Lecture 4: Law of Large Numbers and Central Limit Theorem

Ma Xuejun

School of Mathematical Sciences

Soochow University

https://xuejunma.github.io

Outline

Review

- Definition of convergence
- Relationship between 4 types of convergence

$$\left.\begin{array}{l} X_n \xrightarrow{a.s.} X \\ \\ X_n \xrightarrow{L_r} X \end{array}\right\} \Rightarrow X_n \xrightarrow{p} X \Rightarrow X_n \xrightarrow{d} X$$

- Stochastic orders: O_p, o_p

Term 1

- The Weak/Strong Law of Large Numbers
- The Central Limit Theorem

Some comments

Theorem 1

Let $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots$ and $\boldsymbol{X}$ be random p-vectors,

1. **(The Portmanteau Theorem)** $\boldsymbol{X}_n \xrightarrow{d} \boldsymbol{X}$ *is equivalent to the following condition:* $E[g(\boldsymbol{X}_n)] \to E[g(\boldsymbol{X})]$ *for every bounded continuous function* g.
2. *(***Levy-Cramer continuity theorem)** *Let* $\phi_{\boldsymbol{X}_1}, \phi_{\boldsymbol{X}_2}, \ldots \phi_{\boldsymbol{X}}$ *be the character functions of* $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots$ *and* $\boldsymbol{X}$ *respectively.* $\boldsymbol{X}_n \xrightarrow{d} \boldsymbol{X}$ *iff* $\lim_{n\to\infty} \phi_{\boldsymbol{X}_n}(\boldsymbol{t}) = \phi_{\boldsymbol{X}}(\boldsymbol{t})$ *for all* $\boldsymbol{t} \in \mathcal{R}^p$
3. **(Cramer-Wold device)** $\boldsymbol{X}_n \xrightarrow{d} \boldsymbol{X}$ *iff* $\boldsymbol{c}^\top \boldsymbol{X}_n \xrightarrow{d} \boldsymbol{c}^\top \boldsymbol{X}$ *for every* $\boldsymbol{c} \in \mathcal{R}^p$

Example

- $X_n \sim Uniform\{\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}$, $X_n \xrightarrow{d} X \sim ?$
- If $g(x) = x^{10}$, then $E[g(X_n)] \rightarrow$

Law of Large Numbers

- Now that we have methods to describe the limit of a sequence of random variables
- Recall the motivating problem: Sample mean $\bar{X}_n$ converges to EX intuitively.
- Question: what is this convergence?Is it convergence in distribution.? probability? a.s.? L_r?

Weak Law of Large Numbers (WLLN)

Theorem 2 (WLLN)

Let $\{X_n\} = X_1, X_2, .., $ be a sequence of independently and identically distributed (i.i.d.) r.v.s such that $E|X_1| < \infty$, Then

$$\bar{X}_n \xrightarrow{P} EX_1$$

- The condition $E|X_1| < \infty$ is to assure the existence of EX_1.
- The theorem can be extended to many dependence structures, such as Markov chains.
- The theorem can be extended to cases that $X_i's$ are not identical, but share the same 1st and 2nd moments.
- According to properties for convergence in probability, for any cont. function $g(\cdot)$,

$$g(\bar{X}_n) = g(\frac{1}{n}\sum_{i=1}^{n} X_i) \xrightarrow{P} g(\mu)$$

.

Frame TitleSketch of Proof of WLLN

Note that when the limit is a constant, convergence in probability is equivalent with convergence in distribution. To prove convergence in distribution, we only need to show

$$\Phi_{X_n}(t) \to \Phi_{EX_1} = e^{itEX_1}$$

We will use the following result without any proof. For a r.v. X with finite first moment, we have

$$\Phi_X(X) = 1 + itEX + o(t)$$

Proof:

$$\begin{aligned}
\Phi_{\bar{X}_n} =& E[exp(it\bar{X}_n)] = E[exp(it\frac{1}{n}\sum_{i=1} nX_i)] \\
=& \prod_{i=1}^{n} E[exp(itX_i/n)] \\
=& (E[exp(itX_1/n)])^n = \Phi_{X_1}^n(t/n)
\end{aligned}$$

Let $n \to \infty$,then

$$\Phi_{\bar{X}_n}(t) = (1 + itEX_i/n + o(1/n))^n \to e^{itEX_1}$$

Strong Law of Large Numbers (WLLN)

Theorem 3 (SLLN)

Let $X_n = X_1, X_2, ...,$ be a sequence of independently and identically distributed (i.i.d.) r.v.s such that $E|X_1| < \infty$,then

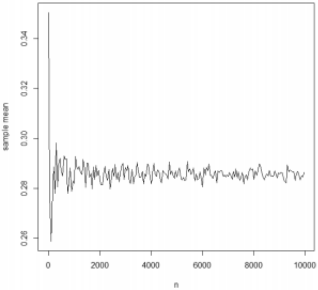
$$\bar{X}_n \xrightarrow{a.s.} EX_1$$

- The conditions can be relaxed. Identical distributions are not required, but there are still constraints on the second moment
- Stronger requirements than WLLN to assure better results The proof is beyond the scope of this course
- LLN: when n gets larger, the distribution of sample mean is more concentrated around EX_1.

Example of LLN: Calculate Expectation

Recall: $EX = \inf_{\infty}^{\infty} xf(x)dx$, where $f(x)$ is pdf of X.

- Generate n samples with pdf $f(x)$, and calculate the $\bar{X}_n$. When n is very large, $EX \approx \bar{X}_n$
- Example: Beta distribution with parameters $a = 2$ and $b = 2$, $EX = \int_0^1 \frac{\tau(7)}{\tau(2)\tau(5)} x^{2-1}(1-x)^{5-1}dx$. Hard to Calculate!



```
     [label={Beta(2,5)},fontsize=\scriptsize]
rm(list=ls())
n.vec <- seq(1, 104, 50)
n.len <- length(n.vec)
mean.full <- NULL
for(i in 1:n.len){
mean.full[i] <- mean(rbeta(n=n.vec[i], shape1=2,shape2=5))
 }
 plot(n.vec, mean.full, type="l",xlab = "n",
 ylab = "sample mean")
 abline(h=2/7,lwd=1,col="blue")
```

- Whats more, according to the continuous mapping theorem, $g(\bar{X}_n) \rightarrow g(E(X)), e.g., [E(X)]^2 \approx \bar{X}_n^2$

Example of LLN: Calculate Expectation

- LLN can also be used to find $E(g(X))$,where $g(\cdot)$ is a function
- Generate n i.i.d. samples $\{X_i\}_{i=1}^n$ with pdf $f(x)$,and let $Y_i = g(X_i)$.Then.$\bar{Y}_n \to E[g(X)]$.When n is very large,$E(g(X)) \approx \bar{Y}_n$
- Example: Beta distribution with parameters $a = 2, b = 5$. $Y = X^2, Z = 2X + 1, W = e^X$

Examples of Using LLN: Integration

- Suppose we wish to calculate

$$\int_0^1 g(x)dx$$

where $g(x)$ may be complicated and the integration in not easy to compute.

- Relate the integration with expectation. We need a density function. Let $X \sim Unif(0,1)$,then the pdf of x is 1 on [0,1]. For function $g(\cdot)$,

$$Eg(X) = \int_0^1 g(X) \cdot 1dx = \int_0^1 g(x)dx$$

Procedure (apply the method in previous slide for mean):

- Generate n i.i.d samples $X \sim Unif(0,1)$, and calculate $g(X_i)$ correspondingly
- Compute $Eg(X) \sim g(\bar{X}_i) = \frac{1}{n}\sum_{i=1}^{n} g(X_i)$
- This method is called Monte Carlo method.

Motivation

Suppose that a fair coin is tossed 100 times. What is the probability that the total number of heads is no smaller than 60?
Let X be the total number of heads, then $X \sim Bin(100, 0.5)$.We are interested in $P(X \geq 60)$

- Calculate directly means calculating 40 probs $\{p(X = i)\}_{i=60,61,..}$and take the summation.COMPLICATED.
- X can be seen as the summation of 100 Bernoulli trials with $p = 0.5$ and limit theorems can be applied.
    - With LLN, we only know $X/100 \xrightarrow{P} 0.5$,CANNOT get $P(X \geq 60)$
    - New Limit Theorem is required to describe the behaviour of X more accurately.

Central Limit Theorem (CLT)

Let $\{X_n\} = X_1, X_2...$ be a sequence of independently and identically distributed (i.i.d.) r.v.s such that $EX_1^2 < \infty$.Let $\sigma^2 = Var(X_1)$ and $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$,then

$$\sqrt{n}[\bar{X}_n - EX_1] = \sqrt{n}[\frac{1}{n}\sum_{i=1}^{n}(X_i - EX_1)] \overset{d}{rightarrow} N(0, \sigma^2)$$

- $EX_1^2 < \infty$ is a regular condition to assure the existence of EX_1 and h$Var(X_1)$
- It means that $\bar{X}_n$ can be approximated by a normal distribution, no matter what the distribution for X_i is.
- Here, $n^{-0.5}$ is the convergence rate. Or, say,$\bar{X}_n = O_p(n^{-0.5})$. If we use $n^{0.5+\delta}$ with $\delta > 0$, then there is no meaningful result; if we use $n^{0.5-\delta}$ then it converges to 0.

Comments on CLT

CLT is the most important theorem in statistics

- CLT means that, the sample mean will be approximately normally distributed for large sample sizes, regardless of the distribution of the samples
- Many statistics (say, $\bar{X}_n$, $\bar{X}_n^2$ have distributions that are approximately normal, even the population distribution is not normal ($\Leftarrow$ The dist. of statistics can be approximated)
- Statistical inference can be derived based on normality, provided the sample size is large
- n practice, it gives a very rough guideline to approximate $\bar{X}_n$ when n is large (a few hundreds or even more)
- However, the convergence is the weakest convergence, converge in distribution. With the result, for statistics (e.g., $\bar{X}_n$), we can only calculate

$$P(\bar{X}_n \geq a), P(\bar{X}_n \leq a), P(a \leq \bar{X}_n \leq b)$$

Comparison Between LLN and CLT

	LLN	CLT
Results	Focus on $\bar{X}_n$	Focus on $\frac{\bar{X}_n-\mu}{\sigma/\sqrt{n}}$
Convergence	In probability	In distribution
Interpretation	$\bar{X}_n$ converges to μ	The rate $\bar{X}_n$ converges to μ
Usage	Monte Carlo Method	Statistical Inference

Table 1: Caption

- Compare to real numbers, LLN means that

$$\frac{2\sqrt{n}+1}{\sqrt{n}} \to 2$$

CLT mean that

$$\sqrt{n}(\frac{2\sqrt{n}+1}{\sqrt{n}} - 2) \to 1$$

.

CLT Example

Let $X_i \sim i.i.d.Exp(1), i = 1, 2, \ldots$ We know that $E[X_1] = Var(X_1) = 1$,and so the sample mean converges to 1.How many samples we need so that our error is at most 10
The target is, to figure out n, so that $P(0.9 \leq \bar{X}_n \leq 1.1) \geq 0.95$. For large n, with CLT, we have $\sqrt{n}(\bar{X}_n - 1) \xrightarrow{d} N(0, 1)$.Therefore, we may use standard normal distribution to approximate the probability. Then,

$$\begin{aligned} P(0.9 \leq \bar{X}_n \leq 1.1) &= P(-0.1 \leq \bar{X}_n - 1 \leq 0.1) \\ &= P(-0.1\sqrt{n} \leq \sqrt{n}(\bar{X}_n - 1) \leq 0.1\sqrt{n}) \\ &\approx \Phi(0.1\sqrt{n}) - \Phi(-0.1\sqrt{n}) \\ &= 2\Phi(0.1\sqrt{n}) - 1 \leq 0.95 \end{aligned}$$

Check the normal table, and we can find $n > 384$.

Homework

1. suppose $X_1, \ldots, X_n$ are iid with mean μ, variance σ^2, and $EX_1^4 < \infty$, then
$$\sqrt{n}(S_n^2 - \sigma^2) \xrightarrow{d} N(0, \mu_4 - \sigma^4)$$
where where μ_4 denotes the centered fourth moment of X_1.
2. (Multivariate CLT for iid case) Let $\boldsymbol{X}_i$ be iid random p-vectors with mean $\boldsymbol{\mu}$ and and covariance matrix $\boldsymbol{\Sigma}$. Then
$$\sqrt{n}(\bar{\boldsymbol{X}}_n - \boldsymbol{\mu}) \xrightarrow{d} N_p(0, \boldsymbol{\Sigma})$$