Ve401 Probabilistic Methods in Engineering

Sample Final Exam Exercises

The following exercises have been compiled from past final exams of Ve401. An exam will usually consist of 25-30 Points worth of such exercises to be completed in 100 minutes. In the actual exam, necessary tables of values of distributions will be provided. A PDF file with all necessary tables has been made available on Canvas. For certain of these exercises, Mathematica will be needed - in those exams, the sue fo Mathematica was permitted.

Multiple Choice

Exercise 1.

In the following exercises, mark the boxes corresponding to true statements with a cross (☒). In each case, exactly one of the provided statements is true.

i) In linear regression, a large value of R^2 indicates that

- ☐ The regression is significant, i.e., it is not likely that the coefficients $\beta_1, \ldots, \beta_p$ all vanish.
- ☐ Our fitted model will do very well when making predictions and finding confidence intervals.
- ☒ Our model explains a large proportion of the observed variation in the measured response.
- ☐ Our model is close to the true model for $\mu_{Y|x}$.

ii) Suppose that a Fisher test of the null hypothesis

$$H_0 \colon \mu \leq \mu_0$$

yields a very small P-value. Which of the following statements will be true?

- ☐ It is likely that the true value of μ is much larger than μ_0.
- ☒ Data was obtained that was very unusual, if the assumption is made that H_0 is true.
- ☐ It is unlikely that H_0 is true, given the data that was obtained.
- ☐ The rejection of H_0 is unlikely to be a mistake.

iii) Suppose that you are performing regression using the model $\mu_{Y|x} = \beta_0 + \beta_1 x + \beta_2 x^2$ and that you obtain a value of R^2 close to one, based on a large sample size. Which of the following statements will be true?

- ☐ There is evidence that $\beta_0 \neq 0$.
- ☐ There is evidence that $\beta_1 \neq 0$.
- ☐ There is evidence that $\beta_2 \neq 0$.
- ☒ There is evidence that $|\beta_1| + |\beta_2| \neq 0$.

iv) Suppose that you perform a test comparing a mean μ to a null value μ_0

$$H_0 \colon \mu \leq \mu_0,$$

After obtaining your data and from it the sample mean $\overline{X}$, you find that $\overline{X} > \mu_0$ and calculate a P-value of 0.3%. Mark all of the following statements that you think are correct.

- ☐ There is a 99.7% chance that H_0 is true.
- ☐ There is a 0.3% chance that H_0 is true.
- ☒ If H_0 were true, there would be at most a 0.3% chance of obtaining a value of $\overline{X}$ equal or greater to the one measured.
- ☐ If H_0 were false, there would be at least a 99.7% chance of obtaining a value of $\overline{X}$ equal to the one measured or greater.

(4 Marks)

Hypothesis Tests for Location and Dispersion

Exercise 2.
A manufacturer of precision measuring instruments claims that the standard deviation in the use of an instrument is not more than 0.00002 inch. An analyst, who is unaware if the claim, uses the instrument eight times and obtains a sample standard deviation of 0.00005 inch.

i) Using $\alpha = 0.01$, is there evidence that the manufacturer's claim is not justified?

ii) What is the power of the test if the true standard deviation equals 0.00004 inch?

iii) What is the smallest sample size that can be used to detect a true standard deviation of 0.00004 inch or more with a probability of at least 0.95? Use $\alpha = 0.01$.

(2+1+1 Marks)

Solution 2.

i) We test the hypothesis

$$H_0 \colon \sigma \leq 0.00002.$$

at $\alpha = 0.01$. If H_0 is true, the statistic

$$X^2_{n-1} = (n-1)\frac{S^2}{\sigma_0^2}$$

follows a chi-squared distribution with $n - 1 = 7$ degrees of freedom. **(1/2 Mark)** The critical value is $\chi^2_{0.01,7} = 18.5$. **(1/2 Mark)** The value of the statistic is

$$x_7^2 = 7 \cdot \frac{25 \cdot 10^{-8}}{4 \cdot 10^{-8}} = 43.75.$$

(1/2 Mark) Since this exceeds the critical value, we can reject H_0 at the 1% level of significance. **(1/2 Mark)** There is evidence that the manufacturer's claim is not justified.

ii) We use the OC curve for the right-tailed chi-squared test. The abscissa parameetr is

$$\lambda = \frac{S}{\sigma_0} = 2$$

and the sample size is $n = 8$. We read off $\beta \approx 0.34$, so the power is approximately $1 - \beta = 0.66$.

iii) Again, we use the OC chart with $\lambda = 2$ and $\beta = 1 - 0.95 = 0.05$. A sample size of $n = 20$ is sufficient to achieve the power stated.

Exercise 3.
The diameters of bolts are known to have a standard deviation of 0.0001 inch. A random sample of 10 bolts yields an average diameter of 0.2546 inch.

i) Test the hypothesis that the true mean diameter of bolts equals 0.255 inch, using $\alpha = 0.05$.

ii) What size sample would be necessary to detect a true mean bolt diameter of 0.2552 inch or more with a probability of at least 0.90, assuming $\alpha = 0.05$?

(2+2 Marks)

Solution 3.

i) We test $H_0 \colon \mu = 0.255$ at $\alpha = 0.05$. We will use the statistic

$$Z = \frac{\overline{X} - 0.255}{\sigma/\sqrt{n}},$$

which follows a standard normal distribution if H_0 is true. For $\alpha = 5\%$, we will reject H_0 if $|Z| > z_{0.025} = 1.96$. Now

$$z = \frac{0.2546 - 0.255}{0.0001/\sqrt{10}} = -12.65.$$

Since $|-4| > 1.96$, we reject H_0, i.e., the true mean diamater is different from 0.255 inch.

ii) Since in our case

$$d = \frac{\mu - \mu_0}{\sigma} = \frac{0.2552 - 0.255}{0.0001} = 2,$$

we can see from the OC curve that in our case $n = 3$ is sufficient.

Exercise 4.

A company wants to test whether a new assembly line procedure increases the physical stress on its workers. It selects eleven workers to work for one day using each of the assembly line procedures. At the end of each day, their pulse frequency is measured:

Procedure 1	X	63	65	71	75	72	75	68	74	62	73	72
Procedure 2	Y	80	78	96	87	88	96	82	83	77	79	71

It is thought that the median pulse frequency is higher in Procedure 2 than in Procedure 1.

i) Formulate H_0.

ii) Use the Wilcoxon signed rank test at the 5% level of significance to determine whether you can reject H_0.

iii) Use a paired T-test at the 5% level of significance to determine whether you can reject H_0.

(2 + 2 + 2 Marks)

Solution 4.

i) Denote by M_X the median pulse frequency in procedure 1 and by M_Y the median pulse frequency in procedure 2. Then we have

$$H_0 \colon M_Y \leq M_X \qquad \text{or} \qquad H_0 \colon M_Y - M_X \leq 0.$$

(We are trying to find evidence to support the hypothesis that the median pulse frequency is higher in Procedure 2 than in Procedure 1.)

ii) We perform a paired test and consider M_{Y-X}. Then we test

$$H_0 \colon M_{Y-X} \leq 0.$$

We will reject H_0 if $|W_-|$ is small. **(1/2 Mark)** We calculate $Y - X$:

$Y - X$	17	13	25	12	16	21	14	9	15	6	-1

(1/2 Mark) We can see from the table of $Y - X$ that there is only a single negative value of $Y - X$, which has rank 1. Therefore,

$$|W_-| = 1.$$

(1/2 Mark) and $W = |W_-| = 1$. **(1/2 Mark)** According to the table for the Wilcoxon signed-rank test, we reject H_0 at the 5% level of significance if $W < 14$, so we can here reject H_0. **(1/2 Mark)**

iii) For the purposes of the t-test, we assume that the medians are equal to the means, i.e., $\mu_X = M_X$, $\mu_Y = M_Y$. If H_0 is true, $\mu_{Y-X} = 0$ and

$$\frac{\hat{\mu}_{Y-X} - \mu_{Y-X}}{\sqrt{S^2_{Y-X}/n}} = \frac{\hat{\mu}_{Y-X}}{\sqrt{S^2_{Y-X}/n}}$$

satisfies the T-distribution with $\gamma = 10$ degrees of freedom. **(1/2 Mark)** By the table for the T-distribution, at $\alpha = 5\%$ level of significance and $\gamma = 10$, the critical value is 1.812.

In our case, the sample mean of $Y - X$ is

$$\hat{\mu}_{Y-X} = \overline{Y - X} = \frac{1}{11}(17 + 13 + 25 + 12 + 16 + 21 + 14 + 9 + 15 + 6 - 1) = 13.36.$$

(1/2 Mark) The sample variance is

$$S^2_{Y-X} = \frac{1}{10}\sum_{i=1}^{11}(Y_i - X_i - \overline{Y - X})^2 = 49.85.$$

(1/2 Mark) We obtain

$$\frac{\hat{\mu}_{Y-X}}{\sqrt{S^2_{Y-X}/n}} = 6.28.$$

Since $6.28 > 1.812$, we can reject H_0.**(1/2 Mark)**

Exercise 5.

In a hardness test, a steel ball is pressed into the material being tested at a standard load. The diameter of the indentation is measured, which is related to the hardness. Two types of steel balls are available, and their performance is compared on 10 randomly selected specimens. The hypothesis that the two steel balls give the same expected hardness measurement is to be tested at a significance level of $\alpha = 0.05$. Each specimen is tested twice, once with each ball. The results are given below:

Specimen	1	2	3	4	5	6	7	8	9	10
Ball x	75	46	57	43	58	38	61	56	64	65
Ball y	52	41	43	47	32	49	52	44	57	60

Use each of the following methods to test the hypothesis

i) A pooled T-test (assume that the variances are unequal).

ii) A Wilcoxon signed rank test.

iii) A paired T-test.

Compare the results obtained by each of the above tests. What assumptions are necessary for the validity of each test? What is your final conclusion regarding the hypothesis?
(2+2+2+3 Marks)

Solution 5.

i) We first compute the sample means and variances:

$$\overline{x} = 56.3, \qquad \overline{y} = 47.7$$
$$s^2_X = 125.344, \qquad s^2_Y = 67.122$$

(1/2 Mark) The value of the pooled test statistic is

$$T_\gamma = \frac{(\overline{x} - \overline{y})}{\sqrt{s^2_x/10 + s^2_y/10}} = 1.96.$$

(1/2 Mark) The degrees of freedom for this test are

$$\gamma = \frac{\left(s^2_x/10 + s^2_y/10\right)^2}{\frac{(s^2_x/10)^2}{9} + \frac{(s^2_x/10)^2}{10}} = 16.49$$

rounded down to 16. **(1/2 Mark)** Since $t_{0.025,16} = 2.120 > 1.96$, we do not have enough evidence to reject H_0. **(1/2 Mark)**

Assumptions: X, Y both follow a normal distribution. (The sample size is too small for the Central Limit theorem to be applicable.) **(1/2 Mark)**

ii) The Wilcoxon statistics are

```
Diff = SortBy[X - Y, Abs]

{-4, 5, 5, 7, 9, -11, 12, 14, 23, 26}

W+ = 0;
For[i = 1, i ≤ Length[Diff], i++, If[Positive[Diff[[i]]], W+ = W+ + i,]];
W+

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W- = 0;
For[i = 1, i ≤ Length[Diff], i++, If[Negative[Diff[[i]]], W- = W- + i,]];
W-

7
```

(1/2 Mark) so the test statistic is $W = \min(W_+, |W_-|) = 7$. **(1/2 Mark)** For a two-sided test at $\alpha = 0.05$ the critical value is 8, **(1/2 Mark)** so there is enough evidence to reject H_0. **(1/2 Mark)**

Assumptions: $X - Y$ follows a symmetric distribution so that the mean is equal to the median. **(1/2 Mark)**

iii) For the paired T-test we calculate the sample mean and variance of $D = X - Y$:

$$\overline{d} = 8.6, \qquad s_d^2 = 124.7.$$

(1/2 Mark) We test $H_0 \colon D = 0$. The statistic used is

$$T_9 = \frac{\overline{d}}{\sqrt{s_d^2/10}} = 2.435$$

(1/2 Mark) The critical value is $t_{0.025,9} = 2.262$. **(1/2 Mark)** Since the value of the test statistic exceeds this, we reject H_0. **(1/2 Mark)**

Assumptions: X, Y both follow a normal distribution. (The sample size is too small for the Central Limit theorem to be applicable.) **(1/2 Mark)**

Based on the test results, we can reject the null hypothesis. **(1/2 Mark)** While the pooled test was not powerful enough to do so, by the elimination of extraneous factor through pairing we were able to collect enough evidence to reject H_0. **(1 Mark)**

Exercise 6.

A product developer is interested in reducing the drying time of a primer paint. Two formulations of the paint are tested; formulation 1 is the standard chemistry, and formulation 2 has a new drying ingredient that should reduce the drying time. From experience, it is known that the standard deviation of drying time is 8 minutes, and this inherent variability should be unaffected by the addition of the new ingredient. Ten specimens are painted with formulation 1, and another 10 specimens are painted with formulation 2; the 20 specimens are painted in random order.

i) The two sample average drying times are $\overline{x}_1 = 121$ minutes and $\overline{x}_2 = 112$ minutes. Perform a significance test to judge the effectiveness of the new ingredient. What is the P-value of the test? What conclusions can you draw about the effectiveness of the new ingredient?

ii) If the true difference in mean drying times is as much as 10 minutes, find the sample sizes required to detect this difference with probability at least 0.90, assuming the hypothesis test is conducted with $\alpha = 0.01$.

(3+3 Marks)

Solution 6.

i) We test $H_1 \colon \mu_2 < \mu_1$, $H_0 \colon \mu_2 \geq \mu_1$. **(1/2 Mark)** The test statistic

$$Z = \frac{\overline{X}_1 - \overline{X}_2}{\sqrt{\sigma^2/n_1 + \sigma^2/n_1}}$$

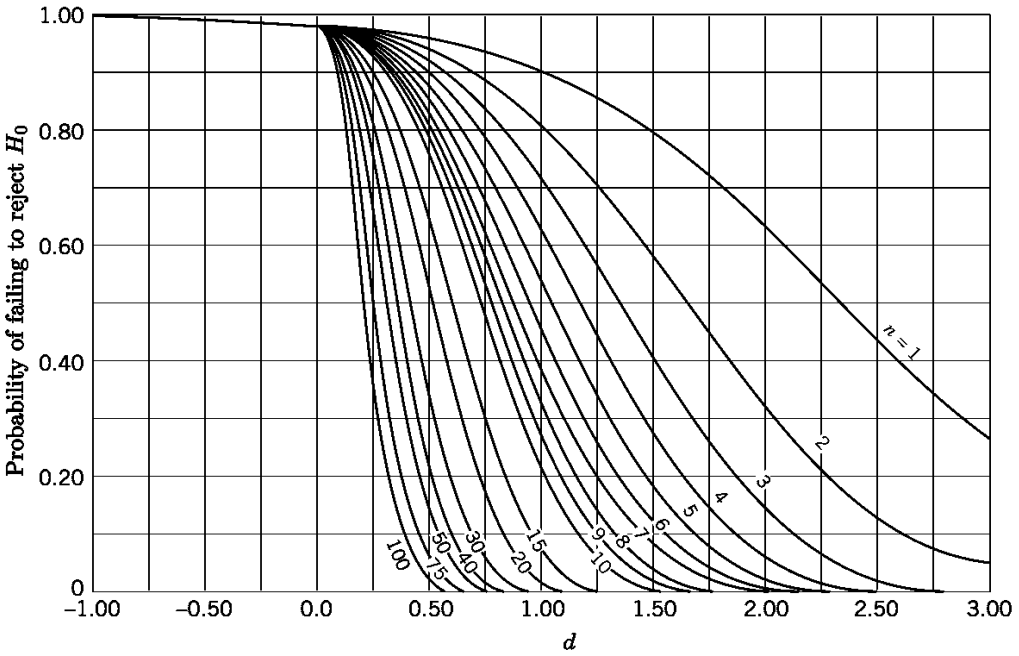
follows (at least approximately) a standard normal distribution. We will reject H_0 if Z is too large to have occurred by chance if H_0 is true. **(1/2 Mark)**

The observed value of Z is

$$z = \frac{121 - 112}{8\sqrt{1/10 + 1/10}} = \frac{9\sqrt{10}}{8\sqrt{2}} = 2.52.$$

(1/2 Mark) The probability of observing this large or a larger result if $\mu_1 = \mu_2$ is $0.5 - 0.4941 = 0.0059$. This is the P-value of the test. **(1 Mark)** Since the P-value is significantly less than 1%, we can conclude that there is evidence that the new drying ingredient reduces the drying time. **(1/2 Mark)**

ii) We use the OC chart for a one-sided test based on the normal distribution:



with $d = (\mu_1 - \mu_2)/\sqrt{\sigma_1^2 + \sigma_2^2} = 10/\sqrt{128} = 10/(8\sqrt{2}) = 0.88$ **(1/2 Mark)** and $\beta = 0.1$. **(1/2 Mark)** This gives a minimum sample size of about $n_1 = n_2 = 17$. **(2 Marks for any number greater than 15 and not greater than 20)**.

Exercise 7.

A polymer is manufactured in a batch chemical process. Viscosity measurements are normally made on each batch, and long experience with the process has indicated that the variability in the process is fairly stable. The chemical process uses a catalyst, Catalyst A, which is intended to be replaced by the more environmentally friendly Catalyst B if the viscosity is not markedly influenced by the change in catalyst.

Let μ_A denote the mean viscosity of the polymer using Catalyst A and μ_B be the mean viscosity using Catalyst B. It is hoped that the change in catalyst will not influence the mean viscosity, but if it turns out to do so significantly, then Catalyst A will not be replaced.

i) Roughly 95% of the time, Catalyst A will lead to a polymer viscosity in the range $\mu_A \pm 2\sigma$. If μ_B differs from μ_A by at most σ, what percentage of the polymer viscosity will at most fall outside of the range $\mu_A \pm 2\sigma$?

This percentage determined in 1. is considered acceptable and catalyst A will be replaced if Catalyst B changes the mean viscosity of the polymer by less than 1 standard deviation.

ii) Formulate an appropriate hypothesis test to decide whether Catalyst A should be replaced.

iii) Given sample sizes $n_A = n_B = 20$ for the viscosities using Catalysts A and B, respectively, find the power of the test.

Pilot data yield the following viscosities:

Catalyst A	708, 732, 731, 677, 748, 702, 696, 692, 716, 729, 697, 681, 704, 740, 710, 687, 731, 704, 702, 698
Catalyst B	761, 708, 727, 730, 737, 702, 752, 758, 718, 712, 750, 747, 723, 698, 763, 756, 707, 716, 715, 732

iv) Given the above data, is the null hypothesis rejected at $\alpha = 1\%$? Quote all relevant statistics and critical values.

v) Find a 99% confidence interval on the difference in mean batch viscosity resulting from the process change.

vi) What is your conclusion regarding the catalyst change? Comment on the results of 5. and 6. above. How likely is it that you have reached the wrong conclusion?

(2+1+2+3+1+3) Marks

Solution 7.

i) We need to calculate $P[X > 2] + P[X < -2]$ for X following a normal distribution with mean $\mu = 1$ and $\sigma^2 = 1$. **(1/2 Mark)** These correspond to $P[Z > 1] + P[Z < -3]$ for a standard normal distribution, **(1/2 Mark)** and we have $P[Z > 1] + P[Z < -3] = 1 - 0.8413 + 1 - 0.9987 = 0.1600$. At most 16% of the plymer will have viscosity outside the stated range. **(1 Mark)**

ii) We conduct a Neyman-Pearson test,

$$H_0 \colon \mu_1 = \mu_2, \qquad\qquad H_1 \colon |\mu_1 - \mu_2| > \sigma.$$

(1 Mark)

iii) We use the OC curve for a two-sided T-test with $n^* = 2n - 1 = 39$ **(1/2 Mark)** and

$$d = \frac{|\mu_A - \mu_B|}{2\sigma} = \frac{\sigma}{2\sigma} = \frac{1}{2}.$$

(1/2 Mark) Either of the following is acceptable:

- Using the curve for $\alpha = 0.05$, we read off $\beta \approx 0.135$, so the power would be roughly 86%.
- Using the curve for $\alpha = 0.01$, we read off $\beta \approx 0.335$, so the power would be approximately 66%.

(1 Mark)

iv) We find

$$\overline{x}_A = 709.25, \qquad s_A^2 = 399.566, \qquad \overline{x}_B = 730.6, \qquad s_B^2 = 447.2$$

(1/2 Mark) and

$$s_p^2 = \frac{1}{2}(s_A^2 + s_B^2) = 423.383, \qquad\qquad s_p = 20.5763.$$

(1/2 Mark) The value of the test statistic is

$$\frac{\overline{x}_A - \overline{x}_B}{s_p\sqrt{\frac{1}{n_A} + \frac{1}{n_B}}} = -3.28119.$$

(1 Mark) The critical value of the T-distribution with $n_A + n_B - 2 = 38$ degrees of freedom is $t_{38,0.005} = 2.71156$. **(1/2 Mark)** Since the observed value of the test statistic is larger than this value, we reject H_0 and therefore accept H_1. **(1/2 Mark)**

v) A 99% confidence interval is given by

$$\begin{aligned}\mu_A - \mu_B &= \overline{x}_A - \overline{x}_B \pm t_{38,0.005} s_p \sqrt{\frac{2}{20}} \\ &= -21.35 \pm 17.6435\end{aligned}$$

(1 Mark)

vi) The result of the hypothesis test, performed at $\alpha = 1\%$, $\beta \approx 33.5\%$, was to accept H_1, i.e., that a change in catalyst does affect the mean viscosity by at least one standard deviation. The probability of having reached the wrong conclusion is not more than 1%. **(1 Mark)**

At the same time, the 99% confidence interval for the difference in means is centered roughly at the value of s_p, so it may well be that the difference in means is one standard deviation or more, but it could also be less than that. The half-width of the interval is about the same size as the estimated difference in means. **(1 Mark)** While it would be prudent to not change catalysts, further analysis with a larger sample size is required for greater certainty. **(1 Mark)**

Chi-Squared Goodness-of-Fit Tests

Exercise 8.

The second midterm exam of the course Ve401 in Spring 2010 had 25 marks in total. The students taking the exam obtained the following marks:

> 0, 2, 4.5, 5.5, 8, 8.5, 9, 9.5, 10, 10, 10.5, 10.5, 11, 11.5, 11.5, 12, 12, 12, 12.5, 13, 13, 13.5, 13.5, 14, 14, 14, 14.5, 14.5, 15, 15, 15, 15, 15.5, 15.5, 15.5, 16, 16, 16, 16.5, 16.5, 16.5, 16.5, 16.5, 17, 17.5, 17.5, 17.5, 17.5, 17.5, 18, 18, 18, 18.5, 18.5, 18.5, 18.5, 18.5, 18.5, 18.5, 19, 19, 19, 19, 19, 19, 19, 19, 19.5, 19.5, 20, 20, 20, 20.5, 20.5, 20.5, 21, 21, 21, 21, 21, 21, 21.5, 21.5, 21.5, 21.5, 21.5, 21.5, 21.5, 22, 22, 22, 22.5, 22.5, 22.5, 22.5, 23.5, 23.5, 23.5

Let $S = \{\text{number of marks obtained in the exam}\}$ be the sample space for the the trial "student takes the second midterm exam in Ve401" and consider the random variable $X\colon S \to \mathbb{R}$, $X(s) = 25 - s$. In other words, X gives the difference between the total marks and the marks obtained by a random student.

i) The above data can be used to obtain a random sample of size $n = 98$ from X. Note again that $X(0) = 25$, $X(2) = 23, \ldots, X(23.5) = 1.5$. Plot a stem-and-leaf diagram for the values obtained for X. The stems should have integer units, i.e., there should be 25 stems.

ii) The shape of the stem-and-leaf diagram ressembles that of a chi-squared distribution. Assuming that X follows a chi-squared distribution, find a method-of-moments estimate for the degrees of freedom (rounded to one decimal point).

iii) Use a chi-squared goodness-of-fit test to test the hypotheses

$$H_0\colon X \text{ follows a chi-squared distribution,}$$
$$H_1\colon X \text{ does not follow a chi-squared distribution}$$

at $\alpha = 5\%$. When dividing the positive real axis into categories (intervals), it is not necessary for each interval to have the same expected number of values falling into it. But you should still make sure that the expected numbers are large enough for the chi-squared test to be applicable. If the estimated degrees of freedom are not an integer, interpolate the chi-squared table values linearly to obtain the interval boundaries.

(2+2+4 Marks)

Solution 8.

i) The stem-and-leaf diagram is shown below

Stem	Leaves
0	
1	555
2	5555
3	0005555555
4	000000555
5	00055
6	000000005555555
7	00055555
8	055555
9	000555
10	000055
11	00055
12	005
13	00055
14	055
15	005
16	05
17	0
18	
19	5
20	5
21	
22	
23	0
24	
25	0

ii) The chi-squared distribution X with γ degrees of freedom is a gamma distribution with $\beta = 2$ and $\alpha = \gamma/2$. Therefore, its expectation is equal to $\mathrm{E}[X] = \alpha\beta = \gamma$. The method-of-moments estimator for n is then simply the sample mean,

$$\hat{\gamma} = \overline{X}.$$

For the given sample, $\hat{\gamma} = 8.3$.

iii) Since we need to use the given chi-squared table, we must choose our categories carefully. Since $n = 98$, we should make sure that at most one interval corresponds to a probability of 5% or less and all categories should have expectations greater than 1. number. We choose the categories in the following way:

Interval boundary	a_1	a_2	a_3	a_4	a_5	a_6	a_7
$P[X < a_i]$	0.025	0.10	0.25	0.50	0.75	0.90	1
E_i	2.45	7.35	14.7	24.5	24.5	14.7	9.8

where E_i is the expected number of values in the interval $[a_{i-1}, a_i]$ One of the expected values is less than five, but this is less than 20% of the six categories. We interpolate the following values for the intervals from the chi-squared table:

γ	0.025	0.10	0.25	0.50	0.75	0.90
8	2.18	3.49	5.07	7.34	10.2	13.4
9	2.70	4.17	5.9	8.34	11.4	14.7
8.3	2.34	3.69	5.31	7.64	10.56	13.79

Therefore, we have the following results

Category	Interval	E_i	O_i
1	$[0, 2.34)$	2.45	3
2	$[2.34, 3.69)$	7.35	14
3	$[3.69, 5.31)$	14.7	12
4	$[5.31, 7.64)$	24.5	25
5	$[7.64, 10.56)$	24.5	18
6	$[10.56, 13.79)$	14.7	13
7	$[13.79, \infty)$	9.8	13

We have the statistic

$$\sum_{i=1}^{7} \frac{(E_i - O_i)^2}{E_i} = 9.612$$

which follows a chi-squared distribution with $7 - 1 - 1 = 5$ degrees of freedom. Since the critical value is $\chi^2_{0.05,5} = 11.1$, there is not enough evidence to reject H_0 at the 5% level of significance. We have no reason to doubt the assumption that the exam marks follow a chi-squared distribution.

Exercise 9.
A study is conducted to test for independence between air quality and air temperature. These data were obtained from records on 200 randomly selected days over the last few years.

	Air Quality		
Temperature	Poor	Fair	Good
Below average	1	3	24
Average	12	28	76
Above average	12	14	30

Do these data indicate an association between these variables? Explain, based on the P-value of the test.
(3 Marks)

Solution 9.
We have the following sums and expected values (in parentheses):

	Air Quality			
Temperature	Poor	Fair	Good	$n_{k\cdot}$
Below average	1 (3.5)	3 (6.3)	24 (18.2)	28
Average	12 (14.5)	28 (26.1)	76 (75.4)	116
Above average	12 (7.0)	14 (12.6)	30 (36.4)	56
$n_{\cdot k}$	25	45	130	200

(1/2 Mark) None of the expected frequencies is smaller than one and only one is smaller than 5. Therefore, we may use our statistical methods. **(1/2 Mark)** The null hypothesis is

$$H_0 \text{: no association between air quality and temperture.}$$

and the test statistic is

$$\sum_{i,j=1}^{3} \frac{(n_{ij} - \hat{E}_{ij})^2}{\hat{E}_{ij}} = 10.789$$

(1/2 Mark) which follows a χ^2 distribution with $(3 - 1)(3 - 1) = 4$ degrees of freedom. **(1/2 Mark)** Since the probability of obtaining a value of less than 9.49 is 95%, the P-value of the test is a little less than 5%. **(1/2 Mark)** Thus we reject H_0. The data indicate an association between air quality and temperature. **(1/2 Mark)**

Linear Regression

Exercise 10.
Suppose we have the following data:

x	1.0	1.0	3.3	3.3	4.0	4.0	4.0	5.6	5.6	5.6	6.0	6.0	6.5	6.5
y	1.6	1.8	1.8	2.7	2.6	2.6	2.2	3.5	2.8	2.1	3.4	3.2	3.4	3.9

i) Perform a linear regression for y as a function of x.

ii) Test the model for lack of fit at an $\alpha = 0.05$ level of significance.

(2+3 Marks)

Solution 10.
We first calculate

$$n = 14, \qquad \sum_{i=1}^{n} x_i = 62.4, \qquad \sum_{i=1}^{n} y_i = 37.6,$$
$$\sum_{i=1}^{n} x_i^2 = 322.6, \qquad \sum_{i=1}^{n} y_i^2 = 107.76, \qquad \sum_{i=1}^{n} x_i y_i = 181.94.$$

Then

$$S_{xx} = \sum_{i=1}^{n}(x_i - \overline{x})^2 = \frac{1}{n}\Big(n\sum_{i=1}^{n} x_i^2 - \Big(\sum_{i=1}^{n} x_i\Big)^2\Big) = 44.234$$
$$S_{yy} = \sum_{i=1}^{n}(y_i - \overline{y})^2 = \frac{1}{n}\Big(n\sum_{i=1}^{n} y_i^2 - \Big(\sum_{i=1}^{n} y_i\Big)^2\Big) = 6.777$$
$$S_{xy} = \sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y}) = \frac{1}{n}\Big(n\sum_{i=1}^{n} x_i y_i - \Big(\sum_{i=1}^{n} x_i\Big)\Big(\sum_{i=1}^{n} y_i\Big)\Big) = 14.3514$$

i) We assume the model $\mu_{Y|x} = \beta_0 + \beta_1 x$. The estimators for β_0 and β_1 are

$$b_1 = \frac{S_{xy}}{S_{xx}} = 0.324, \qquad b_0 = \overline{y} - b_1\overline{x} = 1.240.$$

It follows that the linear regression line is

$$\mu_{Y|x} = 1.24 + 0.324x,$$

ii) First we note that

$$\mathrm{SSE} = S_{yy} - b_1 S_{xy} = 2.120.$$

Calculating $\mathrm{SSE}_{\mathrm{pe}}$ is comparatvely simple in this case, since we have two repeated measurements for most values of the regressor:

$$\begin{aligned}\mathrm{SSE}_{\mathrm{pe}} &= \sum_{i=1}^{4}\sum_{j=1}^{2}(Y_{ij} - \overline{Y}_i)^2 + \sum_{i=5}^{6}\sum_{j=1}^{3}(Y_{ij} - \overline{Y}_i)^2 \\ &= \frac{1}{2}\sum_{i=1}^{4}(Y_{i1} - Y_{i2})^2 + 2(2.6 - 2.467)^2 + (2.2 - 2.467)^2 + 2\cdot 0.7^2 \\ &= 1.66\end{aligned}$$

Furthermore,

$$\mathrm{SSE}_{\mathrm{lf}} = \mathrm{SSE} - \mathrm{SSE}_{\mathrm{pe}} = 2.12 - 1.66 = 0.46.$$

We test

H_0: the linear regression model is appropriate,
H_1: the linear regression model is not appropriate.

using the statistic

$$F_{k-2,n-k} = F_{4,8} = \frac{\mathrm{SSE}_{\mathrm{lf}}/(k-2)}{\mathrm{SSE}_{\mathrm{pe}}/(n-k)} = \frac{8}{4} \cdot \frac{0.46}{1.66} = 0.554 < 1.$$

We reject H_0 if the value of the statistic is too large, indicating a significantly larger lack-of-fit error compared to pure error. However, since the value is actually less than one, we do not reject H_0. (Formally, the P-value of the test is greater than 5% because $P[F_{4,8} > 3.84] = 5\%$.) We conclude there is no evidence to indicate that linear regression is not appropriate.

Exercise 11.
A chemical engineer is investigating the effect of process operating temperature on product yield. The study results in the following data:

Temperature (° C)	100	120	140	160	180
Yield (%)	45	54	66	74	85

Fit a linear regression model and find

i) a 95% confidence interval for the slope;

ii) a 95% confidence interval for the intercept;

iii) a 95% confidence interval for the mean yield at 130° C;

iv) a 95% prediction interval for the yield at 130° C.

(4 × 2 Marks)

Solution 11.
We calculate

$$n = 5, \qquad \sum x_k = 700, \qquad \sum y_k = 324,$$
$$\sum x_k^2 = 102000, \qquad \sum y_k^2 = 21998, \qquad \sum x_k y_k = 47360,$$
$$\overline{x} = 140, \qquad S_{xx} = 4000.$$

We assume the model $\mu_{Y|x} = \beta_0 + \beta_1 x$. The estimators for β_0 and β_1 are

$$b_1 = \frac{n\sum x_k y_k - (\sum x_k)(\sum y_k)}{n\sum x_k^2 - (\sum x_k)^2} = \frac{1}{2}, \qquad b_0 = \frac{1}{n}\sum y_k - \frac{b_1}{n}\sum x_k = -\frac{26}{5}$$

We need the estimator for the variance,

$$S^2 = \mathrm{SSE}/(n-2) = \mathrm{SSE}/3,$$

where the sum of squares error is

$$\mathrm{SSE} = \sum (y_k - b_0 - b_1 x_k)^2 = 2.8.$$

Thus $s^2 = 0.933$, $s = 0.966$. For a 95% confidence interval we need $t_{0.025,3} = 3.182$, using a $T_{n-2} = T_3$-distribution. Now 95% confidence intervals for the slope and intercept are given by

$$b_1 \pm t_{\alpha/2,n-2} s/\sqrt{S_{xx}} = 0.5 \pm 3.182 \cdot 0.966/\sqrt{4000} = 0.5 \pm 0.049,$$
$$b_0 \pm t_{\alpha/2,n-2} s\sqrt{\sum x_k^2}/\sqrt{nS_{xx}} = 0 - 5.2 \pm \sqrt{102000/5} \cdot 3.182 \cdot 0.966/\sqrt{4000} = -5.2 \pm 6.950,$$

For the confidence interval for $\mu_{Y|130}$ we need the term

$$t_{\alpha/2,n-2} s\sqrt{\frac{1}{n} + \frac{(130-\bar{x})^2}{S_{xx}}} = 1.46$$

An confidence interval for $\mu_{Y|130}$ is given by

$$\hat{\mu}_{Y|4} \pm 1.46 = b_0 + b_1 \cdot 130 \pm 1.46 = 59.8 \pm 1.46.$$

For the prediction interval for $\mu_{Y|130}$ we need the term

$$t_{\alpha/2,n-2}s\sqrt{1+\frac{1}{n}+\frac{(130-\bar{x})^2}{S_{xx}}} = 3.40,$$

so

$$\widehat{Y \mid 130} = 59.8 \pm 3.40.$$

Exercise 12.
You have applied a quadratic regression model to a data set of $n = 7$ points. Your computer algebra system tells you that $R^2 = 0.781$. Is the regression significant at the $\alpha = 5\%$ level?
(3 Marks)

Solution 12.
We test

$$H_0\text{: regression not significant.}$$

We have $k = 2$ (quadratic model), and we know that

$$\begin{aligned} F_{k,n-k-1} &= \frac{n-k-1}{k}\frac{\text{SSR}}{\text{SSE}} = \frac{n-k-1}{k}\frac{\text{SSR}/S_{yy}}{\text{SSE}/S_{yy}} = \frac{n-k-1}{k}\frac{\text{SSR}/S_{yy}}{1-\text{SSR}/S_{yy}} \\ &= \frac{n-k-1}{k}\frac{R^2}{1-R^2} = \frac{4}{2}\cdot\frac{0.781}{0.219} \\ &= 7.132 = F_{2,4} \end{aligned}$$

According to the table, the critical value the $\alpha = 5\%$ level is 6.944, so we can reject H_0.

Exercise 13.
Consider the following data, which result from an experiment to determine the effect of $x =$ test time in hours at a particular temperature on $y =$ change in oil viscosity.

x	0.50	1.00	1.50	2.00	2.50
y	-0.51	-2.09	-6.03	-9.28	-17.12

i) Fit the model $\mu_{y|x} = \beta_0 + \beta_1 x + \beta_2 x^2$ to the data.

ii) Test whether the regression is significant at a 5% level.

iii) Give a 90% prediction interval for $y \mid 0$.

iv) Use an F-test to test whether a linear model is sufficient at a 5% level of significance; give SSE_{full} and $\text{SSE}_{\text{reduced}}$.

In order to solve this exercise without the use of a computer you may use that

$$(X^TX)^{-1} = \begin{pmatrix} 4.6 & -6.6 & 2. \\ -6.6 & 10.6857 & -3.4286 \\ 2. & -3.4286 & 1.1429 \end{pmatrix}$$

where X is the model determination matrix. The entries in the matrix have been rounded; make sure you use all the given decimal places in your calculations, otherwise your results will be off.
(3+3+2+3 Marks)

Solution 13.

i) From

$$\hat{\beta} = b = (X^TX)^{-1}X^Ty.$$

we find

$$b = \begin{pmatrix} 4.6 & -6.6 & 2. \\ -6.6 & 10.6857 & -3.4286 \\ 2. & -3.4286 & 1.1429 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0.5 & 1. & 1.5 & 2. & 2.5 \\ 0.25 & 1. & 2.25 & 4. & 6.25 \end{pmatrix} \begin{pmatrix} -0.51 \\ -2.09 \\ -6.03 \\ -9.28 \\ -17.12 \end{pmatrix} = \begin{pmatrix} -0.798 \\ 2.06361 \\ -3.38477 \end{pmatrix}$$

This gives the regression curve

$$\mu_{Y|x} = -0.798 + 2.064x - 3.385x^2.$$

ii) We test

$$H_0\colon \beta_1 = \beta_2 = \ldots = \beta_p = 0$$

using the statistic

$$F_{p,n-p-1} = F_{2,2} = \frac{\mathrm{SSR}/2}{\mathrm{SSE}/2} = \frac{\mathrm{SSR}}{\mathrm{SSE}}.$$

For $\alpha = 5\%$ the critical region of the statistic is the interval $[19, \infty)$. We have

$$S_{yy} = \sum_{i=1}^{n} y_i^2 - \frac{1}{n}\Big(\sum_{i=1}^{n} y_i\Big)^2 = 174.782$$

$$\mathrm{SSR} = b_0 \sum_{i=1}^{n} y_i + \sum_{j=1}^{2}\sum_{i=1}^{n} b_j x_i^j y_i - \frac{1}{n}\Big(\sum_{i=1}^{n} y_i\Big)^2 = 173.649$$

so $\mathrm{SSE} = (S_{yy} - \mathrm{SSR}) = 1.133$. Then the value of the statistic is

$$F_{2,2} = \frac{\mathrm{SSR}/2}{\mathrm{SSE}/2} = \frac{173.649}{1.133} > 19.0,$$

so we reject H_0. The regression is significant.

iii) The estimator for the variance is

$$\mathrm{MSE} = \frac{\mathrm{SSE}}{n-p-1} = 0.567.$$

We set

$$x_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

A 90% prediction interval is then given by

$$Y \mid 0 = \hat{Y} \mid 0 \pm t_{\alpha/2,n-p-1}\sqrt{\mathrm{MSE}(1 + x_0^T (X^T X)^{-1} x_0)}.$$

where $t_{\alpha/2,n-p-1} = t_{0.05,2} = 2.920$ and we use the point estimate

$$\hat{Y} \mid 0 = -0.798 + 2.064 \cdot 0 - 3.385 \cdot 0^2 = -0.798.$$

Now $x_0^T (X^T X)^{-1} x_0 = 4.6$, so the prediction interval is

$$Y \mid 0 = \hat{Y} \mid 0 \pm t_{0.05,2}\sqrt{\mathrm{MSE}(1 + x_0^T (X^T X)^{-1} x_0)} = -0.798 \pm 5.20.$$

iv) A (reduced) linear model gives

$$\mu_{Y|x} = 5.117 - 8.082x.$$

with $\mathrm{SSE}_{\mathrm{reduced}} = 3.828$. We have seen above that $\mathrm{SSE}_{\mathrm{full}} = 1.133$. We test

$$H_0\colon \text{reduced model is sufficient,} \qquad H_1\colon \text{full model is needed.}$$

at $\alpha = 5\%$. The test statistic

$$F_{p-m,n-p-1} = F_{1,6} = \frac{p-m}{n-p-m}\frac{\mathrm{SSE}_{\mathrm{reduced}} - \mathrm{SSE}_{\mathrm{full}}}{\mathrm{SSE}_{\mathrm{full}}} = \frac{1}{5-2-1}\frac{\mathrm{SSE}_{\mathrm{reduced}} - \mathrm{SSE}_{\mathrm{full}}}{\mathrm{SSE}_{\mathrm{full}}}$$

follows an F distribution with 1 and 2 degrees of freedom; therefore, the critical interval for rejection of H_0 is $(18.5, \infty)$. Now the statistic takes the value

$$\frac{1}{2}\frac{\mathrm{SSE}_{\mathrm{reduced}} - \mathrm{SSE}_{\mathrm{full}}}{\mathrm{SSE}_{\mathrm{full}}} = \frac{1}{2}\frac{3.828 - 1.133}{1.133} = 1.18 < 18.5,$$

so we fail to reject H_0 at the stated level of significance. There is no evidence that the linear model is not sufficient.

Exercise 14.
Consider the following data:

x	1	2	3	4	5	6	7
y	8	17	29	34	46	42	52

Fit a model of the form $\mu_{Y|x} = \beta_0 + \beta_1 x + \beta_2 x^2$. You may use that

$$(X^T X)^{-1} = \frac{1}{7}\begin{pmatrix} 17 & -9 & 1 \\ -9 & 67/12 & -2/3 \\ 1 & -2/3 & 1/12 \end{pmatrix},$$

where X is the model determination matrix. What is the value of R^2 for this model?
(3+2 Marks)

Solution 14.
Note first that

$$X^T y = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 4 & 9 & 16 & 25 & 36 & 49 \end{pmatrix} \begin{pmatrix} 8 \\ 17 \\ 29 \\ 34 \\ 46 \\ 42 \\ 52 \end{pmatrix} = \begin{pmatrix} 228 \\ 1111 \\ 6091 \end{pmatrix}$$

From

$$\hat{\beta} = b = (X^T X)^{-1} X^T y.$$

we find

$$b = \frac{1}{7}\begin{pmatrix} 17 & -9 & 1 \\ -9 & 67/12 & -2/3 \\ 1 & -2/3 & 1/12 \end{pmatrix} \begin{pmatrix} 228 \\ 1111 \\ 6091 \end{pmatrix} = \begin{pmatrix} -32/7 \\ 155/12 \\ -61/84 \end{pmatrix}$$

This gives the regression curve

$$\mu_{Y|x} = -\frac{32}{7} + \frac{155}{12}x - \frac{61}{84}x^2.$$

To find R^2 we note that

$$\begin{aligned} S_{yy} &= \sum_{i=1}^{7} y_i^2 - \frac{1}{7}\Big(\sum_{i=1}^{7} y_i\Big)^2 = 1507.71 \\ \mathrm{SSR} &= b_0 \sum_{i=1}^{7} y_i + b_1 \sum_{i=1}^{7} x_i y_i + b_2 \sum_{i=1}^{7} x_i^2 y_i - \frac{1}{7}\Big(\sum_{i=1}^{7} y_i\Big)^2 \\ &= \langle X^T y, b\rangle - \frac{1}{7}\Big(\sum_{i=1}^{7} y_i\Big)^2 = 1458.62 \end{aligned}$$

so $R^2 = \mathrm{SSR}/S_{yy} = 0.967$.

Exercise 15.
Suppose that in the simple linear regression model $Y = \beta_0 + \beta_1 x + E$ it is known that $\beta_0 = 0$, i.e., the model to be fitted is

$$\mu_{Y|x} = \beta_1 x.$$

i) Derive a least-squares estimator $\hat{\beta}_1$ for β_1.

ii) Do you expect a confidence interval for β_1 in the model $\mu_{Y|x} = \beta_1 x$ to be larger or smaller than in the model $\mu_{Y|x} = \beta_0 + \beta_1 x$? Explain!

iii) Find the distribution of $\hat{\beta}_1$ and derive a confidence interval for β_1.

(2 + 1 + 2 Marks)

Solution 15.

i) We are fitting a model with model determination matrix

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Suppose we have a sample of n measurements (x_i, y_i), $i = 1, \ldots, n$. The least-squares estimator is

$$b = b_1 = (X^T X)^{-1} X^T Y = \frac{\langle X, Y \rangle}{\langle X, X \rangle} = \frac{\sum_{i=1}^n x_i y_i}{\sum_{i=1}^n x_i^2}$$

(1 Marks)

ii) The confidence interval in the model $\mu_{Y|x} = \beta_1 x$ will be larger because the variance of b_1 will be larger. This is because a single parameter b_1 must account for all of the variation of the responses. In the model $\mu_{Y|x} = \beta_0 + \beta_1 x$ both b_0 and b_1 can vary a little bit if one of the y_i is changed, but in the model $\mu_{Y|x} = \beta_1 x$ the estimator b_1 must account for the same change of y_i all by itself, hence it varies more.

(Any answer is fine as long as it is accompanied by a plausible explanation.) **(1 Mark)**

iii) The estimator b_1 follows a normal distribution with mean β_1 and variance

$$\sigma^2 (X^T X)^{-1} = \frac{\sigma^2}{\sum_{i=1}^n x_i^2}.$$

(1 Mark) The confidence interval

$$b_1 \pm \frac{z_{\alpha/2}\sigma}{\sqrt{\sum_{i=1}^n x_i^2}}$$

(1 Mark) is acceptable as an answer, but we could also find the hat matrix

$$H = X(X^T X)^{-1} X^T = (h_{ij})$$

where

$$h_{ij} = \frac{x_i x_j}{\sum_{k=1}^n x_k^2}.$$

Then

$$\mathrm{SSE} = \langle Y, (1 - H)Y \rangle = \sum_{i=1}^n y_i^2 - \frac{1}{\sum_{k=1}^n x_k^2} \sum_{i,j=1}^n x_i y_i x_j y_j.$$

Setting $S^2 = \mathrm{SSE}\,/(n-1)$ we have the confidence interval

$$b_1 \pm \frac{t_{\alpha/2, n-1} S}{\sqrt{\sum_{i=1}^n x_i^2}}$$

Exercise 16.

Given repeated measurements in simple linear regression, we are able to decompose the error sum of squares $\mathrm{SS_E}$ into the components due to pure error $\mathrm{SS_{E,pe}}$ and due to lack-of-fit error $\mathrm{SS_{E,lf}}$,

$$\mathrm{SS_E} = \mathrm{SS_{E,pe}} + \mathrm{SS_{E,lf}}.$$

Let Y_{ij} denote the jth observation of $Y \mid x_i$, where $j = 1, \ldots, n_i$, and define

$$\overline{Y}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} Y_{ij}.$$

Then

$$\mathrm{SS_{E,pe}} = \sum_{i=1}^k \sum_{j=1}^{n_i} (Y_{ij} - \overline{Y}_i)^2$$

and $\mathrm{SS_{E,lf}} = \mathrm{SS_E} - \mathrm{SS_{E,pe}}$.

i) Show that

$$\mathrm{SS}_{\mathrm{E,lf}} = \sum_{i=1}^{k} n_i(\overline{Y}_i - \widehat{Y}_i)^2,$$

where $\widehat{Y}_i = b_0 + b_i x_i$.

ii) Explain in words (no formulas!) what $\mathrm{SS}_{\mathrm{E,pe}}$ represents, based on the above sum. You should write 1-3 sentences.

iii) Explain in words (no formulas!) what $\mathrm{SS}_{\mathrm{E,lf}}$ represents, based on the above sum. You should write 1-3 sentences.

iv) The following data represents the plasma level of polyamine (Y) in 25 children aged $0-4$ years old (x):

x	Y				
0	20.12	16.1	10.21	11.24	13.35
1	8.75	9.45	13.22	12.11	10.38
2	9.25	6.87	7.21	8.44	7.55
3	6.45	4.35	5.58	7.12	8.1
4	5.15	6.12	5.7	4.25	7.98

Perform a linear regression for the model $\mu_{Y|x} = \beta_0 + \beta_1 x$ on this data and test for lack of fit. State and compare the values for $\mathrm{SS}_{\mathrm{E,pe}}$ and $\mathrm{SS}_{\mathrm{E,lf}}$.

v) Fit a quadratic regression model $\mu_{Y|x} = \beta_0 + \beta_1 x + \beta_2 x^2$ to the above data. Is the linear model sufficient or does the quadratic model improve the fit significantly?

vi) Calculate and compare the values of R^2 for both the linear and the quadratic model. Comment on the result.

vii) In general, if repeated measurements are available, will the maximum achievable value of R^2 be greater or smaller than for data without repeated measurements? Why?

viii) Sketch the above data together with the quadratic model. Do you see any potential issues or problems with the fitted model?

(2 + 2 + 2 + 4 + 3 + 2 + 2 + 3 Marks)

Solution 16.

i) We have

$$\begin{aligned}
\mathrm{SS}_{\mathrm{E}} &= \sum_{i=1}^{n}\big(Y_{ij} - \widehat{Y}_i\big)^2 \\
&= \sum_{i=1}^{k}\sum_{j=1}^{n_i}\big(Y_{ij} - \overline{Y}_i + \overline{Y}_i - \widehat{Y}_i\big)^2 \\
&= \sum_{i=1}^{k}\sum_{j=1}^{n_i}\big(Y_{ij} - \overline{Y}_i\big)^2 + \sum_{i=1}^{k} n_i\big(\overline{Y}_i - \widehat{Y}_i\big)^2 + 2\sum_{i=1}^{k}\big(\overline{Y}_i - \widehat{Y}_i\big)\sum_{j=1}^{n_i}\big(Y_{ij} - \overline{Y}_i\big) \\
&= \mathrm{SS}_{\mathrm{E,pe}} + \mathrm{SS}_{\mathrm{E,lf}}
\end{aligned}$$

Since

$$\sum_{j=1}^{n_i}\big(Y_{ij} - \overline{Y}_i\big) = \sum_{j=1}^{n_i} Y_{ij} - n_i\overline{Y}_i = 0,$$

we obtain the result. **(2 Marks)**

ii) The $\mathrm{SS}_{\mathrm{E,pe}}$ represents the variability of the response variable without reference to any particular model. **(1 Mark)** It will be small if the variance of the response is generally small and large otherwise. **(1 Mark)**

iii) The $\mathrm{SS}_{\mathrm{E,lf}}$ represents the weighted square difference between the mean response estimated solely from local measurements and the mean response estimated from the global model. **(1 Mark)** It will be small if the model prediction for the mean is consistent with the local estimates, large otherwise. **(1 Mark)**

iv) A simple linear regression gives

$$\widehat{\mu}_{X|x} = 13.24 - 2.119x.$$

(1 Mark) The sum of squares error is $\mathrm{SS_E} = 116.606$ **(1/2 Mark)** and the pure error is $\mathrm{SS_{E,pe}} = 97.536$. **(1/2 Mark)**

To test for lack of fit, we use the F-statistic for $k = 5$, $n = 25$:

$$F_{3,20} = \frac{19.07/3}{97.536/20} = 1.3035$$

(1 Mark) The critical value for $\alpha = 0.05$ is is $f_{3,20;0.05} = 3.10$. **(1/2 Mark)** Since the test statistic has a smaller value, we are unable to conclude that there is evidence that the linear model is inappropriate. **(1/2 Mark)**

v) We fit a quadratic model, resulting in

$$\widehat{\mu}_{X|x} = 14.2769 - 4.19271x + 0.518429x^2$$

(1 Mark) To test for sufficiency of the linear model, we find the 95% confidence interval for β_2 in the quadratic model,

$$\beta_2 \in -0.00417573, 1.04103$$

Since this interval includes zero, we are (barely) unable to reject $H_0 \colon \beta_2 = 0$ at the $\alpha = 5\%$ level. **(1 Mark)** There is not much evidence that the quadratic term is needed. **(1 Mark)**

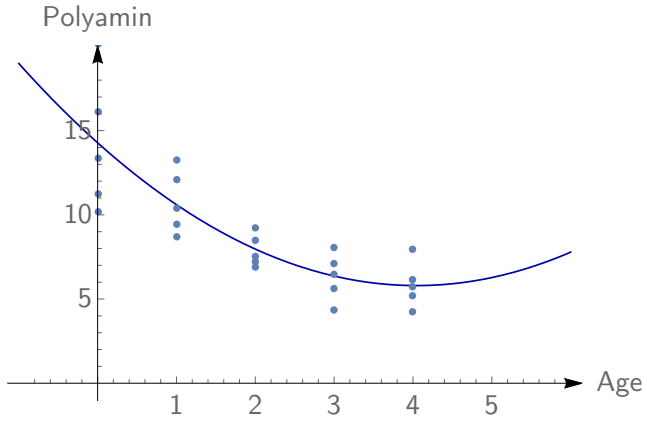
vi) We have

$$R^2_{\mathrm{lin}} = 0.658, \qquad R^2_{\mathrm{quad}} = 0.713.$$

(1 Mark) As expected, the value of R^2 is better for the quadratic model, but only slightly so. The quadratic model does not do a significantly better job at explaining the observed variation in the response data than the linear model. **(1 Mark)**

vii) In general, when repeated measurements are taken, the regression curve can not be arbitrarily close to two (or more) measurements at the same value of x at the same time. Therefore, it is not possible to force R^2 to be as close to unity as one wants by increasing the number of parameters. Hence, in general, the value of R^2 will be smaller for regressions with repeated measurements. **(2 Marks)**

viii) The data with the quadratic model is shown below:



(1 Mark) An important issue with the quadratic model is that it includes a trend for the polyamin levels to increase beyond age 4. If the regression curve were used to extrapolate beyond the age range for which measurements are available, there would be little to no evidence for the extrapolation to be valid while the predicted behavior exhibits a large qualitative difference to the measured behavior. Therefore, this is a "dangerous" model to use. **(2 Marks)**