QF620 Stochastic Modelling
Mid-term Revision Pack

1 Practice Questions

1. Let W_t denote a standard Brownian motion. Derive the stochastic differential equations for dX_t using Itô's formula for the following processes:

   (a) $X_t = t + W_t^2$

   (b) $X_t = 5^{W_t}$

   (c) $X_t = e^{\int_0^t uW_u\, du}$. ***Hint: The Riemann integral is a function of time,*** $f(t) = \int_0^t uW_u du$.

   (d) $X_t = \frac{W_t}{Z_t}$ where $dW_t dZ_t = \rho\, dt$

2. Solve the following stochastic differential equations:

   (a) $dX_t = 5X_t dt + 0.1X_t dW_t$.

   (b) $dX_t = \mu X_t dt + \sigma dW_t$. ***Hint: Apply Itô's formula to*** $f(t, x) = e^{-\mu t}x$.

   (c) $dX_t = \frac{1}{4}(X_t - 1)dW_t$. ***Hint: Displaced-diffusion process.***

3. Let W_t denote a standard Brownian motion.

   (a) Evaluate $\mathbb{E}[W_t^2]$ and $\mathbb{E}[W_t^4]$.

   (b) Evaluate the expectation $\mathbb{E}[|W_T|]$, where $|\cdot|$ denote the absolute value.

   (c) Determine the mean and variance of the following integrals:

      i.
      $$\int_t^T u\, dW_u$$

      ii.
      $$\int_t^T W_u\, du$$

4. Let W_t denote a standard Brownian motion. Derive the stochastic differential equations for dX_t using Itô's formula for the following processes:

   (a) $X_t = W_t^3 - 3tW_t$

   (b) $X_t = t^2 W_t - 2\int_0^t uW_u du$

   (c) $X_t = W_t \tilde{W}_t$ where W_t and $\tilde{W}_t$ are independent Brownian motions

   (d) $X_t = W_t Z_t$ where $dW_t dZ_t = \rho dt$

5. Consider the following 2 stochastic differential equations

$$\begin{cases} dX_t = \sigma_X X_t dW_t^X \\ dY_t = \sigma_Y Y_t dW_t^Y \end{cases}$$

where W_t^X and W_t^Y are two standard-Brownian motions.

(a) Solve the two SDEs using Itô's formula, and write down the expression for $X_T Y_T$.

(b) If W_t^X and W_t^Y are independent, i.e. $\text{Cov}(W_T^X, W_T^Y) = 0$, evaluate $\mathbb{E}[X_T Y_T]$.

(c) If W_t^X and W_t^Y are correlated, i.e. $\text{Cov}(W_T^X, W_T^Y) = \rho T$, evaluate $\mathbb{E}[X_T Y_T]$.

Hint: Combine $W_T^X + W_T^Y$ into a single normally distributed random variable, then apply the moment generating function method to evaluate the expectation.

6. Let W_t denote a Brownian motion. Evaluate the stochastic integral

$$I_t = \int_0^t W_u \, dW_u,$$

and proceed to find the mean and variance of the stochastic integral I_t.

7. Determine whether the following processes X_t are martingales

(a) $X_t = W_t + 4t$

(b) $X_t = W_t^2$

(c) $X_t = W_t Z_t$ where W_t and Z_t are independent Brownian motion

8. Consider a stock price following the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where W_t is a Brownian motion under the real world probability measure, and a risk-free bond following the differential equation

$$dB_t = r B_t dt.$$

Determine the following

(a) $\mathbb{E}[S_T]$, where the expectation is taken under the real world measure.

(b) $\mathbb{E}^*[S_T]$, where the expectation is taken under the risk-neutral measure associated to the risk-free bond numeraire.

9. Let W_t denote a standard Brownian motion. The stochastic variable X_t follows the process

$$dX_t = \mu X_t dt + \sigma X_t dW_t.$$

Derive the stochastic differential equation for dY_t if:

(a)

$$Y_t = 10^{W_t}$$

(b)

$$Y_t = t^2 + W_t^2$$

(c)

$$Y_t = \frac{X_t}{W_t}$$

(d)

$$Y_t = \frac{t}{W_t}$$

10. Use Itô's formula to show that

$$\int_0^t W_u^2 \, dW_u = \frac{W_t^3}{3} - \int_0^t W_u \, du.$$

2 Suggested Solutions

1. (a) Let $f(t,x) = t + x^2$, we have $\frac{\partial f}{\partial t} = 1$, $\frac{\partial f}{\partial x} = 2x$, and $\frac{\partial^2 f}{\partial x^2} = 2$. By Itô's formula, we obtain

$$dX_t = dt + 2W_t dW_t + \frac{1}{2} \cdot 2dt = 2dt + 2W_t dW_t. \quad \triangleleft$$

(b) Let $f(W_t) = 5^{W_t}$, we have $f'(W_t) = 5^{W_t} \log(5)$, and $f''(W_t) = 5^{W_t}(\log(5))^2$. By Itô's formula, we obtain

$$dX_t = X_t \log(5) dW_t + \frac{1}{2} X_t (\log(5))^2 dt. \quad \triangleleft$$

(c) Let $f(t) = e^{\int_0^t uW_u \, du}$, we have $f'(t) = e^{\int_0^t uW_u \, du} \cdot tW_t$. By Itô's formula, we obtain

$$dX_t = tW_t X_t dt. \quad \triangleleft$$

(d) Let $f(w,z) = \frac{w}{z}$, we have

$$\frac{\partial f}{\partial w} = \frac{1}{z}, \quad \frac{\partial^2 f}{\partial w^2} = 0, \quad \frac{\partial f}{\partial z} = -\frac{w}{z^2}, \quad \frac{\partial^2 f}{\partial z^2} = \frac{2w}{z^3}, \quad \frac{\partial^2 f}{\partial w \partial z} = -\frac{1}{z^2}.$$

By Itô's formula, we obtain

$$\begin{aligned} dX_t &= \frac{1}{Z_t} dW_t - \frac{W_t}{Z_t^2} dZ_t + \frac{1}{2} \frac{2W_t}{Z_t^3} dt - \frac{1}{Z_t^2} \rho dt \\ &= \left(\frac{W_t}{Z_t^3} - \frac{1}{Z_t^2} \rho \right) dt + \frac{1}{Z_t} dW_t - \frac{W_t}{Z_t^2} dZ_t. \quad \triangleleft \end{aligned}$$

2. (a) Consider the sde

$$dX_t = \mu X_t dt + \sigma X_t dW_t.$$

Applying Itô's formula to $\log(X_t)$ and integrating both sides, the solution is given by

$$X_T = X_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma W_T}.$$

Substituting $\mu = 5$ and $\sigma = 0.1$, we obtain

$$X_T = X_0 e^{4.995T + 0.1W_T} \quad \triangleleft$$

(b)

$$\begin{aligned} d(e^{-\mu t} X_t) &= -\mu e^{-\mu t} X_t dt + e^{-\mu t} dX_t \\ &= \sigma e^{-\mu t} dW_t \\ X_T &= X_0 e^{\mu T} + \sigma \int_0^T e^{-\mu(s-T)} dW_s. \quad \triangleleft \end{aligned}$$

(c) This is a displaced-diffusion stochastic differential equation. Let $Y_t = f(X_t) = \log(X_t - 1)$, we have

$$f'(X_t) = \frac{1}{X_t - 1}, \qquad f''(X_t) = -\frac{1}{(X_t - 1)^2}.$$

By Itô's Lemma,

$$dY_t = \frac{1}{4} dW_t - \frac{1}{32} dt.$$

Integrating both sides and simplifying, we have

$$X_T = 1 + (X_0 - 1) e^{-\frac{T}{32} + \frac{W_T}{4}}. \quad \triangleleft$$

3. (a) Given that $W_t \sim N(0,t)$, let $X \sim N(0,1)$, we have

$$\mathbb{E}[W_t^2] = \mathbb{E}[tX^2] = t$$
$$\mathbb{E}[W_t^4] = \mathbb{E}[t^2X^4] = 3t^2. \quad \triangleleft$$

(b) First we note that $W_T \sim N(0,T)$. Let $X \sim N(0,1)$, we have

$$\begin{aligned}
\mathbb{E}\Big[|W_T|\Big] &= \mathbb{E}\Big[\sqrt{T}|X|\Big] \\
&= \frac{\sqrt{T}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |x| e^{-\frac{x^2}{2}}\, dx \\
&= \frac{\sqrt{T}}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} -xe^{-\frac{x^2}{2}}\, dx + \int_{0}^{\infty} xe^{-\frac{x^2}{2}}\, dx\right] \\
&= \frac{\sqrt{T}}{\sqrt{2\pi}} \times 2 \times \underbrace{\int_{0}^{\infty} xe^{-\frac{x^2}{2}}\, dx}_{1} \\
&= \sqrt{\frac{2T}{\pi}}. \quad \triangleleft
\end{aligned}$$

(c) i.

$$\begin{aligned}
\mathbb{E}\left[\int_t^T u\, dW_u\right] &= 0 && \because \text{Stochastic integrals have zero mean.} \\
V\left[\int_t^T u\, dW_u\right] &= \int_t^T u^2 du && \because \text{Itô's Isometry} \\
&= \frac{T^3 - t^3}{3} \quad \triangleleft
\end{aligned}$$

ii. Integration of W_t across dt is a Riemann integral, and can be expressed as infinite sum. We show that it can be rearranged into a weighted sum of normal random variable with 0 mean, and hence the expectation is 0:

$$\mathbb{E}\left[\int_t^T W_u\, du\right] = 0$$

For variance, we have

$$\begin{aligned}
V\left[\int_t^T W_u\, du\right] &= \mathbb{E}\left[\int_t^T W_u\, du \times \int_t^T W_s\, ds\right] \\
&= \mathbb{E}\left[\int_t^T \int_t^T W_u W_s\, du\, ds\right] \\
&= \int_t^T \int_t^T \mathbb{E}[W_u W_s]\, du\, ds \\
&= \int_t^T \int_t^T \mathrm{Cov}(W_u, W_s)\, du\, ds \\
&= \int_t^T \int_t^T \min\{u,s\}\, du\, ds \\
&= \int_t^T \left(\int_t^s u\, du + \int_s^T s\, du\right) ds \\
&= \frac{T^3}{3} + \frac{2t^3}{3} - t^2 T \quad \triangleleft
\end{aligned}$$

4. We apply Itô's formula to derive the stochastic differential equations.

(a) Let $f(t,x) = x^3 - 3tx$, first evaluate the partial derivatives

$$\frac{\partial f}{\partial t} = -3x, \quad \frac{\partial f}{\partial x} = 3x^2 - 3t, \quad \frac{\partial^2 f}{\partial x^2} = 6x.$$

The s.d.e. is given by

$$\begin{aligned} dX_t &= -3W_t dt + (3W_t^2 - 3t)dW_t + \frac{1}{2} \cdot 6 \cdot W_t dt \\ &= 3(W_t^2 - t)dW_t. \quad \triangleleft \end{aligned}$$

(b) Let $f(t, W_t) = t^2 W_t - 2\int_0^t uW_u \, du$, first evaluate the partial derivatives

$$\frac{\partial f}{\partial t}(t, W_t) = 2tW_t - 2tW_t = 0, \quad \frac{\partial f}{\partial x}(t, W_t) = t^2, \quad \frac{\partial^2 f}{\partial x^2}(t, W_t) = 0.$$

The s.d.e. is given by

$$dX_t = t^2 dW_t. \quad \triangleleft$$

(c) Let $f(W_t, \tilde{W}_t) = W_t \tilde{W}_t$, first evaluate the partial derivatives, the s.d.e. is given by

$$dX_t = \tilde{W}_t dW_t + W_t d\tilde{W}_t. \quad \triangleleft$$

(d) Let $f(W_t, Z_t) = W_t Z_t$, first evaluate the partial derivatives, the s.d.e. is given by

$$dX_t = Z_t dW_t + W_t dZ_t + \rho dt. \quad \triangleleft$$

5. (a) The solutions to the pair of stochastic differential equations are given by

$$X_T = X_0 e^{-\frac{\sigma_X^2 T}{2} + \sigma_X W_T^X}$$
$$Y_T = Y_0 e^{-\frac{\sigma_Y^2 T}{2} + \sigma_Y W_T^Y}$$

hence

$$X_T Y_T = X_0 e^{-\frac{\sigma_X^2 T}{2} + \sigma_X W_T^X} Y_0 e^{-\frac{\sigma_Y^2 T}{2} + \sigma_Y W_T^Y}.$$

(b) Suppose W_t^X and W_t^Y are independent Brownian motions. First we observe that

$$\mathbb{E}[\sigma_X W_T^X + \sigma_Y W_T^Y] = 0$$

and

$$V[\sigma_X W_T^X + \sigma_Y W_T^Y] = \sigma_X^2 T + \sigma_Y^2 T.$$

And so

$$\sigma_X W_T^X + \sigma_Y W_T^Y \sim N\Big(0, \sigma_X^2 T + \sigma_Y^2 T\Big).$$

Now let $Z = \sigma_X W_T^X + \sigma_Y W_T^Y \sim N\Big(0, \sigma_X^2 T + \sigma_Y^2 T\Big)$, we have

$$\begin{aligned}\mathbb{E}\left[e^{\sigma_X W_T^X + \sigma_Y W_T^Y}\right] &= \mathbb{E}\left[e^Z\right] \\ &= \exp\left(\frac{\sigma_X^2 T + \sigma_Y^2 T}{2}\right).\end{aligned}$$

Hence

$$\begin{aligned}\mathbb{E}[X_T Y_T] &= \mathbb{E}\left[X_0 e^{-\frac{\sigma_X^2 T}{2} + \sigma_X W_T^X} Y_0 e^{-\frac{\sigma_Y^2 T}{2} + \sigma_Y W_T^Y}\right] \\ &= X_0 e^{-\frac{\sigma_X^2 T}{2}} Y_0 e^{-\frac{\sigma_Y^2 T}{2}} \mathbb{E}\left[e^{\sigma_X W_T^X + \sigma_Y W_T^Y}\right] \\ &= X_0 e^{-\frac{\sigma_X^2 T}{2}} Y_0 e^{-\frac{\sigma_Y^2 T}{2}} e^{\frac{\sigma_X^2 T + \sigma_Y^2 T}{2}} \\ &= X_0 Y_0.\end{aligned}$$

(c) Now W_t^X and W_t^Y are correlated with a correlation coefficient of ρ. Again we observe that

$$\mathbb{E}[\sigma_X W_T^X + \sigma_Y W_T^Y] = 0$$

and

$$V[\sigma_X W_T^X + \sigma_Y W_T^Y] = \sigma_X^2 T + \sigma_Y^2 T + 2\sigma_X \sigma_Y \rho T.$$

And so

$$\sigma_X W_T^X + \sigma_Y W_T^Y \sim N\Big(0, \sigma_X^2 T + \sigma_Y^2 T + 2\sigma_X \sigma_Y \rho T\Big).$$

Now let $Z = \sigma_X W_T^X + \sigma_Y W_T^Y \sim N\Big(0, \sigma_X^2 T + \sigma_Y^2 T + 2\sigma_X\sigma_Y\rho T\Big)$, we have

$$
\begin{aligned}
\mathbb{E}\left[e^{\sigma_X W_T^X + \sigma_Y W_T^Y}\right] &= \mathbb{E}\left[e^Z\right] \\
&= \exp\left(\frac{\sigma_X^2 T + \sigma_Y^2 T + 2\sigma_X\sigma_Y\rho T}{2}\right).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mathbb{E}[X_T Y_T] &= \mathbb{E}\left[X_0 e^{-\frac{\sigma_X^2 T}{2} + \sigma_X W_T^X} Y_0 e^{-\frac{\sigma_Y^2 T}{2} + \sigma_Y W_T^Y}\right] \\
&= X_0 e^{-\frac{\sigma_X^2 T}{2}} Y_0 e^{-\frac{\sigma_Y^2 T}{2}} \mathbb{E}\left[e^{\sigma_X W_T^X + \sigma_Y W_T^Y}\right] \\
&= X_0 e^{-\frac{\sigma_X^2 T}{2}} Y_0 e^{-\frac{\sigma_Y^2 T}{2}} e^{\frac{\sigma_X^2 T + \sigma_Y^2 T + 2\sigma_X\sigma_Y\rho T}{2}} \\
&= X_0 Y_0 e^{\sigma_X\sigma_Y\rho T}.
\end{aligned}
$$

6. First use Itô's lemma to show that

$$I_t = \int_0^t W_u dW_u = \frac{1}{2}W_t^2 - \frac{1}{2}t$$

Then proceed to evaluate

$$\begin{aligned}
&\mathbb{E}[I_t] = 0 \\
&V[I_t] = \mathbb{E}[X_t^2] - \mathbb{E}[X_t]^2 = \mathbb{E}\left[\int_0^t W_u^2 du\right] = \frac{1}{2}t^2 \quad \text{by Itô's isometry}
\end{aligned}$$

7. (a) Apply Itô's lemma to the function $X_t = f(t, W_t) = W_t + 4t$, we have

$$dX_t = 4dt + dW_t \quad \Rightarrow \quad \mathbb{E}[dX_t] = 4t \neq 0.$$

So this process is not a martingale.

(b) Apply Itô's lemma to the function $X_t = f(W_t) = W_t^2$, we have

$$dX_t = dt + 2W_t dW_t \quad \Rightarrow \quad \mathbb{E}[dX_t] = dt \neq 0.$$

So this process is not a martingale.

(c) Apply Itô's lemma to the function $X_t = f(W_t, Z_t)$, we have

$$\begin{aligned}
dX_t &= W_t dZ_t + Z_t dW_t + dW_t dZ_t \\
&= W_t dZ_t + Z_t dW_t \\
\Rightarrow \quad \mathbb{E}[dX_t] &= 0.
\end{aligned}$$

Since W_t and Z_t are independent Brownian motions. So this process is a martingale.

8. (a) W_t is a standard Brownian motion under the real world probability measure. The solution to the stock price sde is given by

$$S_T = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma W_T\right].$$

Taking expectation under the real world probability measure gives

$$\mathbb{E}[S_T] = S_0 \exp(\mu T)$$

(b) If we want to take the expectation under the risk-neutral measure associated to the risk-free bond numeraire, we need to apply Girsanov's theorem to write down the stock price sde. Under the risk-neutral measure associated to the risk-free bond numeraire, the asset ratio $\frac{S_t}{B_t}$ is a martingale. Applying Itô's lemma to $X_t = f(S_t, B_t) = \frac{S_t}{B_t}$, we have

$$dX_t = (\mu - r)X_t dt + \sigma X_t dW_t.$$

Since this process is expected to be a martingale under $\mathbb{Q}^B$, we can write

$$\begin{aligned} dX_t &= (\mu - r)X_t dt + \sigma X_t dW_t \\ &= \sigma X_t \left(dW_t + \frac{\mu - r}{\sigma}\right) \\ &= \sigma X_t dW_t^B \end{aligned}$$

where we've used Girsanov's theorem to define a new Brownian motion under the risk-neutral probability measure associated to the risk-free bond numeraire:

$$dW_t^B = dW_t + \frac{\mu - r}{\sigma} dt.$$

Substituting this to the stock price sde, we obtain

$$\begin{aligned} & dS_t = rS_t dt + \sigma S_t dW_t^B \\ \Rightarrow \quad & S_T = S_0 \exp\left[\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T\right]. \end{aligned}$$

Hence

$$\mathbb{E}^*[S_T] = S_0 \exp(rT).$$

9. (a) Let $Y_t = f(W_t) = 10^{W_t}$, we work out the following partial derivatives

$$f'(W_t) = 10^{W_t} \cdot \log(10), \qquad f''(W_t) = 10^{W_t} \cdot (\log(10))^2.$$

By Itô's Lemma, we have

$$\begin{aligned} dY_t &= f'(W_t)dW_t + \frac{1}{2}f''(W_t)(dW_t)^2 \\ &= \frac{(\log(10))^2}{2}10^{W_t}dt + 10^{W_t}\log(10)dW_t. \quad \triangleleft \end{aligned}$$

(b) Let $Y_t = f(t, W_t) = t^2 + W_t^2$, we work out the following partial derivatives

$$f_t(t, W_t) = 2t, \qquad f_x(t, W_t) = 2W_t, \qquad f_{xx}(t, W_t) = 2.$$

By Itô's Lemma, we have

$$\begin{aligned} dY_t &= f_t(t, W_t)dt + f_x(t, W_t)dW_t + \frac{1}{2}f_{xx}(t, W_t)(dW_t)^2 \\ &= (1 + 2t)dt + 2W_t dW_t. \quad \triangleleft \end{aligned}$$

(c) Let $Y_t = f(X_t, W_t) = \frac{X_t}{W_t}$, we work out the following partial derivatives

$$f_w(X_t, W_t) = -\frac{X_t}{W_t^2}, \quad f_{ww}(X_t, W_t) = \frac{2X_t}{W_t^3}, \quad f_x(X_t, W_t) = \frac{1}{W_t},$$
$$f_{xx}(X_t, W_t) = 0, \quad f_{xw}(X_t, W_t) = -\frac{1}{W_t^2}.$$

By Itô's Lemma, we have

$$\begin{aligned} dY_t &= f_w\, dW_t + \frac{1}{2}f_{ww}\, (dW_t)^2 + f_x\, dX_t + \frac{1}{2}f_{xx}\, (dX_t)^2 + f_{xw}\, dX_t dW_t \\ &= \left(\frac{X_t}{W_t^3} - \frac{\sigma X_t}{W_t^2} + \frac{\mu X_t}{W_t}\right)dt + \left(\frac{\sigma X_t}{W_t} - \frac{X_t}{W_t^2}\right)dW_t. \quad \triangleleft \end{aligned}$$

(d) Let $Y_t = f(t, W_t) = \frac{t}{W_t}$, we work out the following partial derivatives

$$f_t(t, W_t) = \frac{1}{W_t}, \qquad f_x(t, W_t) = -\frac{t}{W_t^2}, \qquad f_{xx}(t, W_t) = \frac{2t}{W_t^3}.$$

By Itô's Lemma, we have

$$\begin{aligned} dY_t &= f_t(t, W_t)dt + f_x(t, W_t)dW_t + \frac{1}{2}f_{xx}(t, W_t)(dW_t)^2 \\ &= \left(\frac{1}{W_t} + \frac{t}{W_t^3}\right)dt - \frac{t}{W_t^2}dW_t. \quad \triangleleft \end{aligned}$$

10. Let $X_t = f(W_t) = \frac{W_t^3}{3}$, we have

$$f'(W_t) = W_t^2, \qquad f''(W_t) = 2W_t.$$

Applying Itô's lemma, we obtain the following stochastic differential equation

$$dX_t = W_t^2 dW_t + W_t dt.$$

Integrating both sides and rearranging, we obtain

$$\int_0^t W_u^2 \, dW_u = \frac{W_t^3}{3} - \int_0^t W_u \, du. \quad \triangleleft$$