Support Vector Machine (SVM)

Shusen Wang

Project a Point onto a Hyperplane

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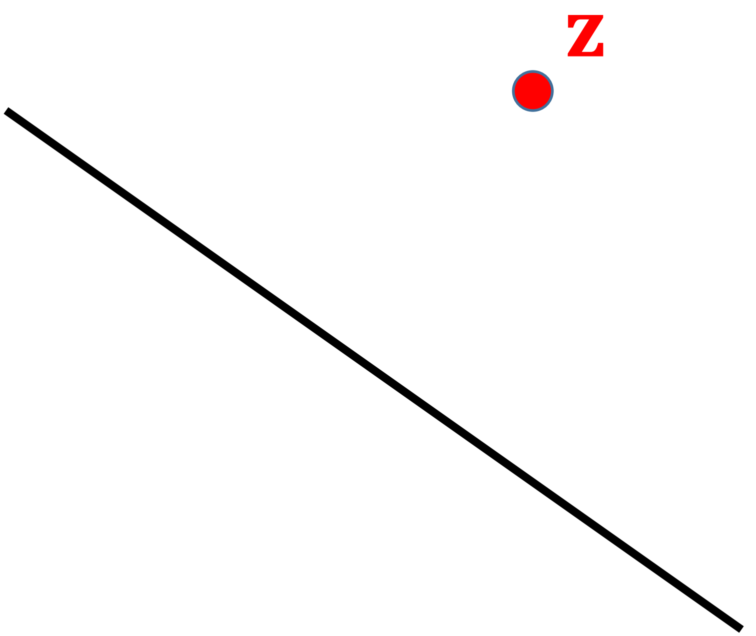
Question: how to project $\mathbf{z}$ onto the hyperplane?

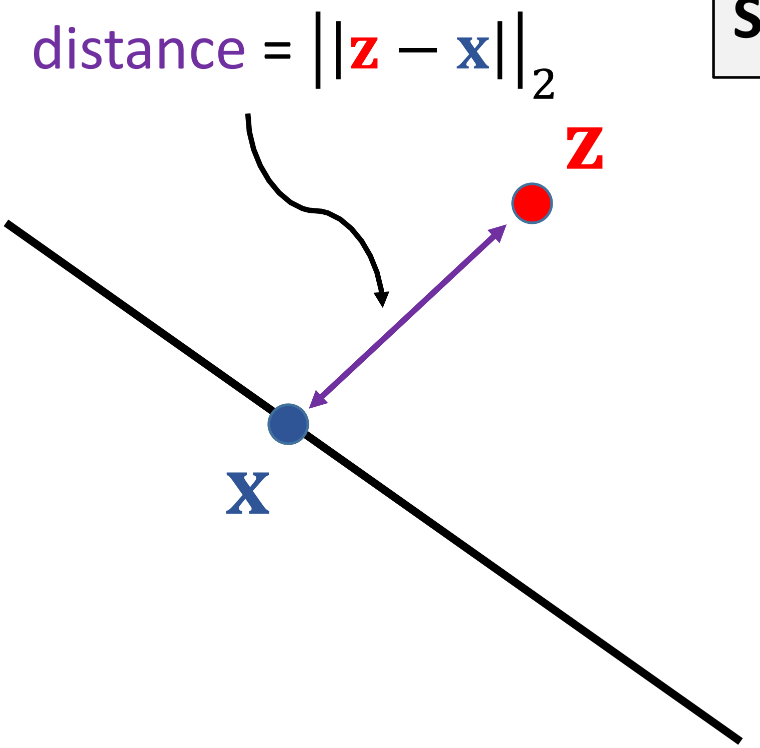


Hyperplane $\mathbf{w}^T\mathbf{x} + b = 0$

Project a Point onto a Hyperplane

Question: how to project $\mathbf{z}$ onto the hyperplane?

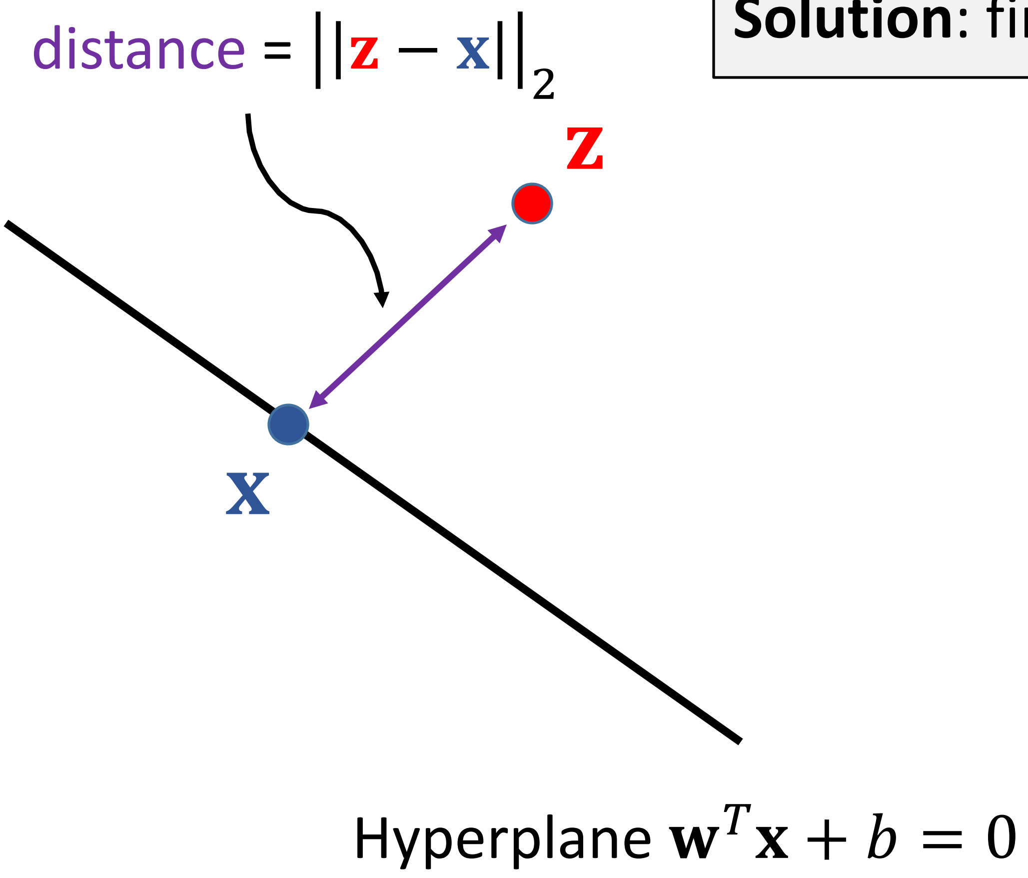
Solution: find $\mathbf{x}$ on the hyperplane such that $||\mathbf{z}-\mathbf{x}||_2^2$ is minimized.



Hyperplane $\mathbf{w}^T\mathbf{x}+b=0$

- $\min_{\mathbf{x}} ||\mathbf{z}-\mathbf{x}||_2^2$; s.t. $\mathbf{w}^T\mathbf{x}+b=0$

Project a Point onto a Hyperplane



Question: how to project $\mathbf{z}$ onto the hyperplane?

Solution: find $\mathbf{x}$ on the hyperplane such that $||\mathbf{z}-\mathbf{x}||_2^2$ is minimized.

- $\min_{\mathbf{x}} ||\mathbf{z}-\mathbf{x}||_2^2;$ s.t. $\mathbf{w}^T\mathbf{x}+b=0$
- Solve the problem using the KKT conditions:

$$\begin{cases} \dfrac{\partial ||\mathbf{z}-\mathbf{x}||_2^2}{\partial \mathbf{x}} + \lambda \dfrac{\partial\,(\mathbf{w}^T\mathbf{x}+b)}{\partial \mathbf{x}} = 0; \\ \mathbf{w}^T\mathbf{x}+b=0. \end{cases}$$

- Solution: $\mathbf{x} = \mathbf{z} - \dfrac{\mathbf{w}^T\mathbf{z}+b}{||\mathbf{w}||_2^2}\mathbf{w}$

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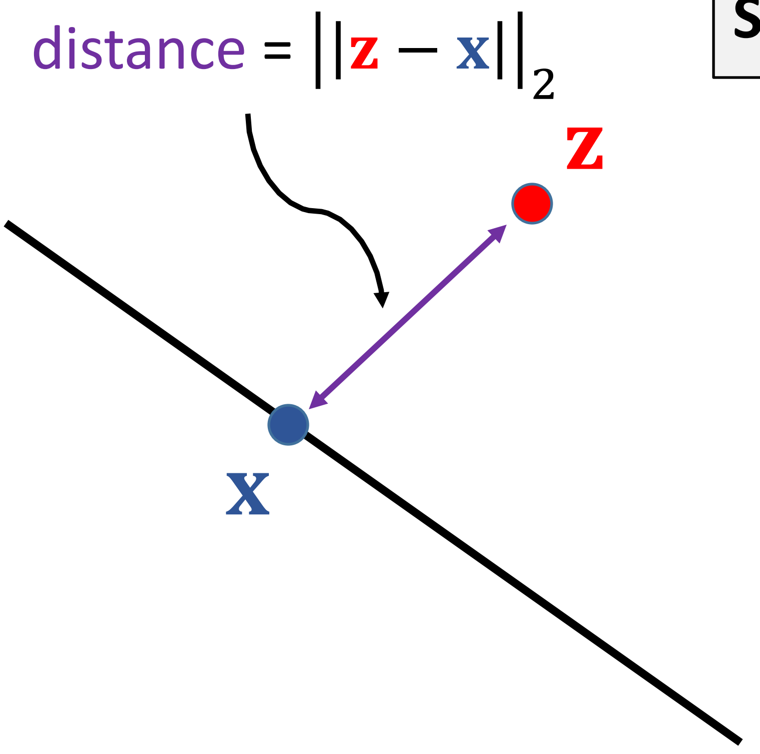

Hyperplane $\mathbf{w}^T\mathbf{x}+b=0$

- Solution: $\mathbf{x}=\mathbf{z}-\frac{\mathbf{w}^T\mathbf{z}+b}{||\mathbf{w}||_2^2}\mathbf{w}$
- The ℓ_2 distance between $\mathbf{z}$ and the hyperplane is

$$\boxed{||\mathbf{z}-\mathbf{x}||_2=\frac{|\mathbf{w}^T\mathbf{z}+b|}{||\mathbf{w}||_2}.}$$

Support Vector Machine (SVM)

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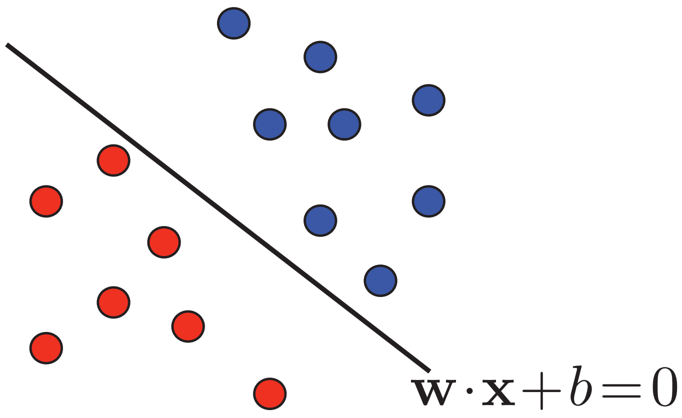
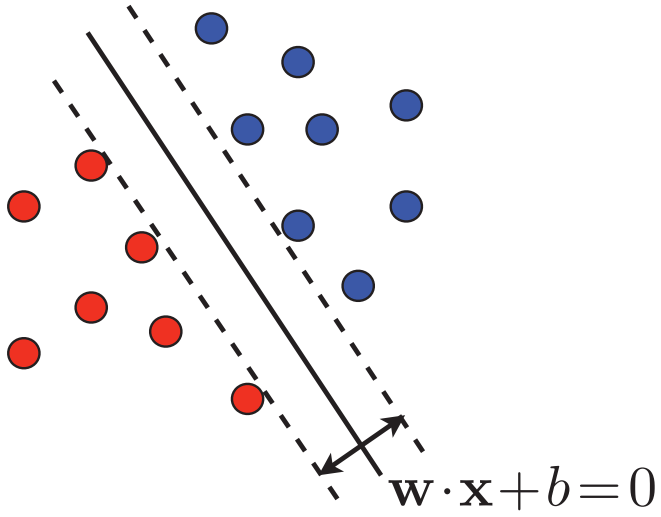
Separate data by a hyperplane (assume the data are separable)





An arbitrary hyperplane.

The hyperplane that maximizes the margin.

The figure is from the book “*Foundations of Machine Learning*”

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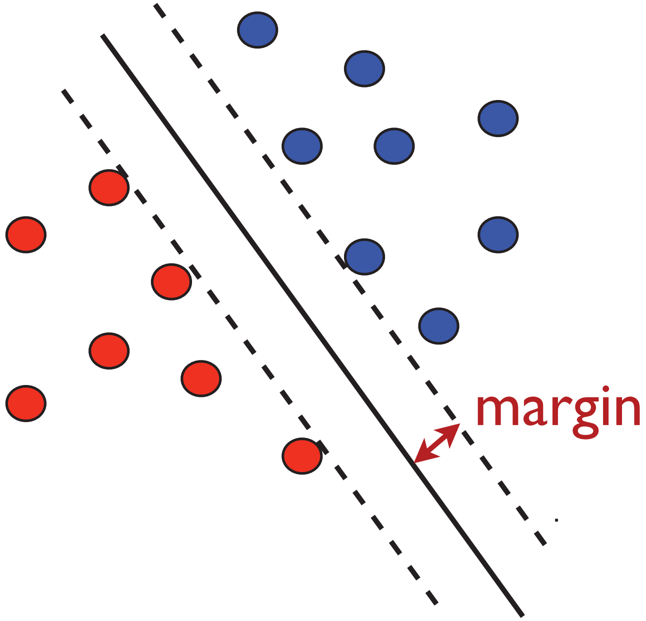

Hyperplane $\mathbf{w}^T\mathbf{x} + b = 0$

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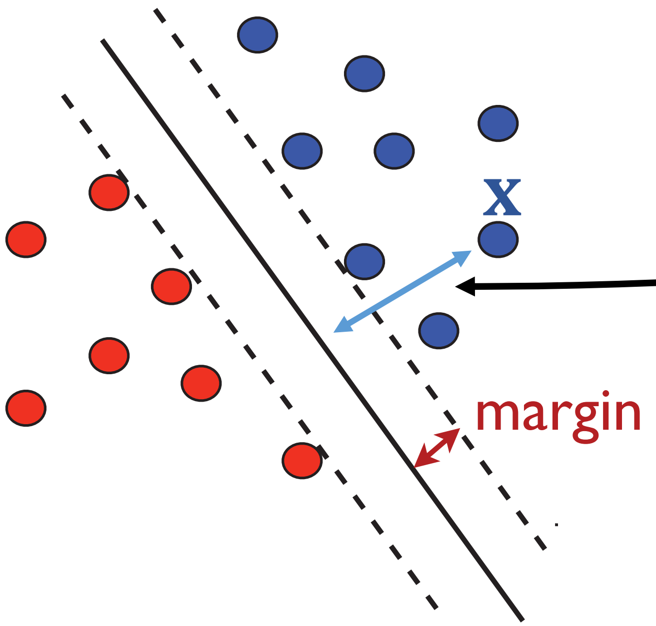
Support Vector Machine (SVM)

Separate data by a hyperplane (assume the data are separable)



Hyperplane $\mathbf{w}^T\mathbf{x} + b = 0$

- The distance between any feature vector, $\mathbf{x}$, and the hyperplane is

$$\text{dist} = \frac{|\mathbf{w}^T\mathbf{x}+b|}{||\mathbf{w}||_2}.$$

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Support Vector Machine (SVM)

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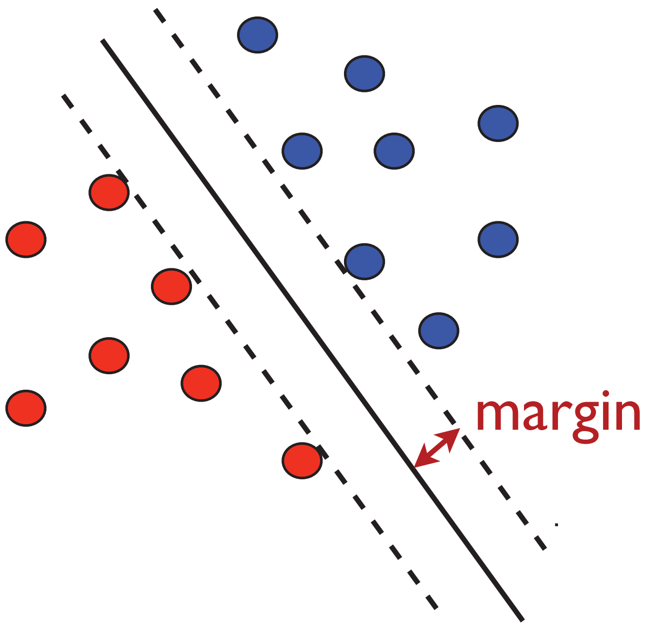

Hyperplane $\mathbf{w}^T\mathbf{x} + b = 0$

- The distance between any feature vector, $\mathbf{x}$, and the hyperplane is
$$\text{dist} = \frac{|\mathbf{w}^T\mathbf{x}+b|}{||\mathbf{w}||_2}.$$
- The margin is the smallest distance:
$$\min_j \frac{|\mathbf{w}^T\mathbf{x}_j+b|}{||\mathbf{w}||_2}$$

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Support Vector Machine (SVM)

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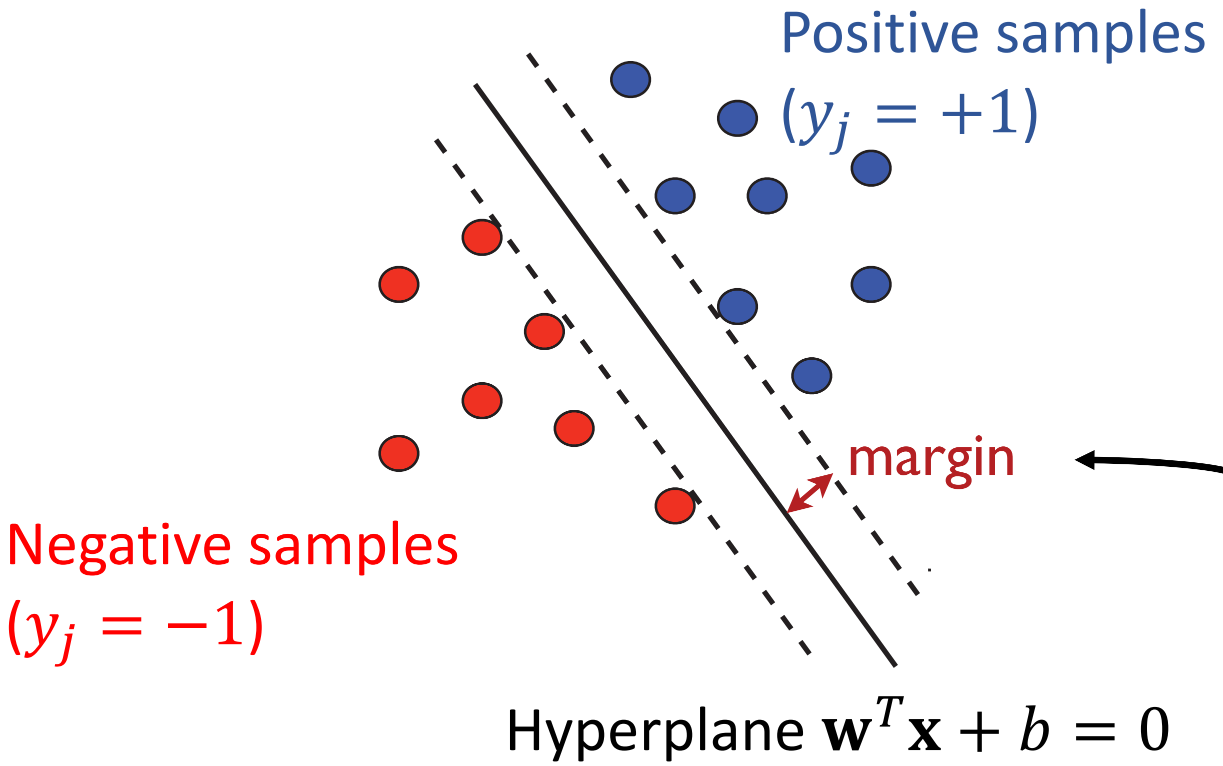

- The distance between any feature vector, $\mathbf{x}$, and the hyperplane is

$$\text{dist} = \frac{|\mathbf{w}^T\mathbf{x}+b|}{||\mathbf{w}||_2}.$$

- The margin is the smallest distance:

$$\min_j \frac{|\mathbf{w}^T\mathbf{x}_j+b|}{||\mathbf{w}||_2} = \min_j \frac{y_j(\mathbf{w}^T\mathbf{x}_j+b)}{||\mathbf{w}||_2}$$

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Support Vector Machine (SVM)

Margin = $\min_j \frac{y_j(\mathbf{w}^T\mathbf{x}_j + b)}{||\mathbf{w}||_2}$; we want to maximize the margin.

Support Vector Machine (SVM)

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Define $\bar{\mathbf{x}}_j = [\mathbf{x}_j; 1] \in \mathbb{R}^{d+1}$

Define $\bar{\mathbf{w}} = [\mathbf{w}, b] \in \mathbb{R}^{d+1}$

➔ $\boxed{\mathbf{x}_j^T\mathbf{w} + b = \bar{\mathbf{x}}_j^T\bar{\mathbf{w}}}$

Support Vector Machine (SVM)

Margin = $\min_j \frac{y_j \mathbf{w}^T \mathbf{x}_j}{||\mathbf{w}||_2}$; we want to maximize the margin.

Support Vector Machine (SVM): $\max_{\mathbf{w}} \min_j \frac{y_j \mathbf{w}^T \mathbf{x}_j}{||\mathbf{w}||_2}$

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Support Vector Machine (SVM)

Support Vector Machine (SVM): $\max_{\mathbf{w}} \min_{j} \frac{y_j \mathbf{w}^T \mathbf{x}_j}{||\mathbf{w}||_2}$

$$\operatorname*{argmax}_{\mathbf{w}} \min_{j} \frac{y_j \mathbf{w}^T \mathbf{x}_j}{||\mathbf{w}||_2} = \operatorname*{argmax}_{\mathbf{w}} \frac{\min_{j} y_j \mathbf{w}^T \mathbf{x}_j}{||\mathbf{w}||_2}$$

$$= \operatorname*{argmax}_{\mathbf{w}} \frac{1}{||\mathbf{w}||_2}, \quad \text{s.t.} \quad \left(\min_{j} \; y_j \mathbf{w}^T \mathbf{x}_j\right) = 1$$

$$= \operatorname*{argmin}_{\mathbf{w}} ||\mathbf{w}||_2^2, \quad \text{s.t.} \quad \left(\min_{j} \; y_j \mathbf{w}^T \mathbf{x}_j\right) = 1$$

$$= \operatorname*{argmin}_{\mathbf{w}} ||\mathbf{w}||_2^2, \quad \text{s.t.} \quad y_j \mathbf{w}^T \mathbf{x}_j \geq 1 \text{ for all } j$$

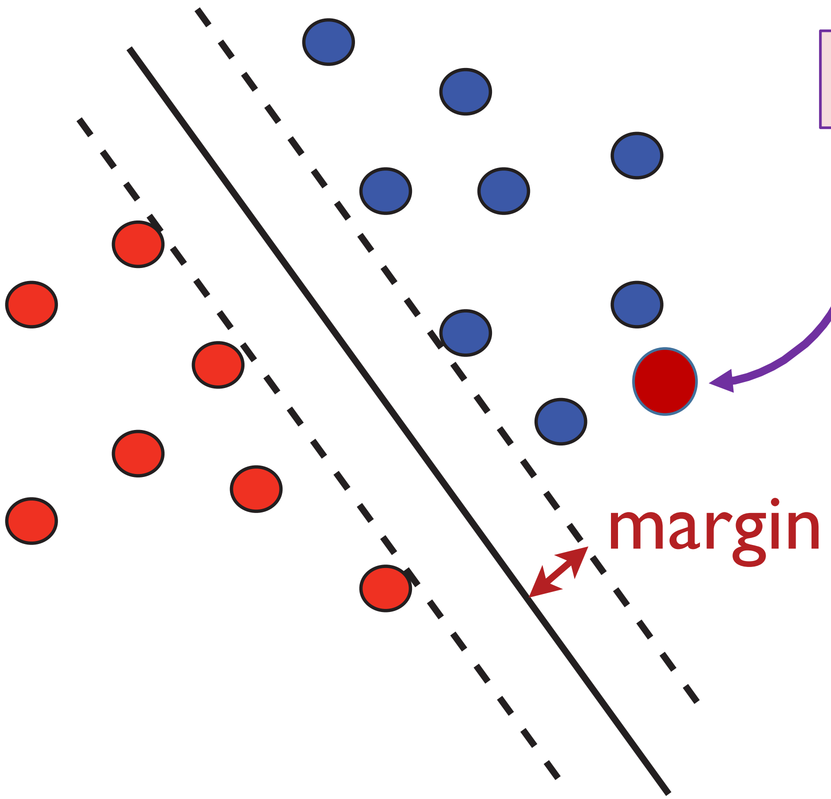
Support Vector Machine (SVM)

$$\min_{\mathbf{w}} ||\mathbf{w}||_2^2, \quad \text{s.t.} \quad y_j \mathbf{w}^T \mathbf{x}_j \geq 1 \text{ for all } j \in \{1, \cdots, n\}.$$

Equivalent form of SVM

Support Vector Machine (SVM)

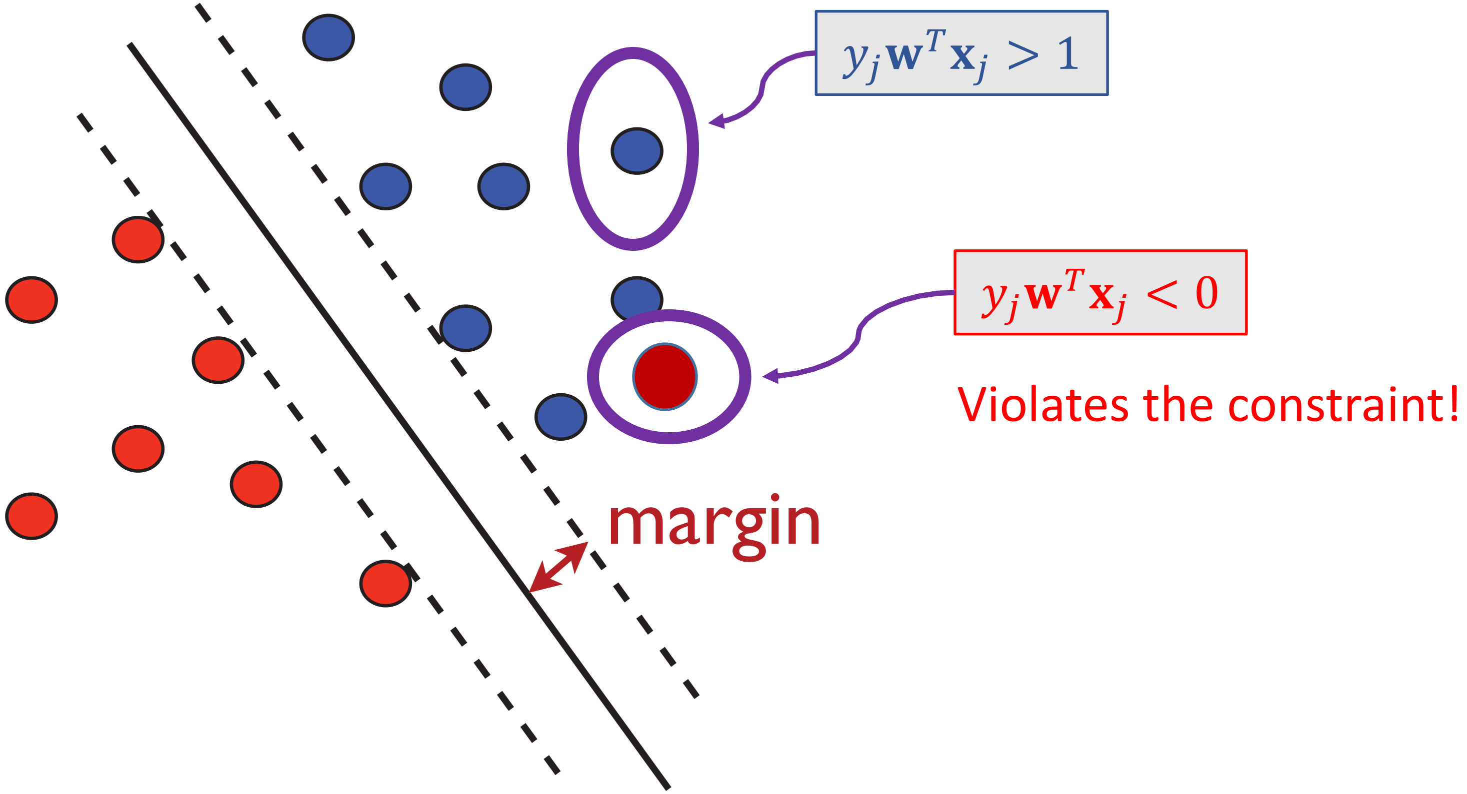
$$\min_{\mathbf{w}} ||\mathbf{w}||_2^2, \quad \text{s.t.} \quad y_j \mathbf{w}^T \mathbf{x}_j \geq 1 \text{ for all } j \in \{1, \cdots, n\}.$$



What if the data is inseparable?

Support Vector Machine (SVM)

$$\min_{\mathbf{w}} \|\mathbf{w}\|_2^2, \quad \text{s.t.} \quad y_j \mathbf{w}^T \mathbf{x}_j \geq 1 \text{ for all } j \in \{1, \cdots, n\}.$$



Support Vector Machine (SVM)

$$\min_{\mathbf{w}} \left|\left|\mathbf{w}\right|\right|_2^2, \quad \text{s.t.} \quad 1 - y_j \mathbf{w}^T \mathbf{x}_j \leq 0 \ \text{ for all } j \in \{1, \cdots, n\}.$$

Relax

$$\min_{\mathbf{w}, \xi_j} \left|\left|\mathbf{w}\right|\right|_2^2 + \lambda \sum_j \left[\xi_j\right]_+, \quad \text{s.t.} \quad 1 - y_j \mathbf{w}^T \mathbf{x}_j = \xi_j \ \text{ for all } j \in \{1, \cdots, n\}.$$

- $\left[\xi_j\right]_+ = \max\{\xi_j, 0\}$

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- $[\xi_j]_+ = \max\{\xi_j, 0\}$
- $\xi_j \leq 0$ means the constraint $1 - y_j \mathbf{w}^T \mathbf{x}_j \leq 0$ is satisfied
  ➔ no penalty!
- $\xi_j > 0$ means the constraint is violated (because the data is inseparable)
  ➔ penalize the violation ξ_j.

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Relax

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Equivalent

$$\min_{\mathbf{w}, b} ||\mathbf{w}||_2^2 + \lambda \sum_j [1 - y_j \mathbf{w}^T \mathbf{x}_j]_+ .$$

Comparisons

SVM: $\min_{\mathbf{w}} ||\mathbf{w}||_2^2 + \lambda \sum_j g(y_j \mathbf{w}^T \mathbf{x}_j)$.

Hinge loss: $g(z) = [1 - z]_+$.

Comparisons

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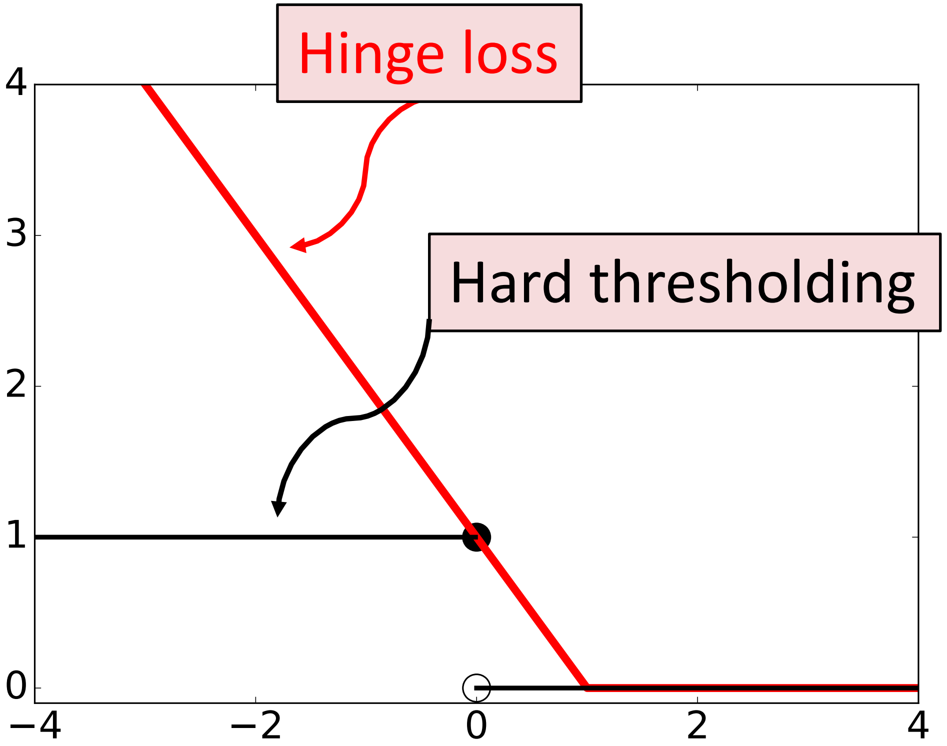

Hard thresholding: $h(z) = \begin{cases} 1, & \text{if } z < 0; \\ 0, & \text{if } z \geq 0. \end{cases}$

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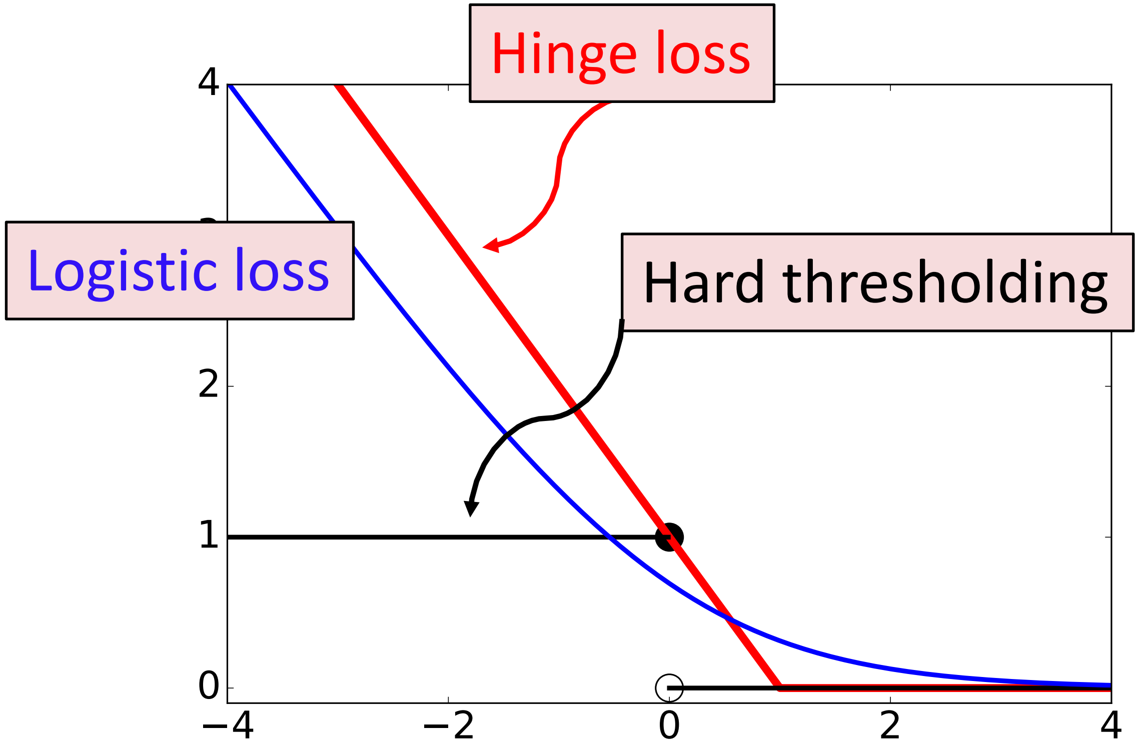
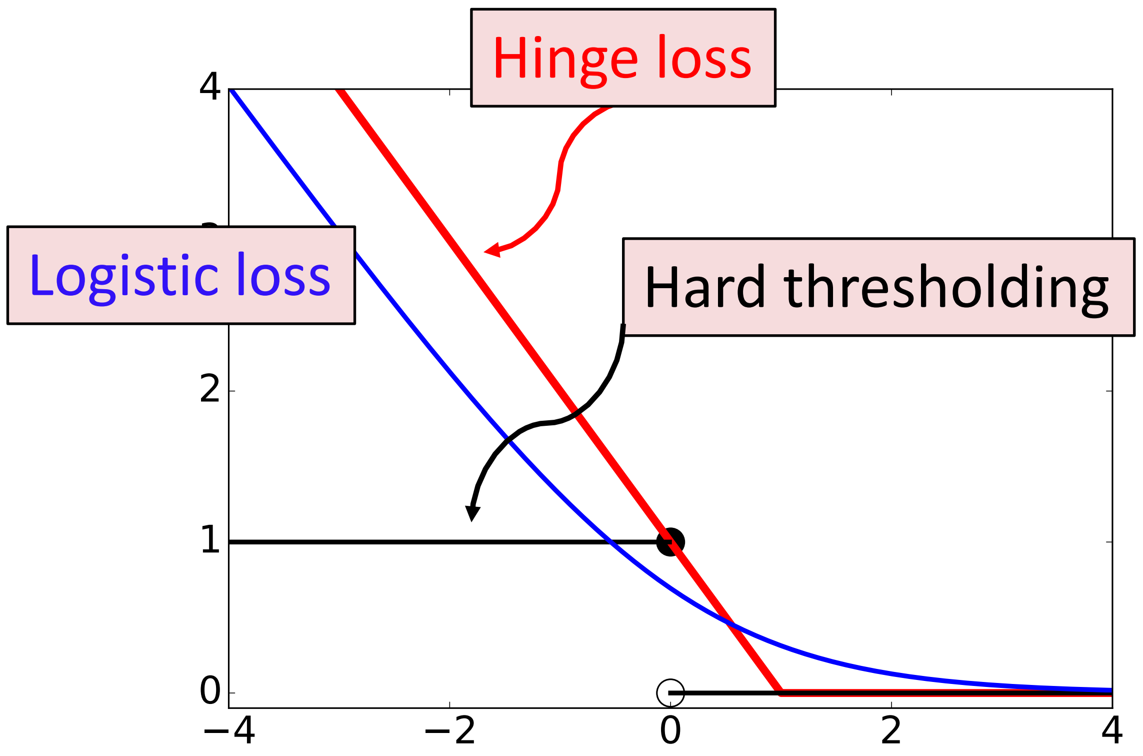


Hard thresholding: $h(\mathrm{z}) = \begin{cases} 1, & \text{if } z < 0; \\ 0, & \text{if } z \geq 0. \end{cases}$

Logistic loss: $l(\mathrm{z}) = \log(1 + e^{-\mathrm{z}})$.

Comparisons



- Convexity
  - Hinge loss and logistic loss are convex.
  - Global optima can be efficiently found.
- Smoothness
  - Hinge loss is non-smooth.
  - Logistic loss is smooth.
- Logistic regression is easier to solve than SVM.
  - GD for logistic regression has linear convergence.
  - Algorithms for SVM have sub-linear convergence.