Report on the manuscript SINUM M150419 entitled "Stabilization-Free Virtual Element Methods"

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I want to highlight some major and minor issues.

Major issues:

1. the English presentation is unfortunately very dodgy and should be revised before any type of resubmission;

2. an important question is whether you have in mind specific occasions where stabilization free virtual elements are superior to stabilized ones. More precisely, your procedure is sensibly more expensive and complicated than that of the standard VEM: you need a simplicial tessellation of each element; you project on BDM-type polynomial spaces; the theoretical setting itself is rather involved. If there are no apparent advantages in some specific occasion, the stabilization-free approach seems to be rather lame;

3. lines 17-18; when you discuss the results in references [9,10,20], I am not quite sure that the polynomial degree of the projection depends *only* on the number of vertices. In fact, in an updated version of the preprint [19], it is possible to see that also the shape of the polygon plays a role in the choice of such a polynomial degree;

4. line 22; "arbitrary dimension" only refers to the nonconforming version of the scheme. Please, better highlight this fact;

5. I appreciate the fact that you "preview" equation (1.1) in the introduction, as it is the lynchpin of the forthcoming analysis. Yet, it would be beneficial a brief comment on the hidden constants. For instance, you should state on what such constants depend on;

6. lines 41-43; you should underline that the polynomial spaces onto which you project are based on a regular simplicial tessellation of the elements of the mesh;

7. lines 50-51; when you define the function ϕ, are you sure that the normal component of the trace is a polynomial only on the faces of K? Don't you also need this property to be valid on the faces of the simplicial tessellation of K?

8. line 138; you may wish to recall the degrees of freedom of this space, even though they are well known;

9. line 305; reference [21, Lemma 10] for the polynomial inverse inequality is not proper; in fact, the inverse estimate follows from standard polynomial inverse estimates on simplices (refer, e.g., to Verfürth's book), the regularity of the element, and standard polynomial inverse estimates;

10. line 305; reference [11, (2.18)] is also a standard scaled trace inequality (refer to any PDE textbook);

11. line 308-309; you also use the definition of negative norm (which I cannot find), an integration by parts, and a Cauchy-Schwarz inequality;

12. line 310; "other side" —>"lower bound";

13. lines 311-312; are you sure about the definition of ϕ_1? What about the degrees of freedom on internal faces?

14. elaborate more the equivalence (3.26);
15. you should write that the eq. in 322 is proven based on the inequality on lines 315-316;
16. line 323; the operator I_K^{div} should be defined explicitly;
17. eq. (3.28); can you elaborate more on the first inequality? Moreover, the second inequality follows from the bound on line 322; you should underline this;
18. after line 336, can you state the difference between the space defined on line 336 and that on line 263?
19. the definition of the DoFs in eqs. (4.3) and (4.4) is wrong: you should first pick bases of bulk and face polynomial spaces and define the moments accordingly;
20. line 381; the space $\mathbb{P}_p \setminus \mathbb{P}_{p-2}$ is *not* defined as an orthogonal complement, but rather as *any* completion of the smaller space into the larger one. The two definitions are equivalent if you use orthogonal polynomial bases, which I humbly think is not what you are actually doing;
21. you may wish to shorten up or even remove the proof of Lemma 4.1, which is standard in the virtual element literature;
22. line 406; "inequality" —>"inequalities"; moreover, mention that you use both $H^1 - L^2$ and L^2 boundary-L^2 bulk inverse inequalities;
23. line 408; is [11, (2.15)] the proper reference for the Poincaré-Friedrichs inequality? I think is a standard result and you may wish to cite any standard PDE book;
24. eq. (4.12); if you complete the sequence of identities and inequalities with $h_k^2|v|_{1,K}^2$, you get indeed a norm equivalence (and not only an upper bound);
25. lines 445 and 456; explain from where the term $Q_0^k(div\phi)$ appears;
26. provide more details on the inequality on lines 457-458;
27. eq. (4.16); why do you cite [15, (4.16)]? It probably follows from [Brenner, SINUM 2003] and the definition of the nonconforming space;
28. line 484; the first inequality is probably a "$\approx$";
29. line 486; "uni-solvent" —>"well posed"?
30. the definition of the DoFs in eqs. (5.1), (5.2), and (5.3) is wrong: you should first pick bases of bulk and face polynomial spaces and define the moments accordingly;
31. you may wish to reduce or even drop the proof or Theorem 5.3, which appears to be a standard Strang-type result for the VEM;
32. in the numerical section; I agree that it is important to check the rate of convergence of the method;

    However, the convergence does not automatically guarantee the invertibility of the local stiffness matrices, as imposing the Dirichlet boundary conditions may have a "stabilizing effect". So, you should investigate the number of zero eigenvalues of the local stiffness matrix.

    For instance, pick three different hexagonal elements with a fixed degree of accuracy, e.g., $k = 3$: the first being a regular hexagon; the second being a quasi-regular hexagon (small perturbation of the previous element); the third being a square with an edge containing two hanging nodes. Check whether the stiffness matrix *always* has only one zero eigenvalue, i.e., the method is indeed stabilization free;

33. my feeling is that the proposed stabilization-free approach is much more expensive than the standard one. Thus, you should check the assembling time of the stabilization-free and standard VEMs, on: (*i*) a (reasonably large) sequence of meshes; (*ii*) fixing a mesh, increasing the degree of accuracy, e.g., up to $k = 10$.

34. along the same avenue, you should also compare the condition number of the global matrices obtained with the two approaches in the two situations (*i*) and (*ii*) above. Furthermore, you may also wish to consider sequences of meshes with “collapsing elements” (refine a mesh and change the aspect ratio), and check the condition numbers again. As a suggestion, since the condition number should be computed scaling the diagonal of the matrix, you can empirically check such conditioning by testing the two approaches on a patch test. The conditioning can be estimated checking the growth of the error for the patch test.

Minor issues (I will not highlight all the typos but only few of them as an example):

- line 3; “Poisson” —>“the Poisson”;
- line 24; “space” —>“the space”;
- line 31; “space” —>“that the space”;
- line 43; “usual” —>“the usual”;
- line 49; “space” —>“the space”;
- line 68; “. And” —>“and”;
- line 81; “banach” —>“Banach”;
- line 82; “and $\mathbb{K}$, where $\mathbb{K}$” —>“and $\mathbb{K}$”;
- line 105; “surface” —>“the surface”;
- line 107; “smooth” —>“a smooth”;
- line 262; “shape” —>“the shape”;
- line 347; “discrete” —>“the discrete”;
- lines 465-466; “DoF” —>“DoFs”; “is” —>“are”; “vanishes” —>“vanish”;
- line 511; “inequlities” —>“inequalities”;
- line 549; “DoFs” —>“the DoFs”.