial_n(\Pi_k^K v)\|_{0,F},
\end{aligned}
$$

which combined with the inverse inequality for polynomials implies

$$
h_K^2 |\Pi_k^K v|_{1,K}^2 \lesssim \|Q_{k-2}^K v\|_{0,K}^2 + \sum_{F\in\mathcal{F}(K)} h_F \|Q_{k-1}^F v\|_{0,F}^2.
$$

Thanks to the Poincaré-Friedrichs inequality [11, (2.15)] and (4.6),

$$
\begin{aligned}
\|\Pi_k^K v\|_{0,K}^2 &\lesssim h_K |\Pi_k^K v|_{1,K}^2 + h_K^{2-d} \left| \int_{\partial K} v \,\mathrm{d}s \right|^2 \\
&= h_K^2 |\Pi_k^K v|_{1,K}^2 + h_K^{2-d} \left| \sum_{F\in\mathcal{F}(K)} \int_F Q_0^F v \,\mathrm{d}s \right|^2 \\
&\lesssim h_K^2 |\Pi_k^K v|_{1,K}^2 + \sum_{F\in\mathcal{F}(K)} h_F \|Q_0^F v\|_{0,F}^2.
\end{aligned}
$$

Hence (4.10) follows from the last two inequalities.

Finally (4.11) holds from (4.8) and (4.10). ∎

LEMMA 4.3. *It holds the norm equivalence*

$$
h_K^2 |v|_{1,K}^2 \lesssim \|v\|_{0,K}^2 \eqsim \|Q_{k-2}^K v\|_{0,K}^2 + \sum_{F\in\mathcal{F}(K)} h_F \|Q_{k-1}^F v\|_{0,F}^2 \quad \forall\, v \in V_k(K). \tag{4.12}
$$

Proof. Since $\Delta v \in \mathbb{P}_k(K)$ and $\partial_n v|_F \in \mathbb{P}_{k-1}(F)$, we get from the integration by parts that

$$
\begin{aligned}
|v|_{1,K}^2 &= -(\Delta v, Q_k^K v)_K + \sum_{F\in\mathcal{F}(K)} (\partial_n v, Q_{k-1}^F v)_F \\
&\leq \|\Delta v\|_{0,K} \|Q_k^K v\|_{0,K} + \sum_{F\in\mathcal{F}(K)} \|\partial_n v\|_{0,F} \|Q_{k-1}^F v\|_{0,F}.
\end{aligned}
$$

Applying the similar argument as in Lemma 4.1, we obtain

$$
h_K^2 |v|_{1,K}^2 \lesssim \|Q_k^K v\|_{0,K}^2 + \sum_{F\in\mathcal{F}(K)} h_F \|Q_{k-1}^F v\|_{0,F}^2.
$$

Then it follows from (4.11) that

$$
\|v\|_{0,K}^2 \lesssim \|Q_{k-2}^K v\|_{0,K}^2 + \sum_{F\in\mathcal{F}(K)} h_F \|Q_{k-1}^F v\|_{0,F}^2.
$$

The other side $\|Q_{k-2}^K v\|_{0,K}^2 + \sum_{F\in\mathcal{F}(K)} h_F \|Q_{k-1}^F v\|_{0,F}^2 \lesssim \|v\|_{0,K}^2$ holds from the trace inequality and the inverse inequality (4.9). ∎

4.2. Local inf-sup condition and norm equivalence. With the help of the macro element space $\mathbb{V}_{k-1}^{\mathrm{div}}(K)$, we will present a norm equivalence of space $\nabla V_k(K)$, which is vitally important to design stabilization-free virtual element methods.

LEMMA 4.4. *It holds the inf-sup condition*

$$\|\nabla v\|_{0,K} \lesssim \sup_{\boldsymbol{\phi}\in\mathbb{V}_{k-1}^{\mathrm{div}}(K)} \frac{(\boldsymbol{\phi}, \nabla v)_K}{\|\boldsymbol{\phi}\|_{0,K}} \quad \forall\, v \in V_k(K). \tag{4.13}$$

Consequently

$$\|Q_{K,k-1}^{\mathrm{div}}\nabla v\|_{0,K} \eqsim \|\nabla v\|_{0,K} \quad \forall\, v \in V_k(K). \tag{4.14}$$

Proof. Clearly the norm equivalence (4.14) follows from the local inf-sup condition (4.13). We will focus on the proof of (4.13). Without loss of generality, assume $v \in V_k(K)\cap L_0^2(K)$. Based on DoFs (3.29)-(3.31), take $\boldsymbol{\phi}\in\mathbb{V}_{k-1}^{\mathrm{div}}(K)$ such that

$$\begin{aligned}
(\boldsymbol{\phi}\cdot\boldsymbol{n}, q)_F &= h_K^{-1}(v,q)_F && \forall\, q\in\mathbb{P}_{k-1}(F) \text{ on each } F\in\mathcal{F}(K),\\
(\operatorname{div}\boldsymbol{\phi}, q)_K &= -h_K^{-2}(v,q)_K && \forall\, q\in\mathbb{P}_{\max\{k-2,0\}}(K)/\mathbb{R},\\
(\boldsymbol{\phi}, \boldsymbol{q})_K &= 0 && \forall\, \boldsymbol{q}\in\operatorname{div}\operatorname{skw}\mathring{\boldsymbol{V}}_k^{d-2}(K).
\end{aligned}$$

Then $(\boldsymbol{\phi}\cdot\boldsymbol{n})|_F = h_K^{-1}Q_{k-1}^F v$ for $F\in\mathcal{F}(K)$. Since $\operatorname{div}\boldsymbol{\phi}\in\mathbb{P}_{\max\{k-2,0\}}(K)$, we have $\operatorname{div}\boldsymbol{\phi} - Q_0^K(\operatorname{div}\boldsymbol{\phi}) = -h_K^{-2}Q_{k-2}^K v$. Apply the integration by parts to get

$$\begin{aligned}
(\boldsymbol{\phi}, \nabla v)_K &= -(\operatorname{div}\boldsymbol{\phi}, v)_K + (\boldsymbol{\phi}\cdot\boldsymbol{n}, v)_{\partial K}\\
&= -(\operatorname{div}\boldsymbol{\phi} - Q_0^K(\operatorname{div}\boldsymbol{\phi}), v)_K + \sum_{F\in\mathcal{F}(K)} (\boldsymbol{\phi}\cdot\boldsymbol{n}, Q_{k-1}^F v)_F\\
&= h_K^{-2}\|Q_{k-2}^K v\|_{0,K}^2 + \sum_{F\in\mathcal{F}(K)} h_K^{-1}\|Q_{k-1}^F v\|_{0,F}^2.
\end{aligned}$$

By the norm equivalence (4.12), we get

$$\|\nabla v\|_{0,K}^2 \lesssim h_K^{-2}\|Q_{k-2}^K v\|_{0,K}^2 + \sum_{F\in\mathcal{F}(K)} h_K^{-1}\|Q_{k-1}^F v\|_{0,F}^2 = (\boldsymbol{\phi}, \nabla v)_K. \tag{4.15}$$

On the other hand, it follows from the integration by parts that

$$\begin{aligned}
\|Q_0^K(\operatorname{div}\boldsymbol{\phi})\|_{0,K} &\lesssim h_K^{d/2}\big|Q_0^K(\operatorname{div}\boldsymbol{\phi})\big| \lesssim h_K^{-d/2}\big|(\operatorname{div}\boldsymbol{\phi}, 1)_K\big| = h_K^{-d/2}\big|(\boldsymbol{\phi}\cdot\boldsymbol{n}, 1)_{\partial K}\big|\\
&\lesssim \sum_{F\in\mathcal{F}(K)} h_F^{-1/2}\|\boldsymbol{\phi}\cdot\boldsymbol{n}\|_{0,F}.
\end{aligned}$$

Employing the norm equivalence (3.32), we acquire

$$\begin{aligned}
\|\boldsymbol{\phi}\|_{0,K} &\eqsim h_K\|\operatorname{div}\boldsymbol{\phi}\|_{0,K} + \sum_{F\in\mathcal{F}(K)} h_F^{1/2}\|\boldsymbol{\phi}\cdot\boldsymbol{n}\|_{0,F}\\
&\lesssim h_K\|\operatorname{div}\boldsymbol{\phi} - Q_0^K(\operatorname{div}\boldsymbol{\phi})\|_{0,K} + \sum_{F\in\mathcal{F}(K)} h_F^{1/2}\|\boldsymbol{\phi}\cdot\boldsymbol{n}\|_{0,F}\\
&\lesssim h_K^{-1}\|Q_{k-2}^K v\|_{0,K} + \sum_{F\in\mathcal{F}(K)} h_F^{-1/2}\|Q_{k-1}^F v\|_{0,F}^2.
\end{aligned}$$

Then we obtain from the norm equivalence (4.12) and the Poincaré-Friedrichs inequality [11, (2.14)] that

$$\|\boldsymbol{\phi}\|_{0,K} \lesssim h_K^{-1}\|v\|_{0,K} \lesssim \|\nabla v\|_{0,K}.$$

Finally we conclude (4.13) from (4.15) and the last inequality. ∎

4.3. Discrete method. Define the global nonconforming virtual element space

$$V_h := \{v_h \in L^2(\Omega) : v_h|_K \in V_k(K) \text{ for each } K \in \mathcal{T}_h,$$
$$\text{DoF (4.3) is single-valued for } F \in \mathcal{F}_h, \text{ and vanishes for } F \in \mathcal{F}_h^\partial\}.$$

We have the discrete Poincaré inequality (cf. [15, (4.16)])

$$\|v_h\|_0 \lesssim |v_h|_{1,h} \quad \forall\, v_h \in V_h. \tag{4.16}$$

Based on the weak formulation (4.2), we propose a stabilization free virtual element method for problem (4.1) as follows: find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = (f, Q_h v_h) \quad \forall\, v_h \in V_h, \tag{4.17}$$

where the discrete bilinear form

$$a_h(u_h, v_h) := (Q_{h,k-1}^{\text{div}} \nabla_h u_h, Q_{h,k-1}^{\text{div}} \nabla_h v_h) + \alpha(Q_h u_h, Q_h v_h).$$

Remark 4.5. By introducing $\boldsymbol{\phi}_h = Q_{h,k-1}^{\text{div}} \nabla_h u_h$, the virtual element method (4.17) can be rewritten as the following primal mixed virtual element method: find $\boldsymbol{\phi}_h \in \mathbb{V}_{h,k-1}^{\text{div}}$ and $u_h \in V_h$ such that

$$\begin{aligned} (\boldsymbol{\phi}_h, \boldsymbol{\psi}_h) - (\boldsymbol{\psi}_h, \nabla_h u_h) &= 0 && \forall\, \boldsymbol{\psi}_h \in \mathbb{V}_{h,k-1}^{\text{div}}, \\ (\boldsymbol{\phi}_h, \nabla_h v_h) + \alpha(Q_h u_h, Q_h v_h) &= (f, Q_h v_h) && \forall\, v_h \in V_h. \end{aligned}$$

It follows from the discrete Poincaré inequality (4.16) that

$$a_h(u_h, v_h) \lesssim |u_h|_{1,h} |v_h|_{1,h} \quad \forall\, u_h, v_h \in H_0^1(\Omega) + V_h. \tag{4.18}$$

Lemma 4.6. *It holds the coercivity*

$$|v_h|_{1,h}^2 \lesssim a_h(v_h, v_h) \quad \forall\, v_h \in V_h. \tag{4.19}$$

Proof. Due to (4.14), we have

$$\sum_{K\in\mathcal{T}_h} \|\nabla_h v_h\|_{0,K}^2 \lesssim \sum_{K\in\mathcal{T}_h} \|Q_{K,k-1}^{\text{div}} \nabla v_h\|_{0,K}^2 \le a_h(v_h, v_h) \quad \forall\, v_h \in V_h,$$

which implies the coercivity (4.19). □

Theorem 4.7. *The stabilization free virtual element method* (4.17) *is uni-solvent.*

Proof. Thanks to the boundedness (4.18) and the coercivity (4.19), we conclude the result from the Lax-Milgram lemma [23]. □

4.4. Error analysis.

Theorem 4.8. *Let $u \in H_0^1(\Omega)$ be the solution of problem* (4.1), *and $u_h \in V_h$ be the solution of the virtual element method* (4.17). *Assume $u \in H^{k+1}(\Omega)$ and $f \in H^{k-1}(\Omega)$. Then*

$$|u - u_h|_{1,h} \lesssim h^k (|u|_{k+1} + |f|_{k-1}). \tag{4.20}$$

Proof. Take any $v_h \in V_h$. Recall the consistency error estimate in [15, Lemma 5.5]

$$a(u, v_h - u_h) + (f, v_h - u_h) \lesssim h^k |u|_{k+1} |v_h - u_h|_{1,h}.$$

Then

$$\begin{aligned} & a(u, v_h - u_h) - (f, Q_h(v_h - u_h)) \\ &= a(u, v_h - u_h) + (f, v_h - u_h) + (f - Q_h f, v_h - u_h) \\ &= a(u, v_h - u_h) + (f, v_h - u_h) + (f - Q_h f, v_h - u_h - Q_h^0(v_h - u_h)) \\ &\lesssim h^k (|u|_{k+1} + |f|_{k-1}) |v_h - u_h|_{1,h}. \end{aligned}$$

By the definitions of $a_h(\cdot, \cdot)$ and $a(\cdot, \cdot)$, it follows from the discrete Poincaré inequality (4.16) that

$$\begin{aligned} & a_h(v_h, v_h - u_h) - a(u, v_h - u_h) \\ &= (Q^{\mathrm{div}}_{h,k-1} \nabla_h v_h, Q^{\mathrm{div}}_{h,k-1} \nabla_h (v_h - u_h)) - (\nabla u, \nabla_h (v_h - u_h)) \\ &\quad + \alpha(Q_h v_h, Q_h(v_h - u_h)) - \alpha(u, v_h - u_h) \\ &= (Q^{\mathrm{div}}_{h,k-1} \nabla_h v_h - \nabla u, \nabla_h (v_h - u_h)) + \alpha(Q_h v_h - u, v_h - u_h) \\ &\lesssim (\|\nabla u - Q^{\mathrm{div}}_{h,k-1} \nabla_h v_h\|_0 + \|u - Q_h v_h\|_0) |v_h - u_h|_{1,h}. \end{aligned}$$

Summing the last two inequlities, we get from the coercivity (4.19) and (4.17) that

$$\begin{aligned} |v_h - u_h|^2_{1,h} \lesssim a_h(v_h - u_h, v_h - u_h) &= a_h(v_h, v_h - u_h) - (f, Q_h(v_h - u_h)) \\ &\lesssim h^k (|u|_{k+1} + |f|_{k-1}) |v_h - u_h|_{1,h} \\ &\quad + (\|\nabla u - Q^{\mathrm{div}}_{h,k-1} \nabla_h v_h\|_0 + \|u - Q_h v_h\|_0) |v_h - u_h|_{1,h}, \end{aligned}$$

which implies

$$|v_h - u_h|_{1,h} \lesssim h^k (|u|_{k+1} + |f|_{k-1}) + \|\nabla u - Q^{\mathrm{div}}_{h,k-1} \nabla v_h\|_0 + \|u - Q_h v_h\|_0.$$

Since $\mathbb{P}_{k-1}(K; \mathbb{R}^d) \subseteq \mathbb{V}^{\mathrm{div}}_{k-1}(K)$ for $K \in \mathcal{T}_h$, we have $Q^{\mathrm{div}}_{h,k-1}(Q_h^{k-1} \nabla u) = Q_h^{k-1} \nabla u$. Hence

$$\begin{aligned} \|\nabla u - Q^{\mathrm{div}}_{h,k-1} \nabla_h v_h\|_0 &\le \|\nabla u - Q^{\mathrm{div}}_{h,k-1} \nabla u\|_0 + \|Q^{\mathrm{div}}_{h,k-1}(\nabla u - \nabla_h v_h)\|_0 \\ &= \|\nabla u - Q_h^{k-1} \nabla u - Q^{\mathrm{div}}_{h,k-1}(\nabla u - Q_h^{k-1} \nabla u)\|_0 \\ &\quad + \|Q^{\mathrm{div}}_{h,k-1}(\nabla u - \nabla_h v_h)\|_0 \\ &\le \|\nabla u - Q_h^{k-1} \nabla u\|_0 + |u - v_h|_{1,h}. \end{aligned}$$

Similarly, we have

$$\|u - Q_h v_h\|_0 \le \|u - Q_h u\|_0 + \|Q_h(u - v_h)\|_0 \le \|u - Q_h u\|_0 + \|u - v_h\|_0.$$

By combining the last three inequalities, we obtain

$$|v_h - u_h|_{1,h} \lesssim h^k (|u|_{k+1} + |f|_{k-1}) + \|u - v_h\|_{1,h},$$

which together with the triangle inequality yields

$$|u - u_h|_{1,h} \lesssim h^k (|u|_{k+1} + |f|_{k-1}) + \inf_{v_h \in V_h} \|u - v_h\|_{1,h}.$$

At last, (4.20) follows from the approximation of V_h [15]. ∎

5. Stabilization-free conforming virtual element method. In this section we will develop a stabilization-free conforming virtual element method for the second order elliptic problem (4.1) in two dimensions.

For polygon $K \subset \mathbb{R}^2$, let $\mathcal{V}(K)$ be the set of all vertices of K. And we overload notation $\mathcal{E}(K)$ to denote the set of all edges of K in this section.

5.1. H^1-conforming virtual element. Recall the H^1-conforming virtual element in [22, 1, 6, 7]. The degrees of freedom are given by

$$v(\delta) \quad \forall\ \delta \in \mathcal{V}(K), \tag{5.1}$$

$$\frac{1}{|e|}(v,q)_e \quad \forall\ q \in \mathbb{P}_{k-2}(e), e \in \mathcal{E}(K), \tag{5.2}$$

$$\frac{1}{|K|}(v,q)_K \quad \forall\ q \in \mathbb{P}_{k-2}(K). \tag{5.3}$$

And the space of shape functions is

$$\begin{aligned} V_k(K) := \big\{ v \in H^1(K) : \Delta v \in \mathbb{P}_k(K), v|_{\partial K} \in H^1(\partial K), v|_e \in \mathbb{P}_k(e)\ \ \forall\ e \in \mathcal{E}(K), \\ \text{and } (v - \Pi_k^K v, q)_K = 0 \quad \forall\ q \in \mathbb{P}_k(K)/\mathbb{P}_{k-2}(K)\big\}, \end{aligned}$$

where Π_k^K is defined by (4.5)-(4.6). It holds $\mathbb{P}_k(K) \subseteq V_k(K)$.

For $v \in V_k(K)$, the H^1 projection $\Pi_k^K v$ and the L^2 projection $Q_k^K v = \Pi_k^K v + Q_{k-2}^K v - Q_{k-2}^K \Pi_k^K v$ are computable using only DoFs (5.1)-(5.3). We have the norm equivalence of space $V_k(K)$ (cf. [22, Lemma 4.7] and [14, 11, 5]), that is for $v \in V_k(K)$, it holds

$$h_K^2 |v|_{1,K}^2 \lesssim \|v\|_{0,K}^2 \eqsim \|Q_{k-2}^K v\|_{0,K}^2 + \sum_{\delta \in \mathcal{V}(K)} h_K^2 |v(\delta)|^2 + \sum_{e \in \mathcal{E}(K)} h_K \|Q_{k-2}^e v\|_{0,e}^2. \tag{5.4}$$

Employing the same argument as in Lemma 4.4, from (5.4), we get the norm equivalence

$$\|Q_{K,k}^{\mathrm{div}} \nabla v\|_{0,K} \eqsim \|\nabla v\|_{0,K} \quad \forall\ v \in V_k(K). \tag{5.5}$$

Remark 5.1. When $k \geq 2$, we can replace $Q_{K,k}^{\mathrm{div}}$ by the L^2-orthogonal projection operator onto space $\mathbb{V}_{k,k-2}^{\mathrm{div}}(K)$, where

$$\begin{aligned} \mathbb{V}_{k,k-2}^{\mathrm{div}}(K) :=& \{\boldsymbol{\phi} \in \mathbb{V}_k^{\mathrm{div}}(K) : \operatorname{div} \boldsymbol{\phi} \in \mathbb{P}_{k-2}(K)\} \\ =& \{\boldsymbol{\phi} \in \boldsymbol{V}_k^{BDM}(K) : \operatorname{div} \boldsymbol{\phi} \in \mathbb{P}_{k-2}(K), \boldsymbol{\phi} \cdot \boldsymbol{n}|_e \in \mathbb{P}_k(e)\ \ \forall\ e \in \mathcal{E}(K)\}. \end{aligned}$$

5.2. Discrete method. Define the global conforming virtual element space

$$V_h := \{v_h \in H_0^1(\Omega) : v_h|_K \in V_k(K) \text{ for each } K \in \mathcal{T}_h\}.$$

Based on the weak formulation (4.2), we propose a stabilization free virtual element method for problem (4.1) as follows: find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = (f, Q_h v_h) \quad \forall\ v_h \in V_h, \tag{5.6}$$

where the discrete bilinear form

$$a_h(u_h, v_h) := (Q_{h,k}^{\mathrm{div}} \nabla u_h, Q_{h,k}^{\mathrm{div}} \nabla v_h) + \alpha (Q_h u_h, Q_h v_h).$$

It is obvious that

$$a_h(u,v) \lesssim |u|_1|v|_1 \quad \forall\, u,v \in H_0^1(\Omega). \tag{5.7}$$

And the norm equivalence (5.5) implies the coercivity

$$|v_h|_1^2 \lesssim a_h(v_h,v_h) \quad \forall\, v_h \in V_h. \tag{5.8}$$

Therefore the stabilization-free virtual element method (5.6) is uni-solvent.

Remark 5.2. By introducing $\boldsymbol{\phi}_h = Q_{h,k}^{\mathrm{div}}\nabla u_h$, the virtual element method (5.6) can be rewritten as the following primal mixed virtual element method: find $\boldsymbol{\phi}_h \in \mathbb{V}_{h,k}^{\mathrm{div}}$ and $u_h \in V_h$ such that

$$\begin{aligned}(\boldsymbol{\phi}_h,\boldsymbol{\psi}_h) - (\boldsymbol{\psi}_h,\nabla u_h) &= 0 && \forall\, \boldsymbol{\psi}_h \in \mathbb{V}_{h,k}^{\mathrm{div}},\\ (\boldsymbol{\phi}_h,\nabla v_h) + \alpha(Q_hu_h,Q_hv_h) &= (f,Q_hv_h) && \forall\, v_h \in V_h.\end{aligned}$$

5.3. Error analysis.

THEOREM 5.3. *Let $u \in H_0^1(\Omega)$ be the solution of problem* (4.1), *and $u_h \in V_h$ be the solution of the virtual element method* (5.6). *Assume $u \in H^{k+1}(\Omega)$ and $f \in H^{k-1}(\Omega)$. Then*

$$|u-u_h|_1 \lesssim h^k(|u|_{k+1} + |f|_{k-1}). \tag{5.9}$$

Proof. Take any $v_h \in V_h$. Applying the same argument as in Theorem 4.8, we have

$$\|\nabla u - Q_{h,k}^{\mathrm{div}}\nabla v_h\|_0 + \|u - Q_hv_h\|_0 \lesssim \|\nabla u - Q_h^{k-1}\nabla u\|_0 + \|u-Q_hu\|_0 + |u-v_h|_1,$$

$$a_h(v_h,v_h-u_h) - a(u,v_h-u_h) \lesssim \|\nabla u - Q_{h,k}^{\mathrm{div}}\nabla v_h\|_0|v_h-u_h|_1 + \|u-Q_hv_h\|_0\|v_h-u_h\|_0.$$

Combining the last two inequalities gives

$$a_h(v_h,v_h-u_h) - a(u,v_h-u_h) \lesssim (\|\nabla u - Q_h^{k-1}\nabla u\|_0 + \|u-Q_hu\|_0 + |u-v_h|_1)|v_h-u_h|_1.$$

By the coercivity (5.8), (5.6), (4.2) and the error estimate of Q_h^0,

$$\begin{aligned}|v_h-u_h|_1^2 &\lesssim a_h(v_h-u_h,v_h-u_h) = a_h(v_h,v_h-u_h) - (f,Q_h(v_h-u_h))\\ &\lesssim a_h(v_h,v_h-u_h) - a(u,v_h-u_h) + (f-Q_hf,v_h-u_h)\\ &\lesssim a_h(v_h,v_h-u_h) - a(u,v_h-u_h) + h\|f-Q_hf\|_0|v_h-u_h|_1.\end{aligned}$$

Hence we get from the triangle inequality and the last two inequalities that

$$\begin{aligned}|u-u_h|_1 &\le |u-v_h|_1 + |v_h-u_h|_1\\ &\lesssim \|\nabla u - Q_h^{k-1}\nabla u\|_0 + \|u-Q_hu\|_0 + |u-v_h|_1 + h\|f-Q_hf\|_0.\end{aligned}$$

By the arbitrariness of $v_h \in V_h$, we derive

$$|u-u_h|_1 \lesssim \|\nabla u - Q_h^{k-1}\nabla u\|_0 + \|u-Q_hu\|_0 + h\|f-Q_hf\|_0 + \inf_{v_h\in V_h}|u-v_h|_1.$$

At last, (5.9) follows from the last inequality, the error estimates of Q_h^{k-1} and Q_h, and the approximation of V_h [22, 14, 11, 5]. ∎

FIG. 1. *Convex polygon mesh $\mathcal{T}_0$ (left) and non-convex polygon mesh $\mathcal{T}_1$ (right).*

6. Numerical results.

In this section, we will numerically verify the convergence of the nonconforming virtual element method (4.17) and the conforming virtual element method (5.6). The numerical example is implemented by using the FEALPy package [25]

Consider the second order elliptic problem (4.1) with $\alpha = 2$ on rectangular domain $\Omega = (0, 1) \times (0, 1)$. The exact solution and source term are given by

$$u = \sin(\pi x)\sin(\pi y), \quad f = (2\pi^2 + 2)\sin(\pi x)\sin(\pi y).$$

The rectangular domain Ω is partitioned by the convex polygon mesh $\mathcal{T}_0$ and non-convex polygon mesh $\mathcal{T}_1$, respectively, In both virtual element methods (4.17) and (5.6), we choose $k = 1, 2, 5$. The numerical results of the nonconforming virtual element method (4.17) on meshes $\mathcal{T}_0$ and $\mathcal{T}_1$ mesh are shown in Figure 2. We can see that $\|u - Q_h u_h\|_0 = O(h^{k+1})$ and $\|\nabla u - Q^{\mathrm{div}}_{h,k-1}\nabla_h u_h\|_0 = O(h^k)$, which coincide with Theorem 4.8. And the numerical results of the conforming virtual element method (5.6) are presented in Figure 3. Again $\|u - Q_h u_h\|_0 = O(h^{k+1})$ and $\|\nabla u - Q^{\mathrm{div}}_{h,k}\nabla u_h\|_0 = O(h^k)$, which confirm the theoretical convergence rate in Theorem 5.3.

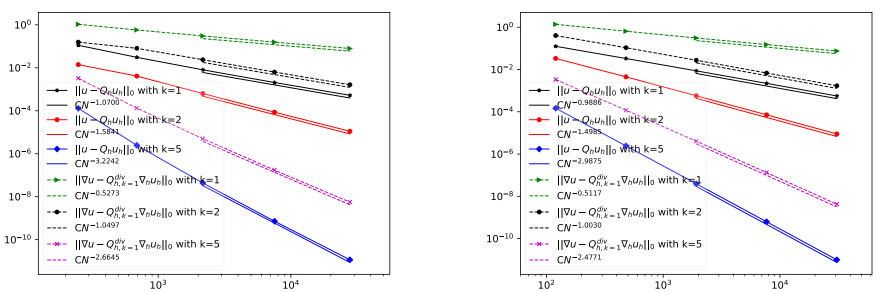

FIG. 2. *Errors $\|u - Q_h u_h\|_0$ and $\|\nabla u - Q^{\mathrm{div}}_{h,k-1}\nabla_h u_h\|_0$ of nonconforming virtual element method* (4.17) *on $\mathcal{T}_0$ (left) and $\mathcal{T}_1$ (right) with $k = 1, 2, 5$.*

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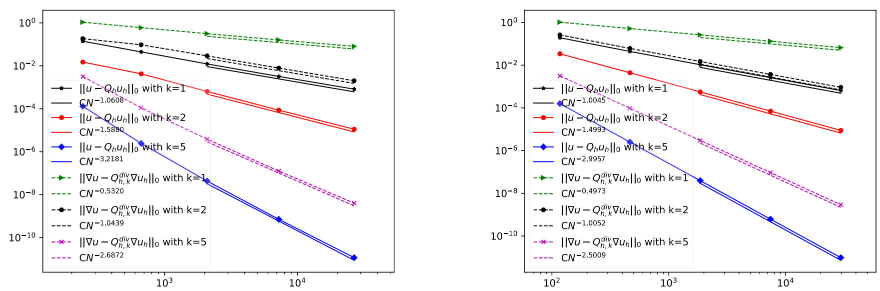

FIG. 3. *Errors* $\|u - Q_h u_h\|_0$ *and* $\|\nabla u - Q_{h,k}^{\mathrm{div}} \nabla u_h\|_0$ *of conforming virtual element method* (5.6) *on* $\mathcal{T}_0$ *(left) and* $\mathcal{T}_1$ *(right) with* $k = 1, 2, 5$.

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