My Story of Ployhedral Theory of Mixed-integer Programming

Xue Sen

March 22, 2022

1 Basic Form of

Problems of the form

$$\begin{cases} Max \quad cx + hy \\ \qquad Ax + Gy \leq b \\ \qquad x \geq 0 \quad integral \\ \qquad y \geq 0, \end{cases} \tag{1}$$

where $c \in Q^n$, $h \in Q^p$, the matrices $A \in Q^{m \cdot n}$, $G \in Q^{m \cdot p}$ and $b \in Q^m$. Set S of feasible solution to the problem above is called a mixed integer set when $p \geq 1$.

2 Fundamental Definition

Here we explain some concepts of Ployhedral Theory.

Polyhedron : a ployhedron in R^n is a set of the form $P = \{x \in R^n : Ax \leq b\}$ where A is a real matrix and b a real vector.

Rational Ployhedron : a Rational Ployhedron is a ployhedron and its A and b are rational.

Ployhedron Cone : a ployhedron cone is a ployhedron of the form $\{x \in R^n : Ax \leq 0\}$

Convex Hull : a convex hull of a set S is condensed as conv(S). Conv(S)=$\{x \in R^n x = \sum_{i=1}^{k} \lambda_i x^i, \quad where\ k \geq 1\ and\ \sum \lambda_i = 1, \quad \lambda_i > 0, \quad x^i \in S\}$

Proporties of convex hull:

Ploytope : convex hull of a finite set of points in R^n is called a ploytope.

Conic Hull : the conic hull of a nonempty set $S \in R^n$ is cone(S)=$\{x \in R^n \quad x = \sum_{i=1}^{k} \lambda_i x^i \quad wherek \geq 1and\lambda > 0, \quad x^i \in S\}$. If S is a finite set then cone(S) is said to be finitely generated.

Ray : Given a cone C and $r \in C - \{0\}$, the set cone(r)=$\{\lambda r : \lambda \geq 0\}$ is called a ray of C.

Pointed Cone : We say a cone C is pointed if for every $r \in C - \{0\}$, $-r \notin C$.

3 Projections

Let P$\subseteq R^{n+p}$. The projection of P onto the x-space R^n is

$$Proj_x P = \{x \in R^n : \exists y \in R^p with (x, y) \in P\}$$

4 Valid inequalities

An inequality $cx \leq \sigma$ is valid for the set $P \subseteq R^n$ if $cx \leq \sigma$ is satisfied by every point in P.

5 Facets

Let $P = \{x \in R^n : Ax \leq b\}$ be a polyhedron. A face of P is a set of the form:

$$F = P \cap \{x \in R^n : cx = \sigma\}$$

where $cx \leq \sigma$ is a valid inequality for P.

A face is itself a polyhedron.

A face is said proper if it is nonempty and properly contained in P. Maximal proper faces of P are called facets.

6 Relationship of Feasible

Theorem 1. *A system of linear equation Ax=b is infeasible* $\Longleftrightarrow$ $uA = 0$, $ub < 0$ *is feasible.*

Proof: using Gaussian elimination on Ax=b.

Theorem 2. *A system of linear inequalities Ax≤b is infeasible* $\Longleftrightarrow$ *the system* $uA = 0$, $ub < 0$, *u≥0 is feasible.*

Theorem 3. *Linear Programming Duality:*

$$P = \{x : Ax \leq b\} \quad and \quad D = \{u : uA = c, u \geq 0\}$$

if P and D are both nonempty, then

$$max\{cx : x \in P\} = min\{ub : u \in D\}$$

Example:

The proof of the form of duality using Lagrange relaxation:

Suppose a integer programming problem:

$$Min \quad cx$$

$$S.t.$$

$$Ax \geq b$$

$$x \geq 0$$

We assume the lagrange function:

$$L(u) = cx - u^T(Ax - b) \qquad u \geq 0$$

$$Min\{L(u)\} \leq cx^* - u^T(Ax^T - b) \leq cx^*$$

Convert the problem as find the maximum lower bound of the original problem.

$$Max\{u^T b + Min_{x \geq 0}\{(c - u^T A)x\}\}$$

if $c - u^T A < 0$, then $Min\{(c - u^T A)x\} = -\infty$

So we need only another situation

$$Max \quad u^T b$$

$$s.t. \quad u^T A \leq c$$

$$u \geq 0$$

Theorem 4. *Minkowski-Weyl theorem for polyhedra:*

For a subset P of R^n*, the following two conditions are equivalent:*

1. *P is a polyhedron, i.e., there is a matrix A and a vector b such that* $P = \{x \in R^n : \quad Ax \leq b\}$
2. *There exist vectors* $v^1, ..., v^p, r^1, ..., r^q$ *such that*

$$P = conv(v^1, ..., v^p) + cone(r^1, ..., r^q)$$

PRIMAL	minimize	maximize	DUAL
constraints	$\geq b_i$	≥ 0	**variables**
	$\leq b_i$	≤ 0	
	$= b_i$	free	
variables	≥ 0	$\leq c_j$	**constraints**
	≤ 0	$\geq c_j$	
	free	$= c_j$	

Figure 1: Dual Problem

7 Union of polyhedra

Let $P_i = \{x \in R^n : A_i x \leq b^i\}, \quad i = 1, ..., k.$

$\overline{conv}(\cup_{i=1}^k P_i)$ is the smallest closed convex set that contains $\cup_{i=1}^k P_i$.

We can rise an example to show that $conv(P_1 \cup P_2)$ may not be a closed set.

Theorem 5. *According to Minkowski-Weil's Theorem, let $P_i = Q_i + C_i$ be nonempty polyhedra i=1,...,k. Then $Q = conv(\cup_{i=1}^k Q_i)$ is a polytope, $C = conv(\cup_{i=1}^k C_i)$ is a finitely generated cone.*

$$\overline{conv}(\cup_{I=1}^k P_i) = Q + C$$

8 Split disjunctions

Let $P = \{(x, y) \in R^n \times R^p : Ax + Gy \leq b\}$ and let $S = P \cap (Z^n \times R^p)$. For $\pi \in Z^n$ and $\pi_0 \in Z$, define

$$\Pi_1 = P \cap \{(x, y) : \pi x \leq \pi_0\}$$

$$\Pi_2 = P \cap \{(x, y) : \pi x \geq \pi_0 + 1\}$$

Clearly $S \subseteq \Pi_1 \cup \Pi_2$ and therefore $conv(S) \subseteq conv(\Pi_1 \cup \Pi_2)$.

9 One-side splits, Chvatal inequalities

Let P and S be the same as the last section. Let $\pi \in Z^n, \quad z = max\{\pi x (x, y) \in P\}$. A split defined by $(\pi, \pi_0) \in Z^n \times Z$ is a one-side split for P if

$$\pi_0 \leq z \leq \pi_0 + 1$$

10 Gomory's mixed-integer inequalities

Let $P = \{(x, y) \in R_+^n \times R_+^p : Ax + Gy \leq b\}$ and let $S = P \cap (Z^n \times R^p)$. Here P is defined by a system of inequalities together with nonnegativity constraints. Any system of linear programming can be converted into a system of this type.

Consider

$$Ax + Gy + Is = b, x, y, s \geq 0 \tag{2}$$

For $\lambda \in Q^m$, Consider

$$\sum_{j=1}^{n} a_j^\lambda x_j + \sum_{j=1}^{p} g_j^\lambda y_j + \sum_{i=1}^{m} \lambda_i s_i = \beta^\lambda \tag{3}$$

11 Decomposition

11.1 Lagrangian

The conventional statement of a general problem(P) of optimization from this point of view is

$$\begin{aligned}&\text{minimize } f_0(x) \text{ over all } x \in X\\ &\text{such that } f_i(x) \begin{cases} \leq 0 & \text{for } i = 1, \dots, s, \\ = 0 & \text{for } i = s+1, \dots, m. \end{cases}\end{aligned} \tag{4}$$

We'll refer to the convex case of (P) when the objective and inequality constraint functions $f_0, f_1, ..., f_s$ are convex and the equality constraint functions $f_{s+1}, ..., f_m$ are affine(linear-plus-constant).

Optimality conditions for (P) involve the Lagrangian function.

$$L(x, y) = f_0(x) + y_1 f_1(x) + \cdots + y_m f_m(x) \text{ for } x \in X \text{ and } y \in Y \tag{5}$$

where

$$Y = \mathbb{R}^s_+ \times \mathbb{R}^{m-s} = \{y = (y_1, \dots, y_m) \mid y_i \geq 0 \text{ for } i \in [1, s]\} \tag{6}$$

f has the Lagrangian representation

$$f(x) = \sup_{y \in Y} L(x, y) = \sup_{v \in Y} \{f_0(x) + y_1 f_1(x) + \ldots + y_m f_m(x)\} \text{ for } x \in X, \tag{7}$$

we can state the problem(D)

$$\text{maximize } g(y) = \inf_{x \in X} \{f_0(x) + y_1 f_1(x) + \ldots + y_m f_m(x)\} \text{ over } y \in Y \tag{8}$$

Theorem 6. *In the convex case of $(\mathcal{P})$, the existence for $\bar{x}$ of a multiplier vector $\bar{y}$ satisfying the Lagrange multiplier rule $(\mathcal{L})$ is sufficient for $\bar{x}$ to be a globally optimal solution to $(\mathcal{P})$. The vectors $\bar{y}$ that appear in this condition along with $\bar{x}$ are then precisely the optimal solutions to the dual problem $(\mathcal{D})$, and the optimal values in the two problems agree: one has*

$$\min(\mathcal{P}) = \max(\mathcal{D}) \tag{9}$$

12 Test Data

Test set	Type	Size	Problem class	Ref. Origin
MIPLIB	Mixed	30		Mixed
CORAL	Mixed	38		Mixed
MILP	Mixed	37		Mixed
ENLIGHT	IP	7		Combinatorial game
http://miplib.zib.de/contrib/AdrianZymolkal				
ALU	IP	25		Infeasible chip verification
http://miplib.zib.de/contrib/ALU/				
FCTP	MBP	16		Fixed charge transportation
ACC	BP	7		Sports scheduling
FC	MBP	20		Fixed charge network flow
ARCSET	IP/MIP	23		Capacitated network design
MIK	MIP	41		Mixed integer knapsack