1 Modal analysis of the IEA 10 MW blade

1.1 Modal shape analysis

1.1.1 Complex modal shape

The FEM governing equation of a single blade can be written as:

$$[M]_{[N\times N]}\{\ddot{x}\}_{[N\times 1]} + [C]_{[N\times N]}\{\dot{x}\}_{[N\times 1]} + [K]_{[N\times N]}\{x\}_{[N\times 1]} = \{f\}_{[N\times 1]} \quad (1)$$

where $[M]$,$[C]$ and $[K]$ are the mass, damping and stiffness matrix, respectively. N is the structural DOFs.

The Eq. (1) can be written in the state space as:

$$\{\dot{y}\}_{2N\times 1} = [A]_{2N\times 2N}\{y\}_{2N\times 1} + [B]_{2N\times N}\{f\}_{N\times 1}, \quad (2)$$

where

$$\{y\} = \begin{Bmatrix} \{x\} \\ \{\dot{x}\} \end{Bmatrix}_{[2N\times 1]}, \quad \{\dot{y}\} = \begin{Bmatrix} \{\dot{x}\} \\ \{\ddot{x}\} \end{Bmatrix}_{[2N\times 1]},$$
$$[A] = \begin{bmatrix} 0 & I \\ -[M]^{-1}[K] & -[M]^{-1}[C] \end{bmatrix}, \quad [B] = \begin{bmatrix} 0 \\ [M]^{-1} \end{bmatrix},$$

$\{y\}$ is the state vector, $\{\dot{y}\}$ is the state vector derivative, $[A]$ is the state matrix, and $[B]$ is the input matrix.

The eigenvalue analysis of $[A]$ is:

$$\begin{aligned} &\begin{bmatrix} 0 & I \\ -[M]^{-1}[K] & -[M]^{-1}[C] \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \lambda\mathbf{x} \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{x} \\ \lambda\mathbf{x} \end{bmatrix} \\ &\Rightarrow -\omega^2\mathbf{x} - \lambda[M]^{-1}[C]\mathbf{x} = \lambda^2\mathbf{x} \quad ([C] = 2\zeta\omega[M]) \\ &\Rightarrow \lambda^2 + 2\zeta\omega\lambda + \omega^2 = 0, \end{aligned} \quad (3)$$

where ζ is the damping ratio, ω is the undamped natural frequency. Then we can get the conjugate eigenvalue pair as $-\zeta\omega_n \pm i\sqrt{1-\zeta^2}\omega_n$, where n

represents the order number. The eigenvalues can be expressed as:

$$
\begin{aligned}
[\Lambda] &= \begin{bmatrix} \lambda_1 & & & & & \\ & \ddots & & & 0 & \\ & & \lambda_N & & & \\ & & & \lambda_1^* & & \\ & 0 & & & \ddots & \\ & & & & & \lambda_N^* \end{bmatrix} \\
&= \begin{bmatrix} \sigma_1 + j\omega_1 & & & & & \\ & \ddots & & & 0 & \\ & & \sigma_N + j\omega_N & & & \\ & & & \sigma_1 - j\omega_1 & & \\ & 0 & & & \ddots & \\ & & & & & \sigma_N - j\omega_N \end{bmatrix}.
\end{aligned} \tag{4}
$$

And eigenvectors can be expressed as:

$$
\begin{aligned}
[\Psi'] &= \begin{bmatrix} \Psi_1' & \Psi_2' & \cdots & \Psi_N' & \Psi_1'^* & \Psi_2'^* & \cdots & \Psi_N'^* \end{bmatrix} \\
&= \begin{bmatrix} \Psi & \Psi^* \\ \Psi\Lambda & \Psi^*\Lambda^* \end{bmatrix}_{[2N\times 2N]},
\end{aligned} \tag{5}
$$

where $(-)^*$ represents the conjugate value. For the free vibration case, the Eq. (2) can be written as:

$$
\{\dot{y}\}_{2N\times 1} - [A]_{2N\times 2N}\{y\}_{2N\times 1} = 0, \tag{6}
$$

For the complex modal shape, the eigenvector $[\Psi']$ is normalized as:

$$
\begin{aligned}
[\Psi']^T[\Psi'] &= [I], \\
[\Psi']^T[A][\Psi'] &= \mathrm{diag}[a_i, a_i^*] \quad (a_i = \lambda_i),
\end{aligned} \tag{7}
$$

the proof process is:

$$
\begin{aligned}
&([A] - \lambda_r[I])\,[\Psi_r'] = 0 \Rightarrow [\Psi_s']^T\,([A] - \lambda_r[I])\,[\Psi_r'] = 0 \quad (r^{th}\text{ order}) \\
&[\Psi_r']^T\,([A] - \lambda_s[I])\,[\Psi_s'] = 0 \Rightarrow [\Psi_s']^T\,\big([A]^T - \lambda_s[I]\big)\,[\Psi_r'] = 0 \quad (s^{th}\text{ order}) \\
&\Rightarrow [\Psi_s']^T[A][\Psi_r'] = 0 \quad ([A]^T = [A], \lambda_s \neq \lambda_r).
\end{aligned} \tag{8}
$$

Then we have the complex modal weight $\{Y'\}$ as:

$$\{y\} = \begin{Bmatrix} \{x\} \\ \{\dot{x}\} \end{Bmatrix} = \begin{bmatrix} \Psi & \Psi^* \\ \Psi\Lambda & \Psi^*\Lambda^* \end{bmatrix} \begin{Bmatrix} \{Y\} \\ \{Y\}^* \end{Bmatrix} = [\Psi'] \{Y'\}, \tag{9}$$

bring to the Eq. (6), we can get:

$$\{\dot{Y}'\} - \mathrm{diag}[a_i, a_i^*]\{Y'\} = 0 \Rightarrow \dot{Y}_r - a_r Y_r = 0 \quad (a_r = \lambda_r), \tag{10}$$

then we can get:

$$Y_r = Y_{r0} e^{\lambda_r t} \Rightarrow \begin{Bmatrix} \{Y\} \\ \{Y^*\} \end{Bmatrix} = \mathrm{diag}[e^{\lambda_i t}, e^{\lambda_i^* t}] \begin{Bmatrix} \{Y_0\} \\ \{Y_0^*\} \end{Bmatrix}, \tag{11}$$

finally we can get:

$$\{x\} = \Psi \mathrm{diag}[e^{\lambda_i t}]\{Y_0\} + \Psi^* \mathrm{diag}[e^{\lambda_i^* t}]\{Y_0\}^*, \tag{12}$$

for the l location, the response is:

$$x_l(t) = \sum_{r=1}^{N} \psi_{lr} e^{\lambda_r t} y_{r0} + \sum_{r=1}^{N} \psi_{lr}^* e^{\lambda_r^* t} y_{r0}^*, \tag{13}$$

and we have the modal shape, initial condition and complex eigenvalues expressed as:

$$\psi_{lr} = \eta_{lr} e^{j\gamma_{lr}}, \quad y_{r0} = T_r e^{j\theta_r}, \quad \lambda_r = \alpha_r + j\beta_r, \tag{14}$$

so the free vibration at l location is:

$$x_l(t) = \sum_{r=1}^{N} \eta_l r T_r e^{\alpha_r t} [e^{j(\beta_r t + \gamma_{lr} + \theta_r)} + e^{-j(\beta_r t + \gamma_{lr} + \theta_r)}] = 2 \sum_{r=1}^{N} \eta_l r T_r e^{\alpha_r t} \cos(\beta_r t + \gamma_{lr} + \theta_r) \tag{15}$$

In OpenFAST, the linearization method is applied to estimate the state space matrix [1]:

$$\begin{aligned} \Delta\dot{x} &= A\Delta x + B\Delta u^+ \\ \Delta y &= C\Delta x + D\Delta u^+ \end{aligned} \tag{16}$$

1.1.2 Undamped modal shape

The complex modes take into account the damping effect, and the undamped modal shape can be obtained based on K and M:

$$K\Phi = \omega^2 M\Phi, \tag{17}$$

which is introduced in most dynamics of structures related books. Rewrite the governing equation as:

$$[M]_{[N\times N]}\{\ddot{x}\}_{[N\times 1]} + [C]_{[N\times N]}\{\dot{x}\}_{[N\times 1]} + [K]_{[N\times N]}\{x\}_{[N\times 1]} = \{f\}_{[N\times 1]}, \tag{18}$$

where the x can be expressed based on modal decomposition as:

$$\{x\} = \sum_r [\Phi_r]\{q_r(t)\}, \tag{19}$$

where $[\Phi_r]$ and $\{q_r\}$ is the r^{th} order modal shape and modal weight. Left multiplying the $[\Phi_r]^T$, we can get the governing equation in modal space as:

$$\begin{aligned}
&\begin{bmatrix} \tilde{M}_{11} & & & \\ & \tilde{M}_{22} & & \\ & & \ddots & \\ & & & \tilde{M}_{nn} \end{bmatrix} \begin{bmatrix} \ddot{q}_1(t) \\ \ddot{q}_2(t) \\ \vdots \\ \ddot{q}_n(t) \end{bmatrix} + \begin{bmatrix} \tilde{C}_{11} & & & \\ & \tilde{C}_{22} & & \\ & & \ddots & \\ & & & \tilde{C}_{nn} \end{bmatrix} \begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \\ \vdots \\ \dot{q}_n(t) \end{bmatrix} \\
&+ \begin{bmatrix} \tilde{K}_{11} & & & \\ & \tilde{K}_{22} & & \\ & & \ddots & \\ & & & \tilde{K}_{nn} \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_n(t) \end{bmatrix} = \begin{bmatrix} \tilde{F}_1 \\ \tilde{F}_2 \\ \vdots \\ \tilde{F}_n \end{bmatrix},
\end{aligned} \tag{20}$$

can be written as:

$$m_r\ddot{q}_r(t) + c_r\dot{q}_r(t) + k_r q_r(t) = F_r \quad (r = 1, 2, \cdots, n), \tag{21}$$

where m_r, c_r, k_r and F_r are modal mass, modal damping, modal stiffness, and modal force, respectively.

1.2 Structural dissipation energy analysis

1.2.1 Modal shape comparison

The complex model shape, undamped modal shape based on OpenFAST and modal shape from DTU [2] result comparison is shown in Fig. 1.

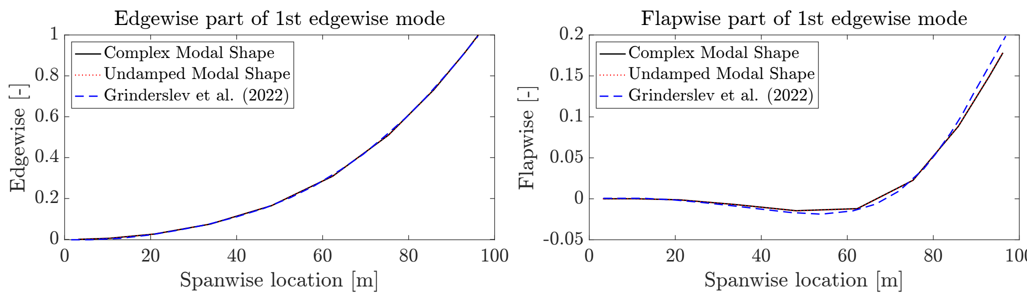

Figure 1: Modal shape comparison

1.2.2 Structural dissipation estimation

There are two ways to estimate the structural dissipation energy. The first method is:

$$\Pi_s = \int_0^T [C]\{\dot{x}\}\{\dot{x}\}dt, \tag{22}$$

which computes the dissipation energy based on damping force integration. Based on undamped modal shape, the second method is:

$$\Pi_s = \int_0^T cq^2 w_n^2 \cos^2(\omega t)dt = 2\pi\zeta m_r q^2 w_n^2, \tag{23}$$

which computes the dissipation energy in modal space.

In reference paper [2], they try to develop a relation as:

$$P_{\text{STRUC}} = F_{\text{STRUC}} \cdot \delta_{\text{DAMP}} \cdot A^2, \tag{24}$$

where P_{STRUC} is the dissipation power, δ_{DAMP} is the damping ratio related to the standard model, A is the maximum edgewise motion in the 1st edgewise mode, and F_{STRUC} is the coefficient we need to determine.

So there exist three ways to estimate the structural dissipation energy: 1. the first method based on undamped modal shape; 2. the first method based on complex modal shape; 3. modal space method based on undamped modal shape.

Table 1 displays the dissipation power comparison. DTU gets the value as 540 by linear regression based on simulation result.

Table 1: Dissipation Power comparison

	x	y	z	θ_x	θ_y	θ_z	Totoal power (F)
method 1	2.78	427.26	14.10	18.37	-3.48	-8.61	450.39
method 2	2.27	427.20	14.17	18.06	-3.23	-8.54	449.93
method 3	/	/	/	/	/	/	450.52

References

[1] Jason M Jonkman, Alan D Wright, Greg J Hayman, and Amy N Robertson. Full-system linearization for floating offshore wind turbines in openfast. In *International Conference on Offshore Mechanics and Arctic Engineering*, volume 51975, page V001T01A028. American Society of Mechanical Engineers, 2018.

[2] Christian Grinderslev, Niels Nørmark Sørensen, Georg Raimund Pirrung, and Sergio González Horcas. Multiple limit cycle amplitudes in high-fidelity predictions of standstill wind turbine blade vibrations. *Wind Energy Science*, 7(6):2201–2213, 2022.