Coadjoint Orbits of Extensions of Lie Groups

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1 Introduction

This project aimed to study the properties of Lie groups using extensions. We can denote a general extension, where $N \trianglelefteq G$ and the lowercase letters are the corresponding Lie algebras, as the following:

$$N \to G \to G/N$$

$$\mathfrak{n} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{n}.$$

Extensions are a means of studying "larger" groups by understanding properties of "smaller" groups, i.e., understanding G by looking at N or G/N. To verify these conclusions, I compared the results given by the extension to results I found using linear Poisson geometry and coadjoint orbits.

This write-up will be structured based on examples. The first example, an extension involving $GL(2)$, will have its results thoroughly explained and applied. The subsequent examples will apply the results with a brief explanation.

2 $GL(2)$ and $PGL(2)$

For our first example and explanation of results, we look at the following extension:

$$\mathbb{R} \hookrightarrow GL(2) \twoheadrightarrow PGL(2).$$

2.1 Rewriting the Extension

Recall that

$$PGL(2, \mathbb{R}) \cong GL(2, \mathbb{R})/\{\lambda I\}$$

and that

$$\{\lambda I\} \cong \mathbb{R}^{\times}.$$

We therefore have the extension

$$\mathbb{R}^{\times} \to GL(2) \to PGL(2)$$

$$\mathbb{R} \to \mathfrak{gl}(2) \to \mathfrak{pgl}(2).$$

Since $\mathfrak{pgl}(2) \cong \mathfrak{gl}(2)/\mathbb{R}$, each coset is of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

Taking one particular representative of the coset (for $\lambda = -\frac{1}{2}(a+d)$),

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} -\frac{1}{2}(a+d) & 0 \\ 0 & -\frac{1}{2}(a+d) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(a-d) & b \\ c & \frac{1}{2}(d-a) \end{pmatrix},$$

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thus giving an isomorphism from $\mathfrak{pgl}(2)$ to $\mathfrak{sl}(2)$. So, the extension is

$$\mathbb{R} \to \mathfrak{gl}(2) \to \mathfrak{sl}(2).$$

The map on the right ($\mathfrak{gl}(2) \to \mathfrak{sl}(2)$) is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{2}(a-d) & b \\ c & \frac{1}{2}(d-a) \end{pmatrix},$$

and the dual of this map is

$$f(x,y,z) \mapsto f\left(\frac{1}{2}(a-d), b, c\right).$$

2.2 Finding Casimirs

To calculate the Casimir functions on $\mathfrak{gl}(2)^*$, we must find functions that Poisson-commute with all the basis elements. Since we are in a matrix Lie algebra, we will use the standard Lie bracket for matrices where $[A,B] = AB - BA$. Consider the following basis for $\mathfrak{gl}(2)$:

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cong x,\ \{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cong y,\ \{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cong z,\ \{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cong w, \right\}.$$

Using the fact that the Poisson bracket here is the Lie bracket, we calculate the following:

$$\{x,y\} = y,\ \{x,z\} = -z,\ \{x,w\} = 0,$$
$$\{y,z\} = x - w,\ \{y,w\} = y,\ \{z,w\} = -z.$$

Recall that for any smooth f, g, the Poisson bracket takes the following form:

$$\{f,g\} = \sum_{i,j} c_{ij} \left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} - \frac{\partial g}{\partial x_i}\frac{\partial f}{\partial x_j}\right).$$

To calculate the c_{ij}, we can compare the equation given by the general form of the Poisson bracket to the brackets we found earlier. For example,

$$\begin{aligned} y &= \{x,y\} \\ &= c_{xy}(1-0) + c_{xz}(0) + c_{xw}(0) + c_{yz}(0) + c_{yw}(0) + c_{xw}(0) \\ &= c_{xy}. \end{aligned}$$

All other c_{ij} can be found as such. So, we get that

$$\begin{aligned} \{f,g\} = y &\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial y}\right) - z\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial z} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial z}\right) \\ &+ (x-w)\left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial g}{\partial y}\frac{\partial f}{\partial z}\right) + y\left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial w} - \frac{\partial g}{\partial y}\frac{\partial f}{\partial w}\right) \\ &- z\left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial w} - \frac{\partial g}{\partial z}\frac{\partial f}{\partial w}\right). \end{aligned}$$

Now, in order for a function H to Poisson commute with each of the basis elements, we must have

$$\begin{aligned} 0 &= \{H,x\} = -y\left(\frac{\partial H}{\partial y}\right) + z\left(\frac{\partial H}{\partial z}\right), \\ 0 &= \{H,x\} = y\left(\frac{\partial H}{\partial x}\right) - (x-w)\left(\frac{\partial H}{\partial z}\right) - y\left(\frac{\partial H}{\partial w}\right), \\ 0 &= \{H,x\} = -z\left(\frac{\partial H}{\partial x}\right) + (x-w)\left(\frac{\partial H}{\partial y}\right) + z\left(\frac{\partial H}{\partial w}\right), \\ 0 &= \{H,x\} = y\left(\frac{\partial H}{\partial y}\right) - z\left(\frac{\partial H}{\partial z}\right). \end{aligned}$$

Two such H that satisfy the above are

$$yz - xw$$

and

$$x + w.$$

Note 1. *Other functions may satisfy the above criteria. For example, we also get $H = yz + \frac{1}{2}x^2 + \frac{1}{2}w^2$ as a possibility. However,*

$$yz + \frac{1}{2}x^2 + \frac{1}{2}w^2 = yz - xw + \frac{1}{2}(x+w)^2.$$

In other words, any Casimir will be any combination of $yz - xw$ and $x + w$. Geometrically, the leaves will be the intersection of $yz - xw$ and $x + w$.

2.3 The Geometric View Using Symplectic Leaves

Given this context, our first step in using the extension to learn about $\mathfrak{gl}(2)$ through $\mathfrak{sl}(2)$ is comparing symplectic leaves under T^*. Take the maps

$$\mathfrak{gl}(2) \cong \mathbb{R}^4 \xrightarrow{T} \mathfrak{sl}(2) \cong \mathbb{R}^3,$$

$$\mathfrak{gl}(2)^* \xleftarrow{T^*} \mathfrak{sl}(2)^*$$

Let $E_1, ..., E_4$ be the standard basis for $\mathfrak{gl}(2)$ and $F_1, ..., F_4$ the corresponding dual basis for $\mathfrak{gl}(2)^*$. Every $h \in \mathfrak{gl}(2)^*$ must have the form

$$h = aF_1 + bF_2 + cF_3 + dF_4.$$

We thus have an isonmorphism between $\mathfrak{gl}(2)^*$ and $\mathbb{R}^4$. Likewise, let $e_1, ..., e_3$ be the standard basis for $sl(2)$ and $f_1, ..., f_3$ the corresponding dual basis for $\mathfrak{sl}(2)^*$. Then, every $g \in sl(2)^*$ has the form

$$g = xf_1 + yf_2 + zf_3,$$

and we have an isomorphism between $\mathfrak{sl}(2)^*$ and $\mathbb{R}^3$.

Now, recall that the symplectic leaves on $\mathfrak{sl}(2)^*$ are the subset of $\mathbb{R}^3$ given by

$$\{4yz + x^2\}.$$

Through our isomorphism, we see that this subset of $\mathbb{R}^3$ is isomorphic to the subset of $\mathfrak{sl}(2)^*$ given by the $g = xf_1 + yf_2 + zf_3$ such that x, y, z satisfy $\{4yz + x^2 = k\}$ for some $k \in \mathbb{R}$ (i.e., in the level set). Now we find the image of this subset under T^*.

Take a general $g \in sl(2)^*$. Then,

$$T^*g = xT^*f_1 + yT^*f_2 + zT^*f_3.$$

Looking at the first term,

$$f_1 \circ T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = f_1 \begin{pmatrix} \frac{1}{2}(a-d) & b \\ c & \frac{1}{2}(d-a) \end{pmatrix} = \frac{1}{2}(a-d).$$

This implies that $T^*f_1 = \frac{1}{2}(F_1 - F_4)$ since F_1 "picks out" a and F_4 d. Doing the same for f_2, f_3, f_4,

$$T^*g = \frac{x}{2}(F_1 - F_4) + yF_2 + zF_3 = \frac{x}{2}F_1 + yF_2 + zF_3 - \frac{x}{2}F_4.$$

Therefore, any arbitrary g must map to something of the above form. In our case, we have x, y, z satisfying $\{4yz + x^2 = k\}$.

Then, through the isomorphism between $\mathfrak{gl}(2)^*$ and $\mathbb{R}^4$, this function can be viewed as

$$\left(\frac{x}{2}, y, z, -\frac{x}{2}\right).$$

Now, recall that the symplectic leaves on $\mathfrak{gl}(2)^*$ are $h = aF_1 + bF_2 + cF_3 + dF_4$ such that $\{bc - ad = k'\}$ (once again using the fact that $\mathfrak{gl}(2)^* \cong \mathbb{R}^4$). From $T^*g = \frac{x}{2}F_1 + yF_2 + zF_3 - \frac{x}{2}F_4$, we can relabel to get

$$a = \frac{x}{2},\ \ b = y,\ \ c = z,\ \ d = -\frac{x}{2}$$

or

$$x = 2a = -2d,\ \ y = b,\ \ z = c$$

where x, y, z satisfy $\{4yz + x^2 = k\}$. Then,

$$\{4yz + x^2 = 4bc + (2a)(-2d) = 4bc - 4ad = k\}$$

and therefore

$$\{bc - ad = \frac{k}{4}\},$$

so that T^*g is contained in the symplectic leaves on $\mathfrak{gl}(2)^*$. Moreover, it is specifically the portion of the leaves on $\mathfrak{gl}(2)$ where $a = -d$.

Now that we've seen the geometric map of symplectic leaves of $\mathfrak{gl}(2)^*$ and $\mathfrak{sl}(2)^*$, how can we see the "functional" interpretation? That is, what can we know about the Casimir on $\mathfrak{sl}(2)^*$ through the Casimir on $\mathfrak{gl}(2)^*$?

First, by a Casimir on $\mathfrak{gl}(2)^*$, we mean that the Casimir is in the "dual" of $\mathfrak{gl}(2)^*$, or more precisely, in $C^\infty(\mathfrak{gl}(2)^*)$. So, we can take the dual map of T^*:

$$\mathfrak{gl}(2)^* \xleftarrow{T^*} \mathfrak{sl}(2)^*,$$

$$C^\infty(\mathfrak{gl}(2)^*) \xrightarrow{\text{dual of } T^*} C^\infty(\mathfrak{sl}(2)^*).$$

In the following sections, we explore properties of the dual of T^*.

2.4 Poisson Bracket Preservation for Linear Functions

Firstly, does the dual of T^* preserve the Poisson bracket? In other words, we want to check the following for $f, g \in C^\infty(\mathfrak{gl}(2)^*)$:

$$(\text{dual of } T^*)(\{f, g\}) = \{(\text{dual of } T^*)(f), (\text{dual of } T^*)(g)\}.$$

Recall from Appendix A that the dual of T^* is T itself (but extended to nonlinear functions as well). By Appendix B, we can represent any smooth function as a polynomial (and thus as an expression composed of linear functions), and showing Poisson bracket preservation for this polynomial implies Poisson bracket preservation for the smooth function. In this section, we will show bracket preservation for linear functions (to clarify, by linear functions, we mean the linear functions in $C^\infty(\mathfrak{gl}(2)^*)$). In the next section, 2.5, we will extend this to all smooth functions.

Since any linear function can be written as a linear combination of the basis, it is sufficient to show that

$$T(\{f, g\}) = \{T(f), T(g)\}$$

for linear f, g in the basis of $\mathfrak{gl}(2) \subseteq C^\infty(\mathfrak{gl}(2)^*)$ (By this, I mean that the basis for $\mathfrak{gl}(2)$ acts as a basis of all the linear functions in $C^\infty(\mathfrak{gl}(2)^*)$.

First, we have the trivial case that $f = g$, which implies that $\{f, g\} = 0$ and $\{T(f), T(g)\} = 0$. Now, take

$$f = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$g = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Recall that

$$T : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{2}(a-d) & b \\ c & \frac{-1}{2}(a-d) \end{pmatrix},$$

so

$$T(f) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

and

$$T(g) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Since the Poisson bracket for matrices amounts to the standard Lie bracket (i.e., for matrices A, B, $\{A, B\} = AB - BA$) the right side our Poisson preservation equation yields

$$T \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and the left side

$$\{ \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We can continue to check for any combination of basis elements to see that there is indeed Poisson bracket preservation for all linear functions.

2.5 Poisson Bracket Preservation in General

We can extend this to all smooth functions on $\mathfrak{gl}(2)^*$. Let there be polynomials p_n and q_m such that $p_n \to h$ and $q_m \to k$ for $h, k \in C^\infty(\mathfrak{gl}(2)^*)$. Then, by Appendix B,

$$\{p_n, q_m\} \to \{h, k\},$$

so

$$T(\{p_n, q_m\}) = \{T(p_n), T(q_m)\}$$

implies that

$$(\text{dual of } T^*)(\{h, k\}) = \{(\text{dual of } T^*)(h), (\text{dual of } T^*)(k)\}.$$

2.6 Mapping Casimirs to Casimirs

To summarize what we have so far:

$$\mathfrak{gl}(2) \xrightarrow{T} \mathfrak{sl}(2),$$

$$\mathfrak{gl}(2)^* \xleftarrow{T^*} \mathfrak{sl}(2)^*,$$

$$C^\infty(\mathfrak{gl}(2)^*) \xrightarrow{\text{dual of } T^*} C^\infty(\mathfrak{sl}(2)^*).$$

Now, we look at something more specific than preserving the Poisson bracket: preserving Casimir functions. That is, does the dual of T^* send a Casimir function on $\mathfrak{gl}(2)^*$ to a Casimir function on $\mathfrak{sl}(2)^*$?

Recall from Appendix A that the dual of T^* is T itself (but extended to nonlinear functions as well). First, let's understand $C^\infty(\mathfrak{gl}(2)^*)$ more. We can think of it as functions of functions. For example, given linear $v, w \in \mathfrak{gl}(2) \subseteq C^\infty(\mathfrak{gl}(2)^*)$, we get more elements of $C^\infty(\mathfrak{gl}(2)^*)$ by multiplying the linear functions, say $vw + w^2$. Letting $h \in \mathfrak{gl}(2)^*$, this would give

$$(vw + w^2)(h) = v(h)w(h) + w(h)w(h).$$

Then, applying the dual of T^* to $vw + w^2$ gives

$$(\text{dual of } T^*)(vw + w^2) = T(vw + w^2) = T(v)T(w) + T(w)T(w) \in C^\infty(\mathfrak{sl}(2)^*).$$

Applied to our example, given the Casimir $yz - xw$ (where $x, y, z, w \in C^\infty(\mathfrak{gl}(2)^*)$) we get

$$\begin{aligned}(\text{dual of } T^*)(yz - xw) &= T(y)T(z) - T(x)T(w)\\ &= \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} - \begin{pmatrix}\frac{1}{2} & 0\\ 0 & -\frac{1}{2}\end{pmatrix}\begin{pmatrix}-\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix}\end{aligned}$$

where each matrix is in $C^\infty(\mathfrak{sl}(2)^*)$ (in other words, each $T(x)$... is a function that comes from a matrix).

Now, given a dual basis $\{f_1, f_2, f_3\}$ corresponding to the basis

$$\left\{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ -1 & 0\end{pmatrix}\right\}$$

of $\mathfrak{sl}(2)$, any element of $\mathfrak{sl}(2)^*$ is of the form $g = af_1 + bf_2 + cf_3$. By the definition of a corresponding dual basis, evaluating, say $\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$ on g will give b. Therefore,

$$\begin{aligned}\left(\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} - \begin{pmatrix}\frac{1}{2} & 0\\ 0 & -\frac{1}{2}\end{pmatrix}\begin{pmatrix}-\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix}\right)(g) &= \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}(g)\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}(g) - \frac{1}{4}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}(g)\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}(g)\\ &= bc + \frac{1}{4}a^2,\end{aligned}$$

which is the Casimir we found earlier, showing that the dual of T^* preserves Casimirs.

Note 2. *When does Poisson bracket preservation imply Casimir preservation? Let N, M be smooth manifolds, let $\phi : C^\infty(N) \to C^\infty(M)$ be linear, and let $f, g \in C^\infty(N)$. Suppose that the Poisson bracket is preserved under ϕ, that is,*

$$\phi(\{f, g\}) = \{\phi(f), \phi(g)\}.$$

Suppose f is a Casimir in $C^\infty(N)$. Then, by definition, $\{f, g\} = 0$ so that $\phi(\{f, g\}) = 0$ by linearity. For the right side to be zero, this implies that the right must be $\{\phi^*(f), 0\}$. However if ϕ^* is not surjective we may not be able to get a g such that $\phi^*(g) = 0$.

As a side-note, recall that the dual map from $sl(2)^*$ to $gl(2)^*$ (when we were looking at $GL(2)$ and $PGL(2)$) was injective and not surjective. This could perhaps be the reason why plugging the Casimir into the dual map gave $4bc - \frac{1}{4}(a-d)^2$ which doesn't match the Casimir for $gl(2)$. However the dual map from $gl(2)^*$ to $sl(2)^*$ was surjective (but gave the adjoint action...). Go through and see if these are bracket preserving in general.

3 $SO(4)$

For a more geometric example, consider the following extension:

$$S_L \hookrightarrow SO(4) \twoheadrightarrow SO(4)/S_L.$$

Definition 3. Left and right **isoclinic rotations** (denoted S_L and S_R, respectively) are a type of 4-dimensional double rotations where the angles of rotation are equal.

3.1 Rewriting the Extension

We have

$$S_R \cong S_L \cong \mathbb{H} \cong SU(2),$$
$$SO(4) \cong S_L \times S_R,$$
$$\mathfrak{su}(2) \cong \mathfrak{so}(3).$$

Therefore, we rewrite the extension as follows:

$$SU(2) \to SO(4) \to S_R,$$
$$SU(2) \to SO(4) \to SU(2),$$
$$\mathfrak{su}(2) \to \mathfrak{so}(4) \to \mathfrak{su}(2),$$
$$\mathfrak{so}(3) \xrightarrow{S} \mathfrak{so}(4) \xrightarrow{T} \mathfrak{so}(3).$$

The maps are given explicitly by

$$S : \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \mapsto \frac{-1}{2} \begin{pmatrix} 0 & a & b & -c \\ -a & 0 & -c & -b \\ -b & c & 0 & a \\ c & b & -a & 0 \end{pmatrix}$$

and

$$T : \begin{pmatrix} 0 & x & y & z \\ -x & 0 & u & v \\ -y & -u & 0 & w \\ -z & -v & -w & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & x-w & z-u \\ w-x & 0 & y+v \\ u-z & -y-v & 0 \end{pmatrix}.$$

Thus, the dual maps are

$$S^* : f(x,y,z,u,v,w) \mapsto f\left(\frac{-a}{2}, \frac{-b}{2}, \frac{c}{2}, \frac{c}{2}, \frac{b}{2}, \frac{-a}{2}\right)$$

and

$$T^* : f(a,b,c) \mapsto f(x-w, z-u, y+v).$$

A Double Dual Spaces Proof Used in 2.4 and 2.6

Here, I will provide a more rigorous proof for seeing T^{**} as T. In Section 2.6, we view $\mathfrak{gl}(2) \subseteq C^\infty(\mathfrak{gl}(2)^*)$. For a general vector space U, this would mean seeing $U \subseteq U^{**}$. To write this more precisely, let $\iota_U : U \to U^{**}$ be the inclusion map where, for $u \in U$ and $f \in U^*$,

$$\iota_U(u)(f) = f(u).$$

Let U, V be vector spaces and $T : U \to V$ be a Lie algebra homomorphism. Let $u \in U$. We show that

$$\iota_V \circ T(u) = T^{**} \circ \iota_U(u).$$

(In our loose notation, this is just $T = T^{**}$).

It is sufficient to show that

$$\iota_V \circ T = \iota_U \circ T^*.$$

Let $f \in V^*$. Then,

$$(\iota_V \circ T(u))(f) = f(T(u)).$$

Furthermore,

$$\begin{aligned}(\iota_U(u) \circ T^*)(f) &= \iota_U(u)(f \circ T)\\ &= (f \circ T)(u)\\ &= f(T(u)).\end{aligned}$$

Therefore, $\iota_V \circ T(u) = T^{**} \circ \iota_U(u)$.

We can extend this result to all smooth functions on U^*, that is,

$$\text{dual of } T^* : C^\infty(U^*) \to C^\infty(V^*)$$

where we define

$$\text{dual of } T^*(g) = g \circ T^*$$

for $g \in C^\infty(U^*)$. In other words, the dual of T^* is exactly T (or, more precisely, $\iota_V \circ T(u) = (\text{dual of } T^*) \circ \iota_U(u)$).

B Poisson Bracket Convergence Proof Used in 2.4

First, we show that the function $\{\cdot, g\} : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ (where the domain is under the W^1 norm and the codomain under the sup norm) is continuous, i.e., for each $\epsilon > 0$, there exists $\delta > 0$ such that if $\|f - h\|_{W^1} < \delta$, then $\|\{f, g\} - \{h, g\}\|_\infty < \epsilon$.

Let $\epsilon > 0$. Let

$$\delta = \frac{\epsilon}{\sup_{x\in K} \left\| \begin{pmatrix} \frac{\partial g}{\partial x_2}(x) + \dots + \frac{\partial g}{\partial x_n}(x) \\ -\frac{\partial g}{\partial x_1}(x) + \frac{\partial g}{\partial x_3}(x) + \dots + \frac{\partial g}{\partial x_n}(x) \\ \vdots \\ -\frac{\partial g}{\partial x_1}(x) - \frac{\partial g}{\partial x_2}(x) - \dots - \frac{\partial g}{\partial x_{n-2}}(x) + \frac{\partial g}{\partial x_n}(x) \end{pmatrix} \right\|}.$$

Note that we can set δ to be this because g is fixed. I eventually write the denominator as $\sup_{x\in K} \| \begin{pmatrix} \vdots \end{pmatrix} \|$.

Rewriting the expression, we get

$$\|f - h\|_{W^1} = \sup_{x\in K} |f(x) - h(x)| + \sup_{x\in K} \| \begin{pmatrix} \frac{\partial(f-h)}{\partial x_1}(x) \dots \frac{\partial(f-h)}{\partial x_n}(x) \end{pmatrix} \|$$

where the second expression is the supremum of the operator norm of the Jacobian.

Suppose $\|f-h\|_{W^1} < \delta$. Then,

$$
\begin{aligned}
\|\{f,g\}-\{h,g\}\|_\infty &= \|\{f-h,g\}\|_\infty \\
&= \|\frac{\partial(f-h)}{\partial x_1}\left(\frac{\partial g}{\partial x_2}+...+\frac{\partial g}{\partial x_n}\right)+\frac{\partial(f-h)}{\partial x_2}\left(-\frac{\partial g}{\partial x_1}+\frac{\partial g}{\partial x_3}+...+\frac{\partial g}{\partial x_n}\right) \\
&+...+\frac{\partial(f-h)}{\partial x_{n-1}}\left(-\frac{\partial g}{\partial x_1}-\frac{\partial g}{\partial x_2}-...-\frac{\partial g}{\partial x_{n-2}}+\frac{\partial g}{\partial x_n}\right)\|_\infty \\
&= \left\|\begin{pmatrix}\frac{\partial(f-h)}{\partial x_1} & ... & \frac{\partial(f-h)}{\partial x_n}\end{pmatrix}\begin{pmatrix}\frac{\partial g}{\partial x_2}+...+\frac{\partial g}{\partial x_n} \\ -\frac{\partial g}{\partial x_1}+\frac{\partial g}{\partial x_3}+...+\frac{\partial g}{\partial x_n} \\ \vdots \\ -\frac{\partial g}{\partial x_1}-\frac{\partial g}{\partial x_2}-...-\frac{\partial g}{\partial x_{n-2}}+\frac{\partial g}{\partial x_n}\end{pmatrix}\right\|_\infty \\
&= \sup_{x\in K}\left|\begin{pmatrix}\frac{\partial(f-h)}{\partial x_1}(x) & ... & \frac{\partial(f-h)}{\partial x_n}(x)\end{pmatrix}\begin{pmatrix}\frac{\partial g}{\partial x_2}(x)+...+\frac{\partial g}{\partial x_n}(x) \\ -\frac{\partial g}{\partial x_1}(x)+\frac{\partial g}{\partial x_3}(x)+...+\frac{\partial g}{\partial x_n}(x) \\ \vdots \\ -\frac{\partial g}{\partial x_1}(x)-\frac{\partial g}{\partial x_2}(x)-...-\frac{\partial g}{\partial x_{n-2}}(x)+\frac{\partial g}{\partial x_n}(x)\end{pmatrix}\right| \\
&\leq \sup_{x\in K}\left\|\begin{pmatrix}\frac{\partial(f-h)}{\partial x_1}(x) & ... & \frac{\partial(f-h)}{\partial x_n}(x)\end{pmatrix}\right\|\sup_{x\in K}\|\begin{pmatrix}\vdots\end{pmatrix}\| \qquad (1) \\
&\leq \sup_{x\in K}\left\|\begin{pmatrix}\frac{\partial(f-h)}{\partial x_1}(x) & ... & \frac{\partial(f-h)}{\partial x_n}(x)\end{pmatrix}\right\|\sup_{x\in K}\|\begin{pmatrix}\vdots\end{pmatrix}\| + \sup_{x\in K}\|\begin{pmatrix}\vdots\end{pmatrix}\|\sup_{x\in K}|f(x)-h(x)| \\
&= \sup_{x\in K}\|\begin{pmatrix}\vdots\end{pmatrix}\|\left(\sup_{x\in K}\left\|\begin{pmatrix}\frac{\partial(f-h)}{\partial x_1}(x) & ... & \frac{\partial(f-h)}{\partial x_n}(x)\end{pmatrix}\right\| + \sup_{x\in K}|f(x)-h(x)|\right) \\
&= \sup_{x\in K}\|\begin{pmatrix}\vdots\end{pmatrix}\|(\|f-h\|_{W^1}) \\
&< \left(\sup_{x\in K}\|\begin{pmatrix}\vdots\end{pmatrix}\|\right)\frac{\epsilon}{\sup_{x\in K}\|\begin{pmatrix}\vdots\end{pmatrix}\|} \\
&= \epsilon,
\end{aligned}
$$

where (3) is the supremum of the operator norm times the norm of the vector with all the sums of the partial derivatives written in the previous line. The inequality holds since for linear $T:\mathbb{R}^n\to\mathbb{R}$, $|T(x)|\leq\|T\|\|x\|$.

Therefore, the function $\{\cdot,g\}$ is continuous (with the respective norms).

To show that the function $\{g,\cdot\}$ is continuous, we can use the same δ and see that

$$
\begin{aligned}
\|\{g,f-h\}\|_\infty &= \|-\{f-h,g\}\|_\infty \\
&= \|\{f-h,g\}\|_\infty.
\end{aligned}
$$

The rest of the proof follows.

By the Sequential Characterization of Continuity, if $p_n\to f$, then $\{p_n,g\}\to\{f,g\}$ and $\{g,p_n\}\to\{g,f\}$ (or if $q_k\to g$, then $\{f,q_k\}\to\{f,g\}$).

Now, we show that if $p_n\to f$ and $q_k\to g$, then $\{p_n,q_k\}\to\{f,g\}$. First,

$$\{p_n,q_k\}\to\{f,q_k\}$$

by convergence in the first component. Then,

$$\{f,q_k\}\to\{f,g\}$$

by convergence in the second component.