Lecture 8: Point estimation: Methods of Evaluating Estimators

Ma Xuejun

School of Mathematical Sciences

Soochow University

https://xuejunma.github.io

Outline

1. Review

2. Unbiasedness
   - Evaluation of Estimators:Bias and Variance
   - Uniform Minimum Variance Unbiased Estimator

3. Mean Square Error
   - Bias-Variance Tradeoff
   - Mean Square Error

Evaluation of Estimators

- We already discussed several types of estimators and the computing issue
- We can also define any statistic to be an estimator
- Which is better? Which is worse?

Evaluation of Estimators

There are plenty of ways to evalute.Here are some popular used criteria.

- Bias and Variance
  - Unbiased estimator
  - Cramer-Rao Lower Bound
  - Rao-Blackwell Theorem
- Mean squared error (MSE)
  - Trade-off between bias and variance
  - Loss function
  - Mean squared error
- Minimax Theory
- Large sample theory
  - Consistency
  - Efficiency

Unbiasedness

■ Say that $\hat{\theta} = w(X_1, \cdots, X_n)$ is an estimator of θ,then it would be good if it satifies that

$$E[\hat{\theta}] = \theta$$

Unbiased Estimator

Let $\hat{\theta}$ be an estimator of a parameter θ.Then the bias of $\hat{\theta}$ is defined as

$$Bias(\hat{\theta};\theta) = E_{\theta}[\hat{\theta}] - \theta$$

If $Bias(\hat{\theta}) = 0$,then we say $\hat{\theta}$ is unbias.

- $E_\theta[\hat{\theta}]$ means the expectation of $\hat{\theta}$ when the underlying parameter equals to θ.
- The bias is a function of θ.For unbiased estimators,the bias is a function that always equals to 0.

Unbiasedness: Example

■ Let $X_1, \cdots, X_n \sim Exp(\lambda)$. Estimate λ.
Recall that the MLE for exponential distribution is $1/\bar{X}_n$.Let the estimator be $\hat{\lambda} = 1/\bar{X}_n$.Note that $n\bar{X}_n \backsim Gamma(n, \lambda)$. Therefore,the bias is

$$Bias(\hat{\lambda}, \lambda) = \frac{n}{n-1}\lambda - \lambda = \frac{1}{n-1}\lambda$$

Therefore,the MLE $\hat{\lambda}$ is a biased estimator.However,when $n \to \infty$,the bias is close to 0.
■ Let $X_1, \cdots, X_n \sim N(\mu, \sigma^2)$.Find the bias for the sample variance.The sample variance is $\frac{1}{n-1}(X_i - \bar{X}_n)^2$.The bias is

$$Bias(\hat{\sigma}^2, \sigma^2) = E[\frac{1}{n-1}(X_i - \bar{X}_n)^2] - \sigma^2 = 0$$

So,the sample variance is unbiased estimator.

Variance

- In the previous normal example,we show that the bias for sample variance is 0.
- If we take the estimator as $\tilde{\sigma}^2 = \frac{n}{n-1}(X_1^2 - \bar{X}_n^2)$,then
  $$E[\tilde{\sigma}^2] - \sigma^2 = \frac{n}{n-1}[E[X_1^2] - E[\bar{X}_n^2]] - \sigma^2 = \frac{n}{n-1} \times \frac{n-1}{n}\sigma^2 - \sigma^2 = 0,$$
  which is also unbiased.
- Which estimator is better? the sample variance or $\tilde{\sigma}^2$?

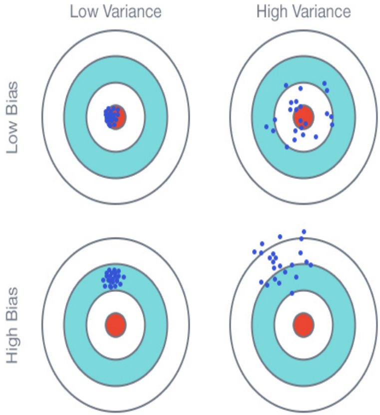
Variance

Let $\hat{\theta}$ be an estimator of the parameter θ.Then the variance of $\hat{\theta}$ is defined as

$$Var(\hat{\theta};\theta) = Var_\theta(\hat{\theta}).$$

- Targeting at θ,the estimator with smaller variance is better.
- For the previous example, the variance for sample variance is $2\sigma^4/(n-1)$,but for $\tilde{\sigma}^2$ is approximately σ^4.So,the sample variance is a better estimator.

Bias-Variance



Uniform Minimum Variance Unbiased Estimator

- Obviously,one unbiased estimator with smallest variance is the best unbiased estimator.
- However,recall that $Var(\hat{\theta};\theta)$ is a function about θ.
- It is possible that for some θ_1,$Var(\hat{\theta_1};\theta_1) < Var(\hat{\theta_2};\theta_1)$,but for another θ_2,$Var(\hat{\theta_1};\theta_2) > Var(\hat{\theta_2};\theta_2)$.
- The best unbiased estimator would be one estimator that for any other estimator W,$Var(\hat{\theta};\theta) < Var(W;\theta)$ holds for all $\theta \in \Theta$

Definition 7.3.7: Uniform Minimum Variance Unbiased Estimator

An estimator W^* of $\tau(\theta)$ is the best unbiased estimator if $E[W^*;\theta] = \tau(\theta)$ for every θ and for any other unbiased estimator W,we have

$$Var(W^*;\theta) \leq Var(W;\theta), \qquad \theta \in \Theta.$$

W^* is called the minimum variance unbiased estimator (UMVUE) for $\tau(\theta)$.

UMVUE

- Does the UMVUE exist?
  - Not necessarily.It is possible that UMVUE does not exist.
- How to prove one estimator is UMVUE?
  - There is a lower bound for the variance of unbiased estimators.If there is one unbiased estimator with variance approching the lower bound,then it is UMVUE.
- How to find the UMVUE?

Cramer-Rao Lower Bound

Theorem 7.3.9 Cramer-Rao Lower Bound

Let $X_1, \cdots, X_n$ with joint density $f(x_1, x_2, \cdots, x_n; \theta)$ and let $W(X_1, \cdots, X_n) : X^n \to \mathbb{R}$ be an estimator with

$$\frac{d}{d\theta}(E[W(X_1, \cdots, X_n; \theta)]) = \int \frac{\partial}{\partial\theta}[W(x_1, x_2, \cdots, x_n) f(x_1, x_2, \cdots, x_n; \theta] dx,$$

and $Var(W(X_1, \cdots, X_n); \theta) < +\infty$,then

$$Var(W(X_1, \cdots, X_n); \theta) \geq \frac{(\frac{d}{d\theta}(E[W(X_1, \cdots, X_n; \theta)]))^2}{E_\theta\{[\frac{\partial}{\partial\theta}\log(f(X_1, \cdots, X_n; \theta)))]^2\}}$$

The condition can be written as

$$\begin{aligned}\frac{d}{d\theta}(E[W(X_1, \cdots, X_n); \theta]) &= \frac{d}{d\theta}\int W(x_1, x_2, \cdots, x_n) f(x_1, x_2, \cdots, x_n; \theta) dx \\ &= \int \frac{\partial}{\partial\theta}[W(x_1, x_2, \cdots, x_n) f(x_1, x_2, \cdots, x_n; \theta)] dx.\end{aligned}$$

Remark: The integral and the derivative is exchangeable.It is satisfied under regular conditions.

Cramer-Rao Lower Bound

Corollary 7.3.10 Corollary: Unbiased Estimators

Let $X_1, \cdots, X_n$ with joint density $f(x_1, x_2, \cdots, x_n; \theta)$ and let $W : X^n \to \mathbb{R}$ be an estimator of $\tau(\theta)$.Suppose the conditions hold,then

$$Var(W;\theta) \geq \frac{\left(\frac{d}{d\theta}E[W(X_1,\cdots,X_n;\theta)]\right)^2}{nE_\theta[[\frac{\partial}{\partial\theta}\log(f(X;\theta))]^2]} = \frac{\tau'(\theta)^2}{nE_\theta[[\frac{\partial}{\partial\theta}\log(f(X;\theta))]^2]}$$

- The lower biund does not depend on the estimator.It is the lower bound for all estimators.
- The lower biund is a function of the parameter θ
- If there is an estimator W^*,which achieves the lower bound for every θ,then this estimator W^* is UMVUE.
- No need to prove $Var(W^*;\theta) \leq Var(W;\theta)$ for all W.

Score and Fisher Information

- An important item here is $E_\theta[[\frac{\partial}{\partial\theta}\log(f(X_1,\cdots,X_n;\theta)))]^2]$
- Actually,we hve some notions and lemmas w.r.t.this quantity

Score function

Let $X_1,\cdots,X_n$ be with joint density $f(x_1,x_2,\cdots,x_n;\theta)$ .The score function is the derivative of the log-likelihood function,which is

$$S_n(\theta)=\frac{\partial}{\partial\theta}\log(f(X_1,\cdots,X_n;\theta))$$

If $X_1,\cdots,X_n$ are i.i.d.with density $f(x;\theta)$,then the score function equals to

$$\frac{\partial}{\partial\theta}\log\prod_{i=1}^{n}f(X_i;\theta)=\sum_{i=1}^{n}\frac{\partial}{\partial\theta}\log f(X_i;\theta)$$

Score function

Lemma

Under regularity conditions,

$$E_\theta[S_n(\theta)] = 0$$

Proof The expectation of score function is

$$\begin{aligned}
E_\theta[S_n(\theta)] &= \int \frac{\partial \log(f(x_1,\cdots,x_n;\theta))}{\partial\theta} f(x_1,\cdots,x_n;\theta)dx_1\cdots dx_n \\
&= \int \frac{\frac{\partial}{\partial\theta} f(x_1,\cdots,x_n;\theta)}{f(x_1,\cdots,x_n;\theta)} f(x_1,\cdots,x_n;\theta)dx_1\cdots dx_n \\
&= \int \frac{\partial}{\partial\theta} f(x_1,\cdots,x_n;\theta)dx_1\cdots dx_n \\
&= \frac{\partial}{\partial\theta}\int f(x_1,\cdots,x_n;\theta)dx_1\cdots dx_n = \frac{\partial}{\partial\theta}1 = 0
\end{aligned}$$

Note If θ mismatches,it may not hold.It is possible $E_{\theta_1}[S_n(\theta_2)] \neq 0$

Fisher Information

Fisher Information

Let $X_1, \cdots, X_n$ be with with joint density $f(x_1, x_2, \cdots, x_n; \theta)$. The Fisher information is the variance of the score function,which is

$$I_n(\theta) = \mathbf{Var}_\theta(S_n(\theta)) = E\left\{\left[\frac{\partial}{\partial\theta}\log f(X_1, \cdots, X_n; \theta)\right]^2\right\}$$

If $X_1, \cdots, X_n$ are i.i.d.with density $f(x; \theta)$,then the Fisher information is

$$I_n(\theta) = nI(\theta)$$

where $I(\theta)$ is the Fisher information for single observation.

- $S_n(\theta) = 0$,yet $Var_\theta(S_n(\theta))$ is a function of θ
- **Proof** in i.i.d.case,the score function is $S_n(\theta) = \sum_{i=1}^{n} \log f(X_i; \theta)$

$$I_n(\theta) = Var_\theta(S_n(\theta)) = Var(\sum_{i=1}^{n} \frac{\partial}{\partial\theta}\log f(X_i; \theta))$$

$$= \sum_{i=1}^{n} Var(\frac{\partial}{\partial\theta}\log f(X_i; \theta)) = nVar(\frac{\partial}{\partial\theta}\log f(X_1; \theta)) = nI(\theta)$$

Fisher Information

- According to the Cramer-Rao lower bound,all the unbiased estimator for θ has variance larger than $1/I_n(\theta)$.So $I_n(\theta)$ gives us the bound for the information we can get from the data. That's why we call is as Information.
- Another statement of Cramer-Rao lower bound

Corollary:Unbiased Estimators

Let $X_1, \cdots, X_n$ be i.i.d.samples with density $f(x;\theta)$ and let $W : X^n \to \mathbb{R}$ be an unbiased estimator of $\tau(\theta)$.Suppose the conditions hold,then

$$Var(W;\theta) \geq \frac{\tau'(\theta)^2}{I_n(\theta)} = \frac{\tau'(\theta)^2}{nI(\theta)}$$

- Obviously,the variance will converge to 0 when n increases.
- The best unbiased estimator has convergence rate at $1/\sqrt{n}$.

Fisher Information

Lemma: Fisher Information

Under regularity conditions,

$$I_n(\theta) = E\Big\{[\tfrac{\partial}{\partial\theta}\log f(X_1,\cdots,X_n;\theta)]^2\Big\} = -E[\tfrac{\partial^2}{\partial\theta^2}\log f(X_1,\cdots,X_n;\theta)]$$

Proof. In short,we denote $X = (X_1, X_2, \cdots, X_n)$.For the L.H.S,there is

$$E[[\tfrac{\partial}{\partial\theta}\log f(X;\theta)]^2] = E[\tfrac{1}{(f(X;\theta))^2}(\tfrac{\partial}{\partial\theta}f(X;\theta))^2]$$

For the R.H.S,we have

$$\begin{aligned}
-E\Big[\frac{\partial^2}{\partial\theta^2}\log f(X;\theta)\Big] &= -E\Big[\frac{\partial}{\partial\theta}\frac{1}{f(X;\theta)}\frac{\partial f(X;\theta)}{\partial\theta}\Big] \\
&= E\Big[\frac{1}{(f(X;\theta))^2}[\frac{\partial}{\partial\theta}f(X;\theta)]^2\Big] - E\Big[\frac{1}{f(X;\theta)}\frac{\partial f(X;\theta)}{\partial\theta}\Big] \\
&= E\Big[\frac{1}{(f(X;\theta))^2}[\frac{\partial}{\partial\theta}f(X;\theta)]^2\Big] - \int\frac{\partial^2 f(X;\theta)}{\partial\theta^2}dx \\
&= L.H.S - \frac{\partial^2}{\partial\theta^2}\int f(X;\theta)dx^1 = L.H.S
\end{aligned}$$

[1] true for an exponential family

Example of CRLB-Poisson

Suppose $X_1, \cdots, X_n$ from a iid sample from Poisson distribution,

$$f(x|\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}$$

Find the CRLB for $\hat{\lambda}$.
Solution For the Poisson distribution,

$$l(\lambda) = X \ln \lambda - \lambda - \ln X!$$

$$l'(\lambda) = \frac{X}{\lambda} - 1 \qquad l''(\lambda) = \frac{X}{\lambda^2}$$

$$I(\lambda) = \frac{E[X]}{\lambda^2} = \frac{1}{\lambda}$$

Finally,we have the CRLB $\frac{\lambda}{n}$.
Recall that the MLE for Poisson example is $\bar{X}_n$,with expectation λ and variance $\frac{\lambda}{n}$.So the MLE is UMVUE for Poisson distribution.

- **Example 7.3.12** $\bar{X}$ is UMVUE for λ if $X_1, \cdots, X_n$ are i.i.d. Poisson(λ). From Theorem 7.3.9, we have for any unbiased estimator $W(\mathbf{X})$ of λ.

$$\begin{aligned} \text{Var}_\lambda W(\boldsymbol{X}) &\geq \frac{1}{-nE_\lambda\left[\frac{\partial^2}{\partial\lambda^2}\log f(\boldsymbol{x}|\lambda)\right]} \qquad (4.1) \\ \log f(\boldsymbol{x}|\lambda) &= \log\left[e^{-\lambda}\frac{\lambda^x}{x!}\right] = -\lambda + x\log\lambda - \log x! \\ \frac{\partial^2}{\partial\lambda^2}\log f(\boldsymbol{x}|\lambda) &= -x\frac{1}{\lambda^2}. \end{aligned}$$

Therefore,

$$-E_\lambda\left[\frac{\partial^2}{\partial\lambda^2}\log f(\boldsymbol{x}|\lambda)\right] = \frac{1}{\lambda^2}E_\lambda X = \frac{1}{\lambda}$$

.

(4.1) Becomes $\text{Var}_\lambda(W(\boldsymbol{X})) \geq \frac{\lambda}{n}$.
$\text{Var}_\lambda(\bar{X}) = \frac{\lambda}{n}$, so $\bar{X}$ is UMVUE

- **Example 7.3.13 (Unbiased Estimator for Scale Parameter)** Let $X_1, \cdots, X_n$ be i.i.d. with pdf $f(x|\theta) = \frac{1}{\theta}$, $0 < x < \theta$. Since $\frac{\partial}{\partial\lambda} \log f(x|\theta) = -\frac{1}{\theta}$, we have

$$E_\theta \left[\frac{\partial}{\partial\lambda} \log f(x|\theta) \right] = \frac{1}{\theta^2}$$

So if W is unbiased for θ, then

$$\text{Var}_\theta(W) \geq \frac{\sigma^2}{n}.$$

- On the other hand, $Y = \max(Y_1, \cdots, Y_n)$ is a sufficient statistic. $f_Y(y|\theta) = ny^{n-1}/\theta^n$, $0 < y < \theta$. So

$$E_\theta Y = \int_0^\theta y \cdot \frac{ny^{n-1}}{\theta^n} dy = \frac{n}{n+1}\theta,$$

showing that $\frac{n+1}{n}Y$ is an unbiased estimator of θ.

$$\begin{aligned}
\mathsf{Var}_\theta\left(\frac{n+1}{n}Y\right) &= \left(\frac{n+1}{n}\right)^2 \mathsf{Var}_\theta(Y) \\
&= \left(\frac{n+1}{n}\right)^2 \left[E_\theta Y^2 - (EY)^2\right] \\
&= \left(\frac{n+1}{n}\right)\left[\frac{n}{n+2}\theta^2 - \left(\frac{n}{n+1}\theta\right)^2\right] \\
&= \frac{1}{n(n+2)}\theta^2,
\end{aligned}$$

which is uniformly smaller than θ^2/n.

- Cramér-Rao lower bound Theorem is not applicable to this pdf since

$$\begin{aligned}
\frac{d}{d\theta}\int_0^\theta h(x)f(x|\theta)dx = \frac{d}{d\theta}\int_0^\theta h(x)\frac{1}{\theta}dx
&= \int_0^\theta h(x)\frac{\partial}{\partial\theta}\frac{1}{\theta}dx + \frac{h(\theta)}{\theta} \\
&\neq \int_0^\theta h(x)\frac{\partial}{\partial\theta}f(x|\theta)dx.
\end{aligned}$$

Example of CRLB-Normal

Example

Let $X_1, \cdots, X_n$ be a random sample from the $N(\mu, \sigma^2)$ distribution. Find the CRLB and,in case 1 and 2. check whether it is equalled,for the variance of an biased estimator of

- μ when σ^2 is known,
- σ^2 whenμ is known
- μ when σ^2 is unknown
- σ^2 whenμ is unknown

Example of CRLB-Normal

Solution:The sample joint pdf is

$$f_X(X|\theta) = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} \exp(-\tfrac{1}{2}(x_i-\mu)^2/\sigma^2)$$

and

$$\log f_X(X|\theta) = -\tfrac{n}{2}\log(2\pi) - \tfrac{n}{2}\log(\sigma^2) - \tfrac{1}{2} = \sum_{i=1}^{n}(x_i-\mu)^2/\sigma^2$$

1.When σ^2 is known $\theta = \mu$ and

$$\log f_X(X|\theta) = -\tfrac{n}{2}\log(2\pi) - \tfrac{n}{2}\log(\sigma^2) - \tfrac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^2/\sigma^2$$

$$S(X) = \tfrac{\partial}{\partial\theta}\log f_X(X|\theta) = \sum_{i=1}^{n}(x_i-\mu)^2/\sigma^2 = \tfrac{n}{\sigma^2}[\bar{x}-\theta]$$

where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$,is a n unbiased estimator of $\theta = \mu$ whose variance equals the CRLB and that $\frac{n}{\sigma^2} = I(\theta)$ i.e.CRLB=$\frac{\sigma^2}{n}$.Thus $\bar{X}$ is UMVUE.

Example of CRLB-Normal

2.When μ is known but σ^2 is unknown,$\theta = \sigma^2$ and

$$\log f_X(X|\theta) = -\tfrac{n}{2}\log(2\pi) - \tfrac{n}{2}\log(\theta) - \tfrac{1}{2\theta}\sum_{i=1}^{n}(x_i - \mu)^2/\sigma^2$$

Hence

$$S(X) = \frac{\partial}{\partial\theta}\log f_X(X|\theta) = -\frac{n}{2\theta} + \frac{1}{2\theta}\sum_{i=1}^{n}(x_i - \mu)^2/\sigma^2$$

$$= \frac{n}{2\theta^2}[\frac{1}{2}\sum_{i=1}^{n}(x_i - \mu)^2 - \theta]$$

$\frac{1}{2}\sum_{i=1}^{n}(x_i - \mu)^2$ is an unbiased estimator of $\tau = \sigma^2$ and $\frac{n}{2\theta^2} = I(\theta)$ i.e.the $CRLB = \frac{2\theta^2}{n} = \frac{2\sigma^4}{n}$

Example of CRLB-Normal

3.and 4. Case both μ and σ^2 is unknown
here $\theta = \binom{\mu}{\sigma^2} = \binom{\theta_1}{\theta_2}$ i.e.$\theta_1 = \mu$ and $\theta_2 = \sigma^2$

$$f_X(X|\theta) = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} \exp(-\frac{1}{2}(x_i - \mu)^2/\sigma^2) \quad \propto \theta_2^{-n/2} \exp(\frac{1}{2\theta_2} \sum_{i=1}^{n} (x_i - \theta_1)^2/\sigma$$

and

$$\log f_X(X|\theta) = -\tfrac{n}{2} \log \theta_2 - \tfrac{1}{2\theta_2} \sum_{i=1}^{n} (x_i - \theta_1)^2/\sigma^2$$

Thus

$$= \tfrac{\partial}{\partial \theta} \log f_X(X|\theta) = \tfrac{1}{\theta_2} \sum_{i=1}^{n} (x_i - \theta_1)/\sigma^2$$

$$\tfrac{\partial^2}{\partial \theta^2} \log f_X(X|\theta) = -\tfrac{n}{\theta_2}$$

$$\tfrac{\partial^2}{\partial \theta^2 \theta^1} \log f_X(X|\theta) = -\tfrac{1}{\theta_2^2} \sum_{i=1}^{n} (x_i - \theta_1)$$

$$\tfrac{\partial}{\partial \theta^2} \log f_X(X|\theta) = -\tfrac{n}{2\theta_2} + \tfrac{1}{2\theta_2^2} \sum_{i=1}^{n} (x_i - \theta_1)^2$$

Example of CRLB-Normal

$$\frac{\partial^2}{\partial\theta^2}\log f_X(X|\theta) = \frac{n}{2\theta_2^2} - \frac{1}{\theta_2^3}\sum_{i=1}^{n}(x_i-\theta_1)^2$$

Consequently

$$I_{11}(\theta) = -E(-\frac{n}{\theta_2}) = \frac{n}{\theta_2}$$

$$I_{12}(\theta) = -E(-\frac{1}{\theta_2^2}\sum_{i=1}^{n}(x_i-\theta_1)) = 0$$

$$I_{22}(\theta) = -E(\frac{n}{2\theta_2} - \frac{1}{\theta_2^3}\sum_{i=1}^{n}(x_i-\theta_1)^2) = \frac{n}{2\theta_2^2}$$

Example of CRLB-Normal

i.e

$$I(\theta) = \begin{bmatrix} \frac{n}{\theta_2} & 0 \\ 0 & \frac{n}{2\theta_2^2} \end{bmatrix}$$

and

$$[I(\theta)]^{-1} = J(\theta) = \begin{bmatrix} \frac{\theta_2}{n} & 0 \\ 0 & \frac{2\theta_2^2}{n} \end{bmatrix} = \begin{bmatrix} \frac{\sigma^2}{n} & 0 \\ 0 & \frac{2\sigma^4}{n} \end{bmatrix}$$

Consequently,for unbiased estimators $\hat{\mu}, \hat{\sigma^2}$ of μ and σ^2 respectively

$$Var(\hat{\mu}) \geq \frac{\sigma^2}{n}$$

and

$$Var(\hat{\sigma^2}) \geq \frac{2\sigma^4}{n}$$

The vector case

■ For the normal example,e=we consider $\theta = (\mu, \sigma^2)$,where the unknown parameter is a vector.

■ Let $\theta = (\theta_1, \cdots, \theta_k)$,then the score function is

$$S_n(\theta) = (\frac{\partial}{\partial\theta_1 l(\theta)}, \frac{\partial}{\partial\theta_2 l(\theta)}, \cdots, \frac{\partial}{\partial\theta_k l(\theta)})^T$$

$E[S_n(\theta)] = 0$ still holds.

■ The Fisher information is now a $k \times k$ matrix,actually,the covariance matrix for $S_n(\theta)$,that

$$I_n = E_\theta[S_n(\theta)(S_n(\theta))^T],$$

For the (r, s) element of I_n,there is $I_n(r, s) = -E_\theta[\frac{\partial^2 l(\theta)}{\partial\theta_r \partial\theta_s}]$.So,under regular conditions,I_n equals to the expectation of the Hessian matrix for $-l(\theta)$.

Bias-Variance Tradeoff

- UMVUE has uniformly minimum variance of all unbiased estimators
- Recall the original formula of Cramer-Rao Lower Bound,that

$$Var(W(X_1, X_2, \cdots, X_m); \theta) \geq \frac{(\frac{d}{d\theta}(E_\theta[W(X_1, \cdots, X_m)]))^2}{E_\theta[[\frac{\partial}{\partial\theta}\log(f(X_1, \cdots, X_n; \theta))]^2]}$$

- For the lower bound ,note that the denominator does not changes for whatever $E[W]$ is.The numerator depends on $E_\theta[W]$.
- If W is unbiaesd,then $E_\theta[W] = \tau(\theta)$,so the numerator is always $[\tau'(\theta)]^2$.If W is biased,then $E_\theta[W] = \tau(\theta) + Bias$,which may induce a smaller lower bound.
- Example:take $W = 0$,then $E[W] = 0$ with large bias,but the variance is 0.

- For estimation,we should consider both the bias and the variance.
- How to comnine them?

Description of our problem:what is our goal?

- Goal: We want to estimate θ (or $\tau(\theta)$) with the random sample $X_1, \cdots, X_n$.
- For any estimator $\hat{\theta}$,it differs from θ by $\hat{\theta} - \theta$
- We hope $\hat{\theta} - \theta$ can be small in most cases
- We can evaluate the error by $(\hat{\theta} - \theta)^2$,then the overall loss can be evaluated by

$$E_\theta[(\hat{\theta} - \theta)^2]$$

- It combines the bias and the variance.We call it as Mean Squared Error.

Mean Squared Error

Definition: Mean Squared Error(MSE)

Let $\hat{\theta}$ be an estimator of a parameter θ.The Mean Squared Error(MSE)of $\hat{\theta}$ is

$$E_\theta[(\hat{\theta}-\theta)^2]$$

- According to the definition,
  $MSE(\hat{\theta}) = E_\theta[(\hat{\theta}-\theta)^2] = (E_\theta[\hat{\theta}-\theta])^2 + Var_\theta(\hat{\theta}-\theta) = Bias^2 + Var_\theta(\hat{\theta})$
  MSE combines the variance and the bias of one estimator.
- When $\hat{\theta}$ is unbiased,then $Bias = 0$ and its MSE equals to its variance,which is bounded by CRLB.
- Given the estimator $\hat{\theta}$,the MSE is a function of θ.
- Obviously,for any $\tau(\theta)$,and an estimator W of $\tau(\theta)$,we can also define the MSE as $E[(W-\tau(\theta))^2]$.

- Example 7.3.3 (Normal MSE) Let $X_1, X_2, \cdots, X_n$ be i.i.d. $N(\mu, \sigma^2)$. Then statistics $\bar{X}$ and S^2 are both unbiased.

$$\begin{aligned} MSE(\bar{X}) &= E(\bar{X}-\mu)^2 = \text{Var}(\bar{X}) = \sigma^2/n \\ E(S^2-\sigma^2)^2 &= \text{Var}(S^2) = \frac{2\sigma^4}{n-1}[2] \end{aligned}$$

[2]
$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1) \Rightarrow \text{Var}\left(\frac{(n-1)S^2}{\sigma^2}\right) = 2(n-1)$$

- Example 7.3.4 Maximum Likelihood estimator of σ^2 is $\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 = \frac{n-1}{n}S^2$.

$$\begin{aligned}
\text{Var}\left(\frac{n-1}{n}S^2\right) &= \frac{(n-1)^2}{n^2}\cdot\frac{2\sigma^4}{n-1} = \frac{2(n-1)}{n^2}\sigma^4 \\
MSE\left(\frac{n-1}{n}S^2\right) &= \left(\frac{n-1}{n}ES^2 - \sigma^2\right)^2 + \frac{2(n-1)}{n^2}\sigma^4 \\
&= \sigma^4\left(\frac{n-1}{n} - 1\right)^2 + \frac{2(n-1)}{n^2}\sigma^4 \\
&= \sigma^4\frac{2n-1}{n^2}
\end{aligned}$$

Since

$$\frac{2n-1}{n^2} < \frac{2}{n-1},$$

So in this case MLE has smaller MSE than the unbiased estimator S^2.

Remark While MSE is a reasonable measurement for location parameters, it may not be a good to compare estimators of scale parameters with MSE.

Example 7.3.5 (MSE of binomial Bayes Estimator) $X_1, \cdots, X_n \sim$ Bernoulli(p).

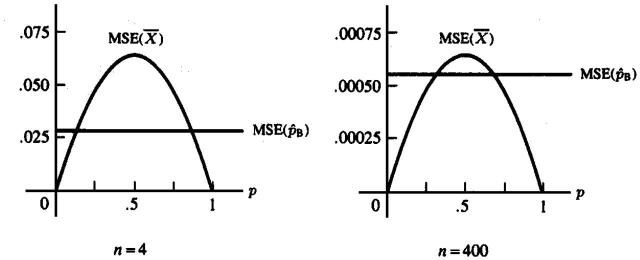
- Let $\hat{p} = \frac{X_1 + \cdots + X_n}{n}$. $E_p(\hat{p} - p)^2 = \text{Var}_p(\bar{X}) = \frac{p(1-p)}{n}$.
- Let $\hat{p}_B = \frac{Y+\alpha}{\alpha+\beta+n}$ be the Bayes estimator. Here $Y = \sum_{i=1}^n X_i$

$$
\begin{aligned}
MSE(\hat{p}) &= \text{Var}_p(\hat{p}_B) + (\text{Bias}_p(\hat{p}_B))^2 \\
&= \text{Var}\left(\frac{Y+\alpha}{\alpha+\beta+n}\right) + \left(E_p\left(\frac{Y+\alpha}{\alpha+\beta+n}\right) - p\right)^2 \\
&= \frac{np(1-p)}{(\alpha+\beta+n)^2} + \left(\frac{np+\alpha}{\alpha+\beta+n} - p\right)^2
\end{aligned}
$$

In the absence of good prior information about p, we might choose α and β to make the MSE of $\hat{p}_B$ constant. Choose $\alpha = \beta = \sqrt{n/4}$ gives

$$
\hat{p}_B = \frac{Y + \sqrt{n/4}}{n + \sqrt{n}}, \; E(\hat{p}_B - p)^2 = \frac{n}{4(n+\sqrt{n})^2}
$$

Figure 7.3.1 Comparison of $MSE(\hat{p})$ and $MSE(\hat{p}_B)$ for sample size $n = 4$ and $n = 400$ in Example 7.3.5



- For small n, $\hat{p}_B$ is the better choice (unless there is a strong belief that p is near 0 or 1)
- For large n, $\hat{p}$ is the better choice (unless there is a strong belief that p is close to $\frac{1}{2}$)