Lecture 1: Review of Basic Probability

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Outline

- Syllabus
- Brief review of basic probability and statistics
  - Why is a random variable?
  - Transformations; independence; expectation
  - Important distributions
  - Some statistics

Terms

- Sample space; Measure; Random variable
- Transformation; Independence; Expectation; Conditional expectation;Variance retur Standard deviation; Moment Generating Function;Characteristic function
- Common distributions
- Sample mean;Sample variance; Sample distribution
- Moment inequalities

Sample space and Measure

What do we mean by randomness?

- We construct an experiment, yet the result of the experiment has many possibilities.
    - Flip a coin, the result can be either head or tail
- Although we can not know the result beforehand, we do have some information about the result.
    - Approximately, there is equal chance for a head and a tail
- Randomness: the uncertainty of experiment results

Question: How to describe our information?

Sample space and Measure

- Information 1. Possible outcomes

Definition 1 (Sample space (Outcome space))

Let Ω be a sample space, which is a set containing all possible outcomes.

- Information 2. Probabilities for these possible outcomes
  - $\sigma - field\mathcal{F}$: a set of subsets of Ω which satisfies 3 rules.
    - Measurable space:$(\Omega,\mathcal{F})$
    - Event(measurable sets): element of $\mathcal{F}$
  - Probability measure P: for any element in the σ-field,assign it a probability, indicating the chance this event will happen
- $(\Omega;\mathcal{F};P)$ (Probability space, measure space)is our information about the possible outcomes of this experiment.In short, we write it as the sample space Ω with probability P,or just Ω if there is no confusion.

Definition 2 (σ-field)

Let $\mathcal{F}$ be a collection of subsets of a sample space.$\mathcal{F}$ is called a σ-field (or σ-algebra) if and only if it has the following properties.

- The empty set $\phi \in \mathcal{F}$.
- If $A \in \mathcal{F}$,then the complement $A^c \in \mathbb{F}$
- If $A_i \in \mathcal{F}$,$i = 1, 2, ...$,then their union $\cup A_i \in \mathcal{F}$.

- Measurable space:$(\Omega, \mathcal{F})$
- Event (measurable sets): element of $\mathcal{F}$
- $\sigma(A) = \{\phi, A, A^c, \Omega\}$.
- Flip a coin, the result can be either head or tail
  $\Omega = \{H, T\}$,$\mathcal{F} = \{...\}$

Definition 3 (Measure)

Let Measurable space $(\Omega, \mathcal{F})$, A be a measurable space.A set function v defined on $\mathcal{F}$ is called a measure if and only if it has the following properties.

- $0 \leq v(A) \leq \infty$,for any $A \in \mathcal{F}$
- $v(\phi) = 0$
- If $A_i \in \mathcal{F}$,$i = 1, 2, ...$,and A_i's are disjoint,i.e. $A_i \cap A_j = \varnothing$ for any $i \neq j$,then $v(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} v(A_i)$

- measure space:$(\Omega, \mathcal{F}, v)$
- probability measure $v(\Omega) = 1$.We usually denote it by P instead of v, $(\Omega, \mathcal{F}, P)$.
- Flip a coin, the result can be either head or tail $\Omega = \{H, T\}$,$\mathcal{F} = \{...\}$
  - $v(A) = |A|$ the number of elements in $A \in (F)$.
  - $P(A) = \frac{|A|}{|\Omega|}$

Random Variables

What is of interest?

- Manufacturers Ω: all the combinations of good light bulbs and defective light bulbs. Need: proportion of defective light bulbs from a lot
- Market researchers Ω: survey results of all consumers for one product. Need: preference of all consumers about this product, with a scale 1-10.

Our interest:

- Not the details of Ω, but a special measurable characteristic of the outcomes!
- A random variable, is a mapping from Ω to R, which draws the measurable characteristic of interest

Example: an opinion poll. 50 people;1: agree;0 disagree:

- Ω has 2^{50} elements.
- interest: the number of people who agree out of 50. $X =$ number of 1s recorded out of 50. $\mathbb{X} = \{0, 1, 2, ..., 50\}$

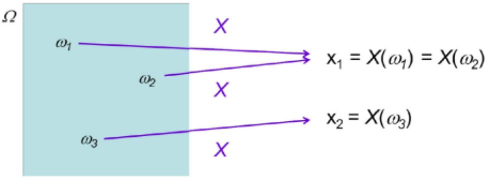
Random Variable

Definition 4 (Random Variable)

- Let $(\Omega, \mathcal{F})$ and $(\mathcal{R}, \mathcal{B})$($\mathcal{B}$:Borel σ-field) be measurable spaces
- X is a function from Ω to $\mathcal{R}$.The function X is called a random variable(r.v.;measurable function)if and only if

$$X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} \subset \mathcal{F}$$

for any $B \in \mathcal{B}$.



Random Variable

- Suppose we have a sample space

$$\Omega = \{\omega_1, ..., \omega_n\}$$

  with a probability function P.
- We defined a random variable X with range $\mathbb{X} = \{x_1, ..., x_m\}$.
- We write

$$P_X(X = x_i) = P(\{\omega_j \in \Omega : X(\omega_j) = x_i\})$$

$$P_X(X \in B) = P(\{\omega \in \Omega : X(\omega) \in B\})$$

  where P_X is an induced probability function χ.
- Notations:
  - Upper-case letters $X, Y, Z...$ to denote r.v.'s
  - Lower-case letters $x, y, z...$ to denote their possible values.

Example:Random variable

Example 1.4.3

- Consider the experiment of tossing a coin three times.
- H:Head; T:Tail.
- X:the number of heads obtained in the three tosses.

ω	HHH	HHT	HTH	THH	TTH	THT	HTT	TTT
$X(\omega)$	3	2	2	2	1	1	1	0

- $\mathbb{X} = \{0, 1, 2, 3\}$.The induced probability function on χ is given by

x	0	1	2	3
$P_X(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$P_X(X = 1) = P(\{HTT, THT, TTH\}) = \frac{3}{8}$$

.

Cumulative Density Function

Definition 1.5.1 Cumulative Density Function

The cumulative distribution function (CDF) of a random variable is defined by

$$F(x) = P(X \leq x); -\infty < x < \infty$$

For all CDF's;there is

- $F(x)$ is right-continuous.At each x,$\lim_{n\to\infty} F(y_n) = F(x)$ for any sequence $y_n \to x$ with $y_n > x$.
- $F(x)$ is non-decreasing.
- $\lim_{x\to\infty} F(x) = 0$,$\lim_{x\to\infty} F(x) = 1$.

Any $F(x)$ satisfying Properties 1-3 is a CDF for some random variable.

Example: Logistic distribution.

Example 1.5.5

$$F_X(x) = \frac{1}{1+e^{-x}}$$

- $\lim_{x\to-\infty} F_X(x) = 0$
- $\lim_{x\to\infty} F_X(x) = 1$
- $$f_X(x) = \frac{d}{dx}F_X(x) = \frac{e^{-x}}{(1+e^{-x})^2} > 0$$

Discrete v.s. Continuous r.v.

- If X is discrete,then its probability mass function(pmf) is

$$p_X(x) = p(x) = P(X = x)$$

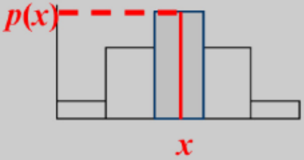
- If X is continuous,then its probability density function(pdf) satisfies

$$P(X \in A) = \int_A f_X(x)dx = \int_A f(x)dx = \int_A dF(x)$$

  and $f_X(x) = f(x) = F'(x)$.

- We say that X and Y have the same distribution(i.e. $X \overset{D}{=} Y$) if $P(X \in A) = P(Y \in A)$for all A. $X \overset{D}{=} Y$ if and only if $F_X(t) = F_Y(t)$

	RANDOM VARIABLE, X	
Type	**Discrete**	**Continuous**
Values	A finite/countable set of numbers $x_1, x_2, x_3, \dots$	All numbers in an interval
Probability	Probability Mass Function, p *pmf* $P(X=x) = p(x)$	Probability Density Function, f *pdf* $P(a < X < b) = \begin{bmatrix} \text{area} \\ \text{under the} \\ \text{graph of } f \\ \text{over } (a,b) \end{bmatrix}$



Transformation

Given a r.v. X with density function $f_X(\cdot)$,it is often that we are interested in a transformation $Y = g(X)$which is defined as a known function g (either one-to-to or many-to-one) of X.

- Obviously,the composite function $g \circ X$ defines a new r.v. Y from Ω to R.
- Let $Y = g(X)$.

$$\begin{aligned} P(Y \in A) &= P(g(X) \in A) \\ &= P(X \in g^{-1}(A)) \end{aligned} \tag{1}$$

where $g^{-1}(A) = x \in R, g(x) \in A$.In particular,

$$F_Y(y) = Pr\{Y \in y\} = P(X \in g^{-1}(-\infty, y])$$

If X has pdf $f_X(x)$,then

$$F_Y(y) = \int_{g^{-1}(-\infty,y]} f_X(x)dx = \int_{\{x:g(x)\leq y\}} f_X(x)dx$$

Example 2.1.2

Suppose X has a uniform distribution on the interval $(0, 2\pi)$,that is

$$f_X(x) = \begin{cases} 1/2\pi, & 0 < x < 2\pi, \\ 0, & otherwise \end{cases} \tag{2}$$

Consider $Y = sin^2(X)$

$$\begin{aligned} P(Y \le y) &= P(X \le x_1) + P(x_2 \le X \le x_3) + P(X \ge x_4) \\ &= 2P(X \le x_1) + 2P(x_2 \le X \le \pi) \end{aligned} \tag{3}$$

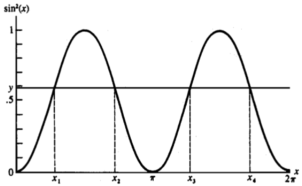


Figure 2.1.1. *Graph of the transformation* $y = \sin^2(x)$ *of Example 2.1.2*

- If g is increasing,

$$F_Y(y) = F_X(g^{-1}(y))$$

.

- If g is decreasing,

$$F_Y(y) = 1 - F_X(g^{-1}(y))$$

.

Theorem 2.1.5

Let X have probability distribution function(pdf) $f_X(x)$ and $Y = g(X)$,where g is a monotone function.Let

$$\mathcal{Y} = \{y : g^{-1}(y) \quad is \quad a \quad possible \quad value \quad of \quad X\}$$

. Suppose $f_X(x)$ is continuous and that $g^{-1}(y)$ has a continuous derivative on $\mathcal{Y}$.Then the pdf on Y is given by

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y))|\frac{d}{dy}g^{-1}(y)|, & y \in \mathcal{Y}, \\ 0, & otherwise \end{cases} \tag{4}$$

Example 2.1.4

$X \sim f_X(x) = 1I(0 < x < 1), F_X(x) = x. Y = g(x) = -\log x$,find its distribution.

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$X \sim f_X(x) = 1I(0 < x < 1), F_X(x) = x. Y = g(x) = -\log x$, find its distribution.

Proof:

- $Y = g(x) = -\log x \Rightarrow x = e^{-y}, g^{-1}(y) = e^{-y}$
- g is decreasing function.

$$\frac{d}{dx} g(x) = \frac{d}{dx}(-logx) = \frac{-1}{x} < 0, 0 < x < 1$$

- $$\begin{aligned} F_Y(y) = P_Y(Y \leq y) &= P_X(g(x) \leq y) \\ &= P_X(X \geq g^{-1}(y)) \\ &= 1 - P_X(X \leq g^{-1}(y)) \\ &= 1 - e^{-y} \end{aligned}$$

Example 2.1.6

Let

$$f_X(x) = \frac{1}{(n-1)!\beta^n} x^{n-1} e^{-x/\beta}, 0 < x < \infty$$

be the Gamma pdf $Y = 1/X$.Find the pdf of Y

Example 2.1.6

Let

$$f_X(x) = \frac{1}{(n-1)!\beta^n} x^{n-1} e^{-x/\beta}, 0 < x < \infty$$

be the Gamma pdf $Y = 1/X$.Find the pdf of Y

Proof.$g^{-1}(y) = 1/y, \mathcal{Y} = (0, \infty), |\frac{d}{dy} g^{-1}(y)| = 1/y^2$.Therefore for all $y > 0$,

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y)) |\frac{d}{dy} g^{-1}(y)| \\ &= \frac{1}{(n-1)!\beta^n} \left(\frac{1}{y}\right)^{n-1} e^{-1/(\beta y)} \frac{1}{y^2} \\ &= \frac{1}{(n-1)!\beta^n} \left(\frac{1}{y}\right)^{n+1} e^{-1/(\beta y)} \end{aligned} \tag{5}$$

- A special case of a pdf known as the inverted Gamma distribution.

Theorem 2.1.8

Let X have pdf $f_X(x)$,let $Y = g(X)$.Suppose there exists a partition $A_0, A_1, \cdots, A_k$ such that $P(X \in A_0) = 0$ and $f_X(x)$ is continuous on each A_i.

$$P(X \in \cup_{i=1}^k A_i) = 1.$$

Futher,we have $g(\cdot)$ is monotone if restricted to $A_i, i = 1, 2, \cdots, k$.Let

$$g_i^{-1}(y) = \{x \in A_i : g(x) = y\}$$

and assume $g_i^{-1}(y)$ has continuous derivative on $\mathcal{Y}$ for each i.Then

$$f_Y(y) = \begin{cases} \sum_{i=1}^k f_X(g_i^{-1}(y))|\frac{d}{dy}g_i^{-1}(y)|, & y \in \mathcal{Y}, \\ 0, & otherwise \end{cases}$$

- Remark Unfortunately,I found the above Theorem has very little practical use.

Example 2.1.9

Let $X \sim N(0,1)$, $Y = X^2$,we may use the above theorem to find the pdf of Y.

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Proof:

- $g(x) = x^2$ is monotone on $(-\infty, 0)$ and on $(0, \infty)$.
- $\mathcal{Y} = (0, \infty)$.

$$
\begin{aligned}
A_0 &= \{0\} \\
A_1 &= (-\infty, 0), g_1(x) = x^2 . g_1^{-1}(y) = -\sqrt{y} \\
A_2 &= (0, \infty), g_2(x) = x^2, g_1^{-1}(y) = \sqrt{y}
\end{aligned}
$$

The pdf Y is

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \Phi(\sqrt{y}) \frac{1}{2} \frac{1}{\sqrt{y}} + \Phi(-\sqrt{y}) \frac{1}{2} \frac{1}{\sqrt{y}} = \frac{1}{\sqrt{y}} \phi(\sqrt{y})$$

Probability integral transform

Theorem 2.1.10 Probability integral transform

Let X have continuous cdf $F_X(x)$ and define the random variable Y as $Y = F_X(X)$.Then Y is uniformly distributed on $(0,1)$,that id

$$P(Y \leq y) = y, 0 < y < 1.$$

- $F_X^{-1}(\tau) = inf\{x : F_X(x) \geq \tau\}$
- Proof:

$$\begin{aligned}
P_Y(Y \leq y) &= P_X(F_X(x) \leq y) \\
&= P_X(F_X^{-1}[F_X(x)] \leq F_X^{-1}(y)) \\
&= P_X(X \leq F_X^{-1}(y)) \\
&= F_X(F_X^{-1}(y)) \\
&= y
\end{aligned}$$

Theorem 4.2.10

Two r.v.'s X and Y are independent if and only if

$$P(X \in A; Y \in B) = P(X \in A)P(Y \in B)$$

for all A and B.

- $F(x, y) = F(x)F(y)$ for any x and y, $f(x, y) = f(x)f(y)$ or $p(x, y) = p(x)p(y)$
- When X and Y are independent,$h(x)$ and $g(Y)$ are also independent,if h and g are well-defined functions.

Expection

- Definition:

$$E(X) = \sum_x xp(x)$$

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx$$

- Properties:
  - $E(g(X)) = \int_{-\infty}^{\infty} g(x)f(x)dx$
  - $E(g(X,Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f(x,y)dxdy$
  - If $X_1, ..., X_n$ are independent,then

$$E(\prod_{i=1}^{n} X_i) = \prod_{i=1}^{n} E(X_i)$$

- Example 2.2.2

$X \sim exp(\lambda)$,

$$f_X(x) = \frac{1}{\lambda} e^{-x/\lambda} x > 0.$$

Then $E[X] = \lambda$.

- Example 2.2.3

$X \sim$ Binomial(n,p),

$$f_X(x) = \binom{n}{x} p^x (1-p)^{n-x} \quad x = 0, 1, \cdots .$$

Then $E[X] = np$.

- Example 2.2.4

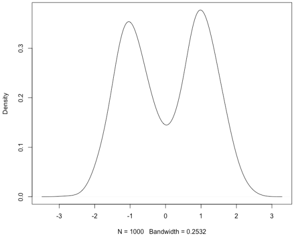
$X \sim$ Cauchy,

$$f_X(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2} \quad -\infty < x < \infty.$$

Then $E[X]$ is not defined!(or do not exist).

- Mixed normal distribution

$$X = 0.5N(-1, 0.5^2) + 0.5N(1, 0.5^2)$$



Mixed normal distribution

```
rm(list=ls())
n <- 1000
comp <- sample(c(0, 1), size = n, prob = c(0.5, 0.5),
               replace = T)
x <- rnorm(n, mean = ifelse(comp == 0, -1, 1),
           sd = ifelse(comp == 0, 0.5, 0.5))
plot(density(x), main="")
```

- Theorem 2.2.5 Let X be a r.v. and let a, b and c be constants.Then for any functions $g_1(x)$ and $g_2(x)$ whose expecations exist.

  (1) $E[ag_1(X) + bg_2(X) + c] = aE[g_1(X)] + b[g_2(X)] + c$

  (2) If $g_1(x) \geq 0$ for all x,then $E[g_1(X)] \geq 0$.

  (3) If $g_1(x) \geq g_2(x)$for all x,then $E[g_1(X)] \geq E[g_2(X)]$

  (4) If $a \leq g_1(x) \leq b$for all x,then $a \leq E[g_1(x)] \leq b$

- Example 2.2.6

$E(X)$ is the "center" of a distribution(or its r.v.) in the sense that

$$\min_b E(X-b)^2 = E[X - EX]^2.$$

- **Homework**

$$\min_b E\rho_\tau(X-b)$$

Remark:$\rho_\tau(t) = \tau t I(t \geq 0) + (\tau - 1)tI(t \leq 0)$.

Variance & Standard Deviation

- Motivation:Describe the "spread" of r.v.
- Definition. $Var(x) = E[(x-\mu)^2]$, where $\mu = E(X)$, $sd(X) = \sqrt{Var(x)}$.
- Properties.
  - $Var(X) = E(X^2) - [E(X)]^2$
  - If $X_1, ..., X_n$ are independent,then

    $$Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} Var(X_i)$$

  - The covariance is

    $$Cov(X,Y) = E[(X-E(X))(Y-E(Y))] = E(XY) - E(X)E(Y)$$

    and the correlation coefficient is

    $$Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$$

  - For any two r.v.s with variance existed,

    $$Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$$

Conditional Expectation

- Conditional Expectation of X when Y is given as y is that
  - $\mathbf{E}(X|Y=y) = \sum_x x p_{X|Y}(x|y)$ for discrete r.v.
  - $\mathbf{E}(X|Y=y) = \int_{-\infty}^{\infty} f_{X|Y}(x|y)dx$ for cont. r.v.
  - Interpretation: Note that $X|Y=y$ is a new r.v., $\mathbf{E}(X|Y=y)$ is the expectation on this r.v.
- Law of Total Expectation

$$\mathbf{E}[\mathbf{E}(X|Y)] = \mathbf{E}(X)$$

- Law of Total Variance

$$Var(X) = Var[\mathbf{E}(X|Y)] + \mathbf{E}[Var(X|Y)]$$

Theorem 4.4.3

if X and Y are any two r.v.s, then

$$\mathbf{E}(X) = \mathbf{E}\Big[\mathbf{E}(X|Y)\Big]$$

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Proof:

$$\begin{aligned}
\mathbf{E}X &= \int\int xf(x,y)dxdy \\
&= \int \Big[\int xf(x|y)dx\Big] f_Y(y)dy \\
&= \int \mathbf{E}(X|y) f_Y(y)dy = \mathbf{E}\Big[\mathbf{E}(X|Y)\Big]
\end{aligned}$$

In general, the conditional expectation $\mathbf{E}[X|Y]$ can by defined as a r.v. $g(Y)$ such that

$$\mathbf{E}[(X-g(Y))^2] = \inf_{among \quad all \quad reasonable \quad function \quad h} \mathbf{E}[(X-h(Y))^2]$$

or $\mathbf{E}[X|Y]$ is the function of Y which is "closest" to X in terms of mean square error.

Example 4.4.1

$Y \sim$ Number of eggs lay by a mother fish,and $X \sim$ Number of survivors(young fish).On the average,how many eggs will survive?

Then it is reasonable to assume

$$Y \sim Poisson(\lambda)$$

$$X|Y \sim Binomial(Y, p)$$

So,

$$\begin{aligned} EX &= E[E(X|Y)] \\ &= E(pY) \\ &= p\lambda \end{aligned}$$

Example 4.4.5

$$X|Y \sim Binomial(Y, p)$$
$$Y|A \sim Poisson(\Lambda)$$
$$\Lambda \sim exponential(\beta)$$

Proof:

$$\begin{aligned} E[X] &= E[E(X|Y)] \\ &= pE(Y) \\ &= pE[E(Y|\Lambda)] \\ &= pE[\Lambda] \\ &= p\beta. \end{aligned}$$

Theorem 4.4.7

For any two random variables X and Y

$$Var(X) = Var[\mathbf{E}(X|Y)] + \mathbf{E}[Var(X|Y)]$$

provided that the expectation exist.

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Proof:

$$\begin{aligned}
Var(X) &= \mathbf{E}\Big\{[X - \mathbf{E}(X|Y) + \mathbf{E}(X|Y) - \mathbf{E}]^2\Big\} \\
&= \mathbf{E}\Big\{[X - \mathbf{E}(X|Y)]^2 + [\mathbf{E}(X|Y) - \mathbf{E}X]^2 \\
&+ 2[X - \mathbf{E}(X|Y)][\mathbf{E}(X|Y) - \mathbf{E}X]\Big\} \\
&= \mathbf{E}\{[X - \mathbf{E}(X|Y)]^2\} + \mathbf{E}\{[\mathbf{E}(X|Y) - \mathbf{E}X]^2\} \\
&= \mathbf{E}[Var(X|Y)] + Var[\mathbf{E}(X|Y)]
\end{aligned}$$

$$\mathbf{E}\Big\{2[X - \mathbf{E}(X|Y)][\mathbf{E}(X|Y) - \mathbf{E}X]\Big\} = \mathbf{E}[\mathbf{E}(Z|Y)]$$

Moment Generating Function and Characteristic Function

- Moment Generating Function(MGF)
  - Definition:$M_X(t) = E(e^{tX})$:a function of t,not r.v.
  - If $Y = aX + b, M_Y(t) = e^{bt}M_X(at)$
  - If X and Y are independent,then $M_{X+Y}(t) = M_X(t)M_Y(t)$
- Characteristic Function
  - Definition:$\phi_X(t) = E[e^{itX}]$:a function of t;$i = \sqrt{-1}$.
  - Bounded:$|\phi(t)| \leq 1$
  - If X and Y are independent,then $\phi_{X+Y}(t) = \phi_X(t)\phi_Y(t)$.

An example of two distribution functions but with the same moments.

Example 2.3.10

Consider the two pdfs given by

$$f_1(x) = \frac{1}{\sqrt{2\pi}x} e^{-(\log x)^2/2}, 0 \le x < \infty,$$
$$f_2(x) = f_1(x)[1 + sin(2\pi \log x)], 0 \le x < \infty.$$

Then it can be shown if $X_1 \sim f_1(x)$,

$$E[X_1^r] = e^{r^2/2}, r = 0, 1, ...$$

Now suppose that $X_2 \sim f_2(x)$,we have for $r = 0, 1, ...$

$$\begin{aligned} E[X_2^r] &= \int_0^\infty x^r f_1(x)[1 + sin(2\pi \log x)]dx \\ &= E[X_1^r] + \int_0^\infty x^r f_1(x) sin(2\pi \log x)dx \end{aligned}$$

$$
\begin{aligned}
&\int_0^{\infty} x^r \frac{1}{\sqrt{2\pi x}} 2e^{-(\log x)^2} \sin(2\pi \log x) dx \quad y = \log x - r \\
&= \int_{-\infty}^{\infty} e^{(y+r)r} \frac{1}{\sqrt{2\pi}} e^{-(y+r)^2/2} sin(2\pi(y+r)) dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(y^2 - r^2)} sin(2\pi y) dy \cdot cos(2\pi r) \\
&\quad + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(y^2 - r^2)} cos(2\pi y) dy \cdot sin(2\pi r) \\
&= 0 \quad r = 0, 1, \ldots
\end{aligned}
$$

since $e^{-\frac{1}{2}(y^2 - r^2)} sin(2\pi y)$ is an odd function.

However,we have the following theorem.

Theorem 2.3.11

Let $F_X(x)$ and $F_Y(y)$ be two cdfs all of whose moments exist.

1. If F_X and F_Y have bounded support,then $F_X(u) = F_Y(u)$ for all u iff $EX^r = EY^r$ for all $r = 0, 1, 2, \cdots$
2. If the moment generating functions functions exist and $M_X(t) = M_Y(t)$ for all t in some neighborhood of 0,then $F_X(u) = F_Y(u)$ for all u.

Differentiating Under An Integral Sign

If a, b are finite and $f(x,\theta)$ is differential with respect to θ.Then we have

$$\frac{d}{d\theta}\int_a^b f(x,\theta)dx = \int_a^b \frac{\partial}{\partial\theta} f(x,\theta)dx$$

But in statistics,we often need to evaluate $\frac{d}{d\theta}\int_{-\infty}^{\infty} f(x,\theta)dx$,which may or may not be $\int_{-\infty}^{\infty}\frac{\partial}{\partial\theta} f(x,\theta)dx$.

Theorem 2.4.2

Suppose the function $h(x,y)$ is continuous at y_0 for each x,and there exists a function $g(x)$ satisfying
a)$|h(x,y)| \leq g(x)$,for all x and y;
b)$\int_{-\infty}^{\infty} g(x)dx < \infty$.
Then

$$\lim_{y\to y_0}\int_{-\infty}^{\infty} h(x,y)dx = \int_{-\infty}^{\infty}\lim_{y\to y_0} h(x,y)dx$$

.

Apply the above Thheorem to the differentiation case, then we have

- Theorem 2.4.3 Suppose $f(x,\theta)$ is differentiale at $\theta = \theta_0$,and there exists a function $g(x,\theta_0)$ and a constant $\delta > 0$ such that
a)$|\frac{f(x,\theta_0+\Delta)-f(x,\theta_0)}{\Delta}| \le g(x,\theta_0)$,for all x and $|\Delta| \le \delta$;
b)$\int_{-\infty}^{\infty} g(x,\theta_0)dx < \infty$. Then

$$\frac{d}{d\theta} f(x,\theta)dx|_{\theta=\theta_0} = \int_{-\infty}^{\infty} \left[\frac{\partial}{\partial\theta} f(x,\theta)|_{\theta=\theta_0}\right] dx \qquad (*)$$

- Corollary Suppose that there exists $\delta > 0$ and function $g(x,\theta)$ such that $|\frac{\partial}{\partial\theta} f(x,\theta)|_{\theta=\theta'}| \le g(x,\theta)$,for all θ' with $|\theta' - \theta| < \delta$,and $\int_{-\infty}^{\infty} g(x,\theta)dx < \infty$.Then $(*)$ holds.

- Remark Finding bound $g(x,\theta)$ is cumbersome.We need to know that differentiating under the integral sign is not always automatic.In most situations,we just do it!!
- Example 2.4.6$X \sim N(\mu, 1)$,

$$M_X(t) = E(e^tX) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-(x-\mu)^2/2} dx,$$
$$\frac{d}{dt} M_X(t) = \frac{d}{dt} E(e^{tX}) = E(\frac{\partial}{\partial t} e^{tX}) = E(Xe^{tX}).$$

For the exchange of operation of differentiation and summation,we have

- Theorem 2.4.8 Suppose that the series $\sum_{x=0}^{\infty} h(\theta, x)$ converges for all θ in an interval $(a, b)$and
  1. $\frac{\partial}{\partial\theta} h(\theta, x)$ is continuous in θ for each x;
  2. $\sum_{x=0}^{\infty} \frac{\partial}{\partial\theta} h(\theta, x)$ converges uniformly on every closed bounded subinterval of (a, b).

  Then

$$\frac{d}{d\theta} \sum_{x=0}^{\infty} h(\theta, x) = \sum_{x=0}^{\infty} \frac{\partial}{\partial\theta} h(\theta, x)$$

.

- Theorem 2.4.10 Suppose that the series $\sum_{x=0}^{\infty} h(\theta, x)$ converges uniformly on $[a, b]$ and that,for each x,$h(\theta, x)$ is a continuous function of θ.Then

$$\int_a^b \sum_{x=0}^{\infty} h(\theta, x) d\theta = \sum_{x=0}^{\infty} \int_a^b h(\theta, x) d\theta$$

.

Important Distribution

- Discrete distributions:
  - Bernoulli r.v.:$X \sim Bernoulli(p)$,$p(1) = p$,$p(0) = 1 - p$,$p(x) = 0$ if $x \neq 0$ and $x \neq 1$.It can be written as $p^x(1-p)^{1-x}$for $x = 0, 1$.
  - Binomial r.v.:$X \sim Binomial(n, p)$,$p(x) = \begin{pmatrix} n \\ x \end{pmatrix} p^x q^{n-x}$,$x = 0, 1, 2, ..., n$.Summation of n Bernoulli random variables.
  - Poisson r.v.:$X \sim Pois(\lambda)$,$p(x) = \frac{\lambda^x}{x!}e^{-\lambda}$,$x = 0, 1, 2, ..., n$.
- Continuous distribution
  - Uniform r.v.:$X \sim Unif(a, b)$, $f(x) = \frac{1}{b-a}$, $x \in (a, b)$
  - Exponential r.v.:$X \sim Exp(\lambda)$, $f(x) = \lambda e^{-\lambda x}$
  - Normal r.v.:$X \sim N(\mu, \sigma^2)$, $f(x) = \frac{1}{\sqrt{2\pi}\sigma} exp(-\frac{(x-\mu)^2}{2\sigma^2})$

Multivariate Normal Distribution

- The d random vector $X \sim N(\mu, \Sigma)$,

$$f(x) = \frac{1}{(2\pi)^{d/2}|\Sigma|^{\frac{1}{2}}} exp\Big(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\Big)$$

- $AX + b \sim N(A\mu + b, A\Sigma^{-1}A^T)$
- Conditional distribution.

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \quad \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

then

$$X_1|X_2 \sim N(\mu_1 + \Sigma_{11}\Sigma_{22}^{-1}(X_2 - \mu_2), \Sigma_{11} - \sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})$$

Statistics

- Sample mean $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$
- Sample variance:$S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$
- Sampling distribution of $\bar{X}_n$:$G_n(t) = P(\bar{X}_n \leq t)$

When it is normal:

- If $X \sim N(\mu, \Sigma^2)$,then $\bar{X}_n$ and S_n^2 are independent,

$$\bar{X}_n \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

$$\frac{(n-1)S_n^2}{\sigma^2} \sim \chi^2_{n-1}$$

Moment inequalities

Lemma 4.7.1

Let a and b be any two positive numbers,and let p and q be any positive numbers satisfying

$$\frac{1}{p}+\frac{1}{q}=1$$

. Then

$$\frac{1}{p}a^p+\frac{1}{q}b^q \geq ab$$

with equality holds if and only if $a^p=b^q$.

- Proof:Consider for fixed $b(or a)$,

$$g(a)=\frac{1}{p}a^p+\frac{1}{q}b^q-ab$$

with equality if and only if $a^p=b^q$.

Theorem 4.7.2(Holder's Inequality)

Let X and Y be any two random variables.Let p and q be any positive numbers satisfying

$$\frac{1}{p}+\frac{1}{q}=1$$

Then

$$|E(XY)|\leq E|XY|\leq (E|X|^p)^{\frac{1}{p}}(E|Y|^q)^{\frac{1}{q}}$$

Proof:In the inequality (1),let

$$a=\frac{|X|}{(E|X|^p)^{\frac{1}{p}}},b=\frac{|Y|}{(E(Y)^q)^{\frac{1}{q}}}$$

After some simplification,and take expectation on the two sides of the inequality. The result can been obtained.

- Theorem 4.7.3(Cauchy-Schwarz Inequality)

For any two random variables X and Y,

$$|E(XY)| \leq E|XY| \leq (E|X|^2)^{\frac{1}{2}}(E|Y|^2)^{\frac{1}{2}}$$

- Example 4.7.4 (Covariance Inequality))

If X and Y have means μ_X and μ_Y , and variances σ_X^2 and σ_Y^2 , respectively.We can apply the Cauchy-Schwarz Inequality to get

$$(Cov(X,Y))^2 \leq \sigma_X^2 \dot{\sigma}_Y^2$$

- Example

Let $p > 1$,then apply Holders Inequality.For any random variables X,

$$E|X| \leq \{E|X|^p\}^{\frac{1}{p}}$$

If $1 < r < s$,we have (Liapounovs Inequality)

$$(E|X|^r)^{\frac{1}{r}} \leq (E|X|^s)^{\frac{1}{s}}$$

- Proof Let q be such that $\frac{1}{p} + \frac{1}{q} = 1$,then

$$E|X| = E|X|\dot{1} \leq (E|X|^p)^{\frac{1}{p}}\dot{(}E1^q)^{\frac{1}{q}} = (E|X|^p)^{\frac{1}{p}}$$

.

- Proof Let s be such that $s = pr$,then $s > 1$.

$$E(|X|^r) \leq (E|X|^r)^p)^{\frac{1}{p}}$$

.

Theorem 4.7.5(Minkowskis Inequality)

Let X and Y be any two random variables.Then for $1 < p < \infty$

$$[E|X+Y|^p]^{\frac{1}{p}} \leq (E|X|^p)^{\frac{1}{p}} + (E|Y|^p)^{\frac{1}{p}}$$

Proof:

$$\begin{aligned} E|X+Y|^p &= E(|X+Y||X+Y|^{p-1}) \\ &\leq E(|X||X+Y|^{p-1}) + E(|Y||X+Y|^{p-1}) \end{aligned} \tag{6}$$

Using Holder's Inequality,

$$E(|X||X+Y|^{p-1}) \leq (E|X|^p)^{\frac{1}{p}}[E|X+Y|^{q(p-1)}]^{\frac{1}{q}} \tag{7}$$

where q is such that $\frac{1}{p}+\frac{1}{q}=1$ or $\frac{1}{q}=1-\frac{1}{p}$,i.e.$q=\frac{p}{p-1}$ or $q(p-1)=p$.Similarly,

$$E(|Y||X+Y|^{p-1}) \leq (E|Y|^p)^{\frac{1}{p}}[E|X+Y|^{q(p-1)}]^{1/q} \tag{8}$$

So combine () and () with (),divide through by $[E(|X+Y|^{q(p-1)})]^{1/q}$,we have

$$E|X+Y|^p \leq (E|X+Y|^p)^{\frac{p-1}{p}}[(E|X|^p)^{\frac{1}{p}} + (E|Y|^p)^{\frac{1}{p}}]$$

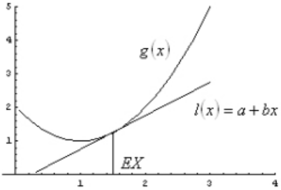
Theorem 4.7.7(Jensens Inequality)

For any random variable X, if $g(x)$ is a convex function, then

$$Eg(X) \geq g(EX)$$

- Equality holds if and only if, for any line $a + bx$ that is tangent to $g(x)$ at $x = EX$, $P(g(X) = a + bX) = 1$.
- If $g(x)$ is linear, $g(EX) = a + bEX = Eg(X)$.

Remark For any twice differentiable function $g(x)$, it is convex if $g``(x)\ 0$ for all x.



Example 1 (An inequality for means)

Let $a_1, a_2, \cdots, a_n$ be n non-negative numbers. Define

$$a_A = \frac{1}{n}(a_1 + a_2 + \cdots + a_n)$$

$$a_G = [a_1 a_2 \cdots a_n]^{1/n} a_H = \frac{1}{\frac{1}{n}(\frac{1}{a_1} + \cdots + \frac{1}{a_n})}$$

An inequality relating these means is

$$a_H \leq a_G \leq a_A$$

.

Remark The above inequality gives a reason for Maximum Likelihood Estimation(MLE).

Proof:

Let X be a random variable with range $a_1, ..., a_n$, and $P(X = a_i) = 1/n, n = 1, ..., n$. Since $logx$ is a concave function, $E \log X \leq \log(EX)$, hence

$$\log a_G = \frac{1}{n}\sum_{i=1}^{n} \log a_i = E \log X \leq \log(EX)$$
$$= \log\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right) = \log a_A$$

So, $a_G \leq a_A$. Furthermore,

$$\log \frac{1}{a_H} = \log\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{a_i}\right) = E \log\frac{1}{X} \geq E\left(\log\frac{1}{X}\right) = -\log(EX)$$
$$= -\log a_G = \log\left(\frac{1}{a_G}\right).$$

So, $a_G \geq a_H$.

Markovs Inequality

Theorem 5 (Markovs(Chebyshevs) Inequality)

- *If g is strictly increasing and positive on* $(0, \infty)$, $g(x) = g(-x)$.
- X *is a r.v. such that* $E[g(X)] < \infty$*,then for each* $a > 0$

$$P(|X| \geq a) \leq \frac{E[g(X)]}{g(a)}$$

Proof:

$$\begin{aligned} E[g(X)] &\geq E[g(X) I_{\{g(X) \geq g(a)\}}] \\ &\geq g(a) E[g(X) I_{\{g(X) \geq g(a)\}}] \\ &= g(a) E[I_{|X| \geq a}] \\ &= g(a) P(|X| \geq a) \end{aligned}$$

Some special cases: Markovs Inequality

$$g(x) = |x| \Rightarrow P(|X| \geq a) \leq \frac{E|X|}{a}$$

$$g(x) = x^p \Rightarrow P(|X| \geq a) \leq \frac{E|g(X^p)|}{a^p}$$

$$g(x) = x^2 \Rightarrow P(|X - EX| \geq a) \leq \frac{Var(X)}{a^2}$$

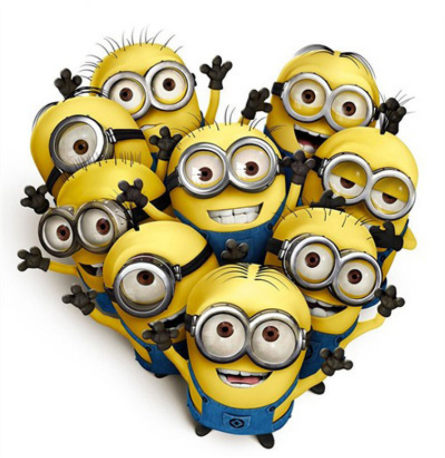
$$g(x) = e^{t|x|} \Rightarrow P(|X| \geq a) \leq \frac{E[e^{t|X|}]}{e^{ta}}$$

for some constant $t \geq 0$

Homework

- If $\mu = EX \geq 0$ and $0 \leq \lambda \leq 1$,then
$$P(X > \lambda\mu) \geq \frac{(1-\lambda)^2\mu^2}{EX^2}$$

Consequently, if $E|Y| = 1, P(|Y| > \lambda) \geq (1-\lambda)^2/EY^2$ (This gives a lower bound complementing Chebyshevs inequality.)



Thanks !