QF620 Additional Examples
Session 7: Equivalent Martingale Measure

1 Examples

1. Consider the following stochastic differential equation

$$dX_t = 0.16dt + 4.563dW_t,$$

where W_t is a $\mathbb{P}$-Brownian motion. Is there a probability measure $\tilde{\mathbb{P}}$ under which the drift rate of X_t is -0.16 instead?

2. Consider the following stochastic differential equation

$$dY_t = \sigma dW_t,$$

where W_t is a $\mathbb{P}$-Brownian motion. Express the stochastic differential equation of Y_t using a $\tilde{\mathbb{P}}$-Brownian motion so that it drifts at the risk-free rate r.

3. In the real-world probability measure $\mathbb{P}$, a process Z_t follows the stochastic differential equation

$$dZ_t = \mu Z_t dt + \sigma Z_t dW_t,$$

where W_t is a $\mathbb{P}$-Brownian motion. Find a way to express Z_t in another measure $\tilde{\mathbb{P}}$ such that the drift of Z_t under $\tilde{\mathbb{P}}$ is νdt instead of μdt.

4. In the real-world probability measure $\mathbb{P}$, the stock price process S_t follows the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where W_t is a $\mathbb{P}$-Brownian motion. What is the stochastic differential equation followed by the forward price process F_t in the real-world measure?

5. Show that by changing measure to the risk-neutral measure $\mathbb{Q}$, the forward price process becomes a martingale.

6. You have \$10 and will be tossing a coin twice. Each time you get a head, ***your wealth increases by \$2***, and each time you get a tail, ***your wealth decreases by \$2***. You believe that the coin is fair, and hence the probability of obtaining a tail or a head is the same at $\frac{1}{2}$. Your opponent believes that the coin is biased and that the probability of getting a head is $\frac{1}{4}$, while the probability of getting a tail is $\frac{3}{4}$. Let X_2 denote the value of your wealth after two tosses. Let $\mathbb{Q}$ denote your probability measure and let $\mathbb{P}$ denote your opponent's probability measure. What is your expectation of your wealth after two tosses, i.e. $\mathbb{E}^{\mathbb{Q}}[X_2]$? What is your opponent's expectation of your wealth after two tosses, i.e. $\mathbb{E}^{\mathbb{P}}[X_2]$?

7. In the same setting as the previous question, evaluate

$$\mathbb{E}^{\mathbb{Q}}\left[X_2\frac{d\mathbb{P}}{d\mathbb{Q}}\right] \qquad \text{and} \qquad \mathbb{E}^{\mathbb{P}}\left[X_2\frac{d\mathbb{Q}}{d\mathbb{P}}\right].$$

8. Let $\mathbb{P}$ denote the real-world probability measure, and W_t denote a $\mathbb{P}$-Brownian motion. A stock price process follows the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t.$$

In the market, there is also a risk-free bond whose price process follows the differential equation

$$dB_t = rB_t dt.$$

Let $\mathbb{Q}$ denote the risk-neutral probability measure. What is $\mathbb{E}^{\mathbb{P}}[S_T]$ and $\mathbb{E}^{\mathbb{Q}}[S_T]$?

9. Let $\mathbb{P}$ denote the real-world probability measure and $\mathbb{Q}$ denote the risk-neutral probability measure. Let W_t be a $\mathbb{P}$-Brownian motion and W_t^B be a $\mathbb{Q}$-Brownian motion. We have

$$\begin{cases} dS_t = \mu S_t dt + \sigma S_t dW_t \\ dS_t = rS_t dt + \sigma S_t dW_t^B. \end{cases}$$

Show that $\mathbb{E}^{\mathbb{P}}\left[S_T\frac{d\mathbb{Q}}{d\mathbb{P}}\right] = \mathbb{E}^{\mathbb{Q}}[S_T] = S_0e^{rT}$.

10. In a similar setting as the previous question, now show that $\mathbb{E}^{\mathbb{Q}}\left[S_T\frac{d\mathbb{P}}{d\mathbb{Q}}\right] = \mathbb{E}^{\mathbb{P}}[S_T] = S_0e^{\mu T}$.

11. Consider the following stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where W_t is a $\mathbb{P}$-Brownian motion in the real-world probability measure. Determine the probability of the event $\{S_T > K\}$ under the real-world probability measure.

12. Consider two tradable assets in the financial market:

$$\begin{cases} dS_t = \mu S_t dt + \sigma S_t dW_t \\ dB_t = rB_t dt \end{cases}$$

where W_t is a $\mathbb{P}$-Brownian motion in the real-world probability measure. What is the probability of the event $\{S_T > K\}$ under the risk-neutral probability measure $\mathbb{Q}^*$ where the numeraire is the risk-free bond?

13. A European cash-or-nothing digital option pays

$$\mathbb{1}_{K_1<S_T<K_2} = \begin{cases} \$1, & K_1 < S_T < K_2 \\ \$0, & \text{otherwise} \end{cases}$$

on the expiry date T. Derive a valuation formula for this option.

14. Under the risk-neutral measure $\mathbb{Q}^*$ associated to the risk-free bond as the choice of numeraire, the stock price follows the following stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dW_t^*,$$

where W_t^* is a $\mathbb{Q}^*$-Brownian motion. Derive the valuation formula for an option paying $(S_T^2 - K)^+$ on expiry date T.

15. **Discussion** We know that under the risk-neutral measure, all asset ratios are martingales, as long as the expectation is taken under the risk-neutral measure associated with the numeraire. So for instance, we have worked out in Week 5 that $\frac{B_t}{S_t}$ is a martingale as long as we take the expectation under the risk-neutral measure associated with the stock numeraire, i.e. $\mathbb{Q}^S$. To this end, we have shown, using Itô's Formula, that with $Y_t = \frac{B_t}{S_t} = f(B_t, S_t)$, we have

$$\begin{aligned} dY_t &= (r - \mu + \sigma^2)Y_t dt - \sigma Y_t dW_t \\ &= -\sigma Y_t \left(dW_t - \frac{r - \mu + \sigma^2}{\sigma} dt \right) \\ &= -\sigma Y_t dW_t^S, \end{aligned}$$

where W_t^S is a $\mathbb{Q}^S$-Brownian motion, and the measure $\mathbb{Q}^S$ is related to the real-world measure $\mathbb{P}$ by the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}^S}{d\mathbb{P}} = \exp\left[-\frac{1}{2}\kappa^2 T - \kappa W_T \right], \quad \kappa = -\frac{r - \mu + \sigma^2}{\sigma},$$

and

$$dW_t^S = dW_t - \frac{r - \mu + \sigma^2}{\sigma} dt.$$

So under $\mathbb{Q}^S$ measure, the stock price follows the following stochastic differential equation

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t \\ &= \mu S_t dt + \sigma S_t \left(dW_t^S + \frac{r - \mu + \sigma^2}{\sigma} dt \right) \\ &= (r + \sigma^2) S_t dt + \sigma S_t dW_t^S. \end{aligned}$$

This sde can be solved (Itô's Formula to $f(S_t) = \log S_t$) and the solution is given by

$$S_T = S_0 e^{\left(r + \frac{\sigma^2}{2}\right)T + \sigma W_T^S}.$$

The expectation under the $\mathbb{Q}^S$ measure is given by

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}^S}[S_T] &= \mathbb{E}^{\mathbb{Q}^S}\left[S_0 e^{\left(r + \frac{\sigma^2}{2}\right)T + \sigma W_T^S} \right] \\ &= S_0 e^{(r + \sigma^2)T}. \end{aligned}$$

Under the risk-neutral measure $\mathbb{Q}^*$ associated to the risk-free bond B_t as the numeraire, the stock price follows the following stochastic differential equation

$$dS_t = r S_t dt + \sigma S_t dW_t^*.$$

The solution is given by

$$S_T = S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T^*}.$$

The Radon-Nikodym $\frac{d\mathbb{Q}^S}{d\mathbb{Q}^*}$ is given by

$$\frac{d\mathbb{Q}^S}{d\mathbb{Q}^*} = \frac{S_T/S_0}{B_T/B_0} = \frac{S_0 \exp\left[\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T^*\right]/S_0}{B_0 e^{rT}/B_0} = \exp\left(-\frac{1}{2}\sigma^2 T + \sigma W_T^* \right).$$

If we take the expectation under the $\mathbb{Q}^S$ measure, we see that we obtain

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}^S}[S_T] = \mathbb{E}^{\mathbb{Q}^*}\left[S_T \times \frac{d\mathbb{Q}^S}{d\mathbb{Q}^*}\right] &= \mathbb{E}^{\mathbb{Q}^*}\left[S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma W_T^*} \times e^{-\frac{1}{2}\sigma^2 T+\sigma W_T^*}\right] \\
&= S_0 e^{(r-\sigma^2)T}\mathbb{E}^{\mathbb{Q}^S}\left[e^{2\sigma W_T^*}\right] \\
&= S_0 e^{(r-\sigma^2)T} e^{\frac{4\sigma^2 T}{2}} \\
&= S_0 e^{(r+\sigma^2)T}.
\end{aligned}
$$

This is consistent with what we would expect to see. $\triangleleft$

2 Suggested Solutions

1. Note that

$$\begin{aligned} dX_t &= 0.16dt + 4.563dW_t \\ &= -0.16dt + 0.32dt + 4.563dW_t \\ &= -0.16dt + 4.563\left(dW_t + \frac{0.32}{4.563}dt\right) \end{aligned}$$

By Girsanov's Theorem, under the Radon-Nikodym derivative

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \exp\left(-\kappa W_T - \frac{1}{2}\kappa^2 T\right) \qquad \text{where} \qquad \kappa = \frac{0.32}{4.563},$$

there exists a probability measure $\tilde{\mathbb{P}}$ which is equivalent to $\mathbb{P}$ and $\tilde{W}_t$ is a $\tilde{\mathbb{P}}$-Brownian motion, and we have

$$\tilde{W}_t = W_t + \kappa t \quad \Rightarrow \quad d\tilde{W}_t = dW_t + \kappa dt.$$

So

$$dX_t = -0.16dt + 4.563d\tilde{W}_t. \quad \lhd$$

2. We proceed as follow:

$$\begin{aligned} dY_t = \sigma dW_t &= rdt - rdt + \sigma dW_t \\ &= rdt + \sigma\left(dW_t - \frac{r}{\sigma}dt\right). \end{aligned}$$

By Girsanov's Theorem, under the Radon-Nikodym derivative

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \exp\left(-\kappa W_T - \frac{1}{2}\kappa^2 T\right) \qquad \text{where} \qquad \kappa = -\frac{r}{\sigma},$$

there exists a probability measure $\tilde{\mathbb{P}}$ which is equivalent to $\mathbb{P}$ and $\tilde{W}_t$ is a $\tilde{\mathbb{P}}$-Brownian motion, and we have

$$\tilde{W}_t = W_t + \kappa t \quad \Rightarrow \quad d\tilde{W}_t = dW_t + \kappa dt.$$

So

$$\begin{aligned} dY_t &= rdt + \sigma\left(dW_t - \frac{r}{\sigma}dt\right) \\ &= rdt + \sigma d\tilde{W}_t. \quad \lhd \end{aligned}$$

3. First we write

$$\begin{aligned} dZ_t &= \mu Z_t dt + \sigma Z_t dW_t \\ &= \nu Z_t dt - \nu Z_t dt + \mu Z_t dt + \sigma Z_t dW_t \\ &= \nu Z_t dt + \sigma Z_t\left(dW_t + \frac{\mu - \nu}{\sigma}dt\right). \end{aligned}$$

By Girsanov's Theorem, under the Radon-Nikodym derivative

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \exp\left(-\kappa W_T - \frac{1}{2}\kappa^2 T\right) \qquad \text{where} \qquad \kappa = \frac{\mu - \nu}{\sigma},$$

there exists a probability measure $\tilde{\mathbb{P}}$ which is equivalent to $\mathbb{P}$ and $\tilde{W}_t$ is a $\tilde{\mathbb{P}}$-Brownian motion, and we have

$$\begin{aligned} dZ_t &= \nu Z_t dt + \sigma Z_t \left(dW_t + \frac{\mu - \nu}{\sigma} dt \right) \\ &= \nu Z_t dt + \sigma Z_t d\tilde{W}_t. \quad \triangleleft \end{aligned}$$

4. The forward price is defined as $F_t = S_t e^{r(T-t)} = f(t, S_t)$, where $f : \mathbb{R}^2 \to \mathbb{R},\ f(t,x) = xe^{-r(T-t)}$. The partial derivatives are given by

$$\frac{\partial f}{\partial t}(t, S_t) = -rF_t, \qquad \frac{\partial f}{\partial x}(t, S_t) = e^{r(T-t)}, \qquad \frac{\partial^2 f}{\partial x^2}(t, S_t) = 0.$$

Applying Itô's Formula, we obtain

$$\begin{aligned} dF_t &= \frac{\partial f}{\partial t}(t, S_t)dt + \frac{\partial f}{\partial x}(t, S_t)dS_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, S_t)(dS_t)^2 \\ &= -rF_t dt + e^{r(T-t)}(\mu S_t dt + \sigma S_t dW_t) \\ &= (\mu - r)F_t dt + \sigma F_t dW_t. \quad \triangleleft \end{aligned}$$

5. As we've discussed, in the risk-neutral measure, the stock price follows

$$dS_t = rS_t dt + \sigma S_t dW_t^B,$$

where W_t^B is a $\mathbb{Q}$-Brownian motion. The measure change is achieved via the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\kappa W_T - \frac{1}{2}\kappa^2 T \right) \qquad \text{where} \qquad \kappa = \frac{\mu - r}{\sigma},$$

and W_T is a $\mathbb{P}$-Brownian motion. This allows us to define

$$W_t^B = W_t + \kappa t \quad \Rightarrow \quad dW_t^B = dW_t + \kappa dt$$

as a $\mathbb{Q}$-Brownian motion. Substituting, we have

$$\begin{aligned} dF_t &= (\mu - r)F_t dt + \sigma F_t dW_t \\ &= (\mu - r)F_t dt + \sigma F_t \left(dW_t^B - \frac{\mu - r}{\sigma} dt \right) \\ &= (\mu - r)F_t dt - (\mu - r)F_t dt + \sigma F_t dW_t^B \\ &= \sigma F_t dW_t^B. \quad \triangleleft \end{aligned}$$

6. The following binomial trees sketch the wealth process X and the Radon-Nikodym derivative process $\frac{d\mathbb{Q}}{d\mathbb{P}}$:

$$\begin{tikzcd} & & 14 \\ & 12 \arrow[ur] \arrow[dr] & \\ 10 \arrow[ur] \arrow[dr] & & 10 \\ & 8 \arrow[ur] \arrow[dr] & \\ & & 6 \end{tikzcd} \qquad \begin{tikzcd} & & 4 \\ & 2 \arrow[ur] \arrow[dr] & \\ 1 \arrow[ur] \arrow[dr] & & \frac{4}{3} \\ & \frac{2}{3} \arrow[ur] \arrow[dr] & \\ & & \frac{4}{9} \end{tikzcd}$$

Taking expectations, we have

$$\mathbb{E}^{\mathbb{Q}}[X_2] = \frac{1}{2} \cdot \frac{1}{2} \times 14 + 2 \times \frac{1}{2} \cdot \frac{1}{2} \times 10 + \frac{1}{2} \cdot \frac{1}{2} \times 6 = 10$$
$$\mathbb{E}^{\mathbb{P}}[X_2] = \frac{1}{4} \cdot \frac{1}{4} \times 14 + 2 \times \frac{1}{4} \cdot \frac{3}{4} \times 10 + \frac{3}{4} \cdot \frac{3}{4} \times 6 = 8. \quad \triangleleft$$

7. Radon-Nikodym derivative allows us to evaluate expectation with a change of measure, so that $\mathbb{E}^{\mathbb{P}}[X_2] = \mathbb{E}^{\mathbb{Q}}\left[X_2 \frac{d\mathbb{P}}{d\mathbb{Q}}\right]$ and $\mathbb{E}^{\mathbb{Q}}[X_2] = \mathbb{E}^{\mathbb{P}}\left[X_2 \frac{d\mathbb{Q}}{d\mathbb{P}}\right]$. This can be readily verified as follow:

$$\mathbb{E}^{\mathbb{Q}}\left[X_2 \frac{d\mathbb{P}}{d\mathbb{Q}}\right] = \frac{1}{2} \cdot \frac{1}{2} \times 14 \times \frac{1}{4} + 2 \times \frac{1}{2} \cdot \frac{1}{2} \times 10 \times \frac{3}{4} + \frac{1}{2} \cdot \frac{1}{2} \times 6 \times \frac{9}{4} = 8$$
$$\mathbb{E}^{\mathbb{P}}\left[X_2 \frac{d\mathbb{Q}}{d\mathbb{P}}\right] = \frac{1}{4} \cdot \frac{1}{4} \times 14 \times 4 + 2 \times \frac{1}{4} \cdot \frac{3}{4} \times 10 \times \frac{4}{3} + \frac{3}{4} \cdot \frac{3}{4} \times 6 \times \frac{4}{9} = 10. \quad \triangleleft$$

8. Solving the stochastic differential equation of the stock price process directly under the real-world probability measure by applying Itô's formula to the function $f : \mathbb{R} \to \mathbb{R},\ f(x) = \log(x)$, with $X_t = \log S_t = f(S_t)$, we have

$$S_T = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right) T + \sigma W_T\right].$$

Taking expectation under the $\mathbb{P}$ measure, we obtain

$$\mathbb{E}^{\mathbb{P}}[S_T] = S_0 e^{\mu T}.$$

Under the risk-neutral measure $\mathbb{Q}$, the price process $\frac{S_t}{B_t}$ is a martingale. We are able to identify the Radon-Nikodym derivative that allows us to attain this measure change

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\kappa W_T - \frac{1}{2}\kappa^2 T\right) \qquad \text{where} \qquad \kappa = \frac{\mu - r}{\sigma},$$

and a $\mathbb{Q}$-Brownian motion W_t^B, where

$$dW_t^B = dW_t + \frac{\mu - r}{\sigma} dt.$$

Substituting, we obtain

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t \\ &= \mu S_t dt + \sigma S_t \left(dW_t^B - \frac{\mu - r}{\sigma} dt\right) \\ &= r S_t dt + \sigma S_t dW_t^B. \end{aligned}$$

Solving this stochastic differential equation (again by applying Itô's Formula to $X_t = \log S_t$), we obtain

$$S_T = S_0 \exp\left[\left(r - \frac{\sigma^2}{2}\right) T + \sigma W_T^B\right].$$

Taking expectation under the $\mathbb{Q}$ measure, we obtain

$$\mathbb{E}^{\mathbb{Q}}[S_T] = S_0 e^{rT}. \quad \triangleleft$$

9. The 2^{nd} inequality, $\mathbb{E}^{\mathbb{Q}}[S_T] = S_0 e^{rT}$, should be clear. To show that $\mathbb{E}^{\mathbb{P}}\left[S_T \frac{d\mathbb{Q}}{d\mathbb{P}}\right]$ yields the same result, first we note that

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-\kappa W_T - \frac{1}{2}\kappa^2 T} = e^{-\frac{\mu-r}{\sigma}W_T - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T}.$$

We then proceed as follow (remember W_t is a $\mathbb{P}$-Brownian motion):

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[S_T \frac{d\mathbb{Q}}{d\mathbb{P}}\right] &= \mathbb{E}^{\mathbb{P}}\left[\overbrace{S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma W_T}}^{S_T} \overbrace{e^{-\frac{\mu-r}{\sigma}W_T - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T}}^{\frac{d\mathbb{Q}}{d\mathbb{P}}}\right] \\
&= S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)T} e^{-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T} \mathbb{E}^{\mathbb{P}}\left[e^{\left(\sigma - \frac{\mu-r}{\sigma}\right)W_T}\right] \\
&= S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)T} e^{-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T} e^{\frac{\left(\sigma - \frac{\mu-r}{\sigma}\right)^2 T}{2}} \\
&= S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)T} e^{-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T} e^{\frac{\left(\sigma^2 - 2(\mu-r) + \left(\frac{\mu-r}{\sigma}\right)^2\right)T}{2}} \\
&= S_0 e^{rT}. \quad \triangleleft
\end{aligned}$$

10. Same as before, the challenge is to show that $\mathbb{E}^{\mathbb{Q}}\left[S_T \frac{d\mathbb{P}}{d\mathbb{Q}}\right] = S_0 e^{\mu T}$. Note the Radon-Nikodym derivative

$$\begin{aligned}
&\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-\kappa W_T - \frac{1}{2}\kappa^2 T} = e^{-\frac{\mu-r}{\sigma}W_T - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T} \\
\Rightarrow \quad &\frac{d\mathbb{P}}{d\mathbb{Q}} = e^{\frac{\mu-r}{\sigma}W_T + \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T}.
\end{aligned}$$

Also, note the relationship between the $\mathbb{P}$- and $\mathbb{Q}$-Brownian motion:

$$W_t^B = W_t + \kappa T = W_t + \frac{\mu - r}{\sigma}T \quad \Rightarrow \quad W_t = W_t^B - \frac{\mu - r}{\sigma}T,$$

and hence the Radon-Nikodym derivative can be written (in terms of W_t^B) as

$$\begin{aligned}
\frac{d\mathbb{P}}{d\mathbb{Q}} &= e^{\frac{\mu-r}{\sigma}W_T + \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T} \\
&= e^{\frac{\mu-r}{\sigma}(W_T^B - \frac{\mu-r}{\sigma}T) + \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T} \\
&= e^{\frac{\mu-r}{\sigma}W_T^B - \left(\frac{\mu-r}{\sigma}\right)^2 T + \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T} \\
&= e^{\frac{\mu-r}{\sigma}W_T^B - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T}
\end{aligned}$$

We proceed to evaluate the expectation (remember W_t^B is a $\mathbb{Q}$-Brownian motion):

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[S_T \frac{d\mathbb{P}}{d\mathbb{Q}}\right] &= \mathbb{E}^{\mathbb{Q}}\left[\overbrace{S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T^B}}^{S_T} \overbrace{e^{\frac{\mu-r}{\sigma}W_T^B - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T}}^{\frac{d\mathbb{P}}{d\mathbb{Q}}}\right] \\
&= S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T} e^{-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T} \mathbb{E}^{\mathbb{Q}}\left[e^{\left(\sigma + \frac{\mu-r}{\sigma}\right)W_T^B}\right] \\
&= S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T} e^{-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T} e^{\frac{\left(\sigma + \frac{\mu-r}{\sigma}\right)^2 T}{2}} \\
&= S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T} e^{-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T} e^{\frac{\left(\sigma^2 + 2(\mu-r) + \left(\frac{\mu-r}{\sigma}\right)^2\right)T}{2}} \\
&= S_0 e^{\mu T}. \quad \triangleleft
\end{aligned}$$

11. The solution is given by

$$S_T = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma W_T}$$

The probability of the event $\{S_T > K\}$ under the real-world probability measure $\mathbb{P}$ is therefore given by

$$\begin{aligned} S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma W_T} &> K \\ \Rightarrow \quad x &> \frac{\log \frac{K}{S_0} - \left(\mu - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = x^*. \end{aligned}$$

So

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{S_T > K}\right] &= \frac{1}{\sqrt{2\pi}} \int_{x^*}^{\infty} e^{-\frac{x^2}{2}} dx \\ &= \Phi(\infty) - \Phi(x^*) \\ &= 1 - \Phi(x^*) \\ &= \Phi(-x^*) \qquad \left(\because 1 - \Phi(x) = \Phi(-x)\right) \\ &= \Phi\left(\frac{\log \frac{S_0}{K} + \left(\mu - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right). \quad \lhd \end{aligned}$$

12. Under the probability measure $\mathbb{Q}^*$, the stock price follows the stochastic differential equation (show it)

$$dS_t = rS_t dt + \sigma S_t dW_t^*.$$

The solution to this stochastic differential equation is given by

$$S_T = S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T^*}.$$

To determine the probability of the event $\{S_T > K\}$ under $\mathbb{Q}^*$, the inequality to satisfy is

$$\begin{aligned} S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T} &> K \\ \Rightarrow \quad x &> \frac{\log \frac{K}{S_0} - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = x^*. \end{aligned}$$

So

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}^*}\left[\mathbb{1}_{S_T > K}\right] &= \frac{1}{\sqrt{2\pi}} \int_{x^*}^{\infty} e^{-\frac{x^2}{2}} dx \\ &= \Phi(\infty) - \Phi(x^*) \\ &= 1 - \Phi(x^*) \\ &= \Phi(-x^*) \qquad \left(\because 1 - \Phi(x) = \Phi(-x)\right) \\ &= \Phi\left(\frac{\log \frac{S_0}{K} + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right). \quad \lhd \end{aligned}$$

13. S_t follows the stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dW_t^*$$

where W_t^* is a $\mathbb{Q}^*$-Brownian motion. The solution is given by

$$S_T = S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T^*}.$$

Let V_t denote the value of this digital option at time t, the ratio $\frac{V_t}{B_t}$ is a martingale under the $\mathbb{Q}^*$ measure. We have

$$\begin{aligned} \frac{V_0}{B_0} &= \mathbb{E}^{\mathbb{Q}^*}\left[\frac{V_T}{B_T}\right] \\ V_0 &= e^{-rT}\mathbb{E}^{\mathbb{Q}^*}\left[\mathbb{1}_{K_1 < S_T < K_2}\right] \end{aligned}$$

The inequalities to satisfy are:

$$\begin{array}{ccccc} K_1 & < & S_T & < & K_2 \\ K_1 & < & S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}x} & < & K_2 \end{array}$$

The left-hand inequality is

$$\begin{aligned} & K_1 < S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}x} \\ \Rightarrow \quad x_L^* = & \frac{\log\frac{K_1}{S_0} - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} < x. \end{aligned}$$

The right-hand inequality is

$$\begin{aligned} S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}x} &< K_2 \\ \Rightarrow \quad x &< \frac{\log\frac{K_2}{S_0} - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = x_H^*. \end{aligned}$$

And so we can evaluate the expectation

$$\begin{aligned} V_0 &= e^{-rT}\mathbb{E}^{\mathbb{Q}^*}\left[\mathbb{1}_{K_1 < S_T < K_2}\right] \\ &= \frac{e^{-rT}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathbb{1}_{K_1 < S_T < K_2} e^{-\frac{x^2}{2}}dx \\ &= \frac{e^{-rT}}{\sqrt{2\pi}}\int_{-x_L^*}^{x_H^*} e^{-\frac{x^2}{2}}dx \\ &= e^{-rT}\Big[\Phi(x_H^*) - \Phi(x_L^*)\Big]. \quad \triangleleft \end{aligned}$$

14. Solving the SDE

$$dS_t = rS_t dt + \sigma S_t dW_t,$$

we obtain

$$S_T = S_0 \exp\left[\left(r - \frac{1}{2}\sigma^2\right)T + \sigma W_T\right],$$

and hence

$$S_T^2 = S_0^2 \exp\left[\left(2r - \sigma^2\right)T + 2\sigma W_T\right].$$

The option can be valued as follow:

$$V = e^{-rT}\mathbb{E}\left[(S_T^2 - K)^+\right] = \frac{e^{-rT}}{\sqrt{2\pi}} \int_{x^*}^{\infty} \left[S_0^2 e^{(2r-\sigma^2)T + 2\sigma\sqrt{T}x} - K\right] e^{-\frac{x^2}{2}}\, dx,$$

where x^* is given by

$$\begin{aligned} S_0^2 e^{(2r-\sigma^2)T + 2\sigma\sqrt{T}x^*} - K > 0 \\ x^* > \frac{\log\left(\frac{K}{S_0^2}\right) - (2r-\sigma^2)T}{2\sigma\sqrt{T}}. \end{aligned}$$

Carrying on with the integration by completing the square for the first part, we obtain

$$\begin{aligned} V &= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{x^*}^{\infty} S_0^2 e^{(2r-\sigma^2)T} e^{2\sigma^2 T} e^{-\frac{(x-2\sigma\sqrt{T})^2}{2}}\, dx - \frac{Ke^{-rT}}{\sqrt{2\pi}} \int_{x^*}^{\infty} e^{-\frac{x^2}{2}}\, dx \\ &= S_0^2 e^{(r+\sigma^2)T} \Phi\left(\frac{\log\frac{S_0^2}{K} + (2r+3\sigma^2)T}{2\sigma\sqrt{T}}\right) - Ke^{-rT}\Phi\left(\frac{\log\frac{S_0^2}{K} + (2r-\sigma^2)T}{2\sigma\sqrt{T}}\right). \quad \triangleleft \end{aligned}$$