On the separation of Cuts

Sen

Zhejiang University

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Overview

1. A brief introduction of valid inequalities families
   - lift-and-project cuts
   - Gomory mixed integer cuts
   - split cuts
   - cover inequalities
2. A introduction of implementing the cuts
   - zeor-half cuts
3. An idea of separation heuristic and GNN

Valid inequalities and Cuts for MIP

$$
\begin{aligned}
\max \quad & cx + hy \\
& Ax + Gy \leq b \\
& x \geq 0 \quad \text{integral} \\
& y \geq 0
\end{aligned} \tag{1}
$$

Valid inequalities and Cuts for MIP

$P := \{(x, y) \in R_+^n \times R_+^p : Ax + Gy \leq b\}$.
$S := P \bigcap (Z_+^n \times R_+^p)$.

$$\begin{aligned} \max \quad & cx + hy \\ & (x, y) \in \text{conv}(S) \end{aligned} \tag{2}$$

- There are usually numerous inequality for conv(S)!
- An inequality is said to be valid for a set if it is satisfied by every point in this set.
- A cut with respect to a point $(x, y) \notin \text{conv}(S)$ is a valid inequality for conv(S) that is violated by (x, y).

Separation

Definition

Given a set of feasible solutions to an MIP problem X, a formulation P for X, and a family of valid inequalities $\mathcal{F}$, the separation problem for a given point $(x^*, y^*) \in P$ is to

1. either prove that there is no valid inequality in $\mathcal{F}$ that cuts off (x^*, y^*)
2. or to find a valid inequality $\alpha x + \beta y \leq \gamma$ from $\mathcal{F}$ that cuts off (x^*, y^*), i.e. where $\alpha x^* + \beta y^* > \gamma$

Lift-and-Project(1990s)

For 0-1 linear programming

$$\begin{aligned} \min \quad & cx \\ & Ax \geq b \\ & x_j \in \{0,1\} \text{ for } j = 1, \ldots, n \\ & x_j \geq 0 \quad \text{ for } j = n+1, \ldots, n+p \end{aligned} \tag{3}$$

$P := \{x \in R_{+}^{n+p} : Ax \geq b\}$.
$S := \{x \in \{0,1\}^n \times R_{+}^{p} : Ax \geq b\}$.

Lift-and-Project

- Step 0: Select binary variable $j \in \{1, \ldots, n\}$.
- Step 1: Generate the nonlinear system $x_j(Ax - b) \geq 0, (1 - x_j)\,(Ax - b) \geq 0$.
- Step 2: Linearize the system by substituting y_i for $x_i x_j, i \neq j$, and x_j for x_j^2. Call this polyhedron M_j.
- Step 3: Project M_j onto the x-space. Let P_j be the resulting polyhedron.

$$P_j = \text{conv}\left\{(Ax \geq b, x_j = 0) \cup (Ax \geq b, x_j = 1)\right\}$$

Lift-and-Project Cuts

$$M_j := \begin{array}{l} \{x \in \mathbb{R}_+^{n+p}, y \in \mathbb{R}_+^{n+p} : \\ Ay - bx_j \geq 0, \quad Ax + bx_j - Ay \geq b, \quad y_j = x_j\} \end{array} \tag{4}$$

$$\downarrow$$

$$M_j := \begin{array}{l} \{x \in \mathbb{R}_+^{n+p}, y \in \mathbb{R}_+^{n+p-1} : \\ A_j y + \left(a^j - b\right) x_j \geq 0, Ax + \left(b - a^j\right) x_j - A_j y \geq b\} \end{array} \tag{5}$$

$$\downarrow$$

$$M_j = \left\{x \in \mathbb{R}_+^{n+p}, y \in \mathbb{R}_+^{n+p-1} : \tilde{B}_j x + A_j y \geq 0, \quad \tilde{A}_j x - A_j y \geq b\right\} \tag{6}$$

Lift-and-Project Cuts

$$P_j = \left\{ x \in \mathbb{R}_+^{n+p} : \left(u\tilde{B}_j + v\tilde{A}_j \right) x \geq vb \text{ for all } (u,v) \in Q \right\} \tag{7}$$

where $Q := \{(u,v) : uA_j - vA_j = 0, \quad u \geq 0, v \geq 0\}$

- For a given $\bar{x}$, the cut generation LP is

$$\begin{gathered} \max vb - \left(u\tilde{B}_j + v\tilde{A}_j \right) \bar{x} \\ uA_j - vA_j = 0 \\ u \geq 0, v \geq 0 \end{gathered} \tag{8}$$

Gomory Mixed Integer Cuts(1960s)

Consider a single equality constraint for a MILP:

$$S := \left\{ (x, y) \in \mathbb{Z}_+^n \times \mathbb{R}_+^p : \sum_{j=1}^{n} a_j x_j + \sum_{j=1}^{p} g_j y_j = b \right\} \tag{9}$$

Let $b = \lfloor b \rfloor + f_0$ where $0 < f_0 < 1$.
Let $a_j = \lfloor a_j \rfloor + f_j$ where $0 \leq f_j < 1$. Then

$$\sum_{j=1}^{n} (\lfloor a_j \rfloor + f_j) x_j + \sum_{j=1}^{p} g_i y_j = \lfloor b \rfloor + f_0 \tag{10}$$

$$\sum_{f_j \leq f_0} f_j x_j + \sum_{f_j > f_0} (f_j - 1) x_j + \sum_{j=1}^{p} g_j y_j = k + f_0 \tag{11}$$

Gomory Mixed Integer Cuts

where k is some integer. Since $k \leq -1$ or $k \geq 0$, any $x \in S$ satisfies the disjunction

$$\sum_{f_j \leq f_0} \frac{f_j}{f_0} x_j - \sum_{f_j > f_0} \frac{1 - f_j}{f_0} x_j + \sum_{j=1}^{p} \frac{g_j}{f_0} y_j \geq 1 \tag{12}$$

OR

$$-\sum_{f_j \leq f_0} \frac{f_j}{1 - f_0} x_j + \sum_{f_j > f_0} \frac{1 - f_j}{1 - f_0} x_j - \sum_{j=1}^{p} \frac{g_j}{1 - f_0} y_j \geq 1 \tag{13}$$

Gomory Mixed Integer Cuts

Gomory mixed integer inequality:

$$\sum_{f_j \le f_0} \frac{f_j}{f_0} x_j + \sum_{f_j > f_0} \frac{1 - f_j}{1 - f_0} x_j + \sum_{g_j > 0} \frac{g_j}{f_0} y_j - \sum_{g_j < 0} \frac{g_j}{1 - f_0} y_j \ge 1 \tag{14}$$

Gomory Mixed Integer Cuts

$$
\begin{gathered}
\max z = 5.5x_1 + 2.1x_2 \\
-x_1 + x_2 \leq 2 \\
8x_1 + 2x_2 \leq 17 \\
x_1, x_2 \geq 0 \\
x_1, x_2 \text{ integer.}
\end{gathered}
$$

Gomory Mixed Integer Cuts

$$
\begin{aligned}
z \qquad\quad + 0.58x_3 + 0.76x_4 &= 14.08 \\
x_2 + 0.8x_3 + 0.1x_4 &= 3.3 \\
x_1 \qquad - 0.2x_3 + 0.1x_4 &= 1.3 \\
x_1, x_2, x_3, x_4 &\geq 0
\end{aligned}
$$

For inequality:

$$x_2 + 0.8x_3 + 0.1x_4 = 3.3$$

Generate GMI cut:

$$\frac{1 - 0.8}{1 - 0.3}x_3 + \frac{0.1}{0.3}x_4 \geq 1$$

Split Cuts (1990s)

$P := \{x \in \mathbb{R}^n : Ax \leq b\}$
$S := \{x \in P : x_j \in \mathbb{Z}, j \in I\}$

- Given a vector $\pi \in Z^n$ such that $\pi_j = 0$ for all $j \in P \backslash I$, the scalar product is integer for all $x \in S$. Thus, for any $\pi_0 \in Z$, it follows that every $x \in S$ satisfies exactly one of the terms of the disjunction $\pi x \leq \pi_0$ or $\pi x \geq \pi_0 + 1$. We refer to the latter as a split disjunction, and say that a vector $(\pi, \pi 0) \in Z^n \times Z$ such that $\pi_j = 0$ for all all $j \in P \backslash I$ is a split.
- Given P and I, an inequality $\alpha x \leq \beta$ is a split inequality if there exists a split (π, π_0) such that $\alpha x \leq \beta$ is valid for both sets

$$\Pi_1 := P \cap \{x : \pi x \leq \pi_0\}$$
$$\Pi_2 := P \cap \{x : \pi x \geq \pi_0 + 1\}$$

Split Cuts

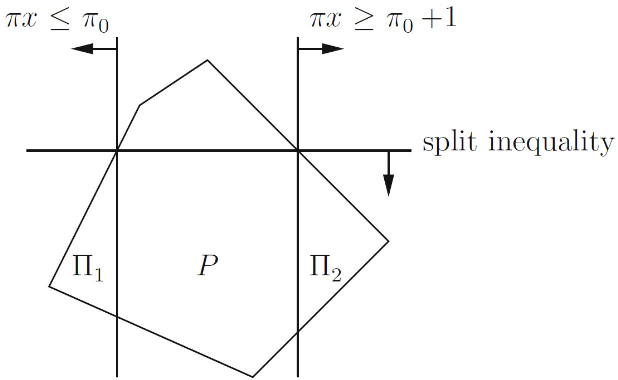

Figure: split cuts

Split Cuts

Theorem

Let $P := \{(x, y) \in R^n_+ \times R^p_+ : Ax + Gy \leq b\}$ *be a rational polyhedron and let* $S := P \cap (Z^n \times R^p)$. *The split closure relative to* P *is identical to the Gomory mixed integer closure relative to* P.

Cover Inequalities

- The 0,1 knapsack set $K := \left\{x \in \{0,1\}^n : \sum_{j=1}^n a_j x_j \leq b\right\}$
- A cover is a subset $C \subseteq N$ such that $\sum_{j \in C} a_j > b$ and it is minimal if $\sum_{j \in C \setminus \{k\}} a_j \leq b$ for all $k \in C$. For any cover C, the cover inequality associated with C is

$$\sum_{j \in C} x_j \leq |C| - 1$$

  and it is valid for $\text{conv}(K)$.

Separation of Cover Inequalities

Separation problem

$$\zeta = \min \left\{ \sum_{j \in C} (1 - \bar{x}_j) : C \text{ is a cover for } K \right\}$$

If $\zeta \geq 1$, then $\bar{x}$ satisfies all the cover inequalities for K. If $\zeta < 1$, then an optimal cover yields a violated cover inequality.
Assuming that $a_i \in Z$ for $i \in N$, and b are integer, problem above can be formulated as the following integer program

$$\zeta = \min \sum_{j=1}^{n} (1 - \bar{x}_j) z_j$$

$$\sum_{j=1}^{n} a_j z_j \geq b + 1$$

$$z \in \{0, 1\}^n$$

Zero-Half Cuts

- A Chvátal-Gomory (CG) cut is a valid inequality for P_I of the form $\lambda^T Ax \leq \lfloor \lambda^T b \rfloor$, where $\lambda \in R_+^m$ is such that $\lambda^T A \in \mathbb{Z}^n$.
- CG cuts can equivalently be obtained in the following way. Let $\mu \in \mathbb{Z}_+^m$ and $q \in \mathbb{Z}_+$ be such that $\mu^T A \equiv 0 (\bmod q)$ and $\mu^T b = kq + r$ with $k \in \mathbb{Z}$ and $r \in \{1, \ldots, q-1\}$. Then the mod-q cut $\mu^T Ax \leq \mathrm{kq}$ is a valid inequality for P_I. When q=2, we get zero-half cuts.
- Zero-half cuts provide tight approximation of P_I.

A simple example of zero-half cut

Zero-half cuts are based on the observation that when the left-hand side of an inequality consists of integral variables and integral coefficients, then the right-hand side can be rounded down to produce a zero-half cut.

$$
\begin{array}{llllll}
x_1 & +2x_2 & +x_3 & +3x_4 & & \leq 8 \\
x_1 & & +3x_3 & +x_4 & +2x_5 & \leq 5
\end{array}
\tag{15}
$$

Add together we get

$$
\begin{array}{llllll}
2x_1 & +2x_2 & +4x_3 & +4x_4 & +2x_5 & \leq 13
\end{array}
\tag{16}
$$

Zero-Half Cuts

Divide constraint (10) by 2

$$x_1 \quad +x_2 \quad +2x_3 \quad +2x_4 \quad +x_5 \quad \leq 6.5 \tag{17}$$

Since x_i is integer for $i = \{1, 2, 3, 4, 5\}$, the RHS can be round down to integer, then we get a zero-haf cut.

$$x_1 \quad +x_2 \quad +2x_3 \quad +2x_4 \quad +x_5 \quad \leq 6 \tag{18}$$

Separation Problem of Zero-Half Cuts

For a given relaxation solution x^*, the separation problem is

$$z_{SEP} = \min\{{s^*}^T \mu : \mu \in F(\bar{A}, \bar{b})\}$$

Where $s^* = b - Ax^*$,

$$\bar{A} = A(mod2), \ \ \bar{b} = b(mod2)$$
$$F(\bar{A}, \bar{b}) = \{\mu \in \{0,1\}^m : \mu^T \bar{b} = 1(mod2), \ \ \mu^T \bar{A} = 0(mod2)\}.$$

Separation Problem of Zero-Half Cuts

In the example above

$$\bar{A} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\bar{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mu = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Separation Heuristic

- Let $G = (V \cup \{q\}, E)$, a undirected multigraph in which vertex $j \in V$ presents column j in $\bar{A}$ and edge e_i denote row $i \in M$, M is the set of rows of $\bar{A}$.
- For each edge, its weight is s_i^*. the edge is labeled odd if $\bar{b}_i = 1$ and even otherwise.

Separation Heuristic

Let $O_i := \{j \in N : \bar{a}_{ij} = 1\}$, for all $i \in M$

- **Consider a scenario that** $|O_i| \leq 2$ **for all** $i \in M$.
- The edge of row i connects the two vertices h and k such that $O_i = \{h, k\}$ (if $O_i = \{h\}$, then let the edge connect vertex h to the special vertex q)
- There is a one-to-one correspondence between the 0-1 vectors $\mu \in F(\bar{A}, \bar{b})$ and the odd Eulerian cycles (an odd Eulerian cycle is an Eulerian cycle contains an odd number of odd edges).
- The separation problem is then equivalent to finding a minimum-weight odd Eulerian cycle.

Separation Heuristic

Let $S_j := \{i \in M : \bar{a}_{ik} = 1 \iff k = j\}$, for all $j \in N$

- **Consider the general scenario that** $|O_i| \geq 0$ **for all** $i \in M$.
- For each pair of vertices h, k in G, there may be more than one edge connecting h and k, but we only need the odd and even edges with minimum weight which are denoted as $odd(h, k)$ and $even(h, k)$.
- For row i with $|O_i| = 1$, $\delta_j^p := \min\left\{s_i^* : i \in S_j, \bar{b}_i = p\right\}$ for $p = 0, 1$, $odd(h, q) = \delta_h^1$
- For row i with $|O_i| = 2$, we set $odd(h, k) = \min\{odd(h, k), s_i^*\}$
- For row i with $|O_i| \geq 3$, consider each pairs $h, k \in O_i$. Update $odd(h, k) = \min\{odd(h, k), best^1\}$.

Loop

For j $\in O_i \setminus \{h, k\}$: $(Set\ best^1 = s_i^*)$
old _ best^p = $best^p$ (p=0,1)
best^p = $\min\{old\_best^p + \delta_j^0, old\_best^{1-p} + \delta_j^1\}$ (p=0,1)

Separation Heuristic

- A heuristic is implemented to find minimum-weight odd Eulerian cycles.
- The quality of cut $\alpha x \leq \alpha_0$ are measured by

$$Q(x^*, \alpha, \alpha_0) = \frac{|\alpha x^* - \alpha_0|}{||\alpha||} \tag{19}$$

- The generated cuts are added in a CUT_POOL, in which they are sorted by measurement of quality and similarity, and then added to the current LP.

An idea of separation heuristic and GNN

In practice, we usually care about the following questions:

1. The heuristic to find cuts.
2. The number of cuts we add.
3. The choice of "good" cuts.

An idea of separation heuristic and GNN

Use a heuristic to generate a set of cuts denoted by C.
Formulate a MILP.

$$\begin{aligned} \min_y \; & max_x(c^T x + \epsilon \textstyle\sum_{i=1}^{|C|} y_i) \\ \text{s.t.} \; & (Original\ \ Constraints) \\ & \alpha_i x \le \alpha_i^0 y_i + (1 - y_i)M \\ & \qquad\qquad for\ \ i \in C \\ & x \in R^{n+p} \\ & y \in \{0,1\}^{|C|} \end{aligned} \tag{20}$$

- The solution of this MILP can be used as labels of the generated cuts. Then the problem can be considered as a binary classification problem.
- A GNN model may be used to predict which cut we should add.
- Experiment is needed.

References

Giuseppe Andreello, Alberto Caprara, Matteo Fischetti. (2007)
Embedding $\{0, \frac{1}{2}\}$-Cuts in a Branch-and-Cut Framework: A Computational Study.
NFORMS Journal on Computing 19(2):229-238.

Cornuéjols, G. (2008)
Valid inequalities for mixed integer linear programs.
Math. Program. 112, 3–44

Conforti, M., Cornuejols, G. and Zambelli, G.,(2014)
Integer programming. 1st ed.
Cham: Springer, pp.195-315.

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