Jacobian and the
Change of Variables in Double Integrals

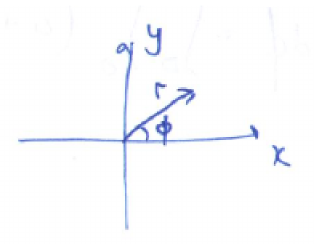
In our discussion of the integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2} r\,dr\,d\phi = \pi,$$

we have performed a change of variables

$$dx\,dy = r\,dr\,d\phi$$

to change from Cartesian (planar) to polar coordinates, with the relationships:

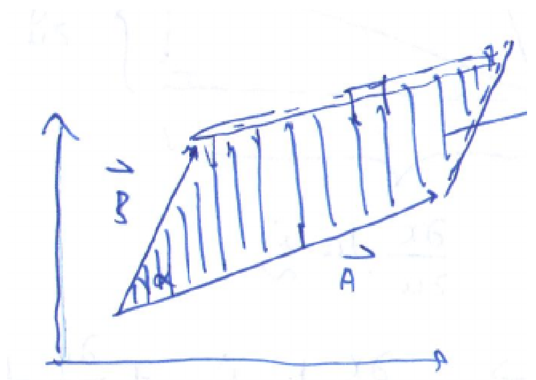


$$\begin{cases} x = r\cos\phi \\ y = r\sin\phi \end{cases}$$

This supplementary note covers the mathematics behind this step.

Vector Algebra—Area of a Parallelogram

The area of parallelogram described by two vectors $\vec{A}$ and $\vec{B}$ is given by the vector cross-product Area = $\left|\vec{A}\times\vec{B}\right|$.



Suppose

$$\begin{cases} \vec{A} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k} \\ \vec{B} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k} \end{cases}$$

Its area is given by

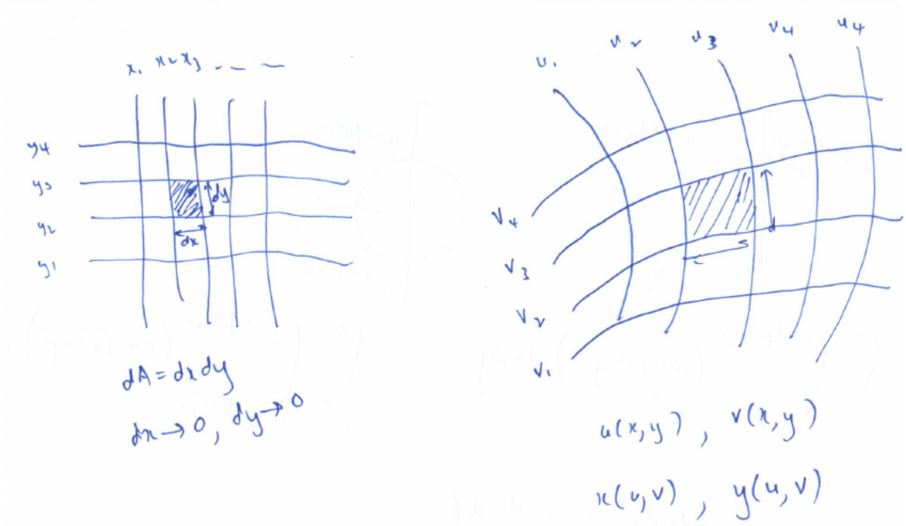
$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

Evaluating the determinant yields

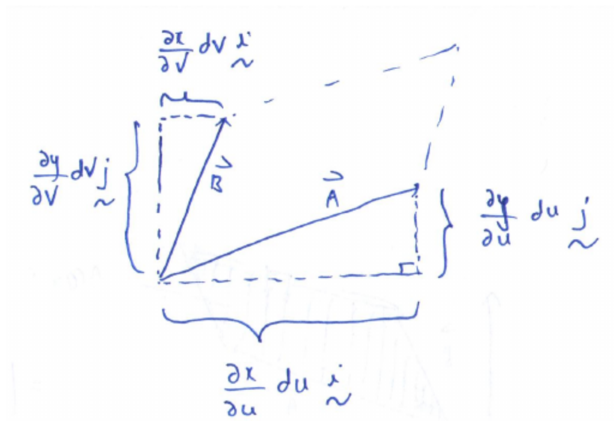
$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = a_yb_z\hat{i} + a_zb_x\hat{j} + a_xb_y\hat{k} - a_yb_x\hat{k} - a_xb_z\hat{j} - a_zb_y\hat{i}$$

Effect of Changing Coordinates

When we change the coordinates of the infinitesimal area $dA = dx \cdot dy$ in the Cartesian coordinates (x, y) into another coordinate system (u, v), we need to account for the change in the shape of this area. The sketch below provides an illustration of the intuition behind the change:



In short, a rectangular shape in the (x, y) coordinates becomes a parallelogram under the (u, v) coordinates. We make use of the vector algebra above to calculate this area:



Note that

$$\begin{cases} \vec{A} = \dfrac{\partial x}{\partial u} du \hat{\imath} + \dfrac{\partial y}{\partial u} du \hat{\jmath} \\ \vec{B} = \dfrac{\partial x}{\partial v} dv \hat{\imath} + \dfrac{\partial y}{\partial v} dv \hat{\jmath} \end{cases}$$

The cross-product is given by

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial x}{\partial u} du & \frac{\partial y}{\partial u} du & 0 \\ \frac{\partial x}{\partial v} dv & \frac{\partial y}{\partial v} dv & 0 \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} du\, dv = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} du\, dv\, \hat{k}.$$

The area is given by

$$\left| \vec{A} \times \vec{B} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} du\, dv.$$

Jacobian

Having established the infinitesimal area under the new coordinate, we write

$$dA = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} du\ dv = \left| \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right| du\ dv.$$

We call this the Jacobian of (x, y) with respective to (u, v), i.e.

$$J = \frac{\partial(x,y)}{\partial(u,v)} \equiv \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

In summary, we have established that

$$dx\ dy = |J|\ du\ dv.$$

Coming back to our Cartesian to polar coordinates transformation, we have

$$x = r\cos\phi, \qquad y = r\sin\phi.$$

We can think of it as $u \to r$ and $v \to \phi$. Their derivatives are given by

$$\frac{\partial x}{\partial r} = \cos\phi, \quad \frac{\partial x}{\partial \phi} = -r\sin\phi, \quad \frac{\partial y}{\partial r} = \sin\phi, \quad \frac{\partial y}{\partial \phi} = r\cos\phi,$$

giving us the Jacobian

$$J = \begin{vmatrix} \cos\phi & \sin\phi \\ -r\sin\phi & r\cos\phi \end{vmatrix} = r\cos^2\phi + r\sin^2\phi = r,$$

leading to

$$dx\ dy = r\ dr\ d\phi.$$