Project — Part I (Analytical Option Formulae)

Consider the following European options:

- Vanilla call/put
- Digital cash-or-nothing call/put
- Digital asset-or-nothing call/put

Derive and implement the following models to value these options in Python:

1. Black-Scholes model $S_T = S_0 e^{(r - \frac{\sigma^2}{2})T + \sigma\sqrt{T}x}$
2. Bachelier model
3. Black76 model
4. Displaced-diffusion model

Vanilla call $V_c = e^{-rT} E[(S_T - K)^+]$

put $V_p = e^{-rT} E[(K - S_T)^+]$

Digital cash-or-nothing call $V_c = e^{-rT} E[\mathbb{1}_{S_T > K}]$

put $V_p = e^{-rT} E[\mathbb{1}_{S_T < K}]$

Digital asset-or-nothing call $V_c = e^{-rT} E[\mathbb{1}_{S_T > K} \cdot S_T]$

$V_p = e^{-rT} E[\mathbb{1}_{S_T < K} \cdot S_T]$

Black-scholes model $C_{van} = S_0 \Phi(d_1) - K e^{-rT} \Phi(d_2)$

$$d_1 = \frac{\log \frac{S_0}{K} + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

$P_{van} = K e^{-rT} \Phi(-d_2) - S_0 \Phi(-d_1)$

$d_2 = d_1 - \sigma\sqrt{T}$

$C_{DCN} = cash \times e^{-rT} \Phi(d_2)$

$P_{DCN} = cash \times e^{-rT} \Phi(-d_2)$

$C_{DAN} = S_0 \Phi(d_1)$

$P_{DAN} = S_0 \Phi(-d_1)$

Bachelier

Bachelier Model

$$C_{van} = e^{-rT}\left[(S_0 - K)\,\Phi(-x^*) + \sigma\sqrt{T}\,\phi(-x^*) \right] \qquad x^* = \frac{K - S_0}{\sigma\sqrt{T}}$$

$$P_{van} = e^{-rT}\left[(K - S_0)\,\Phi(x^*) + \sigma\sqrt{T}\,\phi(x^*) \right]$$

$$C_{DCN} = e^{-rT}\,\Phi(-x^*) \times cash$$

$$P_{DCN} = e^{-rT}\,\Phi(x^*) \times cash$$

$$C_{DAN} = e^{-rT}\left[S_0\,\Phi(-x^*) + \sigma\sqrt{T}\,\phi(-x^*) \right]$$

$$P_{DAN} = e^{-rT}\left[S_0\,\Phi(x^*) - \sigma\sqrt{T}\,\phi(x^*) \right]$$

B76

B76: $F_T = F_0 e^{-\frac{\sigma^2 T}{2} + \sigma W_T}$

BS: $S_T = S_0 e^{(r-\frac{\sigma^2}{2})T + \sigma W_T}$

$x > \dfrac{\log\left(\frac{K}{F_0}\right) + \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}} = x^* = d_1$

$d_2 = d_1 - \sigma\sqrt{T} = \dfrac{\log\left(\frac{K}{F_0}\right) - \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}$

Vanilla:

$$V_c = e^{-rT} E[(F_T - K)^+] = e^{-rT} \int_{x^*}^{\infty} \frac{1}{\sqrt{2\pi}} (F_0 e^{-\frac{\sigma^2 T}{2} + \sigma\sqrt{T} x} - K)\, e^{-\frac{x^2}{2}}\, dx$$

$$= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{x^*}^{\infty} F_0 e^{-\frac{x^2 - 2\sigma\sqrt{T}x + \sigma^2 T}{2}}\, dx - K e^{-rT} \Phi(-x^*)$$

$$= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{x^*}^{\infty} F_0 e^{-\frac{(x - \sigma\sqrt{T})^2}{2}}\, d(x - \sigma\sqrt{T}) - K e^{-rT} \Phi(-x^*)$$

$$= F_0 e^{-rT} \Phi(\sigma\sqrt{T} - x^*) - K e^{-rT} \Phi(-x^*)$$

$$= e^{-rT} [F_0 \Phi(-d_2) - K \Phi(-d_1)]$$

$$V_p = e^{-rT} [K \Phi(d_1) - F_0 \Phi(d_2)]$$

$$\boxed{B76(F_0, K, \sigma, r, T) = BS(F_0 e^{-rT}, K, \sigma, r, T)}$$

PCN:

$V_c = e^{-rT} \Phi(-d_1)$

$V_p = e^{-rT} \Phi(d_1)$

DAN:

$V_c = e^{-rT} F_0 \Phi(-d_2)$

$V_p = e^{-rT} F_0 \Phi(d_2)$

DD

Displaced - Diffusion $(F_0, K, r, \sigma, T, \beta) = \text{Black 76} \left(\frac{F_0}{\beta}, K + \frac{1-\beta}{\beta} F_0, r, \sigma\beta, T\right)$

Project — Part II (Model Calibration)

On 1-Dec-2020, the S&P500 (SPX) index value was 3662.45, while the SPDR S&P500 Exchange Traded Fund (SPY) stock price was 366.02. The call and put option prices (bid & offer) over 3 maturities are provided in the spreadsheet:

- `SPX_options.csv`
- `SPY_options.csv`

The discount rate on this day is in the file: `zero_rates_20201201.csv`.

Calibrate the following models to match the option prices:

1. Displaced-diffusion model
2. SABR model (fix $\beta = 0.7$)

Plot the fitted implied volatility smile against the market data.

Report the model parameters:

1. σ, β
2. α, ρ, ν

And discuss how does change β in the displaced-diffusion model and ρ, ν in the SABR model affect the shape of the implied volatility smile.

Project — Part III (Static Replication)

Suppose on 1-Dec-2020, we need to evaluate an exotic European derivative expiring on 15-Jan-2021 which pays:

1. Payoff function:

$$S_T^{1/3} + 1.5 \times \log(S_T) + 10.0$$

2. "Model-free" integrated variance:

$$\sigma^2_{\mathsf{MF}} T = \mathbb{E}\left[\int_0^T \sigma_t^2 \, dt\right]$$

Determine the price of these 2 derivative contracts if we use:

1. Black-Scholes model (what σ should we use?)
2. Bachelier model (what σ should we use?)
3. Static-replication of European payoff (using the SABR model calibrated in the previous question)

Project — Part IV (Dynamic Hedging)

Suppose $S_0 = \$100$, $\sigma = 0.2$, $r = 5\%$, $T = \frac{1}{12}$ year, i.e. 1 month, and $K = \$100$. Use a Black-Scholes model to simulate the stock price. Suppose we sell this at-the-money call option, and we hedge N times during the life of the call option. Assume there are 21 trading days over the month.

The dynamic hedging strategy for an option is

$$C_t = \phi_t S_t - \psi_t B_t,$$

where

$$\phi_t = \frac{\partial C}{\partial S} = \Phi\left(\frac{\log\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right),$$

and

$$\psi_t B_t = -Ke^{-r(T-t)}\Phi\left(\frac{\log\left(\frac{S_t}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right).$$

Project — Part IV (Dynamic Hedging)

Work out the hedging error of the dynamic delta hedging strategy by comparing the replicated position based on ϕ and ψ with the final call option payoff at maturity.

Use 50,000 paths in your simulation, and plot the histogram of the hedging error for $N = 21$ and $N = 84$.

Reference: `http://pricing.free.fr/docs/when_you_cannot_hedge.pdf`

Project Report

Deadline: 15-Nov-23 (Wednesday) noon.

Please submit

- Project report (no more than 10 pages, including title page and appendix)
- Python codes (1 file for each part, 4 files overall)