QF620 Additional Examples
Session 1: Probability Theory and Lattice Models

1 Examples

1. Let X denote a random variable with a mean of μ. Discuss whether the following inequality is always true:

$$\mathbb{E}[X^2] \geq \mu^2.$$

2. Determine the mean, mode and median of the random variable X with following distribution:

   (a) Standard normal distribution where $X \sim N(0, 1)$

   (b) Normal distribution where $X \sim N(\mu, \sigma^2)$

3. The moment generating function for normal distribution is given by

$$M_X(\theta) = \exp\left(\mu\theta + \frac{1}{2}\sigma^2\theta^2\right).$$

   Show that the mean and variance of this distribution is given by μ and σ^2, respectively, by taking the first and second order derivative of the MGF and setting $\theta = 0$.

4. Consider $Z \sim N(0, 1)$, i.e. Z follows a standard normal distribution with a mean of 0 and a variance of 1. In this case, the moment generating function is given by setting $\mu = 0$ and $\sigma^2 = 1$:

$$M_X(\theta) = \exp\left(\frac{\theta^2}{2}\right).$$

   Determine the third and fourth moments of Z, i.e. $\mathbb{E}[Z^3]$ and $\mathbb{E}[Z^4]$.

5. Consider a market with the following securities:

   - Assets A and B, both worth 100 today
   - Asset A will be worth 110 tomorrow with probability 0.9 and 90 otherwise
   - Asset B will be worth 110 tomorrow with probability 0.5 and 90 otherwise
   - Asset C is worth 1 both today and tomorrow.

   (a) How does the prices of at-the-money (struck at 100) call options on A and B compare?

   (b) Price call options on both A and B struck at 90, 100 and 110.

6. An asset is worth 100 today. It will be worth 90, 100 or 110 tomorrow. If the no-arbitrage price for an at-the-money (ATM) call option struck at 100 is 2, what is the no-arbitrage price for a call option struck at 105? Assume interest rate is zero.

7. A stock is worth 100 today, and will be worth either one of 85, 95, 105 and 115 tomorrow. Assume interest rate is zero.

   (a) Derive the no-arbitrage boundary for a call option struck at 100.

   (b) If the call option struck at 100 is worth 5, derive the no-arbitrage boundary for a call option struck at 110.

8. A stock is worth 50 today, and will be worth 40 or 70 tomorrow. Interest rate is zero. What are the risk-neutral probabilities?

9. A stock is worth 50 today, and will be worth 40, 55 or 70 tomorrow. Interest rate is zero. What are the range of values for the risk-neutral probability p_1^*, which denote the risk-neutral probability of the stock moving to 40?

2 Suggested Solutions

1. Yes, this inequality is always valid, since X is a random variable, its variance must be greater than 0, hence

$$
\begin{aligned}
V[X] = \mathbb{E}[X^2] - \mu^2 &\geq 0 \\
\Rightarrow \quad \mathbb{E}[X^2] &\geq \mu^2.
\end{aligned}
$$

2. (a) If $X \sim N(0,1)$, then the mean is given by $\mathbb{E}[X] = 0$. The mode is determined as follow:

$$
\begin{aligned}
\phi(x) &= \frac{1}{\sqrt{2\pi}} \exp\left(-x^2/2\right) \\
\phi'(x) &= \frac{1}{\sqrt{2\pi}} \exp\left(-x^2/2\right) \times (-x) = 0 \qquad \Rightarrow \qquad x = 0.
\end{aligned}
$$

The median is given by

$$
\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{u^2}{2}\right) du = 0.5.
$$

Since the probability density function of the standard normal distribution is symmetric across the y-axis, we can infer that $\Phi(0) = 0.5$.

(b) For $X \sim N(\mu, \sigma^2)$, mean is given by $\mathbb{E}[X] = \mu$. The mode is given by

$$
\phi'(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \times \left(-\frac{x-\mu}{\sigma^2}\right) = 0 \qquad \Rightarrow \qquad x = \mu.
$$

Again due to the symmetric property of normal distribution, we know that $\Phi(\mu) = 0.5$.

3. First we determine the derivatives with respect to θ

$$
\begin{aligned}
M_X(\theta) &= \exp\left(\mu\theta + \frac{1}{2}\sigma^2\theta^2\right) \\
M_X'(\theta) &= (\mu + \sigma^2\theta) \exp\left(\mu\theta + \frac{1}{2}\sigma^2\theta^2\right) \\
M_X''(\theta) &= [\sigma^2 + (\mu + \sigma^2\theta)^2] \exp\left(\mu\theta + \frac{1}{2}\sigma^2\theta^2\right).
\end{aligned}
$$

So the mean and variance of a random variable $X \sim N(\mu, \sigma^2)$ is

$$
\begin{aligned}
\mathbb{E}[X] &= M_X'(0) = \mu \\
V[X] &= M_X''(0) - [M_X'(0)]^2 = \sigma^2. \quad \triangleleft
\end{aligned}
$$

4. We have

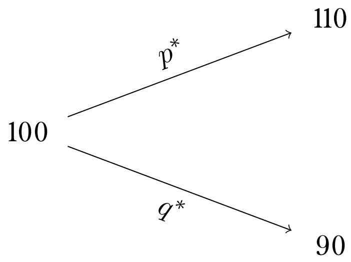
$$
\begin{aligned}
M_Z(\theta) &= e^{\frac{\theta^2}{2}} \\
M_Z'(\theta) &= \theta e^{\frac{\theta^2}{2}} \\
M_Z''(\theta) &= e^{\frac{\theta^2}{2}} + \theta^2 e^{\frac{\theta^2}{2}} \\
M_Z'''(\theta) &= 3\theta e^{\frac{\theta^2}{2}} + \theta^3 e^{\frac{\theta^2}{2}} \\
M_Z''''(\theta) &= 3e^{\frac{\theta^2}{2}} + 6\theta^2 e^{\frac{\theta^2}{2}} + \theta^4 e^{\frac{\theta^2}{2}}
\end{aligned}
$$

Setting $\theta = 0$, we obtain:

$$
\begin{aligned}
\mathbb{E}[Z] &= M_Z'(0) = 0 \\
\mathbb{E}[Z^2] &= M_Z''(0) = 1 \\
\mathbb{E}[Z^3] &= M_Z'''(0) = 0 \\
\mathbb{E}[Z^4] &= M_Z''''(0) = 3
\end{aligned}
$$

5. Securities A and B are both risky assets (e.g. stocks), with the given real world probabilities. Security C is a risk-free bond or risk-free money market account.

   - As we have mentioned in the class, option valuation uses risk neutral probabilities to determine a unique no-arbitrage price. The binomial tree is given by



   Here p^*, q^* are risk neutral probabilities, and obviously $q^* = 1 - p^*$. We determine p^* by recognising that

$$
\begin{aligned}
S_0 = 100 = \frac{1}{1+0}\mathbb{E}[S_1] = \frac{1}{1+0}\left(110p^* + (1-p^*)90\right) \\
\Rightarrow p^* = 0.5.
\end{aligned}
$$

   Option prices on both underlying are worth the same amount:

$$
C_0 = \frac{1}{1+0}\mathbb{E}[C_1] = (110-100)^+ p^* + (90-100)^+(1-p^*) = 5.
$$

   - Call options with strikes at 90 and 110 are worth, respectively

$$
\begin{aligned}
C_0(K=90) = (110-90)^+ p^* + (90-90)^+(1-p^*) = 10 \\
C_0(K=110) = (110-110)^+ p^* + (90-110)^+(1-p^*) = 0.
\end{aligned}
$$

6. This is a trinomial tree. The underlying principles are still the same, and the probabilities has to sum to 1. This is a straightforward question since an option price is given. So we have

$$
\begin{aligned}
C_0(K=100) = 2 = \frac{1}{1+0}\mathbb{E}[C_1] = (110-100)^+ p^* + (100-100)^+ q^* + (90-100)^+(1-p^*-q^*) \\
\Rightarrow p^* = \frac{1}{5}.
\end{aligned}
$$

   We already have enough information to work out the option price struck at 105, this is given by

$$
C_0(K=105) = (110-105)^+ p^* + (100-105)^+ q^* + (90-105)^+(1-p^*-q^*) = 1.
$$

7. (a) To work out the no-arbitrage boundary for a call option struck at 100, we note that we only have one parameter Δ to match 4 market states. Suppose we long Δ amount of stock to hedge against shorting one call option $C(K=100)$, our portfolio could be worth either one of

$$
115\Delta - 15, \quad 105\Delta - 5, \quad 95\Delta, \quad 85\Delta
$$

tomorrow. Since the market is incomplete, there's no unique no-arbitrage price, and we end up with just a boundary instead. We obviously want to obtain the tightest no-arbitrage price bound possible. A general rule of thumb is therefore to ensure that the portfolio is worth the same in the extreme up tick vs. the extreme down tick (minimizing its variation). So we determine Δ as

$$115\Delta - 15 = 85\Delta \quad \Rightarrow \quad \Delta = 0.5.$$

This gives a portfolio worth of either

$$42.5, \quad 47.5 \quad, 47.5, \quad 42.5.$$

Since the worth of the portfolio $S \times \Delta - C$ varies between 42.5 and 47.5, and since interest rate is zero, then today the option must be worth between 2.5 and 7.5.

(b) Next, it is given that this option is worth $C_0(K = 100) = 5$ exactly. Let p_1^*, p_2^*, p_3^*, and p_4^* denote the risk neutral probabilities of the stock moving to 85, 95, 105, and 115, respectively. We have that

$$C_1(K = 100) = p_4^* \times (115 - 100) + p_3^* \times (105 - 100) = 5, \quad \Rightarrow \quad p_4^* = \frac{1 - p_3^*}{3}.$$

Since probabilities must lie between 0 and 1, we see that p_4^* satisfies the inequality

$$0 < p_4^* < \frac{1}{3}.$$

So the no-arbitrage boundary for the call option $C_1(K = 110) = p_4^* \times (115 - 110)$ is

$$0 < C_1(K = 110) < \frac{5}{3}.$$

8. Let p^* denote the risk neutral probability for an up tick. Since interest rate is 0, based on the principle that $\mathbb{E}[S_1] = S_0$, we have

$$p^* \times 70 + (1 - p^*) \times 40 = 50, \quad \Rightarrow \quad p^* = \frac{1}{3}.$$

9. Let p_1^*, p_2^*,and p_3^* denote the risk neutral probability of the stock moving to 40, 55,and 70 respectively. We have the following relationship

$$\begin{aligned} p_3^* \times 70 + p_2^* \times 55 + p_1^* \times 40 &= 50 \\ p_1^* + p_2^* + p_3^* &= 1. \end{aligned}$$

Substituting $p_3^* = 1 - p_1^* - p_2^*$ into the first, it simplifies to

$$p_1^* = \frac{4 - 3p_2^*}{6}.$$

Since $0 < p_2^* < 1$, from here we can infer that

$$\frac{1}{6} < p_1^* < \frac{2}{3}.$$