Bachelier Model (1900)

Louis Bachelier was the first mathematician to use random walk to analyse stock prices in 1900.

In Bachelier model, the stock price process is a **symmetrical random walk**, correspond to a market under **equilibrium**. This follows a **normal distribution**:

$$S_T = S_0 + \sigma W_T, \quad W_T \sim N(0, T).$$

Given this definition, we can then proceed to derive valuation formulas for vanilla European options.

However, a shortcoming of this model is that the lack of a lower bound at 0.

In other words, while this is a reasonable model for interest rates, it leads to non-zero probability for negative stock prices.

Bachelier Model – Arithmetic Brownian Process

The Bachelier model for the stock price process is defined as

$$dS_t = \sigma\, dW_t. \quad \Rightarrow \quad \int_0^T dS_t = \sigma \int_0^T dW_T$$

Integrating this stochastic equation, we can show that the terminal stock price is normally distributed as

$$S_T - S_0 = \sigma (W_T - W_0)$$

$$S_T \sim N(S_0, \sigma^2 T). \qquad S_T = S_0 + \sigma W_T = S_0 + \sigma\sqrt{T} X,$$

$$X \sim N(0,1)$$

Let V_c denote the price of a European call option, we have:

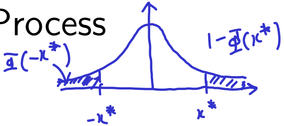
$$\boxed{V_c = e^{-rT}\mathbb{E}[(S_T - K)^+]} = e^{-rT}\mathbb{E}\left[(S_0 + \sigma W_T - K)^+\right]$$
$$= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(S_0 + \sigma\sqrt{T}x - K\right)^+ e^{-\frac{x^2}{2}}\, dx$$

Note that $\left(S_0 + \sigma\sqrt{T}x - K\right)^+ = 0$ whenever $S_0 + \sigma\sqrt{T}x - K < 0$, and will only take on non-zero values when

$$S_0 + \sigma\sqrt{T}x - K > 0 \quad \Rightarrow \quad x > \frac{K - S_0}{\sigma\sqrt{T}} = x^*.$$

$\Phi(\infty) - \Phi(x^*) = 1 - \Phi(x^*) = \Phi(-x^*)$

Bachelier Model – Arithmetic Brownian Process



Hence, we now write

$$
\begin{aligned}
V_c &= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{x^*}^{\infty} \left(S_0 + \sigma\sqrt{T}x - K\right) e^{-\frac{x^2}{2}}\, dx \\
&= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{x^*}^{\infty} (S_0 - K) e^{-\frac{x^2}{2}}\, dx + \frac{e^{-rT}}{\sqrt{2\pi}} \int_{x^*}^{\infty} \sigma\sqrt{T} x e^{-\frac{x^2}{2}}\, dx \\
&= e^{-rT}(S_0 - K)\Big[\Phi(\infty) - \Phi(x^*)\Big] - \frac{e^{-rT}\sigma\sqrt{T}}{\sqrt{2\pi}} \Big[e^{-\frac{x^2}{2}}\Big]_{x^*}^{\infty} \\
&= e^{-rT}\left[(S_0 - K)\Phi(-x^*) + \frac{\sigma\sqrt{T}}{\sqrt{2\pi}} e^{-\frac{(x^*)^2}{2}}\right] \\
&= e^{-rT}\left[(S_0 - K)\Phi(-x^*) + \frac{\sigma\sqrt{T}}{\sqrt{2\pi}} e^{-\frac{(-x^*)^2}{2}}\right] \\
&= e^{-rT}\left[(S_0 - K)\Phi\left(\frac{S_0 - K}{\sigma\sqrt{T}}\right) + \sigma\sqrt{T}\phi\left(\frac{S_0 - K}{\sigma\sqrt{T}}\right)\right].
\end{aligned}
$$

For **at-the-money (ATM)** options, we have $K = S_0$, and this formula reduces to $V_c = e^{-rT}\sigma\sqrt{\frac{T}{2\pi}}$.

$$\int_{x^*}^{\infty} x e^{-\frac{x^2}{2}} dx$$

let $\quad u = \frac{x^2}{2}$

$$\frac{du}{dx} = \frac{2x}{2} = x \quad \Rightarrow du = x\,dx$$

$$\int_{x^*}^{\infty} x e^{-\frac{x^2}{2}} dx = \int e^{-u} du = -\left[e^{-u}\right] = -\left[e^{-\frac{x^2}{2}}\right]_{x^*}^{\infty}$$

$$= -\left[0 - e^{-\frac{x^{*2}}{2}}\right]$$

$$= e^{-\frac{x^{*2}}{2}}$$

$$\text{Call} = e^{-rT}\left[(S_0-K)\,\Phi\left(\frac{S_0-K}{\sigma\sqrt{T}}\right) + \sigma\sqrt{T}\,\phi\left(\frac{S_0-K}{\sigma\sqrt{T}}\right)\right]$$

ATM : $S_0 = K$

$$\text{ATM Call} = e^{-rT}\sigma\sqrt{T}\,\phi(0) = e^{-rT}\sigma\sqrt{T}\cdot\frac{1}{\sqrt{2\pi}}$$

$$\simeq \sigma\sqrt{T}\cdot 0.4$$

$$\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

Black-Scholes Model (1973)

In a landmark 1973 paper, Fischer Black and Myron Scholes introduced the **Black-Scholes model**, which models the stock price as

$$S_T = S_0 \exp\left[\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T\right], \quad W_T \sim N(0,T).$$

Given this definition, we can readily verify that $\mathbb{E}[S_T] = S_0 e^{rT}$. Rearranging, we can write it as

$$\frac{S_T}{S_0} = \exp\left[\left(r - \frac{\sigma^2}{2}\right)T + \sigma N(0,T)\right].$$

Consequently,

$$\log\frac{S_T}{S_0} = \left(r - \frac{\sigma^2}{2}\right)T + \sigma N(0,T) \sim N\left(\left(r - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right).$$

Alternatively, we can also write S_T as

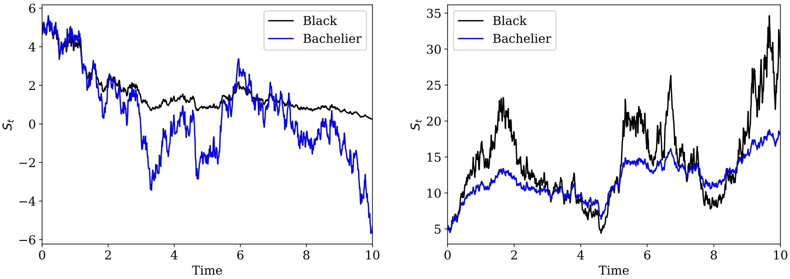
$$\log S_T = \log S_0 + \left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}N(0,1).$$

Black-Scholes *vs.* Bachelier

Below are 2 sample paths (same Brownian motion) from the 2 models:

$$\text{Black-Scholes:} \quad S_{t+\Delta t} = S_t \exp\left[\left(r - \frac{\sigma^2}{2}\right)\Delta t + \sigma \cdot (W_{t+\Delta t} - W_t)\right]$$

$$\text{Bachelier:} \quad S_{t+\Delta t} = S_t + \sigma \cdot (W_{t+\Delta t} - W_t)$$



Question How do the two models compare?

Black-Scholes Model – Geometric Brownian Process

Under the Black-Scholes model, the stock price process follows the stochastic differential equation:

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

Note that a direct integration does not allow us to solve the SDE:

$$S_T = S_0 + \int_0^T rS_t dt + \int_0^T \sigma S_t dW_t.$$

However, we can solve the SDE by first applying **Itô's formula** to the function $X_t = f(S_t) = \log(S_t)$:

$$dX_t = \left(r - \frac{\sigma^2}{2}\right) dt + \sigma dW_t$$

Integrating both sides and substituting for X_t, we arrive at

$$S_T = S_0 \exp\left[\left(r - \frac{\sigma^2}{2}\right) T + \sigma W_T\right].$$

$$dS_t = rS_t\,dt + \sigma S_t\,dW_t$$

$$f(x) = \log(x)$$

$$X_t = f(S_t) = \log(S_t)$$

$$f'(x) = \frac{1}{x}, \quad f''(x) = -\frac{1}{x^2}$$

Itô's formula:

$$\boxed{dX_t = f'(S_t)\,dS_t + \frac{1}{2} f''(S_t)\,(dS_t)^2}$$

$$= \frac{1}{\cancel{S_t}}\left(r\cancel{S_t}\,dt + \sigma \cancel{S_t}\,dW_t\right) - \frac{1}{2}\cdot\frac{1}{\cancel{S_t^2}}\cdot \sigma^2 \cancel{S_t^2}\,dt$$

$$= r\,dt + \sigma\,dW_t - \frac{\sigma^2}{2}\,dt$$

$$\int_0^T dX_t = \int_0^T \left(r - \frac{\sigma^2}{2}\right) dt + \int_0^T \sigma\,dW_t$$

$$X_T - X_0 = \left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T$$

$$\log S_T - \log S_0 = \left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T$$

$$\log \frac{S_T}{S_0} = \left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T$$

$$\frac{S_T}{S_0} = e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T}$$

Black-Scholes Model – Option Pricing

Now let us derive the option pricing formula for a European call option under Black-Scholes model.

$$\begin{aligned} V_c &= e^{-rT}\mathbb{E}[(S_T-K)^+] \\ &= \frac{e^{-rT}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}x}-K\right)^+ e^{-\frac{x^2}{2}}\,dx \end{aligned}$$

Again, the terms in the $(\cdot)^+$ operator will need to be positive for it take non-zero values

$$S_T-K>0 \quad\Rightarrow\quad x>\frac{\log\left(\frac{K}{S_0}\right)-\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}=x^*.$$

Now we can proceed to evaluate the integral

$$V_c=\frac{e^{-rT}}{\sqrt{2\pi}}\int_{x^*}^{\infty}\left(S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}x}-K\right)e^{-\frac{x^2}{2}}\,dx.$$

Black-Scholes Model – Option Pricing

Next, we have:

$$
\begin{aligned}
V_c &= \frac{e^{-\cancel{r}T}}{\sqrt{2\pi}} \int_{x^*}^{\infty} S_0 e^{\left(\cancel{r}-\frac{\sigma^2}{2}\right)T+\overbrace{\sigma\sqrt{T}x}} e^{-\frac{x^2}{2}}\, dx - \frac{e^{-rT}}{\sqrt{2\pi}} \int_{x^*}^{\infty} K e^{-\frac{x^2}{2}}\, dx \\
&= \frac{S_0 e^{-\frac{\sigma^2 T}{2}}}{\sqrt{2\pi}} \int_{x^*}^{\infty} e^{-\frac{x^2-2\sigma\sqrt{T}x}{2}}\, dx - \frac{Ke^{-rT}}{\sqrt{2\pi}} \int_{x^*}^{\infty} e^{-\frac{x^2}{2}}\, dx \\
&= \frac{S_0 e^{-\frac{\sigma^2 T}{2}}}{\sqrt{2\pi}} \int_{x^*}^{\infty} e^{-\frac{\overbrace{x^2-2\sigma\sqrt{T}x+\sigma^2T}-\sigma^2T}{2}}\, dx - \frac{Ke^{-rT}}{\sqrt{2\pi}} \int_{x^*}^{\infty} e^{-\frac{x^2}{2}}\, dx \\
&= \frac{S_0 e^{\cancel{-\frac{\sigma^2 T}{2}}}}{\sqrt{2\pi}} \int_{x^*}^{\infty} e^{-\frac{\overbrace{\left(x-\sigma\sqrt{T}\right)^2}}{2}} e^{\cancel{\frac{\sigma^2 T}{2}}} dx - \frac{Ke^{-rT}}{\sqrt{2\pi}} \int_{x^*}^{\infty} e^{-\frac{x^2}{2}}\, dx \\
&= \frac{S_0}{\sqrt{2\pi}} \int_{x^*}^{\infty} e^{-\frac{\left(x-\sigma\sqrt{T}\right)^2}{2}}\, dx - \frac{Ke^{-rT}}{\sqrt{2\pi}} \int_{x^*}^{\infty} e^{-\frac{x^2}{2}}\, dx
\end{aligned}
$$

where we have used the **completing the square** trick.

$$\int_{x^*}^{\infty} e^{-\frac{(x-\sigma\sqrt{T})^2}{2}}\,dx = \int_{x^*-\sigma\sqrt{T}}^{\infty} e^{-\frac{y^2}{2}}\,dy$$

let $y = x - \sigma\sqrt{T}$

$dy = dx$

$x = x^* \Rightarrow y = x^* - \sigma\sqrt{T}$

$x \to \infty \Rightarrow y \to \infty$

Black-Scholes Model – Option Pricing

Finally, we obtain

$$\begin{aligned} V_c &= S_0\Big[\Phi(\infty) - \Phi(x^* - \sigma\sqrt{T})\Big] - Ke^{-rT}\Phi(-x^*) \\ &= S_0\Phi(-x^* + \sigma\sqrt{T}) - Ke^{-rT}\Phi(-x^*) \\ &= S_0\Phi\left(\frac{\log\frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) - Ke^{-rT}\Phi\left(\frac{\log\frac{S_0}{K} + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) \end{aligned}$$

In many references, it is common to let

$$d_1 = \frac{\log\frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T},$$

leading to

$$V_c = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2).$$

Black Model (1976) – Forward Price Process

So far we have regarded the underlying as the stochastic variable, and derived a stochastic differential equation to describe its price dynamic.

In 1976, Fischer Black proposed **modeling the forward price** instead of the underlying price. We have the definition of the forward price

$F_0 = S_0 e^{rT}$

$$F_t = e^{r(T-t)} S_t \quad = f(t, S_t)$$

and the underlying price process of

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

Applying Itô's formula to the function $f(t,x) = e^{r(T-t)}x$ allows us to write down the stochastic differential equation for the forward price

$$dF_t = \sigma F_t dW_t,$$

which is a more compact equation—it is driftless and is therefore a **martingale**.

$$dS_t = rS_t\,dt + \sigma S_t\,dW_t$$

$$F_t = e^{r(T-t)} \cdot S_t = f(t, S_t)$$

$$f(t,x) = e^{r(T-t)} \cdot x$$

$$f_t = -r e^{r(T-t)} \cdot x$$

$$f_x = e^{r(T-t)}, \quad f_{xx} = 0$$

Itô's formula:

$$dF_t = f_t(t,S_t)\,dt + f_x(t,S_t)\,dS_t + \frac{1}{2} f_{xx}(t,S_t)(dS_t)^2$$

$$= -r e^{r(T-t)} S_t\,dt + e^{r(T-t)} \cdot (rS_t\,dt + \sigma S_t\,dW_t) + 0$$

$$= \sigma S_t e^{r(T-t)}\,dW_t$$

$$= \sigma F_t\,dW_t$$

Black Model (1976) – Forward Price Process

The Black model is defined on the forward price and is given by

$dS_t = r S_t dt + \sigma S_t dW_t$,

$$dF_t = \sigma F_t dW_t.$$

As this is also a geometric process, we can solve this stochastic differential equation by applying Itô's formula to $X_t = f(F_t)$ where $f(x) = \log(x)$.

The solution is given by:

$S_T = S_0 e^{(r-\frac{\sigma^2}{2})T + \sigma W_T}$

$$F_T = F_0 e^{-\frac{\sigma^2 T}{2} + \sigma W_T}$$

Let $D(0,T) = e^{-rT}$ denote the **discount factor**, under this model the price of a European call option is given by

$$V_c = D(0,T)\left[F_0 \Phi\left(\frac{\log\frac{F_0}{K} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) - K\Phi\left(\frac{\log\frac{F_0}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)\right].$$

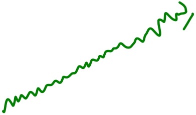
$$dr_t = \kappa(\theta - r_t)\,dt + \sigma\, dW_t$$

mean-reversion speed

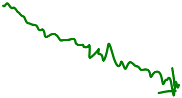
long-run average

$\theta = r_t$: $dr_t = \kappa(\ 0\ )\,dt + \sigma\, dW_t$

$\theta > r_t$: $dr_t = \kappa(+\text{ve})\,dt + \sigma\, dW_t$



$\theta < r_t$: $dr_t = \kappa(-\text{ve})\,dt + \sigma\, dW_t$



Mean-reverting Process – Vasicek Model

The **Ornstein-Uhlenbeck process** is used in solid-state physics to model gas molecules under the influence of pressure and temperature.

Oldrich Vasicek adapted this model in 1977 to model interest rate as a **mean reverting stochastic process**, given by

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t.$$

Applying Itô formula to $X_t = e^{\kappa t} r_t = f(t, r_t)$, we obtain

$$\begin{aligned} d(e^{\kappa t} r_t) &= \kappa e^{\kappa t} r_t dt + e^{\kappa t} dr_t \\ &= \kappa\theta e^{\kappa t} dt + \sigma e^{\kappa t} dW_t. \end{aligned}$$

Integrating both sides from 0 to t, we can obtain a solution to the stochastic differential equation

$$\int_0^t d(e^{\kappa u} r_u) = \int_0^t \kappa\theta e^{\kappa u} du + \int_0^t \sigma e^{\kappa u} dW_u$$

$$r_t = r_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}) + \sigma \int_0^t e^{\kappa(u-t)} dW_u.$$

$$dr_t = \kappa(\theta - r_t)\,dt + \sigma\,dW_t$$

$$X_t = e^{\kappa t}\cdot r_t = f(t, r_t)$$

$$f(t,x) = e^{\kappa t}\cdot x$$

$$f_t = \kappa e^{\kappa t}\cdot x$$

$$f_x = e^{\kappa t}, \quad f_{xx} = 0$$

Itô's formula:

$$dX_t = f_t(t,r_t)\,dt + f_x(t,r_t)\,dr_t + \frac{1}{2} f_{xx}(t,r_t)(dr_t)^2$$

$$= \cancel{\kappa e^{\kappa t}\cdot r_t\,dt} + e^{\kappa t}\left[\kappa(\theta - \cancel{r_t})\,dt + \sigma\,dW_t\right] + 0$$

$$dX_t = \theta\kappa e^{\kappa t}\,dt + \sigma e^{\kappa t}\,dW_t$$

$$\int_0^t dX_u = \theta\kappa \int_0^t e^{\kappa u}\,du + \sigma \int_0^t e^{\kappa u}\,dW_u$$

$$X_t - X_0 = \theta\left[e^{\kappa u}\right]_0^t + \sigma \int_0^t e^{\kappa u}\,dW_u$$

$$e^{\kappa t} r_t - e^{\kappa \cdot 0} \cdot r_0 = \theta\left(e^{\kappa t} - 1\right) + \sigma \int_0^t e^{\kappa u}\, dW_u$$

$$r_t = r_0 e^{-\kappa t} + \theta\left(1 - e^{-\kappa t}\right) + \sigma \int_0^t e^{\kappa(u-t)}\, dW_u$$

Mean-reverting Process – Vasicek Model

Taking expectation on both sides gives us the mean

$$\mathbb{E}[r_t] = r_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}).$$

Recall **Itô's Isometry theorem** states that

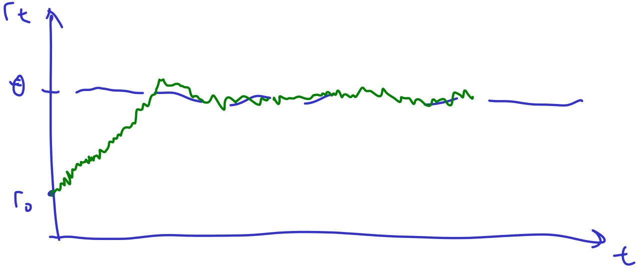
$$\mathbb{E}\left[\left(\int_0^T X_t dW_t\right)^2\right] = \mathbb{E}\left[\int_0^T X_t^2 dt\right].$$

Applying it to our case,

$$\begin{aligned} V[r_t] &= \mathbb{E}\left[\left(\sigma \int_0^t e^{\kappa(u-t)} dW_u\right)^2\right] \\ &= \mathbb{E}\left[\sigma^2 \int_0^t e^{2\kappa(u-t)} du\right] = \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa t}\right). \end{aligned}$$

The distribution of r_t is therefore given by

$$r_t \sim N\left(r_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}), \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa t})\right).$$



r_t
θ
r_0
t

$$\int_0^T dW_t = \lim_{n\to\infty} \sum_{i=1}^{n} \left(W_{t_i} - W_{t_{i-1}} \right)$$

$$= \lim_{n\to\infty} \left[\left(\cancel{W_{t_1}} - W_{t_0} \right) + \left(\cancel{W_{t_2}} - \cancel{W_{t_1}} \right) + \dots + \left(W_{t_n} - \cancel{W_{t_{n-1}}} \right) \right]$$

$$= \lim_{n\to\infty} \left[W_{t_n} - W_{t_0} \right]$$

$$= W_T - W_0$$

Session 6: Valuation Framework and Stochastic Volatility Models

Tee Chyng Wen

QF620 Stochastic Modelling in Finance

Pricing Models *vs* Reporting Models

So far we have been formulating our models as **pricing models**.

⇒ As the name suggests, pricing models are used to price and risk-manage derivatives.

⇒ The **dynamic** of the pricing models ought to conform to the modeler's intuition of the underlying asset's **evolution over time**.

⇒ The **Greeks** of the pricing models should accurately capture the **sensitivities** of the derivatives.

All financial institutions with a trading desk tend to have their own choices of pricing models, with the model parameters (e.g. σ, β, etc.) calibrated to the liquid option markets.

Pricing Models *vs* Reporting Models

Apart from pricing models, many (option) exchanges have also adopted the notion of **reporting models**.

$\Rightarrow$ A reporting model is used merely to report market option prices—these prices are driven by **supply and demand**.

$\Rightarrow$ Since it is often more elegant to report implied volatilities instead of prices (why?), a reporting model is required to perform this **conversion from price to volatility**.

$\Rightarrow$ Reporting models tend to make simplifying assumption about the asset dynamics, given that the primary objective is to arrive at an analytical tractable pricing formula for **price-volatility conversion**.

$\Rightarrow$ This reporting model's parameters (e.g. implied volatilities) can then be displayed on brokers' screens to communicate live option prices.

Option Market Screenshot

Example SPX index option chain, expiration on 15-Oct-2021.

IV: 19.3

CALLS BID SIZE	BID	ASK	ASK SIZE	IMPL. BID VOL.	IMPL. ASK VOL.	STRIKE	PUTS BID SIZE	BID	ASK	ASK SIZE	IMPL. BID VOL.	IMPL. ASK VOL.
20	79.80	81.50	20	22.568%	23.495%	4290	75	13.00	13.40	75	23.071%	23.225%
20	75.80	77.50	20	22.373%	23.271%	4295	75	13.90	14.40	50	22.861%	23.010%
30	71.90	73.50	30	22.200%	23.143%	4300	76	15.00	15.40	21	22.600%	22.817%
30	68.00	69.60	30	21.974%	22.890%	4305	70	16.10	16.50	20	22.432%	22.573%
30	64.20	65.70	30	21.763%	22.656%	4310	70	17.20	17.70	63	22.210%	22.347%
30	60.40	62.00	30	21.566%	22.303%	4315	71	18.40	18.90	43	21.934%	22.135%
38	56.80	58.20	53	21.311%	22.097%	4320	63	19.70	20.20	43	21.737%	21.933%
53	53.10	54.50	38	21.193%	21.835%	4325	20	21.10	21.60	43	21.482%	21.675%
38	49.60	50.90	38	21.139%	21.580%	4330	40	22.50	23.00	43	21.170%	21.422%
42	46.60	47.40	38	20.893%	21.266%	4335	40	23.70	24.60	40	20.924%	21.173%
44	43.20	44.00	39	20.586%	20.954%	4340	41	25.60	26.20	20	20.616%	20.862%
49	39.90	40.70	38	20.337%	20.702%	4345	21	27.30	28.00	20	20.367%	20.610%
45	36.80	37.50	60	20.081%	20.384%	4350	40	29.20	29.80	40	20.111%	20.353%
44	33.70	34.40	60	19.756%	20.118%	4355	41	31.00	31.70	20	19.666%	20.088%
45	30.70	31.40	60	19.479%	19.781%	4360	23	32.80	33.80	20	19.268%	19.751%
69	27.80	28.50	60	19.187%	19.491%	4365	83	34.90	35.90	20	19.036%	19.460%
66	25.10	25.80	60	18.877%	19.183%	4370	43	37.10	38.40	63	18.602%	19.274%
67	22.50	23.20	61	18.544%	18.854%	4375	43	39.50	40.80	30	18.265%	19.009%
62	20.10	20.70	40	18.249%	18.564%	4380	39	42.10	43.40	30	17.964%	18.722%
101	17.70	18.40	60	17.924%	18.248%	4385	30	44.60	46.10	30	17.632%	18.474%
79	15.60	16.20	25	17.634%	17.968%	4390	30	47.40	49.00	30	17.333%	18.201%
86	13.60	14.20	25	17.380%	17.656%	4395	30	50.40	52.10	30	16.997%	17.898%
81	11.80	12.40	50	17.091%	17.381%	4400	30	53.60	55.30	30	16.690%	17.704%
90	10.10	10.70	50	16.765%	17.146%	4405	30	56.90	58.60	30	16.341%	17.410%
145	8.60	9.20	141	16.556%	16.879%	4410	30	60.30	62.10	30	16.022%	17.238%
108	7.30	7.80	141	16.228%	16.576%	4415	30	63.90	65.70	30	15.742%	16.960%
122	6.10	6.60	286	16.042%	16.323%	4420	30	67.70	69.60	30	15.417%	16.827%
90	5.10	5.50	291	15.832%	16.137%	4425	30	71.70	73.60	30	15.038%	16.683%
108	4.20	4.60	42	15.599%	15.935%	4430	20	75.70	77.70	20	14.836%	16.530%
89	3.50	3.80	124	15.469%	15.718%	4435	20	80.00	81.90	20	14.479%	16.489%

Implied Volatility

Based on the observed option prices traded in the market, we can calculate the **implied volatilities**:

$\Rightarrow$ they are defined as the volatility parameter (σ) that we need to substitute into the Black-Scholes formula to match the option prices we observe.

In general, for each strike K, we will need to have an implied volatility parameter σ:

Strikes	Prices	Implied Volatilities
K_1	$C(K_1),\ P(K_1)$	BlackScholes($S,\ K_1,\ r,\ \sigma_{K_1},\ T$)
K_2	$C(K_2),\ P(K_2)$	BlackScholes($S,\ K_2,\ r,\ \sigma_{K_2},\ T$)
K_3	$C(K_3),\ P(K_3)$	BlackScholes($S,\ K_3,\ r,\ \sigma_{K_3},\ T$)
K_4	$C(K_4),\ P(K_4)$	BlackScholes($S,\ K_4,\ r,\ \sigma_{K_4},\ T$)
$\vdots$	$\vdots$	$\vdots$