QF620 Stochastic Modelling
End-term Revision Pack

1 Practice Questions

1. Suppose the stock price follows the stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dW_t^*$$

where W_t^* is a standard Brownian motion under the risk-neutral measure associated with the risk-free bond as numeraire. Derive the valuation formula for an option paying

$$\left(\sqrt{S_T} - K\right)^+$$

on the maturity date T.

2. Suppose the stock price follows the stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dW_t^*$$

where W_t^* is a standard Brownian motion under the risk-neutral measure associated with the risk-free bond as numeraire. We want to derive a valuation formula for an option paying

$$S_T\left(S_T - K\right)^+$$

on the maturity date T. Derive the formula by taking the expectation of the payoff in the $\mathbb{Q}^*$ measure.

3. (a) Consider the stochastic process $Y_t = \exp(\nu t + \sigma W_t)$. What is the necessary relationship between ν and σ for Y_t to be a martingale?

   (b) Consider the stock price process

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

   where W_t is a standard Brownian motion under the real-world measure $\mathbb{P}$, and we also have the risk-free bond process $dB_t = rB_t dt$. What is the Radon-Nikodym derivative $\frac{d\mathbb{Q}^*}{d\mathbb{P}}$ that change from the real-world measure to the risk-neutral measure associated to the risk-free bond as the choice of numeraire?

4. Suppose the stock price follows the process

$$dS_t = rS_t dt + \sigma S_t dW_t^*,$$

where W_t^* is a $\mathbb{Q}^*$-Brownian motion.

(a) Derive a valuation formula for a *forward contract* paying

$$S^{\frac{1}{n}} - K^{\frac{1}{m}},$$

where $n,\ m \in \mathbb{N}$.

(b) Derive a valuation formula for an *option* paying

$$\left(S^{\frac{1}{n}} - K^{\frac{1}{m}}\right)^+,$$

where $n,\ m \in \mathbb{N}$.

2 Suggested Solutions

1. First, solve the sde to obtain

$$\begin{aligned} S_T &= S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma W_T^*} \\ \Rightarrow \quad \sqrt{S_T} &= \sqrt{S_0} e^{\frac{1}{2}\left(r-\frac{\sigma^2}{2}\right)T+\frac{\sigma}{2}W_T^*} \end{aligned}$$

The option is in-the-money when

$$\begin{aligned} \sqrt{S_T} &> K \\ \sqrt{S_0} e^{\frac{1}{2}\left(r-\frac{\sigma^2}{2}\right)T+\frac{\sigma}{2}W_T^*} &> K \\ x &> \frac{2\log\frac{K}{\sqrt{S_0}} - \left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = x^* \end{aligned}$$

Now we can proceed to evaluate the expectation

$$\begin{aligned} V_0 &= e^{-rT}\mathbb{E}^*\left[\left(\sqrt{S_T}-K\right)^+\right] \\ &= e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{x^*}^{\infty}\left(\sqrt{S_T}-K\right)\ e^{-\frac{x^2}{2}}\ dx \\ &= e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{x^*}^{\infty}\sqrt{S_0}e^{\frac{1}{2}\left(r-\frac{\sigma^2}{2}\right)T+\frac{\sigma}{2}\sqrt{T}x}\ e^{-\frac{x^2}{2}}\ dx - Ke^{-rT}\frac{1}{\sqrt{2\pi}}\int_{x^*}^{\infty}e^{-\frac{x^2}{2}}\ dx \\ &= e^{-\frac{rT}{2}}\frac{1}{\sqrt{2\pi}}\sqrt{S_0}e^{-\frac{\sigma^2 T}{4}}\int_{x^*}^{\infty}e^{-\frac{x^2-\sigma\sqrt{T}x+\frac{\sigma^2 T}{4}-\frac{\sigma^2 T}{4}}{2}}\ dx - Ke^{-rT}\Phi(-x^*) \\ &= e^{-\frac{rT}{2}}\frac{1}{\sqrt{2\pi}}\sqrt{S_0}e^{-\frac{\sigma^2 T}{8}}\int_{x^*}^{\infty}e^{-\frac{\left(x-\frac{\sigma\sqrt{T}}{2}\right)^2}{2}}\ dx - Ke^{-rT}\Phi(-x^*) \\ &= e^{-\frac{rT}{2}}\sqrt{S_0}e^{-\frac{\sigma^2 T}{8}}\Phi\left(-x^*+\frac{\sigma\sqrt{T}}{2}\right) - Ke^{-rT}\Phi(-x^*) \quad \triangleleft \end{aligned}$$

2. Under the $\mathbb{Q}^*$ measure, we have

$$S_T = S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma W_T^*}.$$

The option is in-the-money when

$$x > \frac{\log\frac{K}{S_0} - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = x^*$$

Taking the expectation, we obtain

$$\begin{aligned}
V_0 &= e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{x^*}^{\infty}\left(S_T^2 - S_T K\right)\ e^{-\frac{x^2}{2}}\ dx \\
&= e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{x^*}^{\infty} S_0^2 e^{2\left(r-\frac{\sigma^2}{2}\right)T+2\sigma\sqrt{T}x}\ e^{-\frac{x^2}{2}}\ dx - e^{-rT}\frac{K}{\sqrt{2\pi}}\int_{x^*}^{\infty} S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}x}\ e^{-\frac{x^2}{2}}\ dx \\
&= e^{rT}\frac{S_0^2}{\sqrt{2\pi}}e^{-\sigma^2 T}\int_{x^*}^{\infty} e^{-\frac{x^2-4\sigma\sqrt{T}x+4\sigma^2T-4\sigma^2T}{2}}\ dx - KS_0\Phi\left(-x^* + \sigma\sqrt{T}\right) \\
&= e^{rT}\frac{S_0^2}{\sqrt{2\pi}}e^{\sigma^2 T}\int_{x^*}^{\infty} e^{-\frac{\left(x-2\sigma\sqrt{T}\right)^2}{2}}\ dx - KS_0\Phi\left(-x^* + \sigma\sqrt{T}\right) \\
&= e^{rT}S_0^2 e^{\sigma^2 T}\Phi\left(-x^* + 2\sigma\sqrt{T}\right) - KS_0\Phi\left(-x^* + \sigma\sqrt{T}\right) \\
&= S_0^2 e^{(r+\sigma^2)T}\Phi\left(\frac{\log\frac{S_0}{K} + \left(r + \frac{3\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) - S_0 K\Phi\left(\frac{\log\frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) \quad \triangleleft
\end{aligned}$$

3. (a) By Itô's formula,

$$dY_t = \left(\nu + \frac{1}{2}\sigma^2\right) Y_t dt + \sigma^2 Y_t dW_t.$$

So the necessary relationship is

$$\nu + \frac{1}{2}\sigma^2 = 0. \quad \triangleleft$$

(b) Defined $X_t = f(B_t, S_t)$, we apply Itô's formula to obtain

$$\begin{aligned} dX_t &= f_b(B_t, S_t)dB_t + f_x(B_t, S_t)dS_t + \frac{1}{2} f_{xx}(B_t, S_t)(dS_t)^2 \\ &= -\frac{S_t}{B_t^2} rB_t dt + \frac{1}{B_t}(\mu S_t dt + \sigma S_t dW_t) + 0 \\ &= (\mu - r)X_t dt + \sigma X_t dW_t. \end{aligned}$$

Under the risk-neutral measure $\mathbb{Q}^*$ associated with the risk-free bond numeraire B_t, we need X_t to be a martingale, hence

$$dX_t = \sigma X_t \left(dW_t + \frac{\mu - r}{\sigma} dt\right) = \sigma X_t dW_t^*.$$

The Radon-Nikodym derivative is given by

$$\frac{d\mathbb{Q}^*}{d\mathbb{P}} = \exp\left(-\frac{1}{2}\kappa^2 t - \kappa W_t\right)$$

where $\kappa = \frac{\mu - r}{\sigma}$. $\triangleleft$

4. (a) The solution to the stochastic differential equation is given by

$$S_T = S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma W_T^*} \qquad \Rightarrow \qquad S_T^{1/n} = S_0^{1/n} e^{\left(r-\frac{\sigma^2}{2}\right)\frac{T}{n}+\frac{\sigma}{n}W_T^*}.$$

The forward contract is valued as

$$\begin{aligned} f &= e^{-rT}\mathbb{E}^*\left[S_T^{1/n} - K^{1/m}\right] \\ &= e^{-rT}\mathbb{E}^*\left[S_0^{1/n} e^{\left(r-\frac{\sigma^2}{2}\right)\frac{T}{n}+\frac{\sigma}{n}W_T^*} - K^{1/m}\right] \\ &= e^{-rT}\left[S_0^{1/n} e^{\left(r-\frac{\sigma^2}{2}\right)\frac{T}{n}+\frac{\sigma^2 T}{2n^2}} - K^{1/m}\right]. \quad \triangleleft \end{aligned}$$

(b) First we solve for the integration region

$$\begin{aligned} S_T^{1/n} - K^{1/m} &> 0 \\ x^* &> \frac{n \log \frac{K^{1/m}}{S_0^{1/n}} - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = x^*. \end{aligned}$$

The option can be valued as

$$\begin{aligned} V_0 &= \frac{e^{-rT}}{\sqrt{2\pi}}\int_{x^*}^{\infty}\left(S_0^{1/n} e^{\left(r-\frac{\sigma^2}{2}\right)\frac{T}{n}+\frac{\sigma\sqrt{T}}{n}x} - K^{1/m}\right)e^{-\frac{x^2}{2}}\,dx \\ &= \frac{e^{-rT}}{\sqrt{2\pi}}S_0^{1/n}e^{\left(r-\frac{\sigma^2}{2}\right)\frac{T}{n}}\int_{x^*}^{\infty} e^{-\frac{\left(x-\frac{\sigma\sqrt{T}}{n}\right)^2}{2}+\frac{\sigma^2 T}{2n^2}}\,dx - \frac{e^{-rT}}{\sqrt{2\pi}}K^{1/m}\int_{x^*}^{\infty} e^{-\frac{x^2}{2}}\,dx \\ &= S_0^{1/n} e^{-rT+\frac{rT}{n}-\frac{\sigma^2 T}{2n}+\frac{\sigma^2 T}{2n^2}}\,\Phi\left(-x^* + \frac{\sigma\sqrt{T}}{n}\right) - e^{-rT}K^{1/m}\Phi(-x^*). \quad \triangleleft \end{aligned}$$