Lecture 5: Edgeworth Expansion and Models

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Outline

- Review
- The Delta Method
- The Edgeworth Expansion

Terminology

- The Delta Method; The multivariate Delta method
- The Edgeworth Expansion; the expression of the expansion, including the notations related to the expression.

Review

- The Weak Law of Large Numbers: converge in probability/distribution
- The Strong Law of Large Numbers: almost sure convergence
- Use of LLN: Monte Carlo method
- The Central Limit Theorem

The Delta Method

- CLT is for $\bar{X}_n$
- Can we generalize it to $g(\bar{X})$? So the application of CLT is extended.
- Example: what is the limiting dist. for $e^{\bar{X}_n}$?

Theorem 5.5.24: The Delta Method

Suppose that $\frac{\sqrt{n}(Y_n-\mu)}{\sigma} \xrightarrow{d} N(0,1)$, and that $g(\cdot)$ is a differentiable function such that $g'(\mu) \neq 0$, then

$$\frac{\sqrt{n}(g(Y_n)-g(\mu))}{|g'(\mu)|\sigma} \xrightarrow{d} N(0,1)$$

Remark.

- $Y_n = \bar{X}_n$ for the CLT. Here, Y_n is a generlized case. For any Y_n satisfying the convergence rule, we have the delta method.
- In other words, $Y_n \xrightarrow{d} N(\mu, \sigma^2/n)$ implies that $g(Y_n) \xrightarrow{d} N\Big(g(\mu), (g'(\mu))^2\sigma^2/n\Big)$

Proof: The Delta Method

Intuition of proof: We are interested in the term $\frac{\sqrt{n}(g(Y_n)-g(\mu))}{|g'(\mu)|\sigma}$. Since $\frac{\sqrt{n}(Y_n-\mu)}{\sigma}$ converges to normal distribution, we have that $Y_n - \mu = O_p(n^{-0.5})$.
Since g is differentiable at μ, when $y - \mu = o(1)$, there is

$$g(y) = g(\mu) + g'(\mu)(y-\mu) + o(1)(y-\mu)$$

Introduce it into the term of interest,

$$\begin{aligned}
\frac{\sqrt{n}(g(Y_n)-g(\mu))}{|g'(\mu)|\sigma} &= \frac{\sqrt{n}(g(\mu)+g'(\mu)(Y_n-\mu)+rem-g(\mu))}{|g'(\mu)|\sigma} \\
&= \frac{\sqrt{n}(g'(\mu)(Y_n-\mu)+rem)}{|g'(\mu)|\sigma} \\
&= \sqrt{n}\left[\frac{(Y_n-\mu)}{\sigma}\right] + \sqrt{n}|g'(\mu)|\sigma \times rem
\end{aligned}$$

The first term converges to $N(0,1)$ in distribution, and the second term converges to 0. So the summation would converge to $N(0,1)$ in distribution. Note. The

point here is that $g(y) = g(\mu) + g'(\mu)(y-\mu) + o(1)(y-\mu)$ when y changes with a small quantity. Since δ is always used to denote small quantity, so the method is called "The Delta Method".

The Multivariate Delta Method

Suppose that $Y_n = (Y_{n1}, Y_{n2}, \ldots, Y_{nk})$ is a sequence of random vectors such that $\sqrt{n}(Y_n - \mu) \xrightarrow{d} N(0, \Sigma)$. Let $g : R^k \rightarrow R$ and let

$$\begin{pmatrix} \frac{\partial g}{\partial y_1} \\ \vdots \\ \frac{\partial g}{\partial y_k} \end{pmatrix}$$

Let ∇_μ denote $\nabla g(y)$ evaluated at $y = \mu$ and assume that the elements of ∇_μ are nonzero. Then

$$\sqrt{n}\Big(g(Y_n - g(\mu)\Big) \xrightarrow{d} N(0, \nabla_\mu^\top \Sigma \nabla_\mu)$$

- The generalisation of multivariate CLT.

Example 1: The Delta Method

Let $X_1, \ldots, X_n$ be i.i.d with finite mean μ and finite variance σ^2. By the CLT,

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} N(0,1).$$

Let $W_n = e^{\bar{X}_n}$. Thus, $W_n = g(\bar{X}_n)$ where $g(t) = e^t$. Since $g'(t) = e^t$, the delta method implies that

$$\frac{\sqrt{n}(W_n - e^{\mu})}{\sigma e^{\mu}} \xrightarrow{d} N(0,1)$$

Example 2: The Delta Method

Let $X_1, \ldots, X_n$ be i.i.d with finite mean μ and finite covariance matrix Σ. The according to the multivariate CLT, we have

$$\sqrt{n}\left(\begin{bmatrix} \bar{X}_{n1} \\ \bar{X}_{n2} \end{bmatrix} - \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \right) \xrightarrow{d} N(0, \Sigma)$$

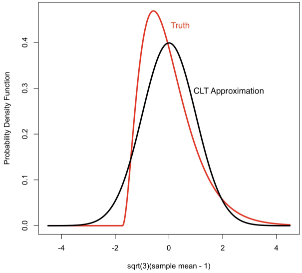
Let $g(s,t) = s^2 + t^2$, then can apply the delta method and we have that

$$\sqrt{n}(\bar{X}_{n1}^2 + \bar{X}_{n2}^2 - \mu_1^2 - \mu_2^2) \xrightarrow{d} 2\mu_1 Z_1 + 2\mu_2 Z_2$$

where $Z = (Z_1, Z_2)^\top \sim N(0, \Sigma)$.

The Edgeworth Expansion

- Note that CLT is a good estimate for large n
- When n is small, it might be away from the truth
- Example: Consider $X_i \overset{i.i.d}{\sim} Gamma(1,1), i = 1,2,3$, We know that $\mathbf{E}X_1 = \mathbf{Var}X_1 = 1$. Let $Z = \sqrt{3}(\bar{X}_3 - 1))$, then according to CLT, $Z \sim N(0,1)$ approximately.



- Since n is small, there is large difference! How to get a good estimate?

The Edgeworth Expansion

Recall that in the proof of CLT, we figured out the characteristic function as

$$\phi_{Z_n}(t) = \phi_Y^n\left(\frac{t}{\sqrt{n}}\right)$$

where $Z_n = \frac{1}{\sigma}\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^n (X_i - \mathbf{E}X_1)\right]$ and $Y = (X_1 - EX_1)/\sigma$. When we take it as $\phi_Y(t/\sqrt{n}) = 1 - \sigma^2 t^2/(2\sigma^2 n) + o(1/n)$, then we have CLT.

What if we consider higher order moments?

- Let κ_j be $j-$th cumulant of Y, where

  $$\kappa_1 = \mathbf{E}Y,\ \kappa_2 = \mathbf{Var}(Y),\ \kappa_3 = \mathbf{E}(Y - \mathbf{E}Y)^3,\ \kappa_4 = \mathbf{E}(Y - \mathbf{E}Y)^4 - 3\mathbf{Var}Y$$

- Consider $K_Y = \log(\phi_Y(t))$ and the Taylor series expansion of $K_Y(t)$ at $t = 0$,

  $$K_Y(t) = \frac{1}{\phi_Y(0)}\frac{d\phi_Y(t)}{dt}\Big|_{t=0} + \frac{1}{\phi_Y(0)}\frac{d^2\phi_Y(t)}{dt^2}\Big|_{t=0} + \cdots + = \sum_{j=1}^{\infty}\kappa_j\frac{(it)^j}{j!}$$

- Since $Y = (X_1 - \mathbf{E}X_1)/\sigma$, we have $\kappa_1 = \mathbf{E}Y = 0$, $\kappa_2 = \mathbf{Var}Y = 1$, so

$$\phi_Y(t) = \exp\left(-\frac{t^2}{2} + \sum_{j=3}^{\infty} \frac{(it)^j}{j!}\right) = \exp\left(-\frac{t^2}{2} + \frac{\kappa_3^3(it)}{3!} + \cdots + \frac{k_j^j}{j|} + \dots\right)$$

- Introduce it into the equation for

$$\phi_{Z_n}(t) = e^{-t^2/2}\left[1 + n^{-1/2} r_i(it) + n^{-1} r_2(it) + o(n^{-1})\right]$$

where $r_1(it) = \kappa_3^3(it)/6$, and $r_2(it) = \frac{1}{24}\kappa_4^4(it) + \frac{1}{72}\kappa_3^{12}(it)$

- Here, we apply the higher order moments of X to depict the characteristic function more clearly

Therefore, we have the Edgeworth expansion:

$$F_{Z_n}(z) = \Phi(z) + \frac{1}{\sqrt{n}} p_1(z)\phi(z) + \frac{1}{n} p_2(z)\phi(z) + o\Big(\frac{1}{n}\Big)$$

where $p_1(z) = -\kappa_3(z^2 - 1)/6$ and
$p_2(z) = -z\Big[\kappa_4(z^2 - 3)/24) + \kappa_3^2(z^4 - 10z^2 + 15)/72\Big]$.

Returning to our example. Consider the rst order Edgeworth expansion, we have

$$P(Z_e \leq z) \approx \Phi(z) + \frac{1}{\sqrt{3}}\phi(z)p_1(z)$$

where $p_1(z) = -\kappa_3(z^2 - 1)/6$ and $\kappa_3 = 2$.