$$\int_0^T \frac{dB_t}{B_t} = \int_0^T r\,dt \Rightarrow [\log B_t]_0^T = [rt]_0^T$$

Application of EMM — Black-Scholes

$$\log B_T - \log B_0 = rT$$

Under the **Black-Scholes economy**, let B_t denote the value of the money-market account with $B_0 = 1$, and let S_t denote the stock price process. The following differential equations described their dynamics:

$$B_T = B_0 e^{rT}$$

$$dB_t = rB_t dt$$
$$dS_t = \mu S_t dt + \sigma S_t dW_t.$$

Here W_t is a $\mathbb{P}$-Brownian motion under the real-world measure, and μ is its (unknown) drift coefficient.

Question Which is the most difficult parameter to estimate among r, μ, and σ?

Source: Google Finance

$\mathbb{P}$

$$dS_t = \mu S_t\,dt + \sigma S_t\,dW_t$$

$$dB_t = r B_t\,dt$$

$$\mu > r$$

$\frac{S_t}{B_t}$ drifts upward

$\mathbb{Q}^*$

$$dS_t = r S_t\,dt + \sigma S_t\,dW_t^*$$

$$dB_t = r B_t\,dt$$

$\frac{S_t}{B_t}$ is a martingale

Application of EMM — Black-Scholes

The value of B_t is strictly positive and can be used as a numeraire. Define the relative price process $X_t = \frac{S_t}{B_t} = f(S_t, B_t)$, we can apply Itô's formula to obtain

$$dX_t = (\mu - r)X_t dt + \sigma X_t dW_t.$$

To identify the equivalent martingale measure we apply **Girsanov's theorem** with $\kappa = \frac{\mu - r}{\sigma}$ to obtain:

$$dW_t^* = dW_t + \frac{\mu - r}{\sigma} dt,$$

where W_t^* is a standard Brownian motion under probability measure $\mathbb{Q}^*$. Here the * notation is used to indicate we have chosen the **risk-free account B_t as our numeraire**, which is the most common choice. Substituting, we obtain

$$\begin{aligned} dX_t &= (\mu - r)X_t dt + \sigma X_t \left(dW_t^* - \frac{\mu - r}{\sigma} dt \right) \\ &= \sigma X_t dW_t^*. \end{aligned}$$

$$dS_t = \mu S_t\,dt + \sigma S_t\,dW_t \longrightarrow \mathbb{P}$$

$$dB_t = r B_t\,dt$$

$$f(b,x) = \frac{x}{b}$$

$$f_b = -\frac{x}{b^2}$$

$$f_x = \frac{1}{b}, \quad f_{xx} = 0$$

let $X_t = \frac{S_t}{B_t} = f(B_t, S_t)$

Itô's formula:

$$dX_t = f_b(B_t, S_t)\cdot dB_t + f_x(B_t, S_t)\,dS_t + \frac{1}{2} f_{xx}(B_t, S_t)(dS_t)^2$$

$$= -\frac{S_t}{B_t^{\cancel{2}}}\cdot r\cancel{B_t}\,dt + \frac{1}{B_t}\cdot\left(\mu S_t\,dt + \sigma S_t\,dW_t\right) + 0$$

$$dX_t = (\mu - r) X_t\,dt + \sigma X_t\,dW_t$$

$$= \sigma X_t\left(dW_t + \frac{\mu - r}{\sigma}\,dt\right) \quad \mathbb{P}$$

$$dX_t = \sigma X_t \, dW_t^*$$

$\mathbb{Q}^*$

$$dW_t^* = dW_t + \frac{\mu - r}{\sigma} dt \qquad \text{(Girsanov)}$$

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t \qquad \mathbb{P}$$

$$dS_t = \mu S_t \, dt + \sigma S_t \left(dW_t^* - \frac{\mu - r}{\sigma} dt \right) \qquad \mathbb{Q}^*$$

$$\therefore \quad = r S_t \, dt + \sigma S_t \, dW_t^*$$

Application of EMM — Black-Scholes

This is the only measure which turns the relative price process into martingale. We can now determine what is the stock price process under this unique **martingale measure** $\mathbb{Q}^*$:

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S \left(dW_t^* - \frac{\mu - r}{\sigma} dt \right) \\ &= r S_t dt + \sigma S_t dW_t^*. \end{aligned}$$

Under the equivalent martingale measure, the drift of the stock μ is irrelevant and is replaced by the risk-free interest rate r. The solution to this stochastic differential equation is

$$S_T = S_0 \exp \left[\left(r - \frac{1}{2} \sigma^2 \right) T + \sigma W_T^* \right].$$

Application of EMM — Black-Scholes

A European call option with strike K and maturing at time T where $V_T = (S_T - K)^+$ can be evaluated by **martingale pricing theorem** as follow

$$
\begin{aligned}
\frac{V_0}{B_0} &= \mathbb{E}^* \left[\frac{V_T}{B_T} \right] = \mathbb{E}^* \left[\frac{(S_T - K)^+}{B_T} \right] \\
&= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[S_0 e^{\left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}x} - K \right]^+ e^{-\frac{x^2}{2}} dx \\
&= S_0 \Phi(d_1) - K e^{-rT} \Phi(d_1 - \sigma\sqrt{T}), \quad d_1 = \frac{\log\left(\frac{S_0}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}.
\end{aligned}
$$

We have already learned how to derive the Black-Scholes **option pricing formula** by evaluating the expectation.

$$\mathbb{Q}^* : \quad \mathbb{E}^*\left[\frac{S_T}{B_T}\right] = \mathbb{E}^*\left[\frac{S_0 e^{(r-\frac{\sigma^2}{2})T + \sigma W_T^*}}{B_0 e^{rT}}\right]$$

$$= \frac{S_0}{B_0} e^{-\frac{\sigma^2 T}{2}} \, \mathbb{E}^*\left[e^{\sigma W_T^*}\right]$$

$$= \frac{S_0}{B_0} e^{-\frac{\sigma^2 T}{2}} \cdot e^{\frac{\sigma^2 T}{2}}$$

$$= \frac{S_0}{B_0}$$

$$\frac{V_0}{B_0} = \mathbb{E}^*\left[\frac{V_T}{B_T}\right] \Rightarrow \frac{V_0}{B_0} = \mathbb{E}^*\left[\frac{V_T}{B_0 e^{rT}}\right]$$

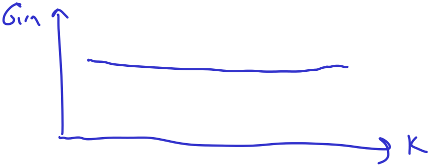
SINGAPORE MANAGEMENT UNIVERSITY

Session 8: Static Replication of European Payoffs

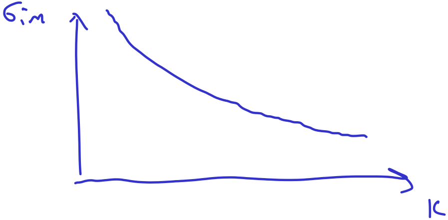
Tee Chyng Wen

QF620 Stochastic Modelling in Finance

$\beta = 1$



$\beta = 0$



SABR :

$$\begin{cases} dF_t = \alpha_t F_t^{\beta} \, dW_t^F \\ d\alpha_t = \nu \alpha_t \, dW_t^{\alpha} \end{cases} \quad dW_t^F \, dW_t^{\alpha} = \rho \, dt$$

α , β , ρ , ν

$\rho > 0$:

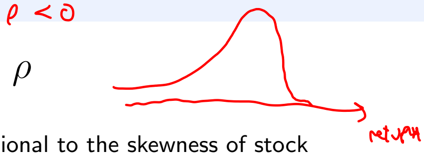
$F_t \nearrow$ $\qquad$ $F_t \searrow$

$\alpha_t \nearrow$ $\qquad$ $\alpha_t \searrow$

$\rho < 0$:

$F_t \nearrow$ $\qquad$ $F_t \searrow$

$\alpha_t \searrow$ $\qquad$ $\alpha_t \nearrow$



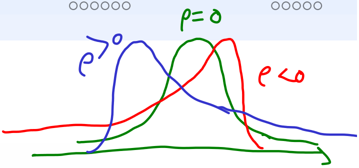
Behavior of Model Parameters – ρ

Implication on Distribution

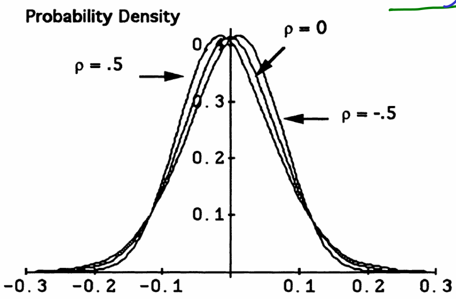
- The correlation parameter ρ is proportional to the skewness of stock returns.
- Intuitively, a negative correlation results in high volatility when the stock price drops, and this spreads the left tail of the probability density. The right tail is associated with low volatility and is not spread out.
- A negative correlation creates a fat left tail and a thin right tail in the stock return distribution.

Implication on Pricing

- This increases the prices of out-of-the-money puts and decreases the prices of out-of-the-money calls relative to the Black-Scholes model price.
- Intuitively, out-of-the-money put options benefit substantially from a fat left tail.
- A positive correlation will have completely opposite effects—it creates a fat right tail and a thin left tail.



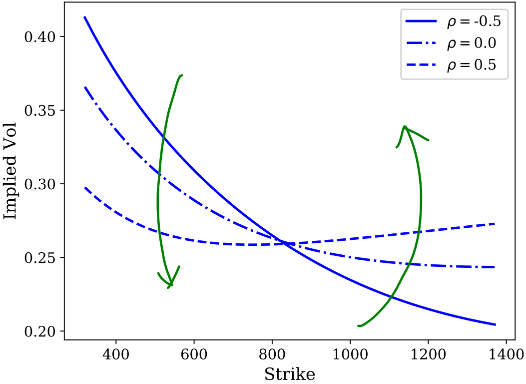
Behavior of Model Parameters – ρ



⇒ **Positive correlation** between stock and volatility is associated with **positive skew** in return distribution.

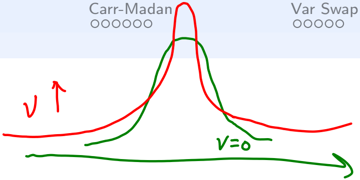
⇒ **Negative correlation** between stock and volatility is associated with **negative skew** in return distribution.

Behavior of Model Parameters – ρ



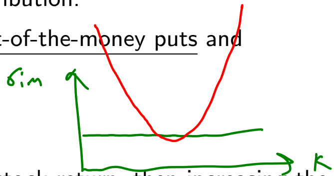
Negative correlation increases the price of out-of-the-money put options and decreases the price of out-of-the-money call options.

Behavior of Model Parameters – ν



Implication on Distribution

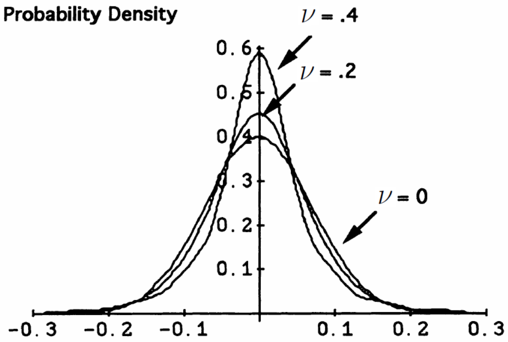
- When the volatility of volatility parameter is 0, we are back to a normal log-return distribution (if $\beta = 0$).
- Otherwise, it increases the kurtosis of stock returns, creating two fat tails in both ends of the distribution.
- This has the effect of raising out-of-the-money puts and out-of-the-money call prices.



Implication on Pricing

- If volatility is uncorrelated with stock return, then increasing the volatility of volatility only increases the kurtosis of spot return.
- In this case, random volatility is associated with increases in the prices of far-from-the-money options relative to near-the-money options.
- In contrast, the correlation of volatility with the spot return produces skewness.

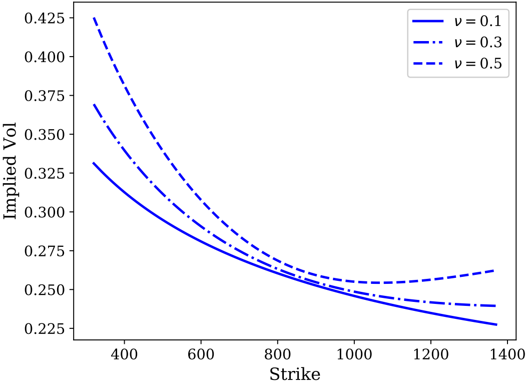
Behavior of Model Parameters – ν



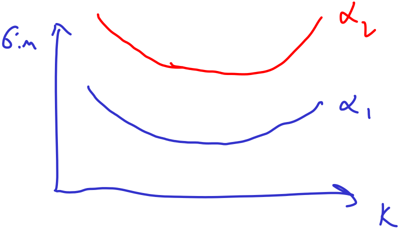
⇒ **Increasing volatility-of-volatility** has the effect of **increasing the kurtosis** of return.

⇒ When the volatility-of-volatility parameter is 0, volatility will be deterministic.

Behavior of Model Parameters – ν

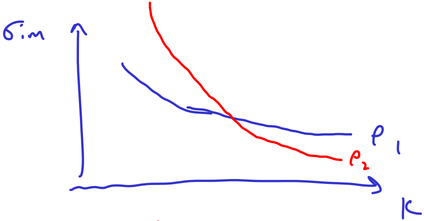


Larger volatility-of-volatility ν increases the price of out-of-the-money call and put options.



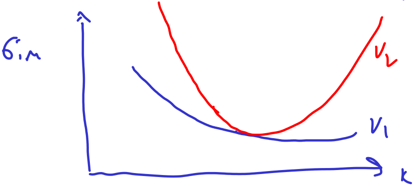
$\alpha_2 > \alpha_1$

level



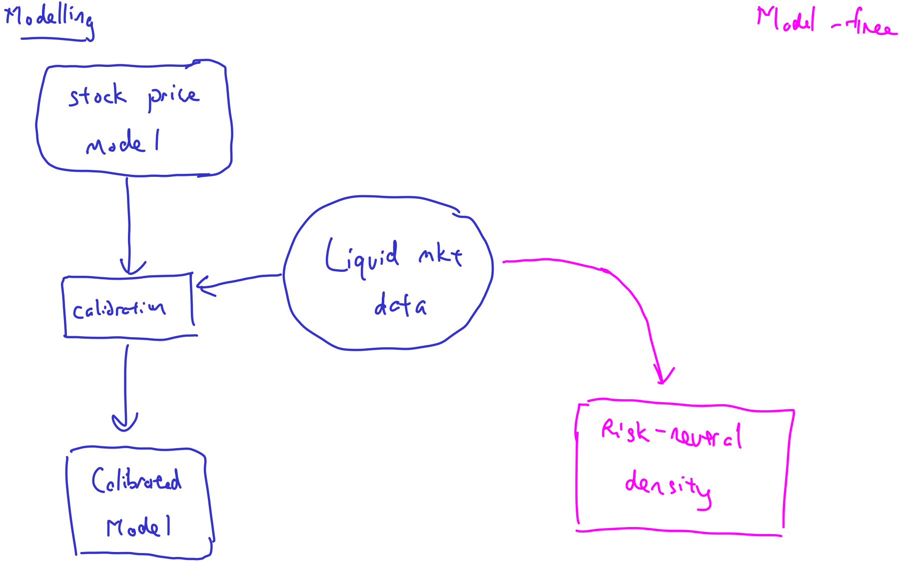
$\rho_2 < \rho_1$

slope



$\nu_2 > \nu_1$

curvature



Modelling
Model-free
stock price model
Liquid mkt data
calibration
Calibrated Model
Risk-neutral density

What is the "model-free" framework?

In a **model-free** formulation, we let $f(s)$ denote the risk-neutral probability density function of the stock price at time T, we can price a vanilla European call option maturing at time T as follows:

$$C(K) = e^{-rT}\mathbb{E}^*[(S_T - K)^+] = e^{-rT}\int_K^\infty (s-K)f(s)\,ds.$$

In earlier modelling approach, we will attempt to specify a model for the stock price process. A typical example is the Black-Scholes model, which will lead to:

$$C(K) = \frac{e^{-rT}}{\sqrt{2\pi}}\int_{x^*}^\infty \left(S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}x} - K\right)e^{-\frac{x^2}{2}}\,dx.$$

We could also have used the Bachelier model, the displaced-diffusion model, or the SABR model.

$\Rightarrow$ Once a model is chosen, the risk-neutral density is also determined, by calibrating the model to market option data.

What is the "model-free" framework?

Suppose we have sufficient liquid option quotes in the market, can we skip over the step of using a model to specify the stock price process, but instead **extract the risk-neutral density function** directly?

same $f(s)$

Market Price	Model-Free Formula
$C(K_1)$	$e^{-rT}\int_{K_1}^{\infty}(s-K_1)f(s)\,ds$
$C(K_2)$	$e^{-rT}\int_{K_2}^{\infty}(s-K_2)f(s)\,ds$
$C(K_3)$	$e^{-rT}\int_{K_3}^{\infty}(s-K_3)f(s)\,ds$
$C(K_4)$	$e^{-rT}\int_{K_4}^{\infty}(s-K_4)f(s)\,ds$
$\vdots$	$\vdots$

Implied Risk-Neutral Density

- Black-Scholes model for European call and put options allow us to determine their prices by taking expectation of the option payoff on maturity, discount back to today.

- The model **assumes a lognormal process for the stock price** following
$$dS_t = rS_t dt + \sigma S_t dW_t^*,$$
under $\mathbb{Q}^*$, where the volatility σ is a model parameter that we need to determine.

- Since the vanilla option market is very liquid, we do not need to rely on any mathematical models to calculate the prices of options.

- Instead, the traded price of these options are published real-time by exchanges globally, and the process can now be reversed—given that an option traded at a particular price, what is the implied volatility that we should substitute into our Black-Scholes formula to give us this price, assuming that the underlying stock price is indeed following a lognormal process?

Implied Risk-Neutral Density

$\sigma_{im}(K, T)$

- One option price allows us to determine one implied volatility for a particular **strike** and **maturity**.

- The market is constantly providing live information about option prices across a wide range of strikes for a given maturity.

- Given this information, we can now bring our analysis to the next level—instead of asking for just one single implied volatility to match one option price, we want to determine, for a given maturity, the **implied risk-neutral distribution**, that allows us to match the market volatility smile or skew.

- To this end, we need to apply **Leibniz's rule**:

$$I(x) = \int_{u(x)}^{v(x)} g(x,t)dt$$

$$\frac{dI(x)}{dx} = g(x, v(x))\frac{dv}{dx} - g(x, u(x))\frac{du}{dx} + \int_{u(x)}^{v(x)} \frac{\partial g(x,t)}{\partial x}dt$$

$$C(K) = \int_{K}^{\infty} (S - K)\, f(S)\, dS$$

$$= \int_{u(K)=K}^{v(K)=\infty} g(K, S)\, dS$$

Implied Risk-Neutral Density

This allows us to **extract the risk-neutral probability density function** from market-traded vanilla option prices.

Let $f(s)$ denote the risk-neutral probability density, we can apply Leibniz's rule to obtain:

$$
\begin{aligned}
C(K) &= e^{-rT}\mathbb{E}[(S_T-K)^+] = e^{-rT}\int_K^\infty (s-K)f(s)\,ds \\
\frac{\partial C(K)}{\partial K} &= e^{-rT}\left[\overset{0}{\cancel{\lim_{x\to\infty}(x-K)f(x)}}\frac{dx}{dK} - \overset{0}{\cancel{(K-K)}}f(K)\frac{dK}{dK} - \int_K^\infty f(s)\,ds\right] \\
&= -e^{-rT}\int_K^\infty f(s)\,ds \\
\frac{\partial^2 C(K)}{\partial K^2} &= -e^{-rT}\left[\overset{0}{\cancel{\lim_{x\to\infty} f(x)}}\frac{dx}{dK} - f(K)\overset{1}{\cancel{\frac{dK}{dK}}} + \int_K^\infty \overset{0}{\cancel{\frac{\partial f(s)}{\partial K}}}\,ds\right] \\
&= e^{-rT}f(K).
\end{aligned}
$$

Implied Risk-Neutral Density

We can also carry out the same procedure to the put options:

$$P(K) = e^{-rT}\mathbb{E}[(K - S_T)^+] = e^{-rT}\int_0^K (K - s)f(s)\,ds$$

These give us

$$\frac{\partial^2 C(K)}{\partial K^2} = e^{-rT}f(K) \qquad \text{and} \qquad \frac{\partial^2 P(K)}{\partial K^2} = e^{-rT}f(K).$$

This is the **Breeden-Litzenberger formula**, which showed in 1978 that the terminal distribution of the stock price implicit in the option prices, also known as the **implied distribution**, can be obtained by differentiating the call & put option prices twice with respect to the strike price.

Subsequently, **Carr and Madan** showed in 1998 that any European payoff can be replicated using a portfolio of cash, forward contracts, and European call & put options.