VIRTUAL ELEMENT METHODS WITHOUT EXTRINSIC STABILIZATION*

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Abstract. Virtual element methods (VEMs) without extrinsic stabilization in arbitrary degree of polynomial are developed for second order elliptic problems, including a nonconforming VEM in arbitrary dimension and a conforming VEM in two dimensions. The key is to construct local $H(\mathrm{div})$-conforming macro finite element spaces such that the associated L^2 projection of the gradient of virtual element functions is computable, and the L^2 projector has a uniform lower bound on the gradient of virtual element function spaces in L^2 norm. Optimal error estimates are derived for these VEMs. Numerical experiments are provided to test the VEMs without extrinsic stabilization.

Key words. virtual element, stabilization, macro finite element, norm equivalence, error analysis

MSC codes. 65N12, 65N22, 65N30

1. Introduction. An additional stabilization term is usually required in the virtual element methods (VEMs) to ensure the coercivity of the discrete bilinear form [12, 13]. The local stabilization term $S_K(\cdot,\cdot)$ has to satisfy

$$c_*|v|_{1,K}^2 \le S_K(v,v) \le c^*|v|_{1,K}^2$$

for v belongs to the non-polynomial subspace of the virtual element space, which influences the condition number of the stiffness matrix and brings in the pollution factor $\frac{\max\{1,c^*\}}{\min\{1,c_*\}}$ in the error estimates [31, 10, 37]. For the a posteriori error analysis on anisotropic polygonal meshes in [5], the stabilization term dominates the error estimator, which makes the anisotropic a posteriori error estimator suboptimal. The stabilization term significantly affects the performance of the VEM for the Poisson eigenvalue problem [15], and improper choices of the stabilization term will produce useless results. Special stabilization terms are designed for a nonlinear elasto-plastic deformation problem [35] and an electromagnetic interface problem in three dimensions [21], which are not easy to be extended to other problems. In short, the stabilization term has to be chosen carefully for different partial differential equations to make the VEM work well, which is arduous and reduces its practicality.

We intend to construct VEMs without extrinsic stabilization in arbitrary degree of polynomial, including a nonconforming VEM in arbitrary dimension and a conforming VEM in two dimensions, in a unified way. The key to construct VEMs without extrinsic stabilization is to find a finite-dimensional space $\mathbb{V}(K)$ for polytope K and a projector Q_K onto the space $\mathbb{V}(K)$ such that

(C1) It holds the norm equivalence

$$\|Q_K\nabla v\|_{0,K} \eqsim \|\nabla v\|_{0,K} \quad \forall\ v \in V_k(K) \tag{1.1}$$

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on shape function space $V_k(K)$ of virtual elements;

(C2) The projection $Q_K\nabla v$ is computable based on the degrees of freedom (DoFs) of virtual elements for $v \in V_k(K)$.

The hidden constants in (1.1) are independent of the size of K, but depend on the degree of polynomials, and the chunkiness parameter and the geometric dimension of K; see Section 2.2 for details. We can choose Q_K as the L^2-orthogonal projector with respect to the inner product $(\cdot,\cdot)_K$. The norm equivalence (1.1) implies that the space $\mathbb{V}(K)$ should be sufficiently large compared with the virtual element space $V_k(K)$. In standard virtual element methods, $Q_{k-1}^K\nabla v$ [14] or $\nabla\Pi_k^K v$ [12, 13, 1, 9] are used, where Q_{k-1}^K is the L^2-orthogonal projector onto the $(k-1)$-th order polynomial space $\mathbb{P}_{k-1}(K;\mathbb{R}^d)$, and Π_k^K is the H^1 projection operator onto the k-th order polynomial space $\mathbb{P}_k(K)$. While only

$$\|Q_{k-1}^K\nabla v\|_{0,K} \lesssim \|\nabla v\|_{0,K}, \quad \|\nabla\Pi_k^K v\|_{0,K} \lesssim \|\nabla v\|_{0,K}$$

hold rather than the norm equivalence (1.1), then the additional stabilization term is usually required to ensure the coercivity of the discrete bilinear form. To remove the additional stabilization term, based on a regular simplicial tessellation of polytope K, we employ k-th order or $(k-1)$-th order $H(\mathrm{div})$-conforming macro finite elements as $\mathbb{V}(K)$ in this paper, and keep the virtual element space $V_k(K)$ as the usual ones.

We first construct $H(\mathrm{div})$-conforming macro finite elements based on a simplicial partition $\mathcal{T}_K$ of polytope K in arbitrary dimension. The shape function space $\mathbb{V}_k^{\mathrm{div}}(K)$ is a subspace of the k-th order Brezzi-Douglas-Marini (BDM) element space on the simplicial partition $\mathcal{T}_K$ for $k \geq 1$ and the lowest order Raviart-Thomas (RT) element space for $k = 0$, with some constraints. To ensure the L^2 projection $Q_{K,k}^{\mathrm{div}}\nabla v$ onto the space $\mathbb{V}_k^{\mathrm{div}}(K)$ is computable for virtual element function $v \in V_k(K)$, we require that $\mathrm{div}\,\boldsymbol{\phi} \in \mathbb{P}_{\max\{k-1,0\}}(K)$ and $\boldsymbol{\phi}\cdot\boldsymbol{n}$ on each $(d-1)$-dimensional face of K is a polynomial for $\boldsymbol{\phi} \in \mathbb{V}_k^{\mathrm{div}}(K)$. Based on these considerations and the direct decomposition of an $H(\mathrm{div})$-conforming macro finite element space related to $\mathbb{V}_k^{\mathrm{div}}(K)$, we propose the unisolvent DoFs for the space $\mathbb{V}_k^{\mathrm{div}}(K)$, and establish the L^2 norm equivalence. By the way, we use the matrix-vector language to review a conforming finite element for differential $(d-2)$-form in [8, 7].

By the aid of the projector $Q_{K,k}^{\mathrm{div}}$, we advance a nonconforming VEM without extrinsic stabilization in arbitrary dimension and a conforming VEM without extrinsic stabilization in two dimensions for second order elliptic problems. Indeed, these VEMs can be equivalently recast as primal mixed VEMs. We prove the norm equivalence (1.1) and the well-posedness of the VEMs without extrinsic stabilization, and derive the optimal error estimates.

Numerical experiments are provided to test the convergence rate, the invertibility of the local stiffness matrices, the assembling time and the condition number of stiffness matrix for the VEMs without extrinsic stabilization, which appear competitive with respect to other existing VEMs.

The idea on constructing VEMs without extrinsic stabilization in this paper is simple, and can be extended to more VEMs and more partial differential equations. Since there is no additional stabilization term, these VEMs will be preferred in the engineering community. More benefits of the VEMs without extrinsic stabilization will be the study of future works. By the way, we refer to [29] for a hybrid high-order method, and [44, 3, 2, 45, 46, 4] for weak Galerkin finite element methods without extrinsic stabilization.

The rest of this paper is organized as follows. Notation and mesh conditions are presented in Section 2. In Section 3, $H(\mathrm{div})$-conforming macro finite elements in arbitrary dimension are constructed. A nonconforming VEM without extrinsic stabilization in arbitrary dimension is developed in Section 4, and a conforming VEM without extrinsic stabilization in two dimensions is devised in Section 5. Some numerical results are shown in Section 6.

2. Preliminaries.

2.1. Notation. Let $\Omega \subset \mathbb{R}^d$ be a bounded polytope. Given a bounded domain $K \subset \mathbb{R}^d$ and a non-negative integer m, let $H^m(K)$ be the usual Sobolev space of functions on K. The corresponding norm and semi-norm are denoted respectively by $\|\cdot\|_{m,K}$ and $|\cdot|_{m,K}$. By convention, let $L^2(K) = H^0(K)$. Let $(\cdot,\cdot)_K$ be the standard inner product on $L^2(K)$. If K is Ω, we abbreviate $\|\cdot\|_{m,K}$, $|\cdot|_{m,K}$ and $(\cdot,\cdot)_K$ by $\|\cdot\|_m$, $|\cdot|_m$ and $(\cdot,\cdot)$, respectively. Let $H_0^m(K)$ be the closure of $\mathcal{C}_0^\infty(K)$ with respect to the norm $\|\cdot\|_{m,K}$, and $L_0^2(K)$ consist of all functions in $L^2(K)$ with zero mean value. For integer $k \geq 0$, notation $\mathbb{P}_k(K)$ stands for the set of all polynomials over K with the total degree no more than k. Set $\mathbb{P}_{-1}(K) = \{0\}$. For a Banach space $B(K)$, let $B(K;\mathbb{X}) := B(K) \otimes \mathbb{X}$ with $\mathbb{X} = \mathbb{R}^d$ and $\mathbb{K}$ being the set of antisymmetric matrices. Denote by Q_k^K the L^2-orthogonal projector onto $\mathbb{P}_k(K)$ or $\mathbb{P}_k(K;\mathbb{X})$. Let $\operatorname{skw}\boldsymbol{\tau} := (\boldsymbol{\tau} - \boldsymbol{\tau}^\intercal)/2$ be the antisymmetric part of a tensor $\boldsymbol{\tau}$. Denote by $\#S$ the number of elements in a finite set S.

Given a d-dimensional polytope K, let $\mathcal{F}(K)$ and $\mathcal{E}(K)$ be the set of all $(d-1)$-dimensional faces and $(d-2)$-dimensional faces of K respectively. For $F \in \mathcal{F}(K)$, denote by $\boldsymbol{n}_{K,F}$ the unit outward normal vector to ∂K, which will be abbreviated as $\boldsymbol{n}_F$ or $\boldsymbol{n}$ if not causing any confusion.

Given a d-dimensional simplex T, let $F_i \in \mathcal{F}(T)$ be the $(d-1)$-dimensional face opposite to vertex $\mathtt{v}_i$, $\boldsymbol{n}_i$ be the unit outward normal to the face F_i, and λ_i be the barycentric coordinate of the point $\boldsymbol{x}$ corresponding to the vertex $\mathtt{v}_i$, for $i = 0, 1, \cdots, d$. Clearly $\{\boldsymbol{n}_1, \boldsymbol{n}_2, \cdots, \boldsymbol{n}_d\}$ spans $\mathbb{R}^d$, and $\{\operatorname{skw}(\boldsymbol{n}_i\boldsymbol{n}_j^\intercal)\}_{1\leq i<j\leq d}$ spans the antisymmetric space $\mathbb{K}$. For $F \in \mathcal{F}(T)$, let $\mathcal{E}(F) := \{e \in \mathcal{E}(T) : e \subset \partial F\}$. For $e \in \mathcal{E}(F)$, denote by $\boldsymbol{n}_{F,e}$ be the unit vector outward normal to ∂F but parallel to F.

Let $\{\mathcal{T}_h\}$ denote a family of partitions of Ω into nonoverlapping simple polytopes with $h := \max_{K\in\mathcal{T}_h} h_K$ and $h_K := \operatorname{diam}(K)$. Denote by $\mathcal{F}_h^r$ the set of all $(d-r)$-dimensional faces of the partition $\mathcal{T}_h$ for $r = 1, \ldots, d$. Set $\mathcal{F}_h := \mathcal{F}_h^1$ for simplicity. Let $\mathcal{F}_h^\partial$ be the subset of $\mathcal{F}_h$ including all $(d-1)$-dimensional faces on $\partial\Omega$. For any $F \in \mathcal{F}_h$, let h_F be its diameter and fix a unit normal vector $\boldsymbol{n}_F$. For a piecewise smooth function v, define

$$\|v\|_{1,h}^2 := \sum_{K\in\mathcal{T}_h} \|v\|_{1,K}^2, \quad |v|_{1,h}^2 := \sum_{K\in\mathcal{T}_h} |v|_{1,K}^2.$$

For domain K, we use $\boldsymbol{H}(\mathrm{div}, K)$ and $\boldsymbol{H}_0(\mathrm{div}, K)$ to denote the standard divergence vector spaces. For a smooth vector function $\boldsymbol{v}$, let $\nabla\boldsymbol{v} := (\partial_i v_j)_{1\leq i,j\leq d}$. On the face $F \in \mathcal{F}_h$, define the surface divergence

$$\operatorname{div}_F \boldsymbol{v} = \operatorname{div}(\boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{n}) = \operatorname{div}\boldsymbol{v} - \partial_n(\boldsymbol{v}\cdot\boldsymbol{n}).$$

Define the surface gradient $\nabla_F v := \nabla v - (\partial_n v)\boldsymbol{n}$ for a smooth function v.

2.2. Mesh conditions. We impose the following conditions on the mesh $\mathcal{T}_h$ in this paper:

(A1) Each element $K \in \mathcal{T}_h$ and each face $F \in \mathcal{F}_h^r$ for $1 \leq r \leq d-1$ is star-shaped with a uniformly bounded chunkiness parameter.

(A2) There exists a quasi-uniform simplicial mesh $\mathcal{T}_h^*$ such that each $K \in \mathcal{T}_h$ is a union of some simplexes in $\mathcal{T}_h^*$.

For $K \in \mathcal{T}_h$, let $\boldsymbol{x}_K$ be the center of the largest ball contained in K. Throughout this paper, we use "$\lesssim \cdots$" to mean that "$\leq C \cdots$", where C is a generic positive constant independent of mesh size h, but may depend on the chunkiness parameter of the polytope, the degree of polynomials k, the dimension of space d, and the shape regularity and quasi-uniform constants of the virtual triangulation $\mathcal{T}_h^*$, which may take different values at different appearances. Let $A \eqsim B$ mean $A \lesssim B$ and $B \lesssim A$.

For polytope $K \in \mathcal{T}_h$, denote by $\mathcal{T}_K$ the simplicial partition of K, which is induced from $\mathcal{T}_h^*$. Let $\mathcal{F}(\mathcal{T}_K)$ and $\mathcal{E}(\mathcal{T}_K)$ be the set of all $(d-1)$-dimensional faces and $(d-2)$-dimensional faces of the simplicial partition $\mathcal{T}_K$ respectively. Set

$$\mathcal{F}^\partial(\mathcal{T}_K) := \{F \in \mathcal{F}(\mathcal{T}_K) : F \subset \partial K\}, \quad \mathcal{E}^\partial(\mathcal{T}_K) := \{e \in \mathcal{E}(\mathcal{T}_K) : e \subset \partial K\}.$$

Hereafter we use T to represent a simplex, and K to denote a general polytope.

3. $H(\mathrm{div})$-Conforming Macro Finite Elements.

In this section we will construct an $H(\mathrm{div})$-conforming macro finite element space $\mathbb{V}_{k-1}^{\mathrm{div}}(K)$ and the corresponding degrees of freedoms in arbitrary dimension, and establish the L^2 norm equivalence for $\boldsymbol{\phi} \in \mathbb{V}_{k-1}^{\mathrm{div}}(K)$

$$\|\boldsymbol{\phi}\|_{0,K} \eqsim h_K \|\operatorname{div} \boldsymbol{\phi}\|_{0,K} + \sup_{\boldsymbol{\psi} \in \operatorname{div} \mathring{\boldsymbol{V}}_k^{d-2}(K)} \frac{(\boldsymbol{\phi}, \boldsymbol{\psi})_K}{\|\boldsymbol{\psi}\|_{0,K}} + \sum_{F \in \mathcal{F}(K)} h_F^{1/2} \|\boldsymbol{\phi} \cdot \boldsymbol{n}\|_{0,F}.$$

The space $\mathbb{V}_{k-1}^{\mathrm{div}}(K)$ and its L^2 norm equivalence will be used to prove the norm equivalence for the virtual element space

$$\|Q_{K,k-1}^{\mathrm{div}} \nabla v\|_{0,K} \eqsim \|\nabla v\|_{0,K} \quad \forall\, v \in V_k(K),$$

where $V_k(K)$ is the nonconforming virtual element space in Section 4, and the conforming virtual element space in Section 5. Here $Q_{K,k-1}^{\mathrm{div}}$ is the computable L^2 projector onto the space $\mathbb{V}_{k-1}^{\mathrm{div}}(K)$.

3.1. $H(\mathrm{div})$-conforming finite elements.

For a d-dimensional polytope $K \in \mathcal{T}_h$ and $k \geq 2$, let

$$\boldsymbol{V}_{k-1}^{\mathrm{BDM}}(K) := \{\boldsymbol{\phi} \in \boldsymbol{H}(\mathrm{div}, K) : \boldsymbol{\phi}|_T \in \mathbb{P}_{k-1}(T; \mathbb{R}^d) \text{ for each } T \in \mathcal{T}_K\}$$

be the local Brezzi-Douglas-Marini (BDM) element space [19, 18, 39], whose degrees of freedom (DoFs) are given by [26]

$$(\boldsymbol{v} \cdot \boldsymbol{n}, q)_F, \quad q \in \mathbb{P}_{k-1}(F), F \in \mathcal{F}(T), \tag{3.1}$$

$$(\operatorname{div} \boldsymbol{v}, q)_T, \quad q \in \mathbb{P}_{k-2}(T)/\mathbb{R}, \tag{3.2}$$

$$(\boldsymbol{v}, \boldsymbol{q})_T, \quad \boldsymbol{q} \in \mathbb{P}_{k-3}(T; \mathbb{K})\boldsymbol{x} \tag{3.3}$$

for $T \in \mathcal{T}_K$. Here $\mathbb{P}_{k-2}(T)/\mathbb{R} := \mathbb{P}_{k-2}(T) \cap L_0^2(T)$, and $\mathbb{P}_{k-3}(T; \mathbb{K})\boldsymbol{x} := \{\boldsymbol{\tau}\boldsymbol{x} : \boldsymbol{\tau} \in \mathbb{P}_{k-3}(T; \mathbb{K})\}$ with $\boldsymbol{x} \in T$ being the independent variable. Define $\mathring{\boldsymbol{V}}_{k-1}^{\mathrm{BDM}}(K) := \boldsymbol{V}_{k-1}^{\mathrm{BDM}}(K) \cap \boldsymbol{H}_0(\mathrm{div}, K)$.

We also need the lowest order Raviart-Thomas (RT) element space [41, 38]

$$\boldsymbol{V}^{\mathrm{RT}}(K) := \{\boldsymbol{\phi} \in \boldsymbol{H}(\mathrm{div}, K) : \boldsymbol{\phi}|_T \in \mathbb{P}_0(T; \mathbb{R}^d) + \boldsymbol{x}\mathbb{P}_0(T) \text{ for each } T \in \mathcal{T}_K\}.$$

The DoFs are given by

$$(\boldsymbol{v}\cdot\boldsymbol{n}, q)_F, \quad q\in\mathbb{P}_0(F), F\in\mathcal{F}(T)$$

for $T\in\mathcal{T}_K$. Define $\mathring{\boldsymbol{V}}^{\mathrm{RT}}(K) := \boldsymbol{V}^{\mathrm{RT}}(K)\cap\boldsymbol{H}_0(\operatorname{div}, K)$.

3.2. Finite element for differential $(d-2)$**-form.** Now recall the finite element for differential $(d-2)$-form, i.e. $H\Lambda^{d-2}$-conforming finite element in [8, 7]. We will present the finite element for differential $(d-2)$-form using the proxy of the differential form rather than the differential form itself as in [8, 7].

By (3.5) in [26], we have the direct decomposition

$$\mathbb{P}_{k-1}(T;\mathbb{R}^d) = \nabla\mathbb{P}_k(T)\oplus\mathbb{P}_{k-2}(T;\mathbb{K})\boldsymbol{x}. \tag{3.4}$$

Recall that [27, (35)]

$$\mathbb{P}_k(T)\cap\ker(I+\boldsymbol{x}\cdot\nabla) = \{0\}. \tag{3.5}$$

Lemma 3.1. *For* $\boldsymbol{w}\in\mathbb{P}_{k-2}(T;\mathbb{K})\boldsymbol{x}$ *satisfying* $(\operatorname{skw}\nabla\boldsymbol{w})\boldsymbol{x} = \boldsymbol{0}$*, it holds* $\boldsymbol{w}=\boldsymbol{0}$*.*

Proof. Since

$$(\operatorname{skw}\nabla\boldsymbol{w})\boldsymbol{x} = \frac{1}{2}(\nabla\boldsymbol{w})\boldsymbol{x} - \frac{1}{2}(\nabla\boldsymbol{w})^{\intercal}\boldsymbol{x} = \frac{1}{2}\nabla(\boldsymbol{w}\cdot\boldsymbol{x}) - \frac{1}{2}(I+\boldsymbol{x}\cdot\nabla)\boldsymbol{w},$$

we acquire from $\boldsymbol{w}\cdot\boldsymbol{x}=0$ that $(I+\boldsymbol{x}\cdot\nabla)\boldsymbol{w}=\boldsymbol{0}$, which together with (3.5) implies $\boldsymbol{w}=\boldsymbol{0}$. ◻

Lemma 3.2. *The polynomial complex*

$$\mathbb{R}\to\mathbb{P}_k(T)\xrightarrow{\nabla}\mathbb{P}_{k-1}(T;\mathbb{R}^d)\xrightarrow{\operatorname{skw}\nabla}\mathbb{P}_{k-2}(T;\mathbb{K}) \tag{3.6}$$

is exact.

Proof. Clearly (3.6) is a complex. It suffices to prove $\mathbb{P}_{k-1}(T;\mathbb{R}^d)\cap\ker(\operatorname{skw}\nabla)\subseteq\nabla\mathbb{P}_k(T)$.

For $\boldsymbol{v}\in\mathbb{P}_{k-1}(T;\mathbb{R}^d)\cap\ker(\operatorname{skw}\nabla)$, by decomposition (3.4), there exist $q\in\mathbb{P}_k(T)$ and $\boldsymbol{w}\in\mathbb{P}_{k-2}(T;\mathbb{K})\boldsymbol{x}$ such that $\boldsymbol{v}=\nabla q+\boldsymbol{w}$. By $\operatorname{skw}\nabla\boldsymbol{v}=\boldsymbol{0}$, we get $\operatorname{skw}\nabla\boldsymbol{w}=\boldsymbol{0}$. Apply Lemma 3.1 to derive $\boldsymbol{w}=\boldsymbol{0}$. Thus, $\boldsymbol{v}=\nabla q\in\nabla\mathbb{P}_k(T)$. ◻

Lemma 3.3. *It holds the decomposition*

$$\mathbb{P}_{k-2}(T;\mathbb{K}) = \operatorname{skw}\nabla\mathbb{P}_{k-1}(T;\mathbb{R}^d)\oplus\big(\mathbb{P}_{k-2}(T;\mathbb{K})\cap\ker(\boldsymbol{x})\big), \tag{3.7}$$

where $\mathbb{P}_{k-2}(T;\mathbb{K})\cap\ker(\boldsymbol{x}) := \{\boldsymbol{\tau}\in\mathbb{P}_{k-2}(T;\mathbb{K}) : \boldsymbol{\tau}\boldsymbol{x}=\boldsymbol{0}\}$.

Proof. Thanks to decomposition (3.4), we have

$$\operatorname{skw}\nabla\mathbb{P}_{k-1}(T;\mathbb{R}^d) = \operatorname{skw}\nabla(\mathbb{P}_{k-2}(T;\mathbb{K})\boldsymbol{x}).$$

By Lemma 3.1, $\operatorname{skw}\nabla\mathbb{P}_{k-1}(T;\mathbb{R}^d)\cap\big(\mathbb{P}_{k-2}(T;\mathbb{K})\cap\ker(\boldsymbol{x})\big)=\{\boldsymbol{0}\}$. Then we only need to check dimensions. Due to complex (3.6),

$$\dim\operatorname{skw}\nabla\mathbb{P}_{k-1}(T;\mathbb{R}^d) = \dim\mathbb{P}_{k-1}(T;\mathbb{R}^d) - \dim\nabla\mathbb{P}_k(T). \tag{3.8}$$

On the other side, by space decomposition (3.4),

$$\dim\mathbb{P}_{k-2}(T;\mathbb{K})\boldsymbol{x} = \dim\mathbb{P}_{k-1}(T;\mathbb{R}^d) - \dim\nabla\mathbb{P}_k(T).$$

Hence, $\dim\operatorname{skw}\nabla\mathbb{P}_{k-1}(T;\mathbb{R}^d) = \dim\mathbb{P}_{k-2}(T;\mathbb{K})\boldsymbol{x}$, which yields (3.7). ◻

By (3.7) and (3.8), it follows

$$\dim \mathbb{P}_{k-2}(T;\mathbb{K})\cap\ker(\boldsymbol{x}) = \dim \mathbb{P}_{k-2}(T;\mathbb{K}) + \dim \nabla\mathbb{P}_k(T) - \dim \mathbb{P}_{k-1}(T;\mathbb{R}^d). \tag{3.9}$$

With the decomposition (3.7) and $\mathbb{P}_{k-1}(F;\mathbb{R}^{d-1}) = \nabla_F P_k(F)\oplus\mathbb{P}_{k-2}(F;\mathbb{K})\boldsymbol{x}$, we are ready to define the finite element for differential $(d-2)$-form. Take $\mathbb{P}_k(T;\mathbb{K})$ as the space of shape functions. The degrees of freedom are given by

$$((\boldsymbol{n}_1^e)^\intercal\boldsymbol{\tau}\boldsymbol{n}_2^e, q)_e, \quad q\in\mathbb{P}_k(e), e\in\mathcal{E}(T), \tag{3.10}$$
$$(\operatorname{div}_F(\boldsymbol{\tau}\boldsymbol{n}), q)_F, \quad q\in\mathbb{P}_{k-1}(F)/\mathbb{R}, F\in\mathcal{F}(T), \tag{3.11}$$
$$(\boldsymbol{\tau}\boldsymbol{n}, \boldsymbol{q})_F, \quad \boldsymbol{q}\in\mathbb{P}_{k-2}(F;\mathbb{K})\boldsymbol{x}, F\in\mathcal{F}(T), \tag{3.12}$$
$$(\operatorname{div}\boldsymbol{\tau}, \boldsymbol{q})_T, \quad \boldsymbol{q}\in\mathbb{P}_{k-3}(T;\mathbb{K})\boldsymbol{x}, \tag{3.13}$$
$$(\boldsymbol{\tau}, \boldsymbol{q})_T, \quad \boldsymbol{q}\in\mathbb{P}_{k-2}(T;\mathbb{K})\cap\ker(\boldsymbol{x}). \tag{3.14}$$

In DoF (3.10), $\boldsymbol{n}_1^e$ and $\boldsymbol{n}_2^e$ are two unit normal vectors of e satisfying $\boldsymbol{n}_1^e\cdot\boldsymbol{n}_2^e = 0$.

LEMMA 3.4. *For $e\in\mathcal{E}(T)$, let $\tilde{\boldsymbol{n}}_1$ and $\tilde{\boldsymbol{n}}_2$ be another two unit normal vectors of e satisfying $\tilde{\boldsymbol{n}}_1\cdot\tilde{\boldsymbol{n}}_2 = 0$. Then*

$$\operatorname{skw}(\tilde{\boldsymbol{n}}_1\tilde{\boldsymbol{n}}_2^\intercal) = \pm\operatorname{skw}(\boldsymbol{n}_1^e(\boldsymbol{n}_2^e)^\intercal).$$

Proof. Notice that there exists an orthonormal matrix $H\in\mathbb{R}^{2\times2}$ such that $(\tilde{\boldsymbol{n}}_1, \tilde{\boldsymbol{n}}_2) = (\boldsymbol{n}_1^e, \boldsymbol{n}_2^e)H$. Then

$$\begin{aligned} 2\operatorname{skw}(\tilde{\boldsymbol{n}}_1\tilde{\boldsymbol{n}}_2^\intercal) = \tilde{\boldsymbol{n}}_1\tilde{\boldsymbol{n}}_2^\intercal - \tilde{\boldsymbol{n}}_2\tilde{\boldsymbol{n}}_1^\intercal &= (\tilde{\boldsymbol{n}}_1, \tilde{\boldsymbol{n}}_2)\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\tilde{\boldsymbol{n}}_1^\intercal\\ \tilde{\boldsymbol{n}}_2^\intercal\end{pmatrix}\\ &= (\boldsymbol{n}_1^e, \boldsymbol{n}_2^e)H\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}H^\intercal(\boldsymbol{n}_1^e, \boldsymbol{n}_2^e)^\intercal. \end{aligned}$$

By a direct computation, $H\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}H^\intercal = \det(H)\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}$. Hence

$$2\operatorname{skw}(\tilde{\boldsymbol{n}}_1\tilde{\boldsymbol{n}}_2^\intercal) = 2\det(H)\operatorname{skw}(\boldsymbol{n}_1^e(\boldsymbol{n}_2^e)^\intercal),$$

which ends the proof. ∎

LEMMA 3.5. *Let $\boldsymbol{\tau}\in\mathbb{P}_k(T;\mathbb{K})$ and $F\in\mathcal{F}(T)$. Assume the degrees of freedom (3.10)-(3.12) on F vanish. Then $\boldsymbol{\tau}\boldsymbol{n}|_F = \boldsymbol{0}$.*

Proof. Due to (3.10), we get $(\boldsymbol{n}_1^e)^\intercal\boldsymbol{\tau}\boldsymbol{n}_2^e|_e = 0$ on each $e\in\mathcal{E}(F)$, which together with Lemma 3.4 indicates $\boldsymbol{n}_{F,e}^\intercal\boldsymbol{\tau}\boldsymbol{n}_F|_e = 0$. By the unisolvence of BDM element on face F, cf. DoFs (3.1)-(3.3), it follows from DoFs (3.11)-(3.12) that $\boldsymbol{\tau}\boldsymbol{n}|_F = \boldsymbol{0}$. ∎

LEMMA 3.6. *For $\boldsymbol{\tau}\in\mathbb{P}_k(T;\mathbb{K})$, $\boldsymbol{\tau}\boldsymbol{n}|_{F_i} = \boldsymbol{0}$ for $i = 1,\ldots,d$, if and only if*

$$\boldsymbol{\tau} = \sum_{1\le i<j\le d}\lambda_i\lambda_j q_{ij}\boldsymbol{N}_{ij} \tag{3.15}$$

for some $q_{ij}\in\mathbb{P}_{k-2}(T)$. Here $\{\boldsymbol{N}_{ij}\}_{1\le i<j\le d}$ denotes the basis of $\mathbb{K}$ being dual to $\{\operatorname{skw}(\boldsymbol{n}_i\boldsymbol{n}_j^\intercal)\}_{1\le i<j\le d}$, i.e.,

$$\boldsymbol{N}_{ij}:\operatorname{skw}(\boldsymbol{n}_l\boldsymbol{n}_m^\intercal) = \delta_{il}\delta_{jm}, \quad 1\le i<j\le d,\ 1\le l<m\le d.$$

Proof. For $1 \leq l \leq d$ but $l \neq i, j$, by the definition of $\boldsymbol{N}_{ij}$, it holds $\boldsymbol{N}_{ij}\boldsymbol{n}_l = \boldsymbol{0}$. Hence, for $\boldsymbol{\tau} = \sum_{1\leq i<j\leq d} \lambda_i\lambda_j q_{ij}\boldsymbol{N}_{ij}$, obviously we have $\boldsymbol{\tau}\boldsymbol{n}|_{F_i} = \boldsymbol{0}$ for $i = 1, \ldots, d$.

On the other side, assume $\boldsymbol{\tau}\boldsymbol{n}|_{F_i} = \boldsymbol{0}$ for $i = 1, \ldots, d$. Express $\boldsymbol{\tau}$ as

$$\boldsymbol{\tau} = \sum_{1\leq i<j\leq d} p_{ij}\boldsymbol{N}_{ij},$$

where $p_{ij} = \boldsymbol{n}_i^{\intercal}\boldsymbol{\tau}\boldsymbol{n}_j \in \mathbb{P}_k(T)$. Therefore, $p_{ij}|_{F_i} = p_{ij}|_{F_j} = 0$, which ends the proof. $\square$

LEMMA 3.7. *The degrees of freedom* (3.10)-(3.14) *are uni-solvent for* $\mathbb{P}_k(T; \mathbb{K})$.

Proof. By $\mathbb{P}_{k-1}(F; \mathbb{R}^{d-1}) = \nabla_F P_k(F) \oplus \mathbb{P}_{k-2}(F; \mathbb{K})\boldsymbol{x}$, the number of degrees of freedom (3.11)-(3.12) is $(d^2 + d)\binom{k+d-2}{k-1} - (d + 1)\binom{k+d-1}{k}$. Using (3.4) and (3.9), the number of degrees of freedom (3.10)-(3.14) is

$$\frac{1}{2}(d^2 + d)\binom{k+d-2}{k} + (d^2 + d)\binom{k+d-2}{k-1} - (d + 1)\binom{k+d-1}{k}$$
$$+ \frac{1}{2}(d^2 + d)\binom{k+d-2}{k-2} + \binom{k+d}{k} - (d + 1)\binom{k+d-1}{k-1} = \frac{1}{2}(d^2 - d)\binom{k+d}{k},$$

which equals to $\dim \mathbb{P}_k(T; \mathbb{K})$.

Assume $\boldsymbol{\tau} \in \mathbb{P}_k(T; \mathbb{K})$ and all the degrees of freedom (3.10)-(3.14) vanish. It holds from Lemma 3.5 that $\boldsymbol{\tau}\boldsymbol{n}|_{\partial T} = \boldsymbol{0}$. Noting that $\boldsymbol{\tau}$ is antisymmetric, we also have $\boldsymbol{n}^{\intercal}\boldsymbol{\tau}|_{\partial T} = \boldsymbol{0}$. On each $F \in \mathcal{F}(T)$, it holds

$$\boldsymbol{n}^{\intercal} \operatorname{div} \boldsymbol{\tau} = \operatorname{div}(\boldsymbol{n}^{\intercal}\boldsymbol{\tau}) = \operatorname{div}_F(\boldsymbol{n}^{\intercal}\boldsymbol{\tau}) + \partial_n(\boldsymbol{n}^{\intercal}\boldsymbol{\tau}\boldsymbol{n}) = \operatorname{div}_F(\boldsymbol{n}^{\intercal}\boldsymbol{\tau}). \tag{3.16}$$

Hence $\boldsymbol{n}^{\intercal} \operatorname{div} \boldsymbol{\tau}|_{\partial T} = 0$. Thanks to DoFs (3.1)-(3.3) for BDM element, we acquire from DoF (3.13) and $\operatorname{div}\operatorname{div} \boldsymbol{\tau} = 0$ that $\operatorname{div} \boldsymbol{\tau} = \boldsymbol{0}$, which together with DoF (3.14) and decomposition (3.7) gives

$$(\boldsymbol{\tau}, \boldsymbol{q})_T = 0 \quad \forall\, \boldsymbol{q} \in \mathbb{P}_{k-2}(T; \mathbb{K}).$$

Applying Lemma 3.6, $\boldsymbol{\tau}$ has the expression as in (3.15). Taking $\boldsymbol{q} = q_{ij} \operatorname{skw}(\boldsymbol{n}_i\boldsymbol{n}_j^{\intercal})$ in the last equation for $1 \leq i < j \leq d$, we get $q_{ij} = 0$. Thus $\boldsymbol{\tau} = \boldsymbol{0}$. $\square$

For polygon $K \in \mathcal{T}_h$, define the local finite element space for differential $(d-2)$-form

$$\boldsymbol{V}_k^{d-2}(K) := \{\boldsymbol{\tau} \in \boldsymbol{L}^2(K; \mathbb{K}) : \boldsymbol{\tau}|_T \in \mathbb{P}_k(T; \mathbb{K}) \text{ for each } T \in \mathcal{T}_K,$$
$$\text{all the DoFs (3.10)-(3.12) are single-valued}\}.$$

Thanks to Lemma 3.5, space $\boldsymbol{V}_k^{d-2}(K)$ is $H\Lambda^{d-2}$-conforming. Define $\mathring{\boldsymbol{V}}_k^{d-2}(K) := \boldsymbol{V}_k^{d-2}(K) \cap \mathring{H}\Lambda^{d-2}(K)$, where $\mathring{H}\Lambda^{d-2}(K)$ is the subspace of $H\Lambda^{d-2}(K)$ with homogeneous boundary condition. Notice that $\boldsymbol{V}_k^{d-2}(K)$ is the Lagrange element space for $d = 2$, and $\boldsymbol{V}_k^{d-2}(K)$ is the second kind Nédélec element space for $d = 3$ [39].

Recall the local finite element de Rham complexes in [8, 7]. For completeness, we will prove the exactness of these complexes.

LEMMA 3.8. *Let* $k \geq 2$. *Finite element complexes*

$$\boldsymbol{V}_k^{d-2}(K) \xrightarrow{\operatorname{div}\operatorname{skw}} \boldsymbol{V}_{k-1}^{\mathrm{BDM}}(K) \xrightarrow{\operatorname{div}} V_{k-2}^{L^2}(K) \to 0, \tag{3.17}$$

$$\boldsymbol{V}_1^{d-2}(K) \xrightarrow{\operatorname{div}\operatorname{skw}} \boldsymbol{V}^{\mathrm{RT}}(K) \xrightarrow{\operatorname{div}} V_0^{L^2}(K) \to 0, \tag{3.18}$$

$$\mathring{\boldsymbol{V}}_k^{d-2}(K) \xrightarrow{\operatorname{div}\operatorname{skw}} \mathring{\boldsymbol{V}}_{k-1}^{\mathrm{BDM}}(K) \xrightarrow{\operatorname{div}} \mathring{V}_{k-2}^{L^2}(K) \to 0, \tag{3.19}$$

$$\mathring{\boldsymbol{V}}_1^{d-2}(K) \xrightarrow{\operatorname{div}\operatorname{skw}} \mathring{\boldsymbol{V}}^{\mathrm{RT}}(K) \xrightarrow{\operatorname{div}} \mathring{V}_0^{L^2}(K) \to 0, \tag{3.20}$$

are exact, where $\mathring{V}_{k-2}^{L^2}(K) := V_{k-2}^{L^2}(K)/\mathbb{R}$, *and*

$$V_{k-2}^{L^2}(K) := \{v \in L^2(K) : v|_T \in \mathbb{P}_{k-2}(T) \text{ for each } T \in \mathcal{T}_K\}.$$

Proof. We only prove complex (3.17), since the argument for the rest complexes is similar. Clearly (3.17) is a complex. We refer to [25, Section 4] for the proof of $\operatorname{div} \boldsymbol{V}_{k-1}^{\mathrm{BDM}}(K) = V_{k-2}^{L^2}(K)$.

Next prove $\boldsymbol{V}_{k-1}^{\mathrm{BDM}}(K) \cap \ker(\operatorname{div}) = \operatorname{div}\operatorname{skw} \boldsymbol{V}_k^{d-2}(K)$. For $\boldsymbol{v} \in \boldsymbol{V}_{k-1}^{\mathrm{BDM}}(K) \cap \ker(\operatorname{div})$, by Theorem 1.1 in [30], there exists $\boldsymbol{\tau} \in \boldsymbol{H}^1(K;\mathbb{K})$ satisfying $\operatorname{div}\boldsymbol{\tau} = \operatorname{div}\operatorname{skw}\boldsymbol{\tau} = \boldsymbol{v}$. Let $\boldsymbol{\sigma} \in \boldsymbol{V}_k^{d-2}(K)$ be the nodal interpolation of $\boldsymbol{\tau}$ based on DoFs (3.10)-(3.14). Thanks to DoF (3.10), it follows from the integration by parts that

$$(\operatorname{div}_F(\boldsymbol{\sigma}\boldsymbol{n}), 1)_F = (\boldsymbol{v}\cdot\boldsymbol{n}, 1)_F \quad \forall\, F \in \mathcal{F}(\mathcal{T}_K),$$

which together with (3.16) and DoF (3.11) that

$$(\boldsymbol{n}^{\intercal} \operatorname{div}\boldsymbol{\sigma}, q)_F = (\operatorname{div}_F(\boldsymbol{\sigma}\boldsymbol{n}), q)_F = (\boldsymbol{v}\cdot\boldsymbol{n}, q)_F \quad \forall\, q \in \mathbb{P}_{k-1}(F), F \in \mathcal{F}(\mathcal{T}_K).$$

Therefore, due to DoF (3.13) and the fact $\operatorname{div}\operatorname{div}\boldsymbol{\sigma} = \operatorname{div}\boldsymbol{v} = 0$, we acquire from the unisolvence of DoFs (3.1)-(3.3) for BDM element that $\boldsymbol{v} = \operatorname{div}\boldsymbol{\sigma} \in \operatorname{div}\operatorname{skw}\boldsymbol{V}_k^{d-2}(K)$.∎

Note that $\operatorname{div}\operatorname{skw} = \operatorname{curl}$ for $d = 2, 3$. For $k \geq 1$, by finite element complexes (3.17)-(3.20), we have

$$\dim \operatorname{div}\operatorname{skw} \boldsymbol{V}_k^{d-2}(K) - \dim \operatorname{div}\operatorname{skw} \mathring{\boldsymbol{V}}_k^{d-2}(K) = \binom{k+d-2}{d-1} \#\mathcal{F}^{\partial}(\mathcal{T}_K) - 1. \tag{3.21}$$

3.3. $H(\operatorname{div})$**-conforming macro finite element.** For each polygon $K \in \mathcal{T}_h$, define the shape function space

$$\boldsymbol{V}_{k-1}^{\operatorname{div}}(K) := \{\boldsymbol{\phi} \in \boldsymbol{V}_{k-1}^{\mathrm{BDM}}(K) : \operatorname{div}\boldsymbol{\phi} \in \mathbb{P}_{k-2}(K)\},$$

for $k \geq 2$, and

$$\boldsymbol{V}_0^{\operatorname{div}}(K) := \{\boldsymbol{\phi} \in \boldsymbol{V}_0^{\mathrm{RT}}(K) : \operatorname{div}\boldsymbol{\phi} \in \mathbb{P}_0(K)\}.$$

Apparently $\mathbb{P}_{k-1}(K;\mathbb{R}^d) \subseteq \boldsymbol{V}_{k-1}^{\operatorname{div}}(K)$, $\boldsymbol{V}_0^{\operatorname{div}}(K) \cap \ker(\operatorname{div}) = \boldsymbol{V}_0^{\mathrm{RT}}(K) \cap \ker(\operatorname{div})$, and $\boldsymbol{V}_{k-1}^{\operatorname{div}}(K) \cap \ker(\operatorname{div}) = \boldsymbol{V}_{k-1}^{\mathrm{BDM}}(K) \cap \ker(\operatorname{div})$ for $k \geq 2$.

In the following lemma we present a direct sum decomposition of space $\boldsymbol{V}_{k-1}^{\operatorname{div}}(K)$.

LEMMA 3.9. *For* $k \geq 1$, *it holds*

$$\boldsymbol{V}_{k-1}^{\operatorname{div}}(K) = \operatorname{div}\operatorname{skw} \boldsymbol{V}_k^{d-2}(K) \oplus (\boldsymbol{x} - \boldsymbol{x}_K)\mathbb{P}_{\max\{k-2,0\}}(K). \tag{3.22}$$

Then the complex

$$\boldsymbol{V}_k^{d-2}(K) \xrightarrow{\operatorname{div}\operatorname{skw}} \boldsymbol{V}_{k-1}^{\operatorname{div}}(K) \xrightarrow{\operatorname{div}} \mathbb{P}_{\max\{k-2,0\}}(K) \to 0$$

is exact.

Proof. We only prove the case $k \geq 2$, as the proof for case $k = 1$ is similar. Since $\operatorname{div} : (\boldsymbol{x} - \boldsymbol{x}_K)\mathbb{P}_{k-2}(K) \to \mathbb{P}_{k-2}(K)$ is bijective [26, Lemma 3.1], we have $\operatorname{div}\operatorname{skw}\boldsymbol{V}_k^{d-2}(K) \cap (\boldsymbol{x} - \boldsymbol{x}_K)\mathbb{P}_{k-2}(K) = \{\boldsymbol{0}\}$. Clearly $\operatorname{div}\operatorname{skw}\boldsymbol{V}_k^{d-2}(K) \oplus (\boldsymbol{x} - \boldsymbol{x}_K)\mathbb{P}_{k-2}(K) \subseteq \boldsymbol{V}_{k-1}^{\mathrm{div}}(K)$.

On the other side, for $\boldsymbol{\phi} \in \boldsymbol{V}_{k-1}^{\mathrm{div}}(K)$, by $\operatorname{div}\boldsymbol{\phi} \in \mathbb{P}_{k-2}(K)$, there exists a $q \in \mathbb{P}_{k-2}(K)$ such that $\operatorname{div}((\boldsymbol{x}-\boldsymbol{x}_K)q) = \operatorname{div}\boldsymbol{\phi}$, i.e. $\boldsymbol{\phi}-(\boldsymbol{x}-\boldsymbol{x}_K)q \in \boldsymbol{V}_{k-1}^{\mathrm{div}}(K)\cap\ker(\operatorname{div}) = \boldsymbol{V}_{k-1}^{\mathrm{BDM}}(K) \cap \ker(\operatorname{div})$. Thanks to finite element complex (3.17), $\boldsymbol{\phi} - (\boldsymbol{x} - \boldsymbol{x}_K)q \in \operatorname{div}\operatorname{skw}\boldsymbol{V}_k^{d-2}(K)$. Thus (3.22) follows. ∎

Based on the space decomposition (3.22) and the degrees of freedom of BDM element, we propose the following DoFs for space $\boldsymbol{V}_{k-1}^{\mathrm{div}}(K)$

$$(\boldsymbol{\phi}\cdot\boldsymbol{n}, q)_F \quad \forall\, q \in \mathbb{P}_{k-1}(F) \text{ on each } F \in \mathcal{F}^{\partial}(\mathcal{T}_K), \tag{3.23}$$

$$(\operatorname{div}\boldsymbol{\phi}, q)_K \quad \forall\, q \in \mathbb{P}_{\max\{k-2,0\}}(K)/\mathbb{R}, \tag{3.24}$$

$$(\boldsymbol{\phi}, \boldsymbol{q})_K \quad \forall\, \boldsymbol{q} \in \operatorname{div}\operatorname{skw}\mathring{\boldsymbol{V}}_k^{d-2}(K) = \operatorname{div}\mathring{\boldsymbol{V}}_k^{d-2}(K). \tag{3.25}$$

LEMMA 3.10. *The set of DoFs* (3.23)-(3.25) *is uni-solvent for space* $\boldsymbol{V}_{k-1}^{\mathrm{div}}(K)$.

Proof. By (3.21) and (3.22), the number of DoFs (3.23)-(3.25) is

$$\begin{aligned}&\binom{k+d-2}{d-1}\#\mathcal{F}^{\partial}(\mathcal{T}_K) + \dim\mathbb{P}_{\max\{k-2,0\}}(K) - 1 + \dim\operatorname{div}\operatorname{skw}\mathring{\boldsymbol{V}}_k^{d-2}(K)\\ &= \dim\operatorname{div}\operatorname{skw}\boldsymbol{V}_k^{d-2}(K) + \dim\mathbb{P}_{\max\{k-2,0\}}(K) = \dim\boldsymbol{V}_{k-1}^{\mathrm{div}}(K).\end{aligned}$$

Assume $\boldsymbol{\phi} \in \boldsymbol{V}_{k-1}^{\mathrm{div}}(K)$ and all the DoFs (3.23)-(3.25) vanish. By the vanishing DoF (3.23), $\boldsymbol{\phi} \in \boldsymbol{H}_0(\operatorname{div}, K)$ and $\operatorname{div}\boldsymbol{\phi} \in L_0^2(K)$. Then it follows from the vanishing DoF (3.24) that $\operatorname{div}\boldsymbol{\phi} = 0$. Thanks to the exactness of complexes (3.19)-(3.20), $\boldsymbol{\phi} \in \operatorname{div}\operatorname{skw}\mathring{\boldsymbol{V}}_k^{d-2}(K)$. Therefore $\boldsymbol{\phi} = \boldsymbol{0}$ holds from the vanishing DoF (3.25). ∎

Remark 3.11. When K is a simplex and $\mathcal{T}_K = \{K\}$, thanks to DoF (3.3) for the BDM element, DoF (3.25) can be replaced by

$$(\boldsymbol{\phi}, \boldsymbol{q})_K \quad \forall\, \boldsymbol{q} \in \mathbb{P}_{k-3}(K;\mathbb{K})\boldsymbol{x}$$

for $k \geq 3$. And DoF (3.25) disappears for $k = 1$ and $k = 2$.

Next we consider the norm equivalence of space $\boldsymbol{V}_{k-1}^{\mathrm{div}}(K)$.

LEMMA 3.12. *For* $\boldsymbol{\phi} \in \boldsymbol{V}_{k-1}^{\mathrm{div}}(K)$*, it holds the norm equivalence*

$$\|\boldsymbol{\phi}\|_{0,K} \eqsim h_K\|\operatorname{div}\boldsymbol{\phi}\|_{0,K} + \sup_{\boldsymbol{\psi}\in\operatorname{div}\mathring{\boldsymbol{V}}_k^{d-2}(K)} \frac{(\boldsymbol{\phi},\boldsymbol{\psi})_K}{\|\boldsymbol{\psi}\|_{0,K}} + \sum_{F\in\mathcal{F}^{\partial}(\mathcal{T}_K)} h_F^{1/2}\|\boldsymbol{\phi}\cdot\boldsymbol{n}\|_{0,F}. \tag{3.26}$$

Proof. By the inverse inequality [28, 42] (see also [34, Lemma 10]),

$$h_K\|\operatorname{div}\boldsymbol{\phi}\|_{0,K} \lesssim \|\operatorname{div}\boldsymbol{\phi}\|_{-1,K},$$

where

$$\|\operatorname{div}\boldsymbol{\phi}\|_{-1,K} = \sup_{v\in H_0^1(K)} \frac{(\operatorname{div}\boldsymbol{\phi}, v)_K}{|v|_{1,K}} = -\sup_{v\in H_0^1(K)} \frac{(\boldsymbol{\phi}, \nabla v)_K}{|v|_{1,K}} \leq \|\boldsymbol{\phi}\|_{0,K}.$$

Then we have

$$h_K\|\operatorname{div}\boldsymbol{\phi}\|_{0,K}\lesssim\|\boldsymbol{\phi}\|_{0,K}. \tag{3.27}$$

For $F\in\mathcal{F}^{\partial}(\mathcal{T}_K)$, there exists a simplex $T\in\mathcal{T}_K$ satisfying $F\subset\partial T$, then apply the trace inequality [33, Theorem 1.5.1.10] (see also [17, (2.18)]) and the inverse inequality to get

$$h_F^{1/2}\|\boldsymbol{\phi}\cdot\boldsymbol{n}\|_{0,F}\lesssim\|\boldsymbol{\phi}\|_{0,T}+h_T|\boldsymbol{\phi}|_{1,T}\lesssim\|\boldsymbol{\phi}\|_{0,T}.$$

This means

$$\sum_{F\in\mathcal{F}^{\partial}(\mathcal{T}_K)}h_F^{1/2}\|\boldsymbol{\phi}\cdot\boldsymbol{n}\|_{0,F}\lesssim\sum_{F\in\mathcal{F}^{\partial}(\mathcal{T}_K)}\|\boldsymbol{\phi}\|_{0,T}\lesssim\|\boldsymbol{\phi}\|_{0,K}.$$

Combining (3.27), the Cauchy-Schwarz inequality and the last inequality yields

$$h_K\|\operatorname{div}\boldsymbol{\phi}\|_{0,K}+\sup_{\boldsymbol{\psi}\in\operatorname{div}\mathring{\boldsymbol{V}}_k^{d-2}(K)}\frac{(\boldsymbol{\phi},\boldsymbol{\psi})_K}{\|\boldsymbol{\psi}\|_{0,K}}+\sum_{F\in\mathcal{F}^{\partial}(\mathcal{T}_K)}h_F^{1/2}\|\boldsymbol{\phi}\cdot\boldsymbol{n}\|_{0,F}\lesssim\|\boldsymbol{\phi}\|_{0,K}.$$

Next we focus on the proof of the lower bound. Again we only prove the case $k\geq 2$, whose argument can be applied to case $k=1$. Take $\boldsymbol{\phi}_1\in\boldsymbol{V}_{k-1}^{\mathrm{BDM}}(K)$ such that $(\boldsymbol{\phi}_1\cdot\boldsymbol{n})|_{\partial K}=(\boldsymbol{\phi}\cdot\boldsymbol{n})|_{\partial K}$, and all the DoFs (3.1)-(3.3) of $\boldsymbol{\phi}_1$ interior to K equal to zero. By the norm equivalence on each simplex T and the vanishing DoFs (3.2)-(3.3), we get

$$\|\boldsymbol{\phi}_1\|_{0,K}^2=\sum_{T\in\mathcal{T}_K}\|\boldsymbol{\phi}_1\|_{0,T}^2\eqsim\sum_{T\in\mathcal{T}_K}\sum_{F\in\mathcal{F}(T)}h_F\|\boldsymbol{\phi}_1\cdot\boldsymbol{n}\|_{0,F}^2=\sum_{F\in\mathcal{F}^{\partial}(\mathcal{T}_K)}h_F\|\boldsymbol{\phi}\cdot\boldsymbol{n}\|_{0,F}^2. \tag{3.28}$$

Due to the vanishing DoF (3.2), it holds that $\operatorname{div}\boldsymbol{\phi}_1=Q_0^T(\operatorname{div}\boldsymbol{\phi}_1)$ for $T\in\mathcal{T}_K$. Then apply the integration by parts and the Cauchy-Schwarz inequality to acquire

$$\|\operatorname{div}\boldsymbol{\phi}_1\|_{0,T}^2=\|Q_0^T(\operatorname{div}\boldsymbol{\phi}_1)\|_{0,T}^2\leq\frac{1}{|T|}\sum_{F\in\mathcal{F}(T)\cap\mathcal{F}^{\partial}(\mathcal{T}_K)}|F|\|\boldsymbol{\phi}\cdot\boldsymbol{n}\|_{0,F}^2\quad\forall\, T\in\mathcal{T}_K. \tag{3.29}$$

Now let $w\in H^1(K)\cap L_0^2(K)$ be the solution of

$$\begin{cases}-\Delta w=\operatorname{div}(\boldsymbol{\phi}-\boldsymbol{\phi}_1) & \text{in } K,\\ \partial_n w=0 & \text{on } \partial K.\end{cases}$$

The weak formulation is

$$(\nabla w,\nabla v)_K=(\operatorname{div}(\boldsymbol{\phi}-\boldsymbol{\phi}_1),v)_K\quad\forall\, v\in H^1(K)\cap L_0^2(K).$$

Obviously we obtain from (3.29) that

$$\|\nabla w\|_{0,K}\lesssim h_K\|\operatorname{div}(\boldsymbol{\phi}-\boldsymbol{\phi}_1)\|_{0,K}\lesssim h_K\|\operatorname{div}\boldsymbol{\phi}\|_{0,K}+\sum_{F\in\mathcal{F}^{\partial}(\mathcal{T}_K)}h_F^{1/2}\|\boldsymbol{\phi}\cdot\boldsymbol{n}\|_{0,F}. \tag{3.30}$$

Let $I_K^{\mathrm{div}}:\boldsymbol{H}_0(\operatorname{div},K)\to\mathring{\boldsymbol{V}}_{k-1}^{\mathrm{BDM}}(K)$ be the local L^2-bounded commuting projection operator in [6, 32], then

$$\|I_K^{\mathrm{div}}\boldsymbol{\psi}\|_{0,K}\lesssim\|\boldsymbol{\psi}\|_{0,K}\quad\forall\,\boldsymbol{\psi}\in\boldsymbol{H}_0(\operatorname{div},K), \tag{3.31}$$

$$\operatorname{div}(I_K^{\operatorname{div}}\boldsymbol{\psi}) = \operatorname{div}\boldsymbol{\psi} \quad \text{for } \boldsymbol{\psi}\in \boldsymbol{H}_0(\operatorname{div},K) \text{ satisfying } \operatorname{div}\boldsymbol{\psi}\in V_{k-2}^{L^2}(K).$$

Recall $V_{k-2}^{L^2}(K) = \{v\in L^2(K) : v|_T \in \mathbb{P}_{k-2}(T) \text{ for } T\in\mathcal{T}_K\}$. Set $\boldsymbol{\phi}_2 = -I_K^{\operatorname{div}}(\nabla w)\in \mathring{\boldsymbol{V}}_{k-1}^{\mathrm{BDM}}(K)$. We have

$$\operatorname{div}\boldsymbol{\phi}_2 = -\operatorname{div}(I_K^{\operatorname{div}}(\nabla w)) = -\Delta w = \operatorname{div}(\boldsymbol{\phi}-\boldsymbol{\phi}_1). \tag{3.32}$$

It follows from (3.31) and (3.30) that

$$\|\boldsymbol{\phi}_2\|_{0,K} = \|I_K^{\operatorname{div}}(\nabla w)\|_{0,K} \lesssim \|\nabla w\|_{0,K} \lesssim h_K\|\operatorname{div}\boldsymbol{\phi}\|_{0,K} + \sum_{F\in\mathcal{F}^\partial(\mathcal{T}_K)} h_F^{1/2}\|\boldsymbol{\phi}\cdot\boldsymbol{n}\|_{0,F}. \tag{3.33}$$

By (3.32), $\boldsymbol{\phi}-\boldsymbol{\phi}_1-\boldsymbol{\phi}_2 \in \mathring{\boldsymbol{V}}_{k-1}^{\mathrm{BDM}}(K)\cap\ker(\operatorname{div})$, which together the exactness of complex (3.19) indicates $\boldsymbol{\phi}-\boldsymbol{\phi}_1-\boldsymbol{\phi}_2\in\operatorname{div}\mathring{\boldsymbol{V}}_k^{d-2}(K)$. Hence

$$\begin{aligned}
\|\boldsymbol{\phi}\|_{0,K} &\lesssim \|\boldsymbol{\phi}_1\|_{0,K} + \|\boldsymbol{\phi}_2\|_{0,K} + \|\boldsymbol{\phi}-\boldsymbol{\phi}_1-\boldsymbol{\phi}_2\|_{0,K}\\
&\lesssim \|\boldsymbol{\phi}_1\|_{0,K} + \|\boldsymbol{\phi}_2\|_{0,K} + \sup_{\boldsymbol{\psi}\in\operatorname{div}\mathring{\boldsymbol{V}}_k^{d-2}(K)} \frac{(\boldsymbol{\phi}-\boldsymbol{\phi}_1-\boldsymbol{\phi}_2,\boldsymbol{\psi})_K}{\|\boldsymbol{\psi}\|_{0,K}}\\
&\lesssim \|\boldsymbol{\phi}_1\|_{0,K} + \|\boldsymbol{\phi}_2\|_{0,K} + \sup_{\boldsymbol{\psi}\in\operatorname{div}\mathring{\boldsymbol{V}}_k^{d-2}(K)} \frac{(\boldsymbol{\phi},\boldsymbol{\psi})_K}{\|\boldsymbol{\psi}\|_{0,K}}.
\end{aligned}$$

Finally, (3.26) holds from (3.28) and (3.33). ∎

Let

$$\mathbb{V}_{k-1}^{\operatorname{div}}(K) := \{\boldsymbol{\phi}\in\boldsymbol{V}_{k-1}^{\operatorname{div}}(K) : (\boldsymbol{\phi}\cdot\boldsymbol{n})|_F\in\mathbb{P}_{k-1}(F) \quad \forall\, F\in\mathcal{F}(K)\}.$$

On each face $F\in\mathcal{F}(K)$, $(\boldsymbol{\phi}\cdot\boldsymbol{n})|_F$ is a polynomial for $\boldsymbol{\phi}\in\mathbb{V}_{k-1}^{\operatorname{div}}(K)$ but $(\boldsymbol{\phi}\cdot\boldsymbol{n})|_F$ is a piecewise polynomial for $\boldsymbol{\phi}\in\boldsymbol{V}_{k-1}^{\operatorname{div}}(K)$. Due to DoFs (3.23)-(3.25) for $\boldsymbol{V}_{k-1}^{\operatorname{div}}(K)$, a set of unisolvent DoFs for $\mathbb{V}_{k-1}^{\operatorname{div}}(K)$ is

$$(\boldsymbol{\phi}\cdot\boldsymbol{n},q)_F \quad \forall\, q\in\mathbb{P}_{k-1}(F) \text{ on each } F\in\mathcal{F}(K), \tag{3.34}$$

$$(\operatorname{div}\boldsymbol{\phi},q)_K \quad \forall\, q\in\mathbb{P}_{\max\{k-2,0\}}(K)/\mathbb{R}, \tag{3.35}$$

$$(\boldsymbol{\phi},\boldsymbol{q})_K \quad \forall\, \boldsymbol{q}\in\operatorname{div}\mathring{\boldsymbol{V}}_k^{d-2}(K). \tag{3.36}$$

As an immediate result of Lemma 3.12, we get the following norm equivalence of space $\mathbb{V}_{k-1}^{\operatorname{div}}(K)$.

Corollary 3.13. *For $\boldsymbol{\phi}\in\mathbb{V}_{k-1}^{\operatorname{div}}(K)$, it holds the norm equivalence*

$$\|\boldsymbol{\phi}\|_{0,K} \eqsim h_K\|\operatorname{div}\boldsymbol{\phi}\|_{0,K} + \sup_{\boldsymbol{\psi}\in\operatorname{div}\mathring{\boldsymbol{V}}_k^{d-2}(K)} \frac{(\boldsymbol{\phi},\boldsymbol{\psi})_K}{\|\boldsymbol{\psi}\|_{0,K}} + \sum_{F\in\mathcal{F}(K)} h_F^{1/2}\|\boldsymbol{\phi}\cdot\boldsymbol{n}\|_{0,F}. \tag{3.37}$$

For later use, let $Q_{K,k-1}^{\operatorname{div}}$ be the L^2-orthogonal projection operator onto $\mathbb{V}_{k-1}^{\operatorname{div}}(K)$ with respect to the inner product $(\cdot,\cdot)_K$. Introduce the discrete spaces

$$\begin{aligned}
\mathbb{V}_{h,k-1}^{\operatorname{div}} &:= \{\boldsymbol{\phi}_h\in\boldsymbol{L}^2(\Omega;\mathbb{R}^d) : \boldsymbol{\phi}_h|_K\in\mathbb{V}_{k-1}^{\operatorname{div}}(K) \text{ for each } K\in\mathcal{T}_h\},\\
\mathbb{P}_l(\mathcal{T}_h) &:= \{q_h\in L^2(\Omega) : q_h|_K\in\mathbb{P}_l(K) \text{ for each } K\in\mathcal{T}_h\}
\end{aligned}$$

with non-negative integer l. For $\boldsymbol{\phi}\in\boldsymbol{L}^2(\Omega;\mathbb{R}^d)$, let $Q_{h,k-1}^{\operatorname{div}}\boldsymbol{\phi}\in\mathbb{V}_{h,k-1}^{\operatorname{div}}$ be determined by $(Q_{h,k-1}^{\operatorname{div}}\boldsymbol{\phi})|_K = Q_{K,k-1}^{\operatorname{div}}(\boldsymbol{\phi}|_K)$ for each $K\in\mathcal{T}_h$. For $v\in L^2(\Omega)$, let $Q_h^l v\in\mathbb{P}_l(\mathcal{T}_h)$ be determined by $(Q_h^l v)|_K = Q_l^K(v|_K)$ for each $K\in\mathcal{T}_h$. For simplicity, the vector version of Q_h^l is still denoted by Q_h^l. And we abbreviate Q_h^k as Q_h if $l=k$.

4. Nonconforming virtual element method without extrinsic stabilization. In this section we will develop a nonconforming VEM without extrinsic stabilization for the second order elliptic problem in arbitrary dimension

$$\begin{cases} -\Delta u + \alpha u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $\Omega \subseteq \mathbb{R}^d$ is a bounded polygon, $f \in L^2(\Omega)$ and α is a nonnegative constant. The weak formulation of problem (4.1) is to find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = (f, v) \quad \forall\, v \in H_0^1(\Omega), \tag{4.2}$$

where the bilinear form $a(u, v) := (\nabla_h u, \nabla_h v) + \alpha(u, v)$ with ∇_h being the piecewise counterpart of ∇ with respect to $\mathcal{T}_h$.

4.1. H^1-nonconforming virtual element. Several H^1-nonconforming virtual elements have been developed in [9, 20, 24, 34]. In this paper we adopt those in [20, 24]. The degrees of freedom are given by

$$\frac{1}{|F|}(v, \phi_i^F)_F, \quad i = 1, \ldots, \dim \mathbb{P}_{k-1}(F), F \in \mathcal{F}(K), \tag{4.3}$$

$$\frac{1}{|K|}(v, \phi_i^K)_K, \quad i = 1, \ldots, \dim \mathbb{P}_{k-2}(K), \tag{4.4}$$

where $\{\phi_i^F\}_{i=1}^{\dim \mathbb{P}_{k-1}(F)}$ is a basis of $\mathbb{P}_{k-1}(F)$, and $\{\phi_i^K\}_{i=1}^{\dim \mathbb{P}_{k-2}(K)}$ a basis of $\mathbb{P}_{k-2}(K)$.

To define the space of shape functions, we need a local H^1 projection operator $\Pi_k^K : H^1(K) \to \mathbb{P}_k(K)$: given $v \in H^1(K)$, let $\Pi_k^K v \in \mathbb{P}_k(K)$ be the solution of the problem

$$(\nabla \Pi_k^K v, \nabla q)_K = (\nabla v, \nabla q)_K \quad \forall\, q \in \mathbb{P}_k(K), \tag{4.5}$$

$$\int_{\partial K} \Pi_k^K v \,\mathrm{d}s = \int_{\partial K} v \,\mathrm{d}s. \tag{4.6}$$

It holds

$$\Pi_k^K q = q \quad \forall\, q \in \mathbb{P}_k(K). \tag{4.7}$$

With the help of operator Π_k^K, the space of shape functions is defined as

$$V_k(K) := \{v \in H^1(K) : \Delta v \in \mathbb{P}_k(K),\, \partial_n v|_F \in \mathbb{P}_{k-1}(F) \text{ for each face } F \in \mathcal{F}(K),$$
$$\text{and } (v - \Pi_k^K v, q)_K = 0 \quad \forall\, q \in \mathbb{P}_{k-2}^{\perp}(K)\},$$

where $\mathbb{P}_{k-2}^{\perp}(K)$ means the orthogonal complement space of $\mathbb{P}_{k-2}(K)$ in $\mathbb{P}_k(K)$ with respect to the inner product $(\cdot, \cdot)_K$. Due to (4.7), it holds $\mathbb{P}_k(K) \subseteq V_k(K)$. DoFs (4.3)-(4.4) are uni-solvent for the shape function space $V_k(K)$.

For $v \in V_k(K)$, the H^1 projection $\Pi_k^K v$ is computable using DoFs (4.3)-(4.4), and the L^2 projection

$$Q_k^K v = \Pi_k^K v + Q_{k-2}^K v - Q_{k-2}^K \Pi_k^K v \tag{4.8}$$

is also computable using DoFs (4.3)-(4.4).

We will prove the inverse inequality and the norm equivalence for the virtual element space $V_k(K)$.

LEMMA 4.1. *It holds the inverse inequality*

$$|v|_{1,K} \lesssim h_K^{-1}\|v\|_{0,K} \quad \forall\, v \in V_k(K). \tag{4.9}$$

Proof. By (A.4) with $m = j = 1$ in [24], it follows that

$$h_K^{1/2}\|\partial_n v\|_{0,\partial K} \lesssim |v|_{1,K} + h_K\|\Delta v\|_{0,K}.$$

Then apply (A.3) in [24] to get

$$h_K\|\Delta v\|_{0,K} + h_K^{1/2}\|\partial_n v\|_{0,\partial K} \lesssim |v|_{1,K} + h_K\|\Delta v\|_{0,K} \lesssim |v|_{1,K}.$$

Employing the integration by parts and the Cauchy-Schwarz inequality, we have

$$|v|_{1,K}^2 = -(\Delta v, v)_K + (\partial_n v, v)_{\partial K} \leq \|\Delta v\|_{0,K}\|v\|_{0,K} + \|\partial_n v\|_{0,\partial K}\|v\|_{0,\partial K}.$$

Combining the last two inequalities gives

$$|v|_{1,K} \lesssim h_K^{-1}\|v\|_{0,K} + h_K^{-1/2}\|v\|_{0,\partial K},$$

which together with the multiplicative trace inequality and the Young's inequality yields (4.9). ∎

LEMMA 4.2. *For $v \in V_k(K)$, we have*

$$\|\Pi_k^K v\|_{0,K}^2 + h_K^2|\Pi_k^K v|_{1,K}^2 \lesssim \|Q_{k-2}^K v\|_{0,K}^2 + \sum_{F\in\mathcal{F}(K)} h_F\|Q_{k-1}^F v\|_{0,F}^2, \tag{4.10}$$

$$\|Q_k^K v\|_{0,K}^2 \lesssim \|Q_{k-2}^K v\|_{0,K}^2 + \sum_{F\in\mathcal{F}(K)} h_F\|Q_{k-1}^F v\|_{0,F}^2. \tag{4.11}$$

Proof. We get from (4.5) and the integration by parts that

$$\begin{aligned}
|\Pi_k^K v|_{1,K}^2 &= (\nabla v, \nabla\Pi_k^K v)_K = -(v, \Delta\Pi_k^K v)_K + (v, \partial_n(\Pi_k^K v))_{\partial K} \\
&= -(Q_{k-2}^K v, \Delta\Pi_k^K v)_K + \sum_{F\in\mathcal{F}(K)} (Q_{k-1}^F v, \partial_n(\Pi_k^K v))_F \\
&\leq \|Q_{k-2}^K v\|_{0,K}\|\Delta\Pi_k^K v\|_{0,K} + \sum_{F\in\mathcal{F}(K)} \|Q_{k-1}^F v\|_{0,F}\|\partial_n(\Pi_k^K v)\|_{0,F},
\end{aligned}$$

which combined with both H^1-L^2 and L^2 boundary-L^2 bulk inverse inequalities for polynomials implies

$$h_K^2|\Pi_k^K v|_{1,K}^2 \lesssim \|Q_{k-2}^K v\|_{0,K}^2 + \sum_{F\in\mathcal{F}(K)} h_F\|Q_{k-1}^F v\|_{0,F}^2.$$

Thanks to the Poincaré-Friedrichs inequality [40] and (4.6),

$$\begin{aligned}
\|\Pi_k^K v\|_{0,K}^2 &\lesssim h_K|\Pi_k^K v|_{1,K}^2 + h_K^{2-d}\left|\int_{\partial K} v\,\mathrm{d}s\right|^2 \\
&= h_K^2|\Pi_k^K v|_{1,K}^2 + h_K^{2-d}\left|\sum_{F\in\mathcal{F}(K)}\int_F Q_0^F v\,\mathrm{d}s\right|^2 \\
&\lesssim h_K^2|\Pi_k^K v|_{1,K}^2 + \sum_{F\in\mathcal{F}(K)} h_F\|Q_0^F v\|_{0,F}^2.
\end{aligned}$$

Hence (4.10) follows from the last two inequalities.

Finally, (4.11) holds from (4.8) and (4.10). ∎

LEMMA 4.3. *It holds the norm equivalence*

$$h_K^2|v|_{1,K}^2 \lesssim \|v\|_{0,K}^2 \eqsim \|Q_{k-2}^K v\|_{0,K}^2 + \sum_{F\in\mathcal{F}(K)} h_F\|Q_{k-1}^F v\|_{0,F}^2 \quad \forall\, v \in V_k(K). \tag{4.12}$$

Proof. Since $\Delta v \in \mathbb{P}_k(K)$ and $\partial_n v|_F \in \mathbb{P}_{k-1}(F)$, we get from the integration by parts that

$$\begin{aligned} |v|_{1,K}^2 &= -(\Delta v, Q_k^K v)_K + \sum_{F\in\mathcal{F}(K)} (\partial_n v, Q_{k-1}^F v)_F \\ &\le \|\Delta v\|_{0,K}\|Q_k^K v\|_{0,K} + \sum_{F\in\mathcal{F}(K)} \|\partial_n v\|_{0,F}\|Q_{k-1}^F v\|_{0,F}. \end{aligned}$$

Applying the similar argument as in Lemma 4.1, we obtain

$$h_K^2|v|_{1,K}^2 \lesssim \|Q_k^K v\|_{0,K}^2 + \sum_{F\in\mathcal{F}(K)} h_F\|Q_{k-1}^F v\|_{0,F}^2.$$

Then it follows from (4.11) that

$$\|v\|_{0,K}^2 \lesssim \|Q_{k-2}^K v\|_{0,K}^2 + \sum_{F\in\mathcal{F}(K)} h_F\|Q_{k-1}^F v\|_{0,F}^2.$$

The other side $\|Q_{k-2}^K v\|_{0,K}^2 + \sum_{F\in\mathcal{F}(K)} h_F\|Q_{k-1}^F v\|_{0,F}^2 \lesssim \|v\|_{0,K}^2$ holds from the trace inequality and the inverse inequality (4.9). ∎

4.2. Local inf-sup condition and norm equivalence. With the help of the macro element space $\mathbb{V}_{k-1}^{\mathrm{div}}(K)$, we will present a norm equivalence for space $\nabla V_k(K)$, which is vitally important to design virtual element methods without extrinsic stabilization.

LEMMA 4.4. *It holds the inf-sup condition*

$$\|\nabla v\|_{0,K} \le C_e \sup_{\boldsymbol{\phi}\in\mathbb{V}_{k-1}^{\mathrm{div}}(K)} \frac{(\boldsymbol{\phi}, \nabla v)_K}{\|\boldsymbol{\phi}\|_{0,K}} \quad \forall\, v \in V_k(K), \tag{4.13}$$

where the constant $C_e \ge 1$ is independent of the mesh size h_K, but depends on the chunkiness parameter of the polytope, the degree of polynomials k, the dimension of space d, and the shape regularity and quasi-uniform constants of the virtual triangulation $\mathcal{T}_h^*$. Consequently,

$$\|Q_{K,k-1}^{\mathrm{div}} \nabla v\|_{0,K} \eqsim \|\nabla v\|_{0,K} \quad \forall\, v \in V_k(K). \tag{4.14}$$

Proof. Clearly the norm equivalence (4.14) follows from the local inf-sup condition (4.13). We will focus on the proof of (4.13). Without loss of generality, assume $v \in V_k(K) \cap L_0^2(K)$. Based on DoFs (3.34)-(3.36), take $\boldsymbol{\phi} \in \mathbb{V}_{k-1}^{\mathrm{div}}(K)$ such that

$$\begin{aligned} (\boldsymbol{\phi}\cdot\boldsymbol{n}, q)_F &= h_K^{-1}(v, q)_F && \forall\, q \in \mathbb{P}_{k-1}(F) \text{ on each } F \in \mathcal{F}(K), \\ (\operatorname{div}\boldsymbol{\phi}, q)_K &= -h_K^{-2}(v, q)_K && \forall\, q \in \mathbb{P}_{\max\{k-2,0\}}(K)/\mathbb{R}, \\ (\boldsymbol{\phi}, \boldsymbol{q})_K &= 0 && \forall\, \boldsymbol{q} \in \operatorname{div}\operatorname{skw} \mathring{\boldsymbol{V}}_k^{d-2}(K). \end{aligned}$$

Then $(\boldsymbol{\phi}\cdot\boldsymbol{n})|_F = h_K^{-1}Q_{k-1}^F v$ for $F\in\mathcal{F}(K)$. Since $\operatorname{div}\boldsymbol{\phi}\in\mathbb{P}_{\max\{k-2,0\}}(K)$, we have $\operatorname{div}\boldsymbol{\phi} - Q_0^K(\operatorname{div}\boldsymbol{\phi}) = -h_K^{-2}Q_{k-2}^K v$. Apply the integration by parts and the fact $v = v - Q_0^K v \in L_0^2(K)$ to get

$$
\begin{aligned}
(\boldsymbol{\phi},\nabla v)_K &= -(\operatorname{div}\boldsymbol{\phi}, v)_K + (\boldsymbol{\phi}\cdot\boldsymbol{n}, v)_{\partial K}\\
&= -(\operatorname{div}\boldsymbol{\phi} - Q_0^K(\operatorname{div}\boldsymbol{\phi}), v)_K + \sum_{F\in\mathcal{F}(K)}(\boldsymbol{\phi}\cdot\boldsymbol{n}, Q_{k-1}^F v)_F\\
&= h_K^{-2}\|Q_{k-2}^K v\|_{0,K}^2 + \sum_{F\in\mathcal{F}(K)} h_K^{-1}\|Q_{k-1}^F v\|_{0,F}^2.
\end{aligned}
$$

By the norm equivalence (4.12), we get

$$
\|\nabla v\|_{0,K}^2 \lesssim h_K^{-2}\|Q_{k-2}^K v\|_{0,K}^2 + \sum_{F\in\mathcal{F}(K)} h_K^{-1}\|Q_{k-1}^F v\|_{0,F}^2 = (\boldsymbol{\phi},\nabla v)_K. \tag{4.15}
$$

On the other hand, it follows from the integration by parts that

$$
\begin{aligned}
\|Q_0^K(\operatorname{div}\boldsymbol{\phi})\|_{0,K} \lesssim h_K^{d/2}\big|Q_0^K(\operatorname{div}\boldsymbol{\phi})\big| \lesssim h_K^{-d/2}\big|(\operatorname{div}\boldsymbol{\phi},1)_K\big| = h_K^{-d/2}\big|(\boldsymbol{\phi}\cdot\boldsymbol{n},1)_{\partial K}\big|\\
\lesssim \sum_{F\in\mathcal{F}(K)} h_F^{-1/2}\|\boldsymbol{\phi}\cdot\boldsymbol{n}\|_{0,F}.
\end{aligned}
$$

Employing the norm equivalence (3.37), we acquire

$$
\begin{aligned}
\|\boldsymbol{\phi}\|_{0,K} &\eqsim h_K\|\operatorname{div}\boldsymbol{\phi}\|_{0,K} + \sum_{F\in\mathcal{F}(K)} h_F^{1/2}\|\boldsymbol{\phi}\cdot\boldsymbol{n}\|_{0,F}\\
&\lesssim h_K\|\operatorname{div}\boldsymbol{\phi} - Q_0^K(\operatorname{div}\boldsymbol{\phi})\|_{0,K} + \sum_{F\in\mathcal{F}(K)} h_F^{1/2}\|\boldsymbol{\phi}\cdot\boldsymbol{n}\|_{0,F}.
\end{aligned}
$$

Noting that $\operatorname{div}\boldsymbol{\phi} - Q_0^K(\operatorname{div}\boldsymbol{\phi}) = -h_K^{-2}Q_{k-2}^K v$ and $(\boldsymbol{\phi}\cdot\boldsymbol{n})|_F = h_K^{-1}Q_{k-1}^F v$ for $F\in\mathcal{F}(K)$, we have

$$
\|\boldsymbol{\phi}\|_{0,K} \lesssim h_K^{-1}\|Q_{k-2}^K v\|_{0,K} + \sum_{F\in\mathcal{F}(K)} h_F^{-1/2}\|Q_{k-1}^F v\|_{0,F}.
$$

Then we obtain from the norm equivalence (4.12) and the Poincaré-Friedrichs inequality [17, (2.14)] that

$$
\|\boldsymbol{\phi}\|_{0,K} \lesssim h_K^{-1}\|v\|_{0,K} \lesssim \|\nabla v\|_{0,K}.
$$

Finally, we conclude (4.13) from (4.15) and the last inequality. ◻

4.3. Discrete method. Define the global nonconforming virtual element space

$$
\begin{aligned}
V_h := \{v_h\in L^2(\Omega) : v_h|_K\in V_k(K) \text{ for each } K\in\mathcal{T}_h,\\
\text{DoFs (4.3) are single-valued for } F\in\mathcal{F}_h, \text{ and vanish for } F\in\mathcal{F}_h^\partial\}.
\end{aligned}
$$

We have the discrete Poincaré inequality [16]

$$
\|v_h\|_0 \leq C_p|v_h|_{1,h} \quad \forall\, v_h\in V_h, \tag{4.16}
$$

where the constant C_p is independent of the mesh size. Hence $|\cdot|_{1,h}$ is indeed a norm for V_h.

Based on the weak formulation (4.2), we propose a virtual element method without extrinsic stabilization for problem (4.1) as follows: find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = (f, Q_h v_h) \quad \forall\, v_h \in V_h, \tag{4.17}$$

where the discrete bilinear form

$$a_h(u_h, v_h) := (Q^{\mathrm{div}}_{h,k-1}\nabla_h u_h, Q^{\mathrm{div}}_{h,k-1}\nabla_h v_h) + \alpha(Q_h u_h, Q_h v_h).$$

Remark 4.5. By introducing $\boldsymbol{\phi}_h = Q^{\mathrm{div}}_{h,k-1}\nabla_h u_h$, the VEM (4.17) can be rewritten as the following primal mixed VEM: find $\boldsymbol{\phi}_h \in \mathbb{V}^{\mathrm{div}}_{h,k-1}$ and $u_h \in V_h$ such that

$$\begin{aligned} (\boldsymbol{\phi}_h, \boldsymbol{\psi}_h) - (\boldsymbol{\psi}_h, \nabla_h u_h) &= 0 && \forall\, \boldsymbol{\psi}_h \in \mathbb{V}^{\mathrm{div}}_{h,k-1}, \\ (\boldsymbol{\phi}_h, \nabla_h v_h) + \alpha(Q_h u_h, Q_h v_h) &= (f, Q_h v_h) && \forall\, v_h \in V_h. \end{aligned}$$

Remark 4.6. The mixed-order HHO method without extrinsic stabilization for the Poisson equation in [29] is equivalent to find $u_h \in W_h$ such that

$$(\nabla_h u_h, \nabla_h v_h) = \sum_{K\in\mathcal{T}_h} (f, Q^K_{k-2} v_h)_K \quad \forall\, v_h \in W_h, \tag{4.18}$$

where

$$\begin{aligned} W_h := \{ v_h \in L^2(\Omega) : v_h|_K \in W_k(K) \text{ for each } K \in \mathcal{T}_h, \\ \text{DoFs (4.3) are single-valued for } F \in \mathcal{F}_h, \text{ and vanish for } F \in \mathcal{F}^{\partial}_h \} \end{aligned}$$

with $W_k(K) := \{ v \in H^1(K) : \Delta v \in \mathbb{P}_{k-2}(K),\, \partial_n v|_F \in \mathbb{P}_{k-1}(F) \text{ for } F \in \mathcal{F}(K)\}$. However, $(\nabla_h u_h, \nabla_h v_h)$ is not computable, and the method (4.18) is not the standard nonconforming VEM in [9]. For the practical computation, a basis of $W_k(K)$ has to be solved approximately as shown in [29, Remark 4.1], then the approximation of the space W_h is no more a virtual element space.

It follows from the discrete Poincaré inequality (4.16) that

$$a_h(u_h, v_h) \lesssim |u_h|_{1,h} |v_h|_{1,h} \quad \forall\, u_h, v_h \in H^1_0(\Omega) + V_h. \tag{4.19}$$

LEMMA 4.7. *It holds the coercivity*

$$|v_h|^2_{1,h} \leq C_e^2 a_h(v_h, v_h) \quad \forall\, v_h \in V_h. \tag{4.20}$$

Proof. Due to (4.14), we have

$$\sum_{K\in\mathcal{T}_h} \|\nabla_h v_h\|^2_{0,K} \leq C_e^2 \sum_{K\in\mathcal{T}_h} \|Q^{\mathrm{div}}_{K,k-1}\nabla v_h\|^2_{0,K} \leq C_e^2 a_h(v_h, v_h) \quad \forall\, v_h \in V_h,$$

which implies the coercivity (4.20). ∎

THEOREM 4.8. *The VEM* (4.17) *is well-posed.*

Proof. Thanks to the boundedness (4.19) and the coercivity (4.20), we conclude the result from the Lax-Milgram lemma [36]. ∎

4.4. Error analysis.

THEOREM 4.9. *Let $u \in H_0^1(\Omega)$ be the solution of problem (4.1), and $u_h \in V_h$ be the solution of the VEM (4.17). It holds that*

$$|u-u_h|_{1,h} \le C_e^2(\|\nabla u - Q_h^{k-1}\nabla u\|_0 + \alpha C_p\|u - Q_h u\|_0) + \inf_{v_h\in V_h}\big((C_e^2+1)|u-v_h|_{1,h} + \alpha C_p\|u-v_h\|_0\big) + C_e^2 \sup_{w_h\in V_h}\frac{a(u,w_h)-(f,Q_h w_h)}{|w_h|_{1,h}}. \tag{4.21}$$

Assume $u \in H^{k+1}(\Omega)$ and $f \in H^{k-1}(\Omega)$. Then

$$|u-u_h|_{1,h} \lesssim h^k(|u|_{k+1} + |f|_{k-1}). \tag{4.22}$$

Proof. Take any $v_h \in V_h$. By the definitions of $a_h(\cdot,\cdot)$ and $a(\cdot,\cdot)$, it follows from the discrete Poincaré inequality (4.16) that

$$\begin{aligned} & a_h(v_h, v_h-u_h) - a(u, v_h-u_h) \\ &= (Q_{h,k-1}^{\mathrm{div}}\nabla_h v_h, Q_{h,k-1}^{\mathrm{div}}\nabla_h(v_h-u_h)) - (\nabla u, \nabla_h(v_h-u_h)) \\ &\quad + \alpha(Q_h v_h, Q_h(v_h-u_h)) - \alpha(u, v_h-u_h) \\ &= (Q_{h,k-1}^{\mathrm{div}}\nabla_h v_h - \nabla u, \nabla_h(v_h-u_h)) + \alpha(Q_h v_h - u, v_h-u_h) \\ &\le (\|\nabla u - Q_{h,k-1}^{\mathrm{div}}\nabla_h v_h\|_0 + \alpha C_p\|u - Q_h v_h\|_0)|v_h-u_h|_{1,h}. \end{aligned}$$

Apply the coercivity (4.20) and (4.17) to get

$$C_e^{-2}|v_h-u_h|_{1,h}^2 \le a_h(v_h-u_h, v_h-u_h) = a_h(v_h, v_h-u_h) - (f, Q_h(v_h-u_h)).$$

Combining the last two inequalities yields

$$|v_h-u_h|_{1,h} \le C_e^2(\|\nabla u - Q_{h,k-1}^{\mathrm{div}}\nabla_h v_h\|_0 + \alpha C_p\|u - Q_h v_h\|_0) + C_e^2\sup_{w_h\in V_h}\frac{a(u,w_h)-(f,Q_h w_h)}{|w_h|_{1,h}}.$$

Since $\mathbb{P}_{k-1}(K;\mathbb{R}^d) \subseteq \mathbb{V}_{k-1}^{\mathrm{div}}(K)$ for $K \in \mathcal{T}_h$, we have $Q_{h,k-1}^{\mathrm{div}}(Q_h^{k-1}\nabla u) = Q_h^{k-1}\nabla u$. Hence

$$\begin{aligned} \|\nabla u - Q_{h,k-1}^{\mathrm{div}}\nabla_h v_h\|_0 &\le \|\nabla u - Q_{h,k-1}^{\mathrm{div}}\nabla u\|_0 + \|Q_{h,k-1}^{\mathrm{div}}(\nabla u - \nabla_h v_h)\|_0 \\ &= \|\nabla u - Q_h^{k-1}\nabla u - Q_{h,k-1}^{\mathrm{div}}(\nabla u - Q_h^{k-1}\nabla u)\|_0 \\ &\quad + \|Q_{h,k-1}^{\mathrm{div}}(\nabla u - \nabla_h v_h)\|_0 \\ &\le \|\nabla u - Q_h^{k-1}\nabla u\|_0 + |u-v_h|_{1,h}. \end{aligned}$$

Similarly, we have

$$\|u - Q_h v_h\|_0 \le \|u - Q_h u\|_0 + \|Q_h(u-v_h)\|_0 \le \|u - Q_h u\|_0 + \|u-v_h\|_0.$$

Then we obtain from the last three inequalities that

$$|v_h-u_h|_{1,h} \le C_e^2\big(\|\nabla u - Q_h^{k-1}\nabla u\|_0 + |u-v_h|_{1,h} + \alpha C_p(\|u-Q_h u\|_0 + \|u-v_h\|_0)\big) + C_e^2\sup_{w_h\in V_h}\frac{a(u,w_h)-(f,Q_h w_h)}{|w_h|_{1,h}}.$$

Thus, we acquire (4.21) from the last inequality and the triangle inequality.

Next we derive estimate (4.22). Recall the consistency error estimate in [24, Lemma 5.5]

$$a(u, w_h) + (f, w_h) \lesssim h^k |u|_{k+1} |w_h|_{1,h} \quad \forall\, w_h \in V_h.$$

Then

$$\begin{aligned} a(u, w_h) - (f, Q_h w_h) &= a(u, w_h) + (f, w_h) + (f - Q_h f, w_h) \\ &= a(u, w_h) + (f, w_h) + (f - Q_h f, w_h - Q_h^0 w_h) \\ &\lesssim h^k (|u|_{k+1} + |f|_{k-1}) |w_h|_{1,h}. \end{aligned}$$

At last, (4.22) follows from (4.21) and the approximation of V_h [24]. ∎

5. Conforming virtual element method without extrinsic stabilization.

In this section we will develop a conforming VEM without extrinsic stabilization for the second order elliptic problem (4.1) in two dimensions.

For polygon $K \subset \mathbb{R}^2$, let $\mathcal{V}(K)$ be the set of all vertices of K. And we overload the notation $\mathcal{E}(K)$ to denote the set of all edges of K in this section.

5.1. H^1-conforming virtual element.

Recall the H^1-conforming virtual element in [22, 1, 12, 13]. The degrees of freedom are given by

$$v(\delta), \quad \delta \in \mathcal{V}(K), \tag{5.1}$$

$$\frac{1}{|e|}(v, \phi_i^e)_e, \quad i = 1, \ldots, k-1, e \in \mathcal{E}(K), \tag{5.2}$$

$$\frac{1}{|K|}(v, \phi_i^K)_K, \quad i = 1, \ldots, \dim \mathbb{P}_{k-2}(K), \tag{5.3}$$

where $\{\phi_i^e\}_{i=1}^{k-1}$ is a basis of $\mathbb{P}_{k-2}(e)$, and $\{\phi_i^K\}_{i=1}^{\dim \mathbb{P}_{k-2}(K)}$ a basis of $\mathbb{P}_{k-2}(K)$. And the space of shape functions is

$$\begin{aligned} V_k(K) := \big\{ v \in H^1(K) : \Delta v \in \mathbb{P}_k(K), v|_{\partial K} \in H^1(\partial K), v|_e \in \mathbb{P}_k(e) \;\; \forall\, e \in \mathcal{E}(K), \\ \text{and } (v - \Pi_k^K v, q)_K = 0 \quad \forall\, q \in \mathbb{P}_{k-2}^{\perp}(K) \big\}, \end{aligned}$$

where Π_k^K is defined by (4.5)-(4.6). It holds $\mathbb{P}_k(K) \subseteq V_k(K)$.

For $v \in V_k(K)$, the H^1 projection $\Pi_k^K v$ and the L^2 projection $Q_k^K v = \Pi_k^K v + Q_{k-2}^K v - Q_{k-2}^K \Pi_k^K v$ are computable using the DoFs (5.1)-(5.3). We have the norm equivalence of space $V_k(K)$ (cf. [22, Lemma 4.7] and [23, 17, 11]), that is for $v \in V_k(K)$, it holds

$$h_K^2 |v|_{1,K}^2 \lesssim \|v\|_{0,K}^2 \eqsim \|Q_{k-2}^K v\|_{0,K}^2 + \sum_{\delta \in \mathcal{V}(K)} h_K^2 |v(\delta)|^2 + \sum_{e \in \mathcal{E}(K)} h_K \|Q_{k-2}^e v\|_{0,e}^2. \tag{5.4}$$

Employing the same argument as in Lemma 4.4, from (5.4), we get the norm equivalence

$$\|Q_{K,k}^{\text{div}} \nabla v\|_{0,K} \eqsim \|\nabla v\|_{0,K} \quad \forall\, v \in V_k(K).$$

Remark 5.1. When $k \geq 2$, we can replace $Q_{K,k}^{\text{div}}$ by the L^2-orthogonal projection operator onto space $\mathbb{V}_{k,k-2}^{\text{div}}(K)$, where

$$\begin{aligned} \mathbb{V}_{k,k-2}^{\text{div}}(K) :=& \{\boldsymbol{\phi} \in \mathbb{V}_k^{\text{div}}(K) : \operatorname{div} \boldsymbol{\phi} \in \mathbb{P}_{k-2}(K)\} \\ =& \{\boldsymbol{\phi} \in \boldsymbol{V}_k^{\text{BDM}}(K) : \operatorname{div} \boldsymbol{\phi} \in \mathbb{P}_{k-2}(K), \boldsymbol{\phi} \cdot \boldsymbol{n}|_e \in \mathbb{P}_k(e) \;\; \forall\, e \in \mathcal{E}(K)\}. \end{aligned}$$

5.2. Discrete method. Define the global conforming virtual element space

$$V_h := \{v_h \in H_0^1(\Omega) : v_h|_K \in V_k(K) \text{ for each } K \in \mathcal{T}_h\}.$$

Based on the weak formulation (4.2), we propose a virtual element method without extrinsic stabilization for problem (4.1) as follows: find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = (f, Q_h v_h) \quad \forall\, v_h \in V_h, \tag{5.5}$$

where the discrete bilinear form

$$a_h(u_h, v_h) := (Q_{h,k}^{\mathrm{div}} \nabla u_h, Q_{h,k}^{\mathrm{div}} \nabla v_h) + \alpha(Q_h u_h, Q_h v_h).$$

The VEM (5.5) is uni-solvent.

By introducing $\boldsymbol{\phi}_h = Q_{h,k}^{\mathrm{div}} \nabla u_h$, the VEM (5.5) can be rewritten as the following primal mixed VEM: find $\boldsymbol{\phi}_h \in \mathbb{V}_{h,k}^{\mathrm{div}}$ and $u_h \in V_h$ such that

$$\begin{aligned} (\boldsymbol{\phi}_h, \boldsymbol{\psi}_h) - (\boldsymbol{\psi}_h, \nabla u_h) &= 0 && \forall\, \boldsymbol{\psi}_h \in \mathbb{V}_{h,k}^{\mathrm{div}}, \\ (\boldsymbol{\phi}_h, \nabla v_h) + \alpha(Q_h u_h, Q_h v_h) &= (f, Q_h v_h) && \forall\, v_h \in V_h. \end{aligned}$$

Applying the standard error analysis for VEMs, we have the following error estimate for the VEM (5.5).

THEOREM 5.2. *Let $u \in H_0^1(\Omega)$ be the solution of problem (4.1), and $u_h \in V_h$ be the solution of the VEM (5.5). Assume $u \in H^{k+1}(\Omega)$ and $f \in H^{k-1}(\Omega)$. Then*

$$|u - u_h|_1 \lesssim h^k(|u|_{k+1} + |f|_{k-1}).$$

6. Numerical results. In this section, we will numerically test the nonconforming virtual element method (4.17) and the conforming virtual element method (5.5), which are abbreviated as SFNCVEM and SFCVEM respectively. For the convenience of narration, we also abbreviate the standard conforming virtual element method in [12] and non-conforming virtual element method in [20] as CVEM and NCVEM respectively. We implement all the experiments by using the FEALPy package [43] on a PC with AMD Ryzen 5 3500U CPU and 64-bit Ubuntu 22.04 operating system. Set the rectangular domain $\Omega = (0,1) \times (0,1)$.

6.1. Verification of convergence. Consider the second order elliptic problem (4.1) with $\alpha = 2$. The exact solution and source term are given by

$$u = \sin(\pi x)\sin(\pi y), \quad f = (2\pi^2 + 2)\sin(\pi x)\sin(\pi y).$$

The rectangular domain Ω is partitioned by the convex polygon mesh $\mathcal{T}_0$ and non-convex polygon mesh $\mathcal{T}_1$ in Fig. 1, respectively. We choose $k = 1, 2, 5$ in both SFNCVEM and SFCVEM. The numerical results of the SFNCVEM on meshes $\mathcal{T}_0$ and $\mathcal{T}_1$ are shown in Fig. 2. We can see that $\|u - Q_h u_h\|_0 = O(h^{k+1})$ and $\|\nabla u - Q_{h,k-1}^{\mathrm{div}} \nabla_h u_h\|_0 = O(h^k)$, which coincide with Theorem 4.9. And the numerical results of the SFCVEM are presented in Fig. 3. Again $\|u - Q_h u_h\|_0 = O(h^{k+1})$ and $\|\nabla u - Q_{h,k}^{\mathrm{div}} \nabla u_h\|_0 = O(h^k)$, which confirm the theoretical convergence rate in Theorem 5.2.

(a) Convex polygon mesh $\mathcal{T}_0$. (b) Non-convex polygon mesh $\mathcal{T}_1$.

FIG. 1. *Convex polygon mesh and non-convex polygon mesh.*

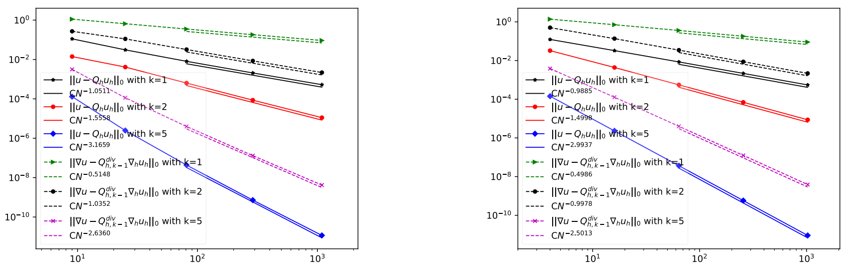


FIG. 2. *Errors* $\|u - Q_h u_h\|_0$ *and* $\|\nabla u - Q_{h,k-1}^{\mathrm{div}} \nabla_h u_h\|_0$ *of nonconforming VEM* (4.17) *on* $\mathcal{T}_0$ *(left) and* $\mathcal{T}_1$ *(right) with* $k = 1, 2, 5$.

6.2. The invertibility of the local stiffness matrices. We construct three different hexagons shown in Fig. 4, and calculate the eigenvalues of local stiffness matrices with $k = 3$ for four virtual element methods. Our numerical results show that, on all the three hexagons, both SFNCVEM and SFCVEM have only one zero eigenvalue. In Tables 1-3, we also present the minimum non-zero eigenvalue, the maximum eigenvalue and the condition number for the local stiffness matrix on different hexagons in Fig. 4, from which we can see that these quantities are comparable for four virtual element methods.

TABLE 1
Comparison of eigenvalues and condition numbers on the regular hexagon.

Method	**Maximum eigenvalue**	**Minimum nonzero eigenvalue**	**Condition number**
NCVEM	975.5693189	0.309674737	3150.303211
CVEM	1012.488116	0.297206358	3406.683909
SFNCVEM	992.5956147	0.318932029	3112.248147
SFCVEM	1011.173331	0.298509692	3387.405362

6.3. Comparison of assembling time. The only difference between the standard VEMs and the VEMs without extrinsic stabilization is the stiffness matrix, so we compare the time consumed in assembling the stiffness matrix of four different VEMs in detail by varying the degree k and the mesh size h respectively. We use the mesh

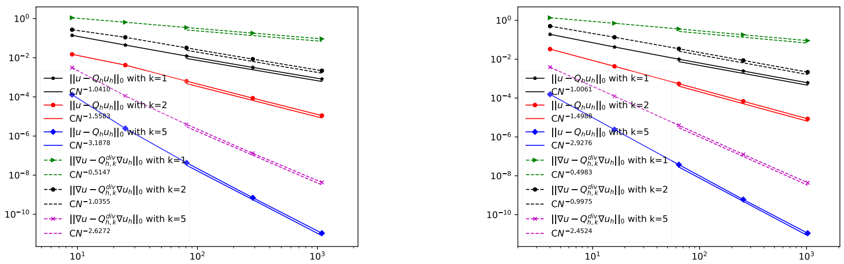
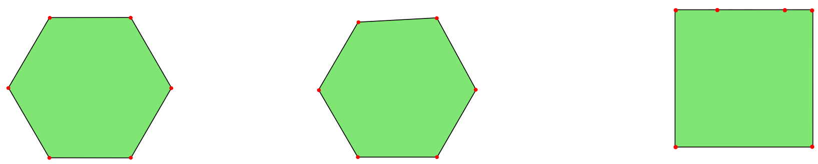


FIG. 3. *Errors* $\|u - Q_h u_h\|_0$ *and* $\|\nabla u - Q_{h,k}^{\mathrm{div}} \nabla u_h\|_0$ *of conforming VEM* (5.5) *on* $\mathcal{T}_0$ *(left) and* $\mathcal{T}_1$ *(right) with* $k = 1, 2, 5$.



(a) Regular hexagon.

(b) Quasi-regular hexagon generated by regular hexagon with a small perturbation.

(c) Square with two hanging nodes.

FIG. 4. *The regular hexagon(Left), the quasi-regular hexagon generated by regular hexagon with a small perturbation (Middle), and the square with two hanging nodes (Right).*

in Fig. 1(a) for this experiment. The results presented in Tables 4 and 5 show that NCVEM, CVEM and SFNCVEM have similar assembling time. However, SFCVEM requires more time due to the projection onto the one-order higher polynomial space.

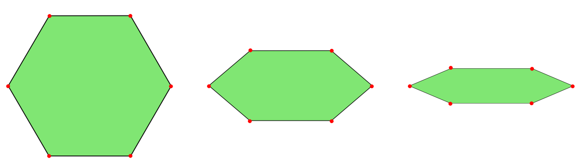


FIG. 5. *The hexagons* H_0, H_1, H_2.

6.4. Condition number of the stiffness matrix. We design two experiments to check the condition number of the stiffness matrices of the four VEMs.

Firstly, we refer to the "collapsing polygons" experiment in [37] and consider a sequence of hexagons $\{H_i\}_{i=0}^{\infty}$, where the vertices of H_i are given by $A_i = (1, 0)$, $B_i = (0.5, a_i)$, $C_i = (-0.5, a)$, $D_i = (-1, 0)$, $E_i = (-0.5, -a_i)$, and $F_i = (0.5, -a_i)$, where $a_i = \frac{\sqrt{3}}{2^{i+1}}$. The hexagons H_0, H_1 and H_2 are drawn in Fig. 5. As shown in Fig. 6 for $k = 8$ and $k = 10$, the condition numbers of stiffness matrices of the VEMs without extrinsic stabilization are smaller than those of the standard methods when i is large.

Table 2
Comparison of eigenvalues and condition numbers on the quasi-regular hexagon.

Method	**Maximum eigenvalue**	**Minimum nonzero eigenvalue**	**Condition number**
NCVEM	935.2883848	0.279027715	3351.955143
CVEM	1014.672395	0.257370621	3942.456177
SFNCVEM	997.4831245	0.282126359	3535.589964
SFCVEM	1047.876056	0.258970708	4046.311124

Table 3
Comparison of eigenvalues and condition numbers on the square with two hanging nodes.

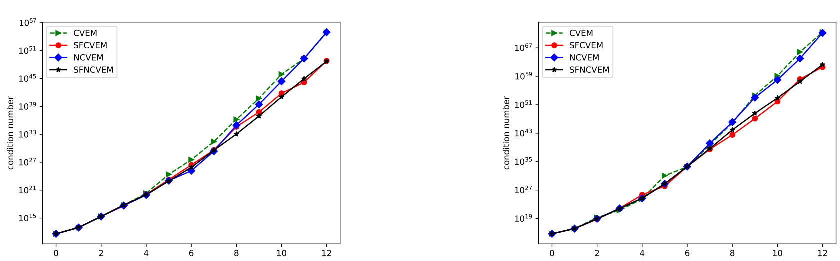
Method	**Maximum eigenvalue**	**Minimum nonzero eigenvalue**	**Condition number**
NCVEM	941.8571938	0.21069027	4470.340249
CVEM	1046.755495	0.200435123	5222.4155
SFNCVEM	986.5963357	0.212761106	4637.108513
SFCVEM	1061.651989	0.202074633	5253.761808

Table 4
Time consumed in assembling stiffness matrix of four VEMs with $h = 0.2$ and different k.

k	2	4	8	10
SFCVEM	0.053684235	0.144996881	1.468627453	2.603836536
SFNCVEM	0.022516727	0.065697193	0.806378841	1.554260015
CVEM	0.021185875	0.059809923	0.600241184	1.160929918
NCVEM	0.0213027	0.061014891	0.596506596	1.129639149

Table 5
Time consumed in assembling stiffness matrix of four VEMs with $k = 5$ and different h.

h	1	0.25	0.0625	0.03125
SFCVEM	0.039689541	0.199015379	1.74412179	4.75462532
SFNCVEM	0.018287182	0.100006819	0.81251812	2.465409517
CVEM	0.018686771	0.087426662	0.781031132	1.983617783
NCVEM	0.018309593	0.096345425	0.767129898	2.159288645



(a) $k = 8$ (b) $k = 10$

Fig. 6. *The condition number of stiffness matrix of four VEMs on $\{H_i\}_{i=0}^{12}$.*

Secondly, we do the patch test for the Laplace equation, i.e. problem (4.1) with $\alpha = 0$ and $f = 0$, but the Dirichlet boundary condition is nonhomogeneous. Take the exact solution $u = 1 + x + y$. Let h_x and h_y be mesh size in the x-direction and y-direction respectively. We examine the behavior of error $\|u - u_h\|_0$ of the four VEMs in the following three cases:
(1) Mesh in Fig. 1(a): Fix $h_x = h_y = 0.2$ but vary $k = 1, 2, \ldots, 10$;
(2) Mesh in Fig. 1(a): Fix $k = 3$ but vary $h_x = h_y = 2^{-i}$ for $i = 1, \ldots, 5$;
(3) Mesh in Fig. 7: Fix $k = 3$ and $h_x = 0.2$, but vary $h_y = 2^{-i}$ for $i = 1, \ldots, 8$.

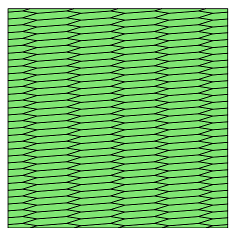


FIG. 7. *The mesh of domain* $(0, 1) \times (0, 1)$ *with* $h_x = 0.2, h_y = 0.03125$.

The errors of the four methods shown in Fig. 8 are similar. Since the error in the patch test grows as the condition number of the stiffness matrix grows, the condition numbers of the stiffness matrix obtained by four methods are comparable.

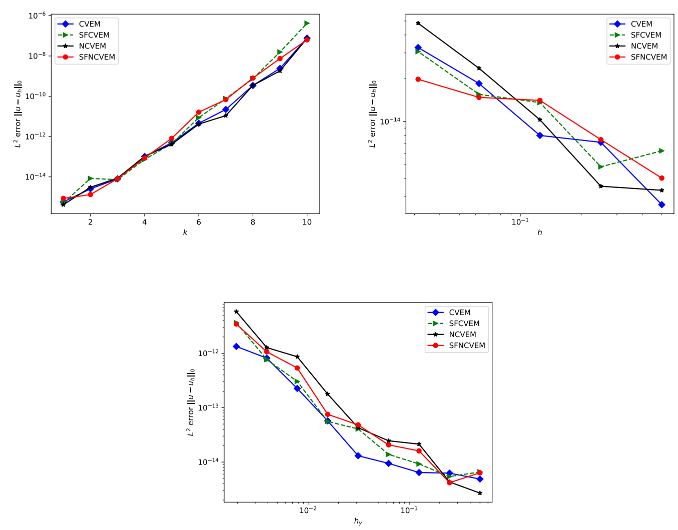


FIG. 8. *The* L^2 *error of patch test.*

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