QF620 Stochastic Modelling in Finance
Assignment 4/4
Due Date: 1-Nov-2023

1. A contract pays (on maturity date T)

$$V_T = \sqrt{S_T}.$$

Derive a valuation formula for the contract at time 0 using

(a) Black-Scholes model

$$dS_t = rS_t dt + \sigma S_t dW_t^*.$$

(b) Static replication approach.

$$\int_0^F h(K)\frac{\partial^2 P(K)}{\partial K^2}dK + \int_F^\infty h(K)\frac{\partial^2 C(K)}{\partial K^2}dK$$

(a)

$$d\log S_t = \frac{1}{S_t}\,dS_t - \frac{1}{2}\frac{1}{S_t^2}(dS_t)^2$$

$$= r\,dt + \sigma\,dW_t^* - \frac{1}{2}\sigma^2 dt$$

$$= (r - \tfrac{1}{2}\sigma^2)\,dt + \sigma\,dW_t^*$$

$$\log\frac{S_T}{S_0} = (r - \tfrac{1}{2}\sigma^2)T + \sigma W_T^*$$

$$S_T = S_0 e^{(r-\frac{1}{2}\sigma^2)T + \sigma W_T^*}$$

$$V_0 = e^{-rT}E(V_T) = e^{-rT} S_0^{\frac{1}{2}} e^{\frac{1}{2}(r-\frac{1}{2}\sigma^2)T} \cdot e^{\frac{1}{8}\sigma^2 T}$$

$$= S_0^{\frac{1}{2}} \cdot e^{-\frac{1}{2}rT - \frac{1}{8}\sigma^2 T}$$

(b) $h(S_T) = \sqrt{S_T}$ $\quad h'(S_T) = \frac{1}{2}S_T^{-\frac{1}{2}}$ $\quad h''(S_T) = -\frac{1}{4}S_T^{-\frac{3}{2}}$

$$V_0 = e^{-rT}h(F) + \int_0^F h''(K)P(K)\,dK + \int_F^\infty h''(K)C(K)\,dK$$

$$= (S_0 e^{-rT})^{\frac{1}{2}} - \frac{1}{4}\left[\int_0^F K^{-\frac{3}{2}} P(K)\,dK + \int_F^\infty K^{-\frac{3}{2}} C(K)\,dK\right]$$