Session 7: Equivalent Martingale Measure

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Intuition behind Measure Change

Suppose we have a normally distributed stochastic process $X_t \sim N(-\kappa t, t)$, its probability density function is given by

$$f(x) = \frac{1}{\sqrt{2\pi t}} \exp\left[-\frac{(x+\kappa t)^2}{2t}\right].$$

Note that the process has a **drift coefficient** of $-\kappa$, which can be either positive or negative. For any bounded function $g(\cdot)$, we have the expectation

$$\mathbb{E}[g(X_t)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx.$$

Next, let us introduce another probability density function without drift:

$$\tilde{f}(x) = \frac{1}{\sqrt{2\pi t}} \exp\left[-\frac{x^2}{2t}\right].$$

Note that we can write the same expectation as:

$$\mathbb{E}[g(X_t)] = \int_{-\infty}^{\infty} g(x) \frac{f(x)}{\tilde{f}(x)} \tilde{f}(x) dx.$$

Intuition behind Measure Change

Since the probability density functions are non-zero, its ratio is well defined, and can be simplify into:

$$\frac{f(x)}{\tilde{f}(x)} = \frac{\frac{1}{\sqrt{2\pi t}}\exp\left[-\frac{(x+\kappa t)^2}{2t}\right]}{\frac{1}{\sqrt{2\pi t}}\exp\left[-\frac{x^2}{2t}\right]} = \exp\left(-\kappa x - \frac{1}{2}\kappa^2 t\right).$$

We can call this the **likelihood ratio**, or more commonly the **Radon-Nikodym derivative** in continuous-time model.

⇒ In discrete-time model, it is simply a ratio of two probabilities

⇒ In continuous-time model, it is a ratio of two probability density functions

To appreciate why it is often referred to as a "derivative", note that:

$$\int g \, d\mu = \int g \, \frac{d\mu}{d\nu} \, d\nu.$$

Intuition behind Measure Change

Let $\mathbb{P}$ denote the probability measure under the PDF $f(x)$, and let $\tilde{\mathbb{P}}$ denote the probability measure under the PDF $\tilde{f}(x)$.

Note that the Radom-Nikodym derivative is **strictly positive**:

$$\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}} = \frac{f(x)}{\tilde{f}(x)} > 0,$$

and that

$$\mathbb{E}^{\tilde{P}}\left[\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}}\right] = 1.$$

The Radon-Nikodym deriative allows us to **change the probability measure** under which the expectation is evaluated:

$$\mathbb{E}^{P}[g(X_t)] = \mathbb{E}^{\tilde{P}}\left[g(X_t)\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}}\right].$$

Note that the two probability measures $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are equivalent (why?).

Girsanov Theorem

Using our definition of $\frac{d\mathbb{Q}}{d\mathbb{P}}$, we can show that if W_t is a standard Brownian motion under $\mathbb{P}$, then it becomes a Brownian motion with a drift coefficient $-\kappa$ under $\mathbb{Q}$, i.e. $W_t^* = W_t + \kappa t$. In addition, W_t^* follows the following $\mathbb{Q}$-Brownian motion properties:

1. $\mathbb{E}^Q[W_t^*] = 0$
2. $\mathbb{E}^Q[e^{\sigma W_t^*}] = e^{\frac{1}{2}\sigma^2 t}$
3. $\mathbb{E}^Q[e^{\sigma(W_{t+s}^* - W_s^*)}|s] = e^{\frac{1}{2}\sigma^2 t}$

Girsanov Theorem

If W_t is a $\mathbb{P}$-Brownian motion and κ_t satisfies $\mathbb{E}^P\left[\exp\left(\frac{1}{2}\int_0^T \kappa_t^2 dt\right)\right] < \infty$, then there exists a measure $\mathbb{Q}$ such that

1. $\mathbb{Q}$ is equivalent to $\mathbb{P}$
2. $\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \kappa_t \, dW_t - \frac{1}{2}\int_0^T \kappa_t^2 \, dt\right)$
3. $W_t^* = W_t + \int_0^t \kappa_u \, du$ is a $\mathbb{Q}$-Brownian motion.

Girsanov Theorem — Example

Example Let W_t denote a $\mathbb{P}$-Brownian motion, and let W_t^* denote a $\mathbb{Q}$-Brownian motion. The probability measures $\mathbb{P}$ and $\mathbb{Q}$ are equivalent and are related by the Radon-Nikodym derivative. Show that

1. $\mathbb{E}^P[W_t] = 0$
2. $\mathbb{E}^Q[W_t] = -\kappa t$
3. $\mathbb{E}^Q[W_t^*] = 0$
4. $\mathbb{E}^P[W_t^*] = \kappa t$

Girsanov Theorem — Example

Example Consider a stochastic process X_t satisfying the following SDE

$$dX_t = \mu X_t dt + \sigma X_t dW_t,$$

where W_t is a $\mathbb{P}$-Brownian motion. Change the measure so that the drift coefficient of X_t is ν instead of μ.

Solution Again, rewriting our SDE in the following format

$$dX_t = \nu X_t dt + \sigma X_t \left(dW_t + \frac{\mu - \nu}{\sigma} dt \right),$$

we let $\kappa = \frac{\mu-\nu}{\sigma}$, and apply Girsanov to get an equivalent measure $\mathbb{Q}$ under which

$$W_t^* = W_t + \frac{\mu - \nu}{\sigma} t$$

is a $\mathbb{Q}$-Brownian motion. The process X_t satisfies the following SDE under this new measure

$$dX_t = \nu X_t dt + \sigma X_t dW_t^*,$$

where W_t^* is a $\mathbb{Q}$-Brownian motion. $\triangleleft$

Before Black-Scholes:

> *Various people developed models of derivatives that are actuarial in that they define the value of an option as the empirical expected discounted value of its payoffs.*
>
> *This value does of course depend on the volatility of the stock. But they don't know what rate of return to use for growing the stock price into the future, and they don't know what rate to use for discounting the payoffs.*
>
> *People who wanted to use this model had to forecast the return of the stock and figure out what discount rate to use as a consequence of its risk. It was personal.*
>
> — Emanuel Derman

Source: A Stylized History of Quantitative Finance

Black-Scholes (1971–3)

> *Hedge to eliminate stock risk from option. Require that hedged portfolio, which is riskless, earns the known riskless rate. Then we get the same formula for the option value as the actuarial one, but where all growth and discount rates are riskless rates.*
>
> *The value of the option does not depend on the expected return of the stock, since that has been hedged away. Instead it depends on the riskless rate, which is known, and on the future volatility of the stock.*

— Emanuel Derman

Source: A Stylized History of Quantitative Finance

Before Black-Scholes:

$$\text{Call} = S_0 \Phi\left(\frac{\log\frac{S_0}{K} + \left(\mu + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) - Ke^{-fT}\Phi\left(\frac{\log\frac{S_0}{K} + \left(\mu - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

Black-Scholes:

$$\text{Call} = S_0 \Phi\left(\frac{\log\frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) - Ke^{-rT}\Phi\left(\frac{\log\frac{S_0}{K} + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

1997 Nobel Prize citation:

> *Robert C. Merton and Myron S. Scholes have, in collaboration with the late Fischer Black, developed a pioneering formula for the valuation of stock options.*
>
> *Their methodology has paved the way for economic valuations in many areas.*
>
> *It has also generated new types of financial instruments and facilitated more efficient risk management in society.*
>
> — The Royal Swedish Academy of Sciences

Black-Scholes Assumptions

The Black-Scholes market model contains two differential equations

$$\begin{cases} dS_t = \mu S_t dt + \sigma S_t dW_t \\ dB_t = r B_t dt \end{cases}$$

The context is that the market contains

1. A **risky asset** S_t, typically a stock price process.
2. A **risk-free asset** B_t, typically a risk-free bond.

Assumptions made include:

1. Underlying is lognormal with constant mean and variance.
2. The risk-free rate r is a constant.
3. No dividend is paid during the life of the option.
4. Short selling is permitted.
5. No risk-free arbitrage opportunities.
6. Trading is possible in continuous time.
7. No transaction costs, no taxes and no trading limits.

Choice of Numeraire

Based on previous discussions on the stock price process, we have established that it is growing at the risk-free rate under the risk-neutral measure:

$$\mathbb{E}_t^*[S_{t+\Delta t}] = S_t e^{r\Delta t}.$$

The expectation notation $\mathbb{E}^*$ is used to indicate that the expectation is evaluated under the risk-neutral measure $\mathbb{Q}^*$. This relationship can be rearranged into

$$\frac{S_t}{e^{rt}} = \mathbb{E}_t^*\left[\frac{S_{t+\Delta t}}{e^{r(t+\Delta t)}}\right].$$

- In words, this means that the best estimate of the price ratio on the subsequent time step is just the price ratio on the current time step.
- The security in the denominator of the price ratio expression is called the **numeraire** security.
- The only requirement for a particular security to qualify as a numeraire security is that it has to be **strictly positive** at all times.
- The risk-free money market account paying an interest of r is a popular choice of numeraire.

Equivalent Martingale Measure

Key concepts:

- In a complete market, any derivative security is attainable. Since we can hedge a derivative product perfectly, the derivative security **loses its randomness** and **behaves like a risk-less bond**.

- So real world probabilities do not come into the picture in a risk-neutral valuation framework at all.

- If we hedge according to our risk-neutral valuation framework, then all risk is eliminated, and the hedged portfolio grows at a risk-free rate.

- Consequently, the hedged portfolio divided by the risk-free bond is a **martingale**.

- Two probabilities measures are equivalent if they agree on what is possible and what is impossible.

Equivalent Martingale Measure

- In other words, if one portfolio is an arbitrage in one measure, then it is an arbitrage in all other equivalent measures.

- If the option price we determined under the risk-neutral measure is **arbitrage-free**, then it is arbitrage-free in the real world.

- If we can express security price processes discounted by a numeraire security as a martingale, then there can be no arbitrage opportunities.

- Under the risk-neutral probabilities associated to this numeraire security, the discounted option price is also a martingale, and we can therefore determine its present value.

- The risk-free money market account $B_t = B_0 e^{rt}$ is a common choice for numeraire (used by Harrison and Kreps (1979)), but **the choice is arbitrary**.

Application of EMM — Black-Scholes

Under the **Black-Scholes economy**, let B_t denote the value of the money-market account with $B_0 = 1$, and let S_t denote the stock price process. The following differential equations described their dynamics:

$$dB_t = rB_t dt$$
$$dS_t = \mu S_t dt + \sigma S_t dW_t.$$

Here W_t is a $\mathbb{P}$-Brownian motion under the real-world measure, and μ is its (unknown) drift coefficient.

Question Which is the most difficult parameter to estimate among r, μ, and σ?

Source: Google Finance

Application of EMM — Black-Scholes

The value of B_t is strictly positive and can be used as a numeraire. Define the relative price process $X_t = \frac{S_t}{B_t} = f(S_t, B_t)$, we can apply Itô's formula to obtain

$$dX_t = (\mu - r)X_t dt + \sigma X_t dW_t.$$

To identify the equivalent martingale measure we apply **Girsanov's theorem** with $\kappa = \frac{\mu - r}{\sigma}$ to obtain:

$$dW_t^* = dW_t + \frac{\mu - r}{\sigma} dt,$$

where W_t^* is a standard Brownian motion under probability measure $\mathbb{Q}^*$. Here the * notation is used to indicate we have chosen the **risk-free account B_t as our numeraire**, which is the most common choice. Substituting, we obtain

$$\begin{aligned} dX_t &= (\mu - r)X_t dt + \sigma X_t \left(dW_t^* - \frac{\mu - r}{\sigma} dt \right) \\ &= \sigma X_t dW_t^*. \end{aligned}$$

Application of EMM — Black-Scholes

This is the only measure which turns the relative price process into martingale. We can now determine what is the stock price process under this unique **martingale measure** $\mathbb{Q}^*$:

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sigma S \left(dW_t^* - \frac{\mu - r}{\sigma} dt \right) \\
&= r S_t dt + \sigma S_t dW_t^*.
\end{aligned}
$$

Under the equivalent martingale measure, the drift of the stock μ is irrelevant and is replaced by the risk-free interest rate r. The solution to this stochastic differential equation is

$$
S_T = S_0 \exp \left[\left(r - \frac{1}{2} \sigma^2 \right) T + \sigma W_T^* \right].
$$

Application of EMM — Black-Scholes

A European call option with strike K and maturing at time T where $V_T = (S_T - K)^+$ can be evaluated by **martingale pricing theorem** as follow

$$
\begin{aligned}
\frac{V_0}{B_0} &= \mathbb{E}^*\left[\frac{V_T}{B_T}\right] = \mathbb{E}^*\left[\frac{(S_T - K)^+}{B_T}\right] \\
&= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[S_0 e^{\left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}x} - K\right]^+ e^{-\frac{x^2}{2}}\, dx \\
&= S_0 \Phi(d_1) - K e^{-rT} \Phi(d_1 - \sigma\sqrt{T}), \quad d_1 = \frac{\log\left(\frac{S_0}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}.
\end{aligned}
$$

We have already learned how to derive the Black-Scholes **option pricing formula** by evaluating the expectation.