Session 5: Stochastic Differential Equations

Tee Chyng Wen

QF620 Stochastic Modelling in Finance

"Comments on the life and mathematical legacy of Wolfgang Doeblin":

> *One may invoke many reasons why the emergence of a specific branch of probability — the study of stochastic process — took a quite tortuous path throughout the 20^{th} century.*
>
> *On one hand, the pioneers were very often quite original mathematicians, such as Bachelier, Lévy, Itô, ..., whose novel ways of looking at things took a long time to be accepted.*

— Bernard Bru and Marc Yor (2002)

On the other hand, perhaps the fact that Brownian motion possesses so many properties, which we summarize as:

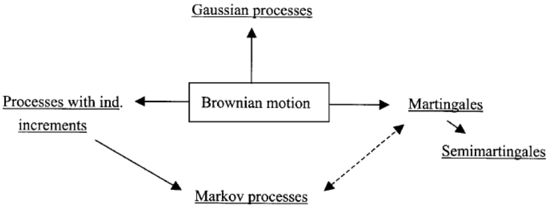
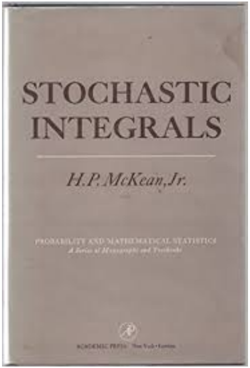


Fig. 1. Brownian motion and related processes

led many authors to develop studies of one or another special class of processes, thus giving a hard time to outsiders.

Itô's calculus took 25 years (1944-1969) to be accepted, the latter year being that of the publication of McKean's marvellous little book: Stochastic Integrals.

Bernard Bru and Marc Yor (2002)



SDEs and Martingale

Example Use Itô's formula to derive the stochastic differential equations of the following processes, and determine which of them are martingales:

1. $X_t = W_t^2$ (no)
2. $X_t = 2 + t + e^{W_t}$ (no)
3. $X_t = W_t^2 + \tilde{W}_t^2$, where W_t and $\tilde{W}_t$ are independent Brownian. (no)
4. $X_t = W_t^2 - t$ (yes)
5. $X_t = W_t^3$ (no)
6. $X_t = e^{\theta W_t - \frac{\theta^2 t}{2}}$ (yes)

Bachelier Model (1900)

Louis Bachelier was the first mathematician to use random walk to analyse stock prices in 1900.

In Bachelier model, the stock price process is a **symmetrical random walk**, correspond to a market under **equilibrium**. This follows a **normal distribution**:

$$S_T = S_0 + \sigma W_T, \quad W_T \sim N(0,T).$$

Given this definition, we can then proceed to derive valuation formulas for vanilla European options.

However, a shortcoming of this model is that the lack of a lower bound at 0.

In other words, while this is a reasonable model for interest rates, it leads to non-zero probability for negative stock prices.

Bachelier Model – Arithmetic Brownian Process

The Bachelier model for the stock price process is defined as

$$dS_t = \sigma\, dW_t.$$

Integrating this stochastic equation, we can show that the terminal stock price is normally distributed as

$$S_T \sim N(S_0, \sigma^2 T).$$

Let V_c denote the price of a European call option, we have:

$$\begin{aligned} V_c = e^{-rT}\mathbb{E}[(S_T - K)^+] &= e^{-rT}\mathbb{E}\left[(S_0 + \sigma W_T - K)^+\right] \\ &= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(S_0 + \sigma\sqrt{T}x - K\right)^+ e^{-\frac{x^2}{2}}\, dx \end{aligned}$$

Note that $\left(S_0 + \sigma\sqrt{T}x - K\right)^+ = 0$ whenever $S_0 + \sigma\sqrt{T}x - K < 0$, and will only take on non-zero values when

$$S_0 + \sigma\sqrt{T}x - K > 0 \quad \Rightarrow \quad x > \frac{K - S_0}{\sigma\sqrt{T}} = x^*.$$

Bachelier Model – Arithmetic Brownian Process

Hence, we now write

$$\begin{aligned}
V_c &= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{x^*}^{\infty} \left(S_0 + \sigma\sqrt{T}x - K \right) e^{-\frac{x^2}{2}}\ dx \\
&= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{x^*}^{\infty} (S_0 - K) e^{-\frac{x^2}{2}}\ dx + \frac{e^{-rT}}{\sqrt{2\pi}} \int_{x^*}^{\infty} \sigma\sqrt{T} x e^{-\frac{x^2}{2}}\ dx \\
&= e^{-rT}(S_0 - K)\Big[\Phi(\infty) - \Phi(x^*)\Big] - \frac{e^{-rT}\sigma\sqrt{T}}{\sqrt{2\pi}} \Big[e^{-\frac{x^2}{2}} \Big]_{x^*}^{\infty} \\
&= e^{-rT} \left[(S_0 - K)\Phi(-x^*) + \frac{\sigma\sqrt{T}}{\sqrt{2\pi}} e^{-\frac{(x^*)^2}{2}} \right] \\
&= e^{-rT} \left[(S_0 - K)\Phi(-x^*) + \frac{\sigma\sqrt{T}}{\sqrt{2\pi}} e^{-\frac{(-x^*)^2}{2}} \right] \\
&= e^{-rT} \left[(S_0 - K)\Phi\left(\frac{S_0 - K}{\sigma\sqrt{T}} \right) + \sigma\sqrt{T}\phi\left(\frac{S_0 - K}{\sigma\sqrt{T}} \right) \right].
\end{aligned}$$

For **at-the-money (ATM)** options, we have $K = S_0$, and this formula reduces to $V_c = e^{-rT}\sigma\sqrt{\frac{T}{2\pi}}$.

Black-Scholes Model (1973)

In a landmark 1973 paper, Fischer Black and Myron Scholes introduced the **Black-Scholes model**, which models the stock price as

$$S_T = S_0 \exp\left[\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T\right], \quad W_T \sim N(0,T).$$

Given this definition, we can readily verify that $\mathbb{E}[S_T] = S_0 e^{rT}$. Rearranging, we can write it as

$$\frac{S_T}{S_0} = \exp\left[\left(r - \frac{\sigma^2}{2}\right)T + \sigma N(0,T)\right].$$

Consequently,

$$\log\frac{S_T}{S_0} = \left(r - \frac{\sigma^2}{2}\right)T + \sigma N(0,T) \sim N\left(\left(r - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right).$$

Alternatively, we can also write S_T as

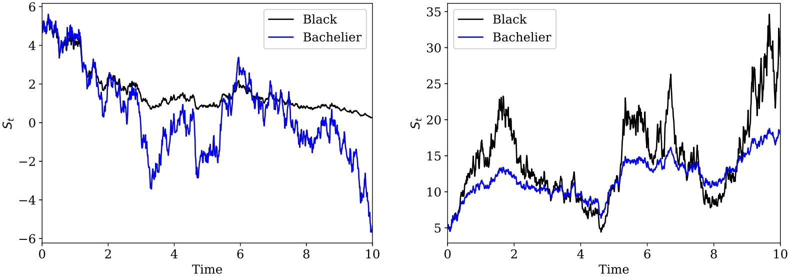
$$\log S_T = \log S_0 + \left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}N(0,1).$$

Black-Scholes *vs.* Bachelier

Below are 2 sample paths (same Brownian motion) from the 2 models:

$$\text{Black-Scholes:} \quad S_{t+\Delta t} = S_t \exp\left[\left(r - \frac{\sigma^2}{2}\right)\Delta t + \sigma \cdot (W_{t+\Delta t} - W_t)\right]$$

$$\text{Bachelier:} \quad S_{t+\Delta t} = S_t + \sigma \cdot (W_{t+\Delta t} - W_t)$$



Question How do the two models compare?

Black-Scholes Model – Geometric Brownian Process

Under the Black-Scholes model, the stock price process follows the stochastic differential equation:

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

Note that a direct integration does not allow us to solve the SDE:

$$S_T = S_0 + \int_0^T rS_t dt + \int_0^T \sigma S_t dW_t.$$

However, we can solve the SDE by first applying **Itô's formula** to the function $X_t = f(S_t) = \log(S_t)$:

$$dX_t = \left(r - \frac{\sigma^2}{2}\right) dt + \sigma dW_t$$

Integrating both sides and substituting for X_t, we arrive at

$$S_T = S_0 \exp\left[\left(r - \frac{\sigma^2}{2}\right) T + \sigma W_T\right].$$

Black-Scholes Model – Option Pricing

Now let us derive the option pricing formula for a European call option under Black-Scholes model.

$$\begin{aligned} V_c &= e^{-rT}\mathbb{E}[(S_T-K)^+] \\ &= \frac{e^{-rT}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(S_0e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}x}-K\right)^+ e^{-\frac{x^2}{2}}\,dx \end{aligned}$$

Again, the terms in the $(\cdot)^+$ operator will need to be positive for it take non-zero values

$$S_T-K>0 \quad\Rightarrow\quad x>\frac{\log\left(\frac{K}{S_0}\right)-\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}=x^*.$$

Now we can proceed to evaluate the integral

$$V_c=\frac{e^{-rT}}{\sqrt{2\pi}}\int_{x^*}^{\infty}\left(S_0e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}x}-K\right)e^{-\frac{x^2}{2}}\,dx.$$

Black-Scholes Model – Option Pricing

Next, we have:

$$
\begin{aligned}
V_c &= \frac{e^{-rT}}{\sqrt{2\pi}}\int_{x^*}^{\infty} S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}x} e^{-\frac{x^2}{2}}\,dx - \frac{e^{-rT}}{\sqrt{2\pi}}\int_{x^*}^{\infty} K e^{-\frac{x^2}{2}}\,dx \\
&= \frac{S_0 e^{-\frac{\sigma^2 T}{2}}}{\sqrt{2\pi}}\int_{x^*}^{\infty} e^{-\frac{x^2-2\sigma\sqrt{T}x}{2}}\,dx - \frac{Ke^{-rT}}{\sqrt{2\pi}}\int_{x^*}^{\infty} e^{-\frac{x^2}{2}}\,dx \\
&= \frac{S_0 e^{-\frac{\sigma^2 T}{2}}}{\sqrt{2\pi}}\int_{x^*}^{\infty} e^{-\frac{x^2-2\sigma\sqrt{T}x+\sigma^2T-\sigma^2T}{2}}\,dx - \frac{Ke^{-rT}}{\sqrt{2\pi}}\int_{x^*}^{\infty} e^{-\frac{x^2}{2}}\,dx \\
&= \frac{S_0 e^{-\frac{\sigma^2 T}{2}}}{\sqrt{2\pi}}\int_{x^*}^{\infty} e^{-\frac{\left(x-\sigma\sqrt{T}\right)^2}{2}} e^{\frac{\sigma^2 T}{2}}\,dx - \frac{Ke^{-rT}}{\sqrt{2\pi}}\int_{x^*}^{\infty} e^{-\frac{x^2}{2}}\,dx \\
&= \frac{S_0}{\sqrt{2\pi}}\int_{x^*}^{\infty} e^{-\frac{\left(x-\sigma\sqrt{T}\right)^2}{2}}\,dx - \frac{Ke^{-rT}}{\sqrt{2\pi}}\int_{x^*}^{\infty} e^{-\frac{x^2}{2}}\,dx
\end{aligned}
$$

where we have used the **completing the square** trick.

Black-Scholes Model – Option Pricing

Finally, we obtain

$$
\begin{aligned}
V_c &= S_0\Big[\Phi(\infty) - \Phi(x^* - \sigma\sqrt{T})\Big] - Ke^{-rT}\Phi(-x^*) \\
&= S_0\Phi(-x^* + \sigma\sqrt{T}) - Ke^{-rT}\Phi(-x^*) \\
&= S_0\Phi\left(\frac{\log\frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) - Ke^{-rT}\Phi\left(\frac{\log\frac{S_0}{K} + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)
\end{aligned}
$$

In many references, it is common to let

$$
d_1 = \frac{\log\frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T},
$$

leading to

$$
V_c = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2).
$$

Black Model (1976) – Forward Price Process

So far we have regarded the underlying as the stochastic variable, and derived a stochastic differential equation to describe its price dynamic.

In 1976, Fischer Black proposed **modeling the forward price** instead of the underlying price. We have the definition of the forward price

$$F_t = e^{r(T-t)} S_t$$

and the underlying price process of

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

Applying Itô's formula to the function $f(t,x) = e^{r(T-t)}x$ allows us to write down the stochastic differential equation for the forward price

$$dF_t = \sigma F_t dW_t,$$

which is a more compact equation—it is driftless and is therefore a **martingale**.

Black Model (1976) – Forward Price Process

The Black model is defined on the forward price and is given by

$$dF_t = \sigma F_t dW_t.$$

As this is also a geometric process, we can solve this stochastic differential equation by applying Itô's formula to $X_t = f(F_t)$ where $f(x) = \log(x)$.

The solution is given by:

$$F_T = F_0 e^{-\frac{\sigma^2 T}{2} + \sigma W_T}$$

Let $D(0,T) = e^{-rT}$ denote the **discount factor**, under this model the price of a European call option is given by

$$V_c = D(0,T)\left[F_0\Phi\left(\frac{\log\frac{F_0}{K} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) - K\Phi\left(\frac{\log\frac{F_0}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)\right].$$

Mean-reverting Process – Vasicek Model

The **Ornstein-Uhlenbeck process** is used in solid-state physics to model gas molecules under the influence of pressure and temperature.

Oldrich Vasicek adapted this model in 1977 to model interest rate as a **mean reverting stochastic process**, given by

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t.$$

Applying Itô formula to $X_t = e^{\kappa t} r_t = f(t, r_t)$, we obtain

$$\begin{aligned} d(e^{\kappa t} r_t) &= \kappa e^{\kappa t} r_t dt + e^{\kappa t} dr_t \\ &= \kappa\theta e^{\kappa t} dt + \sigma e^{\kappa t} dW_t. \end{aligned}$$

Integrating both sides from 0 to t, we can obtain a solution to the stochastic differential equation

$$\int_0^t d(e^{\kappa u} r_u) = \int_0^t \kappa\theta e^{\kappa u} du + \int_0^t \sigma e^{\kappa u} dW_u$$

$$r_t = r_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}) + \sigma \int_0^t e^{\kappa(u-t)} dW_u.$$

Mean-reverting Process – Vasicek Model

Taking expectation on both sides gives us the mean

$$\mathbb{E}[r_t] = r_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}).$$

Recall **Itô's Isometry theorem** states that

$$\mathbb{E}\left[\left(\int_0^T X_t dW_t\right)^2\right] = \mathbb{E}\left[\int_0^T X_t^2 dt\right].$$

Applying it to our case,

$$\begin{aligned} V[r_t] &= \mathbb{E}\left[\left(\sigma \int_0^t e^{\kappa(u-t)} dW_u\right)^2\right] \\ &= \mathbb{E}\left[\sigma^2 \int_0^t e^{2\kappa(u-t)} du\right] = \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa t}\right). \end{aligned}$$

The distribution of r_t is therefore given by

$$r_t \sim N\left(r_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}), \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa t})\right).$$