Affine Algebraic Varieties

Directed Reading Program Spring 2024

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Motivation

Equivalences between geometry and algebra:

Geometry	Algebra
Varieties	Radical Ideals $\leftrightarrow$ Reduced $\mathbb{C}$-Algebras
Irreducible Varieties	Prime Ideals $\leftrightarrow$ Integral Domains
Points	Maximal Ideals $\leftrightarrow$ Fields

Definition.

An **affine algebraic variety** is the common zero set of a collection $\{F_i\}_{i\in I}$ of complex polynomials on $\mathbb{C}^n$. We write

$$V = \mathbb{V}(\{F_i\}_{i\in I}) \subset \mathbb{C}^n$$

for this set of common zeros.

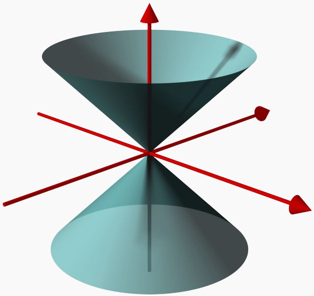
We can also look at the varieties of ideals[1] $I \subset \mathbb{C}[x_1, ..., x_n]$ *(denoted* $\mathbb{V}(I)$*).*

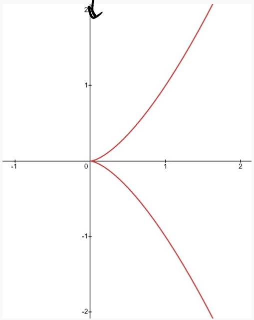
[1] An ideal I of a ring R is an additive subgroup of R such that for every $r \in R$ and every $x \in I$, we have $rx \in I$.

Introduction to Affine Algebraic Varieties

Examples.

1. $\mathbb{V}(0) = \mathbb{C}^n$
2. $\mathbb{V}(1) = \varnothing$ $\qquad /\!\!/ \ \mathbb{V}(x) \cup \mathbb{V}(y)$
3. $\mathbb{V}(xy) = \{x\text{- and } y\text{-axis}\}$
4. $\mathbb{V}(x^2 + y^2 - z^2)$
5. $\mathbb{V}(y^2 - x^3)$





Note. $\mathbb{V}(F_1 \cdot F_2) = \mathbb{V}(F_1) \cup \mathbb{V}(F_2)$.

For ideals I and J, we have $\mathbb{V}(IJ) = \mathbb{V}(I) \cup \mathbb{V}(J)$.

Radical Ideals

Definition.

Let R be a ring. An ideal $I \subset R$ is called **radical** if it is equal to its radical $\sqrt{I}$, where

$$\sqrt{I} := \{f \in R : f^n \in I \text{ for some } n\}.$$

Example. Consider the ideal $I = (x^2) \subset \mathbb{C}[x]$. We have $\sqrt{I} = (x)$.

Note that $I \subseteq \sqrt{I}$.

Ideals

Some facts:

- An ideal $I \subset R$ is radical if and only if R/I is reduced (i.e., given $x \in R/I$, if there exists n such that $x^n = 0$, we must have $x = 0$).
- All maximal ideals[2] are prime.
- All prime ideals[3] are radical.

[2] An ideal $I \subset R$ is maximal if there does not exist a proper ideal J such that $I \subsetneq J$.
[3] An ideal $I \subset R$ is prime if the following holds: if $ab \in I$, then either $a \in I$ or $b \in I$.

Definition.

Let V be an affine algebraic variety in $\mathbb{C}^n$. The set

$$\mathbb{I}(V) = \{f \in \mathbb{C}[x_1, ..., x_n] : \ f(x) = 0 \ \ \forall x \in V\}$$

is an ideal of V.

We call $\mathbb{I}(V)$ the **ideal of all polynomial functions vanishing on** V.

Note. If $V \subseteq W$, then $\mathbb{I}(V) \supseteq \mathbb{I}(W)$.

$\mathbb{I}(V)$ cont.

Fact.

$\underbrace{\mathbb{V}(\underbrace{\mathbb{I}(V)})} = V.$

This tells us that $\mathbb{V}$ is right invertible. Is it left invertible?

Not quite...

$\mathbb{V}$ is only left invertible for radical ideals.

Theorem (Hilbert's Nullstellensatz).

For any ideal $I \subset \mathbb{C}[x_1, ..., x_n]$,

$$\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}.$$

This implies a one-to-one correspondence:

$$\{\text{affine algebraic varieties in } \mathbb{C}^n\} \leftrightarrow \{\text{radical ideals in } \mathbb{C}[x_1, ..., x_n]\}$$
$$V \mapsto \mathbb{I}(V)$$
$$\mathbb{V}(I) \leftarrow\!\shortmid I$$

Application of Nullstellensatz

- Let $I = (x^2) \subset \mathbb{C}[x]$. Find $\sqrt{I}$.

$$\sqrt{I} = \mathbb{I}(\underbrace{\mathbb{V}(x^2)}) = \mathbb{I}(\{0\}) = (x)$$

- Show $(y^2 - x^3) \subseteq (x - \pi^2, y - \pi^3)$.

$$\{(\pi^2, \pi^3)\} \subseteq \mathbb{V}(y^2 - x^3)$$

$$\begin{array}{ccc}
\mathbb{I}\{(\pi^2, \pi^3)\} & \supseteq & \mathbb{I}(\mathbb{V}(\quad)) \\
\| & & \| \\
 & & \sqrt{(y^2 - x^3)} \\
 & & \| \\
(x - \pi^2, y - \pi^3) & = & (y^2 - x^3)
\end{array}$$

{irreducible varieties} $\leftrightarrow$ {prime ideals} .

Theorem.

Let $I \subset \mathbb{C}[x_1, ..., x_n]$ be an ideal. $\mathbb{V}(I)$ is irreducible[4] if and only if $\sqrt{I}$ is prime.

Proof. $(\Leftarrow)$ Sps $\sqrt{I}$ is prime.

$\mathbb{V}(I) = \mathbb{V}(J) \cup \mathbb{V}(K) = \mathbb{V}(JK)$

$\mathbb{I}(\mathbb{V}(I)) = \mathbb{I}(\mathbb{V}(JK))$

$\sqrt{I} = \sqrt{JK} \supseteq JK$

$JK \subseteq \sqrt{I}$. Either $J \subseteq \sqrt{I}$ or $K \subseteq \sqrt{I}$

Either $\mathbb{V}(J) \supseteq \mathbb{V}(\sqrt{I})$

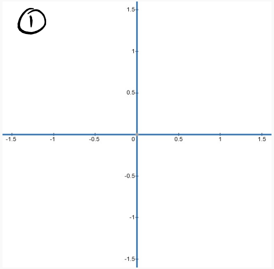
or $\mathbb{V}(K) \supseteq \mathbb{V}(\sqrt{I})$

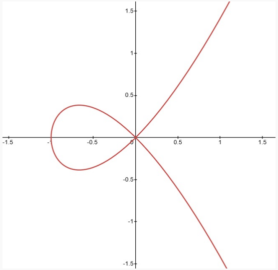
$\mathbb{V}(\sqrt{I}) = \mathbb{V}(I)$

[4] An irreducible variety is one that cannot be written as the union of two proper subvarieties, i.e., if $\mathbb{V}(I) = \mathbb{V}(J) \cup \mathbb{V}(K)$, then either $\mathbb{V}(I) \subseteq \mathbb{V}(J)$ or $\mathbb{V}(I) \subseteq \mathbb{V}(K)$.

Examples.

1. $\mathbb{V}(xy)$: (xy) is not prime; $xy \in (xy)$, but $x \notin (xy)$ and $y \notin (xy)$.
2. $\mathbb{V}(y^2 - x^2(x+1))$: $(y^2 - x^2(x+1))$ is prime and thus the variety is irreducible.





Nullstellensatz Cont.

Corollary.

The maximal ideals of $\mathbb{C}[x_1, ..., x_n]$ are exactly of the form $(x_1 - a_1, \ldots, x_n - a_n)$ for $a_i \in \mathbb{C}$.

Proof Idea. If $m \subset \mathbb{C}[x_1, ..., x_n]$ is a maximal ideal, then $\mathbb{V}(m) = \{(a_1, ..., a_n)\}$ for some point $(a_1, ..., a_n) \in \mathbb{C}^n$.

This implies a one-to-one correspondence:

$$\{\text{points in } \mathbb{C}^n\} \leftrightarrow \{\text{maximal ideals of } \mathbb{C}[x_1, ..., x_n]\}$$

$$(a_1, ..., a_n) \leftrightarrow (x_1 - a_1, ..., x_n - a_n)$$

Definition.

Let $V \subset \mathbb{C}^n$ be an affine algebraic variety. Given a complex polynomial of n variables, the restriction to V gives a map $V \to \mathbb{C}$. This forms a $\mathbb{C}$-algebra

$$\mathbb{C}[x_1, \ldots, x_n]\Big|_V.$$

This is the **coordinate ring of V** and is denoted $\mathbb{C}[V]$.

The Coordinate Ring as a Quotient

Consider the surjective ring homomorphism given by restriction:

$$\mathbb{C}[x_1, ..., x_n] \to \mathbb{C}[x_1, ..., x_n]\Big|_V.$$

The kernel is $\mathbb{I}(V)$. By the isomorphism theorem,

$$\mathbb{C}[x_1, ..., x_n]\Big|_V \cong \frac{\mathbb{C}[x_1, ..., x_n]}{\mathbb{I}(V)}.$$

Coordinate Ring Example

Example. Let $V = \mathbb{V}(x^2 + y^2 - 1)$. Then,

$$\mathbb{C}[x_1, ..., x_n]\Big|_V \cong \frac{\mathbb{C}[x_1, ..., x_n]}{\mathbb{I}(x^2 + y^2 - 1)}.$$

$\equiv 0$

Consider the polynomial $2x^3 + 3x^2 + 2xy^2 - 2$. We have

$$\to \quad 2x^3 + 3x^2 + 2xy^2 - 2 = 2x(x^2 + y^2 - 1) + 3x^2 - 2$$
$$\equiv 3x^2 - 2.$$

$\equiv 0$

$$\{\text{coordinate rings}\} \leftrightarrow \{\text{finitely generated, reduced } \mathbb{C}\text{-algebras}\}.$$

Theorem.

Every finitely generated reduced $\mathbb{C}$-algebra is isomorphic to the coordinate ring of some affine algebraic variety.

Proof. Let S be f.g. red $\mathbb{C}$-alg. Let $s_1, \dots, s_n$ be gen.

$$\left.\begin{array}{c} \mathbb{C}[x_1, \dots, x_n] \to S \\ x_i \mapsto s_i \end{array}\right\} \ker = I$$

$$S \cong \underbrace{\frac{\mathbb{C}[x_1, \dots, x_n]}{I}}_{\text{reduced}}; \quad \Rightarrow I \text{ is radical} \Rightarrow I = \mathbb{I}(\underbrace{V(I)}))$$

Conclusion

1. There is a one-to-one correspondence between **varieties** and **radical ideals** (Hilbert's Nullstellensatz).
2. $\mathbb{V}(I)$ is **irreducible** if and only if $\sqrt{I}$ is **prime**.
3. There is a one-to-one correspondence between **maximal ideals** of $\mathbb{C}[x_1, \ldots, x_n]$ and **points** in $\mathbb{C}^n$ (Corollary of Hilbert's Nullstellensatz).
4. Every **finitely generated reduced $\mathbb{C}$-algebra** is isomorphic to the **coordinate ring** of some affine algebraic variety.

References

Smith, Karen, et al. *An Invitation to Algebraic Geometry.* Springer-Verlag, 2000.