SMU
SINGAPORE MANAGEMENT UNIVERSITY

Session 2: Binomial Tree and the Risk-Neutral Measure

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QF620 Stochastic Modelling in Finance

MGF of Normal Distribution

Let $X \sim N(\mu, \sigma^2)$ denote a random variable following normal distribution with a mean of μ and a variance of σ^2. The moment generating function is

$$M_X(\theta) = \mathbb{E}\left[e^{\theta X}\right] = \exp\left(\mu\theta + \frac{1}{2}\sigma^2\theta^2\right).$$

This can be obtained by the **completing the square** method:

$$\begin{aligned}
\mathbb{E}\left[e^{\theta X}\right] &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{\theta x} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx \\
&= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} \exp\left[\theta x - \frac{x^2 - 2\mu x + \mu^2}{2\sigma^2}\right] dx \\
&= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} \exp\left[-\frac{-2\sigma^2\theta x + x^2 - 2\mu x + \mu^2}{2\sigma^2}\right] dx \\
&= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} \exp\left[-\frac{x^2 - 2(\mu+\sigma^2\theta)x + \mu^2}{2\sigma^2}\right] dx \\
&= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} \exp\left[-\frac{x^2 - 2(\mu+\sigma^2\theta)x + (\mu^2 + 2\mu\sigma^2\theta + \sigma^4\theta^2) - 2\mu\sigma^2\theta - \sigma^4\theta^2}{2\sigma^2}\right] dx
\end{aligned}$$

MGF of Normal Distribution

$$
\begin{aligned}
&= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} \exp\left[-\frac{x^2 - 2(\mu+\sigma^2\theta)x + (\mu+\sigma^2\theta)^2}{2\sigma^2} + \mu\theta + \frac{\sigma^2\theta^2}{2}\right] dx \\
&= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} \exp\left[-\frac{[x-(\mu+\sigma^2\theta)]^2}{2\sigma^2} + \mu\theta + \frac{\sigma^2\theta^2}{2}\right] dx \\
&= e^{\mu\theta + \frac{\sigma^2\theta^2}{2}} \underbrace{\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} \exp\left[-\frac{[x-(\mu+\sigma^2\theta)]^2}{2\sigma^2}\right] dx}_{1} = e^{\mu\theta + \frac{\sigma^2\theta^2}{2}}
\end{aligned}
$$

Hence for $X \sim N(\mu, \sigma^2)$, we have

$$M_X(\theta) = \exp\left(\mu\theta + \frac{1}{2}\sigma^2\theta^2\right).$$

For standard normal random variable $Z \sim N(0,1)$, the MGF simplifies to

$$M_Z(\theta) = \exp\left(\frac{\theta^2}{2}\right).$$

Lognormal Random Variables

Again let $X \sim N(\mu, \sigma^2)$ denote a normally distributed random variable. We say that

$$Y = e^X$$

is a **lognormal random variable**, because

$$\ln(Y) \sim N(\mu, \sigma^2).$$

We can use the MGF we have derived for normal distribution to evaluate expectation of lognormal random variables. For example

$$\mathbb{E}[Y] = \mathbb{E}[e^X] = M_X(1) = \exp\left(\mu + \frac{1}{2}\sigma^2\right).$$

Example In QF, we often use normal distribution to model returns, and lognormal distribution to model asset prices. Why this is a sensible choice?

Changing Probability Measure

A **probability space** is used to model experiments with different possible outcomes.

It is made up of the **sample space** Ω, which is a set of all possible outcomes, and a **probability measure** $\mathbb{P}$, which assigns a probability to each element of Ω.

In short, $(\Omega, \mathbb{P})$ denotes a probability space. A random variable can take either value of Ω, and the likelihood of it taking a specific value $\omega \in \Omega$ is determined by the probability measure $\mathbb{P}$.

In other words, Ω lists the set of possible outcomes, and $\mathbb{P}$ determines the distribution.

Changing the probability measure affects the **likelihood** of each price-path being realized. Given two probability measures, we define the **Radon-Nikodym derivative** of the probability $\mathbb{P}$ with respect to the probability measure $\mathbb{Q}$ as

$$\frac{d\mathbb{P}}{d\mathbb{Q}}(\omega) = \frac{\mathbb{P}(\omega)}{\mathbb{Q}(\omega)},$$

which relates the two probability measure.

Radon-Nikodym Derivative

From the Radon-Nikodym derivative, we can derive $\mathbb{Q}$ from $\mathbb{P}$, or vice versa.

Consider the case where $\mathbb{P}(\omega_i) = 0$, but $\forall i : \mathbb{Q}(\omega_i) \neq 0$. In this case, not all of the ratio $\frac{\mathbb{P}(\omega_i)}{\mathbb{Q}(\omega_i)}$ is defined, hence the Radon-Nikodym derivative does not exist.

If we take them away from our analysis, then we will be losing information: these paths may be $\mathbb{P}$-impossible, but they are $\mathbb{Q}$-possible.

Throwing them away will cause us to lose information about $\mathbb{Q}$. In short, $\frac{d\mathbb{P}}{d\mathbb{Q}}$ is undefined if $\mathbb{Q}$ allows something which $\mathbb{P}$ doesn't, and vice versa. This leads us to the important concept of **equivalence**.

Equivalence of Probability Measure

Two measures $\mathbb{P}$ and $\mathbb{Q}$ are equivalent if they operate on the same sample space Ω and they agree on what is possible and impossible. If ω is any event in the sample space, then

$$\mathbb{P}(\omega) > 0 \quad \Leftrightarrow \quad \mathbb{Q}(\omega) > 0.$$

Radon-Nikodym Derivative

Example A stock is worth \$100 today. Assume interest rate is 0. Consider a two-state model where the stock can increase/decrease its value by \$10.

- A hedge fund manager has a bullish view on this speculative stock, thinking that the stock price will increase with 80% probability per period.
- A market maker takes a risk-neutral view on the same stock, under which the expected stock price two periods later is the same as today.

Note that the fund manager works under an empirical probability measure $\mathbb{P}$, and the market maker works under the risk-neutral measure $\mathbb{Q}$.
The stock process and the Radon-Nikodym process $\frac{d\mathbb{Q}}{d\mathbb{P}}$ are:

$$
\begin{tikzcd}
& & 120 \\
& 110 \arrow[ur] \arrow[dr] & \\
100 \arrow[ur] \arrow[dr] & & 100 \\
& 90 \arrow[ur] \arrow[dr] & \\
& & 80
\end{tikzcd}
\qquad
\begin{tikzcd}
& & \frac{25}{64} \\
& \frac{5}{8} \arrow[ur] \arrow[dr] & \\
1 \arrow[ur] \arrow[dr] & & \frac{25}{16} \\
& \frac{5}{2} \arrow[ur] \arrow[dr] & \\
& & \frac{25}{4}
\end{tikzcd}
$$

Radon-Nikodym Derivative

Let S_i denote the stock price at time period $i = 0, 1, 2$. The hedge fund manager has the following expectation:

$$\mathbb{E}^P[S_2] = 0.8^2 \times 120 + 2 \times 0.8 \times 0.2 \times 100 + 0.2^2 \times 80 = 112$$

The market maker, under risk-neutral measures, expects

$$\mathbb{E}^Q[S_2] = 0.5^2 \times 120 + 2 \times 0.5 \times 0.5 \times 100 + 0.5^2 \times 80 = 100$$

The market maker can **enforce** this expectation by borrowing 100 and buy the stock today. At $t = 2$, the market maker has a debt of exactly 100 to repay.

The hedge fund manager's expectation of 112 is **non-enforceable**—there is risk associated.

Radon-Nikodym derivative gives us a mean to relate this two expectations. We have

$$\mathbb{E}^Q[S_2] = \mathbb{E}^P\left[S_2 \cdot \frac{d\mathbb{Q}}{d\mathbb{P}}\right] \quad \text{and} \quad \mathbb{E}^P[S_2] = \mathbb{E}^Q\left[S_2 \cdot \frac{d\mathbb{P}}{d\mathbb{Q}}\right]$$

Radon-Nikodym Derivative

To see this relationship, note that

$$\begin{aligned}\mathbb{E}^P\left[S_2\cdot\frac{d\mathbb{Q}}{d\mathbb{P}}\right] &= 0.8^2\cdot 120\cdot\frac{25}{64} + 2\cdot 0.8\cdot 0.2\cdot 100\cdot\frac{25}{16} + 0.2^2\cdot 80\cdot\frac{25}{4}\\ &= 100 = \mathbb{E}^Q[S_2].\end{aligned}$$

Notice how the Radon-Nikodym derivative reduces the weight on the higher stock values, and increases the weight on the lower stock values, bringing down the expectation.

Similarly, we have

$$\begin{aligned}\mathbb{E}^Q\left[S_2\cdot\frac{d\mathbb{P}}{d\mathbb{Q}}\right] &= 0.5^2\cdot 120\cdot\frac{64}{25} + 2\cdot 0.5\cdot 0.5\cdot 100\cdot\frac{16}{25} + 0.5^2\cdot 80\cdot\frac{4}{25}\\ &= 112 = \mathbb{E}^P[S_2].\end{aligned}$$

Radon-Nikodym derivative allows us to relate the expectation of one probability measure to the other.

Normal (Arithmetic) Binomial Tree

Suppose that interest rate is 0. Consider a binomial tree model where in each time step the stock moves up or down with the same probability. Denote today's stock price as S_0, on the maturity date T, we want:

1. $\mathbb{E}[S_T] = S_0$
2. $V[S_T] = \sigma^2 T$

To this end, we:

- Divide the time interval from 0 to T into n uniform steps
- For each time step we require a mean of 0 and a variance of $\frac{\sigma^2 T}{n}$.
- Hence, for each time step, the stock moves up or down by $\sigma\sqrt{\frac{T}{n}}$ with probability $\frac{1}{2}$.

Let X_i denote a sequence of independent random variables taking values $+1$ or -1 with equal probability $\frac{1}{2}$, after n steps, the stock will be distributed as

$$S_T = S_0 + \sum_{i=1}^{n} \sigma\sqrt{\frac{T}{n}} X_i.$$

Normal (Arithmetic) Binomial Tree

Central Limit Theorem (CLT)

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with mean μ and variance s^2, and let

$$Y_n = X_1 + X_2 + \cdots + X_n.$$

Then the distribution of

$$Z = \frac{Y_n - n\mu}{\sqrt{ns^2}}$$

converges to a standard normal random variable with $N(0,1)$ as $n \to \infty$.

Suppose we send the limit of $n \to \infty$, CLT states that:

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \quad \sim \quad N(0,1).$$

Hence the distribution of the stock price converges to

$$S_T = S_0 + \sigma\sqrt{T} Z, \quad Z \sim N(0,1).$$

Normal (Arithmetic) Binomial Tree

Example Show that the stock price distribution

$$S_T = S_0 + \sigma\sqrt{T}Z, \quad Z \sim N(0,1)$$

satisfies the requirements of $\mathbb{E}[S_T] = S_0$ and $V[S_T] = \sigma^2 T$.

Non-zero Interest Rates

Suppose the market also consists of a discretely compounded risk-free interest rate r, such that investing \$1 at time $t = 0$ will yield a return of $\$(1 + r)$ one period later.

If a stock is worth S_0 today, and will be worth either S_1^u or S_1^d in the next period, then the following inequality must be satisfied for there to be no-arbitrage

$$S_1^d < S_0(1 + r) < S_1^u.$$

Going through the same replication argument as we have done earlier, under the risk-neutral formulation, we now require the risk-neutral probability to make the stock grow at the risk-free rate, i.e.

$$\mathbb{E}^*[S_1] = p^* S_1^u + (1 - p^*) S_1^d = S_0(1 + r).$$

Rearranging, we have

$$p^* = \frac{S_0(1 + r) - S_1^d}{S_1^u - S_1^d}.$$

Lognormal (Geometric) Binomial Tree

For the equity asset class, we tend to think in terms of a **geometric process** (i.e. multiplicative adjustment) rather than an **arithmetic process** (i.e. additive adjustment).

$$
\begin{tikzcd}
 & S_1^u = u \times S_0 \\
S_0 \arrow[ur] \arrow[dr] & \\
 & S_1^d = d \times S_0
\end{tikzcd}
$$

where

$$u = \frac{S_1(H)}{S_0}, \qquad d = \frac{S_1(T)}{S_0}.$$

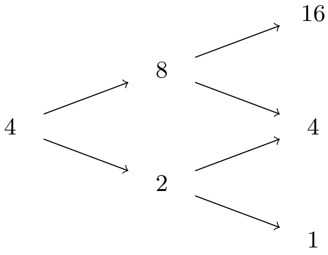
In this case, our risk-neutral probability of an up jump becomes

$$p^* = \frac{S_0(1+r) - S_1^d}{S_1^u - S_1^d} = \frac{(1+r) - d}{u - d}.$$

Under the **Cox-Ross-Rubinstein** formulation, it is common to choose $d = \frac{1}{u}$.

Lognormal (Geometric) Binomial Tree

Example Consider a stock currently worth \$4. Given the lognormal binomial tree parameter of $u = 2$, $d = 0.5$, $r = 25\%$, calculate the price of a European put option struck at \$5 today ($t = 0$), which expires 2 periods later ($t = 2$).



American Options under Binomial Tree

An **American option** can be exercised on any time before expiry. We can use binomial tree to capture the early exercise premium:

$\Rightarrow$ We should only exercise if we make more money by exercising the option.

For European options, risk-neutral expectation is given by

$$V_n^E = \frac{1}{1+r}\mathbb{E}_n^*[V_{n+1}] = \frac{1}{1+r}[p^* \times V_{n+1}^u + q^* \times V_{n+1}^d]$$

For American options, this should become

$$V_n^A = \max\left\{\frac{1}{1+r}[p^* \times V_{n+1}^u + q^* \times V_{n+1}^d],\ (K - S_n)^+\right\}$$

Example Determine the price of an American put option for the same example in the previous page.

Central Limit Theorem for Geometric Process

Central Limit Theorem can be used to determine the distribution of the product of independent random variables X_i. Consider the product

$$W = X_1 \times X_2 \times \cdots X_n,$$

as long as $X_i \in (0, \infty)$, we can write

$$\ln W = \ln X_1 + \ln X_2 + \cdots + \ln X_n.$$

This is just the sum of n independent random variables $\ln X_i$.

As long as each of these random variables has a finite mean and variance, the distribution of $\ln W$ will tend towards a normal distribution under the limit $n \to \infty$.

In other words, W will follow a **lognormal distribution** by Central Limit Theorem.

Lognormal (Geometric) Binomial Tree

Suppose we now model the **log-price process**, and also take interest rate into consideration. We want

1. $\mathbb{E}[\ln(S_T)] = \ln(S_0) + \left(r - \frac{\sigma^2}{2}\right) T$
2. $V[\ln(S_T)] = \sigma^2 T$

Following the same approach as the normal binomial tree, we divide the time interval into n uniform steps, and have the following relationship:

$$\ln(S_i) = \ln(S_{i-1}) + \left(r - \frac{1}{2}\sigma^2\right)\frac{T}{n} + \sigma\sqrt{\frac{T}{n}}X_i,$$

where X_i takes on values ± 1 with equal probability $\frac{1}{2}$. Adding them up, we have

$$\ln(S_T) = \ln(S_0) + \left(r - \frac{1}{2}\sigma^2\right) T + \sigma\sqrt{T}\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i.$$

Lognormal (Geometric) Binomial Tree

Once again, we can apply Central Limit Theorem to show that

$$\ln(S_T) = \ln(S_0) + \left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}Z, \qquad Z \sim N(0,1)$$

$$\Rightarrow S_T = S_0 \exp\left[\left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}Z\right].$$

Example Show that the distribution of the log stock price satisfies the requirement of $\mathbb{E}[\ln(S_T)] = \ln(S_0) + \left(r - \frac{\sigma^2}{2}\right)T$ and $V[\ln(S_T)] = \sigma^2 T$.

Lognormal (Geometric) Binomial Tree

Example If

$$S_T = S_0 \exp\left[\left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}Z\right], \quad Z \sim N(0,1),$$

use the MGF for normal distribution to show that $\mathbb{E}[S_T] = S_0 e^{rT}$, thus satisfying the no-arbitrage condition under the risk-neutral measure.

Changing Measure – Example

Toss a coin for 3 times, and define 2 random variables:

1. X = total number of heads
2. Y = total number of tails

The random variables can be specified without knowing the probability measure (what are they?).

If we specify the probability measure of a fair coin $p = \frac{1}{2}$, we can determine the distribution of X and Y — they have the same distribution.

If we specify the probability measure of a biased coin $p = \frac{2}{3}$, then we will get a different distribution.

Let X be a random variable defined on a finite probability space $(\Omega, \mathbb{P})$, the expectation of X is defined as

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega)\mathbb{P}(\omega).$$

Conditional Expectation

In our binomial pricing model, we have chosen the risk-neutral probability measure

$$p^* = \frac{(1+r)-d}{u-d}, \qquad q^* = \frac{u-(1+r)}{u-d}.$$

At every time-step t_n, we have

$$S_n = \frac{1}{1+r}[p^* S_{n+1}^u + q^* S_{n+1}^d].$$

Using the expectation notation, we can write

$$\mathbb{E}_n^*[S_{n+1}] = p^* S_{n+1}^u + q^* S_{n+1}^d$$
$$\Rightarrow \quad \therefore \; S_n = \frac{1}{1+r}\mathbb{E}_n^*[S_{n+1}].$$

The notation $\mathbb{E}_n^*[S_{n+1}]$ denotes the **conditional expectation** of S_{n+1} based on the information at time t_n.

Conditional Expectation Properties

Linearity: for all constants a and b, we have

$$\mathbb{E}_n[aX + bY] = a\mathbb{E}_n[X] + b\mathbb{E}_n[Y].$$

Extracting known variable: if X only depends on the first n tosses, then

$$\mathbb{E}_n[XY] = X\mathbb{E}_n[Y].$$

Iterated expectation: if $0 \leq n \leq m \leq N$, then

$$\mathbb{E}_n[\mathbb{E}_m[X]] = \mathbb{E}_n[X].$$

Independence: if X only depends on the $(n+1)^{th}$ toss, then

$$\mathbb{E}_n[X] = \mathbb{E}[X].$$

Martingales

We have derived the conditional expectation earlier as follow

$$S_n = \frac{1}{1+r}\mathbb{E}_n^*[S_{n+1}].$$

Dividing both sides by $(1+r)^n$, we obtain

$$\frac{S_n}{(1+r)^n} = \mathbb{E}_n^*\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right].$$

Under the risk-neutral measure, the best estimate based on the information at time t_n of the value of the discounted stock price at time t_{n+1} is the discounted stock price at time t_n:

$$\begin{aligned} M_n &= \mathbb{E}_n^*[M_{n+1}], \quad n = 0, 1, \ldots, N-1 \\ \therefore\ M_0 &= \mathbb{E}^*[M_N] \end{aligned}$$

Martingales

Consider the general binomial model with $0 < d < 1 + r < u$, with the risk-neutral probabilities

$$p^* = \frac{(1+r) - d}{u - d}, \qquad q^* = \frac{u - (1+r)}{u - d}.$$

Under the risk-neutral measure, the discounted stock price is a **martingale**

$$\begin{aligned}
\mathbb{E}_n^*\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right] &= \mathbb{E}_n^*\left[\frac{S_n}{(1+r)^{n+1}}\frac{S_{n+1}}{S_n}\right] \\
&= \frac{S_n}{(1+r)^n}\frac{1}{1+r}\mathbb{E}_n^*\left[\frac{S_{n+1}}{S_n}\right] \\
&= \frac{S_n}{(1+r)^n}\frac{p^* \times u + q^* \times d}{1+r} \\
&= \frac{S_n}{(1+r)^n}.
\end{aligned}$$

Risk-Neutral Pricing Formula

Consider an N-period binomial tree pricing model with $0 < d < 1 + r < u$, and with the risk-neutral probability measure. Let V_N be a derivative security with payoff at time t_N, depending on the outcomes of coin tosses.

For $0 \leq n \leq N$, the discounted price of the derivative security is a martingale:

$$\frac{V_n}{(1+r)^n} = \mathbb{E}_n^* \left[\frac{V_{n+1}}{(1+r)^{n+1}} \right].$$

Furthermore, the price of the derivative security at time t_n is given by the risk-neutral pricing formula

$$V_n = \mathbb{E}_n^* \left[\frac{V_N}{(1+r)^{N-n}} \right].$$

Arbitrage & Complete Markets

- A **complete market** is one in which a derivative product can be **replicated** from more basic instruments, such as cash and the underlying asset.
- This usually involves **dynamically rebalancing** a portfolio of the simpler instruments, according to some formula or algorithm, to replicate the more complicated product, the derivative.
- In an **incomplete market** (e.g. with trading frictions), you cannot replicate the option with simpler instruments.
- In a complete market you can replicate derivatives with the simpler instruments. But you can also turn this on its head so that you can **hedge** the derivative with the underlying instruments to make a risk-free instrument.
- In the binomial model you can replicate an option from stock and cash, or you can hedge the option with the stock to make cash.

Arbitrage & Complete Markets

- In a complete market, all security prices are attainable, and there is no arbitrage opportunities.
- The absence of arbitrage implies the existence of the risk-neutral measure probabilities.
- Under the **real-world measure**, riskier assets tend to have higher expected return than less risky assets.
- Under the **risk-neutral measure**, asset prices grow at the risk-free rate, and the relative value of any two assets is a martingale over time.
- So far we have used the risk-free interest account as the denominator in our ratio. However, this is not the only choice — the choice is in fact arbitrary.

  $\Rightarrow$ Under the risk-neutral measure, all relative price processes should be a **martingale**.