SINGAPORE MANAGEMENT UNIVERSITY

Session 3: Brownian Motion and Martingale

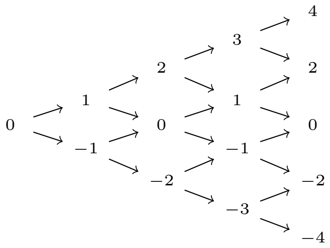
Tee Chyng Wen

QF620 Stochastic Modelling in Finance

Random Walk on a Number Line

A 1-dimensional **symmetric random walk** starts at 0 and goes up or down with an equal probability of 0.5.

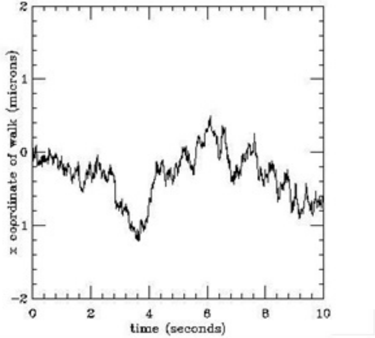
For instance, consider a 4-step 1-d symmetric random walk:

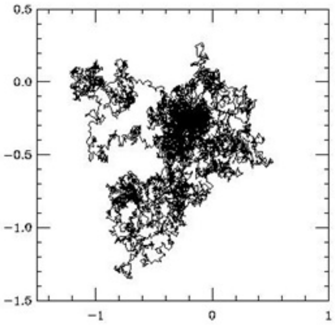


- Total number of paths $= 2^4$.
- Max. $= 4$, Min. $= -4$.
- $S_4 \sim \text{Bin}(4, 0.5)$.

These motions were such as to satisfy me, after frequently repeated observation, that they arose neither from currents in the fluid, nor from its gradual evaporation, but belonged to the particle itself.

— Robert Brown (1773–1858)





Brownian Motion

...the phenomenon of Brownian motion, the apparently erratic movement of tiny particles suspended in a liquid: Einstein showed that these movements satisfied a clear statistical law.

— C. P. Snow (1905–1980)

Si, à l'égard de plusieurs questions traitées dans cette étude, j'ai comparé les résultats de l'observation à ceux de la théorie, ce n'est pas pour vérifier des formules établies par des méthodes mathématiques, mais pour montrer seulement que le marché, à son insu, obéit à une loi qui le domine: la loi de la probabilité.

— Louis Bachelier, Théorie de la spéculation, 1900

Rough translation:

If, regarding several questions analysed in this study, I compared the observed results to those of the theory, it is not to verify the formulas obtained by mathematical methods, but only to show that the market, unwittingly, complies to a law that dominates it: the law of probability.

THE OPERATIONS OF THE STOCK EXCHANGE.

Stock Exchange Operations. — There are two kinds of forward-dated operations[1]:

- Forward contracts[2],
- Options[3].

These operations can be combined in infinite variety, especially since several types of options are dealt with frequently.

LES OPÉRATIONS DE BOURSE.

Opérations de bourse. — Il y a deux sortes d'opérations à terme :

Les opérations fermes ;

Les opérations à prime.

Ces opérations peuvent se combiner à l'infini, d'autant que l'on traite souvent plusieurs sortes de primes.

THÉORIE

DE

LA SPÉCULATION,

PAR M. L. BACHELIER.

Fig. 5.

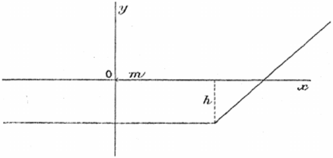


Fig. 7.

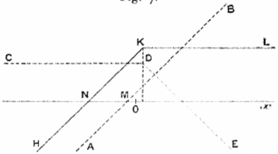
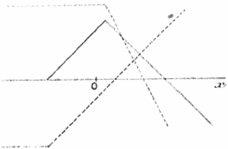


Fig. 10.



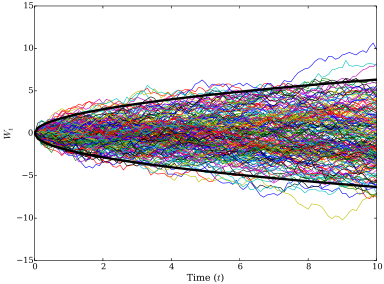
Probabilité de l'opération en blanc...........	0,30
» de bénéfice......................	0,45
» de perte	0,25

Brownian Motion

As early as 1900, **Louis Bachelier**, in his thesis “La Théorie de la Spéculation”, proposed Brownian motion as a model of the fluctuations of stock prices.

Even today it is the building block from which we construct the basic reference model for a continuous time market.

We shall approach this topic by considering Brownian motion as an **infinitesimal random walk** in which smaller and smaller steps are taken at ever more frequent time intervals.



From Random Walk to Brownian Motion

Consider a positive integer n, define a **scaled random walk** $W_n(t)$ to have the following properties

1. $W_n(0) = 0$
2. time spacing is $\frac{t}{n}$
3. up and down jumps equal and of size $\sqrt{\frac{t}{n}}$
4. measure $\mathbb{P}$, given by up and down probabilities everywhere equal to $\frac{1}{2}$

In other words, $X_1, X_2, \ldots$ is a sequence of independent binomial random variables taking values $+1$ or -1 with equal probability, then the value of W_n on the i^{th} step is defined by

$$W_n\left(\frac{i}{n}\cdot t\right) = W_n\left(\frac{i-1}{n}\cdot t\right) + \sqrt{\frac{t}{n}}X_i, \quad 1 \leq i.$$

When n becomes large, W_n will not blow out due to the scaling of $\sqrt{\frac{t}{n}}$. It can be shown that

$$W_n(t) = \sqrt{t}\left(\frac{\sum_{i=1}^{n} X_i}{\sqrt{n}}\right).$$

From Random Walk to Brownian Motion

- The distribution in the brackets tends to $N(0,1)$ by **central limit theorem**. Hence the distribution of $W_n(t)$ tends to $N(0,t)$.
- All the marginal distributions and conditional marginal distributions tend towards the same underlying normal distribution.
- Each random walk W_n has the property that its future movements away from a particular position are independent of where that position is, and indeed independent of its entire history of movements up to that time.
- Additionally, such a future displacement $W_n(s+t) - W_n(s)$ is binomially distributed with zero mean and variance t.
- Once again, the central limit theorem says that all conditional marginals tend towards a normal distribution of the same mean and variance.
- In other words, **Brownian motion is the limit of a scaled random walk** as $n \to \infty$.

Mathematical Definition of Brownian Motion

The actual development of **Brownian motion** as a stochastic process did not appear until 1923 when mathematician **Norbert Weiner** established the modern mathematical framework of what is known today as the Brownian motion random process.

Brownian motion (or **Wiener process**) has the following properties:

1. $W_0 = 0$.
2. W_t is continuous for $t \geq 0$, there are no jumps.
3. It has stationary and independent increments.
4. For $0 \leq s \leq t$, $W_t - W_s$ follows normal distribution where $W_t - W_s \sim N(0, t - s)$.

These are necessary and sufficient conditions for a process to be identified as a Brownian motion.

Brownian motion is an important building block for modeling **continuous-time stochastic processes**. It has become an important framework for modeling financial markets.

Mathematical Definition of Brownian Motion

Example If $Z \sim N(0,1)$, then $X_t = \sqrt{t}Z$ is continuous, and is marginally distributed as $N(0,t)$. Is X_t a Brownian motion?

Solution The increment $X_{s+t} - X_s$ is normally distributed with a mean of 0 and a variance of

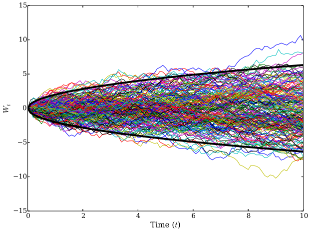
$$\begin{aligned} X_{s+t} - X_s &= \sqrt{s+t}Z - \sqrt{s}Z \\ &\sim \left(\sqrt{s+t} - \sqrt{s}\right) N(0,1) \qquad \because\ Z \text{ is the same RV} \\ &\sim N\left(0, \left(\sqrt{s+t} - \sqrt{s}\right)^2\right) \\ &\sim N\left(0, 2s + t - 2\sqrt{s(s+t)}\right). \end{aligned}$$

- For the process to be a Brownian motion, the increment needs to be of variance t.
- In addition, the increment is not independent of X_s.
- So X_t is not a Brownian motion. $\triangleleft$

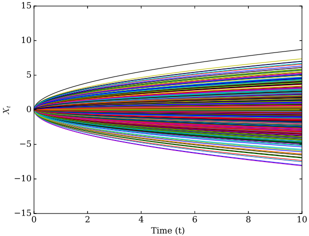
Mathematical Definition of Brownian Motion

- Consider $s < t$, it is important to realize that the condition that the distribution is normal of variance $t - s$ for every t and s and independent of the path up to time s, is much stronger than requiring W_t to be normally distributed with variance t for every t.

- For example, if we let $X_t = \sqrt{t}Z$, where Z is the same draw from a normal distribution for all t then we have that X_t is normally distributed with variance t.



vs



Mathematical Definition of Brownian Motion

Given this formal definition, we can proceed to derive the following statistical properties based on the properties of normal distribution:

$$\begin{aligned}
\mathbb{E}[W_t] &= 0, \quad W_t \sim N(0,t) \\
\mathbb{E}[W_t^2] &= t \\
\mathbb{E}[W_t - W_s] &= 0 \\
V[W_t - W_s] &= \mathbb{E}[(W_t - W_s)^2] = t - s \\
\mathrm{Cov}(W_s, W_t) &= s, \quad s < t
\end{aligned}$$

For $t_1 < t_2 < t_3 < \cdots < t_n$, note that the Brownian motions $W_{t_1}, W_{t_2}, W_{t_3}, \ldots, W_{t_n}$ are **jointly normal** with mean 0 and covariance matrix

$$\begin{bmatrix}
t_1 & t_1 & t_1 & \cdots & t_1 \\
t_1 & t_2 & t_2 & \cdots & t_2 \\
t_1 & t_2 & t_3 & \cdots & t_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
t_1 & t_2 & t_3 & \cdots & t_n
\end{bmatrix}.$$

Mathematical Definition of Brownian Motion

Example Let W_t denote a Brownian process. Conditional on $W_1 > 0$, what is the probability that $W_2 < 0$?

Example Let W_t denote a Brownian process. Determine the probability of $\mathbb{P}(W_1 \times W_2 > 0)$.

Brownian Motion is a Markov Process

We say that a stochastic process exhibits **Markov property** if the conditional distribution of its future state depends only on its present state.

$\Rightarrow$ All relevant information is already subsumed in the present state.

Mathematically, a stochastic process X_t is **Markovian** if

$$\mathbb{E}_t[X_T] = \mathbb{E}[X_T|X_t, X_{t-1}, X_{t-2}, \cdots, X_0] = \mathbb{E}[X_T|X_t]$$

Brownian motion is a Markov process. Suppose $0 < s < t$, note that

$$\begin{aligned}\mathbb{E}_s[W_t] &= \mathbb{E}_s[W_t - W_s + W_s] \\ &= \mathbb{E}_s[W_t - W_s] + \mathbb{E}_s[W_s] \\ &= W_s\end{aligned}$$

This means that to "predict" W_t given all the information up until time s, we only need to consider the value of the process at time s, i.e. W_s.

This should not be surprising since it is an independent increment process. Note that all independent increment processes exhibit Markov property.

Geometric Brownian Motion

Let W_t denote a Brownian motion. We say that

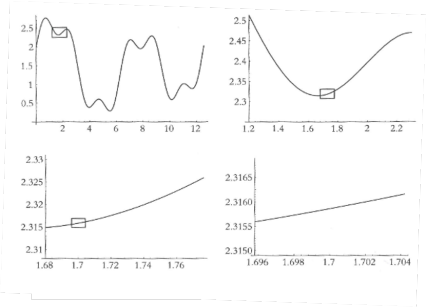
$$X_t = e^{W_t}$$

is a **geometric Brownian motion** (GBM)—a continuous-time stochastic process in which its logarithm follows a Brownian motion. We use this process to model asset prices in the **Black-Scholes model**.

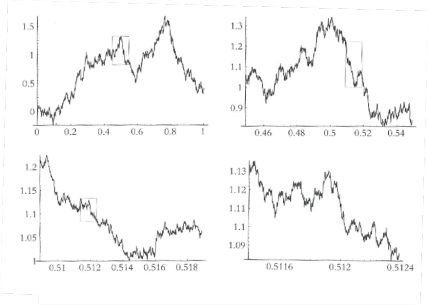
Example Let W_t denote a Brownian motion, and let $X_t = e^{W_t}$. Evaluate the following expectations:

1. $\mathbb{E}[X_T]$
2. $\mathbb{E}[X_T^2]$

Brownian Motion Properties—Differentiability



Smooth functions are differentiable.



Brownian motions are nowhere differentiable, and self-similar.

Brownian Motion Properties—Fractal

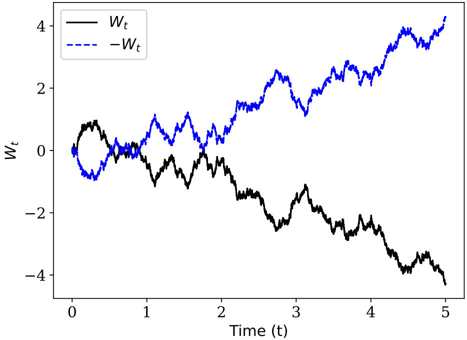
Here's a list of Brownian motion properties that might seem odd at first glance:

- Although W_t is **continuous everywhere**, it is **differentiable nowhere**.
- Brownian motion will eventually hit any and every real value no matter how large or how negative. No matter how far above the axis, it will (with probability one) be back down to zero at some later time.
- Once Brownian motion hits a value, it immediately hits it again **infinitely often**, and will continue to return after arbitrarily large times.
- It doesn't matter what scale you examine Brownian motion on - it looks just the same. Brownian motion is **fractal**.

Brownian Transformation—Reflection

The properties of a Brownian motion are invariant under a number of transformations — it remain a Brownian motion after these transformations.

Brownian motion properties are **invariant** under reflection. In other words, $W_t :\to -W_t$, **reflection along the x-axis** (multiplied by -1) will not affect the properties of W_t as a Brownian motion since the distribution is symmetric around 0.



Brownian Martingales

If M_t is a stochastic process, we say that it is a **martingale** if

$$\forall T \geq t: \quad M_t = \mathbb{E}_t[M_T].$$

In words, the expected value of this process, conditional on information up to t, is equal to its value taken at time t.

Example Let W_t be a Brownian motion, show that W_t is a martingale.

Solution Consider $0 \leq s \leq t$, we can show that

$$\begin{aligned}\mathbb{E}_s[W_t] &= \mathbb{E}_s[W_t - W_s + W_s] \\ &= \mathbb{E}_s[W_t - W_s] + \mathbb{E}_s[W_s] \\ &= W_s. \quad \triangleleft\end{aligned}$$

Brownian Martingales

Example Let W_t be a Brownian motion, show that $W_t^2 - t$ is a martingale.

Solution Consider $0 \leq s \leq t$, we can show that

$$\begin{aligned}\mathbb{E}_s[W_t^2 - t] &= \mathbb{E}_s[(W_t - W_s + W_s)^2] - t \\ &= W_s^2 - s. \quad \triangleleft\end{aligned}$$

Brownian Martingales

Example If W_t is a Brownian motion, show that W_t^3 isn't a martingale.

Solution To show this, we check the conditional expectation

$$\begin{aligned}\mathbb{E}_s[W_t^3] &= \mathbb{E}_s[(W_t - W_s + W_s)^3] \\ &= \mathbb{E}_s[(W_t - W_s)^3 + 3(W_t - W_s)^2 W_s + 3(W_t - W_s)W_s^2 + W_s^3] \\ &= 3(t-s)W_s + W_s^3.\end{aligned}$$

From here we observe that W_t^3 doesn't satisfy the definition of a standard Brownian motion. $\triangleleft$

Brownian Martingales

Recall that if $X \sim N(\mu, \sigma^2)$, then by the Moment Generating Function of normal distribution, we have shown that

$$\mathbb{E}\left[e^{\theta X}\right] = e^{\mu\theta + \frac{1}{2}\sigma^2\theta^2}.$$

Similarly, for standard normal random variable $Z \sim N(0,1)$, the same method will give us

$$\mathbb{E}\left[e^{\theta Z}\right] = e^{\frac{1}{2}\theta^2}.$$

Example Given that $W_t \sim N(0,t)$, evaluate the following expectation:

$$\mathbb{E}\left[e^{\sigma W_t}\right]$$

where $\sigma \in \mathbb{R}$ is a real number.

Brownian Martingales

Example If W_t is a Brownian motion, show that $\exp\left(\sigma W_t - \frac{\sigma^2 t}{2}\right)$ is an **exponential martingale**.

Solution

$$\begin{aligned}
\mathbb{E}_s\left[e^{\sigma W_t - \frac{1}{2}\sigma^2 t}\right] &= \mathbb{E}_s\left[e^{\sigma(W_t - W_s)} \cdot e^{\sigma W_s - \frac{1}{2}\sigma^2 t}\right] \\
&= e^{\sigma W_s - \frac{1}{2}\sigma^2 t} \cdot \mathbb{E}_s\left[e^{\sigma(W_t - W_s)}\right] \\
&= e^{\sigma W_s - \frac{1}{2}\sigma^2 t} \cdot e^{\frac{1}{2}\sigma^2(t-s)} \\
&= e^{\sigma W_s - \frac{1}{2}\sigma^2 s}. \quad \triangleleft
\end{aligned}$$