Quadratic Assignment Problem (QAP)

Problem Description

We are given four sets I, J, K, and L, all of the same cardinality n. The problem involves two cost functions:

- c_{ij}: The cost of assigning person $i \in I$ to job $j \in J$.
- $d_{k\ell}$: The distance between office $k \in K$ and equipment $\ell \in L$.

The objective is to assign each element in I to exactly one element in K and vice versa, and to simultaneously assign each element in J to exactly one element in L and vice versa, minimizing the total cost. The total cost is represented by the product of c_{ij} and $d_{k\ell}$ for corresponding assignments.

Mathematical Formulation

Sets and Parameters

- I: Set of people, $|I| = n$.
- J: Set of jobs, $|J| = n$.
- K: Set of offices, $|K| = n$.
- L: Set of equipment, $|L| = n$.
- c_{ij}: Cost of assigning person $i \in I$ to job $j \in J$.
- $d_{k\ell}$: Distance between office $k \in K$ and equipment $\ell \in L$.

Decision Variables

- $x_{ik} \in \{0, 1\}$: Binary variable, 1 if person $i \in I$ is assigned to office $k \in K$, 0 otherwise.
- $y_{j\ell} \in \{0, 1\}$: Binary variable, 1 if job $j \in J$ is assigned to equipment $\ell \in L$, 0 otherwise.

Objective Function

Minimize the total cost:

$$\min \sum_{i \in I} \sum_{j \in J} \sum_{k \in K} \sum_{\ell \in L} c_{ij} \cdot d_{k\ell} \cdot x_{ik} \cdot y_{j\ell}$$

Constraints

1. **Person to office assignment:**
   Each person must be assigned to exactly one office:

$$\sum_{k \in K} x_{ik} = 1, \quad \forall i \in I$$

2. **Office to person assignment:**
   Each office must be assigned to exactly one person:

$$\sum_{i \in I} x_{ik} = 1, \quad \forall k \in K$$

3. **Job to equipment assignment:**
   Each job must be assigned to exactly one equipment:

$$\sum_{\ell \in L} y_{j\ell} = 1, \quad \forall j \in J$$

4. **Equipment to job assignment:**
   Each equipment must be assigned to exactly one job:

$$\sum_{j \in J} y_{j\ell} = 1, \quad \forall \ell \in L$$