Lecture 5

1. Consider a random sample $X_1, X_2, \cdots, X_n \sim Unif(0, \theta)$.
   (i) Find the estimator for θ through MoM,denoted by $\hat{\theta}_{MM}$.
   (ii)Find the MLE $\hat{\theta}_{MLE}$.
   (iii) What are the expetation and variance of $\hat{\theta_{MM}}$ and $\hat{\theta}_{MLE}$? Which estimator is better?

   (i) Since $X_1, X_2, \cdots, X_n \sim Unif(0, \theta)$,the first order moment is

   $$E[X_i] = \theta/2$$

   Therefore,the data version is

   $$\bar{X_n} = \hat{\theta}_{MM/2}$$

   So the MoM estimator is $2\bar{X_n}$.
   (ii) The likelihood function is

   $$L(\theta; x) = f(\theta; x) = \prod_{i=1}^{n} \tfrac{1}{\theta} 1_{(0,\theta)(x_i)} = \tfrac{1}{\theta^n} 1\{x_{(n)<\theta}\}$$

   Note that $L(\theta)$ is a monotone decreasing function when $\theta > X_{(n)}$.and equals to 0 when $\theta < X_{(n)}$.Therefore,the maximum is achieved at $\theta = X_{(n)}$.The MLE is $\hat{\theta}_{MLE} = X_{(n)}$.
   (iii)

   $$E[\hat{\theta}_{MM}] = 2E[\bar{X_n}] = \theta, \qquad Var[\hat{\theta}_{MM}] = Var2[\bar{X_n}] = \tfrac{4}{n} Var(X_1) = \tfrac{\theta^2}{3n}.$$

   The distribution for $\hat{\theta}_{MLE}$ is

   $$P(\hat{\theta}_{MLE} \le a) = P(X_{(n)} \le a) = \begin{cases} 0, & a \le 0 \\ (a/\theta)^n, & 0 < a < \theta \\ 1, & a \ge \theta \end{cases}$$

   So the pdf is $f_{\hat{\theta}_{MLE}}(a) = na^{n-1}/\theta^n, 0 < a < \theta$.

$$E[\hat{\theta}_{MLE}] = \int_0^\theta a \cdot na^{n-1}/\theta^n da = \frac{n}{n+1}\theta, \quad E[\hat{\theta}_{MLE}^2] = \int_0^\theta a^2 \cdot na^{n-1}/\theta^n da = \frac{n}{n+2}\theta^2$$

$$Var[\hat{\theta}_{MLE}] = E[\hat{\theta}_{MLE}^2] - (E[\hat{\theta}_{MLE}])^2 = \frac{n}{n+2}\theta^2 - (\frac{n}{n+1}\theta)^2 = \frac{n}{(n+1)^2(n+2)}\theta^2.$$

Although $\hat{\theta}_{MLE}$ is a biased estimator,but the bias is very small (with order 1/n),and it has variance at order $1/n^2$,which is much smaller than that of $\hat{\theta}_{MM}$.So,MLE is a better estimator.