



A counterexample to vanishing conjectures for negative K -theory

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Abstract In a 2006 article Schlichting conjectured that the negative K -theory of any abelian category must vanish. This conjecture was generalized in a 2019 article by Antieau, Gepner and Heller, who hypothesized that the negative K -theory of any category with a bounded t -structure must vanish. Both conjectures will be shown to be false.

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0 Introduction

Let \mathcal{E} be any idempotent-complete exact category. We may form the category $\mathbf{Ac}^b(\mathcal{E})$, whose objects are the acyclic bounded cochain complexes in \mathcal{E} . In Lemma 1.2 and Remark 1.3 we prove that $\mathbf{Ac}^b(\mathcal{E})$ always has a bounded t -structure. We eventually show that, if $\mathcal{E} = \mathcal{V}ect(Y)$ is the category of vector bundles on a projective curve Y with only simple nodes as singularities, then there is an injective map $\mathbb{K}_{-1}(\mathcal{E}) \longrightarrow \mathbb{K}_{-2}[\mathbf{Ac}^b(\mathcal{E})]$. Since there are known examples of nodal curves for which $\mathbb{K}_{-1}[\mathcal{V}ect(Y)] \neq 0$, this provides a counterexample to Antieau et al. [1, Conjecture B].

More generally: let \mathcal{E} be any idempotent-complete exact category. In Proposition 3.3 we produce a homotopy fiber sequence

$$\mathbb{K}[\mathbf{Ac}^b(\mathcal{E})] \longrightarrow \mathbb{K}(\mathcal{E}^\oplus) \longrightarrow \mathbb{K}(\mathcal{E})$$

where \mathcal{E}^\oplus is \mathcal{E} with the split exact structure. Thus the vanishing of $\mathbb{K}_n[\mathbf{Ac}^b(\mathcal{E})]$ for $n < 0$ would imply that the natural maps $\mathbb{K}_n(\mathcal{E}^\oplus) \longrightarrow \mathbb{K}_n(\mathcal{E})$ must be isomorphisms for all $n < 0$. It is entirely possible that there are many more counterexamples out there; the one computed in this article is the case of projective nodal curves. More precisely: if $\mathcal{E} = \mathcal{V}ect(Y)$ with Y a projective nodal curve, we prove that $\mathbb{K}_{-1}(\mathcal{E}^\oplus) = 0$. But there are known examples where $\mathbb{K}_{-1}(\mathcal{E}) \neq 0$.

Following Beilinson et al. [4, Proposition 3.1.10], if a triangulated category \mathcal{T} with a t -structure comes from a model and has suitable “filtered” versions, then there is a natural functor $F : \mathbf{D}^b[\mathcal{T}^\heartsuit] \longrightarrow \mathcal{T}^b$, from the bounded derived category of the heart of \mathcal{T} to the bounded part $\mathcal{T}^b \subset \mathcal{T}$. And what’s important here is that the proof goes by a way that lifts to models. If we apply this to $\mathbf{Ac}^b(\mathcal{E})$ we deduce an induced map in K -theory of the form $\mathbb{K}[\mathbf{Ac}^b(\mathcal{E})^\heartsuit] \longrightarrow \mathbb{K}[\mathbf{Ac}^b(\mathcal{E})]$. And to show that the map in K -theory is a homotopy equivalence, it suffices to prove that $F : \mathbf{D}^b[\mathbf{Ac}^b(\mathcal{E})^\heartsuit] \longrightarrow \mathbf{Ac}^b(\mathcal{E})$ is an equivalence of triangulated categories. In Proposition 2.4 we show that the functor $F : \mathbf{D}^b[\mathbf{Ac}^b(\mathcal{E})^\heartsuit] \longrightarrow \mathbf{Ac}^b(\mathcal{E})$ is an equivalence if and only if the exact category \mathcal{E} is hereditary, meaning $\text{Ext}^i(E, E') = 0$ for all $E, E' \in \mathcal{E}$ and $i \geq 2$.

Since the category of vector bundles on a curve is hereditary, we deduce that in our counterexample the map F is an equivalence. Therefore $\mathbb{K}_{-2}[\mathbf{Ac}^b(\mathcal{E})^\heartsuit] \neq 0$, giving a counterexample to Schlichting [19, Conjecture 1 of Section 10], which is also Antieau et al. [1, Conjecture A].

1 The t -structure on the category $\mathbf{Ac}^b(\mathcal{E})$

Notation 1.1 Let \mathcal{E} be an idempotent-complete exact category, and let $\mathbf{K}(\mathcal{E})$ be the category whose objects are the cochain complexes of objects in \mathcal{E} and

whose morphisms are the homotopy equivalence classes of cochain maps. Let $\mathbf{Ac}(\mathcal{E})$ be the full subcategory of acyclic complexes. The full subcategories $\mathbf{Ac}^-(\mathcal{E}) \subset \mathbf{K}^-(\mathcal{E})$, $\mathbf{Ac}^+(\mathcal{E}) \subset \mathbf{K}^+(\mathcal{E})$ and $\mathbf{Ac}^b(\mathcal{E}) \subset \mathbf{K}^b(\mathcal{E})$ are the obvious bounded versions¹. We remind the reader of the definition of acyclicity: a cochain complex

$$\dots \xrightarrow{\partial^{i-2}} E^{i-1} \xrightarrow{\partial^{i-1}} E^i \xrightarrow{\partial^i} E^{i+1} \xrightarrow{\partial^{i+1}} \dots$$

possibly bounded, is declared acyclic if there exist admissible short exact sequences

$$0 \longrightarrow K^i \xrightarrow{\alpha^i} E^i \xrightarrow{\beta^i} K^{i+1} \longrightarrow 0$$

such that $\partial^i = \alpha^{i+1} \circ \beta^i$. The derived categories $\mathbf{D}^?(\mathcal{E})$ are defined to be the Verdier quotients $\mathbf{K}^?(\mathcal{E}) / \mathbf{Ac}^?(\mathcal{E})$, for $?$ being b , $-$, $+$ or the empty restriction.

Note that we are assuming \mathcal{E} idempotent-complete, and [15, Lemma 1.2] proves that $\mathbf{Ac}(\mathcal{E})$ is a thick subcategory of $\mathbf{K}(\mathcal{E})$. The fact that $\mathbf{Ac}^-(\mathcal{E}) \subset \mathbf{K}^-(\mathcal{E})$, $\mathbf{Ac}^+(\mathcal{E}) \subset \mathbf{K}^+(\mathcal{E})$ and $\mathbf{Ac}^b(\mathcal{E}) \subset \mathbf{K}^b(\mathcal{E})$ are all thick subcategories is older; it may essentially be found in Thomason and Trobaugh [22, 1.11.1] (see also Appendix A). See also [15, Remark 1.10] for a brief synopsis of the argument in Thomason-Trobaugh.

We will usually write E^* as a shorthand for the object

$$\dots \xrightarrow{\partial^{i-2}} E^{i-1} \xrightarrow{\partial^{i-1}} E^i \xrightarrow{\partial^i} E^{i+1} \xrightarrow{\partial^{i+1}} \dots$$

in $\mathbf{K}(\mathcal{E})$.

Lemma 1.2 *Let \mathcal{E} be an idempotent-complete exact category, and let $\mathbf{Ac}(\mathcal{E})$ be the subcategory of acyclics as in Notation 1.1. Define the full subcategories*

$$\begin{aligned} \mathbf{Ac}(\mathcal{E})^{\leq 0} &= \{E^* \in \mathbf{Ac}(\mathcal{E}) \mid E^i = 0 \text{ for all } i > 0\} \\ \mathbf{Ac}(\mathcal{E})^{\geq 0} &= \{E^* \in \mathbf{Ac}(\mathcal{E}) \mid E^i = 0 \text{ for all } i < -2\} \end{aligned}$$

Then the pair $[\mathbf{Ac}(\mathcal{E})^{\leq 0}, \mathbf{Ac}(\mathcal{E})^{\geq 0}]$ define a t -structure on $\mathbf{Ac}(\mathcal{E})$.

Proof The containments $\Sigma \mathbf{Ac}(\mathcal{E})^{\leq 0} \subset \mathbf{Ac}(\mathcal{E})^{\leq 0}$ and $\mathbf{Ac}(\mathcal{E})^{\geq 0} \subset \Sigma \mathbf{Ac}(\mathcal{E})^{\geq 0}$ are obvious from the definition.

¹ In the Introduction we followed the notation of Schlichting [19], where $\mathbf{Ac}^b(\mathcal{E})$ is a model category. In almost all of the article we will follow the notation of Krause [11], where $\mathbf{Ac}^b(\mathcal{E})$ stands for the associated triangulated category.

Now suppose we are given a morphism from an object $E^* \in \mathbf{Ac}(\mathcal{E})^{\leq 0}$ to an object $F^* \in \mathbf{Ac}(\mathcal{E})^{\geq 1}$. We may represent it by a cochain map

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\partial^{-3}} & E^{-2} & \xrightarrow{\partial^{-2}} & E^{-1} & \xrightarrow{\partial^{-1}} & E^0 \longrightarrow 0 \longrightarrow \cdots \\ & & \downarrow & & \downarrow f & & \downarrow g \\ \cdots & \longrightarrow & 0 & \longrightarrow & F^{-1} & \xrightarrow{\tilde{\partial}^{-1}} & F^0 \xrightarrow{\tilde{\partial}^0} F^1 \xrightarrow{\tilde{\partial}^1} \cdots \end{array}$$

The fact that $\tilde{\partial}^0 \circ g = 0$ says that g must factor uniquely through the kernel of the map $\tilde{\partial}^0$, which happens to be the map $\tilde{\partial}^{-1} : F^{-1} \rightarrow F^0$. Thus we may find a (unique) morphism $\theta : E^0 \rightarrow F^{-1}$ with $g = \tilde{\partial}^{-1} \circ \theta$. But now we have the equalities

$$\begin{aligned} \tilde{\partial}^{-1} \circ f &= g \circ \partial^{-1} \\ &= \tilde{\partial}^{-1} \circ \theta \circ \partial^{-1} \end{aligned}$$

where the first comes from the commutativity implied by the cochain map $E^* \rightarrow F^*$, and the second is by precomposing $g = \tilde{\partial}^{-1} \circ \theta$ with ∂^{-1} . And, since $\tilde{\partial}^{-1}$ is a monomorphism (even an admissible monomorphism), it follows that $f = \theta \circ \partial^{-1}$. Thus θ provides a homotopy of the cochain map $E^* \rightarrow F^*$ with the zero map.

Next choose any object $E^* \in \mathbf{Ac}(\mathcal{E})$, that is a complex

$$\cdots \xrightarrow{\partial^{i-2}} E^{i-1} \xrightarrow{\partial^{i-1}} E^i \xrightarrow{\partial^i} E^{i+1} \xrightarrow{\partial^{i+1}} \cdots$$

such that each morphism $\partial^i : E^i \rightarrow E^{i+1}$ has a factorization $E^i \xrightarrow{\beta^i} K^{i+1} \xrightarrow{\alpha^{i+1}} E^{i+1}$ as in Notation 1.1. In particular: we may write $\partial^{-1} : E^{-1} \rightarrow E^0$ as a composite $E^{-1} \xrightarrow{\beta^{-1}} K^0 \xrightarrow{\alpha^0} E^0$. But now consider the cochain maps

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\partial^{-3}} & E^{-2} & \xrightarrow{\partial^{-2}} & E^{-1} & \xrightarrow{\beta^{-1}} & K^0 \longrightarrow 0 \longrightarrow \cdots \\ & & \downarrow \text{id} & & \downarrow \text{id} & & \downarrow \alpha^0 \\ \cdots & \xrightarrow{\partial^{-3}} & E^{-2} & \xrightarrow{\partial^{-2}} & E^{-1} & \xrightarrow{\partial^{-1}} & E^0 \xrightarrow{\partial^0} E^1 \xrightarrow{\partial^1} \cdots \\ & & \downarrow & & \downarrow \beta^{-1} & & \downarrow \text{id} \\ \cdots & \longrightarrow & 0 & \longrightarrow & K^0 & \xrightarrow{\alpha^0} & E^0 \xrightarrow{\partial^0} E^1 \xrightarrow{\partial^1} \cdots \\ & & \downarrow & & \downarrow \text{id} & & \downarrow \text{id} \\ \cdots & \xrightarrow{-\partial^{-2}} & E^{-1} & \xrightarrow{-\beta^{-1}} & K^0 & \longrightarrow & 0 \longrightarrow 0 \longrightarrow \cdots \end{array}$$

and we leave it to the reader to check that this is isomorphic in $\mathbf{Ac}(\mathcal{E})$ to a distinguished triangle $A^* \rightarrow E^* \rightarrow B^* \rightarrow \Sigma A^*$, in which obviously $A^* \in \mathbf{Ac}(\mathcal{E})^{\leq 0}$ and $B^* \in \mathbf{Ac}(\mathcal{E})^{\geq 1}$.

This completes the proof that the pair $[\mathbf{Ac}(\mathcal{E})^{\leq 0}, \mathbf{Ac}(\mathcal{E})^{\geq 0}]$ define a t -structure on $\mathbf{Ac}(\mathcal{E})$. \square

Remark 1.3 Given a triangulated category \mathcal{T} with a t -structure, it is customary to define the subcategories

$$\mathcal{T}^- = \bigcup_{n=1}^{\infty} \mathcal{T}^{\leq -n}, \quad \mathcal{T}^+ = \bigcup_{n=1}^{\infty} \mathcal{T}^{\geq n}, \quad \mathcal{T}^b = \mathcal{T}^- \cap \mathcal{T}^+.$$

In the special case where $\mathcal{T} = \mathbf{Ac}(\mathcal{E})$ and the t -structure is as in Lemma 1.2, the definitions give that

$$[\mathbf{Ac}(\mathcal{E})]^- = \mathbf{Ac}^-(\mathcal{E}), \quad [\mathbf{Ac}(\mathcal{E})]^+ = \mathbf{Ac}^+(\mathcal{E}), \quad [\mathbf{Ac}(\mathcal{E})]^b = \mathbf{Ac}^b(\mathcal{E}),$$

with $\mathbf{Ac}^?(\mathcal{E})$ being as in Notation 1.1. It follows that the t -structure on $\mathbf{Ac}(\mathcal{E})$ restricts to t -structures on $\mathbf{Ac}^?(\mathcal{E})$, with $?$ being each of $-$, $+$ and b . And all four categories have the same heart.

Remark 1.4 The heart of the t -structure of Lemma 1.2 is, by definition, given by the formula $\mathbf{Ac}(\mathcal{E})^\heartsuit = \mathbf{Ac}(\mathcal{E})^{\leq 0} \cap \mathbf{Ac}(\mathcal{E})^{\geq 0}$, and the formula gives that the objects of $\mathbf{Ac}(\mathcal{E})^\heartsuit$ are the admissible short exact sequences

$$0 \longrightarrow E^{-2} \longrightarrow E^{-1} \longrightarrow E^0 \longrightarrow 0$$

in the category \mathcal{E} . Since $\mathbf{Ac}(\mathcal{E})^\heartsuit$ is a full subcategory of $\mathbf{K}(\mathcal{E})$, the morphisms in $\mathbf{Ac}(\mathcal{E})^\heartsuit$ are homotopy equivalence classes of cochain maps

$$\begin{array}{ccccccc} 0 & \longrightarrow & E^{-2} & \longrightarrow & E^{-1} & \longrightarrow & E^0 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & F^{-2} & \longrightarrow & F^{-1} & \longrightarrow & F^0 \longrightarrow 0 \end{array}$$

With this description it isn't immediately obvious where this construction comes from, let alone why this category must be abelian.

Remark 1.5 The abelian category $\mathbf{Ac}(\mathcal{E})^\heartsuit$ isn't new, it may be found in Schlichting [19, Lemma 9 of Section 11]. In Schlichting's presentation this category doesn't come as the heart of some t -structure, instead it is described as a subcategory of the category $\mathbf{Eff}(\mathcal{E}) \subset \mathbf{Mod}\text{-}\mathcal{E}$, whose objects are the effaceable functors in the category $\mathbf{Mod}\text{-}\mathcal{E} = \mathbf{Hom}(\mathcal{E}^{\mathrm{op}}, \mathcal{A})$ of additive functors $\mathcal{E}^{\mathrm{op}} \rightarrow \mathcal{A}$.

Remark 1.6 It might help to consider the special case where \mathcal{E} is an abelian category. The Yoneda map $Y : \mathcal{E} \rightarrow \text{mod-}\mathcal{E}$ embeds \mathcal{E} fully faithfully into the category $\text{mod-}\mathcal{E} \subset \text{Hom}(\mathcal{E}^{\text{op}}, \mathcal{A})$ of finitely presented functors $\mathcal{E}^{\text{op}} \rightarrow \mathcal{A}$. Recall: a functor $F : \mathcal{E}^{\text{op}} \rightarrow \mathcal{A}$ is *finitely presented* if there exists an exact sequence

$$Y(A) \xrightarrow{Y(f)} Y(B) \longrightarrow F \longrightarrow 0$$

which can be thought of as a finite presentation of F in the abelian category $\text{Hom}(\mathcal{E}^{\text{op}}, \mathcal{A})$. Auslander's work tells us that the functor $Y : \mathcal{E} \rightarrow \text{mod-}\mathcal{E}$ has an exact left adjoint $\Lambda : \text{mod-}\mathcal{E} \rightarrow \mathcal{E}$. The way to compute $\Lambda(F)$ is to choose a finite presentation as above, and define $\Lambda(F)$ to be the cokernel of the map $f : A \rightarrow B$. With $\text{eff}(\mathcal{E})$ defined to be the full subcategory of $\text{mod-}\mathcal{E}$ annihilated by the functor Λ , Auslander's formula [2, page 205] goes on to tell us that \mathcal{E} is the Gabriel quotient of $\text{mod-}\mathcal{E}$ by the Serre subcategory $\text{eff}(\mathcal{E})$, see also Krause [11, Theorem 2.2]. In symbols Auslander's formula is

$$\frac{\text{mod-}\mathcal{E}}{\text{eff}(\mathcal{E})} = \mathcal{E}.$$

Krause [11, Corollary 3.2] goes on to give a derived category version of Auslander's formula, in the derived category the formula becomes

$$\frac{\mathbf{D}^b(\text{mod-}\mathcal{E})}{\mathbf{D}_{\text{eff}(\mathcal{E})}^b(\text{mod-}\mathcal{E})} = \mathbf{D}^b(\mathcal{E}).$$

Here $\mathbf{D}_{\text{eff}(\mathcal{E})}^b(\text{mod-}\mathcal{E})$ is the kernel of the functor $\Lambda : \mathbf{D}^b(\text{mod-}\mathcal{E}) \rightarrow \mathbf{D}^b(\mathcal{E})$, the map induced on derived categories by the exact functor of $\Lambda : \text{mod-}\mathcal{E} \rightarrow \mathcal{E}$. Concretely the objects of $\mathbf{D}_{\text{eff}(\mathcal{E})}^b(\text{mod-}\mathcal{E})$ are the bounded cochain complexes in $\text{mod-}\mathcal{E}$ whose cohomology is in $\text{eff}(\mathcal{E})$.

Now the category $\text{mod-}\mathcal{E}$ has enough projectives, in fact the projective objects of $\text{mod-}\mathcal{E}$ are precisely the essential image of the functor $Y : \mathcal{E} \rightarrow \text{mod-}\mathcal{E}$. Not only that: every object in $\text{mod-}\mathcal{E}$ has projective dimension ≤ 2 . To see this take an object $F \in \text{mod-}\mathcal{E}$ and let

$$Y(A) \xrightarrow{Y(f)} Y(B) \longrightarrow F \longrightarrow 0$$

be a finite presentation of F . If K is the kernel in \mathcal{E} of the map $f : A \rightarrow B$, then the sequence

$$0 \longrightarrow Y(K) \longrightarrow Y(A) \xrightarrow{Y(f)} Y(B) \longrightarrow F \longrightarrow 0$$

is easily seen to be exact in $\text{Hom}(\mathcal{E}^{\text{op}}, \mathcal{A})$, and it exhibits a projective resolution of F in the category $\text{mod-}\mathcal{E}$ of length ≤ 2 . Thus every object in $\mathbf{D}^b(\text{mod-}\mathcal{E})$ is isomorphic to a bounded projective resolution, and we obtain an equivalence of triangulated categories

$$\mathbf{K}^b(\mathcal{E}) \cong \mathbf{D}^b(\text{mod-}\mathcal{E}).$$

The inverse image of $\mathbf{D}_{\text{eff}(\mathcal{E})}^b(\text{mod-}\mathcal{E})$ under this equivalence is the category $\mathbf{Ac}^b(\mathcal{E})$ of Notation 1.1, see Krause [11, top of page 674]. Of course the category $\mathbf{D}_{\text{eff}(\mathcal{E})}^b(\text{mod-}\mathcal{E})$ has an obvious, standard t -structure with heart $\text{eff}(\mathcal{E})$. Thus what we have really done in Lemma 1.2 is prove that this t -structure on $\mathbf{Ac}^b(\mathcal{E})$ exists for every exact category, there is no need to assume the category \mathcal{E} abelian in order to produce the t -structure.

And for an abelian category \mathcal{E} we have an equivalence of categories $\mathbf{Ac}(\mathcal{E})^\heartsuit \cong \text{eff}(\mathcal{E})$. Thus for abelian categories \mathcal{E} , the heart of our new t -structure agrees with Auslander's old subcategory $\text{eff}(\mathcal{E}) \subset \text{mod-}\mathcal{E}$.

2 The natural map $\mathbf{D}^b[\mathbf{Ac}(\mathcal{E})^\heartsuit] \rightarrow \mathbf{Ac}^b(\mathcal{E})$

Let \mathcal{T} be a triangulated category with a t -structure, and let \mathcal{T}^\heartsuit be the heart. Under mild hypotheses on \mathcal{T} , the inclusion $\mathcal{T}^\heartsuit \hookrightarrow \mathcal{T}^b$ can be naturally factored as $\mathcal{T}^\heartsuit \rightarrow \mathbf{D}^b(\mathcal{T}^\heartsuit) \xrightarrow{F} \mathcal{T}^b$, for a triangulated functor $F : \mathbf{D}^b(\mathcal{T}^\heartsuit) \rightarrow \mathcal{T}^b$. The reader can find the most general known version of such a result in [16, Theorem 5.1]. But in this article a more useful version will be the older Beilinson et al. [4, Proposition 3.1.10], since the proof presented there comes as a quasifunctor of model categories and hence induces a map in K -theory.²

It becomes natural to wonder when the functor F is an equivalence. The next Lemma is a slight variant of [4, Proposition 3.1.16], and gives a necessary and sufficient condition. A further variant of [4, Proposition 3.1.16] may be found in Chen et al. [9, Theorem 2.9].

Lemma 2.1 *Let \mathcal{T} be a triangulated category with a t -structure, let \mathcal{T}^\heartsuit be the heart of the t -structure, and let $F : \mathbf{D}^b(\mathcal{T}^\heartsuit) \rightarrow \mathcal{T}^b$ be the natural map.*

² It is natural to wonder if the functor $F : \mathbf{D}^b(\mathcal{T}^\heartsuit) \rightarrow \mathcal{T}^b$ depends on the choice of enhancement, and the answer is: Not much. An enhancement of the triangulated category allows one to give it the much-weaker structure of "good morphisms of triangles" as in [16]. In order to construct the functor F on all of $\mathbf{D}^b(\mathcal{T}^\heartsuit)$ one needs to extend from complexes of length n to complexes of length $n + 1$, and this requires constructing a mapping cone. In the presence of an enhancement this is choice-free, but as the reader can see in the proof of [16, Theorem 5.1], in the presence of the new axioms the choice is rigid enough to permit us to choose the triangle uniquely up to canonical isomorphism.

For a similar argument, but with the detail spelt out much more completely than in the sketchy [16, Section 5], the reader is referred to [18, Appendix E].

The functor F is an equivalence of categories if and only if³ every object $t \in \mathcal{T}^{\leq 0} \cap \mathcal{T}^b$ admits a triangle $a \rightarrow t \rightarrow b$ with $a \in \mathcal{T}^\heartsuit$ and $b \in \mathcal{T}^{\leq -1}$.

Proof Let us start with the necessity: if the functor F is an equivalence then it suffices to produce the triangle in the category $\mathbf{D}^b(\mathcal{T}^\heartsuit)$. The object $t \in \mathbf{D}^b(\mathcal{T}^\heartsuit)^{\leq 0}$ is isomorphic to a cochain complex

$$\cdots \longrightarrow T^{-4} \longrightarrow T^{-3} \longrightarrow T^{-2} \longrightarrow T^{-1} \longrightarrow T^0 \longrightarrow 0 \longrightarrow \cdots$$

with $T^i \in \mathcal{T}^\heartsuit$, and the cochain maps

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & T^0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \parallel & & \downarrow & & \\ \cdots & \longrightarrow & T^{-4} & \longrightarrow & T^{-3} & \longrightarrow & T^{-2} & \longrightarrow & T^{-1} & \longrightarrow & T^0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \parallel & & \parallel & & \parallel & & \parallel & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & T^{-4} & \longrightarrow & T^{-3} & \longrightarrow & T^{-2} & \longrightarrow & T^{-1} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

produce the desired triangle $a \rightarrow t \rightarrow b$.

Now for the sufficiency: we assume the existence of triangles $a \rightarrow t \rightarrow b$ as in the Lemma, and need to prove the functor F an equivalence.

Let $n \geq 1$ be an integer, let A and B be objects of \mathcal{T}^\heartsuit , and choose any morphism $f : A \rightarrow \Sigma^n B$. Form the triangle $\Sigma^{n-1} B \rightarrow t \rightarrow A \rightarrow \Sigma^n B$. The object t belongs to $\mathcal{T}^{\leq 0} \cap \mathcal{T}^b$, and by hypothesis there exists a triangle $a \rightarrow t \rightarrow b$ with $a \in \mathcal{T}^\heartsuit$ and $b \in \mathcal{T}^{\leq -1}$. Now let $H : \mathcal{T} \rightarrow \mathcal{T}^\heartsuit$ be the usual homological functor. The exact sequence $H^0(a) \rightarrow H^0(t) \rightarrow H^0(b) = 0$ tells us that $a = H^0(a) \rightarrow H^0(t)$ is an epimorphism in \mathcal{T}^\heartsuit , while the exact sequence $H^0(t) \rightarrow H^0(A) \rightarrow H^0(\Sigma^n B) = 0$ says that $H^0(t) \rightarrow H^0(A) = A$ is also an epimorphism. We conclude that the composite $a \rightarrow t \rightarrow A$ is an epimorphism in \mathcal{T}^\heartsuit , and the composite $a \rightarrow t \rightarrow A \xrightarrow{f} \Sigma B$ obviously vanishes. Since the epimorphism $a \rightarrow A$ can be constructed for every $f : A \rightarrow \Sigma^n B$, Beilinson et al. [4, Proposition 3.1.16] teaches us that F must be an equivalence of categories. \square

We will soon be applying Lemma 2.1 to the case of $F : \mathbf{D}^b[\mathbf{Ac}(\mathcal{E})^\heartsuit] \rightarrow \mathbf{Ac}^b(\mathcal{E})$. Before proceeding we quickly recall

Reminder 2.2 Let \mathcal{E} be an idempotent-complete exact category. Our notion of “acyclic complexes” is designed in such a way that they go to acyclic

³ The reader might wish to compare the necessary and sufficient condition above with Lurie’s notion of 0 -complicial. See [12, Definition C.5.3.1 and Proposition C.5.3.2].

complexes under every exact embedding of \mathcal{E} as a subcategory of an abelian category. There exists an exact embedding $i : \mathcal{E} \rightarrow \mathcal{A}$, with \mathcal{A} abelian, and such that

- (i) The functor i reflects admissible short exact sequences.
- (ii) A morphism $f : x \rightarrow y$ is an admissible monomorphism in \mathcal{E} if and only if $i(f)$ is a monomorphism in \mathcal{A} .
- (iii) A morphism $f : x \rightarrow y$ is an admissible epimorphism in \mathcal{E} if and only if $i(f)$ is an epimorphism in \mathcal{A} .

Thomason and Trobaugh [22, Lemma A.7.15] proves that the Gabriel-Quillen embedding $i : \mathcal{E} \rightarrow \mathcal{A}$ satisfies not only (i) but also (iii). Dually we obtain an embedding $i' : \mathcal{E} \rightarrow \mathcal{B}$ satisfying (i) and (ii). To achieve (i), (ii) and (iii) we take the embedding of \mathcal{E} into $\mathcal{A} \times \mathcal{B}$.

For an embedding $i : \mathcal{E} \rightarrow \mathcal{A}$ satisfying (i), (ii) and (iii), if a bounded cochain complex $T \in \mathbf{D}^b(\mathcal{E})$ has the property that $i(T)$ is acyclic outside an interval $[a, b]$, then T is isomorphic in $\mathbf{D}^b(\mathcal{E})$ to a cochain complex

$$0 \longrightarrow T^a \longrightarrow T^{a+1} \longrightarrow \dots \longrightarrow T^{b-1} \longrightarrow T^b \longrightarrow 0$$

which vanishes outside the interval $[a, b]$.

Definition 2.3 An idempotent-complete exact category \mathcal{E} is called *hereditary* if in the category $\mathbf{D}^b(\mathcal{E})$ the maps $E \rightarrow \Sigma^n F$ vanish whenever $E, F \in \mathcal{E}$ and $n \geq 2$.

Proposition 2.4 Let \mathcal{E} be an idempotent-complete exact category, let $\mathbf{Ac}(\mathcal{E})$ be the homotopy category of acyclic complexes as in Notation 1.1, and let the t -structure on $\mathbf{Ac}(\mathcal{E})$ be as in Lemma 1.2. Let $F : \mathbf{D}^b[\mathbf{Ac}(\mathcal{E})^\heartsuit] \rightarrow \mathbf{Ac}^b(\mathcal{E})$ be the natural functor from the bounded derived category of the heart to the bounded part of the t -structure on $\mathbf{Ac}(\mathcal{E})$, the subcategory $[\mathbf{Ac}(\mathcal{E})]^b = \mathbf{Ac}^b(\mathcal{E})$.

Then the functor F is an equivalence if and only if the category \mathcal{E} is hereditary.

Remark 2.5 The term “hereditary” goes back to Cartan and Eilenberg [8, Section I.5], where a ring R is called hereditary if every R -module has projective dimension ≤ 1 . The rationale was that submodules of projective modules inherit the property of being projective. In Definition 1.2 of Helmut Lenzig’s 1964 PhD thesis, an abelian category is declared “hereditary” if it has global dimension ≤ 1 . In Definition 2.3 we simply extended the classical term to cover exact categories.

Proof Suppose the functor F is an equivalence, let $n \geq 2$ be an integer and choose a morphism $f : A \rightarrow \Sigma^n B$ in $\mathbf{D}^b(\mathcal{E})$. We need to show that f

vanishes. Complete f in $\mathbf{D}^b(\mathcal{E})$ to a triangle $A \xrightarrow{f} \Sigma^n B \rightarrow T$. The object T is such that every exact embedding $i : \mathcal{E} \rightarrow \mathcal{A}$ takes T to a complex acyclic outside the interval $[-n, -1]$, and Remark 2.2 tells us that T is isomorphic in $\mathbf{D}^b(\mathcal{E})$ to a cochain complex

$$\cdots \longrightarrow 0 \longrightarrow T^{-n} \longrightarrow T^{-n+1} \longrightarrow \cdots \longrightarrow T^{-2} \longrightarrow T^{-1} \longrightarrow 0 \longrightarrow \cdots$$

And the triangle $\Sigma^n B \rightarrow T \rightarrow \Sigma A$ can be represented by cochain maps

$$\begin{array}{ccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & B & \longrightarrow & 0 \longrightarrow \cdots \\ & & & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & 0 & \longrightarrow & T^{-n} & \longrightarrow & T^{-n+1} \longrightarrow \cdots \\ & & & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \longrightarrow \cdots \end{array}$$

This being a triangle means that the sequence

$$\cdots \longrightarrow 0 \longrightarrow B \longrightarrow T^{-n} \longrightarrow \cdots \longrightarrow T^{-1} \longrightarrow A \longrightarrow 0 \longrightarrow \cdots$$

is an object $t \in \mathbf{Ac}(\mathcal{E})^{\leq 0} \cap \mathbf{Ac}^b(\mathcal{E})$. Since we are assuming that F is an equivalence, there exists in $\mathbf{Ac}(\mathcal{E})$ a triangle $a \xrightarrow{\varphi} t \rightarrow b$, with $a \in \mathbf{Ac}(\mathcal{E})^\heartsuit$ and $b \in \mathbf{Ac}(\mathcal{E})^{\leq -1}$.

The morphism $\varphi : a \rightarrow t$ in the category $\mathbf{Ac}(\mathcal{E})$ may be represented by a cochain map

$$\begin{array}{ccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & A^{-2} & \longrightarrow & A^{-1} \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & T^{-3} & \longrightarrow & T^{-2} & \longrightarrow & T^{-1} \longrightarrow \cdots \end{array}$$

The distinguished triangle $a \xrightarrow{\varphi} t \rightarrow b$ tells us that b is homotopy equivalent to $\text{Cone}(\varphi)$, the mapping cone on the cochain map φ . Requiring that $b \cong \text{Cone}(\varphi)$ should belong to $\mathbf{Ac}(\mathcal{E})^{\leq -1}$ amounts to saying that the morphism $A^0 \oplus T^{-1} \rightarrow A$ must be a split epimorphism in \mathcal{E} .

Now the morphism $a \rightarrow t$ is isomorphic in $\mathbf{Ac}(\mathcal{E})$ to the cochain map

$$\begin{array}{ccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & A^{-2} & \longrightarrow & A^{-1} \oplus T^{-1} \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & T^{-3} & \longrightarrow & T^{-2} & \longrightarrow & T^{-1} \longrightarrow \cdots \end{array}$$

and thus, without changing the isomorphism class of the map $\varphi : a \rightarrow t$ in the category $\mathbf{Ac}(\mathcal{E})$, we may assume that the cochain map is such that $A^0 \rightarrow A$ is a split epimorphism. Choose a splitting, meaning choose a morphism $g : A \rightarrow A^0$ such that the composite $A \rightarrow A^0 \rightarrow A$ is the identity. Now form in \mathcal{E} the pullback square

$$\begin{array}{ccc} B^{-1} & \longrightarrow & A \\ \downarrow & & \downarrow g \\ A^{-1} & \longrightarrow & A^0 \end{array}$$

Then the composite

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & A^{-2} & \longrightarrow & B^{-1} & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \parallel & & \downarrow & & \downarrow g & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & A^{-2} & \longrightarrow & A^{-1} & \longrightarrow & A^0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & T^{-3} & \longrightarrow & T^{-2} & \longrightarrow & T^{-1} & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

is a cochain map between acyclic complexes

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & A^{-2} & \longrightarrow & B^{-1} & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \parallel & & \downarrow & & \\ \cdots & \longrightarrow & T^{-3} & \longrightarrow & T^{-2} & \longrightarrow & T^{-1} & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

Coming back to the category $\mathbf{D}^b(\mathcal{E})$: the cochain maps

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & A^{-2} & \longrightarrow & B^{-1} & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & T^{-n} & \longrightarrow & \cdots & \longrightarrow & T^{-3} & \longrightarrow & T^{-2} & \longrightarrow & T^{-1} & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

can be viewed as morphisms $A \rightarrow \Sigma^{-1}T \rightarrow A$ composing to the identity. The triangle $\Sigma^{-1}T \rightarrow A \xrightarrow{f} \Sigma^n B \rightarrow T$ is split, and the map $f : A \rightarrow \Sigma^n B$ that we started out with must vanish in $\mathbf{D}^b(\mathcal{E})$.

Suppose now that in the category $\mathbf{D}^b(\mathcal{E})$ any morphism $A \rightarrow \Sigma^2 B$ vanishes, whenever $A, B \in \mathcal{E}$. Any object in $t \in \mathbf{Ac}(\mathcal{E})^{\leq 0} \cap \mathbf{Ac}^b(\mathcal{E})$ of the form

$$0 \longrightarrow T^{-3} \longrightarrow T^{-2} \longrightarrow T^{-1} \longrightarrow T^0 \longrightarrow 0$$

gives rise to a morphism $T^0 \rightarrow \Sigma^2 T^{-3}$ in the category $\mathbf{D}^b(\mathcal{E})$, and it is classical that this morphism will vanish in $\mathbf{D}^b(\mathcal{E})$ if and only if there is a cochain map of acyclic complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & C^{-2} & \longrightarrow & C^{-1} & \longrightarrow & T^0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & T^{-3} & \longrightarrow & T^{-2} & \longrightarrow & T^{-1} & \longrightarrow & T^0 & \longrightarrow & 0 \end{array}$$

This produces for us a morphism $a \rightarrow t$ with $a \in \mathbf{Ac}(\mathcal{E})^\heartsuit$ whose mapping cone lies in $\mathbf{Ac}(\mathcal{E})^{\leq -1}$.

If $t \in \mathbf{Ac}(\mathcal{E})^{\leq 0} \cap \mathbf{Ac}^b(\mathcal{E})$ is more general, meaning of the form

$$0 \longrightarrow T^{-n} \longrightarrow T^{-n+1} \longrightarrow \dots \longrightarrow T^{-1} \longrightarrow T^0 \longrightarrow 0$$

we may apply the above to the complex

$$0 \longrightarrow K \longrightarrow T^{-2} \longrightarrow T^{-1} \longrightarrow T^0 \longrightarrow 0$$

with K the image of $T^{-3} \rightarrow T^{-2}$; since t is acyclic this image is in \mathcal{E} . This produces for us a cochain map

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & C^{-2} & \longrightarrow & C^{-1} & \longrightarrow & T^0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & K & \longrightarrow & T^{-2} & \longrightarrow & T^{-1} & \longrightarrow & T^0 & \longrightarrow & 0 \end{array}$$

and hence also a cochain map

$$\begin{array}{ccccccccc} \dots & \longrightarrow & 0 & \longrightarrow & C^{-2} & \longrightarrow & C^{-1} & \longrightarrow & T^0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \parallel & & \\ \dots & \longrightarrow & T^{-3} & \longrightarrow & T^{-2} & \longrightarrow & T^{-1} & \longrightarrow & T^0 & \longrightarrow & 0 \end{array}$$

which may be viewed as a morphism $\varphi : a \rightarrow t$ in $\mathbf{Ac}(\mathcal{E})$, with $a \in \mathbf{Ac}(\mathcal{E})^\heartsuit$ and such that the mapping cone of φ belongs to $\mathbf{Ac}(\mathcal{E})^{\leq -1}$. \square

3 The K -theoretic consequences

Remark 3.1 Let \mathcal{E} be an idempotent-complete exact category. In Notation 1.1 we recalled the categories $\mathbf{Ac}^?(\mathcal{E}) \subset \mathbf{K}^?(\mathcal{E})$ and the Verdier quotient $\mathbf{D}^?(\mathcal{E}) = \mathbf{K}^?(\mathcal{E}) / \mathbf{Ac}^?(\mathcal{E})$, where $?$ is any of $b, +, -$ or the empty restriction. Now a special case is the exact category \mathcal{E}^\oplus . This means we take any idempotent-complete additive category \mathcal{E} , and give it the exact structure where the admissible exact sequences are the split short exact sequences.

Specializing to the case of \mathcal{E}^\oplus the general definitions of Notation 1.1, an acyclic complex $E^* \in \mathbf{Ac}(\mathcal{E}^\oplus)$ is a cochain complex

$$\cdots \xrightarrow{\partial^{i-2}} E^{i-1} \xrightarrow{\partial^{i-1}} E^i \xrightarrow{\partial^i} E^{i+1} \xrightarrow{\partial^{i+1}} \cdots$$

where there exist split short exact sequences

$$0 \longrightarrow K^i \xrightarrow{\alpha^i} E^i \xrightarrow{\beta^i} K^{i+1} \longrightarrow 0$$

such that $\partial^i = \alpha^{i+1} \circ \beta^i$. This makes $E^i \cong K^i \oplus K^{i+1}$, and the complex E^* can be decomposed as a direct sum of complexes

$$\cdots \longrightarrow 0 \longrightarrow K^i \xrightarrow{\text{id}} K^i \longrightarrow 0 \longrightarrow \cdots$$

which vanish except in degrees $(i-1)$ and i . Hence all objects in $\mathbf{Ac}^?(\mathcal{E}^\oplus)$ are contractible, and are isomorphic to zero in the homotopy category $\mathbf{K}^?(\mathcal{E}^\oplus) = \mathbf{K}^?(\mathcal{E})$. This makes the Verdier quotient

$$\mathbf{D}^?(\mathcal{E}^\oplus) = \mathbf{K}^?(\mathcal{E}^\oplus) / \mathbf{Ac}^?(\mathcal{E}^\oplus) = \mathbf{K}^?(\mathcal{E}) / 0 = \mathbf{K}^?(\mathcal{E}).$$

Remark 3.2 Now it's time to move on to the K -theoretic consequences, which means we need to consider model categories as well as the associated triangulated categories. In the remainder of this section we follow the conventions of Schlichting [19]. Thus \mathcal{M} will be a category of models, $\mathbf{D} : \mathcal{M} \longrightarrow \mathcal{T}$ will be a functor from \mathcal{M} to the category \mathcal{T} of small triangulated categories, and \mathbb{K} will be a functor from \mathcal{M} to spectra. And we will assume that if $M' \longrightarrow M \longrightarrow M''$ is an exact sequence in \mathcal{M} then

$$\mathbb{K}(M') \longrightarrow \mathbb{K}(M) \longrightarrow \mathbb{K}(M'')$$

is a homotopy fibration. Recall: the sequence $M' \longrightarrow M \longrightarrow M''$ is declared to be exact in \mathcal{M} if the categories $\mathbf{D}(\mathcal{M}')$, $\mathbf{D}(\mathcal{M})$ and $\mathbf{D}(\mathcal{M}'')$ are all idempotent-complete, and (1) the functor $\mathbf{D}(M') \longrightarrow \mathbf{D}(M)$ is fully faithful, (2) the

composite $\mathbf{D}(M') \rightarrow \mathbf{D}(M) \rightarrow \mathbf{D}(M'')$ vanishes, and (3) the natural map $\mathbf{D}(M)/\mathbf{D}(M') \rightarrow \mathbf{D}(M'')$ is fully faithful, with $\mathbf{D}(\mathcal{M}'')$ being the idempotent-completion of the essential image of $\mathbf{D}(M)/\mathbf{D}(M')$.

The next result was presaged in Schlichting [19, Proposition 2 in Section 11]. What's incorrect about Schlichting's proof of his old proposition could be rephrased as saying that the natural functor

$$\mathbf{D}^b[\mathbf{Ac}(\mathcal{E})^\heartsuit] \longrightarrow \mathbf{Ac}^b(\mathcal{E})$$

need not in general be an equivalence; see Proposition 2.4.

We begin with the easy

Proposition 3.3 *Let \mathcal{E} be an idempotent-complete exact category, and let \mathcal{E}^\oplus be the category \mathcal{E} but with the split exact structure. Then there is a homotopy fibration of non-connective K -theory-spectra*

$$\mathbb{K}(M') \longrightarrow \mathbb{K}(\mathcal{E}^\oplus) \longrightarrow \mathbb{K}(\mathcal{E})$$

where $M' \in \mathcal{M}$ satisfies $\mathbf{D}(M') = \mathbf{Ac}^b(\mathcal{E})$.

Proof Let \mathcal{M} be the category of biWaldhausen complicial categories as in Thomason and Trobaugh [22]. We consider the sequence $M' \rightarrow M \rightarrow M''$ in \mathcal{M} where, in the notation of Schlichting [19]—which is in conflict with our notation—one would write $M' = \mathbf{Ac}^b(\mathcal{E})$, $M = \mathbf{Ch}^b(\mathcal{E})$ and $M'' = (\mathbf{Ch}^b(\mathcal{E}, \mathbf{Ac}^b(\mathcal{E})))$. The conflict of notation is that in [19] $\mathbf{Ac}^b(\mathcal{E})$ is an object of \mathcal{M} , whereas our $\mathbf{Ac}^b(\mathcal{E})$ is what in Schlichting's notation would be $\mathbf{D}[\mathbf{Ac}^b(\mathcal{E})]$. Our notation, which [as we have said] clashes with Schlichting, follows Krause [11]. It is impossible to choose a notation which agrees with every previous article in the literature.

We remind the reader: Schlichting's notation means that the objects and morphisms of M , M'' are the same, both categories have for objects the bounded complexes of objects in \mathcal{E} and the morphisms are the cochain maps. The weak equivalences in M are the homotopy equivalences, whereas the weak equivalences in M'' are the cochain maps whose mapping cones are acyclic. The objects of M' are the acyclic complexes, and the weak equivalences are as in M . The natural functor $\mathbf{D} : \mathcal{M} \rightarrow \mathcal{T}$ takes the sequence $M' \rightarrow M \rightarrow M''$ to the sequence of triangulated categories $\mathbf{Ac}^b(\mathcal{E}) \rightarrow \mathbf{K}^b(\mathcal{E}) \rightarrow \mathbf{D}^b(\mathcal{E})$, where $\mathbf{Ac}^b(\mathcal{E})$ is to be understood in our notation, it is a triangulated category.

The categories $\mathbf{Ac}^b(\mathcal{E})$, $\mathbf{K}^b(\mathcal{E})$ and $\mathbf{D}^b(\mathcal{E})$ are all known to be idempotent-complete: in the case of $\mathbf{D}^b(\mathcal{E})$ this is by [3, Theorem 2.8]. For $\mathbf{K}^b(\mathcal{E})$, Reminder 3.1 tells us that that $\mathbf{K}^b(\mathcal{E}) = \mathbf{D}^b(\mathcal{E}^\oplus)$, reducing us to the previous case. And for $\mathbf{Ac}^b(\mathcal{E}) \subset \mathbf{K}^b(\mathcal{E})$ this is because we know $\mathbf{Ac}^b(\mathcal{E})$ to be a

thick subcategory of the idempotent-complete triangulated category $\mathbf{K}^b(\mathcal{E})$, we already mentioned the thickness of $\mathbf{Ac}^b(\mathcal{E})$ as a subcategory of $\mathbf{K}^b(\mathcal{E})$ in Notation 1.1. Hence the sequence $M' \rightarrow M \rightarrow M''$ is exact in \mathcal{M} . Therefore the functor \mathbb{K} takes $M' \rightarrow M \rightarrow M''$ to a homotopy fibration. In this homotopy fibration we have that $\mathbb{K}(M'') = \mathbb{K}(\mathcal{E})$, just because $\mathbf{D}(M'') = \mathbf{D}^b(\mathcal{E})$. And $\mathbb{K}(M) = \mathbb{K}(\mathcal{E}^\oplus)$, on the grounds that $\mathbf{D}(M) = \mathbf{K}^b(\mathcal{E}) = \mathbf{D}^b(\mathcal{E}^\oplus)$. Thus our homotopy fibration becomes $\mathbb{K}(M') \rightarrow \mathbb{K}(\mathcal{E}^\oplus) \rightarrow \mathbb{K}(\mathcal{E})$. \square

Remark 3.4 Proposition 3.3 was straightforward, but in combination with Lemma 1.2 it becomes remarkable. The homotopy fiber on the map $\mathbb{K}(\mathcal{E}^\oplus) \rightarrow \mathbb{K}(\mathcal{E})$ is identified with $\mathbb{K}(M')$, and $\mathbf{D}(M') \cong \mathbf{Ac}^b(\mathcal{E})$ has a bounded t -structure.

Antieau et al. [1, Conjecture B on page 244], if true, would imply that $\mathbb{K}_{-n}(M') = 0$ for all $n > 0$, and we would deduce that the natural map $\mathbb{K}_{-n}(\mathcal{E}^\oplus) \rightarrow \mathbb{K}_{-n}(\mathcal{E})$ would have to be an isomorphism for all $n > 0$. But this will be shown to be false, see Example 5.2 below. As it happens in Example 5.2 the exact category \mathcal{E} will be hereditary, and Proposition 2.4 informs us that the natural map $F : \mathbf{D}^b[\mathbf{Ac}(\mathcal{E})^\heartsuit] \rightarrow \mathbf{Ac}^b(\mathcal{E})$ is an equivalence of categories. From Beilinson et al. [4, proof of Proposition 3.1.10] we know that the map F can be realized as $\mathbf{D}(f)$ for some suitable morphism in \mathcal{M} , and hence induces an isomorphism in K -theory. It follows that $\mathbb{K}_{-n}[\mathbf{Ac}(\mathcal{E})^\heartsuit]$ is also nonzero for some $n > 0$, contradicting [1, Conjecture A], which is a restatement of an older conjecture due to Schlichting [19, Conjecture 1 of Section 10].

There exists an abelian category $\mathbf{Ac}(\mathcal{E})^\heartsuit$ with non-vanishing negative K -theory.

4 Vector bundles on nodal curves

We begin by recalling classical facts about smooth projective curves.

Remark 4.1 Fix an algebraically closed field k , and let X be a smooth, projective curve over k . We allow X to have more than one connected component. A vector bundle on X will mean a locally free sheaf, locally of finite rank. Of course: the rank may depend on the connected component we're at.

Each vector bundle \mathcal{V} on X gives rise to three continuous functions

$$\text{rank} : X \rightarrow \mathbb{N}, \quad \text{degree} : X \rightarrow \mathbb{Z}, \quad \text{slope} : X \rightarrow \mathbb{Q}.$$

These continuous functions assign a number to each connected component of X ; for the rank this number is a positive integer, the degree may be any integer, and the slope is a rational number. The rank is obvious. The degree of a line bundle, on a connected component $X_i \subset X$, is the usual degree—the number

of zeros minus the number of poles of a rational section. For a vector bundle \mathcal{V} of rank n the degree of \mathcal{V} is defined to be the degree of the line bundle $\wedge^n \mathcal{V}$, and of course it will depend on the component. And the slope—whose early history will be recalled in Remark 4.2—is defined by the formula

$$\text{slope}(\mathcal{V}) = \frac{\text{degree}(\mathcal{V})}{\text{rank}(\mathcal{V})},$$

and we repeat that this formula yields a rational number for each component of X .

We remind the reader that, if $f : \mathcal{V} \rightarrow \mathcal{V}'$ is an injective map of vector bundles of equal rank, then $\text{degree}(\mathcal{V}) \leq \text{degree}(\mathcal{V}')$ with equality if and only if f is an isomorphism. Since for us this fact is key we quickly recall the argument: assume X connected, and let $n = \text{rank}(\mathcal{V}) = \text{rank}(\mathcal{V}')$. The determinant $\wedge^n f : \wedge^n \mathcal{V} \rightarrow \wedge^n \mathcal{V}'$ is an injective map between line bundles, and may be viewed as a regular, nonzero section of the line bundle $\mathcal{L} = (\wedge^n \mathcal{V}') \otimes (\wedge^n \mathcal{V})^{-1}$. Being a regular section of \mathcal{L} , the determinant $\wedge^n f$ has no poles. The degree of \mathcal{L} can be computed as the number of zeros of $\wedge^n f$ and must be non-negative—therefore $\text{degree}(\mathcal{V}') - \text{degree}(\mathcal{V}) = \text{degree}(\mathcal{L}) \geq 0$. Equality is equivalent to $\wedge^n f$ having no zeros, which happens if and only if $\wedge^n f : \wedge^n \mathcal{V} \rightarrow \wedge^n \mathcal{V}'$ is an isomorphism. But this is equivalent to $f : \mathcal{V} \rightarrow \mathcal{V}'$ being an isomorphism.

And for us the most important consequence is: any injective endomorphism of a vector bundle \mathcal{V} must be an isomorphism.

Remark 4.2 Since I've been asked about the history: it's traditional to assume X irreducible, which we will do in this Remark. Identifying both $H^0(X)$ and $H^2(X)$ with \mathbb{Z} , we have that $\text{rank}(\mathcal{V}) = \text{ch}_0(\mathcal{V})$ and $\text{degree}(\mathcal{V}) = \text{ch}_1(\mathcal{V})$ are the zeroth and first Chern characters of \mathcal{V} and go back a long way. The quotient $\mu(\mathcal{V}) = \text{degree}(\mathcal{V})/\text{rank}(\mathcal{V})$ was first explicitly introduced in print in 1969 by Narasimhan and Ramanan [14, beginning of Section 2]. The name “slope” for the rational number $\mu(\mathcal{V})$ came later—Ramanan tells me that it was coined by Quillen, but the first occurrence I can find in print is in the 1977 article by Shatz [21, Section 2]. Of course in some sense it all goes back Mumford's 1962 ICM talk [13, Definition on page 529], which discusses the formation of the moduli space of stable vector bundles.

Discussion 4.3 *Now it's time to move on to singular curves—but for simplicity the only singularities we allow are simple nodes.*⁴ *Let Y be a nodal projective*

⁴ The argument we are about to sketch hinges on the relation between a vector bundle \mathcal{V} , on a singular curve Y , and its pullback $\pi^* \mathcal{V}$ via the normalization map $\pi : X \rightarrow Y$. By confining attention to simple double points we make our life easier—the map $\pi : X \rightarrow Y$ is especially straightforward then. But the argument extends, and the reader wishing to treat the general

curve, and let $\pi : X \rightarrow Y$ be the normalization. Then X is a smooth projective curve as in [Reminder 4.1](#). A vector bundle \mathcal{V} on Y , which is a locally free sheaf on Y locally of finite rank, pulls back to a vector bundle $\pi^*\mathcal{V}$ on X . And by [Reminder 4.1](#) the vector bundle $\pi^*\mathcal{V}$ has associated to it rank, degree and slope functions.

Let $\{p_1, p_2, \dots, p_n\}$ be the singular points of Y . Then each p_i has two distinct inverse images in X ; let us call them p'_i and p''_i . Thus for each i we have a commutative diagram of schemes

$$\begin{array}{ccc} \mathrm{Spec}(k) & \xrightarrow{\alpha_i} & X \\ \beta_i \downarrow & \searrow \gamma_i & \downarrow \pi \\ X & \xrightarrow{\pi} & Y \end{array}$$

where the image of α_i is p'_i and the image of β_i is p''_i . The composite $\gamma_i = \pi\alpha_i = \pi\beta_i$ has image $p_i = \pi(p'_i) = \pi(p''_i)$. For any coherent sheaf \mathcal{W} on X we have units of adjunction $\eta(\alpha_i) : \mathcal{W} \rightarrow \alpha_{i*}\alpha_i^*\mathcal{W}$ and $\eta(\beta_i) : \mathcal{W} \rightarrow \beta_{i*}\beta_i^*\mathcal{W}$. Now let \mathcal{V} be a coherent sheaf on Y . There is the unit of adjunction $\eta(\pi) : \mathcal{V} \rightarrow \pi_*\pi^*\mathcal{V}$, and we may combine with the units of adjunction above to obtain a commutative square

$$\begin{array}{ccc} \gamma_{i*}\gamma_i^*\mathcal{V} = \pi_*\alpha_{i*}\alpha_i^*\pi^*\mathcal{V} = \pi_*\beta_{i*}\beta_i^*\pi^*\mathcal{V} & \xleftarrow{\eta(\alpha_i)} & \pi_*\pi^*\mathcal{V} \\ \eta(\beta_i) \uparrow & & \uparrow \eta(\pi) \\ \pi_*\pi^*\mathcal{V} & \xleftarrow{\eta(\pi)} & \mathcal{V} \end{array}$$

And it is a classical fact that, for \mathcal{V} a vector bundle on Y , the sequence

$$0 \rightarrow \mathcal{V} \xrightarrow{\eta(\pi)} \pi_*\pi^*\mathcal{V} \xrightarrow{\bigoplus_{i=1}^n [\eta(\alpha_i) - \eta(\beta_i)]} \bigoplus_{i=1}^n \gamma_{i*}\gamma_i^*\mathcal{V} \rightarrow 0$$

is an exact sequence of sheaves on Y . Moreover: this sequence can be used to construct vector bundles on Y . A vector bundle \mathcal{V} on Y is uniquely determined by the vector bundle $\mathcal{W} = \pi^*\mathcal{V}$ on X , together with the isomorphisms $\alpha_i^*\mathcal{W} \cong \beta_i^*\mathcal{W}$ that arise from the canonical isomorphism $\alpha_i^*\pi^*\mathcal{V} \cong \beta_i^*\pi^*\mathcal{V}$. Concretely: given any vector bundle \mathcal{W} on X and, for each i , an isomorphism $\alpha_i^*\mathcal{W} \cong \beta_i^*\mathcal{W}$, we define \mathcal{V} to be the kernel of the map of sheaves

Footnote 4 continued

case is referred to Serre [[20](#), Chapter IV, Section 1] for a discussion of the normalization map $\pi : X \rightarrow Y$ for arbitrary singular curves Y .

$$\pi_* \mathcal{W} \xrightarrow{\bigoplus_{i=1}^n [\eta(\alpha_i) - \eta(\beta_i)]} \bigoplus_{i=1}^n [\gamma_{i*} \alpha_i^* \mathcal{W} \cong \gamma_{i*} \beta_i^* \mathcal{W}]$$

where the isomorphism $\gamma_{i*} \alpha_i^* \mathcal{W} \cong \gamma_{i*} \beta_i^* \mathcal{W}$ is by applying the functor γ_{i*} to the chosen isomorphism $\alpha_i^* \mathcal{W} \cong \beta_i^* \mathcal{W}$. By construction \mathcal{V} is a coherent sheaf on Y . And checking that \mathcal{V} is a vector bundle on Y , with the natural map $\pi^* \mathcal{V} \rightarrow \mathcal{W}$ an isomorphism, is local in Y in the flat topology. Hence we may do it separately at each singular point $p_i \in Y$, and simplify the argument by first completing at p_i . This we leave to the reader.

Moreover: morphisms of vector bundles $f : \mathcal{V} \rightarrow \mathcal{V}'$ on Y are uniquely determined by the “descent data” above. The morphism f gives rise to a map of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{V} & \xrightarrow{\eta(\pi)} & \pi_* \pi^* \mathcal{V} & \xrightarrow{\bigoplus_{i=1}^n [\eta(\alpha_i) - \eta(\beta_i)]} & \bigoplus_{i=1}^n [\gamma_{i*} \alpha_i^* \pi^* \mathcal{V} = \gamma_{i*} \beta_i^* \pi^* \mathcal{V}] \longrightarrow 0 \\ & & \downarrow f & & \downarrow \pi_* \pi^* f & & \downarrow \bigoplus_{i=1}^n [\gamma_{i*} \alpha_i^* \pi^* f = \gamma_{i*} \beta_i^* \pi^* f] \\ 0 & \longrightarrow & \mathcal{V}' & \xrightarrow{\eta(\pi)} & \pi_* \pi^* \mathcal{V}' & \xrightarrow{\bigoplus_{i=1}^n [\eta(\alpha_i) - \eta(\beta_i)]} & \bigoplus_{i=1}^n [\gamma_{i*} \alpha_i^* \pi^* \mathcal{V}' = \gamma_{i*} \beta_i^* \pi^* \mathcal{V}'] \longrightarrow 0 \end{array}$$

and hence the commutative square on the right uniquely determines the vertical arrow on the left.

Remark 4.4 Just to clarify: everything in Discussion 4.3 is known. See Theorem 1.3 on page 18 of Igor Burban’s 2003 PhD thesis for a complete and thorough treatment, and Burban and Kreussler [7, Theorem 5.1.4] for a published account. For a partial list of articles on related topics, going back to the 1960s, see Burban and Drozd [6, beginning of Chapter 2].

And now the time has come to prove something.

Lemma 4.5 *Let k be an algebraically closed field, let Y be a projective curve over k with only simple nodes, and let \mathcal{V} be a vector bundle on Y . Then any self-map $f : \mathcal{V} \rightarrow \mathcal{V}$ leads to a canonical decomposition $\mathcal{V} = \mathcal{V}' \oplus \mathcal{V}''$ such that*

- (i) *f decomposes as $f = f' \oplus f''$ for maps $f' : \mathcal{V}' \rightarrow \mathcal{V}'$ and $f'' : \mathcal{V}'' \rightarrow \mathcal{V}''$.*
- (ii) *The map f' is an isomorphism while the map f'' is nilpotent.*

Remark 4.6 The beginning of the proof below, which will treat the case where Y is smooth, has similarities with the proof of Fitting's Lemma, see Jacobson [10, pp. 113-114]. Since the category of coherent sheaves on Y isn't Artinian the arguments aren't identical; nevertheless the reader might wish to compare the two proofs.

Proof We first treat the case where Y is smooth. The coherent sheaves $\text{Ker}(f^n)$ are an increasing sequence of subsheaves of the coherent sheaf \mathcal{V} , hence must stabilize. There exists an integer $N \gg 0$ such that, for all $n \geq N$, the inclusion $\text{Ker}(f^n) \subset \text{Ker}(f^{n+1})$ is an isomorphism.

Now: for each integer $n > 0$ we have a short exact sequence of coherent sheaves

$$0 \longrightarrow \text{Ker}(f^n) \xrightarrow{\psi_n} \text{Ker}(f^{2n}) \longrightarrow \text{Ker}(f^n) \cap \text{Im}(f^n) \longrightarrow 0$$

and if $n \geq N$ the map ψ_n is an isomorphism. We conclude that, as long as $n \geq N$, we must have $\text{Ker}(f^n) \cap \text{Im}(f^n) = 0$.

Next note that $\text{Ker}(f^n)$ and $\text{Im}(f^n)$ are coherent subsheaves of \mathcal{V} . They must be torsion-free, and torsion-free coherent sheaves on a smooth curve are vector bundles. Choose any $n \geq N$ and consider the natural composable morphisms

$$\text{Im}(f^n) \longrightarrow \mathcal{V} \longrightarrow \text{Im}(f^n)$$

The kernel of the composite is zero, since the map $\mathcal{V} \longrightarrow \text{Im}(f^n)$ has kernel $\text{Ker}(f^n)$ which intersects $\text{Im}(f^n) \subset \mathcal{V}$ trivially. Thus the composite is an injective endomorphism of the vector bundle $\text{Im}(f^n)$ and must be an isomorphism. We conclude that $\text{Im}(f^n)$ is a direct summand of \mathcal{V} , more precisely we learn that the inclusion $\text{Im}(f^n) \longrightarrow \mathcal{V}$ provides a splitting of the short exact sequence

$$0 \longrightarrow \text{Ker}(f^n) \longrightarrow \mathcal{V} \longrightarrow \text{Im}(f^n) \longrightarrow 0.$$

This gives us our canonical decomposition $\mathcal{V} = \text{Im}(f^n) \oplus \text{Ker}(f^n)$. Obviously the map $f : \mathcal{V} \longrightarrow \mathcal{V}$ takes $\text{Im}(f^n) \subset \mathcal{V}$ to itself and takes $\text{Ker}(f^n) \subset \mathcal{V}$ to itself, that is $f : \mathcal{V} \longrightarrow \mathcal{V}$ decomposes as $f' \oplus f''$ for a unique $f' : \text{Im}(f^n) \longrightarrow \text{Im}(f^n)$ and a unique $f'' : \text{Ker}(f^n) \longrightarrow \text{Ker}(f^n)$. Obviously f'' is nilpotent, more precisely $(f'')^n = 0$. And the kernel of the map f' is

$$\text{Ker}(f) \cap \text{Im}(f^n) \subset \text{Ker}(f^n) \cap \text{Im}(f^n) = 0,$$

which shows that the map $f' : \text{Im}(f^n) \longrightarrow \text{Im}(f^n)$ is an injective endomorphism of the vector bundle $\text{Im}(f^n)$ and hence an isomorphism.

Now for the case where Y is allowed to have simple nodes as singularities. Let $f : \mathcal{V} \rightarrow \mathcal{V}$ be an endomorphism of the vector bundle \mathcal{V} , and let $\pi : X \rightarrow Y$ be the normalization of Y . Then $\pi^* f : \pi^* \mathcal{V} \rightarrow \pi^* \mathcal{V}$ is an endomorphism of the vector bundle $\pi^* \mathcal{V}$ on X , and by the above there exists a decomposition of $\pi^* \mathcal{V}$ as $\pi^* \mathcal{V} = \mathcal{W}' \oplus \mathcal{W}''$ such that

- (i) There exists an integer $n \gg 0$ with $\mathcal{W}' = \text{Im}(\pi^* f^n)$ and $\mathcal{W}'' = \text{Ker}(\pi^* f^n)$.
- (ii) The map $\pi^* f : \mathcal{W}' \oplus \mathcal{W}'' \rightarrow \mathcal{W}' \oplus \mathcal{W}''$ is equal to $\varphi' \oplus \varphi''$ for some morphisms $\varphi' : \mathcal{W}' \rightarrow \mathcal{W}'$ and $\varphi'' : \mathcal{W}'' \rightarrow \mathcal{W}''$.
- (iii) The map φ' is an isomorphism while the map φ'' satisfies $(\varphi'')^n = 0$.

With $\alpha_i : \text{Spec}(k) \rightarrow X$ and $\beta_i : \text{Spec}(k) \rightarrow X$ as in Discussion 4.3, we have induced decompositions $\alpha_i^* \pi^* \mathcal{V} = \alpha_i^* \mathcal{W}' \oplus \alpha_i^* \mathcal{W}''$ and $\beta_i^* \pi^* \mathcal{V} = \beta_i^* \mathcal{W}' \oplus \beta_i^* \mathcal{W}''$ such that

- (i) $\alpha_i^* \pi^* f = \alpha_i^* \varphi' \oplus \alpha_i^* \varphi''$ and $\beta_i^* \pi^* f = \beta_i^* \varphi' \oplus \beta_i^* \varphi''$.

Of course we have canonical isomorphisms $\rho_i : \alpha_i^* \pi^* \mathcal{V} \rightarrow \beta_i^* \pi^* \mathcal{V}$, and these isomorphisms ρ_i must be such that the squares below commute

$$\begin{array}{ccc} \alpha_i^* \pi^* \mathcal{V} & \xrightarrow{\alpha_i^* \pi^* f} & \alpha_i^* \pi^* \mathcal{V} \\ \rho_i \downarrow & & \downarrow \rho_i \\ \beta_i^* \pi^* \mathcal{V} & \xrightarrow{\beta_i^* \pi^* f} & \beta_i^* \pi^* \mathcal{V} \end{array}$$

Raising the horizontal maps to the n^{th} power, the squares

$$\begin{array}{ccc} \alpha_i^* \pi^* \mathcal{V} & \xrightarrow{\alpha_i^* \pi^* f^n} & \alpha_i^* \pi^* \mathcal{V} \\ \rho_i \downarrow & & \downarrow \rho_i \\ \beta_i^* \pi^* \mathcal{V} & \xrightarrow{\beta_i^* \pi^* f^n} & \beta_i^* \pi^* \mathcal{V} \end{array}$$

must also commute. But these squares can be rewritten as

$$\begin{array}{ccc} \alpha_i^* \mathcal{W}' \oplus \alpha_i^* \mathcal{W}'' & \xrightarrow{[\alpha_i^*(\varphi')^n] \oplus 0} & \alpha_i^* \mathcal{W}' \oplus \alpha_i^* \mathcal{W}'' \\ \rho_i \downarrow & & \downarrow \rho_i \\ \beta_i^* \mathcal{W}' \oplus \beta_i^* \mathcal{W}'' & \xrightarrow{[\beta_i^*(\varphi')^n] \oplus 0} & \beta_i^* \mathcal{W}' \oplus \beta_i^* \mathcal{W}'' \end{array}$$

The fact that φ' is an isomorphism means that so are $\alpha_i^*(\varphi')^n$ and $\beta_i^*(\varphi')^n$. And the commutativity of the square forces the map ρ_i to take the kernel

of the top horizontal map to the kernel of the bottom horizontal map, and the image of the top horizontal map to the image of the bottom horizontal map. That is: the isomorphism $\rho_i : \alpha_i^* \mathcal{W}' \oplus \alpha_i^* \mathcal{W}'' \rightarrow \beta_i^* \mathcal{W}' \oplus \beta_i^* \mathcal{W}''$ must split as the direct sum $\rho'_i \oplus \rho''_i$, for isomorphisms $\rho'_i : \alpha_i^* \mathcal{W}' \rightarrow \beta_i^* \mathcal{W}'$ and $\rho''_i : \alpha_i^* \mathcal{W}'' \rightarrow \beta_i^* \mathcal{W}''$.

And this allows us to descend to Y ; the maps $\varphi' : \mathcal{W}' \rightarrow \mathcal{W}'$ and $\varphi'' : \mathcal{W}'' \rightarrow \mathcal{W}''$, together with the descent data given by the isomorphisms ρ'_i and ρ''_i , allow us to uniquely define vector bundles $\mathcal{V}', \mathcal{V}''$ on Y , as well as endomorphisms $f' : \mathcal{V}' \rightarrow \mathcal{V}'$ and $f'' : \mathcal{V}'' \rightarrow \mathcal{V}''$, so that $\pi^* f' : \pi^* \mathcal{V}' \rightarrow \pi^* \mathcal{V}'$ agrees with $\varphi' : \mathcal{W}' \rightarrow \mathcal{W}'$ and $\pi^* f'' : \pi^* \mathcal{V}'' \rightarrow \pi^* \mathcal{V}''$ agrees with $\varphi'' : \mathcal{W}'' \rightarrow \mathcal{W}''$. The uniqueness forces $f' \oplus f''$ to agree with $f : \mathcal{V} \rightarrow \mathcal{V}$. And the fact that f' is an isomorphism and $(f'')^n = 0$ can be checked after pulling back to X . \square

Remark 4.7 We should perhaps spell out what we meant in Lemma 4.5, when we said that the decomposition of \mathcal{V} as $\mathcal{V} = \mathcal{V}' \oplus \mathcal{V}''$ is “canonical”.

The fact that $f : \mathcal{V} \rightarrow \mathcal{V}$ decomposes as $f = f' \oplus f''$, with $f' : \mathcal{V}' \rightarrow \mathcal{V}'$ an isomorphism and $f'' : \mathcal{V}'' \rightarrow \mathcal{V}''$ nilpotent, makes the decomposition unique. Choose an $n \gg 0$ so that $(f'')^n = 0$; then $f^n : \mathcal{V} \rightarrow \mathcal{V}$ has kernel \mathcal{V}'' and image \mathcal{V}' . Thus we can write the formulas

$$\mathcal{V}' = \bigcap_{n=1}^{\infty} \text{Im}(f^n) \quad \text{and} \quad \mathcal{V}'' = \bigcup_{n=1}^{\infty} \text{Ker}(f^n),$$

which show that these subbundles are canonically unique—what isn’t immediate from the formulas is that these are vector bundles and that their direct sum is \mathcal{V} .

Suppose we are given a commutative square of maps of vector bundles on Y

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{f} & \mathcal{V} \\ \rho \downarrow & & \downarrow \rho \\ \mathcal{W} & \xrightarrow{g} & \mathcal{W} \end{array}$$

Then obviously

$$\rho \left(\bigcap_{n=1}^{\infty} \text{Im}(f^n) \right) \subset \bigcap_{n=1}^{\infty} \text{Im}(g^n) \quad \text{and} \quad \rho \left(\bigcup_{n=1}^{\infty} \text{Ker}(f^n) \right) \subset \bigcup_{n=1}^{\infty} \text{Ker}(g^n)$$

This means that, in the decomposition $\mathcal{V} = \mathcal{V}' \oplus \mathcal{V}''$ that comes from f and the decomposition $\mathcal{W} = \mathcal{W}' \oplus \mathcal{W}''$ that comes from g , we must have the

compatibility that $\rho : \mathcal{V} \longrightarrow \mathcal{W}$ must split as $\rho = \rho' \oplus \rho''$ for a unique choice of $\rho' : \mathcal{V}' \longrightarrow \mathcal{W}'$ and $\rho'' : \mathcal{V}'' \longrightarrow \mathcal{W}''$.

Lemma 4.8 *Let k be an algebraically closed field, let Y is a projective curve over k with only simple nodes, and let \mathcal{V}^* be a cochain complex of vector bundle on Y . Then any cochain map $f^* : \mathcal{V}^* \longrightarrow \mathcal{V}^*$ leads to a canonical decomposition $\mathcal{V}^* = \mathcal{V}_1^* \oplus \mathcal{V}_2^*$ such that*

- (i) *f^* decomposes as $f^* = f_1^* \oplus f_2^*$ for maps $f_1^* : \mathcal{V}_1^* \longrightarrow \mathcal{V}_1^*$ and $f_2^* : \mathcal{V}_2^* \longrightarrow \mathcal{V}_2^*$.*
- (ii) *The map f_1^* is an isomorphism while the map f_2^* is locally nilpotent. By locally nilpotent we mean that, for any integer $i \in \mathbb{Z}$, there exists an integer n_i (which may depend on i) for which $f_2^i : \mathcal{V}_2^i \longrightarrow \mathcal{V}_2^i$ satisfies $(f_2^i)^{n_i} = 0$.*

Moreover: if f is null homotopic then the complex \mathcal{V}_1^ must be contractible.*

Proof The existence of the canonical decomposition is by Remark 4.7; in each degree i the map $f^i : \mathcal{V}^i \longrightarrow \mathcal{V}^i$ leads to a decomposition $\mathcal{V}^i = \mathcal{V}_1^i \oplus \mathcal{V}_2^i$, and the differential $\partial^i : \mathcal{V}^i \longrightarrow \mathcal{V}^{i+1}$ must split as a direct sum $\partial^i = \partial_1^i \oplus \partial_2^i$ for suitable $\partial_1^i : \mathcal{V}_1^i \longrightarrow \mathcal{V}_1^{i+1}$ and $\partial_2^i : \mathcal{V}_2^i \longrightarrow \mathcal{V}_2^{i+1}$.

It remains to prove the “moreover” statement. Suppose therefore that f^* is null homotopic. Then the isomorphism $f_1^* : \mathcal{V}_1^* \longrightarrow \mathcal{V}_1^*$ can be factored as the composite

$$\mathcal{V}_1^* \xrightarrow{i} \mathcal{V}^* \xrightarrow{f} \mathcal{V}^* \xrightarrow{p} \mathcal{V}_1^*$$

where $\mathcal{V}_1^* \xrightarrow{i} \mathcal{V}^* \xrightarrow{p} \mathcal{V}_1^*$ are the canonical inclusion and projection from the direct sum. Thus the fact that f is null homotopic forces the isomorphism $f_1^* : \mathcal{V}_1^* \longrightarrow \mathcal{V}_1^*$ to also be. \square

Proposition 4.9 *Let k be an algebraically closed field, let Y be a projective curve over k with only simple nodes, and let $\mathbf{K}[\text{Vect}(Y)]$ be the homotopy category of cochain complexes of vector bundles on Y . Then every idempotent in $\mathbf{K}[\text{Vect}(Y)]$ splits.*

Proof Choose an idempotent in $\mathbf{K}[\text{Vect}(Y)]$, and let it be represented by the cochain map $e^* : \mathcal{V}^* \longrightarrow \mathcal{V}^*$. The assumption that e^* is idempotent in $\mathbf{K}[\text{Vect}(Y)]$ means that e^* and $(e^*)^2$ are homotopic.

By Lemma 4.8 we may decompose \mathcal{V}^* , along the map $e^* : \mathcal{V}^* \longrightarrow \mathcal{V}^*$, as $\mathcal{V}^* = \mathcal{V}_1^* \oplus \mathcal{V}_2^*$ is such a way that

- (i) The map e^* may be written as $e_1^* \oplus e_2^*$, for cochain maps $e_1^* : \mathcal{V}_1^* \longrightarrow \mathcal{V}_1^*$ and $e_2^* : \mathcal{V}_2^* \longrightarrow \mathcal{V}_2^*$.

(ii) The map e_1^* is an isomorphism while the map e_2^* is locally nilpotent.

The fact that $(e^*)^2 - e^*$ is null homotopic forces the direct summands $(e_1^*)^2 - e_1^*$ and $(e_2^*)^2 - e_2^*$ to be null homotopic. Since e_1^* is an isomorphism and $e_1^*(e_1^* - 1)$ is null homotopic we deduce that e_1^* is homotopic to the identity.

It remains to show that e_2^* is null homotopic. Replacing e^* by e_2^* , we are reduced to showing

(i) Suppose $e^* : \mathcal{V}^* \rightarrow \mathcal{V}^*$ is a cochain map, and assume that e^* is locally nilpotent and that $e^* - (e^*)^2$ is null homotopic. Then e^* is null homotopic.

Observe the formal equalities

$$\begin{aligned} e^* &= \sum_{i=1}^{\infty} \left[(e^*)^i - (e^*)^{i+1} \right] \\ &= [e^* - (e^*)^2] \cdot \left[\sum_{i=0}^{\infty} (e^*)^i \right] \end{aligned}$$

The infinite sums make sense since the local nilpotence guarantees that in each degree the sums are finite. Thus we have produced a factorization of the cochain map $e^* : \mathcal{V}^* \rightarrow \mathcal{V}^*$ as the composite of two cochain maps

$$\mathcal{V}^* \xrightarrow{\sum_{i=0}^{\infty} (e^*)^i} \mathcal{V}^* \xrightarrow{e^* - (e^*)^2} \mathcal{V}^*$$

where the second is null homotopic. Hence e^* is null homotopic. \square

Remark 4.10 The proof of Proposition 4.9 generalizes, it works to show that idempotents split not only in the unbounded homotopy category $\mathbf{K}[\mathcal{V}ect(X)]$, but also in the bounded subcategories $\mathbf{K}^b[\mathcal{V}ect(X)]$, $\mathbf{K}^-[\mathcal{V}ect(X)]$ and $\mathbf{K}^+[\mathcal{V}ect(X)]$. But for the categories \mathbf{K}^b , \mathbf{K}^- and \mathbf{K}^+ the fact that idempotents split is not new: the reader can construct proofs using the methods of [5, Proposition 3.1]. See the proof of [5, Proposition 3.4] for an outline. For a later proof that is more K -theoretical see Balmer and Schlichting [3, Section 2]. More precisely: the idempotent-completeness of $\mathbf{K}^-(\mathcal{E}) = \mathbf{D}^-(\mathcal{E}^\oplus)$ and of $\mathbf{K}^+(\mathcal{E}) = \mathbf{D}^+(\mathcal{E}^\oplus)$ may be found in [3, Lemma 2.4], while the idempotent-completeness of $\mathbf{K}^b(\mathcal{E}) = \mathbf{D}^b(\mathcal{E}^\oplus)$ follows from [3, Theorem 2.8].

5 The counterexample

Throughout this section k will be a fixed algebraically closed field. If Y is a projective curve over k with only simple nodes as singularities, then $\mathcal{V}ect(Y)$

will be the category of vector bundles over Y . And now we return to the K -theoretic implications of what we have proved.

Remark 5.1 The K -theoretic import of Proposition 4.9 is that $\mathbb{K}_{-1}[\mathcal{V}ect(Y)^\oplus] = 0$. This follows from Schlichting [19, Corollary 6 of Section 9]. We briefly remind the reader: in the category \mathcal{M} of model categories we have a commutative square

$$\begin{array}{ccc} \mathbf{Ch}^b[\mathcal{V}ect(Y)] & \longrightarrow & \mathbf{Ch}^-[\mathcal{V}ect(Y)] \\ \downarrow & & \downarrow \\ \mathbf{Ch}^+[\mathcal{V}ect(Y)] & \longrightarrow & \mathbf{Ch}[\mathcal{V}ect(Y)] \end{array}$$

The functor $\mathbf{D} : \mathcal{M} \rightarrow \mathcal{T}$ takes this to the commutative square

$$\begin{array}{ccc} \mathbf{K}^b[\mathcal{V}ect(Y)] & \longrightarrow & \mathbf{K}^-[\mathcal{V}ect(Y)] \\ \downarrow & & \downarrow \\ \mathbf{K}^+[\mathcal{V}ect(Y)] & \longrightarrow & \mathbf{K}[\mathcal{V}ect(Y)] \end{array}$$

and recalling that, by Reminder 3.1, we have $\mathbf{K}^?(\mathcal{E}) = \mathbf{D}^?(\mathcal{E}^\oplus)$ for $?$ any of $b, +, -$ or the empty restriction, this puts us squarely in the situation of Schlichting [19, Corollary 6 of Section 9]. The connective part of the functor \mathbb{K} takes the commutative square in \mathcal{M} to the homotopy cartesian square

$$\begin{array}{ccc} \mathbb{K}_{\geq 0}(\mathbf{Ch}^b[\mathcal{V}ect(Y)]) & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{K}_{\geq 0}(\mathbf{Ch}[\mathcal{V}ect(Y)]) \end{array}$$

making $\mathbb{K}_{\geq 0}(\mathbf{Ch}[\mathcal{V}ect(Y)])$ a delooping of $\mathbb{K}_{\geq 0}(\mathbf{Ch}^b[\mathcal{V}ect(Y)]) \cong \mathbb{K}_{\geq 0}[\mathcal{V}ect(Y)^\oplus]$. And $\mathbb{K}_{-1}[\mathcal{V}ect(Y)^\oplus]$ can therefore be computed as \mathbb{K}_0 of the idempotent completion of $\mathbf{D}(\mathbf{Ch}[\mathcal{V}ect(Y)]) = \mathbf{K}[\mathcal{V}ect(Y)]$. Proposition 4.9 tells us that the triangulated category $\mathbf{K}[\mathcal{V}ect(Y)]$ is idempotent-complete, and since it has vanishing \mathbb{K}_0 we deduce that $\mathbb{K}_{-1}[\mathcal{V}ect(Y)^\oplus] = 0$.

Example 5.2 As in Remark 5.1 we let Y be a projective curve over k with simple nodes as singularities. Now recall the homotopy fibration of Proposition 3.3. With $\mathcal{E} = \mathcal{V}ect(Y)$ we have an exact sequence $\mathbb{K}_{-1}(\mathcal{E}^\oplus) \rightarrow$

$\mathbb{K}_{-1}(\mathcal{E}) \longrightarrow \mathbb{K}_{-2}(M')$, where $\mathbf{D}(M') = \mathbf{Ac}^b(\mathcal{E})$ is the homotopy category of bounded acyclic cochain complexes.

Remark 5.1 gives the vanishing of $\mathbb{K}_{-1}(\mathcal{E}^\oplus)$, and the sequence of Proposition 3.3 becomes $0 \longrightarrow \mathbb{K}_{-1}(\mathcal{E}) \longrightarrow \mathbb{K}_{-2}(M')$. This shows that $\mathbb{K}_{-2}(M')$ contains $\mathbb{K}_{-1}(\mathcal{E})$ as a submodule. But there are known examples of nodal curves with non-vanishing \mathbb{K}_{-1} ; see Weibel [25, Exercise III.4.4]. Hence there are examples of model categories $M' \in \mathcal{M}$, with non-vanishing negative K -theory, and such that $\mathbf{D}(M') = \mathbf{Ac}^b(\mathcal{E})$ has a bounded t -structure.

Note: as presented in Weibel's book the nodal curves U for which $\mathbb{K}_{-1}[\mathcal{Vect}(U)] \neq 0$ are affine. But the Mayer-Vietoris exact sequence, for assembling $\mathbb{K}[\mathcal{Vect}(Y)]$ from Zariski open covers, permits us to pass to the compactification of these affine curves, which can be chosen to have only simple nodes as singularities and also have non-vanishing $\mathbb{K}_{-1}[\mathcal{Vect}(Y)]$. The Mayer-Vietoris sequence was proved to be exact by Weibel [24, Main Theorem] for reduced quasiprojective varieties with isolated singularities, and in general by Thomason and Trobaugh [22, Theorem 8.1].⁵

The non-vanishing of $\mathbb{K}_{-1}[\mathcal{Vect}(Y)]$ can also be deduced from the Mayer-Vietoris sequence for the conductor square; see Pedrini and Weibel [17, Theorem A.3].

Remark 5.3 In Remark 3.4 we promised the reader that, for the $\mathcal{E} = \mathcal{Vect}(Y)$ of Example 5.2, the category \mathcal{E} will be proved hereditary. It is time to deliver on the promise.

If $\mathcal{V}, \mathcal{V}'$ are vector bundles on Y then there is a spectral sequence, converging to $\mathrm{Ext}^{p+q}(\mathcal{V}, \mathcal{V}')$, whose E^2 term has entries $H^p[Y, \mathcal{E}xt^q(\mathcal{V}, \mathcal{V}')]$. Now as \mathcal{V} is a locally free sheaf the local Ext sheaves $\mathcal{E}xt^q(\mathcal{V}, \mathcal{V}')$ vanish when $q > 0$. And $H^p[Y, \mathcal{E}xt^0(\mathcal{V}, \mathcal{V}')] vanishes if $p > 1$, just because Y is a curve.$

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⁵ For the discerning, careful reader who checks the reference to [22]: the only schemes Y considered in the current article have been quasiprojective, they have ample line bundles, this makes them satisfy the resolution property, and the natural map $\mathbf{D}^b[\mathcal{Vect}(Y)] \longrightarrow \mathbf{D}^{\mathrm{perf}}(Y)$ is an equivalence of categories.

Thomason and Trobaugh [22] allows pathological schemes which do not satisfy the resolution property, and for such schemes the right K -theory to work with, the one for which the Mayer-Vietoris sequence is exact, is the K -theory that comes from perfect complexes and not the one corresponding to $(\mathbf{Ch}^b[\mathcal{Vect}(Y)], \mathbf{Ac}^b[\mathcal{Vect}(Y)])$. It isn't known how pathological a scheme Y has to be for this to be an issue. Non-separated schemes can fail to satisfy the resolution property, but the reader can check Totaro [23, Question 1, page 3 of the Introduction]: for all we know every scheme with an affine diagonal might have the resolution property.

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