Lecture 2: Convergence of Random Variables

Xuejun Ma

School of Mathematical Sciences

Soochow University

https://xuejunma.github.io

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Outline

- Last lecture:review some basic probability concepts; introduce the statistics
- 4 types of convergence
- Relationship between different types of convergence
- Stochastic orders

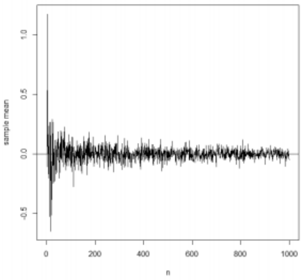
Terms

- Converge in probability; Converge in Lp; converge in quadratic mean; almost sure converge; converge in distribution;
- O_p, o_p

Note: May take 1-2 lectures for this topic.

Look into Sample mean

- Recall:
  Sample mean:$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ Note: When $n \neq m$, X_n and X_m share the same expectation μ but have different distribution.
- Intuitively, when $n \to \infty$,$\bar{X}_n$ is very close to $\mu = E(X)$.



Real data

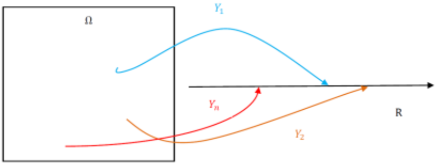
N(0,1)

```
rm(list=ls())
n.vec <- seq(1, 103, 1)
n.len <- length(n.vec)
mean.full <- NULL
for(i in 1:n.len){
mean.full[i] <- mean(rnorm(n.vec[i]))
 }
 plot(n.vec, mean.full, type="l",xlab = "n",
 ylab = "sample mean")
 abline(h=0,lwd=1,col="blue")
```

If $x_1, x_2, ..., x_n, ...$ is an array of numbers, we know how to describe whether they convergence or not. But what if they are random variables? How to describe it?

Generalization

- Let $\{Y_i\}_{i=1}^{\infty} = Y_1, Y_2, ..., Y_n, ...$ denotes a sequence of random variables
- Problem: How to describe the limit of Y_n
- Consider 2 cases:
  - Case 1. $Y_i \sim F$ independently, $i = 1, 2, \ldots$
  - Case 2. $Z_1 = Z_2 = Z_3 = \ldots$, where $Z_1 \sim F$. Let $X \sim F$. Can we say $Y_i \to X$? Can we say $Z_i \to X$. How to differentiate these two cases?
  - Recall: $Y_1, Y_2, ..., Y_n : \omega \to R$. A sequence of functions



Convergence in Probability

Definition 5.5.1: Convergence in Probability

For a sequence of r.v.'s $\{X_n\}_{i=1}^{\infty} = X_1, X_2, ..., X_n, ...$,we say they converge in probability towards the r.v. X(i.e. $X_n \xrightarrow{p} X$ if for any $\epsilon > 0$,

$$\lim_{n\to\infty} P(|X_n - X| \geq \epsilon) = 0$$

.

- The target X has the same sample space with all the $X_i's$.
- X_n are usually dependent,but not identically distribution.
- Practically,find the sequence of events $A_n = \{\omega \in \Omega, |X_n(\omega) - X(\omega)| \geq \epsilon\}$ by obtaining $|X_n - X|$as a new r.v.,and check if $P(A_n) \to 0$ when $n \to \infty$.
- Interpretation: for any ϵ, the event that $|X_n - X|$ has probability smaller than δ when n is large enough. It concerns more about the probability measure and r.v.,instead of the CDF only.

For random p-vectors, $\boldsymbol{X}_1, \boldsymbol{X}_n, \ldots$ and $\boldsymbol{X}$, if

$$\|\boldsymbol{X}_n - \boldsymbol{X}\| \xrightarrow{p} 0,$$

we say $\boldsymbol{X}_n \xrightarrow{P} \boldsymbol{X}$, where $\|\boldsymbol{z}\| = (\sum_{i=1}^{p} z_i^2)^{1/2}$ denotes the Euclidean distance (L2-norm) for $\boldsymbol{z} \in R^p$.

- It is easily to seen that $\boldsymbol{X}_n \xrightarrow{P} \boldsymbol{X}$ iff the corresponding component-wise convergence holds.

Example: Convergence in Probability

- Let X be a r.v. with prob 1 at 1,and $X_n \sim N(1, \frac{1}{n^2})$. According to the property of normal distribution.$X_n - X \sim N(0, \frac{1}{n^2})$,so

$$P(|X_n - X| \geq \epsilon) = P\left(|N(0, \frac{1}{n^2})|\right)$$
$$\leq \frac{1}{n^2\epsilon^2} \leq \delta, n \geq \frac{1}{\epsilon\sqrt{\delta}}$$

So,$X_n \xrightarrow{P} X$.[1]

[1] Chebychevs inequality.

$$P(|X - \mu| \geq \epsilon) \leq \frac{Var(X)}{\epsilon^2}$$

Example: Convergence in Probability

- Let $X_n \sim Ber(0.5)$,and $X \sim Ber(0.5)$,X_n and X are independent.Note for any n,

$$\begin{aligned}
&P(|X_n - X| \geq 1) \\
&= P(\{X_n = 1, X = 0\} \cup \{X_n = 0, X = 1\}) \\
&= P(\{X_n = 1, X = 0\}) + P(\{X_n = 0, X = 1\}) \\
&= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{1}{2} \nrightarrow 0
\end{aligned}$$

So,X_n does NOT converge to X in probability.

Convergence in L_r (rth mean)

Convergence in L_r

For a sequence of r,v,'s $\{X_i\}_{i=1}^{\infty} = X_1, ..., X_n, ...$,we say they converge in Lr towards the r.v. X(i.e. $X_n \xrightarrow{L^r} X$) if for any $\epsilon > 0$,

$$\lim_{n\to\infty} E|X_n - X|^r = 0.$$

where $[E(|X_n - X|^r)]^{\frac{1}{r}}$ is the L^r distance between X_n and X

- The target X has the same sample space with all the X_is
- When $r = 2$, converge in L^2 is also called converge in quadratic mean, i.e.,$X_n \xrightarrow{qm} X$, The convergence in quadratic mean is generally used.
- To show L^r convergence, just figure out an upper bound of $E(|X_n - X|^r)$, and show this upper bound goes to 0.

Example: Convergence in L_2

- Recall the pervious example when X has a point mass at 1, and $X_n \sim N(1, \frac{1}{n^2})$. According to the property of normal distribution.,$X_n - X \sim N(0, \frac{1}{n^2})$,so

$$
\begin{aligned}
E(|X_n - X|^2) &= (E(X_n - X))^2 + Var(X_n - X) \\
&= 0 + \frac{1}{n^2} = \frac{1}{n^2} \to 0.
\end{aligned}
$$

Hence, $X_n \xrightarrow{L^2} X$

Properties: Convergence in L_2

- According to the deviation, if $Var(X_n - X) \to 0$,and $E(X_n - X) \to 0$,then there is
$$E(|X_n - X|^2) = (E(X_n - X))^2 + Var(X_n - X) \to 0$$

Proposition 1

If $Var(X_n - X) \to 0$,and $E(X_n - X) \to 0$,then $X_n \xrightarrow{L^2} X$.

Properties: Convergence in L_2

Proposition 2

Let $0 < s < r < \infty$ if $X_n \xrightarrow{L^r} X$, then $X_n \xrightarrow{L^s} X$

- Recall that with Holder inequality, there is

$$|E(XY)| \leq E|XY| \leq (E|X|^p)^{\frac{1}{p}}(E|Y|^q)^{\frac{1}{q}}$$

  where $\frac{1}{p} + \frac{1}{q} = 1$
- Let $Y = 1, Z = |X_n - X|^r, l = r/s$, and $k = 1/(1 - s/r) > 1$, then

$$\begin{aligned} E\Big(|X_n - X|^s\Big) &= E\Big(|X_n - X|^s \times 1\Big) \\ &\leq \Big[E\Big(|X_n - X|^r\Big)\Big]^{s/r} \times 1^{1/k} \\ &= \Big[E\Big(|X_n - X|^r\Big)\Big]^{s/r} \to 0 \end{aligned}$$

Properties: Convergence in L_2

Proposition 3

Let $0 < r < \infty$ if $X_n \xrightarrow{L^r} X$, then $X_n \xrightarrow{p} X$

Properties: Convergence in L_2

Proposition 3

Let $0 < r < \infty$ if $X_n \xrightarrow{L^r} X$, then $X_n \xrightarrow{p} X$

Proof:

$$P\Big(|X_n - X| \geq \varepsilon\Big) = P\Big(|X_n - X|^r \geq \varepsilon^r\Big)$$
$$\leq \frac{E\Big(|X_n - X|^r\Big)}{\varepsilon^r} \to 0$$

Markov's Inequality : non-negative r.v.

$$P(X \geq a) \leq \frac{E(X)}{a}$$

Markovs Inequality

Markovs(Chebyshevs) Inequality

- If g is strictly increasing and positive on $(0, \infty)$,$g(x) = g(x)$.
- X is a r.v. such that $E[g(X)] < \infty$,then for each $a > 0$

$$P(|X| \geq a) \leq \frac{E[g(X)]}{g(a)}$$

Markovs Inequality

Markovs(Chebyshevs) Inequality

- If g is strictly increasing and positive on $(0,\infty)$,$g(x) = g(x)$.
- X is a r.v. such that $E[g(X)] < \infty$,then for each $a > 0$

$$P(|X| \geq a) \leq \frac{E[g(X)]}{g(a)}$$

Proof:

$$\begin{aligned} E[g(X)] &\geq E[g(X) I_{\{g(X) \geq g(a)\}}] \\ &\geq g(a) E[I_{\{g(X) \geq g(a)\}}] \\ &= g(a) E[I_{\{|X| \geq a\}}] \\ &= g(a) P(|X| \geq a) \end{aligned}$$

Some special cases: Markovs Inequality

$$g(x) = |x| \rightarrow P(|X| \geq a) \leq \frac{E|X|}{a}$$

$$g(x) = x^p \rightarrow P(|X| \geq a) \leq \frac{E[g(X^p)]}{a^p}$$

$$g(x) = x^2 \rightarrow P(|X - EX| \geq a) \leq \frac{Var(X)}{a^2}$$

$$g(x) = e^{t|x|} \rightarrow P(|X| \geq a) \leq \frac{E[e^{t|X|}]}{e^{ta}}$$

for some constant $t \geq 0$

Almost Sure Convergence

Definition 5.5.6

For a sequence of r.v.'s $\{X_{i=1}^{\infty}\} = X_1, ..., X_n, ...$, we say they almost sure convergence to r.v. X(i.e. $X_n \xrightarrow{a.s} X$) if any $\epsilon > 0$,

$$P(\lim_{n\to\infty} X_n = X) = 1 \quad or \quad P(\lim_{n\to\infty} X_n(\omega) = X(\omega)) = 1$$

- The target X has the same sample space with all the $X_i's$.
- $\{X_n\}$ and X are usually dependent
- Practically, to show the a.s. convergence,
  - For each outcome ω,find the sequence $X_1(\omega), X_2(\omega), ...$ (sequence of real numbers) and the real number $X(\omega)$.Figure out whether $\lim_{n\to\infty} X_n(\omega) = X(\omega)$ is true or not.
  - Let the event $A = \{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}$.
  - Check if $P(A) = 1$
- Interpretation: for almost all the outcomes ω! when n is large enough, $|X_n(\omega) - X(\omega)| \le \epsilon$ for any $\epsilon > 0$

Example 1: Almost Sure Convergence

- Let the sample space $\Omega = [0,1]$,with a probability measure that is uniform on this space, i.e. $P([a,b]) = b - a$ for any $0 \leq a \leq b \leq 1$.
- Let

$$X_n(\omega) = \begin{cases} 1, 0 \leq \omega < \frac{n+1}{2n} \\ 0, otherwise \end{cases} \qquad X(\omega) = \begin{cases} 1, 0 \leq \omega < \frac{1}{2} \\ 0, otherwise \end{cases}$$

For each $\omega \in [0,1]$.

- If $\omega \in [0, \frac{1}{2})$,then $X_n(\omega) = 1 = X(\omega)$.
- If $\omega = \frac{1}{2}$,then $X_n(\omega) = 1 \nrightarrow X(\omega) = 0$.
- If $\omega \in (1/2, 1]$,then $X_n(\omega) = 0 = X(\omega)$,when $\frac{n+1}{2n} < \omega$,which is equivalent with $n \geq \frac{1}{2\omega - 1}$.
So,$A = [0, 1/2) \cup (1/2, 1]$.Check $P(A) = 1$?

Example 5.5.7: Almost Sure Convergence

- Let the sample space $\Omega = [0, 1]$,with a probability measure that is uniform on this space, i.e. $P([a, b]) = b - a$ for any $0 \leq a \leq b \leq 1$.
- Define r.v.

$$X_n(\omega) = \omega + \omega^n \quad \textit{and} \quad X(\omega) = \omega$$

  For each $\omega \in [0, 1]$.
- If $\omega \in [0, 1)$,$\omega^n \to 0$,then $X_n(\omega) \to \omega = X(\omega)$.
- If $\omega = 1$,then $X_n(\omega) = 2 \nrightarrow X(\omega) = 1$ for every n
  So,$A = [0, 1)$.Check $P(A) = 1$?

Almost Sure Convergence

- Comparison between almost sure convergence and converge in probability
  - Convergence in probability: for each n, consider $P(|X_n(\omega) - X(\omega)| > \epsilon)$, and check the limit of this probability
  - Almost sure convergence: for each ω, check the limit $\lim_{n\to\infty} X_n(\omega)$, and find the probability of the set that the limit does not equal to $X(\omega)$

Almost Sure Convergence

- Can we express it as the limit of probability?

Theorem 1 (Almost Sure Convergence)

The following statements are equivalent:

- $X_n \xrightarrow{a.s.} X$
- $\forall \epsilon > 0, P(\cap_{k \geq n}\{|X_k - X| < \epsilon\}) \to 1$
- $\forall \epsilon > 0, P(\cup_{k \geq n}\{|X_k - X| \geq \epsilon\}) \to 0$
- $\forall \epsilon > 0,$
$$\lim_{n\to\infty} P(\sup_{k\geq n} |X_k - X| > \epsilon) = 0$$

Here,we consider that set $\cup_{k\geq n}\{|X_k - X| > \epsilon\}$

Property 1: Almost Sure Convergence

Proposition 4

If $X_n \xrightarrow{a.s.} X$, then $X_n \xrightarrow{p} X$

Proof: for any $\varepsilon > 0$

$$\begin{aligned} 0 \leq & P\Big(|X_n - X| \geq \varepsilon\Big) \\ & \leq P\Big(\bigcup_{k=n}^{\infty} |X_k - X| \geq \varepsilon\Big) \\ & = 0 \end{aligned}$$

Hence, $\lim_{n\to\infty} P\Big(|X_n - X| \geq \varepsilon\Big) = 0$, which implies $X_n \xrightarrow{p} X$.

Convergence in Distribution

Definition 5.5.9

Let $\{X_i\}_{i=1}^{\infty} = X_1, X_2, ..., X_n, ...$ be a sequence of r.v.s with CDF $F_1, ..., F_n, ...$,and X be r.v. with CDF F.we say they converges in distribution to r.v. X (i.e. $X_n \xrightarrow{d} X$) if

$$\lim_{n\to\infty} F_n(x) = F(x)$$

at very point at which F is continuous.

- $\{X_n\}$ and X can be dependent or independent
- Convergence:
  - If X is discrete, the convergence stands at points F does not jump
  - If X is cont., the convergence stands at every point
- Convergence in distribution is really the CDFs that converge, not the r.v. Hence it quite different from conv. in prob. or alm. sure conv.

Property 1: Convergence in Distribution

Proposition 5

If $X_n \xrightarrow{P} X$,then $X_n \xrightarrow{d} X$

Proof: Denote $F_n(x) = P(X_n \leq x)$ and $F(x) = P(X \leq x)$.First we have

$$\begin{aligned}F_n(x) =& P(X_n \leq x)\\ &= P(X_n \leq x, |X_n - X| \leq \epsilon) + P(X_n \leq x, |X_n - X| > \epsilon)\\ &\leq P(X \leq x - (X_n - X), |X_n - X| \leq \epsilon) + P(|X_n - X| > \epsilon)\\ &\leq P(X \leq x + \epsilon) + P(|X_n - X| > \epsilon)\\ &= F(x + \epsilon) + P(|X_n - X| > \epsilon)\end{aligned}$$

Or

$$\begin{aligned}F_n(x) =& P(X_n \leq x)\\ &= P(X_n \leq x, X \leq x + \epsilon) + P(X_n \leq x, X > x + \epsilon)\\ &\leq P(X_n \leq x, X \leq x + \epsilon) + P(|X_n - X| > \epsilon)\\ &\leq P(X \leq x + \epsilon) + P(|X_n - X| > \epsilon)\\ &= F(x + \epsilon) + P(|X_n - X| > \epsilon)\end{aligned}$$

On the other hand,

$$\begin{aligned}
F_n(x) =& 1 - P(X_n \geq x) \\
&= 1 - P(X_n \geq x, |X_n - X| \geq \epsilon) - P(X_n \geq x, |X_n - X| \leq \epsilon) \\
&\geq 1 - P(X \geq x - (X_n - X), |X_n - X| \leq \epsilon) - P(|X_n - X| \leq \epsilon) \\
&\geq 1 - P(X \leq x - \epsilon) - P(|X_n - X| \leq \epsilon) \\
&= F(x - \epsilon) - P(|X_n - X| \leq \epsilon)
\end{aligned}$$

Or

$$\begin{aligned}
F_n(x) =& 1 - P(X_n \geq x) \\
&= 1 - P(X_n > x, X \leq x - \epsilon) - P(X_n > x, X > x - \epsilon) \\
&\geq 1 - P(X > x - \epsilon) - P(|X_n - X| \leq \epsilon) \\
&\geq 1 - P(X \leq x - \epsilon) - P(|X_n - X| \leq \epsilon) \\
&= F(x - \epsilon) - P(|X_n - X| \leq \epsilon)
\end{aligned}$$

Combining the two, we have

$$F(x-\epsilon) - P(|X_n - X| \leq \epsilon) \leq F_n(x) \leq F(x+\epsilon) + P(|X_n - X| \leq \epsilon)$$

Letting $n \to \infty$ and since $X_n \xrightarrow{p} X$,

$$F(x-\epsilon) \leq \lim \inf_{n\to\infty} F_n(x) \leq \lim \sup_{n\to\infty} F_n(x) \leq F(x+\epsilon)$$

Recall that F is continuous at x,which means $F(x-\epsilon) \to F(x)$ and $F(x+\epsilon) \to F(x)$ as $\epsilon \to 0$.Hence,

$$F(x) \leq \lim \inf_{n\to\infty} F_n(x) \leq \lim \sup_{n\to\infty} F(x) \leq F(x)$$

Convergence in Distribution

Recall the characteristic function for $X \sim F$ is $\Phi_X(t) = E(e^{it})$.If $\Phi_X(t) = \Phi_Y(t)$ then X and Y have the same distribution.

Theorem: Convergence in Distribution

Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of r.v.s with characteristic functions $\Phi_{X_n}(t)$ and X be a r.v. with the characteristic function $\Phi_X(t)$.Then,

$$X_n \xrightarrow{d} X \Leftrightarrow \lim_{n \to \infty} \Phi_{X_n}(X) = \Phi_X(t)$$

Example: Suppose that $X_n \sim N(\mu + 1/n, \sigma^2 + 1/n)$,then

$$\Phi_{X_{n(t)}} = exp\{(\mu + 1/n^2)t - t^2(\sigma^2 + 1/n)/2\} \to exp\{\mu t - t^2\sigma^2/2\}$$

Note that the limit is the characteristic function for $X \sim N(\mu, \sigma^2)$. So,$X_n \xrightarrow{d} X$.It is easier than the analysis on the CDF of X_n.

Relationship Between 4 Types of Convergence

Some comments

- $$\begin{array}{l} X_n \xrightarrow{a.s.} X \\ \qquad\qquad \Rightarrow X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{d} X \\ X_n \xrightarrow{L_r} \end{array}$$
- If $0 < s < r < \infty$,$X_n \xrightarrow{L_r} X \Rightarrow X_n \xrightarrow{L_s} X$.
- No other implications hold in general.

- (1).(a) If $X_n \xrightarrow{a.s} X$,then $X_n \xrightarrow{P} X$.The converse may not hold.Let

$$P(X_n = 0) = 1 - \frac{1}{n}, P(X_n = 1) = \frac{1}{n}$$

and $X_n's$ are independent.Since
$P(|X_n - 0| > \epsilon) = P(X_n = 1) = n^{-1} \to 0$,Then
$X_n \xrightarrow{P} X$.However,$X_n \not\xrightarrow{a.s.} 0$ since for any $0 < \epsilon < 1$,we have

$$\lim_{n\to\infty} P(\cap_{k\geq n}\{|X_k - 0| < \epsilon\}) = \lim_{n\to\infty} P(\lim_{r\to\infty} \cap_{k\geq n}^{r} \{|X_k| < \epsilon\})$$

$$= \lim_{n\to\infty}\lim_{r\to\infty} P(\cap_{k\geq n}^{r}\{|X_k| < \epsilon\}) = \lim_{n\to\infty}\lim_{r\to\infty}\prod_{k=n}^{r}(1 - \frac{1}{k})$$

$$= \lim_{n\to\infty}\lim_{r\to\infty} \frac{n-1}{n}\frac{n}{n+1}\cdots\frac{r-1}{r} = \lim_{n\to\infty}\lim_{r\to\infty}\frac{n-1}{r} = 0 \neq 1$$

(b) If $X_n \xrightarrow{L_r} X$,then $X_n \xrightarrow{P} X$. The converse may not hold.

$$P(X_n = 0) = 1 - \frac{1}{n}, P(X_n = n) = \frac{1}{n}$$

Then $X_n \xrightarrow{P} 0$ since

$$P(|X_n - 0| > \epsilon) = P(X_n = n) = \frac{1}{n} \to 0$$

. But $EX_n = 1 \neq 0$.

If $X_n \xrightarrow{L_r} X$,then $X_n \xrightarrow{P} X$.The converse may not hold.

$$X \sim N(0,1), X_n = -X \sim N(0,1)$$

Then $X_n \xrightarrow{d} X$.but $X_n \stackrel{P}{\nrightarrow} X$ since

$$P(|X_n - X| > \epsilon) = P(2|X| > \epsilon) \nrightarrow 0$$

.

(2) If $0 < s < r < \infty$,$X_n \xrightarrow{L_r} X \Rightarrow X_n \xrightarrow{L_s} X$.The converse may not hold.

$$P(X_n = 0) = 1 - \frac{1}{n^2}, P(X_n = n) = \frac{1}{n^2}$$

Then $X_n \xrightarrow{L_1} X$ since

$$E|X_n - 0| = \frac{1}{n^2} \cdot n = \frac{1}{n} \to 0$$

But $X_n \xrightarrow{L_2} X$ since

$$E|X_n - 0|^2 = \frac{1}{n^2} \cdot n^2 = 1 \neq 0$$

(3).We now show that "a.s. convergence" and "mean convergence" do not imply each other.

- Let $P(X_n = 0) = 1 - n^{-2}$ and $P(X_n = n^3) = n^{-2}$.Then $X_n \xrightarrow{a.s.} 0$,but $X_n \overset{L_1}{\nrightarrow} 0$.Since

$$\lim_{n\to\infty} P(\cup_{k\geq n}\{|X_k - 0| \geq \epsilon\}) = \lim_{n\to\infty} P(\lim_{r\to\infty} \cup_{k\geq n}^{r}\{|X_k| \geq \epsilon\})$$

$$= \lim_{n\to\infty} \lim_{r\to\infty} P(\cup_{k\geq n}^{r}\{|X_k| \geq \epsilon\}) = \lim_{n\to\infty} \lim_{r\to\infty} \sum_{k=n}^{r} \frac{1}{k^2}$$

$$= \lim_{n\to\infty} \lim_{r\to\infty} (\frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots + \frac{1}{r^2}) \to 0.$$

However,

$$E|X_n - 0| = \frac{1}{n^2} \cdot n^3 \to \infty$$

- $X_n \xrightarrow{L_1} 0$,but $X_n \overset{a.s.}{\nrightarrow} 0$

$$P(X_n = 0) = 1 - \frac{1}{n}, P(X_n = 1) = \frac{1}{n}$$

Properties of Convergence

- $X_n \to X$ and $Y_n \to Y$,then $X_n \pm Y_n \to X + Y$
  - $X_n \xrightarrow{a.s.} X, Y_n \xrightarrow{a.s.} Y$,then $X_n \pm Y_n \to X + Y$,
  - $X_n \xrightarrow{L_r} X, X_n \xrightarrow{L_r} X$,then $X_n + Y_n \xrightarrow{a.s.} X + Y$,
  - $X_n \xrightarrow{P} X, X_n \xrightarrow{P} X$,then $X_n + Y_n \xrightarrow{L_r} X + Y$,
  - $X_n \xrightarrow{d} X, X_n \xrightarrow{d} X$,it is not sure that $X_n + Y_n \xrightarrow{d} X + Y$
- Slutsky's Theorem Let $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} C$,then
  - $X_n + Y_n \xrightarrow{d} X + C$
  - $X_n Y_n \xrightarrow{d} CX$
  - $X_n / Y_n \xrightarrow{d} X/C$ id $C \neq 0$
- The Continuous Mapping Theorem: if $g(\cdot)$ is a continuous function, then
  - $X_n \xrightarrow{a.s.} X$,then $g(X_n) \xrightarrow{a.s.} g(X)$,
  - $X_n \xrightarrow{p} X$,then $g(X_n) \xrightarrow{p} g(X)$,
  - $X_n \xrightarrow{d} X$,then $g(X_n) \xrightarrow{d} g(X)$.

The Continuous Mapping Theorem

Theorem 2 (Continuous Mapping Theorem)

- *Let $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots,$ and $\boldsymbol{X}$ be random p-vectors defined on a probability space*
- *let $g(\cdot)$ be a vector-valued (including real-valued) continuous function defined on R^p.*

If $\boldsymbol{X}_n$ converges to $\boldsymbol{X}$ in probability, almost surely, or in law, then $g(\boldsymbol{X}_n)$ converges to $g(\boldsymbol{X})$in probability, almost surely, or in law, respectively.

Remark The condition that $g(\cdot)$ is continuous function in Theorem can be further relaxed to that $g(\cdot)$ is continuous a.s., i.e., $P(\boldsymbol{X} \in C(g)) = 1$ where $C(g) = \{\boldsymbol{x} : g$ is continuous at $\boldsymbol{x}$ is called the continuity set of g.

Example

- If $X_n \xrightarrow{d} X \sim N(0,1)$, then $1/X_n \xrightarrow{d} 1/X$?
- If $X_n = 1/n$, and

$$g(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Then $X_n \;\; g(X_n) \xrightarrow{d}$?

Stochastic Orders

Recall:

- In mathematics, we use o and O notations to denote the order of terms
- $a_n = o(1)$ means $a_n \to 0$ when $n \to \infty$;$a_n = o(b_n)$ means that $a_n/b_n = o(1)$.
- $a_n = O(1)$ means $|a_n| \leq C$ for some constant $C > 0$,for all large n;$a_n = O(b_n)$ mean $a_n/b_n = O(1)$.

Now we consider the probabilistic version:

o_p

If $X_n \xrightarrow{P} 0$,i.e. $P(|X_n| \geq \epsilon) \to 0$ for every $\epsilon > 0$, then we say that $X_n = o_p(1)$

O_p

We say that $X_n = O_p(1)$,or X_n is bounded in probability,if for any $\epsilon > 0$,there exists $C_\epsilon > 0$,such that

$$P(|X_n| > C_\epsilon) \leq \epsilon$$

Stochastic Orders

Generalisation: Consider a sequence $X_1, X_2, \dots$ of r.v.'s and $a_1, a_2, \dots$, a sequence of positive real numbers,

- For a r.v. X,$X_n \xrightarrow{P} X$ if only if $X_n - X = o_p(1)$
- $X_n = o_p(a_n)$ if only if $a_n^{-1}X_n = o_p(1)$.a_n is the rate.
- $X_n = O_p(a_n)$ if only if $a_n^{-1}X_n = O_p(1)$.a_n is the rate.

Examples:

- If $X_n \sim N(0, \frac{1}{n})$,then $X_n = o_p(1)$ and $X_n = O_p(\frac{1}{\sqrt{n}})$
- If $X_n = o_p(1)$,then $X_n = O_p(1)$
- $O_p(1)o_p(1) = o_p(1), O_p(1)O_p(1) = O_p(1)$
- $O_p(1) + o_p(1) = O_p(1)$
- $O_p(a_n)o_p(b_n) = o_p(a_nb_n), O_p(a_n)O_p(b_n) = O_p(a_nb_n)$
- $(1 + o_p)^{-1} = O_p(1)$
- $o_p(O_p(1)) = o_p(1)$

Homework

1. Show that if $X_n \xrightarrow{d} X$ for a random variable X,then $X_n = O_p(1)$.
2. Let $X, X_1, X_2, \cdots$ be a sequence of random variables.Show that $X_n \xrightarrow{p} X$ as $n \to \infty$ if and only if
$$E(\frac{|X_n - X|}{1+|X_n - X|}) \to 0 \qquad \text{as } n \to \infty.$$
3. Prove that $O_{P(1)} + o_{P(1)} = O_{P(1)}$.
4. Let $X_1, X_2, \ldots$ be iid random variables with $EX_1 = 0, EX_1^2 < \infty$, then $\sqrt{n}\bar{X}_n/S_n \xrightarrow{d} N(0,1)$