Lecture 1: Review of Basic Probability

Ma Xuejun

School of Mathematical Sciences

Soochow University

https://xuejunma.github.io

2020.09.15

We would like to thank Professor Tian Maozai and Professor Wang Wanjie for sending slides and beneficial discussions.

Outline

- Syllabus
- Brief review of basic probability and statistics
  - Why is a random variable?
  - Transformations; independence; expectation
  - Important distributions
  - Some statistics

Terms

- Sample space; Measure; Random variable
- Transformation; Independence; Expectation; Conditional expectation; Variance & Standard deviation; Moment Generating Function; Characteristic function
- Common distributions
- Sample mean; Sample variance; Sample distribution
- Moment inequalities

Sample space and Measure

What do we mean by `randomness`?

- We construct an experiment, yet the result of the experiment has many possibilities.
    - Flip a coin, the result can be either head or tail
- Although we can not know the result beforehand, we do have some information about the result.
    - Approximately, there is equal chance for a head and a tail
- Randomness: the uncertainty of experiment results

Question: How to describe our information?

Sample space and Measure

- Information 1. Possible outcomes

Definition 1.1.1: Sample space (Outcome space)

Let Ω be a sample space, which is a set containing all possible outcomes.

- Information 2. Probabilities for these possible outcomes
  - σ-field $\mathcal{F}$: a set of subsets of Ω which satisfies 3 rules.
    - Measurable space: $(\Omega, \mathcal{F})$
    - Event (measurable sets): element of $\mathcal{F}$
  - Probability measure P: for any element in the σ-field, assign it a probability, indicating the chance this event will happen
- $(\Omega, \mathcal{F}, P)$ (Probability space, measure space)is our information about the possible outcomes of this experiment. In short, we write it as the sample space Ω with probability P, or just Ω if there is no confusion.

σ-field

Let $\mathcal{F}$ be a collection of subsets of a sample space. $\mathcal{F}$ is called a σ-field (or σ-algebra) if and only if it has the following properties.

1. The empty set $\phi \in \mathcal{F}$
2. If $A \in \mathcal{F}$, then the complement $A^c \in \mathcal{F}$.
3. If $A_i \in \mathcal{F}, i = 1, 2, \ldots$, then their union $\cup A_i \in \mathcal{F}$.

- Measurable space: $(\Omega, \mathcal{F})$
- Event (measurable sets): element of $\mathcal{F}$
- $\sigma(A) = \{\phi, A, A^c, \Omega\}$.
- Flip a coin, the result can be either head or tail $\Omega = \{H, T\}$, $\mathcal{F} = \{\ldots\}$

Measure

Let Measurable space $(\Omega, \mathcal{F})$, A be a measurable space. A set function ν defined on $\mathcal{F}$ is called a measure if and only if it has the following properties.

1. $0 \leq \nu(A) \leq \infty$, for any $A \in \mathcal{F}$
2. $\nu(\varnothing) = 0$
3. If $A_i \in \mathcal{F}, i = 1, 2, \ldots$, and A_i's are disjoint, i.e. $A_i \cap A_j = \varnothing$ for any $i \neq j$, then $\nu\Big(\cup_{i=1}^{\infty} A_i \Big) = \sum_{i=1}^{\infty} \nu(A_i)$

- measure space: $(\Omega, \mathcal{F}, \nu)$
- probability measure $\nu(\Omega) = 1$. We usually denote it by P instead of ν, $(\Omega, \mathcal{F}, P)$.
- Flip a coin, the result can be either head or tail $\Omega = \{H, T\}$, $\mathcal{F} = \{\ldots\}$
  - $\nu(A) = |A|$ the number of elements in $A \in (F)$.
  - $$P(A) = \frac{|A|}{|\Omega|}$$

Random Variables

What is of interest?

- Manufacturers Ω: all the combinations of good light bulbs and defective light bulbs. Need: proportion of defective light bulbs from a lot
- Market researchers Ω: survey results of all consumers for one product. Need: preference of all consumers about this product, with a scale 1-10.

Our interest:

- Not the details of Ω, but a special measurable characteristic of the outcomes!
- A random variable, is a mapping from Ω to R, which draws the measurable characteristic of interest

Example: an opinion poll. 50 people; 1: agree; 0 disagree:

- Ω has 2^{50} elements.
- interest: the number of people who agree out of 50. $X =$ number of 1s recorded out of 50. $\mathcal{X} = \{0, 1, 2, \ldots, 50\}$

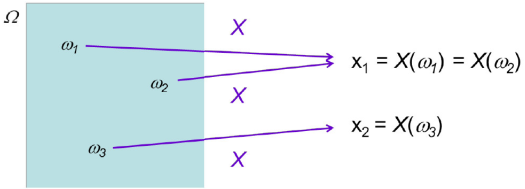
Random Variable

Random Variable

- Let $(\Omega, \mathcal{F})$ and $(R, \mathcal{B})$ ($\mathcal{B}$: Borel σ-field)be measurable spaces
- X is a function from Ω to R. The function X is called a `random variable` (r.v.; measurable function) if and only if

$$X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} \subset \mathcal{F}$$

for any $B \subset \mathcal{B}$.



Random Variable

- Suppose we have a sample space

  $$\Omega = \{\omega_1, \ldots, \omega_n\}$$

  with a probability function P.
- We defined a random variable X with range $\mathcal{X} = \{x_1, \ldots, x_m\}$.
- We write

  $$P_X(X = x_i) = P(\{\omega_j \in \Omega : X(\omega_j) = x_i\})$$
  $$P_X(X \in B) = P(\{\omega \in \Omega : X(\omega) \in B\})$$

  where P_X is an `induced` probability function $\mathcal{X}$.
- Notations:
  - Upper-case letters $X, Y, Z \ldots$ to denote r.v.'s
  - Lower-case letters $x, y, z \ldots$ to denote their possible values.

Example: Random variable

Example 1.4.3

- Consider the experiment of tossing a coin three times.
- H : Head; T : Tail.
- X : the number of heads obtained in the three tosses.

ω	HHH	HHT	HTH	THH	TTH	THT	HTT	TTT
$X(\omega)$	3	2	2	2	1	1	1	0

- $\mathcal{X} = \{0, 1, 2, 3\}$. The induced probability function on $\mathcal{X}$ is given by

x	0	1	2	3
$P_X(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$P_X(X = 1) = P(\{HTT, THT, TTH\}) = \frac{3}{8}.$$

Cumulative Density Function

Definition 1.5.1 Cumulative Density Function

The cumulative distribution function (CDF) of a random variable is defined by

$$F(x) = P(X \leq x); -\infty < x < \infty$$

For all CDF's: there is

1. $F(x)$ is right-continuous. At each x, $\lim_{n\to\infty} F(y_n) = F(x)$ for any sequence $y_n \to x$ with $y_n > x$.
2. $F(x)$ is non-decreasing.
3. $\lim_{x\to-\infty} F(x) = 0, \lim_{x\to\infty} F(x) = 1.$

Any $F(x)$ satisfying Properties 1-3 is a CDF for some random variable.

Example: Logistic distribution.

Example 1.5.5

$$F_X(x) = \frac{1}{1+e^{-x}}$$

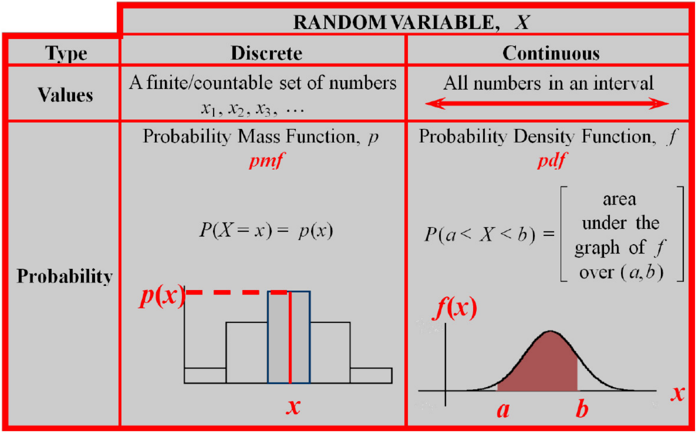
- $\lim_{x\to-\infty} F_X(x) = 0$
- $\lim_{x\to\infty} F_X(x) = 1$
- $$f_X(x) = \frac{d}{dx}F_X(x) = \frac{e^{-x}}{(1+e^{-x})^2} > 0$$

Discrete v.s. Continuous r.v.

- If X is discrete, then its `probability mass function (pmf)` is
$$p_X(x) = p(x) = P(X = x)$$

- If X is continuous, then its `probability density function (pdf)` satisfies
$$P(X \in A) = \int_A f_X(x)dx = \int_A f(x)dx = \int_A dF(x)$$
and $f_X(x) = f(x) = F'(x)$.

- We say that X and Y have the same distribution(i.e. $X \overset{D}{=} Y$) if $P(X \in A) = P(Y \in A)$ for all A. $X \overset{D}{=} Y$) if only if $F_X(t) = F_Y(t)$



	RANDOM VARIABLE, X	
Type	**Discrete**	**Continuous**
Values	A finite/countable set of numbers $x_1, x_2, x_3, \ldots$	All numbers in an interval ↔
Probability	Probability Mass Function, p *pmf* $P(X = x) = p(x)$	Probability Density Function, f *pdf* $P(a < X < b) = \begin{bmatrix} \text{area} \\ \text{under the} \\ \text{graph of } f \\ \text{over } (a,b) \end{bmatrix}$

Transformation

Given a r.v. X with density function $f_X(\cdot)$, it is often that we are interested in a transformation $Y = g(X)$ which is defined as a known function g (either one-to-one or many-to-one) of X.

- Obviously, the composite function $g \circ X$ defines a new r.v. Y from Ω to R.
- Let $Y = g(X)$.

$$\begin{aligned} P(Y \in A) &= P(g(X) \in A) \\ &= P(X \in g^{-1}(A)) \end{aligned}$$

where $g^{-1}(A) = \{x \in R, g(x) \in A\}$. In particular,

$$F_Y(y) = \Pr\{Y \in y\} = P(X \in g^{-1}(-\infty, y])$$

If X has pdf $f_X(x)$, then

$$F_Y(y) = \int_{g^{-1}(-\infty,y]} f_X(x)dx = \int_{\{x:g(x)\leq y\}} f_X(x)dx$$

Example 2.1.2

Suppose X has a uniform distribution on the interval $(0, 2\pi)$, that is

$$f_X(x) = \begin{cases} 1/2\pi, & 0 < x < 2\pi, \\ 0, & \text{otherwise.} \end{cases}$$

Consider $Y = \sin^2(X)$

$$\begin{aligned} P(Y \leq y) &= P(X \leq x_1) + P(x_2 \leq X \leq x_3) + P(X \geq x_4) \\ &= 2P(X \leq x_1) + 2P(x_2 \leq X \leq \pi) \end{aligned}$$

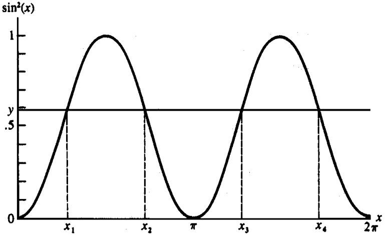


Figure 2.1.1. *Graph of the transformation* $y = \sin^2(x)$ *of Example 2.1.2*

- If g is increasing,

$$F_Y(y) = F_X(g^{-1}(y)).$$

- If g is decreasing,

$$F_Y(y) = 1 - F_X(g^{-1}(y)).$$

Theorem 2.1.5

Let X have probability distribution function (pdf) $f_X(x)$ and $Y = g(X)$, where g is a monotone function. Let

$$\mathcal{Y} = \{y : g^{-1}(y) \text{is a possible value of} X\}.$$

Suppose $f_X(x)$ is continuous and that $g^{-1}(y)$ has a continuous derivative on $\mathcal{Y}$. Then the pdf of Y is given by

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|, & y \in \mathcal{Y}, \\ 0, & \text{otherwise}. \end{cases}$$

Example 2.1.4

$X \sim f_X(x) = 1I(0 < x < 1)$, $F_X(x) = x$. $Y = g(x) = -\log x$, find its distribution.

Proof:

- $Y = g(x) = -\log x \Longrightarrow x = e^{-y}, g^{-1}(y) = e^{-y}$
- g is a decreasing function.

$$\frac{d}{dx} g(x) = \frac{d}{dx}(-\log x) = \frac{-1}{x} < 0, \;\; 0 < x < 1$$

-

$$\begin{aligned} F_Y(y) &= P_Y(Y \leq y) = P_X(g(X) \leq y) \\ &= P_X(X \geq g^{-1}(y)) \\ &= 1 - P_X(X \leq g^{-1}(y)) = 1 - e^{-y} \end{aligned}$$

Example 2.1.6

Let

$$f_X(x) = \frac{1}{(n-1)!\beta^n} x^{n-1} e^{-x/\beta}, \quad 0 < x < \infty$$

be the Gamma pdf $Y = 1/X$. Find the pdf of Y

Proof. $g^{-1}(y) = 1/y, \mathcal{Y} = (0, \infty)$, $\left|\frac{d}{dy} g^{-1}(y)\right| = 1/y^2$. Therefore for all $y > 0$,

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y)) \left|\frac{d}{dy} g^{-1}(y)\right| \\ &= \frac{1}{(n-1)!\beta^n} \left(\frac{1}{y}\right)^{n-1} e^{-1/(\beta y)} \frac{1}{y^2} \\ &= \frac{1}{(n-1)!\beta^n} \left(\frac{1}{y}\right)^{n+1} e^{-1/(\beta y)} \end{aligned}$$

- A special case of a pdf known as the inverted Gamma distribution.

Theorem 2.1.8

Let X have pdf $f_X(x)$, Let $Y = g(X)$. Suppose there exists a partition $A_0, A_1, \cdots, A_k$ such that $P(X \in A_0) = 0$ and $f_X(x)$ is continuous on each A_i.

$$P(X \in \bigcup_{i=1}^{k} A_i) = 1.$$

Further, we have $g(\cdot)$ is monotone if restricted to A_i $i = 1, 2, \cdots, k$. Let

$$g_i^{-1}(y) = \{x \in A_i : g(x) = y\}$$

and assume $g_i^{-1}(y)$ has continuous derivative on $\mathcal{Y}$ for each i. Then

$$f_Y(y) = \begin{cases} \sum_{i=1}^{k} f_X(g_i^{-1}(y))|\frac{d}{dy}g_i^{-1}(y)|, & y \in \mathcal{Y} \\ 0, & \text{otherwise} \end{cases}$$

- **Remark** Unfortunately, I found the above Theorem has very little practical use.

Example 2.1.9

Let $X \sim N(0,1)$, $Y = X^2$. we may use the above theorem to find the pdf of Y.

Proof:

- $g(x) = x^2$ is monotone on $(-\infty, 0)$ and on $(0, \infty)$.
- $\mathcal{Y} = (0, \infty)$.

$$\begin{aligned}
&A_0 = \{0\} \\
&A_1 = (-\infty, 0), \;\; g_1(x) = x^2, \;\; g_1^{-1}(y) = -\sqrt{y} \\
&A_2 = (0, \infty,), \;\; g_2(x) = x^2, \;\; g_1^{-1}(y) = \sqrt{y}
\end{aligned}$$

The pdf Y is

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \Phi(\sqrt{y}) \frac{1}{2} \frac{1}{\sqrt{y}} + \Phi(-\sqrt{y}) \frac{1}{2} \frac{1}{\sqrt{y}} = \frac{1}{\sqrt{y}} \Phi(\sqrt{y})$$

Probability integral transform

Theorem 2.1.10 Probability integral transform

Let X have continuous cdf $F_X(x)$ and define the random variable Y as $Y = F_X(X)$. Then Y is uniformly distributed on $(0, 1)$, that id

$$P(Y \leq y) = y, \;\; 0 < y < 1.$$

- $F_X^{-1}(\tau) = \inf\{x : F_X(x) \geq \tau\}$
- Proof:

$$\begin{aligned} P_Y(Y \leq y) &= P_X(F_X(x) \leq y) \\ &= P_X(F_X^{-1}[F_X(x)] \leq F_X^{-1}(y)) \\ &= P_X(X \leq F_X^{-1}(y)) \\ &= F_X(F_X^{-1}(y)) \\ &= y \end{aligned}$$

Independence

Theorem 4.2.10

Two r.v.'s X and Y are independent if and only if

$$P(X \in A; Y \in B) = P(X \in A)P(Y \in B)$$

for all A and B.

- $F(x, y) = F(x)F(y)$ for any x and y
  $f(x, y) = f(x)f(y)$ or $p(x, y) = p(x)p(y)$
- When X and Y are independent, $h(X)$ and $g(Y)$ are also independent, if h and g are well-defined functions.

Expectation

- Definition:

$$\begin{aligned}\mathbf{E}(X) &= \sum_x xp(x) \\ \mathbf{E}(X) &= \int_{-\infty}^{\infty} xf(x)dx\end{aligned}$$

- Properties:
  - $\mathbf{E}(g(X)) = \int_{-\infty}^{\infty} g(x)f(x)dx$
  - $\mathbf{E}(g(X,Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f(x,y)dxdy$
  - If $X_1, \ldots, X_n$ are independent, then

$$\mathbf{E}\left(\prod_{i=1}^{n} X_i\right) = \prod_{i=1}^{n} \mathbf{E}(X_i)$$

- Example 2.2.2

  $X \sim \exp(\lambda)$,

  $$f_X(x) = \frac{1}{\lambda} e^{-x/\lambda} \quad x > 0.$$

  Then $\mathbf{E}[X] = \lambda$.

- Example 2.2.3

  $X \sim Binomial(n, p)$,

  $$f_X(x) = \binom{n}{x} p^x (1-p)^{n-x} \quad x = 0, 1, \cdots .$$

  Then $\mathbf{E}[X] = np$.

- Example 2.2.4

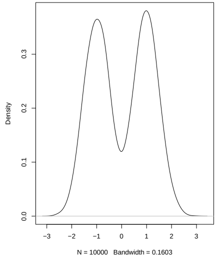
  $X \sim$ Cauchy,

  $$f_X(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2} \quad -\infty < x < \infty.$$

  Then $\mathbf{E}[X]$ is not defindend! (or do not exist).

- Mixed normal distribution

$$X = 0.5N(-1, 0.5^2) + 0.5N(1, 0.5^2)$$



Mixed normal distribution

```
rm(list=ls())
n <- 10000
comp <- sample(c(0, 1), size = n, prob = c(0.5, 0.5),
               replace = T)
x <- rnorm(n, mean = ifelse(comp == 0, -1, 1),
           sd = ifelse(comp == 0, 0.5, 0.5))
plot(density(x), main="")
```

- Theorem 2.2.5 Let X be a r.v. and let a, b, and c be constants. Then for any functions $g_1(x)$ and $g_2(x)$ whose expectations exist.
  (a) $\mathbf{E}[ag_1(X) + bg_2(X) + c] = a\mathbf{E}[g_1(X)] + b[g_2(X)] + c$
  (b) If $g_1(x) \geq 0$ for all x, then $\mathbf{E}[g_1(X)] \geq 0$.
  (c) If $g_1(x) \geq g_2(x)$ for all x, then $\mathbf{E}[g_1(X)] \geq \mathbf{E}[g_2(X)]$
  (d) If $a \leq g_1(x) \leq b$ for all x, then $a \leq \mathbf{E}[g_1(x)] \leq b$

- Example 2.2.6
  $E(X)$ is the "center" of a distribution (or its r,v,) in the sense that

$$\min_b \mathbf{E}(X-b)^2 = \mathbf{E}[X - \mathbf{E}X]^2.$$

- **Homework**

$$\min_b \mathbf{E}\rho_\tau(X-b)$$

Remark: $\rho_\tau(t) = \tau t I(t \geq 0) + (\tau - 1)tI(t < 0)$.

Variance & Standard Deviation

- Motivation: Describe the ”spread” of r.v.
- Definition. $Var(x) = \mathbf{E}[(x-\mu)^2]$, where $\mu = \mathbf{E}(X)$, $sd(X) = \sqrt{Var(x)}$.
- Properties.
  - $Var(X) = \mathbf{E}(X^2) - [\mathbf{E}(X)]^2$
  - If $X_1, \ldots, X_n$ are independent, then

    $$Var\Big(\sum_{i=1}^{n}\Big) = \sum_{i=1}^{n} Var(X_i)$$

  - The covariance is

    $$\mathbf{Cov}(X,Y) = \mathbf{E}[(X - \mathbf{E}(X))(Y - \mathbf{E}(Y))] = \mathbf{E}(XY) - \mathbf{E}(X)\mathbf{E}(Y)$$

    and the correlation coefficient is

    $$Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$$

  - For any two r.v.s with variance existed,

    $$Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$$

Conditional Expectation

- Conditional Expectation of X when Y is given as y is that
  - $\mathbf{E}(X|Y=y) = \sum_x x p_{X|Y}(X|Y)$ for discrete r.v.
  - $\mathbf{E}(X|Y=y) = \int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx$ for cont. r.v.
  - Interpretation: Note that $X|Y=y$ is a new r.v., $\mathbf{E}(X|Y=y)$ is the expectation on this r.v.
- Law of Total Expectation

$$\mathbf{E}\Big[\mathbf{E}(X|Y)\Big] = \mathbf{E}(X)$$

- Law of Total Variance

$$Var(X) = Var\Big[\mathbf{E}(X|Y)\Big] + \mathbf{E}\Big[Var(X|Y)\Big]$$

Theorem 4.4.3

If X and Y are any two r.vs, then

$$\mathbf{E}(X) = \mathbf{E}\Big[\mathbf{E}(X|Y)\Big]$$

Proof:

$$\begin{aligned}
\mathbf{E}X &= \int\int xf(x,y)dxdy \\
&= \int\Big[\int xf(x|y)dx\Big]f_Y(y)dy \\
&= \int \mathbf{E}(X|y)f_Y(y)dy = \mathbf{E}\Big[\mathbf{E}(X|Y)\Big]
\end{aligned}$$

In general, the conditional expectation $\mathbf{E}[X|Y]$ can by defined as a r.v. $g(Y)$ such that

$$\mathbf{E}[(X-g(Y))^2] = \inf_{\text{among all reasonable function } h} \mathbf{E}[(X-h(Y))^2]$$

or $\mathbf{E}[X|Y]$ is the function of Y which is "closest" to X in terms of mean square error.

Example 4.4.1 Hierarchical Model

$Y \sim$ Number of eggs lay by a mother fish, and $X \sim$ Number of survivors (young fish). On the average, how many eggs will survive?

Then it is reasonable to assume

$$\begin{aligned} Y &\sim Poisson(\lambda) \\ X|Y &\sim Binomial(Y, p) \end{aligned}$$

So,

$$\begin{aligned} \mathbf{E}X &= \mathbf{E}\Big[\mathbf{E}(X|Y)\Big] \\ &= \mathbf{E}(pY) \\ &= p\lambda \end{aligned}$$

Example 4.4.5

$$\begin{aligned} X|Y &\sim Binomial(Y,p) \\ Y|\Lambda &\sim Poisson(\Lambda) \\ \Lambda &\sim exponential(\beta) \end{aligned}$$

Proof:

$$\begin{aligned} \mathbf{E}[X] &= \mathbf{E}[\mathbf{E}(X|Y)] \\ &= p\mathbf{E}[Y] \\ &= p\mathbf{E}[\mathbf{E}(Y|\Lambda)] \\ &= p\mathbf{E}[\Lambda] \\ &= p\beta. \end{aligned}$$

Theorem 4.4.7

For any two random variables X and Y,

$$Var(X) = E[Var(X|Y)] + Var[E(X|Y)]$$

provided that the expectation exist.

Proof:

$$\begin{aligned} Var(X) &= \mathbf{E}\Big\{ [X - \mathbf{E}(X|Y) + \mathbf{E}(X|Y) - \mathbf{E}X]^2 \Big\} \\ &= \mathbf{E}\Big\{ [X - \mathbf{E}(X|Y)]^2 + [\mathbf{E}(X|Y) - \mathbf{E}X]^2 \\ &\qquad + 2[X - \mathbf{E}(X|Y)][\mathbf{E}(X|Y) - \mathbf{E}X] \Big\} \\ &= \mathbf{E}\{[X - \mathbf{E}(X|Y)]^2\} + \mathbf{E}\{[\mathbf{E}(X|Y) - \mathbf{E}X]^2\} \\ &= \mathbf{E}[Var(X|Y)] + Var[(\mathbf{E}X|Y)] \end{aligned}$$

Moment Generating Function and Characteristic Function

- Moment Generating Function (MGF)
  - Definition: $M_X(t) = E(e^{tX})$: a function of t, not r.v.
  - If $Y = aX + b$, $M_Y(t) = e^{bt}M_X(at)$
  - If X and Y are independent, then $M_{X+Y}(t) = M_X(t)M_Y(t)$
- Characteristic Function
  - Definition: $\phi_X(t) = E[e^{itX}]$: a function of t; $i = \sqrt{-1}$.
  - Bounded: $\phi(t)| \leq 1$
  - If X and Y are independent, then $\phi_{X+Y}(t) = \phi_X(t)\phi_Y(t)$.

An example of two distribution functions but with the same moments.

Example 2.3.10

Consider the two pdfs given by

$$f_1(x) = \frac{1}{\sqrt{2\pi}x} e^{-(\log x)^2/2}, \quad 0 \leq x < \infty,$$

$$f_2(x) = f_1(x)[1 + \sin(2\pi \log x)], \quad 0 \leq x < \infty,$$

Then it can be shown if $X_1 \sim f_1(x)$,

$$\mathbf{E}[X_1^r] = e^{r^2/2}, \quad r = 0, 1, \cdots ,$$

Now suppose that $X_2 \sim f_2(x)$, we have for $r = 0, 1, \cdots$

$$\begin{aligned} \mathbf{E}[X_2^r] &= \int_0^\infty x^r f_1(x)[1 + \sin(2\pi \log x)]dx \\ &= \mathbf{E}[X_1^r] + \int_0^\infty x^r f_1(x) \sin(2\pi \log x)dx \end{aligned}$$

$$\begin{aligned}
&\int_0^\infty x^r \frac{1}{\sqrt{2\pi}x} e^{-(\log x)^2/2} \sin(2\pi \log x) dx \quad {\color{blue} y = \log x - r} \\
&= \int_{-\infty}^{\infty} e^{(y+r)r} \frac{1}{\sqrt{2\pi}} e^{-(y+r)^2/2} \sin(2\pi(y+r)) dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(y^2-r^2)} \sin(2\pi y) dy \cdot \cos(2\pi r) \\
&\quad + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(y^2-r^2)} \cos(2\pi y) dy \cdot \sin(2\pi r) \\
&= 0 \;\; r = 0, 1, \ldots.
\end{aligned}$$

since $e^{-\frac{1}{2}(y^2-r^2)} \sin(2\pi y)$ is an odd function.[1].

[1] $\sin(A+B) = \sin A \cos B + \sin B \cos A$

However, we have the following theorem.

Theorem 2.3.11

Let $F_X(x)$ and $F_Y(y)$ be two cdfs all of whose moments exist.

(a) If F_X and F_Y have bounded support , then $F_X(u) = F_Y(u)$ for all u iff $EX^r = EY^r$ for all $r = 0, 1, 2, \cdots$

(b) If the moment generating functions exist and $M_X(t) = M_Y(t)$ for all t in some neighborhood of 0 ,then $F_X(u) = F_Y(u)$ for all u.

Differentiating Under An Integral Sign

If a, b are finite and $f(x,\theta)$ is differentiable with respect to θ. Then we have

$$\frac{d}{d\theta}\int_a^b f(x,\theta)dx = \int_a^b \frac{\partial}{\partial\theta} f(x,\theta)dx.$$

But in statistics, we often need to evaluate $\frac{d}{d\theta}\int_{-\infty}^{\infty} f(x,\theta)dx$, which may or may not be $\int_{-\infty}^{\infty}\frac{\partial}{\partial\theta}f(x,\theta)dx$.

Theorem 2.4.2

Suppose the function $h(x,y)$ is continuous at y_0 for each x, and there exists a function $g(x)$ satisfying

a) $|h(x,y)| \le g(x)$, for all x and y;

b) $\int_{-\infty}^{\infty} g(x)dx < \infty$.

Then

$$\lim_{y\to y_0}\int_{-\infty}^{\infty} h(x,y)dx = \int_{-\infty}^{\infty}\lim_{y\to y_0} h(x,y)dx.$$

Apply the above Theorem to the differentiation case, then we have

- Theorem 2.4.3 Suppose $f(x,\theta)$ is differentiable at $\theta = \theta_0$, and there exists a function $g(x,\theta_0)$ and a constant $\delta > 0$ such that
  a) $\left|\frac{f(x,\theta_0+\triangle)-f(x,\theta_0)}{\triangle}\right| \leq g(x,\theta_0)$, for all x and $|\triangle| \leq \delta$;
  b) $\int_{-\infty}^{\infty} g(x,\theta_0)dx < \infty$.
  Then

$$\frac{d}{d\theta}\int_{-\infty}^{\infty} f(x,\theta)dx \mid_{\theta=\theta_0} = \int_{-\infty}^{\infty}\left[\frac{\partial}{\partial\theta}f(x,\theta)\mid_{\theta=\theta_0}\right]dx \quad (*)$$

- Corollary Suppose that there exists $\delta > 0$ and function $g(x,\theta)$ such that $\left|\frac{\partial}{\partial\theta}f(x,\theta)\mid_{\theta=\theta'}\right| \leq g(x,\theta)$, for all θ' with $|\theta'-\theta| < \delta$, and $\int_{-\infty}^{\infty} g(x,\theta)dx < \infty$. Then (*) holds.

- Remark Finding bound $g(x, \theta)$ is cumbersome. We need to know that differentiating under the integral sign is not always automatic. In most situations, we just do it!!
- Example 2.4.6 $X \sim N(\mu, 1)$,

$$M_X(t) = E(e^{tX}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-(x-\mu)^2/2} dx,$$

$$\frac{d}{dt} M_X(t) = \frac{d}{dt} E(e^{tX}) = E(\frac{\partial}{\partial t} e^{tX}) = E(X e^{tX}).$$

For the exchange of operation of differentiation and summation, we have

- Theorem 2.4.8 Suppose that the series $\sum_{x=0}^{\infty} h(\theta, x)$ converges for all θ in an interval (a, b) and

  a) $\frac{\partial}{\partial\theta} h(\theta, x)$ is continuous in θ for each x;

  b) $\sum_{x=0}^{\infty} \frac{\partial}{\partial\theta} h(\theta, x)$ *converges uniformly on every closed bounded subinterval of* (a, b).

  Then

$$\frac{d}{d\theta} \sum_{x=0}^{\infty} h(\theta, x) = \sum_{x=0}^{\infty} \frac{\partial}{\partial\theta} h(\theta, x).$$

- Theorem 2.4.10 Suppose that the series $\sum_{x=0}^{\infty} h(\theta, x)$ converges uniformly on $[a, b]$ and that, for each x, $h(\theta, x)$ is a continuous function of θ. Then

$$\int_a^b \sum_{x=0}^{\infty} h(\theta, x) d\theta = \sum_{x=0}^{\infty} \int_a^b h(\theta, x) d\theta.$$

Important Distributions

- Discrete distributions:
  - Bernoulli r.v.: $X \sim Bernoulli(p)$, $p(1) = p$, $p(0) = 1 - p$, $p(x) = 0$ if $x \neq 0$ and $x \neq 1$. It can also be written as $p^x(1-p)^{1-x}$ for $x = 0, 1$.
  - Binomial r.v.: $X \sim Binomial(n; p)$, $p(x) = \binom{n}{x} p^x q^{n-x}, x = 0, 1, 2, \dots, n$. Summation of n Bernoulli random variables.
  - Poisson r.v.: $X \sim Pois(\lambda)$, $p(x) = \frac{\lambda^x}{x!} e^{-\lambda}$, $x = 0, 1, 2, \dots, n$.
- Continuous distributions:
  - Uniform r.v.: $X \sim Unif(a, b), f(x) = \frac{1}{b-a}, x \in (a, b)$
  - Exponential r.v.: $X \sim Exp(\lambda), f(x) = \lambda e^{-\lambda x}$
  - Normal r.v.: $X \sim N(\mu, \sigma^2), f(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp(-\frac{(x-\mu)^2}{2\sigma^2})$

Multivariate Normal Distribution

- The d random vector $X \sim N(\mu, \Sigma)$,

$$f(x) = \frac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu)^\top \Sigma^{-1}(x-\mu)\right)$$

- $AX + b \sim N(A\mu + b, A\Sigma^{-1}A^\top)$
- Conditional distribution.

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \quad \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

then

$$X_1|Y_2 \sim N\Big(\mu_1 + \Sigma_{11}\Sigma_{22}^{-1}(Y_2 - \mu_2), \Sigma_{11} - \sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\Big)$$

Statistics

- Sample mean $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$
- Sample variance: $S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$
- Sampling distribution of $\bar{X}_n$: $G_n(t) = P(\bar{X}_n \leq t)$

When it is normal:

- If $X \sim N(\mu, \Sigma^2$, then $\bar{X}_n$ and S_n^2 are independent,

$$\bar{X}_n \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

$$\frac{(n-1)S_n^2}{\sigma^2} \sim \chi^2_{n-1}$$

Moment inequalities

Lemma 4.7.1

Let a and b be any two positive numbers, and let p and q be any positive numbers satisfying

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then

$$\frac{1}{p}a^p + \frac{1}{q}b^q \geq ab$$

with equality holds if and only if $a^p = b^q$.

- Proof: Consider for fixed b(*or* a),

$$g(a) = \frac{1}{p}a^p + \frac{1}{q}b^q - ab$$

with equality if and only if $a^p = b^q$.

Theorem 4.7.2 (H$\ddot{o}$lder's Inequality)

Let X and Y be any two random variables. Let p and q be any positive numbers satisfying

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then

$$|\mathbf{E}(XY)| \leq \mathbf{E}|XY| \leq (\mathbf{E}|X|^p)^{\frac{1}{p}} (\mathbf{E}|Y|^q)^{\frac{1}{q}}.$$

Proof: In the inequality (1), let

$$a = \frac{|X|}{(\mathbf{E}|X|^p)^{\frac{1}{p}}}, \quad b = \frac{|Y|}{(\mathbf{E}|Y|^q)^{\frac{1}{q}}}$$

After some simplification, and take expectation on the two sides of the inequality. The result can been obtained.

- Theorem 4.7.3 (Cauchy-Schwarz Inequality)
  For any two random variables X and Y,

$$|\mathbf{E}(XY)| \leq \mathbf{E}|XY| \leq (\mathbf{E}|X|^2)^{\frac{1}{2}}(\mathbf{E}|Y|^2)^{\frac{1}{2}}$$

- Example 4.7.4 (Covariance Inequality) If X and Y have means μ_X and μ_Y, and variances σ_X^2 and σ_Y^2, respectively. We can apply the Cauchy-Schwarz Inequality to get

$$(\mathbf{Cov}(X,Y))^2 \leq \sigma_X^2 \cdot \sigma_Y^2$$

- **Example**
  Let $p > 1$, then apply Hölder's Inequality. For any random variables X,
  $$\mathbf{E}|X| \leq \{\mathbf{E}|X|^p\}^{\frac{1}{p}} \tag{5.1}$$
  If $1 < r < s$, we have (Liapounov's Inequality)
  $$(\mathbf{E}|X|^r)^{\frac{1}{r}} \leq (\mathbf{E}|X|^p)^{\frac{1}{p}} \tag{5.2}$$

- Proof of (**??**) Let q be such that $\frac{1}{p} + \frac{1}{q} = 1$, then
  $$\mathbf{E}|X| = \mathbf{E}|X| \cdot 1 \leq (\mathbf{E}|X|^p)^{1/p} \cdot (\mathbf{E}1^q)^{1/q} = (\mathbf{E}|X|^p)^{1/p}.$$

- Proof of (**??**) Let s be such that $s = pr$, then $s > 1$.
  $$\mathbf{E}(|X|^r) \leq (\mathbf{E}(|X|^r)^p)^{1/p}.$$

Theorem 4.7.5 (Minkowski's Inequality)

Let X and Y be any two random variables. Then for $1 < p < \infty$

$$[\mathbf{E}|X+Y|^p]^{\frac{1}{p}} \le (\mathbf{E}|X|^p)^{\frac{1}{p}} + (\mathbf{E}|Y|^p)^{\frac{1}{p}}$$

Proof:

$$\begin{aligned} \mathbf{E}|X+Y|^p &= \mathbf{E}(|X+Y||X+Y|^{p-1}) \\ &\le \mathbf{E}(|X||X+Y|^{p-1}) + \mathbf{E}(|Y||X+Y|^{p-1}) \qquad (5.3) \end{aligned}$$

Using Hölder's Inequality,

$$\mathbf{E}(|X||X+Y|^{p-1}) \le (\mathbf{E}|X|^p)^{\frac{1}{p}} \left[\mathbf{E}|X+Y|^{q(p-1)}\right]^{\frac{1}{q}} \qquad (5.4)$$

where q is such that $\frac{1}{p} + \frac{1}{q} = 1$ or $\frac{1}{q} = 1 - \frac{1}{p}$, i.e., $q = \frac{p}{p-1}$ or $q(p-1) = p$. Similarly,

$$\mathbf{E}(|Y||X+Y|^{p-1}) \le (\mathbf{E}|Y|^p)^{\frac{1}{p}} \left[\mathbf{E}|X+Y|^{q(p-1)}\right]^{\frac{1}{q}} \qquad (5.5)$$

So combine (**??**) and (**??**) with (**??**), divide through by $[\mathbf{E}(|X+Y|^{q(p-1)})]^{1/q}$, we have

$$\mathbf{E}|X+Y|^p \le (\mathbf{E}|X+Y|^p)^{\frac{p-1}{p}} \left[(\mathbf{E}|X|^p)^{\frac{1}{p}} + (\mathbf{E}|Y|^p)^{\frac{1}{p}}\right]$$

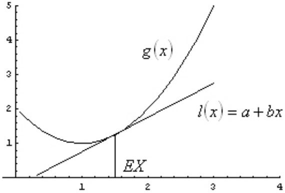
Theorem 4.7.7 (Jensen's Inequality)

For any random variable X, if $g(x)$ is a convex function, then

$$\mathbf{E}g(X) \geq g(\mathbf{E}X)$$

- Equality holds if and only if, for any line $a + bx$ that is tangent to $g(x)$ at $x = \mathbf{E}X$, $P(g(X) = a + bX) = 1$.
- If $g(x)$ is linear, $g(\mathbf{E}X) = a + b\mathbf{E}X = \mathbf{E}g(X)$.

Remark For any twice differentiable function $g(x)$, it is convex iff $g''(x) \geq 0$ for all x.



Example4.7.8 (An inequality for means)

Let $a_1, a_2, \cdots, a_n$ be n non-negative numbers. Define

$$\begin{aligned} a_A &= \frac{1}{n}(a_1 + a_2 + \cdots + a_n) \\ a_G &= [a_1 a_2 \cdots a_n]^{1/n} \\ a_H &= \frac{1}{\frac{1}{n}\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right)} \end{aligned}$$

An inequality relating these means is

$$a_H \le a_G \le a_A.$$

Remark The above inequality gives a reason for Maximum Likelihood Estimation(MLE).

Proof: Let X be a random variable with range $a_1, \ldots, a_n$, and $P(X = a_i) = 1/n, n = 1, \ldots, n$. Since $\log x$ is a concave function, $\mathbf{E} \log X \leq \log(\mathbf{E} X)$, hence

$$\begin{aligned} \log a_G &= \frac{1}{n} \sum_{i=1}^{n} \log a_i = \mathbf{E} \log X \leq \log(\mathbf{E} X) \\ &= \log \left(\frac{1}{n} \sum_{i=1}^{n} a_i \right) = \log a_A \end{aligned}$$

So, $a_G \leq a_A$. Furthermore,

$$\begin{aligned} \log \frac{1}{a_H} &= \log \left(\frac{1}{n} \sum_{i=1}^{n} \frac{1}{a_i} \right) = \mathbf{E} \log \frac{1}{X} \geq \mathbf{E} \Big(\log \frac{1}{X} \Big) = -\log(\mathbf{E} X) \\ &= -\log a_G = \log \left(\frac{1}{a_G} \right). \end{aligned}$$

So, $a_G \geq a_H$.

Markov's Inequality

Markov's(Chebyshev's) Inequality

- If g is strictly increasing and positive on $(0, \infty)$, $g(x) = g(-x)$.
- X is a r.v. such that $E[g(X)] < \infty$, then for each $a > 0$

$$P(|X| \geq a) \leq \frac{E[g(X)]}{g(a)}$$

Proof:

$$\begin{aligned}
E[g(X)] &\geq E[g(X) I_{\{g(X) \geq g(a)\}}] \\
&\geq g(a) E[I_{\{g(X) \geq g(a)\}}] \\
&= g(a) E[I_{\{|X| \geq a\}}] \\
&= g(a) P(|X| \geq a)
\end{aligned}$$

Some special cases: Markov's Inequality

$$g(x) = |x| \Longrightarrow P(|X| \geq a) \leq \frac{E|X|}{a}$$

$$g(x) = x^p \Longrightarrow P(|X| \geq a) \leq \frac{E|g(X^p)|}{a^p}$$

$$g(x) = x^2 \Longrightarrow P(|X - EX| \geq a) \leq \frac{Var(X)}{a^2}$$

$$g(x) = e^{t|x|} \Longrightarrow P(|X| \geq a) \leq \frac{E\left[e^{t|X|}\right]}{e^{ta}}$$

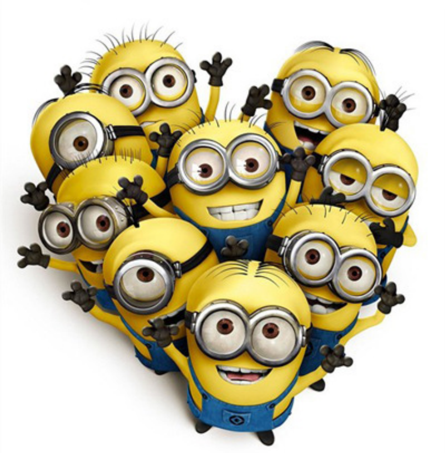
for some constant $t \geq 0$

Homework

1. If $\mu = EX \geq 0$ and $0 \leq \mu < 1$, then

$$P(X > \lambda\mu) \geq \frac{(1-\lambda)^2\mu^2}{EX^2}$$

Consequently, if $E|Y| = 1, P(|Y| > \lambda) \geq (1-\lambda)^2)/EY^2$ (This gives a lower bound complementing Chebyshev's inequality.)



Thank you