

Definition of Stochastic Integrals

What about the second integral? Let us define it using the same approach as the Riemann integral before. With the same notations, we define the **stochastic integral** as

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(t_{i-1}) \times (W_{t_i} - W_{t_{i-1}}).$$

This limit exists in an appropriate sense. With this definition for stochastic integral, we note that

- the result of a stochastic integral is a random variable, as opposed to what we get from Riemann integral.
- in Riemann integral of a function $f(x_i)$, x_i can be any point in the interval $[t_{i-1}, t_i]$, whereas in stochastic integral, x_i must be taken at the left side of each interval (t_{i-1}). This is due to the **previsibility** condition.
- **Previsibility**: the value of $f(x_i)$ is only known at the beginning of the interval. Taking $x_i \neq t_{i-1}$ lead to different results.

$$\int_0^T f(u, \omega_u) du \quad \text{Riemann} \quad I$$

$$\int_0^T f(u, \omega_u) d\omega_u \quad \text{Stochastic} \quad I$$

Properties of Stochastic Integrals

$$V[I_T] = \mathbb{E}[I_T^2] - \mathbb{E}[I_T]^2$$

Consider I_T defined as the stochastic integral:

$$I_T = \int_0^T f(u, W_u) dW_u.$$

Below are the key properties of stochastic integrals I_T :

① $\mathbb{E}[I_T] = 0$

mean

② $\mathbb{E}[I_T^2] = \mathbb{E}\left[\left(\int_0^T f(u, W_u) dW_u\right)^2\right] = \mathbb{E}\left[\int_0^T f(u, W_u)^2 du\right]$

variance

③ If f is a deterministic function, then $I_T \sim N\left(0, \int_0^T f(u)^2 du\right)$.

④ $\mathbb{E}\left[\int_0^T f(u) dW_u \times \int_0^T g(s) dW_s\right] = \mathbb{E}\left[\int_0^T f(u)g(u) du\right]$

⑤ **Itô's Isometry theorem** states that $\mathbb{E}\left[\left(\int_0^T X_t dW_t\right)^2\right] = \mathbb{E}\left[\int_0^T X_t^2 dt\right]$.

Properties of Stochastic Integrals: Proof of (1)

Recall that our first principle definition of the stochastic integrals is given by

$$I_T = \int_0^T f(t, W_t) dW_t = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(t_{i-1}, W_{t_{i-1}}) \times (W_{t_i} - W_{t_{i-1}})$$

Note that Brownian motion's increments are independent, so $(W_{t_i} - W_{t_{i-1}})$ is independent from $W_{t_{i-1}}$. Hence, we have

$$\begin{aligned} \mathbb{E} \left[\int_0^T f(t, W_t) dW_t \right] &= \mathbb{E} \left[\lim_{n \rightarrow \infty} \sum_{i=1}^n f(t_{i-1}, W_{t_{i-1}}) \times (W_{t_i} - W_{t_{i-1}}) \right] \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \mathbb{E} [f(t_{i-1}, W_{t_{i-1}}) \times (W_{t_i} - W_{t_{i-1}})] \\ &\stackrel{\because \perp}{=} \lim_{n \rightarrow \infty} \sum_{i=1}^n \mathbb{E} [f(t_{i-1}, W_{t_{i-1}})] \times \mathbb{E} [(W_{t_i} - W_{t_{i-1}})] \\ &\stackrel{W_{t_i} - W_{t_{i-1}} \sim N(0, t_i - t_{i-1})}{=} \lim_{n \rightarrow \infty} \sum_{i=1}^n \mathbb{E} [f(t_{i-1}, W_{t_{i-1}})] \times 0 = 0 \end{aligned}$$

$$\omega_T \sim N(0, T), \quad \omega_{t+\Delta t} - \omega_t \sim N(0, \Delta t)$$

Properties of Stochastic Integrals: Proof of (2)

Let $\Delta W_{t_{i-1}} = W_{t_i} - W_{t_{i-1}}$ and $\Delta W_{t_{j-1}} = W_{t_j} - W_{t_{j-1}}$, note that

$$\mathbb{E} \left[f(t_{i-1}, W_{t_{i-1}}) \cdot f(t_{j-1}, W_{t_{j-1}}) \cdot \Delta W_{t_{i-1}} \cdot \Delta W_{t_{j-1}} \right] = \begin{cases} \mathbb{E}[f(t_{i-1}, W_{t_{i-1}})^2] \times (t_i - t_{i-1}) & i = j \\ 0 & i \neq j \end{cases}$$

We have

$$\begin{aligned} \mathbb{E} \left[\left(\int_0^T f(t, W_t) dW_t \right)^2 \right] &= \mathbb{E} \left[\int_0^T f(s, W_s) dW_s \times \int_0^T f(u, W_u) dW_u \right] \\ &= \lim_{n \rightarrow \infty} \sum_{i=1, j=1}^n \mathbb{E} \left[f(t_{i-1}, W_{t_{i-1}}) \cdot f(t_{j-1}, W_{t_{j-1}}) \cdot \Delta W_{t_{i-1}} \cdot \Delta W_{t_{j-1}} \right] \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \mathbb{E}[f(t_{i-1}, W_{t_{i-1}})^2] \times (t_i - t_{i-1}) \\ &= \mathbb{E} \left[\int_0^T f(t, W_t)^2 dt \right] \end{aligned}$$

$$i = j \Rightarrow \mathbb{E}[\Delta W_{t_{i-1}}^2]$$

$$= V[\Delta W_{t_{i-1}}] = \Delta t$$

Stochastic Integrals

Example Consider the stochastic integral

$$I_T = \int_0^T W_t dW_t,$$

determine its mean $\mathbb{E}[I_T]$ and variance $V[I_T]$.

$$\mathbb{E}[I_T] = 0$$

$$\begin{aligned} V[I_T] &= \mathbb{E}\left[\left(\int_0^T W_t dW_t\right)^2\right] \\ &= \mathbb{E}\left[\int_0^T W_t^2 dt\right] = \int_0^T \mathbb{E}[W_t^2] dt = \int_0^T t dt \\ &= \frac{T^2}{2} \end{aligned}$$

$$W_t \sim N(0, t)$$

↑ $\mathbb{E}[W_t^2] = t$

$$W_{t+\Delta t} - W_t \sim N(0, \Delta t)$$

↑

$$\mathbb{E}[(W_{t+\Delta t} - W_t)^2]$$

||

$$\mathbb{E}[\Delta W_t^2]$$

||

$$\Delta t$$

Quadratic Variation

Strong Law of Large Numbers

Kolmogorov's strong law states that the average of a sequence of independent random variables having a common distribution will with probability 1 converge to the mean of that distribution:

$$n \rightarrow \infty \Rightarrow \mathbb{P} \left(\frac{X_1 + X_2 + X_3 + \cdots + X_n}{n} = \mu \right) = 1.$$

For example, if X_1, X_2, \dots is a sequence of independent binomial random variables taking values $+1$ or -1 with equal probability, then the Strong Law states that

$$\lim_{n \rightarrow \infty} \frac{X_1 + X_2 + \cdots + X_n}{n} = 0.$$

We can use the Law of Large Numbers to argue that $(\Delta W_t)^2$ and Δt are of the same order, leading to the Box calculus rule of $dW_t \cdot dW_t = dt$.

Quadratic Variation

Suppose W_t is a Brownian motion, and we have the time partition $\{t_0, t_1, t_2, \dots, t_n\}$, where $t_0 = 0$ and $t_n = T$. Consider any two timestamps t_i and t_{i-1} , note that

$$\mathbb{E} \left[\overbrace{(W_{t_i} - W_{t_{i-1}})^2}^{\Delta W_{t_i}} \right] = t_i - t_{i-1}$$

Let us define $\Delta W_t = W_{t_i} - W_{t_{i-1}}$, and $\Delta t = t_i - t_{i-1}$. We have

$$\mathbb{E}[(\Delta W_t)^2] = \Delta t$$

Also note that a quick rearrangement yields

$$\mathbb{E} \left[\frac{(\Delta W_t)^2}{\Delta t} \right] = 1.$$

The Law of Large Numbers asserts that

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{(\Delta W_t)^2}{\Delta t} = 1.$$

Quadratic Variation

Now we write,

$$(\Delta W_t)^2 = \Delta t \cdot \frac{(\Delta W_t)^2}{\Delta t} = \frac{T}{n} \cdot \frac{(\Delta W_t)^2}{\Delta t} = T \cdot \frac{(\Delta W_t)^2}{n \Delta t}$$

Summing up and sending the limit of $n \rightarrow \infty$, we have

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n (\Delta W_t)^2 = T \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{(\Delta W_t)^2}{n \Delta t} = T.$$

Compare this with the expression

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \Delta t = T,$$

and using the differential notation, we arrive at the rule of

$$(dW_t)^2 = dt.$$

Note that this does not mean that every single $(\Delta W_t)^2$ is exactly equal to Δt . But when summed over a time interval, the Law of Large Numbers asserts that it converges to the mean.

Box Calculus Rules

From the discussion above, we see that $(\Delta W_t)^2$ is in the order of Δt , and hence ΔW_t is in the order of $\sqrt{\Delta t}$. Under the limit of $\Delta t \rightarrow 0$, we use differential notation. This yields

$$\begin{aligned}\Delta t &\rightarrow dt \\ (\Delta t)^2 &\rightarrow (dt)^2 = 0 \\ \Delta W_t &\rightarrow dW_t = \sqrt{\Delta t} = dt^{\frac{1}{2}} \\ (\Delta W_t)^2 &\rightarrow dW_t^2 = dt \\ (\Delta t \cdot \Delta W_t) &\rightarrow dt \cdot dW_t = 0 \quad \rightarrow dt^{\frac{3}{2}}\end{aligned}$$

These rules are based on the formalism of **Box Calculus**:

·	dt	dW_t
dt	0	0
dW_t	0	dt

So for example, we can write:

$$dX_t = \mu dt + \sigma dW_t \quad \Rightarrow \quad (dX_t)^2 = \sigma^2 dt.$$

$$dX_t = \mu dt + \sigma dW_t$$

$$(dX_t)^2 = (\mu dt + \sigma dW_t)^2$$

$$= \cancel{\mu^2 (dt)^2} + 2\cancel{\mu \sigma dt dW_t} + \sigma^2 (dW_t)^2$$

$$\sigma^2 dt$$

The History...

Brown, a botanist, discovered the motion of pollen particles in water in 1827.

At the beginning of the 20th century, Brownian motion was studied by Einstein, Perrin and other physicists.

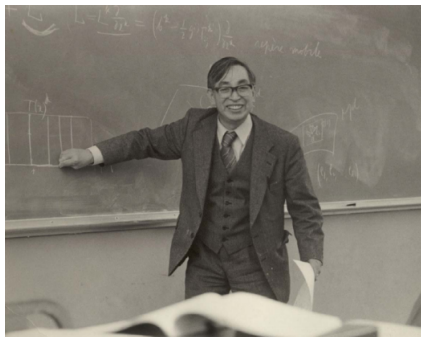
In 1923, against this scientific background, Wiener defined probability measures in path spaces, and used the concept of Lebesgue integrals to lay the mathematical foundations of stochastic analysis.

In 1942, Dr. Itô began to reconstruct from scratch the concept of stochastic integrals, and its associated theory of analysis. He created the theory of stochastic differential equations, which describe motion due to random events.

Kiyosi Itô (1915–2008)

...I finally devised stochastic differential equations, after painstaking solitary endeavours.

— Kiyosi Itô (1915–2008)



Why Itô's Formula?

From a theoretical viewpoint:

- Now that we have defined the stochastic integral, we want to be able to manipulate it without coming back to the definition.
- General rules similar to ordinary calculus chain rules, product rules etc. will be very handy.

From a practical viewpoint:

- We have now defined the stock price process, and knowing that an option price is a function of that random process,
- We want to study and understand the infinitesimal evolution of the option price process as well.

The purpose of computing is insight, not numbers.

— Richard Hamming (1915–1998)

*There is no doubt that the field of quantitative finance
has been thoroughly transformed by the basic insights
provided by Ito's calculus, both on a conceptual and
on a computational level.*

— Hans Föllmer (b. 1941)

Formulation of Itô's Formula

Under ordinary calculus, **Taylor expansion** yields

$$f(t + \Delta t) = f(t) + f'(t)\Delta t + \frac{1}{2!}f''(t)(\Delta t)^2 + \frac{1}{3!}f'''(t)(\Delta t)^3 + \dots$$

If we let $y = f(t)$, a simple rearrangement yields

$$f(t + \Delta t) - f(t) = f'(t)\Delta t + \frac{1}{2!}f''(t)(\Delta t)^2 + \frac{1}{3!}f'''(t)(\Delta t)^3 + \dots$$

$$\Delta y = f'(t)\Delta t + \frac{1}{2!}f''(t)(\Delta t)^2 + \frac{1}{3!}f'''(t)(\Delta t)^3 + \dots$$

Sending the limit of $\Delta t \rightarrow 0$, we obtain

$$\frac{dy}{dt} = f'(t).$$

Formulation of Itô's Formula

Now consider the case for stochastic calculus. We have

$$f(W_t + \Delta W_t) = f(W_t) + f'(W_t)\Delta W_t + \frac{1}{2!}f''(W_t)(\Delta W_t)^2 + \dots$$

If we let $Y_t = f(W_t)$, a simple rearrangement now yields

$$\begin{aligned} f(W_t + \Delta W_t) - f(W_t) &= f'(W_t)\Delta W_t + \frac{1}{2!}f''(W_t)(\Delta W_t)^2 + \dots \\ \Delta Y_t &= f'(W_t)\Delta W_t + \frac{1}{2!}f''(W_t)(\Delta W_t)^2 + \dots \end{aligned}$$

Now when we send the limit of $\Delta t \rightarrow 0$, we arrive at

$$dY_t = f'(W_t)dW_t + \underbrace{\frac{1}{2}f''(W_t)dt}_{\text{extra term}}$$

Formulation of Itô's Formula

Suppose now we model the stock price discrete dynamics as

option
↓ ↓

$$\Delta S_t = \mu \Delta t + \sigma \Delta W_t,$$

and we let $X_t = f(S_t)$, we

have $dS_t = \mu dt + \sigma dW_t$

order
↓

$$f(S_t + \Delta S_t) = f(S_t) + f'(S_t)\Delta S_t + \frac{1}{2!}f''(S_t)(\Delta S_t)^2 + \mathcal{O}((\Delta S_t)^3).$$

The next step is obviously to send the limit of $\Delta t \rightarrow 0$. To that end, we must first analyse the expression

$$(\Delta S_t)^2 = \mu^2 \cdot (\Delta t)^2 + 2 \cdot \mu \cdot \sigma \cdot (\Delta t)(\Delta W_t) + \sigma^2 \cdot (\Delta W_t)^2.$$

Under the limit $\Delta t \rightarrow 0$, we obtain

$$\begin{aligned} dX_t &= f'(S_t)dS_t + \frac{1}{2}f''(S_t)\sigma^2 dt \\ &= \left(\mu f'(S_t) + \frac{\sigma^2}{2}f''(S_t) \right) dt + \sigma f'(S_t) dW_t. \end{aligned}$$

Theorem / Proposition



Lemma



Manuscript



Footnote

Itô's Formula

This leads us to the famous **Itô's formula** (sometimes known as **Itô's lemma**):

Itô's Formula (Function of a Stochastic Process)

If X_t is a stochastic process satisfying

$$dX_t = \mu_t dt + \sigma_t dW_t,$$

and the function $f : \mathbb{R} \rightarrow \mathbb{R}$, f, f', f'' are continuous, then $Y_t = f(X_t)$ is also a stochastic process and is given by

$$dY_t = \left(\mu_t f'(X_t) + \frac{1}{2} \sigma_t^2 f''(X_t) \right) dt + \sigma_t f'(X_t) dW_t.$$

Example Suppose $dX_t = \mu dt + \sigma dW_t$, and $Y_t = X_t^2$. Derive the stochastic differential equation for dY_t .

$$dX_t = \mu_t dt + \sigma_t dW_t$$

$$Y_t = f(X_t)$$

$$dY_t = f'(X_t) \cdot dX_t + \frac{1}{2} f''(X_t) (dX_t)^2$$

$$= f'(X_t) \cdot [\mu_t dt + \sigma_t dW_t] + \frac{1}{2} f''(X_t) \cdot \sigma_t^2 dt$$

$$= \left[\mu_t f'(X_t) + \frac{1}{2} \sigma_t^2 f''(X_t) \right] dt + \sigma_t f'(X_t) dW_t$$

$$dX_t = \mu dt + \sigma dW_t$$

$$Y_t = f(X_t) = X_t^2$$

$$f(x) = x^2$$

$$f'(x) = 2x, \quad f''(x) = 2$$

By Itô's formula:

$$dY_t = f'(X_t) dX_t + \frac{1}{2} f''(X_t) (dX_t)^2$$

$$= 2X_t \cdot (\mu dt + \sigma dW_t) + \frac{1}{2} \times 2 \times \sigma^2 dt$$

$$= (2\mu X_t + \sigma^2) dt + 2\sigma X_t dW_t$$

Itô's Formula

More generally, Itô's Formula also allows us to write down the stochastic differential equation of a function of stochastic processes and time:

Itô's Formula (Function of a Stochastic Process & Time)

If X_t is a stochastic process satisfying

$$dX_t = \mu_t dt + \sigma_t dW_t,$$

and the function $g : \mathbb{R}^2 \rightarrow \mathbb{R}$, $g(t, x)$, g_t , g_x , g_{xx} are continuous, then $Y_t = g(t, X_t)$ is also a stochastic process and is given by

$$dY_t = \left[g_t(t, X_t) + \mu_t g_x(t, X_t) + \frac{1}{2} \sigma_t^2 g_{xx}(t, X_t) \right] dt + \sigma_t g_x(t, X_t) dW_t.$$

Example Suppose $dX_t = \mu dt + \sigma dW_t$, and $Y_t = e^{X_t+t}$. Derive the stochastic differential equation for dY_t .

$$g(t + \Delta t, X_t + \Delta X_t) = g(t, X_t)$$

$$+ \frac{\partial g}{\partial t} \cdot \Delta t + \frac{\partial g}{\partial x} \cdot \Delta X_t$$

$$+ \frac{1}{2} \left[\frac{\partial^2 g}{\partial t^2} \cdot (\Delta t)^2 + 2 \cdot \frac{\partial^2 g}{\partial t \partial x} \Delta t \Delta X_t \right.$$

$$\left. + \frac{\partial^2 g}{\partial x^2} \cdot (\Delta X_t)^2 \right]$$

$$+ \dots$$

$$\underline{\Delta t \rightarrow 0}$$

$$dX_t = \mu_t dt + \sigma_t dW_t$$

$$Y_t = g(t, X_t)$$

$$dY_t = \frac{\partial g}{\partial t} \cdot dt + \frac{\partial g}{\partial x} \cdot dX_t + \frac{1}{2} \cdot \frac{\partial^2 g}{\partial x^2} (dX_t)^2$$

$$= \frac{\partial g}{\partial t} dt + \frac{\partial g}{\partial x} \left(\mu_t dt + \sigma_t dW_t \right) + \frac{1}{2} \frac{\partial^2 g}{\partial x^2} \sigma_t^2 dt$$

$$= \left(\frac{\partial g}{\partial t} + \mu_t \frac{\partial g}{\partial x} + \frac{1}{2} \sigma_t^2 \frac{\partial^2 g}{\partial x^2} \right) dt + \sigma_t \frac{\partial g}{\partial x} dW_t$$

$$dX_t = \mu dt + \sigma dW_t \quad \left| \quad \begin{aligned} g(t, x) &= e^{t+x} \\ g_t &= e^{t+x}, \quad g_x = e^{t+x}, \quad g_{xx} = e^{t+x} \end{aligned} \right.$$

$$Y_t = e^{t+X_t} = g(t, X_t)$$

Ito's formula:

$$dY_t = g_t(t, X_t) dt + g_x(t, X_t) dX_t + \frac{1}{2} g_{xx}(t, X_t) (dX_t)^2$$

$$= e^{t+X_t} dt + e^{t+X_t} (\mu dt + \sigma dW_t) + \frac{1}{2} e^{t+X_t} \cdot \sigma^2 dt$$

$$= \left(1 + \mu + \frac{\sigma^2}{2} \right) Y_t dt + \sigma Y_t dW_t$$

Itô's Formula: Applications

$$f(x) = x^2, \quad f'(x) = 2x, \quad f''(x) = 2$$

Example Show that

$$\int_0^T W_t dW_t = \frac{W_T^2}{2} - \frac{T}{2}$$

by applying Itô formula to $X_t = f(W_t) = W_t^2$.

$$\int x dx = \frac{x^2}{2} + C$$

Itô's :
$$dX_t = f'(W_t) dW_t + \frac{1}{2} \cdot f''(W_t) (dW_t)^2$$

$$\int_0^T dX_t = 2 \int_0^T W_t \cdot dW_t + \int_0^T \frac{1}{2} \times 2 \times dt$$

$$\left[X_t \right]_0^T = 2 \int_0^T W_t dW_t + \left[t \right]_0^T$$

$$X_T - X_0 = 2 \int_0^T W_t dW_t + T$$

$$W_T^2 - 0^2 = 2 \int_0^T W_t dW_t + T$$

$$\int_0^T \omega_t^2 d\omega_t$$

$$X_t = f(\omega_t) = \omega_t^3$$

$$\int x^2 dx = \frac{x^3}{3} + C$$

$$\int w dt = wt + C$$

Itô's Formula: Applications

Example Apply Itô formula to the function $X_t = f(t, W_t) = tW_t$, and show that

$$\int_0^T W_t dt = TW_T - \int_0^T t dW_t = \int_0^T (T-t) dW_t.$$

Use this to show that

$$V \left[\int_0^T W_t dt \right] = \frac{T^3}{3}.$$

$$\begin{aligned} &= \mathbb{E} \left[\left(\int_0^T (T-t) dW_t \right)^2 \right] = \mathbb{E} \left[\int_0^T (T-t)^2 dt \right] \\ &= \int_0^T (T-t)^2 dt \end{aligned}$$

$$X_t = f(t, w_t) = t \cdot w_t$$

$$f(t, x) = t \cdot x$$

$$f_t = x, \quad f_x = t, \quad f_{xx} = 0$$

Ito's:

$$dX_t = f_t(t, w_t) dt + f_x(t, w_t) dw_t + \frac{1}{2} f_{xx}(t, w_t) (dw_t)^2$$

$$\int_0^T dX_t = \int_0^T w_t dt + \int_0^T t dw_t + \cancel{\frac{1}{2} 0 \cdot dt}$$

$$\left[X_t \right]_0^T = \int_0^T w_t dt + \int_0^T t dw_t$$

$$TW_T - 0 = \int_0^T w_t dt + \int_0^T t dw_t$$

$$\int_0^T W_t dt = TW_T - \int_0^T t dW_t$$

$$\boxed{\int_0^T dW_t = W_T}$$

$$= \int_0^T T dW_t - \int_0^T t dW_t$$

$$= \int_0^T (T-t) dW_t$$



Session 5: Stochastic Differential Equations

Tee Chyng Wen

QF620 Stochastic Modelling in Finance

“Comments on the life and mathematical legacy of Wolfgang Doeblin”:

One may invoke many reasons why the emergence of a specific branch of probability — the study of stochastic process — took a quite tortuous path throughout the 20th century.

On one hand, the pioneers were very often quite original mathematicians, such as Bachelier, Lévy, Itô, ..., whose novel ways of looking at things took a long time to be accepted.

— Bernard Bru and Marc Yor (2002)

On the other hand, perhaps the fact that Brownian motion possesses so many properties, which we summarize as:

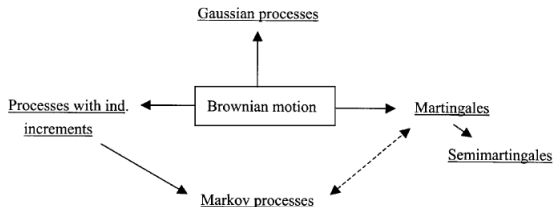
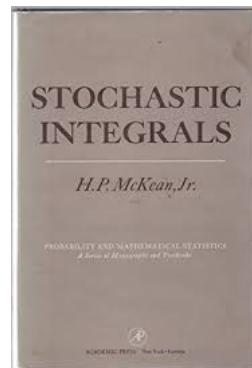


Fig. 1. Brownian motion and related processes

led many authors to develop studies of one or another special class of processes, thus giving a hard time to outsiders.

Itô's calculus took 25 years (1944-1969) to be accepted, the latter year being that of the publication of McKean's marvellous little book: Stochastic Integrals.

Bernard Bru and Marc Yor (2002)



SDEs and Martingale

$$\mathbb{E}_s[X_t] \stackrel{?}{=} X_s$$

Example Use Itô's formula to derive the stochastic differential equations of the following processes, and determine which of them are martingales:

- ① $X_t = W_t^2$ (no)
- ② $X_t = 2 + t + e^{W_t}$ (no)
- ③ $X_t = W_t^2 + \tilde{W}_t^2$, where W_t and \tilde{W}_t are independent Brownian. (no)
- ④ $X_t = W_t^2 - t$ (yes)
- ⑤ $X_t = W_t^3$ (no)
- ⑥ $X_t = e^{\theta W_t - \frac{\theta^2 t}{2}}$ (yes)

$$X_t = W_t^2 + \tilde{W}_t^2$$

$$f(x, y) = x^2 + y^2$$

$$f_x = 2x, \quad f_y = 2y, \quad f_{xy} = 0$$

$$f_{xx} = 2, \quad f_{yy} = 2$$

Ito's:

$$dX_t = f_x(W_t, \tilde{W}_t) dW_t + f_y(W_t, \tilde{W}_t) d\tilde{W}_t$$

$$+ \frac{1}{2} \cdot f_{xx}(W_t, \tilde{W}_t) (dW_t)^2 + \frac{1}{2} \cdot f_{yy}(W_t, \tilde{W}_t) (d\tilde{W}_t)^2$$

$$+ f_{xy}(W_t, \tilde{W}_t) dW_t d\tilde{W}_t$$

$$= 2W_t dW_t + 2\tilde{W}_t d\tilde{W}_t + \frac{1}{2} \cdot 2 \cdot dt + \frac{1}{2} \cdot 2 \cdot dt + 0$$

$$dX_t = 2W_t dW_t + 2\tilde{W}_t d\tilde{W}_t + 2dt$$

$$\mathbb{E}[(\Delta W_t)^2] = \Delta t$$

$$(dW_t)^2 = dt$$

$$\mathbb{E}[\Delta W_t \cdot \Delta \tilde{W}_t] = \text{Cov}(\Delta W_t, \Delta \tilde{W}_t) = 0$$

$$dW_t \cdot d\tilde{W}_t = 0$$