# QF620 Additional Examples
# Session 5: Stochastic Differential Equations

## 1 Examples

1. Solve the following stochastic differential equation

$$dX_t = X_t dW_t, \quad X_0 = 1.$$

*Hint: consider the process of* $\log(X_t)$.

2. Solve the following stochastic differential equation

$$dX_t = (a + X_t)dW_t, \quad X_0 = 0.$$

*Hint: consider the process of* $\log(a + X_t)$.

3. Solve the following stochastic differential equation

$$dX_t = rX_t dt + \sigma dW_t.$$

*Hint: use "integrating factor"* $e^{-rt}$.

4. Consider the stochastic differential equation

$$dS_t = \sigma S_0 dW_t,$$

where $\sigma$ is the volatility and $S_0$ is the stock price today. What is the mean and variance of $S_T$? Is it normally distributed?

5. Consider the stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

What is the mean and variance of $S_T$? Is it normally distributed?

6. Consider the stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

What is the mean and variance of $\log S_T$? Is it normally distributed?

7. Consider the stochastic differential equation

$$dF_t = \sigma(F_t + a)dW_t.$$

Solve for $F_T$.

*Hint: consider the process of* $\log(F_t + a)$.

8. A stochastic process for a stock price is given by

$$S_t = S_0 \exp\left[\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right].$$

Use Itô's Formula to derive the stochastic differential equation for $dS_t$.

9. The stochastic differential equation for the forward price is given by

$$dF_t = \sigma F_t dW_t.$$

Show that $\mathbb{E}[F_T] = F_0$.

10. We know that $\mathbb{E}[e^{\sigma W_T}] = e^{\frac{\sigma^2 T}{2}}$. What about $\mathbb{E}[e^{-\sigma W_T}]$?

11. If we use the following stochastic differential equation to model the evolution of the stock price:

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

What can we say about the process $d\frac{1}{S_t}$? What is $\mathbb{E}\left[\frac{1}{S_T}\right]$?

12. Suppose we use the following stochastic differential equations to model 2 stock price processes ($X_t$ and $Y_t$):

$$\begin{cases} dX_t = rX_t dt + \sigma_X X_t dW_t \\ dY_t = rY_t dt + \sigma_Y Y_t d\tilde{W}_t \end{cases}$$

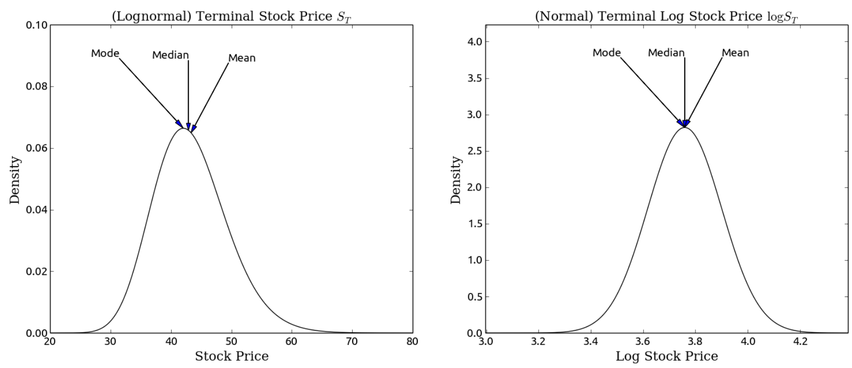
where $W_t \perp \tilde{W}_t$. What can we say about the stochastic differential equation $dZ_t = d(X_t Y_t)$?

13. Consider the stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

What is the stochastic differential equation for $dS_t^2$? What is $\mathbb{E}[S_T^2]$?

14. **Discussion** What is the probability distribution of the terminal stock price $S_T$? If $\log(S_T)$ is normal, it follows that the price $S_T$ is lognormal. Thus, the terminal price $S_T$, conditional on $S_0$, must also be lognormal. The following figure plots the distribution of the stock price and the log stock price.




Example: $S_0 = \$40$, $r = 0.16$, $\sigma = 0.4$, $T = 0.5$.

Note the ordering of the mean, median and mode in the lognormal distribution (i.e. mode $<$ median $<$ mode). In particular, the median of a lognormal distribution is always below its mean.

Since the median is always lying below the mean, it follows that, more often than not, the realised value of a lognormal random variable falls below its expected value. Thus, more often than not, the terminal stock price $S_T$ falls below its expected value.

| | Normal $\log(S_T)$ | Lognormal $S_T$ |
|---|---|---|
| Mean | $\log(S_0) + \left(r - \frac{\sigma^2}{2}\right) T$ | $S_0 e^{rT}$ |
| Median | $\log(S_0) + \left(r - \frac{\sigma^2}{2}\right) T$ | $S_0 e^{(r-\frac{\sigma^2}{2})T}$ |
| Mode | $\log(S_0) + \left(r - \frac{\sigma^2}{2}\right) T$ | $S_0 e^{(r-\frac{3\sigma^2}{2})T}$ |
| Variance | $\sigma^2 T$ | $S_0^2 e^{2rT} \left(e^{\sigma^2 T} - 1\right)$ |
| Range | $-\infty < \log(S_T) < +\infty$ | $0 \leq S_T < +\infty$ |

Black-Scholes assume that the return $\log(S_T)$ is normally distributed, and thus the price ratio $S_T$ is lognormally distributed.

15. **Discussion** Put-call parity states that

$$V_0^c - V_0^p = S_0 - Ke^{-rT}.$$

At what strike level $\bar{K}$ does call and put options have equal price? We see that $V_0^c = V_0^p$ when

$$K = S_0 e^{rT} = \mathbb{E}[S_T], \quad \text{so that} \quad S_0 - \bar{K}e^{-rT} = S_0 - S_0 e^{rT} e^{-rT} = 0.$$

In other words, then the strike price is equal to the forward value of the stock, a call and a put option are worth the same amount. We commonly denote this point as ATMF (at-the-money forward).

A follow up question is: a call option gives you an unlimited potential upside (there is no theoretical upperbound to the stock price level), but a put option gives you a limited potential upside equal to the strike price (when stock price drops to 0). Why do they still worth the same at $\bar{K}$?

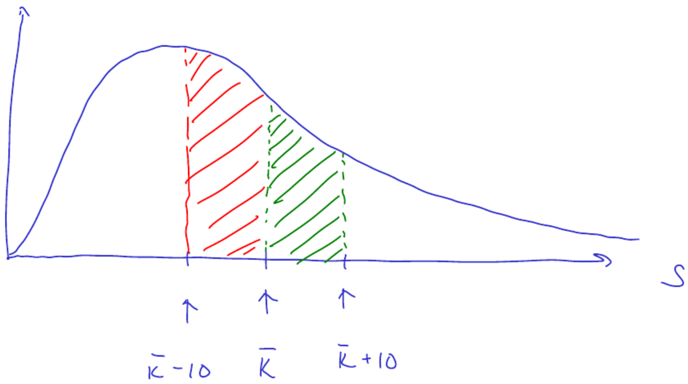
To answer this question, note that in the previous discussion question above, we have established that: *the median of $S_T$ is always lying below the mean, it follows that, more often than not, the realised value of a lognormal random variable falls below its expected value. Thus, more often than not, the terminal stock price $S_T$ falls below its expected value.*

In the financial context, this means that under the risk-neutral probability measure, it is more likely for a call option struck at the ATMF to expire out-of-the money, as compared to a put option struck at the AMTF to expire out-of-the-money. This balances the effect of the call option having theoretical no upperbound to its upside, while the put option has a limited upside.

16. **Discussion** Let $\bar{K}$ denote the ATMF strike price, i.e. $\bar{K} = \mathbb{E}[S_T] = S_0 e^{rT}$. According to the Black-Scholes model, which of the following 2 options is more expensive?

- A call option strike at $\bar{K} + \$10$
- A put option strike at $\bar{K} - \$10$

Note that both options are equally out-of-the money in dollars term, i.e. both strikes are equal distance away from the ATMF. However, as the figure below illustrates, due to the skewness of the lognormal distribution profile, this translate to more "moneyness" being taken away from the put option, as compared to the call option:




When the strike price of both call and put options are equal to $\bar{K} = S_0 e^{rT}$, they have the same price, i.e. $C(\bar{K}) = P(\bar{K})$. Note how with equal horizontal width, a larger area is taken away from the red-shaded region (put-option region), as compared to the green-shaded region (call-option region). This results in $C(\bar{K} + 10) > P(\bar{K} - 10)$.

*Follow-up exercises*:

(a) *Verify this by coding up the Black-Scholes option formula and check the prices by moving the strikes.*

(b) *Test whether the same effect is observed in a Bachelier option pricing model.*

## 2 Suggested Solutions

1. By Itô's formula, consider $Y_t = f(X_t)$, where $f : \mathbb{R} \to \mathbb{R},\ f(x) = \log(x)$, we have

$$\begin{aligned} dY_t &= f'(X_t)dX_t + \frac{1}{2}f''(X_t)(dX_t)^2 \\ &= \frac{1}{X_t}X_t dW_t - \frac{1}{2}dt \\ \therefore \quad Y_T &= W_T - \frac{T}{2} \\ \log(X_T) &= W_T - \frac{T}{2} \quad \Rightarrow \quad X_T = e^{W_T - \frac{T}{2}} \end{aligned}$$

2. The derivatives of the function $f$ are given by

$$f'(X_t) = \frac{1}{a + X_t}, \qquad f''(X_t) = -\frac{1}{(a + X_t)^2}.$$

Let $Y_t = f(X_t) = \log(a + X_t)$. By Itô's formula, we have

$$\begin{aligned} dY_t &= f'(X_t)dX_t + \frac{1}{2}f''(X_t)(dX_t)^2 \\ &= \frac{1}{a + X_t}(a + X_t)dW_t - \frac{1}{2}\frac{1}{(a + X_t)^2}(a + X_t)^2 dt \\ &= dW_t - \frac{1}{2}dt \end{aligned}$$

Integrating both sides from 0 to $T$, we obtain

$$\begin{aligned} \int_0^T dY_t &= \int_0^T dW_t - \int_0^T \frac{1}{2}dt \\ Y_T - Y_0 &= W_T - \frac{T}{2} \\ \log(a + X_T) - \log(a + X_0) &= W_T - \frac{T}{2} \\ \log\left(\frac{a + X_T}{a}\right) &= W_T - \frac{T}{2} \\ X_T &= a\left(e^{W_T - \frac{T}{2}} - 1\right) \end{aligned}$$

3. Using the "integrating factor" $e^{-rt}$, we consider the process $Y_t = X_t e^{-rt} = f(t, X_t)$, where

$$f : \mathbb{R}^2 \to \mathbb{R},\ f(t, x) = xe^{-rt}.$$

Its derivatives are given by

$$f_t = -rxe^{-rt}, \qquad f_x = e^{-rt}, \qquad f_{xx} = 0.$$

By Itô's formula, we have

$$\begin{aligned} dY_t &= f_t(t, X_t)dt + f_x(t, X_t)dX_t + \frac{1}{2}f_{xx}(t, X_t)(dX_t)^2 \\ &= -rX_t e^{-rt}dt + e^{-rt}(rX_t dt + \sigma dW_t) \\ &= \sigma e^{-rt}dW_t \end{aligned}$$

Now integrating both sides from 0 to $T$, we obtain

$$\begin{aligned}
\int_0^T dY_t &= \sigma \int_0^T e^{-rt} dW_t \\
Y_T &= Y_0 + \sigma \int_0^T e^{-rt} dW_t \\
X_T e^{-rT} &= X_0 + \sigma \int_0^T e^{-rt} dW_t \\
X_T &= X_0 e^{rT} + \sigma \int_0^T e^{r(T-t)} dW_t
\end{aligned}$$

4. This simple stochastic differential equation can be readily solved by integrating both sides from 0 to $T$:

$$\begin{aligned}
&\int_0^T dS_u = \sigma S_0 \int_0^T dW_u \\
&S_T - S_0 = \sigma S_0 W_T \\
&\Rightarrow \quad S_T = S_0 + \sigma S_0 W_T.
\end{aligned}$$

Hence the mean of $S_T$ is given by

$$\mathbb{E}[S_T] = \mathbb{E}[S_0 + \sigma S_0 W_T] = S_0,$$

while the variance of $S_T$ is given by

$$V[S_T] = V[S_0 + \sigma S_0 W_T] = \sigma^2 S_0^2 V[W_T] = \sigma^2 S_0^2 T.$$

$S_T$ is normally distributed and $S_T \sim N(S_0, \sigma^2 S_0^2 T)$.

5. First, apply Itô's Formula to solve the stochastic differential equation, and obtain the stochastic process $S_T$ as the solution

$$S_T = S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T}.$$

The mean is given by

$$\begin{aligned}
\mathbb{E}[S_T] &= \mathbb{E}\left[S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T}\right] \\
&= S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T} \mathbb{E}\left[e^{\sigma W_T}\right] \\
&= S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T} e^{\frac{\sigma^2 T}{2}} \\
&= S_0 e^{rT}.
\end{aligned}$$

The variance is given by

$$\begin{aligned}
V[S_T] &= \mathbb{E}[S_T^2] - \mathbb{E}[S_T]^2 \\
&= \mathbb{E}\left[S_0^2 e^{\left(2r - \sigma^2\right)T + 2\sigma W_T}\right] - \mathbb{E}\left[S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T}\right]^2 \\
&= S_0^2 e^{\left(2r - \sigma^2\right)T} \mathbb{E}\left[e^{2\sigma W_T}\right] - S_0^2 e^{2rT} \\
&= S_0^2 e^{\left(2r - \sigma^2\right)T} e^{\frac{4\sigma^2 T}{2}} - S_0^2 e^{2rT} \\
&= S_0^2 e^{(2r + \sigma^2)T} - S_0^2 e^{2rT} \\
&= S_0^2 e^{2rT} \left(e^{\sigma^2 T} - 1\right).
\end{aligned}$$

$S_T$ is not normally distributed (it is lognormally distributed).

6. Applying Itô's Formula to the function $X_t = f(S_t)$, where $f : \mathbb{R} \to \mathbb{R},\ f(x) = \log(x)$, we can derive the stochastic differential equation for $d\log S_t$ as follow:
$$d\log S_t = \left(r - \frac{\sigma^2}{2}\right) dt + \sigma dW_t.$$
Integrating both sides from 0 to $T$, we obtain
$$\begin{aligned}\int_0^T d\log S_u &= \left(r - \frac{\sigma^2}{2}\right) \int_0^T du + \sigma \int_0^T dW_u \\ \log S_T - \log S_0 &= \left(r - \frac{\sigma^2}{2}\right) T + \sigma W_T \\ \log S_T &= \log S_0 + \left(r - \frac{\sigma^2}{2}\right) T + \sigma W_T.\end{aligned}$$
The mean of $\log S_T$ is given by
$$\begin{aligned}\mathbb{E}[\log S_T] &= \mathbb{E}\left[\log S_0 + \left(r - \frac{\sigma^2}{2}\right) T + \sigma W_T\right] \\ &= \log S_0 + \left(r - \frac{\sigma^2}{2}\right) T.\end{aligned}$$
The variance of $\log S_T$ is given by
$$\begin{aligned}V[\log S_T] &= V\left[\log S_0 + \left(r - \frac{\sigma^2}{2}\right) T + \sigma W_T\right] \\ &= V[\sigma W_T] = \sigma^2 T.\end{aligned}$$
Yes $\log S_T$ is normally distributed and $\log S_T \sim N\left(\log S_0 + \left(r - \frac{\sigma^2}{2}\right) T, \sigma^2 T\right)$.

7. Consider the function $X_t = \log(F_t + a) = f(F_t)$, where $f : \mathbb{R} \to \mathbb{R},\ f(x) = \log(x + a)$, its partial derivatives are given by
$$f'(F_t) = \frac{1}{F_t + a}, \qquad f''(F_t) = -\frac{1}{(F_t + a)^2}.$$
Applying Itô's Formula to $f$, we obtain
$$\begin{aligned}dX_t &= f'(F_t)dF_t + \frac{1}{2} f''(F_t)(dF_t)^2 \\ &= \frac{1}{F_t + a}\sigma(F_t + a)dW_t - \frac{1}{2}\frac{1}{(F_t + a)^2}\sigma^2(F_t + a)^2 dt \\ &= -\frac{\sigma^2}{2} dt + \sigma dW_t.\end{aligned}$$
Integrating both sides from 0 to $T$, we obtain
$$\begin{aligned}\int_0^T dX_u &= -\frac{\sigma^2}{2} \int_0^T du + \sigma \int_0^T dW_u \\ X_T - X_0 &= -\frac{\sigma^2 T}{2} + \sigma W_T \\ \log(F_T + a) - \log(F_0 + a) &= -\frac{\sigma^2 T}{2} + \sigma W_T \\ \Rightarrow \quad F_T &= (F_0 + a)e^{-\frac{\sigma^2 T}{2} + \sigma W_T} - a.\end{aligned}$$

8. Consider the function $S_t = S_0 e^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma W_t} = f(t, W_t)$, where $f : \mathbb{R}^2 \to \mathbb{R}$, $f(t,x) = S_0 e^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma x}$, the partial derivatives are given by

$$\frac{\partial f}{\partial t}(t, W_t) = \left(r - \frac{\sigma^2}{2}\right) S_t, \qquad \frac{\partial f}{\partial x}(t, W_t) = \sigma S_t, \qquad \frac{\partial^2 f}{\partial x^2}(t, W_t) = \sigma^2 S_t.$$

Applying Itô's Formula, we obtain

$$\begin{aligned} dS_t &= \frac{\partial f}{\partial t}(t, W_t)dt + \frac{\partial f}{\partial x}(t, W_t)dW_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, W_t)(dW_t)^2 \\ &= \left(r - \frac{\sigma^2}{2}\right) S_t dt + \sigma S_t dW_t + \frac{1}{2}\sigma^2 S_t dt \\ &= rS_t dt + \sigma S_t dW_t. \end{aligned}$$

9. Consider the function $X_t = \log F_t = f(F_t)$, where $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \log(x)$, the partial derivatives are given by

$$f'(F_t) = \frac{1}{F_t}, \qquad f''(F_t) = -\frac{1}{F_t^2}.$$

Applying Itô's Formula, we obtain

$$\begin{aligned} dX_t &= f'(F_t)dF_t + \frac{1}{2}f''(F_t)(dF_t)^2 \\ &= \frac{1}{F_t}\sigma F_t dW_t - \frac{1}{2}\frac{1}{F_t^2}\sigma^2 F_t^2 dt \\ &= -\frac{\sigma^2}{2}dt + \sigma dW_t. \end{aligned}$$

Integrating both sides from 0 to $T$

$$\begin{aligned} \int_0^T dX_u &= -\frac{\sigma^2}{2}\int_0^T du + \sigma \int_0^T dW_u \\ X_T - X_0 &= -\frac{\sigma^2 T}{2} + \sigma W_T \\ \log\frac{F_T}{F_0} &= -\frac{\sigma^2 T}{2} + \sigma W_T \\ F_T &= F_0 e^{-\frac{\sigma^2 T}{2}+\sigma W_T}. \end{aligned}$$

We can show that

$$\mathbb{E}[F_T] = \mathbb{E}\left[F_0 e^{-\frac{\sigma^2 T}{2}+\sigma W_T}\right] = F_0 e^{-\frac{\sigma^2 T}{2}}\mathbb{E}\left[e^{\sigma W_T}\right] = F_0 e^{-\frac{\sigma^2 T}{2}} e^{\frac{\sigma^2 T}{2}} = F_0.$$

10. Using the "completing the square" method, noting that $W_T \sim N(0,T) \sim \sqrt{T}N(0,1)$, we have

$$\begin{aligned}
\mathbb{E}[e^{-\sigma W_T}] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\sigma\sqrt{T}x} e^{-\frac{x^2}{2}} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2+2\sigma\sqrt{T}x}{2}} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2+2\sigma\sqrt{T}x+\sigma^2T-\sigma^2T}{2}} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x+\sigma\sqrt{T})^2}{2}} e^{\frac{\sigma^2T}{2}} dx \\
&= e^{\frac{\sigma^2T}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x+\sigma\sqrt{T})^2}{2}} dx \\
&= e^{\frac{\sigma^2T}{2}}.
\end{aligned}$$

11. Consider $X_t = \frac{1}{S_t} = f(S_t)$, where $f : \mathbb{R} \to \mathbb{R},\ f(x) = \frac{1}{x}$, the partial derivatives are given by

$$f'(S_t) = -\frac{1}{S_t^2}, \qquad f''(S_t) = \frac{2}{S_t^3}.$$

Applying Itô's Formula, we obtain

$$\begin{aligned}
d\frac{1}{S_t} = dX_t &= f'(S_t)dS_t + \frac{1}{2}f''(S_t)(dS_t)^2 \\
&= -\frac{1}{S_t^2}(rS_tdt + \sigma S_t dW_t) + \frac{1}{2}\frac{2}{S_t^3}\sigma^2 S_t^2 dt \\
&= (\sigma^2 - r)X_t dt - \sigma X_t dW_t.
\end{aligned}$$

In order to be able to evaluate the expectation $\mathbb{E}\left[\frac{1}{S_T}\right]$, we first need to solve the stochastic differential equation for $X_t = \frac{1}{S_t}$. Now consider $Y_t = \log X_t = g(X_t)$, where $g : \mathbb{R} \to \mathbb{R},\ g(x) = \log(x)$, the partial derivatives are given by

$$g'(X_t) = \frac{1}{X_t}, \qquad g''(X_t) = -\frac{1}{X_t^2}.$$

Applying Itô's Formula to this function $g$, we obtain

$$\begin{aligned}
dY_t &= g'(X_t)dX_t + \frac{1}{2}g''(X_t)(dX_t)^2 \\
&= \frac{1}{X_t}\left[(\sigma^2 - r)X_t dt - \sigma X_t dW_t\right] - \frac{1}{2}\frac{1}{X_t^2}\sigma^2 X_t^2 dt \\
&= \left(\frac{\sigma^2}{2} - r\right) dt - \sigma dW_t.
\end{aligned}$$

Integrating both sides from 0 to $T$

$$\begin{aligned}
\int_0^T dY_u &= \left(\frac{\sigma^2}{2} - r\right) \int_0^T du - \sigma \int_0^T dW_u \\
Y_T - Y_0 &= \left(\frac{\sigma^2}{2} - r\right) T - \sigma W_T \\
\log \frac{X_T}{X_0} &= \left(\frac{\sigma^2}{2} - r\right) T - \sigma W_T \\
X_T &= X_0 e^{\left(\frac{\sigma^2}{2} - r\right)T - \sigma W_T} \\
\Rightarrow \quad \frac{1}{S_T} &= \frac{1}{S_0} e^{\left(\frac{\sigma^2}{2} - r\right)T - \sigma W_T}.
\end{aligned}$$

Taking expectation, we obtain

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{S_T}\right] &= \mathbb{E}\left[\frac{1}{S_0}e^{\left(\frac{\sigma^2}{2}-r\right)T-\sigma W_T}\right] \\
&= \frac{1}{S_0}e^{\left(\frac{\sigma^2}{2}-r\right)T}\mathbb{E}\left[e^{-\sigma W_T}\right] \\
&= \frac{1}{S_0}e^{\left(\frac{\sigma^2}{2}-r\right)T}e^{\frac{\sigma^2 T}{2}} \\
&= \frac{1}{S_0}e^{\left(\sigma^2-r\right)T}
\end{aligned}$$

12. We shall apply chain rule for stochastic calculus to derive the stochastic differential equation for $dZ_t$. $X_t$ and $Y_t$ are adapted to two independent Brownian motions $W_t$ and $\tilde{W}_t$, Itô's Formula would yield (show this)

$$dZ_t = X_t dY_t + Y_t dX_t.$$

Substituting for $dX_t$ and $dY_t$, we obtain

$$\begin{aligned}
dZ_t = d(X_tY_t) &= X_t(rY_tdt + \sigma_Y Y_t d\tilde{W}_t) + Y_t(rX_tdt + \sigma_X X_t dW_t) \\
&= 2rZ_tdt + Z_t(\sigma_Y d\tilde{W}_t + \sigma_X dW_t).
\end{aligned}$$

13. Consider $X_t = S_t^2 = f(S_t)$, where $f:\mathbb{R}\to\mathbb{R},\ f(x) = x^2$, its partial derivatives are given by

$$f'(S_t) = 2S_t, \qquad f''(S_t) = 2.$$

Applying Itô's Formula to the function $f$, we obtain

$$\begin{aligned}
dS_t^2 = dX_t &= f'(S_t)dS_t + \frac{1}{2}f''(S_t)(dS_t)^2 \\
&= 2S_t(rS_tdt + \sigma S_t dW_t) + \frac{1}{2}\cdot 2\cdot \sigma^2 S_t^2 dt \\
&= (2r+\sigma^2)X_tdt + 2\sigma X_t dW_t.
\end{aligned}$$

We proceed to solve this stochastic differential equation. Consider $Y_t = \log X_t = g(X_t)$, where $g:\mathbb{R}\to\mathbb{R},\ g(x) = \log(x)$, its partial derivatives are given by

$$g'(X_t) = \frac{1}{X_t}, \qquad g''(X_t) = -\frac{1}{X_t^2}.$$

Applying Itô's Formula to $g$, we have

$$\begin{aligned}
d\log X_t = dY_t &= g'(X_t)dX_t + \frac{1}{2}g''(X_t)(dX_t)^2 \\
&= \frac{1}{X_t}[(2r+\sigma^2)X_tdt + 2\sigma X_t dW_t] - \frac{1}{2}\cdot\frac{1}{X_t^2}\cdot 4\cdot\sigma^2 X_t^2 dt \\
&= (2r-\sigma^2)dt + 2\sigma dW_t.
\end{aligned}$$

Integrating both sides from 0 to $T$

$$\begin{aligned}
\int_0^T dY_u &= (2r-\sigma^2)\int_0^T du + 2\sigma\int_0^T dW_u \\
Y_T - Y_0 &= (2r-\sigma^2)T + 2\sigma W_T \\
\log\frac{X_T}{X_0} &= (2r-\sigma^2)T + 2\sigma W_T \\
X_T &= X_0 e^{(2r-\sigma^2)T + 2\sigma W_T}.
\end{aligned}$$

And hence

$$\begin{aligned}\mathbb{E}[S_T^2] = \mathbb{E}[X_T] &= \mathbb{E}\left[X_0 e^{(2r-\sigma^2)T + 2\sigma W_T}\right] \\ &= X_0 e^{(2r-\sigma^2)T}\mathbb{E}\left[e^{2\sigma W_T}\right] \\ &= X_0 e^{(2r-\sigma^2)T} e^{\frac{4\sigma^2 T}{2}} \\ &= S_0^2 e^{(2r+\sigma^2)T}.\end{aligned}$$