Project Description

My SURG 2023 Project (conducted June 29 - August 24) aimed to study the properties of Lie groups using extensions. We can denote a general extension, where $N \trianglelefteq G$ and the lowercase letters are the corresponding Lie algebras, as the following:

$$N \to G \to G/N$$

$$\mathfrak{n} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{n}.$$

Extensions are a means of studying "larger" groups by understanding properties of "smaller" groups, i.e., understanding G by looking at N or G/N. To verify these conclusions, I compared the results given by the extension to results I found using linear Poisson geometry and coadjoint orbits. I primarily looked at specific examples outlined in the *Project Results* section.

Project Progress

While I did come across several general conclusions, most of my time was dedicated to studying five specific examples. I intended to spend two weeks developing general results in my original proposal. However, I decided that it would be more fruitful to understand the examples further so that I could come across more potential results to prove in a general setting.

One complication was the lack of small-dimensional, but non-trivial, examples. Examples that were too large made it difficult to calculate the Casimir or manually find the maps in the extension, but trivial examples gave no helpful conclusions. As a result, my main method of research was to first seek patterns in existing examples and then prove the general form of those conclusions.

Project Results

The general conclusions I came to are best explained in the context of an example (the full list of examples I researched is provided on page 3 of this document). My primary example was

$$\mathbb{R} \to GL(2) \xrightarrow{T} PGL(2).$$

One Casimir function on $\mathfrak{gl}(2)^*$ calculated through Poisson brackets is $yz - xw$.

Firstly, I found that T^* could give us a geometric perspective on the symplectic leaves of $\mathfrak{gl}(2)^*$ given the leaves of $\mathfrak{pgl}(2)^*$. We can view $\mathfrak{pgl}(2)^*$ as $\mathfrak{sl}(2)^*$ and therefore use a Casimir on $\mathfrak{sl}(2)^*$ (such as $4bc + a^2$) as a Casimir on $\mathfrak{pgl}(2)^*$. It turns out that the leaves of $\mathfrak{sl}(2)^*$ map to a part of the leaves of $\mathfrak{gl}(2)^*$: the portion where the first coordinate is the negative of the last. That is, T^* sent the level sets of $4bc + a^2$ to the level sets of $yz - xw$ where $x = -w$.

Then, when studying the map given by the dual of T^*, I found that if the Poisson brackets of linear functions are preserved, then the Poisson brackets for smooth functions are also preserved under a specific norm. That is, given $C^\infty(\mathfrak{gl}(2)^*)$ under the W^1-norm and $C^\infty(\mathfrak{sl}(2)^*)$ under the sup-norm, for $f, g \in C^\infty(\mathfrak{gl}(2)^*)$,

$$(\text{dual of } T^*)\{f, g\} = \{(\text{dual of } T^*)(f), (\text{dual of } T^*)(g)\}.$$

I was able to generalize this proof to functions in $C^\infty(\mathfrak{g}^*)$ for any Lie algebra $\mathfrak{g}$.

I also found that in this example, the dual of T^* mapped a Casimir function to a Casimir function:

$$(\text{dual of } T^*)(yz - xw) = bc + \frac{1}{4}a^2.$$

The key observation that led me to this conclusion was that we could view $\mathfrak{gl}(2) \subseteq C^\infty(\mathfrak{gl}(2)^*)$, where we let $\mathfrak{gl}(2)$ be the set of linear functions on $\mathfrak{gl}(2)^*$. The Casimir preservation property provides an explicit way to find a Casimir on one space ($\mathfrak{sl}(2)^*$ in this example) given a Casimir on another space ($\mathfrak{gl}(2)^*$ in this example).

Finally, in the examples I looked at, I found that calculating the orbits of the coadjoint action of a group on its Lie algebra (the action of G on $\mathfrak{g}$) and the orbits of the coadjoint action of the quotient group on the Lie algebra (the action of G/N on $\mathfrak{g}$) yielded the same results. This leads to another potential way to understand the coadjoint orbits of G given G/N.

After concluding this project, I will focus on proving the general result of Casimir preservation and the conditions under which the preservation holds. To guide my thinking, I can first see if Casimirs are preserved in the other examples I have studied. I will also find other portions (or fibrations) of the symplectic leaf that arise from applying T^*. Finally, I can use the 2-cocycle condition to find more central extensions to study—this would entail "constructing" Lie algebras with certain Lie brackets.

Academic Development

Prior to conducting this project, I was intimidated by the vast area of Lie theory. However, I found more confidence once I focused on taking abstract questions and breaking them down into specific and concrete ones. Then, I was able to set specific goals to help guide my thinking. Developing ideas from specific examples gave me a sense of direction—I was able to understand new concepts by applying them and observing their effects. Furthermore, Professor Cañez guided me through the research process and suggested a key method on how to approach mathematical research: it is often easier to first think of a convenient result and then try to arrive at it, rather than experimenting without a destination. This gave me a way to frame the research as a series of smaller problems.

I also found that I could use some "tricks" to see if I was going in the right direction. For example, by looking at the dimensions of the groups in the extension, we can see which examples would have points as symplectic leaves and which examples would have more complex surfaces.

Personal Development

My main personal takeaway was finding what methods of working would encourage me to self-motivate. I found that the Pomodoro method helped me focus and that taking short breaks allowed me to reset, returning with more enthusiasm towards my work. My mindset also changed from worrying that there would not be an answer to being excited about what interesting results I might find. I now also understand the importance of reviewing material the day of, whether it be content I learned during a meeting with Professor Cañez or reviewing everything (even non-academic) I did that day. Taking time to review and reflect allowed me to feel ready for the next day.

Figures

List of Examples

$$\{I, -I\} \to spin(3) \to SO(3)$$

$$S_L \cong SU(2) \to SO(4) \to SO(4)/S_L$$

$$SL(2) \to GL(2) \to \mathbb{R}^\times$$

$$\mathbb{R} \to \text{Heisenberg} \to \mathbb{R}^2$$