Smooth Manifolds and Lie Groups

Directed Reading Program Fall 2023

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Overview

1. An Introduction to Manifolds
2. Lie Groups
3. A Non-Example: S^2
4. The Hairy Ball Theorem and Parallelizability

Definition (Manifold). Let M be a topological space. M is a manifold if it is

(1) Hausdorff,

(2) second-countable, and

(3) locally Euclidean.

Today, we'll only focus on (3).

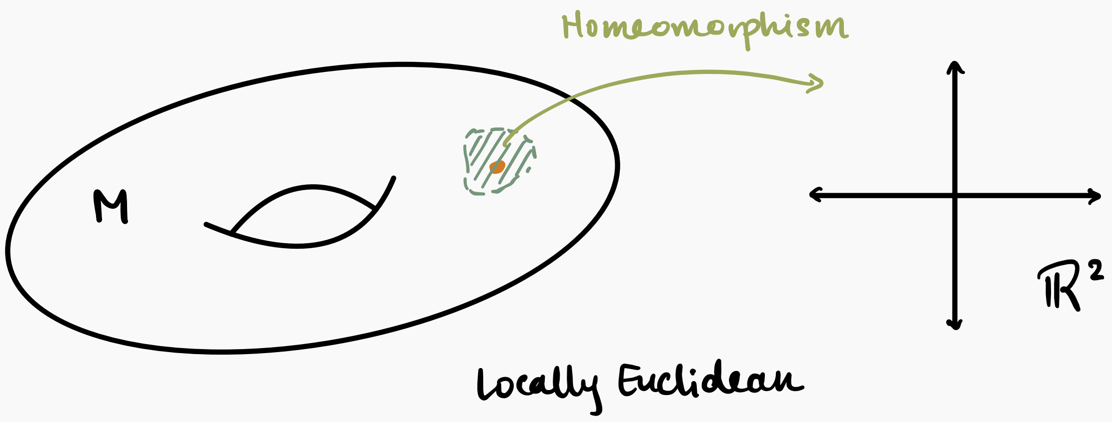
Topological Spaces

Being in a topological space allows us to define continuity of functions.

Definition (Locally Euclidean). Given any point on a manifold M of dimension n, we can find an open set around the point and a homeomorphism (a continuous bijective map with a continuous inverse) between that open set and $\mathbb{R}^n$.

Intuitively...

A manifold is an object that looks like $\mathbb{R}^n$ in a small area around each point.



Smooth Manifolds

We will look at **smooth manifolds**, which will allow us to do calculus on manifolds.

Smooth manifolds "smoothly" look like $\mathbb{R}^n$ locally; instead of finding a continuous map from an open set to $\mathbb{R}^n$, we find a smooth (infinitely differentiable) one.[1]

[1] More precisely, given any two charts on M, the transition map is smooth.

Manifold Examples

- $\mathbb{R}^n$ where we take the identity map as our homeomorphism.
- The torus (we'll touch on this later).
- The 2-dimensional sphere, given by

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

  It is *embedded in* $\mathbb{R}^3$.

Let's provide more details on the sphere.

Example (Sphere)

The sphere is a smooth manifold. Consider the open set $U = \{(x, y, z) : z > 0\}$.

The "Flattening Map"[2]

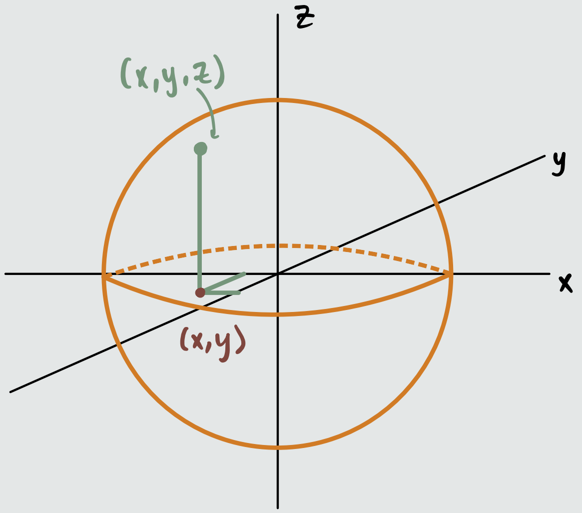
$$\phi : U \to \mathbb{R}^2$$

$$(x, y, z) \mapsto (x, y),$$

with inverse

$$\phi^{-1} : \mathbb{R}^2 \to U$$

$$(x, y) \mapsto \left(x, y, \sqrt{1 - (x^2 + y^2)}\right).$$



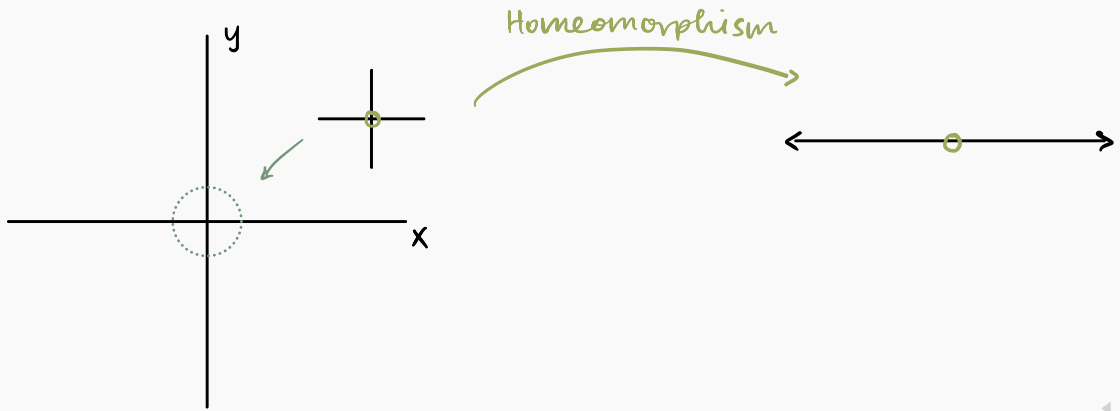
[2]Another map we can use is stereographic projection.

Nonexample

Consider the x- and y-axes (embedded in $\mathbb{R}^2$). This is **not** a manifold.

The key reasoning:

- Homeomorphisms on a space are still homeomorphisms on a subspace of the space.
- Continuous maps preserve connectedness (and homeomorphisms preserve the number of connected components).



Definition (Lie Group). G is a Lie group if it is a smooth manifold that is a group such that

- the multiplication map $m : G \times G \to G$ and
- the inverse map $(\ )^{-1} : G \to G$

are smooth.

Example ($\mathbb{R}^n$)

Consider $\mathbb{R}^n$.

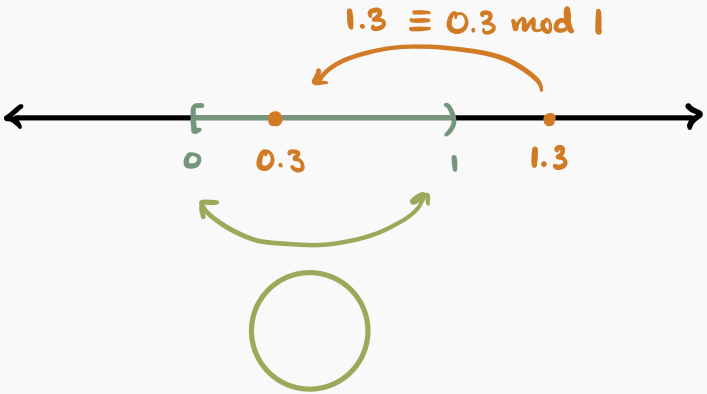
- We've seen that $\mathbb{R}^n$ is a smooth manifold.
- $\mathbb{R}^n$ is a group under addition; in particular, addition and negation (its inverse map) are both smooth.

Example (Tori)

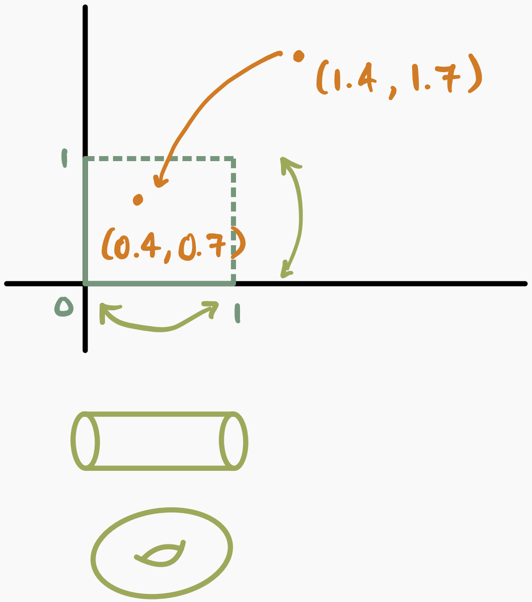
As a group, the n-torus is $\mathbb{R}^n/\mathbb{Z}^n$. Consider the group operation

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1 \mod 1, \ldots, x_n + y_n \mod 1).$$

The 1-Torus



The 2-Torus



Example (Matrix Groups)

- The general linear group: $GL(n) = \{A \in M(n \times n) : M \text{ is invertible}\}$.
- The rotation group: $SO(3) = \{A \in M(3 \times 3) : A^T A = AA^T = I\}$.
- The special linear group: $SL(2) = \{A \in M(2 \times 2) : \det(A) = 1\}$.

Nonexample (Sphere)

We show that the sphere is not a Lie group.

How can you show something is not a Lie group?

It's tricky...

You'd have to show that no group operation on the space will be smooth.

Instead, let's use a much nicer method.

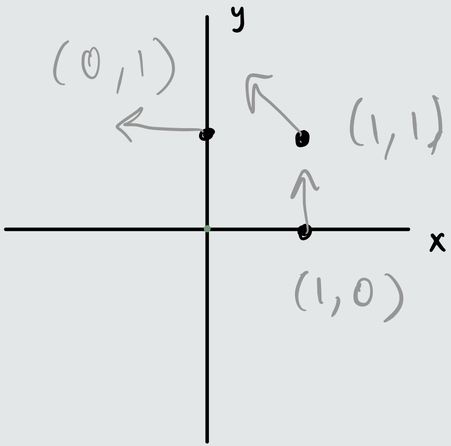
Vector Fields

We create a **vector field** on a smooth manifold by placing tangent vectors at every point on the manifold.

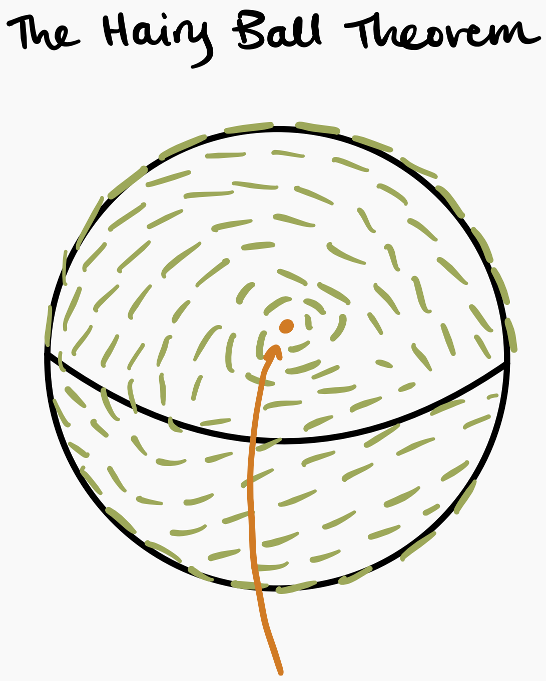
- If the vectors *continuously vary*, the vector field is continuous.
- If the vectors *smoothly vary*, the vector field is smooth.

Example

$$-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$



Theorem (Hairy Ball). Every continuous vector field on a sphere has to vanish somewhere, i.e., there is some point on the sphere whose vector is 0.



The Tangent Bundle of a Lie Group (Parallelizability)

Theorem. Let G be a Lie group of dimension n. Then,

$$TG \cong G \times \mathbb{R}^n.$$

The Tangent Bundle

The **tangent bundle** of a manifold M (written TM) is, loosely speaking, the collection of all smooth vector fields on M.

Diffeomorphic

Two smooth manifolds are **diffeomorphic** ($\cong$) if there exists a smooth, bijective map with a smooth inverse between them. We can *identify* the two spaces with each other.

A Contradiction

Suppose the sphere is a Lie group. Then, by the previous theorem,

$$TS^2 \cong S^2 \times \mathbb{R}^2.$$

Then, there would be a vector field consisting of a *nonzero* constant tangent vector at every point on the sphere. $X = \{(p, (1,1)) : p \in S^2\}$

However, this is impossible under the Hairy Ball Theorem.

"Parallelizability"

The tangent bundle theorem tells us that Lie groups have a *nonvanishing vector field*. This gives us a way to show if something is *not* a Lie group.

Some Fun Facts

It turns out that...

- S^1 and S^3 are Lie groups.
  - S^1 can be endowed with the operation of angle addition.
  - $S^3 \cong SU(2)$.
- S^7 is parallelizable, i.e.,

$$TS^7 \cong S^7 \times \mathbb{R}^7,$$

  but *not* a Lie group (parallelizability is not an "if and only if" relation!).

Conclusion

1. A manifold is locally similar to $\mathbb{R}^n$.
2. A Lie group is a smooth manifold that is also a topological group.
3. The tangent bundle of a Lie group is trivial.