# OBSTRUCTIONS TO THE EXISTENCE OF BOUNDED T-STRUCTURES

#### AMNON NEEMAN

ABSTRACT. In a striking 2019 article, Antieau, Gepner and Heller found K-theoretic obstructions to bounded t-structures. We will survey their work, as well as some progress since. The focus will be on the open problems that arise from this.

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# 0. Introduction

The concept of t-structures on triangulated categories, which is recalled in Definition 3.1, was introduced by Beĭlinson, Bernstein and Deligne [7, Chapter 1], in their attempt to better understand MacPherson's work on intersection homology. In the following decades t-structures rapidly gained attention, and were successfully adapted and employed as a technique applicable across much of mathematics. And bounded t-structures have been of particular interest.

**Example 0.1.** To list a few examples of subjects where bounded *t*-structures have played a pivotal role:

**0.1.1.** In the study of the representations of finite groups of Lie type, Deligne and Lusztig [11] introduced the powerful theory of character sheaves. It has come to play an enormous role in the field, and at its center (at least in modern developments of the

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theory) are bounded perverse t-structures, on derived categories of sheaves over certain varieties.

- **0.1.2.** The theory of stability conditions on triangulated categories, whose origins go back to Bridgeland [9, 10], has turned out to be of great interest in algebraic geometry. It is also growing in prominence in the field of geometric representation theory. And one way to view a stability condition is as a family of bounded t-structures, indexed by the set  $\mathbb{R}$  of real numbers, and satisfying a list of properties we omit.
- **0.1.3.** Motivic homotopy theory, whose origins go back to a string of deeply influential articles by Voevodsky, was developed as an attempt to move forward on Grothendieck's vision of a motivic cohomology theory. More precisely it was inspired by the Beĭlinson-Bloch approach to the problem. Anyway: in the decades that followed Grothendieck's formulation of his program, no one had made any substantial progress on constructing the conjectural abelian category  $\mathcal{A}$  of motives. Voevodsky's idea was that  $\mathbf{D}^b(\mathcal{A})$ , its bounded derived category, should be constructible by homotopy-theoretic methods. And the search for  $\mathcal{A}$  was transformed into the search for a bounded t-structure on  $\mathbf{D}^b(\mathcal{A})$ , with heart  $\mathcal{A}$ .

The conventional wisdom, acquired over three and a half deacades, was that t-structures are plentiful and diverse. There is a plethora of substantially different techniques for constructing them.

**Example 0.2.** Below is a partial list of known methods to produce bounded *t*–structures:

- **0.2.1.** If a triangulated category S is built up of two triangulated categories  $\mathcal{R}$  and  $\mathcal{T}$  by recollement, then any pair of bounded t-structures on  $\mathcal{R}$  and on  $\mathcal{T}$  glue to a bounded t-structure on S. This technique goes back to the birth of the subject in Beĭlinson, Bernstein and Deligne [7, Chapter 1], where it was used to create the perverse t-structures on certain derived categories of sheaves.
- **0.2.2.** One can tilt a *t*-structure with respect to a torsion pair, a technique introduced in the work of Happel, Reiten and Smalø [12]. This has been extensively studied since.
- **0.2.3.** Given a set S of objects in a triangulated category  $\mathcal{T}$ , one can look for the minimal t-structure that the set S generates. By this we mean: the t-structure  $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$  where S is contained in  $\mathcal{T}^{\leq 0}$  and  $\mathcal{T}^{\leq 0}$  is minimal subject to the condition. The idea was introduced by Alonso, Jeremías and Souto [2], and extended by several authors since.

These t-structures are rarely bounded, but can restrict to bounded t-structures on triangulated subcategories of  $\mathfrak{T}$ . A careful study of such examples can be found, for example, in Alonso, Jeremías and Saorín [1].

**0.2.4.** One can construct bounded t-structures on  $\mathfrak{T}$  using Harder-Narasimhan filtrations. This process begins with a central charge, which is a group homomorphism  $Z: K_0(\mathfrak{T}) \longrightarrow \mathbb{C}$ . And then, given any object  $X \in \mathfrak{T}$ , one inductively finds the most

destabilizing subobject of X to produce a filtration. This technique leads to a Bridgeland stability condition. The original paper introducing the method was Bridgeland [9], although the reader might prefer the more elegant, later version due to Bayer [6]. Anyway: at this point the literature on the active subject is immense.

One can therefore imagine the surprise that greeted the 2019 article by Antieau, Gepner and Heller [3], which finds K-theoretic obstructions to the existence of bounded t-structures. Since the aim of this survey is to highlight open questions, let us insert one already.

**Problem 0.3.** In 0.1.3 above we mentioned that the search for a good abelian category  $\mathcal{A}$  of motives, as envisioned by Grothendieck, can be reformulated as the search for the triangulated category  $\mathbf{D}^b(\mathcal{A})$  and a bounded t-structure on it. There are by now several candidates for what should be the right triangulated category  $\mathbf{D}^b(\mathcal{A})$ , and in this active field more candidates are being produced by the day. (OK, this might be an exaggeration, but new candidates pop up with remarkable frequency).

In the light of Antieau, Gepner and Heller [3] we now know, as mentioned above, that there are K-theoretic obstructions to the existence of bounded t-structures. The challenge to the experts becomes to compute these obstructions for the many candidate  $\mathbf{D}^b(\mathcal{A})$ 's, and weed the ones with no chance of working. This means: if an obstruction is nonzero for a candidate  $\mathbf{D}^b(\mathcal{A})$ , then this candidate can safely be discarded. Either there is no way the putative  $\mathbf{D}^b(\mathcal{A})$  could have a bounded t-structure, or else any bounded t-structure on it has to have a heart which is absurd enough to be ruled out as a potential abelian category of motives.

In this survey we will remind the reader of the K-theoretic background, explain in more detail what Antieau, Gepner and Heller [3] proved and conjectured, explain why some of these conjectures seemed highly plausible while others represented a daring leap of faith, describe the progress made on the conjectures since [3] appeared, and (most importantly) highlight the many fascinating questions that remain.

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# 1. A QUICK REMINDER OF ALGEBRAIC K-THEORY

In this section we give a brief and minimalistic overview of algebraic K-theory.

In an abelian category it makes sense to speak of exact sequences. Recall: given a category  $\mathcal{A}$ , being abelian is a property, not a structure. The category  $\mathcal{A}$  is abelian if it has finite limits and colimits, and these satisfy some properties we do not repeat here. Thus if a category  $\mathcal{A}$  happens to be abelian then the exact sequences in it make sense intrinsically.

Now if  $\mathcal{A}$  is an abelian category and  $\mathcal{E} \subset \mathcal{A}$  is a full subcategory, then it also makes sense to speak of exact sequences in  $\mathcal{E}$ . These can be defined to be those diagrams

in  $\mathcal{E}$  which become exact when viewed in  $\mathcal{A}$ . Of course: this definition is no longer intrinsic in  $\mathcal{E}$ , it depends on our chosen embedding into the abelian category  $\mathcal{A}$ . The axiomatic characterization of those  $\mathcal{E}$ 's which admit embeddings into abelian categories, as extension-closed full subcategories, is due to Quillen [22]. We recall

**Reminder 1.1.** An exact category is an additive category  $\mathcal{E}$ , together with a prescribed class of admissible exact sequences

$$A \xrightarrow{f} B \xrightarrow{g} C$$
,

satisfying a list of properties we omit. And it is a theorem that a category  $\mathcal{E}$  is exact if and only if there exists a fully faithful functor  $F:\mathcal{E}\longrightarrow\mathcal{A}$ , where  $\mathcal{A}$  is an abelian category, such that the essential image of F is extension-closed, and a sequence  $A\longrightarrow B\longrightarrow C$  is admissible in  $\mathcal{E}$  if and only if  $0\longrightarrow F(A)\longrightarrow F(B)\longrightarrow F(C)\longrightarrow 0$  is exact in  $\mathcal{A}$ .

**Remark 1.2.** Note that in the exact category  $\mathcal{E}$  the admissible exact sequences are a *structure*, not a *property*. They need to be specified in advance. An additive category  $\mathcal{E}$  can have more than one exact structure, as we will see below.

Let  $\mathcal{F}$  be any additive category. We can turn  $\mathcal{F}$  into an exact category by stipulating that the admissible exact sequences are the "split exact sequences", meaning the isomorphs of diagrams

$$A \xrightarrow{i} A \oplus C \xrightarrow{\pi} C$$

where i is the inclusion and  $\pi$  is the projection. We will denote the additive category  $\mathcal{F}$ , with the split exact structure, by the symbol  $\mathcal{F}^{\oplus}$ . Given any exact category  $\mathcal{E}$ , we can produce a possibly different exact structure by forming  $\mathcal{E}^{\oplus}$ . And we stress that normally one would not expect these two exact structures to agree. This will play a key role in Section 5, starting with Discussion 5.4.

**Remark 1.3.** In Reminder 1.1 we told the reader that we omit the full list of properties that the admissible exact sequences are required to satisfy. But there is one property worth mentioning, even in a minimalistic survey.

In an abelian category A, the diagram

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is a short exact sequence if and only if f is the kernel of g and g is the cokernel of f. In an exact category this is no longer an "if and only if" statement, after all Remark 1.2 teaches us that there cannot exist a characterization of the admissible short exact sequences purely in terms of categorical data from  $\mathcal{E}$ . But it is true that, if the diagram

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is an admissible short exact sequence, then f is the kernel of g and g is the cokernel of f. The reader can see that this follows immediately from the existence of an exact embedding  $F: \mathcal{E} \longrightarrow \mathcal{A}$  as in Reminder 1.1.

**Reminder 1.4.** Let  $\mathcal{E}$  be an essentially small exact category, and let  $F(\mathcal{E})$  be the free abelian group on the isomorphism classes [A] of objects  $A \in \mathcal{E}$ . The *Grothendieck group*  $K_0(\mathcal{E})$  is defined by the formula

$$K_0(\mathcal{E}) = \frac{F(\mathcal{E})}{\langle [A] - [B] + [C] \text{ for every admissible } A \longrightarrow B \longrightarrow C \rangle}$$

In words:  $K_0(\mathcal{E})$  is the quotient of  $F(\mathcal{E})$  by the relations generated by expressions [A] - [B] + [C], where  $A \longrightarrow B \longrightarrow C$  runs over the admissible exact sequences in  $\mathcal{E}$ .

**Definition 1.5.** Let X be a scheme. The special case where  $\mathcal{E}$  is the exact category of vector bundles on X will interest us enough to deserve a symbol: we denote this exact category by  $\operatorname{Vect}(X)$ . The exact structure is that a sequence  $\mathcal{V}' \longrightarrow \mathcal{V} \longrightarrow \mathcal{V}''$ , of vector bundles on X, is declared to be admissible exact if and only if it is exact in the abelian category of all sheaves. And we will adopt the abbreviation

$$K_0(X) := K_0(\operatorname{Vect}(X)).$$

**Reminder 1.6.** For any morphism of schemes  $f: X \longrightarrow Y$ , there is an induced pullback map  $f^*: \mathcal{V}ect(Y) \longrightarrow \mathcal{V}ect(X)$ . Since this is an exact functor (meaning it respects admissible short exact sequences), it induces a group homomorphism  $K_0(Y) \longrightarrow K_0(X)$ .

Now suppose we are given a scheme X, and two open subsets  $U, V \subset X$ . Then there is a commutative square of inclusions

$$U \cap V \longrightarrow U$$

$$\downarrow$$

$$V \longrightarrow X$$

which the functor  $K_0$  takes to a commutative square of group homomorphisms

$$K_0(U \cap V) \longleftarrow K_0(U)$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \qquad$$

We can "fold" this commutative square into

$$K_0(X) \xrightarrow{f} K_0(U) \oplus K_0(V) \xrightarrow{g} K_0(U \cap V)$$

where the composite gf vanishes. And it is classical that, as long as U and V cover X (meaning  $X = U \cup V$ ), this sequence is exact in the middle.

It becomes natural to try to extend to a long exact sequence. And after much work by many people this was done. Assuming X satisfies the resolution property (meaning that there are enough vector bundles on X), there is a way to define positive and negative

algebraic K-theory in such a way that the above extends to a long exact Mayer-Vietoris sequence

$$K_{1}(X) \xrightarrow{\longrightarrow} K_{1}(U) \oplus K_{1}(V) \xrightarrow{\longrightarrow} K_{1}(U \cap V)$$

$$K_{0}(X) \xrightarrow{\longrightarrow} K_{0}(U) \oplus K_{0}(V) \xrightarrow{\longrightarrow} K_{0}(U \cap V)$$

$$K_{-1}(X) \xrightarrow{\longrightarrow} K_{-1}(U) \oplus K_{-1}(V) \xrightarrow{\longrightarrow} K_{-1}(U \cap V)$$

**Remark 1.7.** It goes without saying that, in addition to having long exact sequences, this higher algebraic *K*-theory satisfies many other good properties, which we omit. Its definition and functoriality properties extend well beyond vector bundles on schemes, and even well beyond exact categories.

In this survey, we will use only a tiny portion of the vast general theory. For us, the important observations are:

- **1.7.1.** It is possible to define this higher algebraic K-theory, both positive and negative, for any category of cochain complexes with a *model structure*<sup>1</sup>.
- **1.7.2.** The Mayer-Vietoris long exact sequence, of Reminder 1.6, is a special case of much more general long exact sequences which will come up later.
- **1.7.3.** To each model category as in 1.7.1 one can associate a homotopy category, which is a triangulated category. More formally: there is a functor  $\mathbf{fgt}$ , from the category of model categories to the category of triangulated categories. And there is a theorem saying that, if  $F: \mathcal{M} \longrightarrow \mathcal{N}$  is a morphism of model categories such that  $\mathbf{fgt}(F): \mathbf{fgt}(\mathcal{M}) \longrightarrow \mathbf{fgt}(\mathcal{N})$  is an equivalence of triangulated categories, then the maps  $K_i(F): K_i(\mathcal{M}) \longrightarrow K_i(\mathcal{N})$  are isomorphisms for all  $i \in \mathbb{Z}$ .

It is standard to refer to the model category  $\mathcal{M}$  as an *enhancement* of the triangulated category  $\mathbf{fgt}(\mathcal{M})$ .

## 2. Vanishing theorems for negative K-theory

Next we want to quickly remind the reader that, in a wide range of important special cases, there are vanishing theorems for negative K-theory. And as in Section 1 we begin with the special case of schemes.

<sup>&</sup>lt;sup>1</sup>By model structure we mean either a biWaldhausen complicial category in the sense of Thomason and Trobaugh [26], or a stable infinity category as in Lurie [16].

Reminder 2.1. In Section 1 we left out the history: there was no mention of how and when higher algebraic K-theory was developed, and who contributed what to this achievement. Without changing this substantially, let us note that Chuck Weibel was one of the key players in the study of negative K-theory, with substantial contributions spanning decades. And already in the 1980 article [27, Question 2.9] he formulated what came to be known as Weibel's conjecture.

• If X is a noetherian scheme of dimension  $n < \infty$ , then  $K_i(X) = 0$  for all i < -n. In the following decades much work was done, proving special cases of this conjecture. In full generality the proof may be found in the 2018 article by Kerz, Strunk and Tamme [15, Theorem B]. It took a long time to settle this question.

**Problem 2.2.** It would be interesting to discover the right generalization of this theorem. Presumably there should be a theorem saying that, for some large class of model categories  $\mathcal{M}$ , we have  $K_i(\mathcal{M}) = 0$  for all  $i \ll 0$ . And, like Weibel's conjecture, maybe this general theorem will come with an effective bound, giving an explicit integer n, depending on  $\mathcal{M}$ , such that  $K_i(\mathcal{M}) = 0$  for all i < -n.

I do have a candidate conjecture, but it would require too much notation to introduce it here. And, in any case, the reader is encouraged to use her imagination and come up with conjectures of her own.

In the remainder of the article we will focus on a different set of vanishing conjectures for negative K-theory.

**Reminder 2.3.** By a classical result due to Bass it was known that

(i) For regular, noetherian, finite-dimensional, affine schemes X one has  $K_i(X) = 0$  for all i < 0.

Now recall that, for any scheme and by Definition 1.5,  $K_i(X) = K_i(\text{Vect}(X))$ . The category Vect(X), of vector bundles on the noetherian scheme X, admits an exact embedding into the abelian category Coh(X) of all coherent sheaves on X. And it is a classical theorem due to Quillen that

(ii) For finite-dimensional, regular, noetherian schemes X with enough vector bundles, the natural maps

$$K_i(\operatorname{Vect}(X)) \longrightarrow K_i(\operatorname{Coh}(X))$$

are isomorphisms for all  $i \in \mathbb{Z}$ .

**Reminder 2.4.** Schlichting generalized the known statements of Reminder 2.3. The general vanishing statements he proved, for negative *K*-theory, are as follows:

- (i) Let  $\mathcal{A}$  be an essentially small abelian category. Then  $K_{-1}(\mathcal{A}) = 0$ .
- (ii) Let  $\mathcal{A}$  be a noetherian abelian category. Then  $K_i(\mathcal{A}) = 0$  for all i < 0.

The reader can find (i) in [24, Theorem 6 of Section 10], while (ii) is [24, Theorem 7 of Section 10]. By applying (ii) above to the noetherian abelian category Coh(X), and combining with Quillen's isomorphism of Reminder 2.3(ii), we obtain

(iii) For finite-dimensional, regular, noetherian schemes X with enough vector bundles, the abelian groups  $K_i(X) = K_i(\operatorname{Vect}(X))$  vanish for all i < 0.

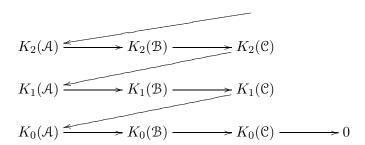
In other words: as an easy corollary of his results, Schlichting was able to generalize the vanishing statement of Reminder 2.3(i) from affine schemes to all schemes. And then he went on to conjecture that even more should be true.

(iv) For any abelian category  $\mathcal{A}$  and all integers i < 0, the groups  $K_i(\mathcal{A})$  vanish. This may be found in [24, Conjecture 1 of Section 10]. The statement (iv) came to be known as Schlichting's conjecture.

**Remark 2.5.** Let us explain why Schlichting's conjecture is plausible. There is an old result of Quillen [22, Section 5, Theorem 4] asserting:

**2.5.1.** Let  $\mathcal{B}$  be an abelian category, and assume  $\mathcal{A} \subset \mathcal{B}$  is a Serre subcategory. Recall: this means that  $\mathcal{A}$  is an abelian subcategory of  $\mathcal{B}$  such that every  $\mathcal{B}$ -subquotient of an object  $A \in \mathcal{A}$  belongs to  $\mathcal{A}$ . Now write  $\mathcal{C}$  for the abelian quotient category  $\mathcal{C} = \mathcal{B}/\mathcal{A}$ .

Then there is a long exact sequence in K-theory



So the challenge becomes to try to show that this long exact sequence is sufficient to define negative K-theory without ever having to leave the world of abelian categories.

The standard methods of homological algebra tell us that this will work as long as there are enough abelian categories  $\mathcal{B}$  with vanishing K-theory. If we could embed every abelian category  $\mathcal{A}$  as a Serre subcategory  $\mathcal{A} \subset \mathcal{B}$ , where the higher K-theory of  $\mathcal{B}$  must vanish for some formal reason, then we'd be in business.

Let me not elaborate here on the technical difficulties that people encountered in their attempts to do this.

#### 3. A REMINDER OF t-STRUCTURES

Before we recall the formal definitions, let us give some intuition. A t-structure on a triangulated category amounts to specifying which objects in the category we declare to be positive, and which objects in the category we declare to be negative. And then some conditions must be satisfied. More formally, we have

**Definition 3.1.** Let  $\mathfrak{T}$  be a triangulated category. A t-structure on  $\mathfrak{T}$  is a pair of full subcategories  $(\mathfrak{T}^{\leq 0}, \mathfrak{T}^{\geqslant 0})$ , with  $\mathfrak{T}^{\leq 0}$  to be thought of as the negative objects and  $\mathfrak{T}^{\geqslant 0}$  to be thought of as the positive objects. And these must satisfy the conditions

- (i)  $\Sigma \mathfrak{I}^{\leqslant 0} \subset \mathfrak{I}^{\leqslant 0}$  and  $\mathfrak{I}^{\geqslant 0} \subset \Sigma \mathfrak{I}^{\geqslant 0}$ .
- (ii)  $\operatorname{Hom}(\Sigma \mathfrak{I}^{\leq 0}, \mathfrak{I}^{\geqslant 0}) = 0.$
- (iii) For every object  $B \in \mathfrak{T}$  there exists a triangle  $A \longrightarrow B \longrightarrow C \longrightarrow in \mathfrak{T}$ , with  $A \in \Sigma \mathfrak{T}^{\leqslant 0}$  and  $C \in \mathfrak{T}^{\geqslant 0}$ .

So much for the general definition of t-structure. Now we furthermore define

(iv) The t-structure  $(\mathfrak{T}^{\leqslant 0}, \mathfrak{T}^{\geqslant 0})$  is declared bounded if, for every object  $X \in \mathfrak{T}$ , there exists an integer n > 0 with  $\Sigma^n X \in \mathfrak{T}^{\leqslant 0}$  and with  $\Sigma^{-n} X \in \mathfrak{T}^{\geqslant 0}$ .

The most classical example is

**Example 3.2.** Let R be an associative ring, and let  $\mathbf{D}(R)$  be the derived category whose objects are cochain complexes of left R-modules. Then the following gives a t-structure on  $\mathbf{D}(R)$ :

$$\mathbf{D}(R)^{\leq 0} = \{X \in \mathbf{D}(R) \mid H^i(X) = 0 \text{ for all } i > 0\},$$
  
 $\mathbf{D}(R)^{\geq 0} = \{X \in \mathbf{D}(R) \mid H^i(X) = 0 \text{ for all } i < 0\}.$ 

This t-structure is not bounded. If we let X be the cochain complex

$$\cdots \longrightarrow R \xrightarrow{0} R \xrightarrow{0} R \xrightarrow{0} R \xrightarrow{0} R \longrightarrow \cdots$$

where in every degree we put the rank-1 free R-module, and all the differentials are zero, then X is an object of  $\mathbf{D}(R)$  but no shift of it is either positive or negative.

But, when restricted to the full subcategory  $\mathbf{D}^b(R)$ , the the t-structure above becomes bounded. The category  $\mathbf{D}^b(R)$  has for objects the cochain complexes that vanish outside a bounded interval.

In Example 0.2 we told the reader that there are many other examples of bounded t-structures, and listed four known, general procedures that produce them.

There is one general theorem about t-structures that we will frequently appeal to, hence we remind the reader:

**Theorem 3.3.** Let  $\mathfrak{T}$  be a triangulated category, and assume that  $(\mathfrak{T}^{\leqslant 0},\mathfrak{T}^{\geqslant 0})$  is a t-structure on  $\mathfrak{T}$ . Then the category  $\mathfrak{T}^{\circlearrowleft}=\mathfrak{T}^{\leqslant 0}\cap\mathfrak{T}^{\geqslant 0}$  is abelian. It is called the heart of the t-structure.

We already mentioned, back in the opening paragraphs of Section 1, that for a category to be abelian is a property, not a structure. A category is abelian if it has finite limits and colimits and these satisfy certain properties. Applying this to  $\mathfrak{T}^{\heartsuit} = \mathfrak{T}^{\leqslant 0} \cap \mathfrak{T}^{\geqslant 0}$ , it follows that there are in  $\mathfrak{T}^{\heartsuit}$  the short exact sequences that come from its being abelian. When we write  $K_i(\mathfrak{T}^{\heartsuit})$  we mean the K-groups with respect to this natural exact structure.

We should perhaps mention that a sequence  $A \longrightarrow B \longrightarrow C$ , in the category  $\mathfrak{T}^{\heartsuit}$ , is short exact in this abelian category if and only if there exists, in the category  $\mathfrak{T}$ , a morphism  $C \longrightarrow \Sigma A$  rendering  $A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A$  an exact triangle.

#### 4. K-Theoretic obstructions to bounded t-structures

We have now arrived at the focus of this survey: we will discuss the amazing results of Antieau, Gepner and Heller [3], the many questions they raise, and the progress since. We begin with the following, which is a restatement of [3, Theorems 1.1 and 1.2].

**Theorem 4.1.** Let  $\mathcal{M}$  be a model category as in 1.7.3, and let  $\mathcal{T} = \mathbf{fgt}(\mathcal{M})$  be the associated triangulated category. Assume that  $\mathcal{T}$  has a bounded t-structure  $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geqslant 0})$ .

Then, with  $K_n(\mathfrak{M})$  as in 1.7.1, the following vanishing results hold:

- (i) Unconditionally we have  $K_{-1}(\mathfrak{M}) = 0$ .
- (ii) If the abelian category  $\mathfrak{T}^{\heartsuit}$  is noetherian, then  $K_n(\mathfrak{M}) = 0$  for all n < 0.

We said at the start that the current survey will highlight the many questions that this fascinating theorem raises. We begin with

**Problem 4.2.** The hypotheses in Theorem 4.1 are about  $\mathcal{T} = \mathbf{fgt}(\mathcal{M})$ , and the conclusions are about  $K_i(\mathcal{M})$ . And there are examples of pairs of model categories  $\mathcal{M}$  and  $\mathcal{N}$ , with  $\mathbf{fgt}(\mathcal{M}) \cong \mathbf{fgt}(\mathcal{N})$  but where  $K_n(\mathcal{M}) \ncong K_n(\mathcal{N})$ .

To elaborate a tiny bit: let k be a field of characteristic p > 0, and let  $W_2(k)$  be the ring of length-two Witt vectors over k. For example: if  $k = \mathbb{Z}/p$  then  $W_2(k) = \mathbb{Z}/p^2$ . One can form the singularity categories  $\mathcal{M} = \operatorname{Sing}(W_2(k))$  and  $\mathcal{N} = \operatorname{Sing}(k[\varepsilon]/\varepsilon^2)$  with the natural model structures, and it is easy to see that they satisfy  $\operatorname{fgt}(\mathcal{M}) \cong \operatorname{fgt}(\mathcal{N})$ . But for  $k = \mathbb{Z}/p$  Schlichting [23] computes that  $K_n(\mathcal{M})$  and  $K_n(\mathcal{N})$  are in general non-isomorphic.

Of course: Schlichting's computations are of positive K-groups. For the pair  $\mathcal{M}, \mathcal{N}$  above it is easy to show that, for all n < 0, we have  $K_n(\mathcal{M}) = 0 = \mathbf{K}_n(\mathcal{N})$ .

This immediately raises the question: to what extent do the negative K-groups  $K_n(\mathcal{M})$  depend on the enhancement  $\mathcal{M}$  of the triangulated category  $\mathbf{fgt}(\mathcal{M})$ ? In particular: is it possible to find a pair of model categories  $\mathcal{M}$  and  $\mathcal{N}$ , with  $\mathbf{fgt}(\mathcal{M}) \cong \mathbf{fgt}(\mathcal{N})$ , with  $K_{-1}(\mathcal{M}) = 0$  and with  $K_{-1}(\mathcal{N}) \neq 0$ ? If such a pair exists then the nonvanishing of  $K_{-1}(\mathcal{N})$ , coupled with Theorem 4.1(i), implies that the triangulated category  $\mathbf{fgt}(\mathcal{N})$  cannot have a bounded t-structure. But the negative K-groups of  $\mathcal{M}$  tell us nothing.

**Remark 4.3.** In Problem 4.2 we mentioned Schlichting's lovely example. The triangulated category  $\mathbf{D_{sg}}(W_2(k)) \cong \mathbf{D_{sg}}(k[\varepsilon]/\varepsilon^2)$  has two enhancements with non-isomorphic K-groups.

The philosopher can look at this fact in two ways. One could take the view that this shows that the passage from  $\mathcal{M}$  to  $\mathbf{fgt}(\mathcal{M})$  loses far too much information, and one should always work with enhancements. But one could also take the opposite view, that there must be something ridiculous about a theory that distinguishes between the two different enhancements of  $\mathbf{D_{sg}}(W_2(k)) \cong \mathbf{D_{sg}}(k[\varepsilon]/\varepsilon^2)$ . After all: the triangulated category in

question is really dumb. The objects are the finite-dimensional vector spaces over the field k, the morphisms are the linear maps, the suspension functor is the identity, and the triangles are the finite direct sums of shifts of the triangle  $0 \longrightarrow k \xrightarrow{\mathrm{id}} k \longrightarrow 0$ .

Since this isn't a treatise on philosophy, let us get back to Theorem 4.1 and its ramifications. The negative K-groups, of an enhancement  $\mathcal{M}$  of the triangulated category  $\mathcal{T}$ , carry information about bounded t-structures on  $\mathcal{T}$ . Surely this needs to be better understood. And so far we have only raised the question of the dependence on the enhancement, which for this survey will be a small issue.

There are two major directions to explore: one can try to improve the K-theoretic results, and one can try to understand better the implications for bounded t-structures. We devote a section to each of those.

# 5. K-Theoretic generalizations of Theorem 4.1

Possible generalizations of Theorem 4.1 were formulated already in Antieau, Gepner and Heller [3]. We recall

**Reminder 5.1.** With the numbering as in the original [3, Introduction], the following conjectures were proposed:

Conjecture A: Let  $\mathcal{A}$  be an essentially small abelian category. Then  $K_n(\mathcal{A}) = 0$  for all n < 0.

Conjecture B: Let  $\mathcal{M}$  be an essentially small model category. If the category  $\mathbf{fgt}(\mathcal{M})$  has a bounded t-structure then  $K_n(\mathcal{M}) = 0$  for all n < 0.

Conjecture C: Let  $\mathcal{M}$  be an essentially small model category. If the category  $\mathbf{fgt}(\mathcal{M})$  has a bounded t-structure, then the natural map

$$K_n(\mathbf{fgt}(\mathfrak{M})^{\heartsuit}) \longrightarrow K_n(\mathfrak{M})$$

is an isomorphism for all  $n \in \mathbb{Z}$ .

Note that Conjecture A was not new, it was simply reiterating the conjecture made by Schlichting more than a decade earlier, see Reminder 2.4(iv).

And about Conjecture C: for  $n \ge 0$  this is a theorem, not a conjecture. One can find variants of the result in [20] and in Barwick [5]. But both results are about connective K-theory, and say nothing about negative K-groups. Thus all three conjectures are really about negative K-theory.

And, after proposing the conjectures in [3, Introduction], Antieau, Gepner and Heller go on to observe that Conjecture B implies the other two conjectures.

**Discussion 5.2.** Already in [3], Conjecture B comes with a plausibility argument—in fact better still, it comes with the outline of a proposed proof. And for the purpose of the discussion that follows I will leave it to the reader to provide enhancements, the plausibility argument will be stated in terms of triangulated categories.

Any time we take an idempotent-complete triangulated category S and a thick subcategory  $R \subset S$ , we may form the Verdier quotient T = S/R. The category T will not in general be idempotent-complete, but we may form the idempotent completion  $T^+$  as in Balmer and Schlichting [4]. And, after enhancing the picture, we obtain a long exact sequence which, in an abuse of notation, we write without mention of the enhancements:

$$K_{-1}(\mathbb{R}) \xrightarrow{K_{-1}(\mathbb{S})} K_{-1}(\mathbb{T}^+)$$
 $K_{-2}(\mathbb{R}) \xrightarrow{K_{-2}(\mathbb{S})} K_{-2}(\mathbb{T}^+)$ 
 $K_{-3}(\mathbb{R}) \xrightarrow{K_{-3}(\mathbb{S})} K_{-3}(\mathbb{T}^+)$ 

By Theorem 4.1(i) we know the vanishing of  $K_{-1}(\mathcal{U})$  for any  $\mathcal{U}$  with a bounded t-structure. So the idea is to prove the theorem by dimension shifting.

Suppose therefore that we are given a triangulated category  $\mathcal{R}$  with a bounded t-structure. Then it suffices to find an algorithm that embeds  $\mathcal{R}$ , as a thick subcategory, in an idempotent-complete triangulated category  $\mathcal{S}$ , and do it in such a way that

- (i) With  $\mathcal{T} = \mathcal{S}/\mathcal{R}$ , the idempotent completion  $\mathcal{T}^+$  has a bounded t-structure.
- (ii) The inclusion  $\mathbb{R} \longrightarrow \mathbb{S}$  induces the zero map  $K_i(\mathbb{R}) \longrightarrow K_i(\mathbb{S})$ .

The reason this would suffice is that, from the long exact sequence, we would deduce that the map  $K_i(\mathfrak{I}^+) \longrightarrow K_{i-1}(\mathfrak{R})$  is surjective. Hence the vanishing of  $K_i(\mathfrak{I}^+)$  would permits us to deduce the vanishing of  $K_{i-1}(\mathfrak{R})$ .

This plausibility argument persuaded me.

Perhaps I should elaborate a little. The first I heard of the beautiful paper [3] was when I happened to be in Boston in April 2018, and Gepner gave a talk about the results at the MIT topology seminar. Later in 2018 Gepner moved to Melbourne, and in Australia he gave a number of talks on the same topic. I attended at least two of these, one in Canberra and one in Sydney.

And each time, after the talk, I would tell anyone who cared to listen that this was a lovely paper, and that Conjecture B had to be true and the plausibility argument should be turned into a proof. After all: we are given a triangulated category  $\mathcal{R}$ , with a bounded t-structure, and out of it we want to cook up another triangulated category  $\mathcal{T}^+$ , also with a bounded t-structure. And in Example 0.2 we learned that there are many known recipes for producing bounded t-structures. How difficult could it possibly be to carry out this program?

In early 2020 Covid hit, and in late March 2020 it arrived in Australia. And we all went into lockdown. Now: in lockdown there is not a lot to do, so I decided to try my

hand at this myself. I set out to prove Conjecture B. All it required was creativity in constructing bounded t-structures, and as we have already said: how hard could this be?

It turns out not to be hard, it's impossible. Being creative with t-structures led to the counterexample of [21], which we will now discuss.

**Example 5.3.** We need to create a triangulated category with a bounded t-structure. There are already many known recipes, but we want one lending itself to computations in K-theory. Here is one that works.

Let  $\mathcal{E}$  be any idempotent-complete exact category, and let  $\mathcal{R} = \mathrm{Ac}^b(\mathcal{E})$  be the category whose objects are the bounded, acyclic complexes in  $\mathcal{E}$ . Recall the definition in the opening paragraph of [19, Section 1]: a complex

$$\cdots \longrightarrow E^{-2} \longrightarrow E^{-1} \longrightarrow E^0 \longrightarrow E^1 \longrightarrow E^2 \longrightarrow \cdots$$

is declared acyclic if each differential  $E^i \longrightarrow E^{i+1}$  can be factored as  $E^i \longrightarrow I^i \longrightarrow E^{i+1}$ , with the resulting  $I^{i-1} \longrightarrow E^i \longrightarrow I^i$  all admissible exact sequences. <sup>2</sup>

This defines the objects in  $\mathcal{R} = Ac^b(\mathcal{E})$ . The morphisms in  $\mathcal{R}$  are the homotopy equivalence classes of cochain maps.

And now for the bounded t-structure. In [21, Lemma 2.1] the reader can find a proof that the following is a t-structure on  $Ac^b(\mathcal{E})$ 

$$\begin{array}{lcl} \operatorname{Ac}^b(\mathcal{E})^{\leqslant 0} &=& \{E^* \in \operatorname{Ac}^b(\mathcal{E}) \mid E^i = 0 \text{ for all } i > 0\} \ , \\ \operatorname{Ac}^b(\mathcal{E})^{\geqslant 0} &=& \{E^* \in \operatorname{Ac}^b(\mathcal{E}) \mid E^i = 0 \text{ for all } i < -2\} \ . \end{array}$$

This t-structure is obviously bounded. The problem becomes to compute the K-theory of the obvious enhancement.

**Discussion 5.4.** The strategy of the proof is simple enough. The category  $\mathcal{R} = \mathrm{Ac}^b(\mathcal{E})$  naturally embeds into the category  $\mathcal{S} = \mathbf{K}^b(\mathcal{S})$ , the homotopy category of bounded cochain complexes in  $\mathcal{E}$ . It is a thick subcategory by [19, Remark 1.10], and  $\mathcal{T} = \mathcal{S}/\mathcal{R}$  is the usual derived category  $\mathbf{D}^b(\mathcal{E})$ . With the standard choice of enhancements this leads, as in Discussion 5.2, to a long exact sequence in K-theory. And in this case it simplifies to

$$0 = K_{-1}(\operatorname{Ac}^{b}(\mathcal{E})) \longrightarrow K_{-1}(\mathcal{E}^{\oplus}) \longrightarrow K_{-1}(\mathcal{E})$$

$$K_{-2}(\operatorname{Ac}^{b}(\mathcal{E})) \xrightarrow{} K_{-2}(\mathcal{E}^{\oplus}) \longrightarrow K_{-2}(\mathcal{E})$$

$$K_{-3}(\operatorname{Ac}^{b}(\mathcal{E})) \xrightarrow{} K_{-3}(\mathcal{E}^{\oplus}) \longrightarrow K_{-3}(\mathcal{E})$$

<sup>&</sup>lt;sup>2</sup>Caution: assume  $\mathcal{E}$  is embedded as an exact subcategory of some abelian category  $\mathcal{A}$ . As in Reminder 1.1 this means that a sequence  $E' \longrightarrow E \longrightarrow E''$  in  $\mathcal{E}$  is admissible if and only if it is short exact in  $\mathcal{A}$ . It does not follow that a cochain complex in  $\mathcal{E}$  of length > 3, which is acyclic in  $\mathcal{A}$ , is also acyclic in  $\mathcal{E}$  as defined in [19, Section 1]. The acyclicity in the ambient  $\mathcal{A}$  in general depends on the embedding.

The vanishing is by Theorem 4.1(i). And the notation, as in Remark 1.2, is that by  $\mathcal{E}^{\oplus}$  we mean the category  $\mathcal{E}$  with the split exact structure. Assuming that Conjecture B is true and combining with Example 5.3, we have the vanishing of  $K_n(\mathrm{Ac}^b(\mathcal{E}))$  for all negative n. And the long exact sequence tells us that this would force the natural maps  $K_n(\mathcal{E}^{\oplus}) \longrightarrow K_n(\mathcal{E})$  to be isomorphisms for all n < 0.

We will be sketching one approach that leads to counterexamples.

**Discussion 5.5.** If we are going to kill a conjecture, then why not kill two with one stone?

With this philosophy in mind, let  $\mathcal{A}$  be the abelian category  $\mathcal{A} = \operatorname{Ac}^b(\mathcal{E})^{\heartsuit}$ . If we could guarantee that the natural map  $K_i(\mathcal{A}) \longrightarrow K_i(\operatorname{Ac}^b(\mathcal{E}))$  is an isomorphism, then an  $\mathcal{E}$  providing a counterexample to Conjecture B would automatically also kill Conjecture A. In the light of 1.7.3, it suffices to prove that the natural map  $\mathbf{D}^b(\mathcal{A}) \longrightarrow \operatorname{Ac}^b(\mathcal{E})$  is an equivalence of triangulated categories.

This isn't true for every  $\mathcal{E}$ . The matter is studied in [21, Section 2], and [21, Proposition 2.4] shows that the natural map  $\mathbf{D}^b(\mathcal{A}) \longrightarrow \mathrm{Ac}^b(\mathcal{E})$  is an equivalence if and only if the category  $\mathcal{E}$  is hereditary. This means: if and only if, for every pair of objects  $E, E' \in \mathcal{E}$ , we have  $\mathrm{Ext}^n(E, E') = 0$  for all n > 1.

If X is an algebraic curve, and  $\mathcal{E} = \mathcal{V}ect(X)$  is the exact category of vector bundles over X, then the category  $\mathcal{E}$  is hereditary.

**Summary 5.6.** With Example 5.3 and Discussions 5.4 and 5.5 in mind, the article [21] restricts attention to the exact category  $\mathcal{E} = \mathcal{V}ect(X)$  with X a projective curve over a field k. The article then proves

- (i) If the singularities of X are no worse than simple nodes, then  $K_{-1}(\mathcal{E}^{\oplus}) = 0$ .
- And this is combined with an appeal to the classical literature, after all
  - (ii) There are known examples of nodal curves X such that, with  $\mathcal{E} = \mathcal{V}ect(X)$ , we have  $K_{-1}(\mathcal{E}) \neq 0$ .

The conjunction of (i) and (ii) with the earlier discussion kills Conjectures A and B.

**Remark 5.7.** Why is it that, after the paper is published and can no longer be changed, we always wish we could edit it? Before I explain the part I would change let me make two points.

(i) There is an inevitability about it. Whatever argument we come up with, and no matter how clever we find it at the time, someone will eventually discover an approach that is shorter, simpler, more elegant and more general. It could be us, or it might be someone else. It can happen quickly or take years. One thing is guaranteed: the passage of time will bring improvements.

As the reader will discover, the part I would rewrite if I could is contained in Summary 5.6(i). In my defense let me say that I found the statement difficult to believe. In the coming paragraphs I will try to explain my skepticism.

As explained in Discussion 5.4: the search for a counterexample to Conjecture B was reduced to looking for exact categories  $\mathcal{E}$ , with  $K_n(\mathcal{E}^{\oplus}) \not\cong K_n(\mathcal{E})$  for some n < 0. To kill Conjectures A and B together, Discussion 5.5 tells us that a hereditary  $\mathcal{E}$  is preferred. And vector bundles over projective curves were a reasonable first place to look, as we will now explain.

We have mentioned that, on any curve X, the exact category  $\operatorname{Vect}(X)$  is hereditary. An irreducible curve has two choices: it is either affine or projective. For affine curves  $X = \operatorname{Spec}(R)$ , we have that  $\operatorname{Vect}(X)$  is equivalent to the category R-proj of finitely generated, projective R-modules. And in the category  $\mathcal{E} = R$ -proj every short exact sequence splits, that is  $\mathcal{E}^{\oplus} = \mathcal{E}$ . For us this eliminates affine curves, on affine curves the map  $K_n(\mathcal{E}^{\oplus}) \longrightarrow K_n(\mathcal{E})$  is guaranteed to be an isomorphism.

Projective curves X have non-split short exact sequences of vector bundles, hence studying  $\mathcal{E} = \mathcal{V}ect(X)$  is natural enough. What surprised me wasn't that  $K_{-1}(\mathcal{E}^{\oplus}) \ncong K_{-1}(\mathcal{E})$ , but that  $K_{-1}(\mathcal{E}^{\oplus}) = 0$ . And the cause for my surprise was that, for affine curves X and with  $\mathcal{E} = \mathcal{V}ect(X)$ , we often have  $K_{-1}(\mathcal{E}^{\oplus}) = \mathbf{K}_{-1}(\mathcal{E}) \not= 0$ . What made projective curves so totally unlike affine curves? I was expecting some difference, but not such a drastic, radical dichotomy.

Because the result seemed suspect to me, I checked it carefully. My second excuse for my stupidity is that

(ii) In the process of carefully proving that a result is true, we can lose sight of the bigger picture. We fail to ask ourselves for the underlying reason. The all-important question should be: *Why* is this surprising result true?

The time has come to stop making excuses, and tell the reader how I would rewrite the article if I could.

If the Almighty Lord granted me a time machine, and permitted me to rewrite [21] and have the improved version published in place of the one currently in print, then instead of Summary 5.6(i) the result would be:

**Theorem 5.8.** Let A be an idempotent-complete additive category. Assume that, for every object  $A \in A$ , the ring Hom(A, A) is artinian. Then  $K_n(A^{\oplus}) = 0$  for all n < 0.

*Proof.* It is known that, if R is any artinian ring, then  $K_n(R) = 0$  for all n < 0; for a reference the reader can see Weibel [28, Theorem 2.3]. Rephrasing this in terms of categories: the category  $\mathcal{B} = R$ -proj, of finitely generated, projective R-modules, has vanishing negative K-theory.

Now choose an object  $A \in \mathcal{A}$ , Applying the above to the ring R = Hom(A, A) gives that the full, additive subcategory  $\text{add}(A) \subset \mathcal{A}$ , of all direct summands of finite direct sums  $A^{\oplus m}$  of A with itself, is such that  $K_n(\text{add}(A)^{\oplus}) = 0$  for all n < 0. But  $\mathcal{A}$  is the filtered union of the full subcategories add(A), making  $K_n(\mathcal{A}^{\oplus})$  the direct limit of the  $K_n(\text{add}(A)^{\oplus})$ . And for n < 0 this yields  $K_n(\mathcal{A}^{\oplus}) = 0$ .

**Problem 5.9.** The emphasis in this survey is on open problems, and Conjecture C is still open. Theorem 5.8 might be relevant, it produces many examples of model categories with bounded t-structures, and whose negative K-theory can be computed and is nonzero.

Let X be any projective scheme and put  $\mathcal{E} = \mathcal{V}ect(X)$ . By Theorem 5.8 we know that  $K_n(\mathcal{E}^{\oplus}) = 0$  for all n < 0, and the exact sequence of Discussion 5.4 tells us that the map  $K_n(\mathcal{E}) \longrightarrow K_{n-1}(\mathrm{Ac}^b(\mathcal{E}))$  is an isomorphism for all n < 0. And there are many examples out there of projective schemes for which  $K_n(\mathcal{E})$  has been computed when n < 0, and is nonzero.

Of course: I have no idea how to go about computing  $K_n(\operatorname{Ac}^b(\mathcal{E})^{\heartsuit})$ , except in the case where  $\dim(X) = 1$  and  $K_n(\operatorname{Ac}^b(\mathcal{E})^{\heartsuit})$  agrees with  $K_n(\operatorname{Ac}^b(\mathcal{E}))$  by Discussion 5.5. But since this provides such a wealth of candidates, one should at least do a reality check on the conjecture. If X and Y are two projective schemes, we can put  $\mathcal{E} = \operatorname{Vect}(X)$  and  $\mathcal{F} = \operatorname{Vect}(Y)$ . The above gives a procedure for computing  $K_n(\operatorname{Ac}^b(\mathcal{E}))$  and  $K_n(\operatorname{Ac}^b(\mathcal{F}))$ . If Conjecture C is true, then these depend only on the heart of the bounded t-structures. Thus an equivalence

$$Ac^b(\mathcal{E})^{\heartsuit} \cong Ac^b(\mathcal{F})^{\heartsuit}$$

would imply that  $K_n(\operatorname{Ac}^b(\mathcal{E})) \cong K_n(\operatorname{Ac}^b(\mathcal{F}))$  for all n. And for n < 0 the discussion shows that this means  $K_n(\mathcal{E}) \cong K_n(\mathcal{F})$ . Or in more classical notation: an equivalence of the hearts of these t-structures would force  $K_n(X) \cong K_n(Y)$  for all n < 0.

This should be checked.

## 6. Categories that can be proved to have no bounded t-structure

In Section 5 the focus was on conjectures aimed at strengthening the vanishing statements in negative K-theory. In the current section we will ask if the results about bounded t-structures can be sharpened.

The title of Antieau, Gepner and Heller's article [3] gives the clue: Theorem 4.1, which is a restatement of [3, Theorems 1.1 and 1.2], provides obstructions to the existence of bounded t-structures. As we mentioned already in Problem 0.3: there are triangulated categories of motives which are conjectured to have bounded t-structures and, armed with Theorem 4.1, we can test these conjectures with a reality check. If their negative K-theory fails to vanish then either they have no bounded t-structures, or these t-structures must have non-noetherian hearts.

Over the decades there has been much work computing the negative K-theory of schemes, and there are many examples out there of schemes with nonvanishing negative K-theory. This immediately leads to [3, Corollary 1.4]. It says:

**Corollary 6.1.** Let X be a noetherian scheme. Then the following holds:

- (i) If  $K_{-1}(X) \neq 0$  then  $\mathbf{D}^{\text{perf}}(X)$  has no bounded t-structure.
- (ii) If  $K_n(X) \neq 0$  for n < -1, then a bounded t-structure on  $\mathbf{D}^{\mathrm{perf}}(X)$  cannot have a noetherian heart.

Being courageous, Antieau, Gepner and Heller go on to formulate [3, Conjecture 1.5]. It predicts

**Conjecture 6.2.** Let X be a finite-dimensional, noetherian scheme. Then the category  $\mathbf{D}^{\mathrm{perf}}(X)$  has a bounded t-structure if and only if X is regular.

**Discussion 6.3.** [3, Conjecture B] was eminently plausible and came with a projected outline of a possible proof. We sketched it in Discussion 5.2.

The evidence for [3, Conjecture 1.5] was thin by comparison. There are many singular noetherian schemes whose negative K-theory vanishes. Recall that all zero-dimensional, noetherian schemes have vanishing negative K-theory—we met this in the proof of Theorem 5.8, where we appealed to a non-commutative generalization. And it is easy to construct zero-dimensional, noetherian schemes with dreadful singularities. In this light, it was a bold leap of faith for Antieau, Gepner and Heller to pass from the meager evidence, provided by Corollary 6.1, to the daring Conjecture 6.2.

Boldness can pay off, this conjecture turns out to be very true—meaning so true that a vast generalization is now a theorem. But let us proceed chronologically: the first progress on the conjecture came in Harry Smith [25]: Smith proved the conjecture for affine X. For affine X there is a classification of the compactly generated t-structures on  $\mathbf{D}_{qc}(X)$  due to Alonso, Jeremías and Saorín [1], and an analysis of which of these compactly generated t-structures restrict to t-structures on  $\mathbf{D}_{coh}^b(X)$ . Using this machinery, Smith was able to prove Conjecture 6.2 for affine X.

In fact Smith proved a stronger statement: if X is singular, affine, irreducible, noetherian and finite-dimensional then the only t-structures on  $\mathbf{D}^{\mathrm{perf}}(X)$  are the trivial ones. This means that, under the hypotheses above, either  $\mathbf{D}^{\mathrm{perf}}(X)^{\leqslant 0} = 0$  or  $\mathbf{D}^{\mathrm{perf}}(X)^{\geqslant 0} = 0$ . Note that this does not generalize beyond the affine case. There are known examples of singular varieties for which  $\mathbf{D}^{\mathrm{perf}}(X)$  has nontrivial semiorthogonal decompositions—which are of course examples of non-trivial t-structures. And when one of the components is admissible, 0.2.1 allows us to glue t-structures on the components, producing even more non-trivial t-structures. There is a rich and growing literature on the possible, nontrivial semiorthogonal decompositions of  $\mathbf{D}^{\mathrm{perf}}(X)$ . For smooth X this literature is immense and goes back decades, it originates with Beilinson [8] and the semiorthogonal decomposition of  $\mathbf{D}^{\mathrm{perf}}(\mathbb{P}^n)$ ; in this case all of the components are admissible and we deduce an abundance of bounded t-structures. For singular X, the study of semiorthogonal decompositions of  $\mathbf{D}^{\mathrm{perf}}(X)$  is a much more recent development, but is an active field: we refer the reader to Kalck, Pavic and Shinder [13] and to Karmazyn, Kuznetsov and Shinder [14] for a tiny sample.

Semiorthogonal decompositions of  $\mathbf{D}^{\mathrm{perf}}(X)$  can give exotic t-structures, both bounded and unbounded. In view of this any classification theorem, of the possible t-structures on  $\mathbf{D}^{b}_{\mathrm{coh}}(X)$  or  $\mathbf{D}^{\mathrm{perf}}(X)$  for X non-affine, would have to be much more delicate than the analysis in Alonso, Jeremías and Saorín [1]. Harry Smith's approach, which proved Conjecture 6.2 in the affine case using the results of [1], does not generalize in any

straightforward way. Nevertheless the conjecture is true unconditionally—not only is it true as originally stated, but so is a major generalization. The following is now a theorem, see [17, Theorem 0.1].

**Theorem 6.4.** Let X be a finite-dimensional, noetherian scheme, and let  $Z \subset X$  be a closed subset. Let  $\mathbf{D}_Z^{\mathrm{perf}}(X)$  be the derived category, whose objects are the perfect complexes on X which restrict to acyclic complexes on the open set X - Z.

The category  $\mathbf{D}_Z^{\mathrm{perf}}(X)$  has a bounded t-structure if and only if Z is contained in the regular locus of X.

Remark 6.5. The proof of Theorem 6.4 is not by computing an obstruction. It uses the metric techniques introduced by the author in the theory of approximable triangulated categories. For a survey of that theory the reader is referred for example to [18]. The current survey is not the right place to expand much on this cryptic comment—in this remark we will confine ourselves to a bare minimum.

Let  $\mathbf{D}_{\mathbf{qc},Z}(X)$  be the derived category, whose objects are complexes of  $\mathfrak{O}_X$ -modules with quasicoherent cohomology supported on Z. This category has a single compact generator, hence a preferred equivalence class of t-structures. Take any t-structure in this preferred equivalence class and view it as a metric on  $\mathbf{D}_{\mathbf{qc},Z}(X)$ , in the sense explained in [18]. This restricts to a metric on the subspace  $\mathbf{D}_Z^{\mathrm{perf}}(X) \subset \mathbf{D}_{\mathbf{qc},Z}(X)$ , and up to equivalence these metrics are independent of any choice. We have a preferred equivalence class of metrics on  $\mathbf{D}_Z^{\mathrm{perf}}(X)$ .

The key to proving Theorem 6.4 turns out to be showing that any bounded t-structure on  $\mathbf{D}_Z^{\mathrm{perf}}(X)$  yields a metric in this preferred equivalence class. And, if the reader will permit us to use yet more of the machinery exposed in [18], there is a process which passes from one triangulated category with a metric to another, it constructs out of S a category  $\mathfrak{S}(S)$ . This can of course be applied to  $S = \mathbf{D}_Z^{\mathrm{perf}}(X)$  with any of the equivalent metrics above. And proof then analyses the category  $\mathfrak{S}(\mathbf{D}_Z^{\mathrm{perf}}(X))$ , and shows that it agrees with  $\mathbf{D}_Z^{\mathrm{perf}}(X)$ .

**Problem 6.6.** Theorem 6.4 is proved by methods which don't fit into this survey. But, now that we know the theorem to be true, it does raise a natural question.

Clearly negative K-theory is not the only obstruction to the existence of bounded t-structures. Given how important bounded t-structures have turned out to be, across wide swaths of mathematics, what is the right obstruction? Can one find an easily computable obstruction to the existence of bounded t-structures, which doesn't vanish on categories such as  $\mathbf{D}_{Z}^{\mathrm{perf}}(X)$ ?

And once again we remind the reader of Problem 0.3. By now there exist a plethora of derived categories of motives, all conjectured to have bounded t-structures. An obstruction that is easily computable on these candidates would be a treasure, it would allow us to weed out the ones with no chance of working.

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CENTRE FOR MATHEMATICS AND ITS APPLICATIONS, MATHEMATICAL SCIENCES INSTITUTE, BUILDING 145, THE AUSTRALIAN NATIONAL UNIVERSITY, CANBERRA, ACT 2601, AUSTRALIA *Email address*: Amnon.Neeman@anu.edu.au