QF620 Stochastic Modelling in Finance
Assignment 2/4
Due Date: 11-Oct-2023

1. Assume that a stock price S_t follows the Black-Scholes lognormal process:

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

   (a) Use Itô's formula to derive the stochastic differential equation for the process S_t^2.

   (b) Solve the stochastic differential equations of S_t and S_t^2.

   (c) Use your solutions to verify the following expectations:

$$\begin{aligned} \mathbb{E}[S_t] &= Se^{rt} \\ \mathbb{E}[S_t^2] &= S^2 e^{(2r+\sigma^2)t}, \end{aligned}$$

2. Digital (or binary) options are characterized by the binary state of their payoffs. A cash-or-nothing digital call option pays \$1 if $S_T > K$ on maturity, and 0 otherwise. We can write the payoff as

$$V_{\text{Cash Digital}}(T) = \mathbb{1}_{S_T > K}.$$

   Using the Black-Scholes model, the stock price follows a lognormal process

$$dS_t = rS_t dt + \sigma S_t dW_t^*.$$

   Derive a valuation formula for this call option:

$$V_{\text{Cash Digital}}(0) = e^{-rT}\mathbb{E}\left[\mathbb{1}_{S_T > K}\right].$$

3. An asset-or-nothing digital call option pays S_T if $S_T > K$ on maturity, and 0 otherwise. We can write the payoff as

$$V_{\text{Asset Digital}}(T) = S_T \mathbb{1}_{S_T > K}.$$

   Again using the Black-Scholes model, derive a valuation formula for the asset-or-nothing digital call option:

$$V_{\text{Asset Digital}}(0) = e^{-rT}\mathbb{E}\left[S_T \mathbb{1}_{S_T > K}\right].$$

1.

(a) $d S_t^2 = 2 S_t dS_t + (dS_t)^2$

$= 2 S_t^2 (r dt + \sigma dW_t) + \sigma^2 S_t^2 dt$

(b) $X_t = \log (S_t)$

$dX_t = \frac{1}{S_t} dS_t - \frac{1}{2S_t^2} (dS_t)^2$

$= r dt + \sigma dW_t - \frac{1}{2}\sigma^2 dt$

$= (r - \frac{1}{2}\sigma^2) dt + \sigma dW_t$

$X_T - X_0 = (r - \frac{1}{2}\sigma^2) T + \sigma W_T$

$\Rightarrow S_t = S_0 e^{(r-\frac{1}{2}\sigma^2)t + \sigma W_t}$

$S_t^2 = S_0^2 e^{2(r-\frac{1}{2}\sigma^2)t + 2\sigma W_t}$

(c) $E(S_t) = S_0 e^{rt - \frac{1}{2}\sigma^2 t} E(e^{\sigma W_t})$

$X \sim N(\mu, \sigma^2) \quad E(e^{\theta X}) = \exp(\mu\theta + \frac{1}{2}\sigma^2\theta^2)$

$\sigma W_t \sim N(0, \sigma^2 t) \quad E(e^{\sigma W_t}) = \exp(\frac{1}{2}\sigma^2 t)$

$\Rightarrow E(S_t) = S_0 e^{rt}$

$E(S_t^2) = S_0^2 e^{2(r-\frac{1}{2}\sigma^2)t + 2\sigma^2 t} = S_0 e^{2rt + \sigma^2 t}$

2.

$E[\mathbb{1}_{S_T > K}] = P(S_T > K) = P[S_0 \exp[(r - \frac{\sigma^2}{2})T + \sigma W_T] > K]$

$= P[e^{\sigma W_T} > \frac{K}{S_0} e^{(\frac{\sigma^2}{2} - r)T}]$

$= P(W_T > \frac{\ln \frac{K}{S_0} + (\frac{\sigma^2}{2} - r)T}{\sigma})$

$= P(\frac{W_T}{\sqrt{T}} > \frac{\ln \frac{K}{S_0} + (\frac{\sigma^2}{2} - r)T}{\sigma\sqrt{T}}$

$= \Phi\left[\frac{(r - \frac{\sigma^2}{2})T - \ln \frac{K}{S_0}}{\sigma\sqrt{T}}\right] = \Phi(d_1)$

$V(0) = e^{-rT} \Phi(d_1)$

3. $E[S_T \mathbb{1}_{S_T>K}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} S_0 e^{(r-\frac{\sigma^2}{2})T + \sigma^2\sqrt{T}x} \mathbb{1}_{S_T>K} e^{-\frac{x^2}{2}} dx$

For $\mathbb{1}_{S_T>K}$ part

$e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}x} > \frac{K}{S_0}$

$(r-\frac{\sigma^2}{2})T + \sigma\sqrt{T}x > \ln\frac{K}{S_0}$

$x^* > \frac{\ln\frac{K}{S_0} - (r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$

$= \frac{1}{\sqrt{2\pi}} \int_{x^*}^{\infty} S_0 e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}x} e^{-\frac{x^2}{2}} dx$

$= \frac{S_0}{\sqrt{2\pi}} e^{(r-\frac{\sigma^2}{2})T} \int_{x^*}^{\infty} e^{\sigma\sqrt{T}x - \frac{x^2}{2}} dx$

$= \frac{S_0}{\sqrt{2\pi}} e^{(r-\frac{\sigma^2}{2})T} \int_{x^*}^{\infty} e^{-\frac{[x^2 - 2\sigma\sqrt{T}x + (\sigma\sqrt{T})^2] - \sigma^2 T}{2}} dx$

$= \frac{S_0}{\sqrt{2\pi}} e^{rT} \int_{x^*}^{\infty} e^{-\frac{(x-\sigma\sqrt{T})^2}{2}} dx$

$= S_0 e^{rT} \Phi(-x^* + \sigma\sqrt{T})$

$V(0) = S_0 \Phi(\sigma\sqrt{T} - x^*) = S_0 \Phi\left(\frac{\ln\frac{S_0}{K} + (r+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)$