Lowest order stabilization free Virtual Element Method for the Poisson equation

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Abstract

We introduce and analyse the first order Enlarged Enhancement Virtual Element Method (E^2VEM) for the Poisson problem. The method has the interesting property of allowing the definition of bilinear forms that do not require a stabilization term. We provide a proof of well-posedness and optimal order a priori error estimates. Numerical tests on convex and non-convex polygonal meshes confirm the theoretical convergence rates.

1 Introduction

In recent years, the study of polygonal methods for solving partial differential equations has received a huge attention. The main reason for this great interest relies in the flexibility of polygonal meshes to discretize domains with high geometrical complexity. A large number of families of polygonal/polyhedral methods has been developed, among them we can list Discontinuous Galerkin Methods [29, 40, 36], Polygonal Finite Elements (PFEM) [44], Mimetic Finite Difference Methods (MFD) [9, 24, 45], Hybrid High Order Methods (HHO) [30, 31, 32], Gradient Discretisation Methods [34, 33], CutFEM [26], other methods that help in circumventing geometrical complexities are Extended FEMs (XFEM) [38], Generalised FEMs (GFEM) [41, 43, 42] as well as Ficticious Domain Methods [35, 5], Immersed Boundary Methods [39], PDE-constrained Optimization Methods [20, 19, 21] and many others. One of the most recent developments in this field is the family of the Virtual Element Methods (VEM). These methods were first introduced in primal conforming form in [6] and were later on applied to most of

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the relevant problems of interest in applications, such as advection-diffusion-reaction equations [7, 15, 18], elastic and inelastic problems [10], plate bending problems [25], parabolic and hyperbolic problems [47, 46], simulations in fractured media [16, 14, 13].

Standard VEM discrete bilinear forms are the sum of a singular part maintaining consistency on polynomials and a stabilizing form enforcing coercivity. In the literature, the stabilization term has been extensively studied, for instance in [11], and remains a somehow arbitrarily chosen component of the method with several possible effects on the stability and conditioning of the method. Moreover, the stabilization term causes issues in many theoretical contexts. The first one that we mention is the derivation of a posteriori error estimates [27, 17], where the stabilization term is always at the right-hand side when bounding the error in terms of the error estimator, both from above and from below. Moreover, the isotropic nature of the stabilization term becomes an issue when devising SUPG stabilizations [15, 18], in problems with anisotropic coefficients, or in the derivation of anisotropic a posteriori error estimates [3]. Finally, other contexts in which the stabilization may induce problems are multigrid analysis [4] and complex non-linear problems [37].

In this work, we introduce a new family of VEM, that we call Enlarged Enhancement Virtual Element Methods (E^2VEM), designed to allow the definition of a coercive bilinear form that involves only polynomial projections. In this framework, it is not required to add an arbitrary stabilizing bilinear form accounting for the non polynomial part of VEM functions. The method is based on the use of higher order polynomial projections in the discrete bilinear form with respect to the standard one [7] and on a modification of the VEM space to allow the computation of such projections. In particular, we extend the enhancement property that is used in the definition of the VEM space ([1], [7]). Indeed, the name of the method comes from this enlarged enhancement property. The degree of polynomial enrichment is chosen locally on each polygon, such that the discrete bilinear form is coercive, and depends on the number of vertices of the polygon. The resulting discrete functional space has the same set of degrees of freedom of the one defined in [7].

The proof of well-posedness is quite elaborate, thus in this paper we choose to deal only with the lowest order formulation and, for the sake of simplicity, we focus on the two dimensional Poisson's problem with homogenous Dirichlet boundary conditions, the extension to general boundary conditions being analogous to what is done for classical VEM. Moreover, the formulation and proofs presented in this work can also be easily extended to the case of a non constant anisotropic diffusion tensor. The extension to an higher order formulation will be the focus of an upcoming work.

The outline of the paper is as follows. In section 2 we state our model problem. In section 3 we introduce the approximation functional spaces and projection operators and we state the discrete problem. Section 4 contains the discussion about the well-posedness of the discrete problem under suitable sufficient conditions on the local projections. In section 5 we prove optimal order H^1 a priori error estimates and in the supplementary materials the L^2 case. Section 6 contains some numerical results assessing the rates of convergence of the method.

Throughout the work, we denote by $(\cdot,\cdot)_\omega$ the standard L^2 scalar product defined on a generic $\omega \subset \mathbb{R}^2$, by $\gamma^{\partial\omega}$ the trace operator, that restricts on the boundary $\partial\omega$ an element of a space defined over $\omega \subset \mathbb{R}^2$. Inside the proofs, we decide to use a single character C for constants, independent of the mesh size, that appear in the inequalities, which means that we suppose to take at each step the maximum of the constants involved.

2 Model Problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded open set. We are interested in solving the following problem:

$$\begin{cases} -\Delta U = f & \text{in } \Omega, \\ U = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Defining $a\colon \mathrm{H}^1_0(\Omega) \times \mathrm{H}^1_0(\Omega) \to \mathbb{R}$ such that,

$$a\left(U, W\right) := \left(\nabla U, \nabla W\right)_\Omega \quad \forall U, W \in \mathrm{H}^1_0(\Omega), \tag{2}$$

then, given $f \in \mathrm{L}^2(\Omega)$, the variational formulation of (1) is given by: find $U \in \mathrm{H}^1_0(\Omega)$ such that,

$$a\left(U, W\right) = \left(f, W\right)_\Omega \quad \forall W \in \mathrm{H}^1_0(\Omega)\,. \tag{3}$$

3 Discrete formulation

In order to define the discrete form of (3), we denote by $\mathcal{M}_h$ a conforming polygonal tessellation of Ω and by E a generic polygon of $\mathcal{M}_h$. We denote by $\#\mathcal{M}_h$ the number of polygons of $\mathcal{M}_h$ and by h the maximum diameter of all the polygons in $\mathcal{M}_h$. Let $\{x_i\}_{i=1}^{N_E^V}$ be the N_E^V vertices of E, $\mathcal{E}_E$ the set of its edges and $\boldsymbol{n}^e = (n^e_x, n^e_y)$ the outward-pointing unit normal vector to the edge e of E. We assume that $\mathcal{M}_h$ satifies the standard mesh assumptions for VEM (see for instance [11, 23]), i.e. $\exists \kappa > 0$ such that

1. for all $E \in \mathcal{M}_h$, E is star-shaped with respect to a ball of radius $\rho \geq \kappa h_E$, where h_E is the diameter of E;

2. for all edges $e \subset \partial E$, $|e| \geq \kappa h_E$.

Notice that the above conditions imply that, denoting by N_E^V the number of vertices of E, it holds

$$\exists N_{\max}^V > 0 \colon \forall E \in \mathcal{M}_h,\ N_E^V \leq N_{\max}^V\,. \tag{4}$$

For any given $E \in \mathcal{M}_h$, let $\mathbb{P}_k(E)$ be the space of polynomials of degree k defined on E. Let $\Pi_{1,E}^{\nabla} : \mathrm{H}^1(E) \to \mathbb{P}_1(E)$ be the $\mathrm{H}^1(E)$-orthogonal operator, defined up to a constant by the orthogonality condition: $\forall\, u \in \mathrm{H}^1(E)$,

$$\left(\nabla\left(\Pi_{1,E}^{\nabla} u - u\right), \nabla p\right)_E = 0 \quad \forall\, p \in \mathbb{P}_1(E)\,. \tag{5}$$

In order to define $\Pi_{1,E}^{\nabla}$ uniquely, we choose any continuous and linear projection operator $\mathrm{P}_0 : \mathrm{H}^1(E) \to \mathbb{P}_0(E)$, whose continuity constant in H^1-norm is independent of h_E and continuous with respect to deformations of the geometry, and we impose $\forall\, u \in \mathrm{H}^1(E)$,

$$\mathrm{P}_0(\Pi_{1,E}^{\nabla} u - u) = 0. \tag{6}$$

Remark 1. *Under the current mesh assumptions, a suitable choice for* P_0 *is the integral mean on the boundary of* E*, i.e.*

$$\mathrm{P}_0(u) := \frac{1}{|\partial E|} \int\limits_{\partial E} \gamma^{\partial E}(u)\, ds \quad \forall u \in \mathrm{H}^1(E).$$

Notice that this is a common choice, see for instance [7].

For any given $E \in \mathcal{M}_h$, let $l \in \mathbb{N}$ be given, as detailed in the next section. Let $\mathcal{EN}_{1,l}^E$ be the set of functions $v \in \mathrm{H}^1(E)$ satisfying

$$(v, p)_E = \left(\Pi_{1,E}^{\nabla} v, p\right)_E \ \forall p \in \mathbb{P}_{l+1}(E)\ . \tag{7}$$

We define the Enlarged Enhancement Virtual Space of order 1 as

$$\mathcal{V}_{1,l}^E := \left\{v \in \mathcal{EN}_{1,l}^E : \Delta v \in \mathbb{P}_{l+1}(E)\,,\ \ \gamma^e(v) \in \mathbb{P}_1(e)\ \forall e \in \mathcal{E}_E,\ \ v \in C^0(\partial E)\right\}.$$

We define as degrees of freedom of this space the values of functions at the vertices of E (see [6, 7]).

Moreover, let $\boldsymbol{\ell} \in \mathbb{N}^{\#\mathcal{M}_h}$ be a vector and denote by $\boldsymbol{\ell}(E)$ the element corresponding to the polygon E, we define the global discrete space as

$$\mathcal{V}_{1,\boldsymbol{\ell}} := \left\{v \in \mathrm{H}_0^1(\Omega) \colon v_{|E} \in \mathcal{V}_{1,l}^E, \text{ where } l = \boldsymbol{\ell}(E)\right\}.$$

Note that $v \in \mathcal{V}_{1,\boldsymbol{\ell}}$ is a continuous function that is a polynomial of degree 1 on each edge of the mesh.

To define our discrete bilinear form, let $\Pi_{l,E}^0 \nabla : \mathrm{H}^1(E) \to [\mathbb{P}_l(E)]^2$ be the $\mathrm{L}^2(E)$-projection operator of the gradient of functions in $\mathrm{H}^1(E)$, defined, $\forall u \in \mathrm{H}^1(E)$, by the orthogonality condition

$$\left(\Pi_{l,E}^0 \nabla u, \boldsymbol{p}\right)_E = (\nabla u, \boldsymbol{p})_E \quad \forall \boldsymbol{p} \in [\mathbb{P}_l(E)]^2\ . \tag{8}$$

Remark 2. *For each function $u \in \mathcal{V}_{1,l}^E$, the above projection is computable given the degrees of freedom of u, applying the Gauss-Green formula and exploiting* (7).

Let $a_h^E \colon \mathcal{V}_{1,l}^E \times \mathcal{V}_{1,l}^E \to \mathbb{R}$ be defined as

$$a_h^E(u,v) := \left(\Pi_{l,E}^0 \nabla u, \Pi_{l,E}^0 \nabla v\right)_E \quad \forall u,v \in \mathcal{V}_{1,l}^E\,,$$

and $a_h \colon \mathcal{V}_{1,\boldsymbol{\ell}} \times \mathcal{V}_{1,\boldsymbol{\ell}} \to \mathbb{R}$ as

$$a_h(u,v) := \sum_{E \in \mathcal{M}_h} a_h^E(u,v) \quad \forall u,v \in \mathcal{V}_{1,\boldsymbol{\ell}}\,. \tag{9}$$

We can state the discrete problem as: find $u \in \mathcal{V}_{1,\boldsymbol{\ell}}$ such that

$$a_h(u,v) = \sum_{E \in \mathcal{M}_h} \left(f, \Pi_{0,E}^0 v\right)_E \quad \forall v \in \mathcal{V}_{1,\boldsymbol{\ell}}\,, \tag{10}$$

where, $\forall E \in \mathcal{M}_h$, $\Pi_{0,E}^0 \colon \mathrm{L}^2(E) \to \mathbb{R}$ is the $\mathrm{L}^2(E)$-projection, defined by

$$\Pi_{0,E}^0 v := \frac{1}{|E|}(v,1)_E \quad \forall v \in \mathcal{V}_{1,l}^E\,. \tag{11}$$

The above projection is computable for any given $v \in \mathcal{V}_{1,l}^E$ exploiting (7).

4 Well-posedness

This section is devoted to prove the well-posedness of the discrete problem stated by (10), under suitable sufficient conditions on $\boldsymbol{\ell}$. The main result is given by Theorem 1, that induces the existence of an equivalent norm on $\mathcal{V}_{1,\boldsymbol{\ell}}$, which implies the well-posedness of (10).

Theorem 1. *Let $E \in \mathcal{M}_h$, $u \in \mathcal{V}_{1,l}^E$ and $l \in \mathbb{N}$ such that*

$$(l+1)(l+2) \geq N_E^V - 1, \tag{12}$$

then

$$\Pi_{l,E}^0 \nabla u = 0 \implies \nabla u_{|E} = 0. \tag{13}$$

We omit in the following the proof of the case of triangles ($N_E^V = 3$ and $l = 0$), indeed this case can be led back to classical results. Then, for technical reasons, the proof of Theorem 1 in the case $N_E^V > 3$ is split into two results, described in Section 4.1 and in Section 4.2, respectively. The proof relies on an auxiliary inf-sup condition that is proved by constructing a suitable Fortin operator, whose existence is guaranteed under condition (12).

4.1 Auxiliary inf-sup condition

In this section, after some auxiliary results, we prove through Proposition 1 that (13) is satisfied if the auxiliary inf-sup condition (24) holds true.

Lemma 1. *Let* $u \in \mathcal{V}^E_{1,l}$*, with* $l \geq 0$*. Then*

$$\Pi^0_{l,E} \nabla u = 0 \implies \Pi^{\nabla}_{1,E} u \in \mathbb{P}_0(E)\,.$$

Proof. Applying (8), we have

$$\Pi^0_{l,E} \nabla u = 0 \implies (\nabla u, \boldsymbol{p})_E = 0 \ \ \forall \boldsymbol{p} \in \left[\mathbb{P}_l(E)\right]^2,$$

that implies

$$(\nabla u, \nabla p)_E = 0 \ \ \forall p \in \mathbb{P}_1(E)\,, \tag{14}$$

thanks to the relation $\nabla \mathbb{P}_1(E) \subseteq \nabla \mathbb{P}_{l+1}(E) \subseteq \left[\mathbb{P}_l(E)\right]^2$. Given (14) and (5),

$$\begin{aligned}\left(\nabla \Pi^{\nabla}_{1,E} u, \nabla p\right)_E = 0 \ \ \forall p \in \mathbb{P}_1(E) &\implies \nabla \Pi^{\nabla}_{1,E} u = 0 \\ &\implies \Pi^{\nabla}_{1,E} u \in \mathbb{P}_0(E)\,.\end{aligned}$$

□

Lemma 2. *Let* $u \in \mathcal{V}^E_{1,l}$*. If* $\Pi^0_{l,E} \nabla u = 0$*, then* (7) *can be rewritten as*

$$(u, p)_E = \mathrm{P}_0\left(u\right) \cdot (1, p)_E \ \ \forall p \in \mathbb{P}_{l+1}(E)\,, \tag{15}$$

where P_0 *is the projection operator chosen in Section 3.*

Proof. Applying Lemma 1 and (6),

$$\Pi^0_{l,E} \nabla u = 0 \implies \Pi^{\nabla}_{1,E} u = \mathrm{P}_0\left(u\right).$$

Then, (7) provides (15). □

We now need to introduce some notations and definitions. First, we denote by $\mathcal{T}_E$ the triangulation of E obtained linking each vertex of E to the centre of the ball with respect to which E is star-shaped, denoted by x_C. Let us define the set of internal edges of the triangulation $\mathcal{T}_E$ as $\mathcal{I}_{\mathcal{E}_E}$. For any $i = 1, \ldots, N^V_E$, let $\tau_i \in \mathcal{T}_E$ be the triangle whose vertices are x_i, x_{i+1} and x_C. We denote by e_i the edge $\overrightarrow{x_C x_i} \in \mathcal{I}_{\mathcal{E}_E}$ and by $\boldsymbol{n}^{e_i}$ the outward-pointing unit normal vector to the edge e_i of τ_i. Then, for each polygon E, we can define the reference polygon $\hat{E}$, such that the mapping $F : \hat{E} \to E$ is given by

$$x = h_E \hat{x} + x_C. \tag{16}$$

Let Σ be the set of all admissible reference polygons, i.e. satisfying the mesh assumptions with the same regularity parameter as the polygons in the mesh.

Lemma 3 ([28, Proof of Lemma 4.9])**.** *Σ is compact.*

Definition 1. *Let $\mathrm{H}^1_{\mathcal{T}}(E)$ be the broken Sobolev space*

$$\mathrm{H}^1_{\mathcal{T}}(E) := \left\{ v \colon v_{|\tau} \in \mathrm{H}^1(\tau)\ \forall \tau \in \mathcal{T}_E \right\}.$$

Let $u \in \mathrm{H}^1_{\mathcal{T}}(E)$, we define $\forall e_i \in \mathcal{I}_{\mathcal{E}_E}$ the jump function $[\![\cdot]\!]_{e_i} : \mathrm{H}^1_{\mathcal{T}}(E) \to \mathrm{L}^2(e_i)$ such that

$$[\![u]\!]_{e_i} := \gamma^{e_i}\left(u_{|\tau_i}\right) - \gamma^{e_i}\left(u_{|\tau_{i-1}}\right).$$

Moreover, $[\![u]\!]_{\mathcal{I}_{\mathcal{E}_E}}$ denotes the vector containing the jumps of u on each $e_i \in \mathcal{I}_{\mathcal{E}_E}$. We endow $\mathrm{H}^1_{\mathcal{T}}(E)$ with the following seminorm and norm : $\forall u \in \mathrm{H}^1_{\mathcal{T}}(E)$,

$$|u|^2_{\mathrm{H}^1_{\mathcal{T}}(E)} := \sum_{\tau \in \mathcal{T}_E} \|\nabla u\|^2_{[\mathrm{L}^2(\tau)]^2} + \sum_{i=1}^{N_E^V} \left\| [\![u]\!]_{e_i} \right\|^2_{\mathrm{L}^2(e_i)}, \tag{17}$$

$$\|u\|^2_{\mathrm{H}^1_{\mathcal{T}}(E)} := |u|^2_{\mathrm{H}^1_{\mathcal{T}}(E)} + \sum_{\tau \in \mathcal{T}_E} \|u\|^2_{\mathrm{L}^2(\tau)}. \tag{18}$$

Definition 2. *Let us define $V \subset \mathrm{H}^1_{\mathcal{T}}(E)$ given by*

$$V := \{ v \in \mathrm{H}^1_{\mathcal{T}}(E) : \forall e_i \in \mathcal{I}_{\mathcal{E}_E},\ [\![v]\!]_{e_i} \in \mathrm{L}^\infty(e_i) \}.$$

Then $\forall v \in V$, we define its seminorm and its norm:

$$|v|^2_V := \sum_{\tau \in \mathcal{T}_E} \|\nabla v\|^2_{[\mathrm{L}^2(\tau)]^2} + \left\| [\![v]\!]_{\mathcal{I}_{\mathcal{E}_E}} \right\|^2_{\mathrm{L}^\infty\left(\mathcal{I}_{\mathcal{E}_E}\right)},$$

$$\|v\|^2_V := |v|^2_V + \sum_{\tau \in \mathcal{T}_E} \|v\|^2_{\mathrm{L}^2(\tau)},$$

where

$$\left\| [\![v]\!]_{\mathcal{I}_{\mathcal{E}_E}} \right\|_{\mathrm{L}^\infty\left(\mathcal{I}_{\mathcal{E}_E}\right)} := \max_{i \in \{1,\dots,N_E^V\}} \left\| [\![v]\!]_{e_i} \right\|_{\mathrm{L}^\infty(e_i)}.$$

Remark 3. *Let us observe that $\mathbb{P}_l(E) \subset V$. Hence, we can use $\|\cdot\|_{[V]^2}$ as a norm for $[\mathbb{P}_l(E)]^2$. Notice that, since $[\mathbb{P}_l(E)]^2 \subset [C^0(E)]^2$, $\left\| [\![\boldsymbol{p}]\!]_{\mathcal{I}_{\mathcal{E}_E}} \right\|_{\mathrm{L}^\infty\left(\mathcal{I}_{\mathcal{E}_E}\right)} = 0$, $\forall \boldsymbol{p} \in [\mathbb{P}_l(E)]^2$.*

Definition 3. *Let $\mathcal{V}_{1,l}^{E,\mathrm{P}_0}$ be the space*

$$\mathcal{V}_{1,l}^{E,\mathrm{P}_0} := \left\{ v \in \mathcal{V}_{1,l}^E : \mathrm{P}_0(v) = 0 \right\}. \tag{19}$$

Definition 4. *Denoting by $\{\psi_i\}_{i=1}^{N_E^V}$ the set of Lagrangian basis functions of $\mathcal{V}_{1,l}^E$, let $\mathcal{Q}(\partial E)$ be the vector space*

$$\mathcal{Q}(\partial E) := \operatorname{span}\left\{ \gamma^{\partial E}(\psi_i - \mathrm{P}_0(\psi_i)) \right\}, \quad \forall i = 1, \dots, N_E^V - 1. \tag{20}$$

Notice that $\forall q \in \mathcal{Q}(\partial E)$, $\exists! v \in \mathcal{V}_{1,l}^{E,\mathrm{P}_0}$ such that $q = \gamma^{\partial E}(v)$.

Definition 5. *Let $\mathcal{R}_{\mathcal{Q}}(E)$ be the vector space, lifting of $\mathcal{Q}(\partial E)$ on E, given by:*

$$\mathcal{R}_{\mathcal{Q}}(E) := \left\{ \bar{q}_{|\tau} \in \mathbb{P}_1(\tau)\ \forall \tau \in \mathcal{T}_E,\ \gamma^{\partial E}(\bar{q}) \in \mathcal{Q}(\partial E),\ \bar{q}(x_C) = 0 \right\}. \tag{21}$$

We note that $\mathcal{R}_{\mathcal{Q}}(E) \subset \mathrm{H}^1_{\mathcal{T}}(E) \cap C^0(E)$. Hence, we use the norm $\|\cdot\|_{\mathrm{H}^1_{\mathcal{T}}(E)}$ defined in (18) *as a norm for $\mathcal{R}_{\mathcal{Q}}(E)$. Notice that $\sum_{i=1}^{N_E^V} \left\| [\![\bar{q}]\!]_{e_i} \right\|_{\mathrm{L}^2(e_i)} = 0$ and $\nabla \bar{q} \in [V]^2$, $\forall \bar{q} \in \mathcal{R}_{\mathcal{Q}}(E)$. We denote by $\{r_j\}_{j=1}^{N_E^V - 1}$ a basis of $\mathcal{R}_{\mathcal{Q}}(E)$.*

Now, we can introduce the bilinear form b which is used in Proposition 1.

Definition 6. *Let $b : \mathcal{R}_{\mathcal{Q}}(E) \times [V]^2 \to \mathbb{R}$, such that $\forall \bar{q} \in \mathcal{R}_{\mathcal{Q}}(E)$, $\forall\, \boldsymbol{v} \in [V]^2$*

$$b(\bar{q}, \boldsymbol{v}) := \int_{\partial E} \bar{q}\, \boldsymbol{v} \cdot n^{\partial E}\, ds. \tag{22}$$

Applying the divergence theorem, we can rewrite the form b:

$$b(\bar{q}, \boldsymbol{v}) = \sum_{\tau \in \mathcal{T}_E} \int_{\tau} \left[\nabla \bar{q}\, \boldsymbol{v} + \bar{q}\, \nabla \cdot \boldsymbol{v} \right] dx - \sum_{i=1}^{N_E^V} \int_{e_i} \gamma^{e_i}(\bar{q})\, [\![\boldsymbol{v}]\!]_{e_i} \cdot \boldsymbol{n}^{e_i} ds. \tag{23}$$

The following lemma gives the continuity of the bilinear form b.

Lemma 4. *Let b be given by* (22)*. Then b is a bilinear form and $\exists C > 0$ independent of h_E such that*

$$b(\bar{q}, \boldsymbol{v}) \leq C \left\| \bar{q} \right\|_{\mathrm{H}^1_{\mathcal{T}}(E)} \left\| \boldsymbol{v} \right\|_{[V]^2} \quad \forall \bar{q} \in \mathcal{R}_{\mathcal{Q}}(E), \forall\, \boldsymbol{v} \in [V]^2 .$$

Proof. The proof of this lemma can be found in the supplementary materials of this paper. $\square$

The following proposition is the first step towards the proof of Theorem 1.

Proposition 1. *Assume $N_E^V > 3$ and let b the continuous bilinear form defined by* (22)*. If $\exists \beta > 0$, independent of h_E, such that*

$$\forall \bar{q} \in \mathcal{R}_{\mathcal{Q}}(E), \qquad \sup_{\boldsymbol{p} \in [\mathbb{P}_l(E)]^2} \frac{b(\bar{q}, \boldsymbol{p})}{\left\| \boldsymbol{p} \right\|_{[V]^2}} \geq \beta \left\| \bar{q} \right\|_{\mathrm{H}^1_{\mathcal{T}}(E)} , \tag{24}$$

then (13) *holds true.*

Proof. Given (8),

$$\Pi^0_{l,E} \nabla u = 0 \implies \left(\nabla u, \boldsymbol{p} \right)_E = 0 \ \ \forall \boldsymbol{p} \in \left[\mathbb{P}_l(E) \right]^2 .$$

Applying Gauss-Green formula, the previous relation becomes

$$(\nabla u, \boldsymbol{p})_E = \left(\gamma^{\partial E}(u)\,, \boldsymbol{p}\cdot n^{\partial E}\right)_{\partial E} - (u, \nabla\cdot\boldsymbol{p})_E = 0 \;\; \forall \boldsymbol{p} \in [\mathbb{P}_l(E)]^2 \,.$$

Since $\nabla\cdot\boldsymbol{p} \in \mathbb{P}_{l-1}(E)$ we apply (15) and we obtain

$$\left(\gamma^{\partial E}(u)\,, \boldsymbol{p}\cdot n^{\partial E}\right)_{\partial E} - \mathrm{P}_0\,(u)\cdot(1, \nabla\cdot\boldsymbol{p})_E = 0 \;\; \forall \boldsymbol{p} \in [\mathbb{P}_l(E)]^2 \,.$$

Then we can apply the divergence theorem and find the relation

$$\left(\gamma^{\partial E}(u - \mathrm{P}_0\,(u))\,, \boldsymbol{p}\cdot n^{\partial E}\right)_{\partial E} = 0 \;\; \forall \boldsymbol{p} \in [\mathbb{P}_l(E)]^2 \,. \tag{25}$$

We have $q = \gamma^{\partial E}(u - \mathrm{P}_0\,(u)) \in \mathcal{Q}(\partial E)$ ($\mathcal{Q}(\partial E)$ defined in (20)). Let $\bar{q} \in \mathcal{R}_{\mathcal{Q}}(E)$ be the lifting of q ($\mathcal{R}_{\mathcal{Q}}(E)$ defined in (21)), then the relation (25) is

$$b(\bar{q}, \boldsymbol{p}) = 0 \;\; \forall \boldsymbol{p} \in [\mathbb{P}_l(E)]^2 \,.$$

Then, since b is a continuous bilinear form, (24) implies $q \equiv 0$. Finally, since $u \in \mathcal{V}^E_{1,l}$, then $u = \mathrm{P}_0\,(u)$. □

4.2 Proof of the inf-sup condition

In this section we show that (24) holds with β independent of h_E. The proof relies on the technique known as Fortin trick [22], that consists in the following two classical results.

Proposition 2 ([22, Proposition 5.4.2])**.** *Assume that there exists an operator $\Pi_E : [V]^2 \to [\mathbb{P}_l(E)]^2$ that satisfies, $\forall \boldsymbol{v} \in [V]^2$,*

$$b(\bar{q}, \Pi_E \boldsymbol{v} - \boldsymbol{v}) = 0 \;\; \forall \bar{q} \in \mathcal{R}_{\mathcal{Q}}(E) \,,$$

and assume that there exists a constant $C_\Pi > 0$, independent of h_E, such that

$$\left\|\Pi_E \boldsymbol{v}\right\|_{[V]^2} \leq C_\Pi \left\|\boldsymbol{v}\right\|_{[V]^2} \;\;\; \forall \boldsymbol{v} \in [V]^2 \,.$$

Assume moreover that $\exists \eta > 0$, independent of h_E such that

$$\inf_{q \in \mathcal{R}_{\mathcal{Q}}(E)} \sup_{\boldsymbol{v} \in [V]^2} \frac{b(q, \boldsymbol{v})}{\left\|q\right\|_{\mathrm{H}^1_{\gamma}(E)} \left\|\boldsymbol{v}\right\|_{[V]^2}} \geq \eta \,. \tag{26}$$

Then the discrete inf-sup condition (24) is satisfied, with $\beta = \frac{\eta}{C_\Pi}$.

Remark 4. *The inf-sup constant β in (24) has to be independent of the mesh size in order to guarantee that the constant in (50), involved in the coercivity of the bilinear form of (10), is independent of the mesh size.*

Remark 5. *The operator Π_E defined in the following is such that the constant C_Π depends on $N^V_{\max}$ and on the continuity constant of P_0, both are bounded independently of h_E by assumption.*

Proposition 3 ([22, Proposition 5.4.4])**.** *Let $\Pi_1, \Pi_2 \in \mathcal{L}([V]^2, [\mathbb{P}_l(E)]^2)$ be such that $\exists c_1, c_2 > 0$,*

$$\|\Pi_1 \boldsymbol{v}\|_{[V]^2} \leq c_1 \|\boldsymbol{v}\|_{[V]^2} \quad \forall \boldsymbol{v} \in [V]^2\,, \tag{27a}$$

$$b(\bar{q}, \Pi_2 \boldsymbol{v} - \boldsymbol{v}) = 0 \quad \forall \bar{q} \in \mathcal{R}_{\mathcal{Q}}(E),\ \forall \boldsymbol{v} \in [V]^2\,, \tag{27b}$$

$$\|\Pi_2 (\mathrm{I} - \Pi_1) \boldsymbol{v}\|_{[V]^2} \leq c_2 \|\boldsymbol{v}\|_{[V]^2} \quad \forall \boldsymbol{v} \in [V]^2\,. \tag{27c}$$

Then, the operator $\Pi_E := \Pi_2 (\mathrm{I} - \Pi_1) + \Pi_1$ satisfies the hyphotesis of Proposition 2.

Following the above results, we have to prove (26) and to show the existence of two operators Π_1, Π_2 satisfying (27a), (27b) and (27c). In the following proposition we achieve the first task.

Proposition 4. *Let $b\colon \mathcal{R}_{\mathcal{Q}}(E) \times [V]^2 \to \mathbb{R}$ be defined by (22). Then, for h_E sufficiently small, the inf-sup condition (26) holds true.*

Proof. Let $\bar{q} \in \mathcal{R}_{\mathcal{Q}}(E)$ be given. Recall that $\nabla \bar{q} \in [V]^2$. Notice that, since $\nabla \bar{q}_{|\tau} \in \mathbb{P}_0(\tau)\ \forall \tau \in \mathcal{T}_E$,

$$\|\nabla \bar{q}\|^2_{[V]^2} = \|\nabla \bar{q}\|^2_{[\mathrm{L}^2(E)]^2} + \left\| [\![\nabla \bar{q}]\!]_{\mathcal{I}_{\mathcal{E}_E}} \right\|^2_{\mathrm{L}^\infty(\mathcal{I}_{\mathcal{E}_E})}\,.$$

Then, since $\|\nabla \bar{q}\|^2_{[V]^2} = 0 \iff \nabla \bar{q} = 0 \iff \bar{q} = 0$, we deduce that, $\forall \bar{q} \in \mathcal{R}_{\mathcal{Q}}(E)$, $\|\nabla \bar{q}\|_{[V^2]}$ is a norm on $\mathcal{R}_{\mathcal{Q}}(E)$. The same argument can be applied to $\|\nabla \bar{q}\|_{[\mathrm{L}^2(E)]^2}$. Then, using the mapping (16) and applying standard arguments about the equivalence of norms in finite dimensional spaces, we have

$$\|\nabla \bar{q}\|_{[\mathrm{L}^2(E)]^2} = \left\| \hat{\nabla} \hat{\bar{q}} \right\|_{[\mathrm{L}^2(\hat{E})]^2} \geq \frac{\displaystyle\min_{\hat{w} \in \mathcal{R}_{\mathcal{Q}}(\hat{E}) : \|\mathrm{dof}(\hat{w})\|_{l_2} = 1} \left\| \hat{\nabla} \hat{w} \right\|_{[\mathrm{L}^2(\hat{E})]^2}}{\sqrt{N^V_E - 1} \displaystyle\max_{i=1,\dots,N^V_E-1} \left\| \hat{\nabla} \hat{\chi}_i \right\|_{[\hat{V}]^2}} \left\| \hat{\nabla} \hat{q} \right\|_{[V]^2}\,, \tag{28}$$

where $\{\hat{\chi}_i\}_{i=1}^{N^V_E-1} \in \mathcal{R}_{\mathcal{Q}}(\hat{E})$ are the Lagrangian functions in the degrees of freedom. Since in the Virtual Element framework there is not an unique reference element $\hat{E}$, independent of E, we have to prove that the constant appearing in (28), in the following denoted by $C(\hat{E})$, is bounded from below independently of $\hat{E}$. By Lemma 3, it is true that the set of all admissible reference elements Σ is a compact set. Moreover, $C(\hat{E})$ is a continuous function on Σ, indeed, there is an isomorphism between $\mathcal{R}_{\mathcal{Q}}(E)$ and $\mathcal{V}^{E,\mathrm{P}_0}_{1,l}$ and P_0 is continuous on Σ by assumption, then we can follow the proofs of [28, Lemma 4.9] and [12, Lemma 4.5]. Then, we have that $\exists m = \min_{\hat{E}\in\Sigma} C(\hat{E}) > 0$, independent of h_E, and it holds true that

$$\|\nabla \bar{q}\|_{[\mathrm{L}^2(E)]^2} \geq m \|\nabla \bar{q}\|_{[V]^2} . \tag{29}$$

Moreover, using (23), we get

$$b(\bar{q}, \nabla \bar{q}) = \|\nabla \bar{q}\|^2_{[\mathrm{L}^2(E)]^2} - \sum_{i=1}^{N_E^V} \int_{e_i} \gamma^{e_i}(\bar{q}) \, [\![\nabla \bar{q}]\!]_{e_i} \cdot \boldsymbol{n}^{e_i} ds , \tag{30}$$

and, since $[\![\nabla \bar{q}]\!]_{e_i} \cdot \boldsymbol{n}^{e_i} \in \mathbb{P}_0(e_i) \; \forall e_i \in \mathcal{I}_{\mathcal{E}_E}$ and $\bar{q}(x_C) = 0$, we get

$$\int_{e_i} \gamma^{e_i}(\bar{q}) \, [\![\nabla \bar{q}]\!]_{e_i} \cdot \boldsymbol{n}^{e_i} ds = \left([\![\nabla \bar{q}]\!]_{e_i} \cdot \boldsymbol{n}^{e_i}\right) \int_{e_i} \gamma^{e_i}(\bar{q}) \, ds = \left([\![\nabla \bar{q}]\!]_{e_i} \cdot \boldsymbol{n}^{e_i}\right) \frac{|e_i|}{2} \bar{q}(x_i) .$$

Then, using the property $\sum_{i=1}^{N_E^V} |\bar{q}(x_i)| \leq C \sqrt{\sum_{\tau\in\mathcal{T}_E} \|\nabla \bar{q}\|^2_{\mathrm{L}^2(\tau)}}$, whose proof can be found in detail in the supplementary materials of this paper, and $|\boldsymbol{n}^{e_i}| = 1 \, \forall i = 1, \dots, N_E^V$,

$$\begin{aligned}
\sum_{e\in\mathcal{I}_{\mathcal{E}_E}} \int_e \gamma^e(\bar{q}) \, [\![\nabla \bar{q}]\!]_e \cdot \boldsymbol{n}^e ds &\leq \left\| [\![\nabla \bar{q}]\!]_{\mathcal{I}_{\mathcal{E}_E}} \right\|_{\mathrm{L}^\infty(\mathcal{I}_{\mathcal{E}_E})} \sum_{i=1}^{N_E^V} \frac{|e_i|}{2} \bar{q}(x_i) \\
&\leq \frac{h_E}{2} \left\| [\![\nabla \bar{q}]\!]_{\mathcal{I}_{\mathcal{E}_E}} \right\|_{\mathrm{L}^\infty(\mathcal{I}_{\mathcal{E}_E})} \sum_{i=1}^{N_E^V} \bar{q}(x_i) \\
&\leq C h_E \left\| [\![\nabla \bar{q}]\!]_{\mathcal{I}_{\mathcal{E}_E}} \right\|_{\mathrm{L}^\infty(\mathcal{I}_{\mathcal{E}_E})} \|\nabla \bar{q}\|_{[\mathrm{L}^2(E)]^2} .
\end{aligned}$$

Then, from (30) we get,

$$b(\bar{q}, \nabla \bar{q}) \geq \|\nabla \bar{q}\|_{[\mathrm{L}^2(E)]^2} \left(\|\nabla \bar{q}\|_{[\mathrm{L}^2(E)]^2} - C h_E \left\| [\![\nabla \bar{q}]\!]_{\mathcal{I}_{\mathcal{E}_E}} \right\|_{\mathrm{L}^\infty(\mathcal{I}_{\mathcal{E}_E})} \right) .$$

Finally, the term in the parentheses can be bounded from below exploiting (29), as follows:

$$\|\nabla \bar{q}\|_{[\mathrm{L}^2(E)]^2} - C h_E \left\| [\![\nabla \bar{q}]\!]_{\mathcal{I}_{\mathcal{E}_E}} \right\|_{\mathrm{L}^\infty(\mathcal{I}_{\mathcal{E}_E})} \geq C_*(1 - h_E) \|\nabla \bar{q}\|_{[V]^2} ,$$

which yields the thesis $\forall h_E \leq h_0$, for any $h_0 < 1$, since we can apply local Poincaré inequalities on each $\tau \in \mathcal{T}_E$, being $\bar{q}(x_C) = 0$, and obtain

$$\|\nabla \bar{q}\|_{[\mathrm{L}^2(E)]^2} \geq C \min\{1, h_E^{-1}\} \|\bar{q}\|_{\mathrm{H}^1_{\mathcal{T}}(E)} .$$

□

Now, let us focus on the operator Π_1 of Proposition 3. This is a best – approximation operator satisfying the Poincaré-type inequality (33). Let $\Pi^0_{0,E}$ be defined in (11).

Lemma 5. *Let Σ be the set of admissible reference elements, it holds*

$$\exists C > 0 : \forall \hat{E} \in \Sigma, \forall \hat{v} \in \mathrm{H}^1_{\mathcal{T}}(\hat{E}), \quad \left\| \hat{v} - \Pi^0_{0,\hat{E}} \hat{v} \right\|_{\mathrm{L}^2(\hat{E})} \leq C \left| \hat{v} \right|_{\mathrm{H}^1_{\mathcal{T}}(\hat{E})} . \tag{31}$$

Proof. The proof of this lemma can be found in the supplementary materials of this paper. □

Proposition 5. *Let $\Pi_1 : [V]^2 \to [\mathbb{P}_l(E)]^2$ be the operator defined $\forall \boldsymbol{v} \in [V]^2$ by*

$$\Pi_1 \boldsymbol{v} = \begin{pmatrix} \Pi^0_{0,E} v_1 \\ \Pi^0_{0,E} v_2 \end{pmatrix} . \tag{32}$$

Then Π_1 satisfies the condition (27a) *and the following inequality holds: $\exists C > 0$ such that $\forall \boldsymbol{v} \in [V]^2$*

$$\left\| \boldsymbol{v} - \Pi_1 \boldsymbol{v} \right\|_{[\mathrm{L}^2(E)]^2} \leq C h_E \left| \boldsymbol{v} \right|_{[V]^2} , \tag{33}$$

where C is independent of h_E.

Proof. Let us notice that

$$\Pi_1 \boldsymbol{v} \in [\mathbb{P}_0(E)]^2 \implies \left\| \Pi_1 \boldsymbol{v} \right\|_{[V]^2} = \left\| \Pi_1 \boldsymbol{v} \right\|_{[\mathrm{L}^2(E)]^2} . \tag{34}$$

Hence, we have

$$\left\| \Pi_1 \boldsymbol{v} \right\|^2_{[\mathrm{L}^2(E)]^2} = \left(\Pi_1 \boldsymbol{v}, \Pi_1 \boldsymbol{v} \right)_{[\mathrm{L}^2(E)]^2} = \left(\Pi_1 \boldsymbol{v}, \boldsymbol{v} \right)_{[\mathrm{L}^2(E)]^2} \leq \left\| \Pi_1 \boldsymbol{v} \right\|_{[\mathrm{L}^2(E)]^2} \left\| \boldsymbol{v} \right\|_{[V]^2} .$$

The condition (27a) is satisfied. In order to prove (33), we can apply standard scaling argument and the property (31) to the norm of each component of $\hat{\boldsymbol{v}} - \hat{\Pi}_1 \hat{\boldsymbol{v}}$, then $\exists C > 0$ such that

$$\begin{aligned} \left\| \boldsymbol{v} - \Pi_1 \boldsymbol{v} \right\|^2_{[\mathrm{L}^2(E)]^2} &= h_E^2 \left\| \hat{\boldsymbol{v}} - \hat{\Pi}_1 \hat{\boldsymbol{v}} \right\|^2_{\left[\mathrm{L}^2\left(\hat{E}\right)\right]^2} \\ &\leq C h_E^2 \left(\left\| \nabla \hat{\boldsymbol{v}} \right\|^2_{\left[\mathrm{L}^2\left(\hat{E}\right)\right]^4} + \sum_{\hat{e} \in \mathcal{I}_{\mathcal{E}_{\hat{E}}}} \left\| [\![\hat{\boldsymbol{v}}]\!]_{\hat{e}} \right\|^2_{[\mathrm{L}^2(\hat{e})]^2} \right) \\ &\leq C h_E^2 \left(\left\| \nabla \boldsymbol{v} \right\|^2_{[\mathrm{L}^2(E)]^4} + h_E^{-1} \sum_{e \in \mathcal{I}_{\mathcal{E}_E}} \left\| [\![\boldsymbol{v}]\!]_{e} \right\|^2_{[\mathrm{L}^2(e)]^2} \right) . \end{aligned}$$

Finally, applying the property $\sum_{i=1}^{N_E^V} \left\| [\![\boldsymbol{v}]\!]_{e_i} \right\|^2_{\mathrm{L}^2(e_i)} \leq C h_E \left\| [\![\boldsymbol{v}]\!]_{\mathcal{I}_{\mathcal{E}_E}} \right\|^2_{\mathrm{L}^\infty\left(\mathcal{I}_{\mathcal{E}_E}\right)}$, we obtain (33). □

In the following, assuming (12), we prove the existence of an operator Π_2 satisfying (27b). First, we need some auxiliary results.

Definition 7. *Let* $\{r_i\}_{i=1}^{N_E^V-1}$ *be a basis of* $\mathcal{R}_{\mathcal{Q}}(E)$*. Let us define the set of linear operators* $D_i : [V]^2 \to \mathbb{R}$ *such that* $\forall \boldsymbol{v} \in [V]^2$

$$D_i(\boldsymbol{v}) := \int_{\partial E} \left(\boldsymbol{v} \cdot \boldsymbol{n}^{\partial E}\right) \gamma^{\partial E}(r_i)\, ds, \quad \forall i = 1, \dots, N_E^V - 1\,.$$

Lemma 6. *If* $(l+1)(l+2) \geq N_E^V - 1$*, there exists a set of functions* $\boldsymbol{\pi}_j \in [\mathbb{P}_l(E)]^2$ *defined by*

$$D_i(\boldsymbol{\pi}_j) = \delta_{ij} \ \ \forall i, j = 1, \dots, N_E^V - 1. \tag{35}$$

Proof. Let $V_l^M(E)$ be the local mixed virtual element space of order l, defined in [8], i.e.

$$V_l^M(E) := \{\boldsymbol{v} \in \mathrm{H}(\mathrm{div}; E) \cap \mathrm{H}(\mathrm{rot}; E) : \gamma^e(\boldsymbol{v} \cdot \boldsymbol{n}^e) \in \mathbb{P}_l(e)\, \forall e \in \mathcal{E}_E,$$
$$\mathrm{div}\boldsymbol{v} \in \mathbb{P}_l(E) \ \text{ and } \mathrm{rot}\boldsymbol{v} \in \mathbb{P}_{l-1}(E)\}.$$

Notice that $[\mathbb{P}_l(E)]^2 \subset V_l^M(E)$. For each $\boldsymbol{v} \in V_l^M(E)$, the degrees of freedom of $\boldsymbol{v}$ are defined by

1. $\int_e \boldsymbol{v} \cdot \boldsymbol{n}^e\, q\, \mathrm{ds}$, $\forall e \in \mathcal{E}_E$, $\forall q \in \mathbb{P}_l(e)$,
2. $\int_E \boldsymbol{v} \cdot \nabla p_l\, \mathrm{dx}$, $\forall p_l \in \mathbb{P}_l(E)$,
3. $\int_E \boldsymbol{v} \cdot \boldsymbol{p}_l^\perp \mathrm{dx}$, $\forall \boldsymbol{p}_l^\perp \in \{\boldsymbol{p}_l^\perp \in [\mathbb{P}_l(E)]^2 : \int_E \boldsymbol{p}_l^\perp \cdot \nabla q\ \mathrm{dx} = 0\, \forall q \in \mathbb{P}_{l+1}(E)\}$.

The number of degrees of freedom defined by the first, the second and the third condition is, respectively, $(l+1)N_E^V$, $\frac{(l+1)(l+2)}{2} - 1$ and $\frac{(l-1)(l+2)}{2} + 1$. Globally, $\dim V_l^M(E) = (l+1)N_E^V + l(l+2)$.

Notice that a possible choice for the basis of $\mathcal{P}_l(\partial E) := \{p \in \mathbb{P}_l(e)\,, \forall e \in \mathcal{E}_E\}$ is composed by the $N_E^V - 1$ basis functions $\{\gamma^{\partial E}(r_i)\}_{i=1}^{N_E^V-1} \subset \mathcal{Q}(\partial E) \subset \mathcal{P}_l(\partial E)$, completed by a choice of linearly independent functions $\{q_i^C\}_{i=N_E^V}^{(l+1)N_E^V} \subset \mathcal{P}_l(\partial E)$. Hence, the first set of degrees of freedom can be split into two groups, i.e.

- $D_i(\boldsymbol{v}) = \int_{\partial E} \boldsymbol{v} \cdot \boldsymbol{n}^{\partial E}\, \gamma^{\partial E}(r_i)\, ds$, $\forall i = 1, \dots, N_E^V - 1$,
- $\int_{\partial E} \boldsymbol{v} \cdot \boldsymbol{n}^{\partial E}\, q_i^C\, \mathrm{ds}$, $\forall i = N_E^V, \dots, (l+1)N_E^V$.

Let $j \in \{1, \dots, N_E^V - 1\}$ and let $V^R(E; j) \subset V_l^M(E)$ be

$$V^R(E; j) := \{\boldsymbol{v} \in V_l^M(E) : D_i(\boldsymbol{v}) = \delta_{ij}\ \forall i = 1, \dots, N_E^V - 1\}. \tag{36}$$

Notice that $\dim V^R(E;j) = \dim V_l^M(E) - (N_E^V - 1)$. Moreover, we define $V^{\perp \mathbb{P}_l}(E) \subset V_l^M(E)$, given by

$$V^{\perp \mathbb{P}_l}(E) := \{\boldsymbol{v} \in V_l^M(E) : \mathrm{dof}(\boldsymbol{v}) \cdot \mathrm{dof}(\boldsymbol{p}) = 0 \, \forall \boldsymbol{p} \in [\mathbb{P}_l(E)]^2\} \tag{37}$$

where $\mathrm{dof}(\boldsymbol{v})$ denotes the vector of degrees of freedom of $\boldsymbol{v} \in V_l^M(E)$. Notice that $\dim V^{\perp \mathbb{P}_l}(E) = \dim V_l^M(E) - (l+1)(l+2)$. Since $(l+1)(l+2) \geq N_E^V - 1$, then $\dim V^R(E;j) \geq \dim V^{\perp \mathbb{P}_l}(E)$ and thus $\exists \boldsymbol{w}_j \in V^R(E;j) \cap [\mathbb{P}_l(E)]^2$ such that

$$\mathrm{dof}(\boldsymbol{w}_j) \cdot \mathrm{dof}(\boldsymbol{\eta}) = 0 \quad \forall \boldsymbol{\eta} \in V^{\perp \mathbb{P}_l}(E) \,. \tag{38}$$

Then we can choose $\boldsymbol{\pi}_j = \boldsymbol{w}_j$. □

In the following proposition we provide a definition of Π_2 and prove an approximation result that is used in Proposition 7.

Proposition 6. *Under the hypothesis of Theorem 1, let us define* $\Pi_2 : [V]^2 \to [\mathbb{P}_l(E)]^2$ *such that* $\forall \boldsymbol{v} \in [V]^2$

$$\Pi_2 \boldsymbol{v} := \sum_{i=1}^{N_E^V - 1} D_i(\boldsymbol{v}) \boldsymbol{\pi}_i \,,$$

where $\boldsymbol{\pi}_i$ *satisfy* (35)*. Then* Π_2 *satisfies* (27b) *and the property* $\exists C > 0 :$ $\forall \boldsymbol{v} \in [V]^2$

$$\left\| \Pi_2 \boldsymbol{v} \right\|_{[V]^2} \leq C \left((1 + h_E^{-1}) \left\| \boldsymbol{v} \right\|_{[\mathrm{L}^2(E)]^2} + (h_E + 1) \left| \boldsymbol{v} \right|_{[V]^2} \right) . \tag{39}$$

Proof. Since

$$\forall \boldsymbol{v} \in [V]^2 \,, \;\; D_i(\Pi_2 \boldsymbol{v}) = D_i(\boldsymbol{v}) \;\; \forall i = 1, \dots, N_E^V - 1, \tag{40}$$

let us check that Π_2 satisfies (27b), indeed by construction $\forall r_i \in \mathcal{R}_{\mathcal{Q}}(E)$, $i = 1, \dots, N_E^V - 1$, $\forall \boldsymbol{v} \in [V]^2$:

$$b(r_i, \Pi_2 \boldsymbol{v} - \boldsymbol{v}) = \int_{\partial E} r_i \, (\Pi_2 \boldsymbol{v} - \boldsymbol{v}) \cdot n^{\partial E} \, dx = D_i(\Pi_2 \boldsymbol{v} - \boldsymbol{v}) = 0.$$

Furthermore, let us consider $\widehat{\Pi_2 \boldsymbol{v}}$ defined on the reference polygon $\hat{E}$. Applying the linearity of the definition of the mapping $F : \hat{E} \to E$, presented in (16), we have

$$\widehat{\Pi_2 \boldsymbol{v}} = \left(\sum_{i=1}^{N_E^V - 1} D_i \left(\Pi_2 \boldsymbol{v} \right) \boldsymbol{\pi}_i \right) \circ F = \sum_{i=1}^{N_E^V - 1} D_i \left(\Pi_2 \boldsymbol{v} \right) \left(\boldsymbol{\pi}_i \circ F \right) \tag{41}$$

Notice that $\boldsymbol{\pi}_i \circ F = \frac{1}{h_E}\widehat{\boldsymbol{\pi}}_i$ $\forall i = 1,\dots,N_E^V - 1$, indeed we have $\forall i = 1,\dots,N_E^V - 1$

$$\hat{D}_j(\boldsymbol{\pi}_i \circ F) = \int\limits_{\partial\hat{E}} (\boldsymbol{\pi}_i \circ F)\cdot\boldsymbol{n}^{\partial\hat{E}}\, \hat{r}_j \, d\hat{s} = \frac{1}{h_E}\int\limits_{\partial E}\left(\boldsymbol{\pi_i}\cdot\boldsymbol{n}^{\partial E}\right) r_j\, ds = \frac{1}{h_E}\delta_{ij} = \frac{1}{h_E}\hat{D}_j(\widehat{\boldsymbol{\pi}}_i).$$

Then, applying the definition of the mapping F (16) and (40) both in E and in $\hat{E}$, we have $\forall i = 1,\dots,N_E^V - 1$

$$D_i(\Pi_2\boldsymbol{v}) = \int\limits_{\partial E}\left(\boldsymbol{v}\cdot\boldsymbol{n}^{\partial E}\right) r_i\, ds = h_E\int\limits_{\partial\hat{E}}\left(\hat{\boldsymbol{v}}\cdot\boldsymbol{n}^{\partial\hat{E}}\right)\hat{r}_i\, d\hat{s} = h_E\hat{D}_i(\hat{\Pi}_2\hat{\boldsymbol{v}}). \tag{42}$$

Applying Lemma 4 on the reference polygon $\hat{E}$, we have $\forall i = 1,\dots,N_E^V - 1$

$$\hat{D}_i(\hat{\Pi}_2\hat{\boldsymbol{v}}) = \hat{D}_i(\hat{\boldsymbol{v}}) = b(\hat{r}_i,\hat{\boldsymbol{v}}) \leq C_b \left\|\hat{r}_i\right\|_{\mathrm{H}^1_{\mathcal{T}}(\hat{E})}\left\|\hat{\boldsymbol{v}}\right\|_{[\hat{V}]^2}. \tag{43}$$

Then, we want to prove the continuity of $\widehat{\Pi_2\boldsymbol{v}}$. Applying (41) and (42), we obtain

$$\left\|\widehat{\Pi_2\boldsymbol{v}}\right\|_{[\hat{V}]^2} \leq \sum_{i=1}^{N_E^V-1}\left|\hat{D}_i\left(\hat{\Pi}_2\hat{\boldsymbol{v}}\right)\right|\left\|\widehat{\boldsymbol{\pi}}_i\right\|_{[\hat{V}]^2} \leq (N_E^V-1)\max_i\left|D_i\left(\hat{\Pi}_2\hat{\boldsymbol{v}}\right)\right|\max_i\left\|\widehat{\boldsymbol{\pi}}_i\right\|_{[\hat{V}]^2}.$$

Applying the mesh assumption (4) and (43), we obtain

$$\left\|\widehat{\Pi_2\boldsymbol{v}}\right\|_{[\hat{V}]^2} \leq C_b N_{\max}^V \max_i\left\|\hat{r}_i\right\|_{\mathrm{H}^1_{\mathcal{T}}(\hat{E})}\max_i\left\|\widehat{\boldsymbol{\pi}}_i\right\|_{[\hat{V}]^2}\left\|\hat{\boldsymbol{v}}\right\|_{[\hat{V}]^2}. \tag{44}$$

We set $C(\hat{E}) := \max_i\left\|\hat{r}_i\right\|_{\mathrm{H}^1_{\mathcal{T}}(\hat{E})}\max_i\left\|\widehat{\boldsymbol{\pi}}_i\right\|_{[\hat{V}]^2}$. This is a continuous function on the set of admissible reference elements Σ, which is a compact set by Lemma 3. Indeed, $\left\|\hat{r}_i\right\|_{\mathrm{H}^1_{\mathcal{T}}(\hat{E})}$ is a continuous function $\forall i = 1,\dots,N_E^V - 1$ on Σ by the same argument used in the proof of Proposition 4. Moreover, by definition, $\widehat{\boldsymbol{\pi}}_i$ depends continuously on the set $\{\hat{r}_i\}_{i=1}^{N_E^V-1}$. Then there exists $M = \max_{\hat{E}\in\Sigma} C(\hat{E}) > 0$. Finally, it results $\exists C = C_b N_{\max}^V M > 0$ such that

$$\left\|\widehat{\Pi_2\boldsymbol{v}}\right\|_{[\hat{V}]^2} \leq C\left\|\hat{\boldsymbol{v}}\right\|_{[\hat{V}]^2}. \tag{45}$$

Then, since $\Pi_2\boldsymbol{v} \in C^0(E)$, we have

$$\left\|\Pi_2\boldsymbol{v}\right\|^2_{[V]^2} = \left\|\Pi_2\boldsymbol{v}\right\|^2_{[\mathrm{L}^2(E)]^2} + \left\|\nabla\Pi_2\boldsymbol{v}\right\|^2_{[\mathrm{L}^2(E)]^4}. \tag{46}$$

Applying (45) and a standard scaling argument, we can analyse the second term as follows:

$$\begin{aligned}\left\|\nabla\Pi_2\boldsymbol{v}\right\|^2_{[\mathrm{L}^2(E)]^4} &= \left\|\hat{\nabla}\widehat{\Pi_2\boldsymbol{v}}\right\|^2_{\left[\mathrm{L}^2\left(\hat{E}\right)\right]^4} \leq \left\|\widehat{\Pi_2\boldsymbol{v}}\right\|^2_{[\hat{V}]^2} \leq C\left\|\hat{\boldsymbol{v}}\right\|^2_{[\hat{V}]^2} \\ &= C\left(h_E^{-2}\left\|\boldsymbol{v}\right\|^2_{[\mathrm{L}^2(E)]^2} + \left\|\nabla\boldsymbol{v}\right\|^2_{\left[\mathrm{L}^2_{\mathcal{T}}(E)\right]^4} + \left\|[\![\boldsymbol{v}]\!]_{\mathcal{I}_{\mathcal{E}_E}}\right\|^2_{\mathrm{L}^\infty\left(\mathcal{I}_{\mathcal{E}_E}\right)}\right).\end{aligned} \tag{47}$$

Moreover, applying similar arguments to the term $\|\Pi_2 \boldsymbol{v}\|^2_{[\mathrm{L}^2(E)]^2}$, we have

$$
\begin{aligned}
\|\Pi_2 \boldsymbol{v}\|^2_{[\mathrm{L}^2(E)]^2} &= h_E^2 \left\|\widehat{\Pi_2 \boldsymbol{v}}\right\|^2_{[\mathrm{L}^2(\hat{E})]^2} \leq h_E^2 \left\|\widehat{\Pi_2 \boldsymbol{v}}\right\|^2_{[\hat{V}]^2} \\
&\leq C h_E^2 \left(h_E^{-2} \|\boldsymbol{v}\|^2_{[\mathrm{L}^2(E)]^2} + \|\nabla \boldsymbol{v}\|^2_{[\mathrm{L}^2_{\mathcal{T}}(E)]^4} + \left\| [\![\boldsymbol{v}]\!]_{\mathcal{I}_{\mathcal{E}_E}} \right\|^2_{\mathrm{L}^\infty(\mathcal{I}_{\mathcal{E}_E})} \right) .
\end{aligned}
\tag{48}
$$

Applying (47) and (48) to (46), we prove (39). □

Finally, we show that the operators Π_1 and Π_2 defined above satisfy (27c).

Proposition 7. *Let* $\Pi_1, \Pi_2 \in \mathcal{L}([V]^2, [\mathbb{P}_l(E)]^2)$ *be given according to Proposition 5 and Proposition 6 respectively, then* (27c) *is satisfied.*

Proof. Applying (39), we have

$$
\|\Pi_2 (I - \Pi_1) \boldsymbol{v}\|_{[V]^2} \leq C \left((1 + h_E^{-1}) \|(I - \Pi_1) \boldsymbol{v}\|_{[\mathrm{L}^2(E)]^2} + (h_E + 1) |(I - \Pi_1) \boldsymbol{v}|_{[V]^2} \right) .
$$

Then, applying (33) to the first term and the property

$$
\Pi_1 \boldsymbol{v} \in [\mathbb{P}_0(E)]^2 \implies |(I - \Pi_1) \boldsymbol{v}|_{[V]^2} = |\boldsymbol{v}|_{[V]^2} ,
$$

to the second one, we have, for h_E sufficiently small,

$$
\|\Pi_2 (I - \Pi_1) \boldsymbol{v}\|_{[V]^2} \leq C (1 + h_E) |\boldsymbol{v}|_{[V]^2} \leq C |\boldsymbol{v}|_{[V]^2} \leq C \|\boldsymbol{v}\|_{[V]^2} .
$$

□

4.3 Coercivity of the discrete bilinear form

In this section we prove the coercivity of the discrete problem defined by (10) with respect to the standard $\mathrm{H}^1_0(\Omega)$ norm, denoted by

$$
\|V\|_{\mathrm{H}^1_0(\Omega)} = \|\nabla V\|_{[\mathrm{L}^2(\Omega)]^2} \quad \forall V \in \mathrm{H}^1_0(\Omega) .
$$

Let

$$
\|v\|_{\boldsymbol{\ell}} := \left(\sum_{E \in \mathcal{M}_h} \left\| \Pi^0_{\boldsymbol{\ell}(E),E} \nabla v \right\|^2_{[\mathrm{L}^2(E)]^2} \right)^{\frac{1}{2}} \qquad \forall v \in \mathcal{V}_{1,\boldsymbol{\ell}} .
$$

We have the following result.

Proposition 8. *Suppose* $\boldsymbol{\ell}$ *satisfies* (12) $\forall E \in \mathcal{M}_h$. *Then,* $\|\cdot\|_{\boldsymbol{\ell}}$ *is a norm on* $\mathcal{V}_{1,\boldsymbol{\ell}}$.

Proof. Let $v \in \mathcal{V}_{1,\boldsymbol{\ell}}$ be given. It is clear from its definition that $\|v\|_{\boldsymbol{\ell}}$ is a semi-norm. Applying Theorem 1 and since $v \in \mathrm{H}_0^1(\Omega)$, we have that

$$\|v\|_{\boldsymbol{\ell}} = 0 \implies \|v\|_{\mathrm{H}_0^1(\Omega)} = 0 \implies v = 0\,.$$

□

Lemma 7. *We have that*

$$\|v\|_{\boldsymbol{\ell}} \leq \|v\|_{\mathrm{H}_0^1(\Omega)} \quad \forall v \in \mathcal{V}_{1,\boldsymbol{\ell}}\,. \tag{49}$$

Moreover, if $\boldsymbol{\ell}(E)$ *satisfies* (12) $\forall E \in \mathcal{M}_h$, *then*

$$\exists c_* > 0\colon\ \|v\|_{\boldsymbol{\ell}} \geq c_* \|v\|_{\mathrm{H}_0^1(\Omega)} \quad \forall v \in \mathcal{V}_{1,\boldsymbol{\ell}}\,, \tag{50}$$

where c_* *does not depend on* h.

Proof. Relation (49) follows immediately by the definition of $\Pi^0_{l,E}$ and an application of the Cauchy-Schwarz inequality. Indeed, let $E \in \mathcal{M}_h$, then

$$\left\|\Pi^0_{l,E}\nabla v\right\|_E^2 = \left(\Pi^0_{l,E}\nabla v, \Pi^0_{l,E}\nabla v\right)_E = \left(\nabla v, \Pi^0_{l,E}\nabla v\right)_E \leq \|\nabla v\|_{[\mathrm{L}^2(E)]^2} \left\|\Pi^0_{l,E}\nabla v\right\|_{[\mathrm{L}^2(E)]^2}\,.$$

On the other hand, by standard scaling arguments we have

$$\|v\|_{\boldsymbol{\ell}}^2 = \sum_{E\in\mathcal{M}_h} \left\|\Pi^0_{l,E}\nabla v\right\|^2_{[\mathrm{L}^2(E)]^2} = \sum_{E\in\mathcal{M}_h} \left\|\hat{\Pi}^0_{l,\hat{E}}\hat{\nabla}\left(\hat{v} - P_0(\hat{v})\right)\right\|^2_{[\mathrm{L}^2(\hat{E})]^2}\,.$$

Notice that $\forall \hat{E} \in \Sigma$, where Σ is the set of admissible reference elements, $\hat{v} - \mathrm{P}_0(\hat{v}) \in \mathcal{V}_{1,l}^{\hat{E},\mathrm{P}_0}$. Moreover, $\forall \hat{w} \in \mathcal{V}_{1,l}^{\hat{E},\mathrm{P}_0}$ both $\left\|\hat{\Pi}^0_{l,\hat{E}}\hat{\nabla}\hat{w}\right\|_{[\mathrm{L}^2(\hat{E})]^2}$ and $\left\|\hat{\nabla}\hat{w}\right\|_{[\mathrm{L}^2(\hat{E})]^2}$ are norms. Then we can apply, by standard arguments about the equivalence of norms on finite dimensional spaces, we obtain $\forall \hat{E} \in \Sigma$

$$\left\|\hat{\Pi}^0_{l,\hat{E}}\hat{\nabla}\hat{w}\right\|_{[\mathrm{L}^2(\hat{E})]^2} \geq C(\hat{E}) \left\|\hat{\nabla}\hat{w}\right\|_{[\mathrm{L}^2(\hat{E})]^2} \tag{51}$$

where

$$C(\hat{E}) = \frac{\min_{\hat{z}\in\mathcal{V}_{1,l}^{\hat{E},\mathrm{P}_0}:\|\mathrm{dof}(\hat{z})\|_{l_2}=1} \left\|\hat{\Pi}^0_{l,\hat{E}}\hat{\nabla}\hat{z}\right\|_{[\mathrm{L}^2(\hat{E})]^2}}{\sqrt{N_E^V - 1}\max_{i=1,\dots,N_E^V-1}\left\|\hat{\nabla}\hat{\psi}_i\right\|_{[\mathrm{L}^2(\hat{E})]^2}}\,. \tag{52}$$

$C(\hat{E})$ is a continuous function on Σ, which is a compact set by Lemma 3. Indeed, $\hat{\Pi}^0_{l,\hat{E}}$ is continuous on Σ, as well as functions in $\mathcal{V}_{1,l}^{\hat{E},\mathrm{P}_0}$ following proofs of [28, Lemma 4.9] and [12, Lemma 4.5]. Moreover, $C(\hat{E}) > 0$, $\forall \hat{E} \in \Sigma$. Indeed, applying Proposition 2, it holds that $\forall \hat{z} \in \mathcal{V}_{1,l}^{\hat{E},\mathrm{P}_0} : \|\mathrm{dof}\,(\hat{z})\|_{l_2} = 1$,

$$\left\|\hat{\Pi}^0_{l,\hat{E}}\hat{\nabla}\hat{z}\right\|^2_{\left[\mathrm{L}^2\left(\hat{E}\right)\right]^2} = \left(\hat{\nabla}\hat{z}, \hat{\Pi}^0_{l,\hat{E}}\hat{\nabla}\hat{z}\right)_{\hat{E}} = \left(\hat{z}, \hat{\Pi}^0_{l,\hat{E}}\hat{\nabla}\hat{z}\cdot \boldsymbol{n}^{\partial\hat{E}}\right)_{\partial\hat{E}} = b(\hat{z}_R, \hat{\Pi}^0_{l,\hat{E}}\hat{\nabla}\hat{z})$$

$$\geq \beta \left\|\hat{\Pi}^0_{l,\hat{E}}\hat{\nabla}\hat{z}\right\|_{\left[\mathrm{L}^2\left(\hat{E}\right)\right]^2} \|\hat{z}_R\|_{\mathrm{H}^1_{\mathcal{T}}(\hat{E})} > 0\,,$$

where $\hat{z}_R$ is the lifting of $\gamma^{\partial\hat{E}}(\hat{z})$ on $\mathcal{R}_{\mathcal{Q}}(\hat{E})$. Then, $\exists m > 0$ such that $m := \min_{\hat{E}\in\Sigma} C(\hat{E})$. Finally, by standard scaling argument we obtain

$$\|v\|^2_{\boldsymbol{\ell}} \geq m^2 \sum_{E\in\mathcal{M}_h} \left\|\hat{\nabla}\left(\hat{v} - \mathrm{P}_0(\hat{v})\right)\right\|^2_{\left[\mathrm{L}^2\left(\hat{E}\right)\right]^2} = m^2 \|v\|_{\mathrm{H}^1_0(\Omega)}\,. \tag{53}$$

□

In the following theorem, we provide a proof of the continuity and the coercivity of the discrete bilinear form. The coercivity property follows from Lemma 7.

Theorem 2. *Let a_h be the bilinear form defined by* (9)*. Then,*

$$a_h\,(w,v) \leq \|w\|_{\mathrm{H}^1_0(\Omega)} \|v\|_{\mathrm{H}^1_0(\Omega)}\,\forall w,v \in \mathcal{V}_{1,\boldsymbol{\ell}}\,. \tag{54}$$

Moreover, suppose $\boldsymbol{\ell}(E)$ satisfies (12) *$\forall E \in \mathcal{M}_h$. Then,*

$$\exists C > 0,\ \textit{independent of } h\colon a_h\,(w,w) \geq C\,\|w\|^2_{\mathrm{H}^1_0(\Omega)}\,\forall w \in \mathcal{V}_{1,\boldsymbol{\ell}}\,. \tag{55}$$

Proof. Let $w, v \in \mathcal{V}_{1,\boldsymbol{\ell}}$ be given. Applying the Cauchy-Schwarz inequality and (49) we get

$$\begin{aligned} a_h\,(w,v) &= \sum_{E\in\mathcal{M}_h} \left(\Pi^0_{\boldsymbol{\ell}(E),E}\nabla w, \Pi^0_{\boldsymbol{\ell}(E),E}\nabla v\right)_E \\ &\leq \sum_{E\in\mathcal{M}_h} \left\|\Pi^0_{\boldsymbol{\ell}(E),E}\nabla w\right\|_{[\mathrm{L}^2(E)]^2} \left\|\Pi^0_{\boldsymbol{\ell}(E),E}\nabla v\right\|_{[\mathrm{L}^2(E)]^2} \\ &\leq \|w\|_{\boldsymbol{\ell}} \|v\|_{\boldsymbol{\ell}} \leq \|w\|_{\mathrm{H}^1_0(\Omega)} \|v\|_{\mathrm{H}^1_0(\Omega)}\,. \end{aligned}$$

Moreover, assuming that $\boldsymbol{\ell}(E)$ satisfies (12) $\forall E \in \mathcal{M}_h$, we can apply the lower bound in (50) and get

$$a_h\,(w,w) = \|w\|^2_{\boldsymbol{\ell}} \geq (c_*)^2 \|w\|^2_{\mathrm{H}^1_0(\Omega)}\,.$$

□

This theorem implies that the bilinear form a_h of the problem (10) satisfies the hypothesis of Lax-Milgram theorem, then the problem admits a unique solution.

5 A priori error estimates

In this section we derive error estimates for the proposed method, in H^1_0 norm and in the standard L^2 norm. First, we recall classical results for Virtual Element Methods concerning the interpolation error and the polynomial projection error (see [27, 7]).

Lemma 8. *Let U be a smooth enough function, then there exists $C > 0$ such that $\forall\, h$, $\exists U_{\mathrm{I}} \in \mathcal{V}_{1,\boldsymbol{\ell}}$ such that*

$$\|U - U_{\mathrm{I}}\|_{\mathrm{L}^2(\Omega)} + h \|U - U_{\mathrm{I}}\|_{\mathrm{H}^1_0(\Omega)} \leq Ch^2 |U|_2 \,. \tag{56}$$

Proof. The proof of this result can be obtained following the same arguments as in [27, Theorem 11]. □

Lemma 9 ([7, Lemma 5.1])**.** *Let U be a smooth enough function, there exist $C_1, C_2 > 0$ such that*

$$\left\|\Pi^0_{\boldsymbol{\ell}} \nabla U - \nabla U\right\|_{\mathrm{L}^2(\Omega)} \leq C_1 h \,|U|_2 \,, \tag{57}$$

$$\left\|\Pi^0_0 U - U\right\|_{\mathrm{L}^2(\Omega)} \leq C_2 h \,\|U\|_{\mathrm{H}^1_0(\Omega)} \,. \tag{58}$$

Theorem 3. *Let $U \in \mathrm{H}^2(\Omega) \cap \mathrm{H}^1_0(\Omega)$ and $f \in \mathrm{L}^2(\Omega)$ be the solution and the right-hand side of* (1)*, respectively. For h sufficiently small, $\exists C > 0$ such that the unique solution $u \in \mathcal{V}_{1,\boldsymbol{\ell}}$ of problem* (10) *satisfies the following error estimate:*

$$\|U - u\|_{\mathrm{H}^1_0(\Omega)} \leq Ch \left(|U|_2 + \|f\|_{\mathrm{L}^2(\Omega)}\right) \,. \tag{59}$$

Proof. Let U_{I} be given by Lemma 8. Applying the triangle inequality, we have

$$\|U - u\|_{\mathrm{H}^1_0(\Omega)} \leq \|U - U_{\mathrm{I}}\|_{\mathrm{H}^1_0(\Omega)} + \|U_{\mathrm{I}} - u\|_{\mathrm{H}^1_0(\Omega)} \,. \tag{60}$$

We deal with the two terms separately. The first one can be bounded applying (56), i.e.

$$\|U - U_{\mathrm{I}}\|_{\mathrm{H}^1_0(\Omega)} \leq Ch \,|U|_2 \,. \tag{61}$$

On the other hand, in order to deal with the second term of (60) let us denote by $\varepsilon = U_{\mathrm{I}} - u$. First, applying the coercivity of the bilinear form a_h (55) and the discrete problem (10), we have that $\exists C > 0$:

$$C \|\varepsilon\|^2_{\mathrm{H}^1_0(\Omega)} \leq a_h(\varepsilon, \varepsilon) = a_h(U_{\mathrm{I}}, \varepsilon) - a_h(u, \varepsilon) = a_h(U_{\mathrm{I}}, \varepsilon) - \sum_{E \in \mathcal{M}_h} \left(f, \Pi^0_{0,E} \varepsilon\right)_E \,. \tag{62}$$

Applying the definition of the L^2 projectors and adding and subtracting terms, i.e. $\Pi^0_{l,E}\nabla U$ and ∇U, we have

$$\begin{aligned}
a_h(\varepsilon,\varepsilon) &= a_h(U_\mathrm{I}-U,\varepsilon)+a_h(U,\varepsilon)-\sum_{E\in\mathcal{M}_h}\left(\Pi^0_{0,E}f,\varepsilon\right)_E\\
&= a_h(U_\mathrm{I}-U,\varepsilon)+\sum_{E\in\mathcal{M}_h}\left(\Pi^0_{l,E}\nabla U-\nabla U,\nabla\varepsilon\right)_E+(\nabla U,\nabla\varepsilon)_E-\left(\Pi^0_{0,E}f,\varepsilon\right)_E\\
&= a_h(U_\mathrm{I}-U,\varepsilon)+\sum_{E\in\mathcal{M}_h}\left(\Pi^0_{l,E}\nabla U-\nabla U,\nabla\varepsilon\right)_E+\left(f-\Pi^0_{0,E}f,\varepsilon\right)_E.
\end{aligned}$$

Let us consider the last three terms separately. The first one can be bounded applying (54) and (56), i.e.

$$a_h(U_\mathrm{I}-U,\varepsilon)\le C\,\|U_\mathrm{I}-U\|_{\mathrm{H}^1_0(\Omega)}\|\varepsilon\|_{\mathrm{H}^1_0(\Omega)}\le Ch\,|U|_2\,\|\varepsilon\|_{\mathrm{H}^1_0(\Omega)}. \tag{63}$$

Applying the Cauchy-Schwarz inequality and (57), the second term can be bounded as follows:

$$\begin{aligned}
\sum_{E\in\mathcal{M}_h}\left(\Pi^0_{l,E}\nabla U-\nabla U,\nabla\varepsilon\right)_E &\le \sum_{E\in\mathcal{M}_h}\left\|\Pi^0_{l,E}\nabla U-\nabla U\right\|_{\mathrm{L}^2(E)}\|\varepsilon\|_{\mathrm{H}^1_0(E)}\\
&\le Ch\,|U|_2\,\|\varepsilon\|_{\mathrm{H}^1_0(\Omega)}.
\end{aligned}\tag{64}$$

The last term can be bounded applying the definition of $\Pi^0_{0,E}$, the Cauchy-Schwarz inequality and (58), i.e.

$$\begin{aligned}
\sum_{E\in\mathcal{M}_h}\left(f-\Pi^0_{0,E}f,\varepsilon\right)_E &= \sum_{E\in\mathcal{M}_h}\left(f,\varepsilon-\Pi^0_{0,E}\varepsilon\right)_E\\
&\le \sum_{E\in\mathcal{M}_h}\|f\|_{\mathrm{L}^2(E)}\left\|\varepsilon-\Pi^0_{0,E}\varepsilon\right\|_{\mathrm{L}^2(E)}\le Ch\,\|f\|_{\mathrm{L}^2(\Omega)}\|\varepsilon\|_{\mathrm{H}^1_0(\Omega)}.
\end{aligned}\tag{65}$$

Finally, applying together (63),(64) and (65) into (62) and simplifying, we have

$$\|\varepsilon\|_{\mathrm{H}^1_0(\Omega)}\le Ch\left(|U|_2+\|f\|_{\mathrm{L}^2(\Omega)}\right). \tag{66}$$

Considering together (61) and (66) we prove (59). □

Theorem 4. *Let $U\in\mathrm{H}^2(\Omega)\cap\mathrm{H}^1_0(\Omega)$ and $f\in\mathrm{H}^1(\Omega)$ be the solution and the right-hand side of* (1)*, respectively. For h sufficiently small, $\exists C>0$ such that the unique solution $u\in\mathcal{V}_{1,\boldsymbol{\ell}}$ of problem* (10) *satisfies the following error estimate:*

$$\|U-u\|_{\mathrm{L}^2(\Omega)}\le Ch^2\left(|U|_2+\|f\|_{\mathrm{H}^1_0(\Omega)}\right). \tag{67}$$

Proof. The proof of this theorem can be found in the supplementary materials of this paper. □

Remark 6. *Denoting by* $\Pi^0_{1,E}$ *the* L^2*-projector from* $\mathrm{L}^2(E)$ *to* $\mathbb{P}_1(E)$*, we can define the discete problem* (10) *as*

$$a_h(u,v) = \sum_{E \in \mathcal{M}_h} \left(f, \Pi^0_{1,E} v\right)_E \quad \forall v \in \mathcal{V}_{1,\boldsymbol{\ell}}\,,$$

and we can require $f \in \mathrm{L}^2(\Omega)$ *so* (67) *still holds as*

$$\|U - u\|_{\mathrm{L}^2(\Omega)} \leq Ch^2 \left(|U|_2 + \|f\|_{\mathrm{L}^2(\Omega)}\right) .$$

6 Numerical Results

Let us consider problem (1) on the unit square with homogeneous Dirichlet boundary conditions and the right-hand side defined such that the exact solution is

$$U_{ex} = \sin(2\pi x)\sin(2\pi y).$$

In the following, we show, in log-log scale plots, the convergence curves of the L^2 and H^1 errors that we measure respectively as follows,

$$\mathrm{L}^2 \text{ error } = \sqrt{\sum_{E \in \mathcal{M}_h} \left\| \Pi^\nabla_{1,E} u - U_{ex} \right\|^2_{\mathrm{L}^2(E)}},$$

$$\mathrm{H}^1 \text{ error } = \sqrt{\sum_{E \in \mathcal{M}_h} \left\| \nabla \Pi^\nabla_{1,E} u - \nabla U_{ex} \right\|^2_{\mathrm{L}^2(E)}},$$

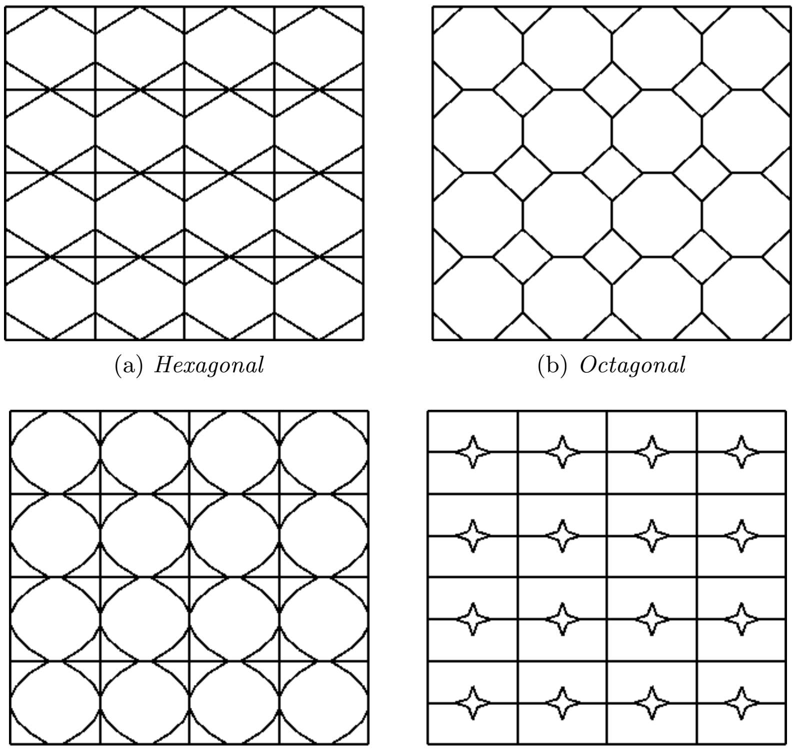
where u is the discrete solution of (10). Then, for each polygon $E \in \mathcal{M}_h$ we choose l such that the sufficient condition (12) is satisfied (see Table 1 for some choices of l).

Table 1: Sufficient l for polygons that have up to 20 edges.

N^V_E	l
3	0
from 4 to 7	1
from 8 to 13	2
from 14 to 20	3

6.1 Meshes

We consider four sequences of meshes for the convergence test. The first sequence, labeled *Hexagonal*, is a tesselation made by hexagons and triangles, as it is shown in Figure 1a. The second sequence, shown in Figure 1b and labeled *Octagonal*, is made by octagons, squares and triangles. Then, the third sequence, labeled *Hexadecagonal*, is made by hexadecagons and concave pentagons, as it is shown in Figure 1c. Finally, the last sequence, labeled *Star Concave*, is a non-convex tessellation made by octagons and nonagons, as it is shown in Figure 1d. In each case we start from a mesh of



(a) *Hexagonal* (b) *Octagonal*

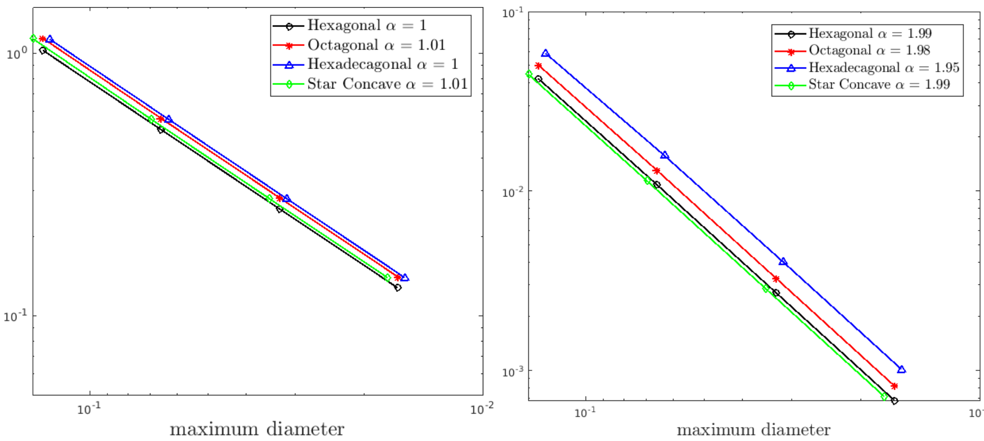
(c) *Hexadecagonal* (d) *Star Concave*

Figure 1: Meshes

$\#\mathcal{M}_h$ polygons then we refine it, obtaining meshes made by $4\#\mathcal{M}_h$, $16\#\mathcal{M}_h$ and $64\#\mathcal{M}_h$ polygons. The first and the third sequence start with $\#\mathcal{M}_h$ equal to 320, the second and the fourth with $\#\mathcal{M}_h$ equal to 164 and 192 respectively.

6.2 Convergence results

For the four mesh sequences, we report the trend of the H^1 and the L^2 errors in Figure 2a and in Figure 2b, respectively, decreasing the maximum diameter of the polygons. In the legends, we report the computed convergence rates with respect to h, denoted by α. We see that we get the expected values for all the meshes, as obtained in (59) and (67).



(a) H^1 *error* (b) L^2 *error*

Figure 2: Logarithmic convergence plots

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A Supplementary materials

A.1 Proof of Lemma 4

In order to show the proof, we have to present a preliminary result.

Lemma 10. *Let $\bar{q} \in \mathcal{R}_{\mathcal{Q}}(E)$. Then $\exists C > 0$, independent of h_E, such that*

$$\sum_{i=1}^{N_E^V} |\bar{q}(x_i)| \leq C \sqrt{\sum_{\tau \in \mathcal{T}_E} \|\nabla \bar{q}\|^2_{\mathrm{L}^2(\tau)}} \,. \tag{68}$$

Proof. We notice that

$$\sum_{i=1}^{N_E^V} |\bar{q}(x_i)| = \frac{1}{2} \sum_{\tau \in \mathcal{T}_E} \left(|\bar{q}(x_{\tau,1})| + |\bar{q}(x_{\tau,2})| \right) \,, \tag{69}$$

where $x_{\tau,1}$ and $x_{\tau,2}$ are the vertices of τ that are on ∂E. We have that

$$\bar{q}_{|\tau} \in \tilde{\mathbb{P}}_1(\tau) = \left\{ p \in \mathbb{P}_1(\tau) : p(x_C) = 0 \right\} \,,$$

and

$$|\bar{q}(x_{\tau,1})| + |\bar{q}(x_{\tau,2})| = \left\| \mathrm{dof}_{\tilde{\mathbb{P}}_1(\tau)}\left(\bar{q}_{|\tau}\right) \right\|_{l_1},$$

having chosen the values at $x_{\tau,1}$ and $x_{\tau,2}$ as set of degrees of freedom on $\tilde{\mathbb{P}}_1(\tau)$ and denoting by $\mathrm{dof}_{\tilde{\mathbb{P}}_1(\tau)}(\cdot)$ the operator returning the vector of such values. Using the mapping (16) we get

$$\left\| \mathrm{dof}_{\tilde{\mathbb{P}}_1(\tau)}\left(\bar{q}_{|\tau}\right) \right\|_{l_1} = \left\| \mathrm{dof}_{\tilde{\mathbb{P}}_1(\hat{\tau})}\left(\hat{\bar{q}}_{|\hat{\tau}}\right) \right\|_{l_1}.$$

The right-hand side of the above equation is a norm on $\tilde{\mathbb{P}}_1(\hat{\tau})$, as well as $\left\| \hat{\nabla}\hat{\bar{q}} \right\|_{\mathrm{L}^2(\hat{\tau})}$. Then, by standard arguments about the equivalence of norms in finite dimensional spaces, we have

$$\left\| \mathrm{dof}_{\tilde{\mathbb{P}}_1(\hat{\tau})}\left(\hat{\bar{q}}_{|\hat{\tau}}\right) \right\|_{l_1} \leq \frac{\sqrt{2}\max_{i=1,2}\left\| \mathrm{dof}_{\tilde{\mathbb{P}}_1(\hat{\tau})}\left(\hat{\chi}_i\right) \right\|_{l_1}}{\min_{\hat{w}\in\tilde{\mathbb{P}}_1(\hat{\tau}):\ \hat{w}(\hat{x}_{\hat{\tau},1})^2+\hat{w}(\hat{x}_{\hat{\tau},2})^2=1}\left\| \hat{\nabla}\hat{w} \right\|_{\mathrm{L}^2(\hat{\tau})}} \left\| \hat{\nabla}\hat{\bar{q}} \right\|_{\mathrm{L}^2(\hat{\tau})},$$

where the $\hat{\chi}_i$ are Lagrangian in the degrees of freedom. Then, $\left\| \mathrm{dof}_{\tilde{\mathbb{P}}_1(\hat{\tau})}\left(\hat{\chi}_1\right) \right\|_{l_1} = \left\| \mathrm{dof}_{\tilde{\mathbb{P}}_1(\hat{\tau})}\left(\hat{\chi}_2\right) \right\|_{l_1} = 1$ and

$$\left\| \mathrm{dof}_{\tilde{\mathbb{P}}_1(\hat{\tau})}\left(\hat{\bar{q}}_{|\hat{\tau}}\right) \right\|_{l_1} \leq \frac{\sqrt{2}}{\min_{\hat{w}\in\tilde{\mathbb{P}}_1(\hat{\tau}):\ \hat{w}(\hat{x}_{\hat{\tau},1})^2+\hat{w}(\hat{x}_{\hat{\tau},2})^2=1}\left\| \hat{\nabla}\hat{w} \right\|_{\mathrm{L}^2(\hat{\tau})}} \left\| \hat{\nabla}\hat{\bar{q}} \right\|_{\mathrm{L}^2(\hat{\tau})}.$$

It can be proved by standard arguments that the constant in the above inequality is continuous with respect to $\hat{\tau}$, since it depends continuously on the deformation of the domain (see the proofs of [28, Lemma 4.9] and [12, Lemma 4.5]). It follows by compactness of the set of admissible reference elements, denoted by Σ, (Lemma 3) that there exists $M > 0$ such that

$$M = \max_{\hat{\tau}\in\Sigma} \frac{\sqrt{2}}{\min_{\hat{w}\in\tilde{\mathbb{P}}_1(\hat{\tau}):\ \hat{w}(\hat{x}_{\hat{\tau},1})^2+\hat{w}(\hat{x}_{\hat{\tau},2})^2=1}\left\| \hat{\nabla}\hat{w} \right\|_{\mathrm{L}^2(\hat{\tau})}},$$

and thus, starting again from (69) and applying the mapping (16), we get

$$\begin{aligned}
\sum_{i=1}^{N_E^V} |\bar{q}(x_i)| &= \frac{1}{2}\sum_{\tau\in\mathcal{T}_E} \left\| \mathrm{dof}_{\tilde{\mathbb{P}}_1(\tau)}\left(\bar{q}_{|\tau}\right) \right\|_{l_1} = \frac{1}{2}\sum_{\hat{\tau}\in\mathcal{T}_{\hat{E}}} \left\| \mathrm{dof}_{\tilde{\mathbb{P}}_1(\hat{\tau})}\left(\hat{\bar{q}}_{|\hat{\tau}}\right) \right\|_{l_1} \\
&\leq \frac{M}{2}\sum_{\hat{\tau}\in\mathcal{T}_{\hat{E}}} \left\| \hat{\nabla}\hat{\bar{q}} \right\|_{\mathrm{L}^2(\hat{\tau})} = \frac{M}{2}\sum_{\tau\in\mathcal{T}_E} \left\| \nabla\bar{q} \right\|_{\mathrm{L}^2(\tau)} \\
&\leq \frac{M\sqrt{N_E^V}}{2}\sqrt{\sum_{\tau\in\mathcal{T}_E} \left\| \nabla\bar{q} \right\|^2_{\mathrm{L}^2(\tau)}},
\end{aligned}$$

and we obtain (68) since N_E^V is uniformly bounded by (4). □

Now, we can present the proof of Lemma 4.

Proof. Let $\bar{q} \in \mathcal{R}_{\mathcal{Q}}(E)$ and $\boldsymbol{v} \in [V]^2$ be given. Starting from (23) and applying the triangular inequality, we have

$$|b(\bar{q},\boldsymbol{v})| \leq \left| \sum_{\tau\in\mathcal{T}_E} \int_\tau \left[\nabla\bar{q}\,\boldsymbol{v} + \bar{q}\,\nabla\cdot\boldsymbol{v}\right]\,dx \right| + \left| \sum_{i=1}^{N_E^V} \int_{e_i} \gamma^{e_i}(\bar{q})\,[\![\boldsymbol{v}]\!]_{e_i}\cdot\boldsymbol{n}^{e_i} ds \right|. \quad (70)$$

Let us consider separately the two terms involved in the inequality. The first part can be analysed applying the properties,

$$\forall \boldsymbol{v} \in [V]^2, \quad \|\nabla\cdot\boldsymbol{v}\|^2_{\mathrm{L}^2(\tau)} \leq 2\,\|\nabla\boldsymbol{v}\|^2_{[\mathrm{L}^2(\tau)]^4}$$

$$\forall \bar{q} \in \mathcal{R}_{\mathcal{Q}}(E), \quad \sum_{\tau\in\mathcal{T}_E}\left(\|\bar{q}\|_{\mathrm{L}^2(\tau)} + \|\nabla\bar{q}\|_{[\mathrm{L}^2(\tau)]^2}\right) \leq \sqrt{2N_E^V}\,\|\bar{q}\|_{\mathrm{H}^1_{\mathcal{T}}(E)}$$

and the mesh assumption (4), as follows

$$\begin{aligned}
\left| \sum_{\tau\in\mathcal{T}_E} \int_\tau \left[\nabla\bar{q}\,\boldsymbol{v} + \bar{q}\,\nabla\cdot\boldsymbol{v}\right]\,dx \right| &\leq \sum_{\tau\in\mathcal{T}_E}\left(\|\nabla\bar{q}\|_{[\mathrm{L}^2(\tau)]^2}\,\|\boldsymbol{v}\|_{[\mathrm{L}^2(\tau)]^2} + \|\bar{q}\|_{\mathrm{L}^2(\tau)}\,\|\nabla\cdot\boldsymbol{v}\|_{\mathrm{L}^2(\tau)}\right) \\
&\leq \sum_{\tau\in\mathcal{T}_E}\|\nabla\bar{q}\|_{[\mathrm{L}^2(\tau)]^2}\left(\|\boldsymbol{v}\|_{[\mathrm{L}^2(\tau)]^2} + \|\nabla\boldsymbol{v}\|_{[\mathrm{L}^2(\tau)]^4}\right) \\
&\quad + \sum_{\tau\in\mathcal{T}_E}\|\bar{q}\|_{\mathrm{L}^2(\tau)}\left(\|\boldsymbol{v}\|_{[\mathrm{L}^2(\tau)]^2} + \sqrt{2}\,\|\nabla\boldsymbol{v}\|_{[\mathrm{L}^2(\tau)]^4}\right) \\
&\leq C\sum_{\tau\in\mathcal{T}_E}\left(\|\boldsymbol{v}\|_{[\mathrm{L}^2(\tau)]^2} + \|\nabla\boldsymbol{v}\|_{[\mathrm{L}^2(\tau)]^4}\right) \\
&\quad \times\left(\|\nabla\bar{q}\|_{\mathrm{L}^2(\tau)} + \|\bar{q}\|_{\mathrm{L}^2(\tau)}\right) \\
&\leq C\,\|\bar{q}\|_{\mathrm{H}^1_{\mathcal{T}}(E)}\sum_{\tau\in\mathcal{T}_E}\left(\|\boldsymbol{v}\|_{[\mathrm{L}^2(\tau)]^2} + \|\nabla\boldsymbol{v}\|_{[\mathrm{L}^2(\tau)]^4}\right).
\end{aligned}$$

Moreover, let us consider the second term of (70), computing exactly the term $\|\gamma^{e_i}(\bar{q})\|_{\mathrm{L}^2(e_i)}$ and applying the properties $\forall\boldsymbol{v}\in[V]^2$

$$\sum_{i=1}^{N_E^V}\left\|[\![\boldsymbol{v}]\!]_{e_i}\right\|_{\mathrm{L}^2(e_i)} \leq \sqrt{2N_E^V}\sqrt{\sum_{i=1}^{N_E^V}\left\|[\![\boldsymbol{v}]\!]_{e_i}\right\|^2_{\mathrm{L}^2(e_i)}}\,,$$

$$\left\|[\![\boldsymbol{v}]\!]_{e_i}\right\|^2_{\mathrm{L}^2(e_i)} \leq h_E\left\|[\![\boldsymbol{v}]\!]_{\mathcal{I}_{\mathcal{E}_E}}\right\|^2_{\mathrm{L}^\infty(\mathcal{I}_{\mathcal{E}_E})}\,,\ \forall e_i\in\mathcal{I}_{\mathcal{E}_E}\,,$$

we have

$$
\begin{aligned}
\left|\sum_{i=1}^{N_E^V} \int_{e_i} \gamma^{e_i}(\bar{q}) \left[\!\left[\boldsymbol{v}\right]\!\right]_{e_i} \cdot \boldsymbol{n}^{e_i} ds\right| &\leq \sum_{i=1}^{N_E^V} \left\|\gamma^{e_i}(\bar{q})\right\|_{\mathrm{L}^2(e_i)} \left\|\left[\!\left[\boldsymbol{v}\right]\!\right]_{e_i} \cdot \boldsymbol{n}^{e_i}\right\|_{\mathrm{L}^2(e_i)} \\
&\leq \sum_{i=1}^{N_E^V} \frac{\sqrt{h_{e_i}}}{\sqrt{3}} \left|\bar{q}(x_i)\right| \left\|\left[\!\left[\boldsymbol{v}\right]\!\right]_{e_i}\right\|_{[\mathrm{L}^2(e_i)]^2} \\
&\leq \frac{h_E}{\sqrt{3}} \left\|\left[\!\left[\boldsymbol{v}\right]\!\right]_{\mathcal{I}_{\mathcal{E}_E}}\right\|_{\mathrm{L}^\infty\left(\mathcal{I}_{\mathcal{E}_E}\right)} \sum_{i=1}^{N_E^V} \left|\bar{q}(x_i)\right| \\
&\leq C h_E \left\|\left[\!\left[\boldsymbol{v}\right]\!\right]_{\mathcal{I}_{\mathcal{E}_E}}\right\|_{\mathrm{L}^\infty\left(\mathcal{I}_{\mathcal{E}_E}\right)} \left\|\bar{q}\right\|_{\mathrm{H}^1_{\mathcal{T}}(E)} ,
\end{aligned}
$$

where we apply Lemma 10 in the last step. Finally, substituting into (70), we obtain

$$
\begin{aligned}
\left|b(\bar{q}, \boldsymbol{v})\right| &\leq C \left\|\bar{q}\right\|_{\mathrm{H}^1_{\mathcal{T}}(E)} \left(\sum_{\tau \in \mathcal{T}_E} \left(\left\|\boldsymbol{v}\right\|_{[\mathrm{L}^2(\tau)]^2} + \left\|\nabla \boldsymbol{v}\right\|_{[\mathrm{L}^2(\tau)]^4} \right) + h_E \left\|\left[\!\left[\boldsymbol{v}\right]\!\right]_{\mathcal{I}_{\mathcal{E}_E}}\right\|_{\mathrm{L}^\infty\left(\mathcal{I}_{\mathcal{E}_E}\right)} \right) \\
&\leq C \left\|\bar{q}\right\|_{\mathrm{H}^1_{\mathcal{T}}(E)} \left\|\boldsymbol{v}\right\|_{[V]^2} .
\end{aligned}
$$

□

A.2 Proof of Lemma 5

Proof. Let $\hat{E} \in \Sigma$. First, we want to prove that $\exists C_\Pi(\hat{E}) > 0$ such that, for all $\hat{v} \in \mathrm{H}^1_{\mathcal{T}}(\hat{E})$,

$$
\int_{\hat{E}} \left|\hat{v} - \Pi^0_{0,\hat{E}} \hat{v}\right| d\hat{x} \leq C_\Pi(\hat{E}) \left(\sum_{\hat{\tau} \in \mathcal{T}_{\hat{E}}} \int_{\hat{\tau}} \left|\nabla \hat{v}\right| d\hat{x} + \sum_{\hat{e} \in \mathcal{I}_{\mathcal{E}_{\hat{E}}}} \int_{\hat{e}} \left|\left[\!\left[\hat{v}\right]\!\right]_{\hat{e}}\right| d\hat{s} \right) , \tag{71}
$$

where $|\nabla \hat{v}|$ denotes the Euclidean norm of $\nabla \hat{v}$. In order to simplify the notation, let us denote by $\mathcal{D}(\hat{v})$ the right hand side of (71), i.e.

$$
\mathcal{D}(\hat{v}) = \sum_{\hat{\tau} \in \mathcal{T}_{\hat{E}}} \int_{\hat{\tau}} \left|\nabla \hat{v}\right| d\hat{x} + \sum_{\hat{e} \in \mathcal{I}_{\mathcal{E}_{\hat{E}}}} \int_{\hat{e}} \left|\left[\!\left[\hat{v}\right]\!\right]_{\hat{e}}\right| d\hat{s} .
$$

By contradiction, suppose

$$
\forall C > 0, \ \exists \hat{v} \in \mathrm{H}^1_{\mathcal{T}}(\hat{E}) \colon \ \left\|\hat{v} - \Pi^0_{0,\hat{E}} \hat{v}\right\|_{\mathrm{L}^1(\hat{E})} > C \, \mathcal{D}(\hat{v}) .
$$

Notice that $\mathcal{D}(\hat{v})$ is a seminorm for all $\hat{v}$ in $\mathrm{H}^1_{\mathcal{T}}(\hat{E})$. In this proof, we consider $\mathrm{H}^1_{\mathcal{T}}(E) \subset \mathrm{L}^1(\hat{E})$ endowed with the L^1-norm. Then, it is possible to define a sequence $\hat{w}_k \in \mathrm{H}^1_{\mathcal{T}}(\hat{E})$ such that, $\forall k \in \mathbb{N}$,

$$
\left\|\hat{w}_k - \Pi^0_{0,\hat{E}} \hat{w}_k\right\|_{\mathrm{L}^1(\hat{E})} > k \mathcal{D}(\hat{w}_k), \quad \left\|\hat{w}_k - \Pi^0_{0,\hat{E}} \hat{w}_k\right\|_{\mathrm{L}^1(\hat{E})} = 1 ,
$$

which means that

$$\mathcal{D}(\hat{w}_k) < \frac{1}{k} \Rightarrow \mathcal{D}(\hat{w}_k) \to 0\,.$$

If we define $\hat{u}_k = \hat{w}_k - \Pi^0_{0,\hat{E}}\hat{w}_k$, we have, since $\Pi^0_{0,\hat{E}}\hat{w}_k$ is constant,

$$\mathcal{D}(\hat{u}_k) \leq \mathcal{D}(\hat{w}_k) \to 0\,. \tag{72}$$

Then, applying (72) and the fact that $\|\hat{u}_k\|_{\mathrm{L}^1(E)} = 1$, we can affirm that the sequence $\hat{u}_k$ is bounded in $\mathrm{H}^1_{\mathcal{T}}(\hat{E})$. Thus, it converges weakly in $\mathrm{H}^1_{\mathcal{T}}(\hat{E})$ to a function $\hat{u}^\star$ up to sub-sequences, i.e.

$$\hat{u}_{k_j} \overset{\mathrm{H}^1_{\mathcal{T}}(\hat{E})}{\rightharpoonup} \hat{u}^\star\,.$$

$\mathrm{H}^1_{\mathcal{T}}(\hat{E})$ is contained in the space of functions of bounded variations, thus it is compactly embedded in $\mathrm{L}^1(\hat{E})$ (see [2, Corollary 3.49]). Then, $\hat{u}_{k_j}$ converges to a function $\hat{u}^{\star\star}$ strongly in $\mathrm{L}^1(\hat{E})$, up to sub-sequences, and by uniqueness of the limit we have $\hat{u}^{\star\star} = \hat{u}^\star$. Let $\hat{u}_{\tilde{k}} = \hat{u}_{k_{j_l}}$ be such that

$$\hat{u}_{\tilde{k}} \overset{\mathrm{H}^1_{\mathcal{T}}(\hat{E})}{\rightharpoonup} \hat{u}^\star, \quad \hat{u}_{\tilde{k}} \overset{\mathrm{L}^1(\hat{E})}{\to} \hat{u}^\star\,.$$

By (72) and the definition of $\mathcal{D}$, $\mathcal{D}(\hat{u}^\star) = 0$, thus $\hat{u}^\star$ is constant. Since $\|\hat{u}^\star\|_{\mathrm{L}^1(\hat{E})} = 1$ then $\hat{u}^\star = \frac{1}{|\hat{E}|}$. It follows that

$$\hat{w}_{\tilde{k}} - \Pi^0_{0,\hat{E}}\hat{w}_{\tilde{k}} \overset{\mathrm{L}^1(E)}{\to} \hat{w}^\star - \Pi^0_{0,\hat{E}}\hat{w}^\star = \frac{1}{\left|\hat{E}\right|}$$

by continuity and linearity of $\Pi^0_{0,\hat{E}}$. By definition of $\Pi^0_{0,\hat{E}}$, this is a contradiction since

$$\hat{w}^\star - \Pi^0_{0,\hat{E}}\hat{w}^\star = const \implies const = 0\,,$$

then, (71) is true. Next, since $\mathrm{H}^1_{\mathcal{T}}(\hat{E})$ is continuously embedded in $\mathrm{L}^2(\hat{E})$ (see [2, Corollary 3.49]), endowing $\mathrm{H}^1_{\mathcal{T}}(\hat{E})$ with the L^1-norm, we have

$$\exists C_I(\hat{E}) > 0\colon \left\|\hat{v} - \Pi^0_{0,\hat{E}}\hat{v}\right\|_{\mathrm{L}^2(\hat{E})} \leq C_I(\hat{E}) \left\|\hat{v} - \Pi^0_{0,\hat{E}}\hat{v}\right\|_{\mathrm{L}^1(\hat{E})}\,. \tag{73}$$

Moreover, by Hölder's inequality and since $h_{\hat{E}} = 1$, we get, exploiting also Young's inequality,

$$\begin{aligned}
\mathcal{D}(\hat{v})^2 &= \left(\sum_{\hat{\tau}\in\mathcal{T}_{\hat{E}}} \int_{\hat{\tau}} |\nabla \hat{v}| \, d\hat{x} + \sum_{\hat{e}\in\mathcal{I}_{\mathcal{E}_{\hat{E}}}} \int_{\hat{e}} |[\![\hat{v}]\!]_{\hat{e}}| \, d\hat{s} \right)^2 \\
&\leq \left(\sum_{\hat{\tau}\in\mathcal{T}_{\hat{E}}} \left(\int_{\hat{\tau}} d\hat{x} \right)^{\frac{1}{2}} \left(\int_{\hat{\tau}} |\nabla \hat{v}|^2 \, d\hat{x} \right)^{\frac{1}{2}} + \sum_{\hat{e}\in\mathcal{I}_{\mathcal{E}_{\hat{E}}}} \left(\int_{\hat{e}} d\hat{s} \right)^{\frac{1}{2}} \left(\int_{\hat{e}} [\![\hat{v}]\!]_{\hat{e}}^2 \, d\hat{s} \right)^{\frac{1}{2}} \right)^2 \\
&\leq 2N_E^V \left(\sum_{\hat{\tau}\in\mathcal{T}_{\hat{E}}} |\hat{\tau}| \, \|\nabla \hat{v}\|^2_{\mathrm{L}^2(\hat{\tau})} + \sum_{\hat{e}\in\mathcal{I}_{\mathcal{E}_{\hat{E}}}} h_{\hat{e}} \, \|[\![\hat{v}]\!]_{\hat{e}}\|^2_{\mathrm{L}^2(\hat{e})} \right) \leq 2N_E^V \, |\hat{v}|^2_{\mathrm{H}^1_{\mathcal{T}}(\hat{E})} \,,
\end{aligned} \tag{74}$$

since $|\hat{\tau}| \leq \left|\hat{E}\right| \leq h^2_{\hat{E}} = 1$ and $h_e \leq h_{\hat{E}} = 1$. From (71), (73) and (74), we obtain that $\exists C(\hat{E}) = \sqrt{2N_E^V C_\Pi(\hat{E}) C_I(\hat{E})} > 0$ such that

$$\left\| \hat{v} - \Pi^0_{0,\hat{E}} \hat{v} \right\|_{\mathrm{L}^2(\hat{E})} \leq C(\hat{E}) \, |\hat{v}|_{\mathrm{H}^1_{\mathcal{T}}(\hat{E})} \,. \tag{75}$$

We are left to show that there exists a constant independent of $\hat{E}$ for which (31) holds. This is true because we can divide by $|\hat{v}|_{\mathrm{H}^1_{\mathcal{T}}(\hat{E})}$, since the case $|\hat{v}|_{\mathrm{H}^1_{\mathcal{T}}(\hat{E})} = 0$ is trivial, and obtain the ratio

$$\frac{\left\| \hat{v} - \Pi^0_{0,\hat{E}} \hat{v} \right\|_{\mathrm{L}^2(\hat{E})}}{|\hat{v}|_{\mathrm{H}^1_{\mathcal{T}}(\hat{E})}} \,,$$

which is a continuous function with respect to $\hat{E}$ and also bounded by (75). Thus, since Σ is a compact set by Lemma 3, then there exists $\hat{E}^\star \in \Sigma$ such that, $\forall \hat{E} \in \Sigma$,

$$\left\| \hat{v} - \Pi^0_{0,\hat{E}} \hat{v} \right\|_{\mathrm{L}^2(\hat{E})} \leq C(\hat{E}^\star) \, |\hat{v}|_{\mathrm{H}^1_{\mathcal{T}}(\hat{E})} \,,$$

thus proving (31) with $C = C(\hat{E}^\star)$. □

A.3 Proof of Theorem 4

Proof. Let us define the auxiliary problem: let $\Psi \in \mathrm{H}^2(\Omega) \cap \mathrm{H}^1_0(\Omega)$ the solution of $a\,(V, \Psi) = (U - u, V)_\Omega \ \forall V \in \mathrm{H}^1_0(\Omega)$. From the definition of Ψ, we get:

$$\exists C > 0 : \quad |\Psi|_2 \leq C \, \|U - u\|_{\mathrm{L}^2(\Omega)} \,, \tag{76}$$

$$\exists C > 0 : \quad \|\Psi\|_{\mathrm{H}^1_0(\Omega)} \leq C \, \|U - u\|_{\mathrm{L}^2(\Omega)} \,. \tag{77}$$

Let us denote by Ψ_I the interpolant of Ψ according to Lemma 8. Applying the auxiliary problem, the discrete problem (10) and the definition of the bilinear form a (2), we have

$$\begin{aligned}
\|U-u\|^2_{\mathrm{L}^2(\Omega)} &= (U-u,U-u)_\Omega = a\left(U-u,\Psi\right)\\
&= a\left(U,\Psi-\Psi_I\right)+a\left(U,\Psi_I\right)-a\left(u,\Psi\right)\\
&= a\left(U,\Psi-\Psi_I\right)+\left(f,\Psi_I\right)_\Omega-a\left(u,\Psi\right)\\
&= a\left(U,\Psi-\Psi_I\right)+\left(f,\Psi_I\right)_\Omega-\left(\sum_{E\in\mathcal{M}_h}\left(f,\Pi^0_{0,E}\Psi_I\right)_E\right)\\
&\qquad +a_h\left(u,\Psi_I\right)-a\left(u,\Psi\right)+a\left(u,\Psi_I\right)-a\left(u,\Psi_I\right)\\
&= a\left(U-u,\Psi-\Psi_I\right)+\left(\sum_{E\in\mathcal{M}_h}\left(f,\Psi_I-\Pi^0_{0,E}\Psi_I\right)_E\right)\\
&\qquad +a_h\left(u,\Psi_I\right)-a\left(u,\Psi_I\right).
\end{aligned} \tag{78}$$

Let us consider the terms of the previous relation separately. First, applying the Cauchy-Schwarz inequality, (56), (58) and (76), we have, for the first term,

$$\begin{aligned}
a\left(U-u,\Psi-\Psi_I\right) &\leq \|U-u\|_{\mathrm{H}^1_0(\Omega)}\|\Psi-\Psi_I\|_{\mathrm{H}^1_0(\Omega)}\\
&\leq Ch\|U-u\|_{\mathrm{H}^1_0(\Omega)}|\Psi|_2 \leq Ch\|U-u\|_{\mathrm{H}^1_0(\Omega)}\|U-u\|_{\mathrm{L}^2(\Omega)},
\end{aligned} \tag{79}$$

and, for the second one,

$$\begin{aligned}
\sum_{E\in\mathcal{M}_h}\left(f,\Psi_I-\Pi^0_{0,E}\Psi_I\right)_E &= \sum_{E\in\mathcal{M}_h}\left(f-\Pi^0_{0,E}f,\Psi_I-\Pi^0_{0,E}\Psi_I\right)_E\\
&\leq \sum_{E\in\mathcal{M}_h}\left\|f-\Pi^0_{0,E}f\right\|_{\mathrm{L}^2(E)}\left\|\Psi_I-\Pi^0_{0,E}\Psi_I\right\|_{\mathrm{L}^2(E)}\\
&\leq Ch|f|_{\mathrm{H}^1(\Omega)}\sum_{E\in\mathcal{M}_h}\left\|\Psi_I-\Pi^0_{0,E}\Psi_I\right\|_{\mathrm{L}^2(E)}.
\end{aligned} \tag{80}$$

Applying the property

$$\forall E\in\mathcal{M}_h,\ \left\|\Psi_I-\Pi^0_{0,E}\Psi_I\right\|_{\mathrm{L}^2(E)} \leq \left\|\Psi_I-\Pi^0_{0,E}\Psi\right\|_{\mathrm{L}^2(E)},$$

(56) and (58) to (80), we obtain

$$\begin{aligned}
\sum_{E\in\mathcal{M}_h}\left(f,\Psi_I-\Pi^0_{0,E}\Psi_I\right)_E &\leq Ch|f|_{\mathrm{H}^1(\Omega)}\sum_{E\in\mathcal{M}_h}\left\|\Psi_I-\Pi^0_{0,E}\Psi\right\|_{\mathrm{L}^2(E)}\\
&\leq Ch|f|_{\mathrm{H}^1(\Omega)}\sum_{E\in\mathcal{M}_h}\left(\|\Psi_I-\Psi\|_{\mathrm{L}^2(E)}+\left\|\Psi-\Pi^0_{0,E}\Psi\right\|_{\mathrm{L}^2(E)}\right)\\
&\leq Ch|f|_{\mathrm{H}^1(\Omega)}\left(h^2|\Psi|_2+h\|\Psi\|_{\mathrm{H}^1_0(\Omega)}\right).
\end{aligned} \tag{81}$$

We can omit higher order terms and apply (77), obtaining

$$\sum_{E \in \mathcal{M}_h} \left(f, \Psi_I - \Pi^0_{0,E} \Psi_I\right)_E \leq Ch^2 \left|f\right|_{\mathrm{H}^1(\Omega)} \left\|U - u\right\|_{\mathrm{L}^2(\Omega)} . \tag{82}$$

Finally, we have to bound $a_h \left(u, \Psi_I\right) - a\left(u, \Psi_I\right)$. Then, applying the orthogonality property of $\Pi^0_{l,E}$, adding and subtracting terms, we have

$$\begin{aligned} a_h \left(u, \Psi_I\right) - a\left(u, \Psi_I\right) &= \sum_{E \in \mathcal{M}_h} \left(\Pi^0_{l,E} \nabla u, \nabla \Psi_I\right)_E - \left(\nabla u, \nabla \Psi_I\right)_E \\ &= \sum_{E \in \mathcal{M}_h} \left(\Pi^0_{l,E} \nabla u - \nabla u, \nabla \Psi_I - \Pi^0_{l,E} \nabla \Psi_I\right)_E \\ &= \sum_{E \in \mathcal{M}_h} \left(\Pi^0_{l,E} \nabla u - \Pi^0_{l,E} \nabla U, \nabla \Psi_I - \Pi^0_{l,E} \nabla \Psi_I\right)_E \\ &\quad + \left(\Pi^0_{l,E} \nabla U - \nabla U, \nabla \Psi_I - \Pi^0_{l,E} \nabla \Psi_I\right)_E \\ &\quad + \left(\nabla U - \nabla u, \nabla \Psi_I - \Pi^0_{l,E} \nabla \Psi_I\right)_E . \end{aligned} \tag{83}$$

Notice that, applying (56) and (57), we have the property $\forall E \in \mathcal{M}_h$:

$$\left\|\nabla \Psi_I - \Pi^0_{l,E} \nabla \Psi_I\right\|_{\mathrm{L}^2(E)} \leq \left\|\nabla \Psi_I - \Pi^0_{l,E} \nabla \Psi\right\|_{\mathrm{L}^2(E)} \leq Ch \left|\Psi\right|_{2,E} .$$

Therefore, applying the continuity of the projection operator and (76), the first and the last term of (83) can be bounded as

$$\begin{aligned} \sum_{E \in \mathcal{M}_h} \left(\Pi^0_{l,E} \nabla u - \Pi^0_{l,E} \nabla U, \nabla \Psi_I - \Pi^0_{l,E} \nabla \Psi_I\right)_E + \left(\nabla U - \nabla u, \nabla \Psi_I - \Pi^0_{l,E} \nabla \Psi_I\right)_E \\ \leq Ch \left\|U - u\right\|_{\mathrm{H}^1_0(\Omega)} \left\|U - u\right\|_{\mathrm{L}^2(\Omega)} . \end{aligned} \tag{84}$$

Similarly, the second term is bounded as

$$\sum_{E \in \mathcal{M}_h} \left(\Pi^0_{l,E} \nabla U - \nabla U, \nabla \Psi_I - \Pi^0_{l,E} \nabla \Psi_I\right)_E \leq Ch^2 \left|U\right|_2 \left\|U - u\right\|_{\mathrm{L}^2(\Omega)} . \tag{85}$$

Finally, applying (79),(82),(84) and (85) to (78) and simplifying, we obtain

$$\left\|U - u\right\|_{\mathrm{L}^2(\Omega)} \leq C \left(h \left\|U - u\right\|_{\mathrm{H}^1_0(\Omega)} + h^2 \left|f\right|_{\mathrm{H}^1(\Omega)} + h^2 \left|U\right|_2\right) .$$

Applying the H^1-estimate (Theorem 3) we obtain the relation (67). □