MIT18.01 Single Variable Calculus

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1 Derivatives

1.1 What is Derivative

Geometric Interpretation

Derivative is the slope of the line tangent to the graph of $f(x)$.

Tangent line: The limit of the secant line (a line drawn between two points on the graph) as the distance between the two points goes to zero.

$$\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \underbrace{\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}}_{difference\ quotient} = \underbrace{f'(x_0)}_{derivative\ of\ f\ at\ x_0}$$

Physical Interpretation

It's a rate of change, $\frac{\Delta y}{\Delta x}$ is average change, while $x \to 0$, it becomes instantaneous rate $\frac{dy}{dx}$.

Example 1 *$f(x) = \frac{1}{x}$, find $f'(x)$.*

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \times \frac{x - (x + \Delta x)}{(x + \Delta x)x} \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \times \frac{-\Delta x}{(x + \Delta x)x} \\
&= \lim_{\Delta x \to 0} -\frac{1}{x^2 + x\Delta x} \\
&= \lim_{\Delta x \to 0} \\
&= -\frac{1}{x^2}
\end{aligned}$$

Example 2 $f(x) = \sin x$, *find* $f'(x)$.

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{\sin(x + \Delta x) - \sin x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x}(\sin x \cos \Delta x + \sin \Delta x \cos x - \sin x) \\
&= \lim_{\Delta x \to 0} \frac{\sin x(\cos \Delta x - 1)}{\Delta x} + \frac{\sin \Delta x \cos x}{\Delta x} \\
&= 0 + \cos x \\
&= \cos x
\end{aligned}$$

Notation

$$\frac{\Delta f}{\Delta x} \to f'(x) \quad (Newton's\ Notation)$$

$$\frac{\Delta d}{\Delta x} \to \frac{dy}{dx} \quad (Leibniz's\ Notation)$$

$$\frac{df}{dx}, f', D\ f$$

1.2 Limits and Continuity

Easy Limits

Just plug in the limit to evaluate.

$$\lim_{\Delta x \to 0} x + 1 = 1$$

Continuity

Left-hand Limit: $\lim_{x \to x_0^-}$

Right-hand Limit: $\lim_{x \to x_0^+}$

$f(x)$ is continuous at x_0 when

$$\lim_{x \to x_0} f(x) = f(x_0)$$

It equals to

1. $\lim_{x \to x_0} f(x_0)$ exists
2. $f(x_0)$ is defined
3. $\lim_{x \to x_0^-} = \lim_{x \to x_0^+}$

Discontinuity

1. Removable Discontinuity: $\lim_{x \to x_0^-} = \lim_{x \to x_0^+}$.
2. Jump Discontinuity: $\lim_{x \to x_0^-} \neq \lim_{x \to x_0^+}$.
3. Infinite Discontinuity: $\lim_{x \to x_0^-} = \pm\infty$, $\lim_{x \to x_0^+} = \pm\infty$.
4. Other Discontinuity: $\sin \frac{1}{x}$, no left or right limit.

Differentiable
Left differential and right differential exist and equal.
Differentiable Implies Continuous

$$\lim_{x \to x_0} (f(x) - f(x_0)) = \lim_{x \to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0} \right] (x - x_0) = f'(x_0) \cdot x_0 = 0$$

But Not vice versa. such as $y = x^{\frac{1}{3}}$ and $y = |x|$.

1.3 Differentiate Fomulas

Specific

$$(\sin x)' = \cos x$$

$$(\cos x)' = -\sin x$$

General

1. Product Rule: $(uv)' = u'v + uv'$.
2. Quotient Rule: $(\frac{u}{v})' = \frac{u'v - uv'}{v^2} (v \neq 0)$.
3. Chain Rule: use new variable names

1.4 Higher Derivatives

$$f''(x) = D^2 f = \frac{d^2 f}{dx^2}$$

1.5 Implicit Differentiation and Inverses

Implicit Differentiation

$$x^2 + y^2 = 1$$
$$2x + 2y\frac{dy}{dx} = 0$$

Inverse Functions

if $f(x) = y$ and $g(y) = x$, then we call g the inverse function of f, also f^{-1}.

f^{-1} as the graph of f reflected about the line $y = x$.

1.6 Exponential and Log, Logarithmic Differentiation, Hyperbolic Functions

How to Find e?

1. let $M(a) = \lim_{x\to 0} \frac{a^{\Delta x}-1}{\Delta x}$
2. so $\frac{d}{dx}a^x = M(a)a^x$
3. let $M(e) = 1$
4. $\frac{d}{dx}e^x = e^x$
5. $\lim_{x\to 0} \frac{e^x-1}{x} = 1$

An Important Limit About e

$$\lim_{x\to\infty}(1 + \frac{1}{x})^x = e$$

How to Differentiate a^x?

- $a^x = e^{x \ln a}$
- use logarithmic differentiation

Hyperbolic Sine and Cosine

$$\sinh x = \frac{e^x - e^{-x}}{2}$$
$$\cosh x = \frac{e^x + e^{-x}}{2}$$
$$\cosh^2(x) - \sinh^2(x) = 1$$

2 Applications of Differentiation

2.1 Linear Approximation

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

geometric significance: the best fit straight line of a function

2.2 Quadratic Approximation

$$f(x) \approx f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2$$

more elaborate than linear approximation
geometric significance: the best fit parabola of a function

2.3 Curve Sketching

- $f' > 0$, function is increasing
- $f' < 0$, function is decreasing
- $f' = 0$, x_0 is critical point, y is the critical value.
- $f'' > 0$, function is convex(concave up)
- $f'' < 0$, function is concave(concave down)
- $f'' = 0$, x_0 is an inflection point

f'' also can tell there is no wiggle in graph
How To Draw a Graph of Function

1. find discontinuities, especially when the value is infinite
2. critical points, $f'(x) = 0$
3. plot the zeros of f, $f(x) = 0$
4. endpoints
5. check local maximum/minimum, critical points and inflection points

Maximum and Minimum

only exists in critical points, endpoints, or points of discontinuity

2.4 Related Rates

see `https://ocw.mit.edu/courses/mathematics/18-01-single-variable-calculus-fall-2`
`lecture-notes/lec12.pdf`.

2.5 Newton's Method

Newton's method is a powerful tool for solving equations of the form $f(x) = 0$ by finding numerical approximations.

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

Warning 1. Newton's Method can find an unexpected root. Warning 2. Newton's Method can fail completely.

2.6 Mean Value Theorem and Inequalities

Mean Value Theorem

If f is differentiable on $a < x < b$, and continuous on $a \leq x \leq b$, then

$$\frac{f(b) - f(a)}{b - a} = f'(c)(a < c < b)$$

2.7 Differentials

$dy = f'(x)dx$

Example 3 *solve* $64.1^{\frac{1}{3}}$

$$\begin{aligned}
y &= x^{\frac{1}{3}} \\
x &= 64, y = 4, dx = 0.1 \\
dy \approx y'dx &= \frac{1}{3}x^{-\frac{2}{3}dx} = \frac{1}{3}64^{-\frac{2}{3}} \times 0.1 \approx 0.002 \\
64.1^{\frac{1}{3}} \approx y + dy &\approx 4 + 0.002 = 4.002
\end{aligned}$$

Indefinite Integral

$$F(x) = \int f(x)dx$$

$$F'(x) = f(x)$$

$$\int \sin x = -\cos x + c$$

Substitution

Example 4 $\int \frac{1}{x \ln x}$
let $u = \ln x$, *then* $\int \frac{dx}{x \ln x} = \int \frac{1}{u} du = \ln u + c = \ln(\ln x) + c$

Advanced Guessing

Example 5 $\int e^{6x}$
Guess e^{6x}

$$\frac{d}{dx} e^{6x} = 6e^{6x}$$

So

$$\int e^{6x} = \frac{1}{6} e^{6x} + c$$

2.8 Differential Equations and Separation of Variables

Using separation of variables.

$$\frac{dy}{dx} = f(x)g(y)$$

$$\frac{dy}{g(y)} = f(x)dx$$

$$H(y) = F(x) + c$$

3 The Definite Integral and Its Applications

3.1 Definite Integrals

Explanations

- the area above the x axis minus the area below the x axis
- cumulative sum
- Riemann Integral: $\sum_{i=1}^{n} f(c_i)\Delta x$
- $$\int_a^b f(x)dx = \lim_{x\to\infty} \sum_{i=1}^{n} f(a + \frac{i(b-a)}{n}\Delta x)$$

3.2 First Fundamental Theorem of Calculus

FTC1, Newton-Leibniz formula
If $f(x)$ is continuous and $F'(x) = f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a) = F(x)\Big|_a^b$$

Intuitive Interpretation of FTC
$x(t)$ is a position; $v(t) = x'(t)$ is the speed or rate of change of x.

$$\int_a^b v(t)dt = x(b) - x(a)$$

Properties of Integrals

1. $\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
2. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$
3. $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$
4. $\int_a^a f(x)dx = 0$
5. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
6. if $f(x) \le g(x)$, then $\int_a^b f(x)dx \le -\int_a^b g(x)dx$ (for estimation)

Substitution of Integrals

Only when u' does not change sign.

$$\int_{x_1}^{x_2} f(x)dx = \int_{u_1}^{u_2} g(u)du \quad u_1 = u(x_1), u_2 = u(x_2)$$

Another Explanation of FTC 1

$$F(b) - F(a) = \int_a^b f(x)dx$$

let $\Delta F = F(b) - F(a)$,

$$\frac{\Delta F}{\Delta x} = \frac{1}{\Delta x}\int_a^b f(x)dx = \frac{1}{b-a}\int_a^b f(x)dx = Average(f) = Average(F')$$

So

$$\Delta F = Average(F')\Delta x$$

and

$$Average(f) = \frac{1}{n}\sum_{i=0}^{n} f(i) \approx \frac{1}{n}\int_0^n f(x)dx$$

$$\int_a^b min(f)dx \leq Average(F')\Delta x = \int_a^b f(x)dx \leq \int_a^b max(f)dx$$

3.3 Second Fundamental Theorem of Calculus

if f is continuous and $G(x) = \int_a^x f(t)dt \quad (a \leq t \leq x)$, then $G'(x) = f(x)$ and $G(a) = 0$

Example 6

$$\frac{d}{dx}\int_0^{x^2} \cos t dt = ?$$

$$u = x^2, F(u) = \int_0^u \cos t dt$$

$$\frac{d}{dx}\int_0^u \cos t dt = F'(u) = \cos u$$

$$\frac{d}{dx}\int_0^{x^2} \cos t dt = \frac{dF(x^2)}{dx} = F'(x^2)\cdot(x^2)' = 2x\cos x^2$$

FTC2 VS MVT

$$\Delta F = Ave(F'(x))\Delta x$$

$$\Delta F = F'(c)\Delta x$$

New Functions/Transcendental Functions

We can use integral to generate new functions, such as $\int_2^x \frac{1}{lnt}dt$.
If telling $L'(x) = \frac{1}{x}$, $L(1) = 0$, then $L(x) = \int_1^x \frac{1}{t}dt$.

3.4 Applications to Logarithms and Geometry

Logarithm

Regard $L(x) = \int_1^x tdt$ as the definition of the logarithm, then we have

$$L'(x) = \frac{1}{x}$$

$$L(1) = 0$$

$$L''(x) = -\frac{1}{x^2} < 0$$

Areas between two curves

$$A = \int_a^b (f(x) - g(x))dx$$

3.5 Volumes by Disks and Shells

Disks

$y = -x^2 + 1 \quad (-1 \leq x \leq 1)$ rotated around the x-axis, calculate the volume.

Shells

$y = x^2$ rotated around the y-axis, calculate the volume.

3.6 Work, Average Value, Probability

Continuous Average

Riemann Sum

$$\frac{(y_1 + y_2 + \cdots + y_n)\Delta x}{b-a} \to \frac{\int_a^b f(x)dx}{b-a}$$

Weighted Average

$$\frac{\int_a^b f(x)w(x)dx}{\int_a^b w(x)dx}$$

Probability
if

$$f(x) = \begin{cases} 0 \\ 1 \end{cases}$$

then, weighted average function becomes the probability.

$$P(x_1 \le x \le x_2) = \frac{\int_{x_1}^{x_2} w(x)dx}{\int_a^b w(x)dx} \quad (a \le x_1 \le x_2 \le b)$$

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} dx = 1$$

3.7 Numerical Integration

Numerical Integration is a way to compute an approximate solution to a definite integral.

1. Riemann Sum: Left Riemann: $\sum\limits_{n=0}^{n-1} y_n \Delta x$, Right Riemann: $\sum\limits_{n=1}^{n} y_n \Delta x$
2. Trapezoidal Rule: $\sum\limits_{n=0}^{n-1} \frac{y_n+y_{n+1}}{2} \Delta x = \frac{LeftRiemann+RightRiemann}{2}$
3. Simpson's Rule: n is even, use parabola to calculate. $\sum\limits_{n=0}^{n-2} \frac{y_n+4y_{n+1}+y_{n+2}}{6} \Delta x$

4 Techniques of Integration

4.1 Trig Substitutions and Trig Integrals

Solving $\int \sin^m x cos^n x dx$

- Either m or n is odd, use $\sin^2 x + \cos^2 x = 1$ to substite to the result of only sin or cos exists, if m is odd, let $u = \cos x$; if n is odd, let $u = \sin x$.

- if both m and n are even, use double-angle formulae to depress the expression.

4.2 Trig Substitution Rule

- $\sqrt{a^2 - x^2} \to x = a\sin\theta \to result = a\cos\theta$
- $\sqrt{a^2 + x^2} \to x = a\tan\theta \to result = a\sec\theta$
- $\sqrt{x^2 - a^2} \to x = a\sec\theta \to result = a\tan\theta$
- $\sqrt{x^2 + 4x} = \sqrt{(x+2)^2 - 4} = \sqrt{(2\sec\theta)^2 - 4} = 2\tan\theta$

If necessarily, finally you should undo trig substitution by drawing a triangle.

4.3 Partial Fractions

Solving $\int \frac{P(x)}{Q(x)}$.
if degree P $<$ degree Q,

1. factor the denominator
2. set up equation, $\frac{A}{x+1} + \frac{B}{x+2}$
3. solve A and B using cover-up method, $\frac{4x-1}{x+2} = A + \frac{B(x-1)}{x+2}$, $A = \frac{4-1}{1+2} = 1$

if the equation has repeated roots, you should calculate the other variable first, and then plug them in to find the values that are relative to the repeated root.

if Q has a quadratic factor, calculate the other variable first ,then **clear the denominator**, finally plug them in to find the rest values.

if degree P $\geq$ degree Q, (improper fraction)

1. use long division to find the quotient and the remainder
2. $\frac{P(x)}{Q(x)} = quotient + \frac{R(x)}{Q(x)}$ where $R(x)$ is the remainder
3. then $\frac{R(x)}{Q(x)}$ comes to the situation where degree P $<$ degree Q

4.4 Integration by Parts

$$(uv)' = u'v + uv'$$
$$uv' = (uv)' - u'v$$
$$uv' = uv - \int u'v$$

4.5 Recurrence Formulas

$$\int (\ln x)^n dx = x(\ln x)^n - n \int (\ln x)^{n-1} dx$$

4.6 Arc Length

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + (\frac{dy}{dx})^2}$$

4.7 Surface Area

Sphere Surface Area, radius is a:

$$\int_{x_1}^{x_2} 2\pi y ds = 2\pi a(x_2 - x_1)$$

4.8 Parametric Equations

$$x = a\cos t, b = a\sin t \rightarrow ds = adt$$

if changing speed, then it becomes $x = a\cos kt, b = a\sin kt$

ds in Parametric Equations

$$\frac{ds}{dt} = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}$$

4.9 Polar Co-ordinates, Area in Polar Co-ordinates

$$Area = \pi r^2 \frac{d\theta}{2\pi} = \frac{1}{2} r^2 d\theta$$

5 Exploring the Infinite

5.1 L 'Hospital's Rule

Only suits **indeterminate form**, that is $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)}$$

growing speed

$$\ln x << x^p << e^x << e^{x^2}$$

$$\frac{1}{\ln x} >> \frac{1}{x^p} >> e^{-x} >> e^{-x^2}$$

5.2 Improper Integrals

$$\int_a^\infty f(x)dx = \lim_{M \to \infty f(x)dx}$$

the expression is converges if the limit exists, or else diverges.

Important Integral

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

Integral Comparison

for powers,

$$\int_1^\infty \frac{1}{x^p} dx$$

if $p \leq 1$, diverges; if $p > 1$, limit $= \frac{1}{p-1}$

if $0 \leq f(x) \leq g(x)$,

- if $\int_a^\infty g(x)dx$ converges, so does $\int_a^\infty f(x)dx$.
- if $\int_a^\infty f(x)dx$ diverges, so does $\int_a^\infty g(x)dx$.

Improper Integrals of the Second Type

$$\int_0^1 \frac{1}{x^p} dx$$

if $p \leq 1$, limit $= \frac{1}{p-1}$; if $p > 1$, diverges.

$$\int_0^1 f(x)dx = lim_{a \to 0^+} \int_a^1 f(x)dx$$

When a integral function contains singularity, it diverges.

5.3 Infinite Series

Geometric Series

$$1 + a + a^2 + \cdots = \frac{1}{1-a} \quad |a| < 1$$

$$S_N = \sum_{i=0}^{N} a_i$$

When N goes infinite, if the limit of this **partial sum** exists, then the series **converges**.

Integral Comparison

Consider a positive, decreasing function $f(x) > 0$.

$$|\sum_{n=1}^{\infty} f(n) - \int_1^{\infty} f(x)dx| < f(1)$$

if $f(x) \sim g(x)$, then $\sum f(n)$ and $\sum g(n)$ eigher both converge or both diverge.

$f(x) \sim g(x)$ means

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = c \quad (0 < c < \infty)$$

Integral Test

1. Limit Comparison

$$\sum_{n=0}^{\infty} \frac{5n+2}{n^3+1} \sim \sum_{n=0}^{\infty} \frac{1}{n^2}$$

2. Ratio Test

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = L$$

1. if $L < 1$, $\sum a_n = L$ converges.

2. if $L > 1$, $\sum a_n = L$ diverges.

3. if $L = 1$, noting.

5.4 Taylor Series

Power Series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

$|x| < R$ where R = radius of convergence, means if $|x| > R$, then $|a_n x^n|$ does not tend to 0. if $a_n = c$, it becomes **geometric series**.

Rules of polynomials apply to series within the radius of convergence.

- Substitution/Algebra
- Differentiation(term by term)
- Integration(term by term)

Taylor's Series

$$f(x) = f(b) + f'(b)(x-b) + \frac{f''(b)}{2}(x-b)^2 + \cdots$$