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Identifying Minimally Infeasible Subsystems of Inequalities

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Given an infeasible system of linear inequalities, we show that the problem of identifying all minimally infeasible subsystems can be reduced to the problem of finding all vertices of a related polyhedron. This results in a shorter enumeration than that performed by a previous method to solve this problem.

Let A be a rational $m \times n$ matrix and let b be a rational m-vector. Suppose that a model developer has determined that the system of linear inequalities $Ax \leq b$ is infeasible. We consider the problem of achieving a consistent system by dropping as few inequalities as possible. In order to do this, the *minimally infeasible subsystems* must be identified. A minimally infeasible subsystem is a subsystem of $Ax \leq b$ that is infeasible, but which could be made feasible by dropping any inequality from it. In order to achieve a consistent set of inequalities, the model developer must drop one constraint from each of the minimally infeasible subsystems.

Given the minimally infeasible subsystems, the problem of finding a maximum cardinality feasible subsystem is an integer covering problem. Let y_i, $i = 1, \ldots, m$ be a variable whose value will be 1 if the ith constraint is chosen to be deleted, and 0 otherwise. Suppose there are r minimally infeasible subsystems and let S_j, $j = 1, \ldots, r$, be the set of indices of the inequalities in the jth minimally infeasible subsystem. Then the solution to the following integer covering problem gives the minimum number of inequalities that must be dropped in order to achieve a consistent system:

$$\min 1^T y$$

subject to

$$\sum_{i \in S_j} y_i \geq 1, \quad j = 1, \ldots, r,$$

$$y_i \geq 0, \quad i = 1, \ldots, m.$$

Any set covering heuristic could be used to find a near maximum cardinality feasible subsystem.

In [7], Jeroslow and Wang have shown that integer programming problems arising from the problem of satisfying Horn Clauses in propositional logic, can be solved as linear programming problems. Thus, the methods described here could be used to identify minimal inconsistencies in a logical rule base constructed using Horn Clauses.

A strongly redundant inequality constraint can be identified by reversing the sense of the inequality and then testing for infeasibility. The minimally infeasible system will, in this case, give a minimal dependency set for the redundant constraint (see [6]).

Methods for diagnosing infeasibility in networks have been studied by Greenberg (see [4] and [5]). In [3], Glover and Greenberg discuss the analysis of infeasibility in integer programming problems through "logical testing."

In general, the number of minimally infeasible subsystems is exponential in the number of constraints. Thus enumerating all minimally infeasible subsystems is unavoidably a computationally intensive process. The method of enumeration we propose is a pivoting method, as is an earlier method proposed by Van Loon.[8] However, whereas Van Loon's method in general enumerates many bases that do not give information about the minimally infeasible subsystems, our method enumerates only bases that do correspond to minimally infeasible subsystems.

Van Loon's method discovers all minimally infeasible subsystems through enumerating the bases of the system $\{(x, s) \in Q^{n+m} \mid Ax + Is = b\}$. In general, a given minimally infeasible subsystem will be enumerated many times, and bases will be enumerated that do not necessarily correspond to minimally infeasible subsystems.

We show that the minimally infeasible subsystems are in one to one correspondence with the vertices of the polyhedron, $P = \{y \in Q^m \mid y^T A = 0,\ y^T b \leq -1,$

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$y \geq 0\}$. This is a simple consequence of Farkas's Lemma and elementary polyhedral theory. Thus enumeration of the minimally infeasible subsystems is accomplished by enumerating the vertices of P. In the absence of degeneracy, the bases enumerated are in one to one correspondence to the minimally infeasible subsystems. If degeneracy is present, several bases may correspond to the same vertex of P (and hence the same minimally infeasible subsystem).

Enumerating the vertices of P involves listing at most (and usually less than) $\binom{m}{n}$ feasible bases. Van Loon's method on the other hand requires the enumeration of $\binom{m+n}{m}$ bases.

1. Identifying Minimally Inconsistent Subsystems

Given a rational vector y, the *support* of y will denote the indices of its nonzero components.

Theorem. *Let A be a rational $m \times n$ matrix and let b be a rational m-vector. Then the indices of the minimally infeasible subsystems of the system $Ax \leq b$ are exactly the supports of the vertices of the polyhedron $P = \{y \in Q^m \mid y^TA = 0, y^Tb \leq -1, y \geq 0\}$.*

An efficient algorithm for finding all extreme points of a polytope or polyhedron is given by Dyer.[1] In the absence of degeneracy, the algorithm, called ENUMERATE, has a worst case time bound of $O(st^2v)$ where s is the number of inequalities, t is the number of variables, and v is the number of extreme points. In the presence of degeneracy, v in the above bound would be replaced by the number of feasible bases.

Example. Consider the following example (from [8]).

$$\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ -1 & -1 \\ 0 & -1 \\ -2 & -1 \end{bmatrix} x \leq \begin{bmatrix} 0 \\ 1 \\ -2 \\ -2 \\ -4 \end{bmatrix} \begin{matrix} (1) \\ (2) \\ (3) \\ (4) \\ (5) \end{matrix}$$

This system has the minimally infeasible subsystems: $\{1, 2, 3\}$, $\{1, 2, 5\}$, and $\{2, 4\}$. In order to find the minimally infeasible subsystems, one must traverse the vertices of the polyhedron

$$P = \left\{ y \mid y^T \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ -1 & -1 \\ 0 & -1 \\ -2 & -1 \end{bmatrix} = 0, \quad y^T \begin{bmatrix} 0 \\ 1 \\ -2 \\ -2 \\ -4 \end{bmatrix} \leq -1, \; y \geq 0 \right\}.$$

After three pivots which are required to find an initial feasible tableau, Dyer's method[1] makes 4 more pivots and completes the visitation of the vertices of P. The three vertices are as follows: i) $y_1 = 1$, $y_2 = 1$, $y_3 = 1$, $y_4 = 0$, $y_5 = 0$; ii) $y_1 = 0$, $y_2 = 1/3$, $y_3 = 0$, $y_4 = 2/3$, $y_5 = 0$; and iii) $y_1 = 4/5$, $y_2 = 3/5$, $y_3 = 0$, $y_4 = 0$, $y_5 = 2/5$. The supports of these three extreme points are the index sets of the three minimally infeasible subsystems given above. Van Loon's method will take 11 pivots to identify all minimally infeasible subsystems.

Before proving the theorem, we need to present the following lemmas. Lemmas 1 and 2 are both simple results in elementary polyhedral theory. For completeness, we have included a proof of Lemma 1. A proof of Lemma 2 can be found in [2] for example.

For ease of notation, we will denote a ray $\{\lambda y \mid \lambda \geq 0\}$, where $y \neq 0$ is a vector, simply by y.

Lemma 1. *Let $K = \{y \in Q^m \mid y^TA = 0, y \geq 0\}$. Let S be the set of extreme rays y of K satisfying $y^Tb < 0$. Without loss of generality scale the elements of S so that $y^Tb = -1$ for all $y \in S$. Then S is equal to the set of extreme points of $P = \{y \in Q^m \mid y^TA = 0, y^Tb \leq -1, y \geq 0\}$.*

Proof. Let $K(S)$ denote the cone generated by S. It is easy to check that S actually forms a set of extreme rays of $K(S)$, and that $P \subseteq K(S)$. Thus it suffices to show that the extreme rays y of $K(S)$ (scaled so $y^Tb = -1$) are exactly the extreme points of P.

Let y be an extreme ray of $K(S)$, scaled so that $y^Tb = -1$, and suppose that it is not an extreme point of P. Then there exist $y_i \in P$, $i = 1, \ldots, k$, and $\lambda_i \geq 0$, $i = 1, \ldots, k$, with $y = \sum_{i=1}^k \lambda_i y_i$ and $\sum_{i=1}^k \lambda_i = 1$. Since $P \subseteq K(S)$, each $y_i \in K(S)$, thus contradicting the fact that y is extreme in $K(S)$.

Now suppose that y is extreme in P, and not extreme in $K(S)$. Then there exist $y_i \in K(S)$, $i = 1, \ldots, k$, and $\lambda_i \geq 0$, $i = 1, \ldots, k$, with $y = \sum_{i=1}^k \lambda_i y_i$. By also scaling the λ_i's, the y_i's can be scaled so that each $y_i{}^Tb = -1$, and thus each $y_i \in P$. Then $-1 = y^Tb = \sum_{i=1}^k \lambda_i y_i{}^Tb = -\sum_{i=1}^k \lambda_i$. Thus $\sum_{i=1}^k \lambda_i = 1$ and we have written y as a convex combination of elements of P. Thus y cannot be extreme in P, which gives a contradiction. ■

We also need to make use of the following version of the well-known Farkas Theorem of the Alternative (see e.g. [2]).

Lemma 2. *Let A be an $m \times n$ rational matrix and let b be a rational m-vector. Then exactly one of the following statements holds:*

1. *there exists $x \in Q^n$ such that $Ax \leq b$;*
2. *there exists $y \in Q^m$ with $y \geq 0$, $y^TA = 0$, and $y^Tb < 0$.*

We now can prove the theorem.

Proof of Theorem. Let K be defined as in the statement of Lemma 1. Then from Lemma 1, it suffices to show that the supports of the extreme rays of K satisfying $y^T b < 0$ index the minimally infeasible subsystems.

Let $A_1 x \leq b_1$ be a subsystem of $Ax \leq b$ which is minimally infeasible, and let m_1 be the number of rows of A_1. Assume that the rows of the system $Ax \leq b$ have been reordered so that the rows from $A_1 x \leq b_1$ are the first m_1 rows. Then by Lemma 2, there exists $w \in Q^{m_1}$ with $y \equiv (w, \underline{0}) \in K$ and $y^T b < 0$. Note that $w > 0$, for if not, applying Lemma 2 to the subset of rows of A_1 corresponding to the nonzero components of w would show that A_1 is not *minimally* infeasible. Thus $w > 0$ and the support of y indexes the rows of the minimally infeasible subsystem $A_1 x \leq b_1$. Now we must show that there is an extreme ray of K having the same property. Suppose y is not an extreme ray. Then there exist $y_i \in K$, $i = 1, \ldots, k$, and $\lambda_i \geq 0$, $i = 1, \ldots, k$, with $y = \sum_{i=1}^{k} \lambda_i y_i$. Without loss of generality we can assume that each of the y_i are extreme rays of K. Further, for at least one of the y_i, (say y_1), $y_1{}^T b < 0$. Note that since each $\lambda_i \geq 0$ and each $y_i \geq 0$ we must have that the support of y_1 is contained in the support of y. However the support of y_1 cannot be *properly* contained in that of y, or as above, $A_1 x \leq b_1$ would not be minimally infeasible. Thus the support of y_1 indexes the rows of the minimally infeasible subsystem $A_1 x \leq b_1$.

Now suppose that y is an extreme ray of K and that $y^T b < 0$. Let $A_1 x \leq b_1$ denote the subsystem of $Ax \leq b$ indexed by the support of y, and let m_1 be the number of rows of A_1. Let $w \in Q^{m_1}$ be the vector consisting of the nonzero components of y. Then $w \geq 0$, $w^T A_1 = 0$, and $w^T b_1 < 0$, so that $A_1 x \leq b_1$ is infeasible, by Lemma 2. Suppose that the subsystem $A_1 x \leq b_1$ is not minimally infeasible. Then, again applying Lemma 2, there is a vector u with $u \geq 0$, $u^T A = 0$, and $u^T b < 0$ whose support is properly contained in that of y. Without loss of generality, scale u so that $y - u \geq 0$. Note that $y - u$ will have at least one nonzero component. Further, $y - u \in K$. Then $y = u + (y - u)$, and thus is a nontrivial combination of elements of K. This contradicts the assumption that y was extreme in K and hence the system $A_1 x \leq b_1$ is minimally infeasible. ∎

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