MIT18.02 Multivariable Calculus

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1 Vectors and Matrices

1.1 Vectors

A vector can be viewd in to different ways: geometrically and algebraically.

Geometric view

A vector is defined as having a magnitude and a direction, the start of the arrow is tail, and the end is denoted as the tip or head.

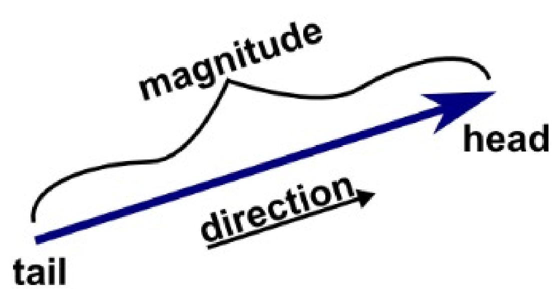

图 1: vector

The vector between two points will be denoted $\overrightarrow{PQ}$. P is the initial point and Q is the terminal point.

The magnitude of the vector $\mathbf{A}$ will be denoted $|\mathbf{A}|$. Magnitude will also be called *length* or *norm*.

Scaling, adding and subtracting vectors

1. Scaling: Scaling a vector means changing its length by a scale factor. Because we use numbers to scale a vector we will often refer to real numbers as *scalars*.

2. Add: Add vectors by placing them head to tail, this can be done in either order.

3. Substract: Subtract vectors either by placing the tail to tail or by adding $\mathbf{A} + (-\mathbf{B})$.

Algebra view

The origin is labeld as O, $O = (0, 0)$ in the plane and $O = (0, 0, 0)$ in the space. If the tail of the vector is placed at origin, we call it *origin vector* and the vector is denoted $A = \langle a_1, a_2 \rangle$.

We can do calculation by using two vectors $\mathbf{i} = \langle 1, 0 \rangle$, $\mathbf{j} = \langle 0, 1 \rangle$

Notations and terminology

1. (a_1, a_2) indicates a point in the plane.

2. $\langle a_1, a_2 \rangle = a_1\mathbf{i} + a_2\mathbf{j}$.

3. for $\mathbf{A} = \langle a_1, a_2 \rangle = a_1\mathbf{i} + a_2\mathbf{j}$, a_1 and a_2 is called the $\mathbf{i}$ and $\mathbf{j}$ components of $\mathbf{A}$.

4. $\overrightarrow{P} = \overrightarrow{OP}$ is the vector from the origin to P.

5. A real number is a *scalar*, you can use it to scale a vector.

Vector algebra using coordinates

1. Magnitude: $|\mathbf{A}| = \sqrt{a_1^2 + a_2^2}$.

2. Addition/Substraction: $\mathbf{A} \pm \mathbf{B} = (a_1 \pm b_1)\mathbf{i} + (a_2 \pm b_2)\mathbf{j}$.

3. $\overrightarrow{PQ} = \overrightarrow{Q} - \overrightarrow{P}$, $\overrightarrow{PQ}$ is the *displacement* from P to Q.

4. in three dimensions, $|\mathbf{A}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

Unit vectors

A unit vector is a vector with unit length. It's denoted $\hat{\mathbf{u}}$.

1.2 Dot product

Dot product is a way of multipling two vectors. It's also called *scalar product* because the result of it is a scalar.

Algebraic definition if $\mathbf{A} = \langle a_1, a_2 \rangle$ and $\mathbf{B} = \langle b_1, b_2 \rangle$ then

$$\mathbf{A} \cdot \mathbf{B} = a_1 b_1 + a_2 b_2$$

Geometric definition

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos\theta$$

$$|\mathbf{A} - \mathbf{B}|^2 = |\mathbf{A}|^2 + |\mathbf{B}|^2 - 2|\mathbf{A}||\mathbf{B}| \cos\theta$$

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$$

When two vectors are perpendicular to each other we say they are *orthogonal*. $\mathbf{A} \perp \mathbf{B} \Leftrightarrow \mathbf{A} \cdot \mathbf{B} = 0$

Dot product and length

Algebraically: $\mathbf{A} \cdot \mathbf{A} = \langle a_1, a_2, a_3 \rangle \cdot \langle a_1, a_2, a_3 \rangle = |\mathbf{A}|^2$

Geometrically: $\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}||\mathbf{A}| \cos\theta = |\mathbf{A}||\mathbf{A}| = |\mathbf{A}|^2$

1.3 Uses of dot product

1. Find the angle between two vectors(either in a plane or space).

2. Judge if two vectors are orthogonal.

3. Using vectors and dot product show the diagonals of a parallelogram have equal lengths if and only if it' s a rectangle.

4. Dot product shows the degree of collinearity, if there is a right-handed coordinate system $\mathbf{i}'$ and $\mathbf{j}'$. To express vector $\mathbf{A}$ in this coordinate system, the coordinate of $\mathbf{A}'$ is $\langle \mathbf{A} \cdot \mathbf{i}', \mathbf{A} \cdot \mathbf{j}' \rangle$.

1.4 Components and Projection

If $\mathbf{A}$ is any vector and $\hat{\mathbf{u}}$ is a unit vector then the component of $\mathbf{A}$ in the direction of $\hat{\mathbf{u}}$ is

$$\mathbf{A} \cdot \hat{\mathbf{u}}.$$

And if θ is the angle between $\mathbf{A}$ and $\hat{\mathbf{u}}$, then

$$\mathbf{A} \cdot \hat{\mathbf{u}} = \mathbf{A} \cos \theta$$

We also call the leg parallel to $\hat{\mathbf{u}}$ the *orthogonal projection* of $\mathbf{A}$ on $\hat{\mathbf{u}}$.

1.5 Areas and Determinants

The value of the determinant is area of the parallelogram formed by $\mathbf{A}$ and $\mathbf{B}$. $\mathbf{A} = \langle a_1, b_1 \rangle$, $\mathbf{B} = \langle a_2, b_2 \rangle$.

$$\det(\mathbf{A}, \mathbf{B}) = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

1.6 Determinant in space

Important facts about $|A|$

1. $|A|$ is multiplied by -1 if we interchange two rows or two columns.

2. $|A| = 0$ if one row or column is all zero, or if two rows or two columns are the same.

3. $|A|$ is multiplied by c, if every element of some row or column is multiplied by c.

4. The value of $|A|$ is unchanged if we add to one row (or column) a constant multiple of another row (resp. column).

The **ij-entry**, written a_{ij}, is the number in the i-th row and j-th column.
The **ij-minor**, written $|A_{ij}|$, is the determinant that' s left after deleting from $|A|$ row and column containing a_{ij}.
The **ij-cofactor**, written A_{ij}, is given as a formula by $A_{ij} = (-1)^{i+j} |A_{ij}|$.

Laplace expansion by cofactors

$$a_{1j} A_{1j} + a_{2j} A_{2j} + a_{3j} A_{3j} = |A|$$

1.7 Cross Product

Cross product is another way of multiplying two vectors. Because the result of this multiplication is another vector it is also called the *vector product*.

$\mathbf{A} = \langle a_1, a_2, a_3\rangle$, $\mathbf{B} = \langle b_1, b_2, b_3\rangle$

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle$$

Algebraic facts

1. $\mathbf{A} \times \mathbf{A} = 0$
2. Anti-commutivity: $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$
3. Distributive law: $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$
4. Non-associativity: $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \neq \mathbf{A} \times (\mathbf{B} \times \mathbf{C})$

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \mathbf{j} \times \mathbf{k} = \mathbf{i}, \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

Geometric description $|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}||\mathbf{B}|\sin\theta$, it is also the area of the parallelogram spanned by $\mathbf{A}$ and $\mathbf{B}$.

The direction of $|\mathbf{A} \times \mathbf{B}|$ is determined by *right hand rule*.

1.8 Equation of a Plane

Judging if point $P(x, y, z)$ is in the plane which points P_1, P_2 and P_3 are in is equal to judging if $\overrightarrow{P_1P} \cdot (\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}) = 0$.

$$V = [a, b, c] = a \cdot (b \times c) = b \cdot (c \times a) = c \cdot (a \times b) = det(a, b, c)$$

So it's equal to $det(\overrightarrow{P_1P}, \overrightarrow{P_1P_2}, \overrightarrow{P_1P_3}) = 0$

1.9 Matrix Algebra

Matrix algebra

A rectangular array of numbers having n rows and n columns is called an $m \times n$ **matrix**.

Two matrices of the same size are *equal* if corresponding entries are equal.

Two special kinds of matrices are the **row-vectors**: the $1 \times n$ matrices; and the **column vectors**: the $m \times 1$ matrices consisting of a column of m numbers.

Matrix operations

1. Scalar multiplication
2. Matrix addition
3. Transposition
4. Matrix multiplication

Laws and properties of matrix multiplication

1. $A(B+C) = AB + AC \quad (A+B)C = AC + BC \quad$ *distributive laws*
2. $(AB)C = A(BC) \quad (cA)B = c(AB)$*associative laws*
3. $I_3 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$
4. In general, $AB \neq BA$,
5. $|AB| = |A|\,|B|$
6. Matrix multiplication can be used to pick out a row or column of a given matrix, such as

$$\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}$$

1.10 Meaning of matrix multiplication

Matrix transforms the unit square into a parallelogram.

$$A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$$

$$A\mathbf{i} = A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}; \quad A\mathbf{j} = A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

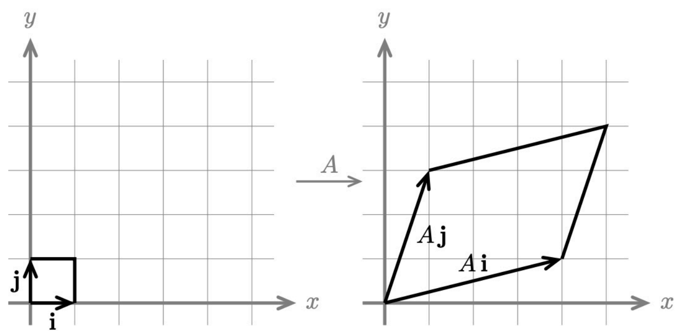

图 2: vector

1.11 Solving Square Systems of Linear Equations; Inverse Matrices

$$MA = I$$
$$A\mathbf{x} = \mathbf{b}$$
$$M(A\mathbf{x}) = M\mathbf{b}$$
$$\mathbf{x} = M\mathbf{b}$$

We can solve $\mathbf{y} = M\mathbf{x}$ by solving $\mathbf{x} = A\mathbf{y}$ where A is the inverse matrix of M.

Inverse Matrices

Let A be an $n \times n$ matrix, with $|A| \neq 0$. Then the inverse of A is an $n \times n$ matrix, written A^{-1}.

$$A^{-1}A = I_n$$

$$A^{-1} = \frac{1}{|A|} adj A = \frac{1}{|A|} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}^T$$

where A_{ij} is the cofactor of the element a_{ij}.

1.12 Equations of Planes II

Planes in point-normal form

If $\mathbf{N}$ is orthogonal to the plane, we call $\mathbf{N}$ the *normal* vector of the plane. And if P_0 is on the plane, then the equation of the plane would be:

$$\begin{aligned} &\mathbf{N} \cdot \overrightarrow{P_0P} = 0 \\ &\Leftrightarrow \langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0 \\ &\Leftrightarrow a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \end{aligned}$$

Example 1 *Find the plane containing the points* $P1 = (1, 2, 3)$*,* $P2 = (0, 0, 3)$*,* $P3 = (2, 5, 5)$*.*

$$\mathbf{N} = \overrightarrow{\mathbf{P_1P_2}} \times \overrightarrow{\mathbf{P_1P_3}} = \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 0 \\ 1 & 3 & 2 \end{pmatrix} = -4\mathbf{i} - \mathbf{j}(-2) + \mathbf{k}(-1) = \langle -4, 2, 1\rangle$$

1.13 Geometry of linear systems of equations

The geometric picture makes this obvious. Here are the three possibilities.

1. The two lines intersect in a point, so there is one solution.
2. The two lines are parallel (and not the same), so there are no solutions.
3. The two lines are the same, so there are an infinite number of solutions.

3 × 3 systems

1. Intersect in a point (1 solution to system).
2. Intersect in a line (∞ solutions).
    (a) Three different planes, the third plane contains the line of intersection of the first two.
    (b) Two planes are the same, the third plane intersects them in a line.
3. Intersect in a plane (∞ solutions)
    (a) All three planes are the same.
4. The planes don' t all intersect at any point (0 solutions).
    (a) The planes are different, but all parallel.
    (b) Two planes are parallel, the third crosses them.
    (c) The planes are different and none are parallel. but the lines of intersection of each pair are parallel.
    (d) Two planes are the same and parallel to the third.

1.14 Solutions to linear systems

1.15 Parametric equations of lines

General parametric equations Given a point (x, y, z), Thinking the position of the point depends on t we written

$$x = x(t), y = y(t), z = z(t)$$

We call t the parameter and the equations for x, y and z are called *parametric equations.*

Parametric equations of lines In general, the line through $P_0 = (x_0, y_0, z_0)$ in the direction of $\mathbf{v}\, \langle v_1, v_2, v_3 \rangle$ has parametrization

$$\langle x, y, z \rangle = \langle x_0 + tv_1, y_0 + tv_2, z_0 + tv_3 \rangle$$

1.16 Intersection of a line and a plane

To find all points of intersection of P with the line, we substitute the formulas for x, y and z into the equation for P and solve for t.

1.17 Parametric Curves