# QF620 Additional Examples
# Session 8: Static Replication of European Payoffs

## 1 Examples

1. Suppose the stock price process follows the stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dW_t^*,$$

where $W_t^*$ is a standard Brownian motion under the risk-neutral measure $\mathbb{Q}^*$ associated with the risk-free bond $B_t$ as numeraire, where $dB_t = rB_t dt$. We would like to value a contract which pays

$$S_T^n$$

on maturity date $T$, where $n \in \mathbb{N}$ is a natural number. Derive a valuation formula for this contract.

2. Consider the same contract as above, determine how we can use static replication to value this contract payoff $h(S_T) = S_T^n$. We have access to the vanilla European call and put options market:

$$\int_0^F h(K) \frac{\partial^2 P(K)}{\partial K^2}\, dK + \int_F^\infty h(K) \frac{\partial^2 C(K)}{\partial K^2}\, dK$$

# 2 Suggested Solutions

1. Solving the stochastic differential equation for the stock price process, we obtain

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_t dW_t^* \\ \Rightarrow \quad S_T &= S_0 \exp\left[\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T^*\right] \\ \Rightarrow \quad S_T^n &= S_0^n \exp\left[n\left(r - \frac{\sigma^2}{2}\right)T + n\sigma W_T^*\right] \end{aligned}$$

   On maturity date $T$, the contract pays

$$V_T = S_T^n,$$

   so using martingale valuation framework, we have

$$\begin{aligned} \frac{V_0}{B_0} &= \mathbb{E}^*\left[\frac{V_T}{B_T}\right] \\ V_0 &= e^{-rT}\mathbb{E}^*\left[S_0^n e^{n\left(r-\frac{\sigma^2}{2}\right)T + n\sigma W_T^*}\right] \\ &= e^{-rT} S_0^n e^{n\left(r-\frac{\sigma^2}{2}\right)T} e^{\frac{n^2\sigma^2 T}{2}} \\ &= S_0^n e^{(n-1)rT} e^{n(n-1)\frac{\sigma^2 T}{2}} \quad \triangleleft \end{aligned}$$

2. The payoff is twice differentiable and is given by

$$h(S_T) = S_T^n, \qquad h'(S_T) = nS_T^{n-1}, \qquad h''(S_T) = n(n-1)S_T^{n-2}.$$

   Using the Carr and Madan static replication formula, we have

$$\begin{aligned} V_0 &= e^{-rT}h(F) + \int_0^F h''(K)P(K)dK + \int_F^\infty h''(K)C(K)dK \\ &= e^{-rT}h(F) + \int_0^F h''(K)P(K)dK + \int_F^\infty h''(K)C(K)dK \\ &= e^{-rT}F_T^n + \int_0^F n(n-1)K^{n-2}P(K)dK + \int_F^\infty n(n-1)K^{n-2}C(K)dK \\ &= e^{-rT}S_0^n e^{nrT} + \int_0^F n(n-1)K^{n-2}P(K)dK + \int_F^\infty n(n-1)K^{n-2}C(K)dK \\ &= S_0^n e^{(n-1)rT} + \int_0^F n(n-1)K^{n-2}P(K)dK + \int_F^\infty n(n-1)K^{n-2}C(K)dK \quad \triangleleft \end{aligned}$$