SINGAPORE MANAGEMENT UNIVERSITY

Session 4: Stochastic Integrals and Itô's Formula

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QF620 Stochastic Modelling in Finance

Building Stochastic Models

Our objective is to formulate a model for the stock price process. Let W_t denote a Brownian motion. We know that just W_t itself is not going to be a particularly good model for a stock, since we will also need to be able to

1. control its **drift** over time, and
2. control its **volatility**.

To that end, with t denoting time, we let

$$S_t = S_0 + \mu t + \sigma W_t, \quad \mu \in \mathbb{R},\ \sigma \in \mathbb{R}.$$

This now looks to be a more reasonable model for a stock price process.

We shall look at the distribution of the stock price S_t under our model. The stock price process is normally distributed as $S_t \sim N(S_0 + \mu t, \sigma^2 t)$. We can verify that:

$$\mathbb{E}[S_t] = S_0 + \mu t.$$
$$V[S_t] = V[\sigma W_t] = \sigma^2 t.$$

Building Stochastic Models

Example Consider the stock model

$$S_t = S_0 + \mu t + \sigma W_t$$

described in the previous page. Show that in this model, there is a non-zero probability for S_t to take on negative values.

Solution Since $S_t \sim N(S_0 + \mu t, \sigma^2 t)$, its probability density function is given by

$$f(s) = \frac{1}{\sqrt{2\pi t}\sigma} \exp\left[-\frac{(s - S_0 - \mu t)^2}{2\sigma^2 t}\right].$$

The probability of the event $S_t < 0$ can then be evaluated as

$$\mathbb{P}(S_t < 0) = \frac{1}{\sqrt{2\pi t}\sigma} \int_{-\infty}^{0} e^{-\frac{(s-S_0-\mu t)^2}{2\sigma^2 t}} ds.$$

If $\mu > 0$, then as t increases this probability will decrease, but it remains a non-zero positive value, since the density for $S_t < 0$ is non-zero.

Building Stochastic Models

A stochastic process is a continuous process that can be written either in the **integral form**

$$S_t = S_0 + \int_0^t \sigma \, dW_u + \int_0^t \mu \, du,$$

or equivalently in the **differential form**

$$dS_t = \mu \, dt + \sigma \, dW_t.$$

In most of the models we encounter in quantitative finance, σ and μ are <u>functions of S_t and t only</u>, so we write

$$dS_t = \mu(t, S_t) \, dt + \sigma(t, S_t) \, dW_t.$$

Suppose μ and σ are both constants, such that $\mu(t, S_t) = \mu$ and $\sigma(t, S_t) = \sigma$, then we can solve the SDE for this simple case and obtain

$$S_t = S_0 + \mu t + \sigma W_t.$$

From Brownian Motions to Stochastic Processes

Brownian motion is the natural candidate to be used to model the evolution of the stock price process S_t. We can write the future stock price as its present price plus a **deterministic** and a **stochastic components**:

$$\underbrace{S_{t+\Delta t}}_{\text{future price}} = \underbrace{S_t}_{\text{present price}} + \underbrace{\mu(t, S_t)\Delta t}_{\text{deterministic}} + \underbrace{\sigma(t, S_t)(W_{t+\Delta t} - W_t)}_{\text{stochastic}}$$

Question Having defined the drift coefficient as $\mu(t, S_t)$, how can the drift term be considered as deterministic, given that S_t is a stochastic process?

From Brownian Motions to Stochastic Processes

By using Brownian motion increment to form the stochastic component of our model, we are effectively using independent normally distributed increment to drive our stock price process.

Now we take the limit of $\Delta t \to 0^+$, and obtain the **stochastic differential equation (SDE)**

$$dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dW_t.$$

To solve this SDE, we wish we could write it in the following format:

$$\frac{dS_t}{dt} = \mu(t, S_t) + \sigma(t, S_t)\frac{dW_t}{dt}.$$

However, this is not feasible given that one of Brownian motion's properties is that it is **nowhere differentiable**.

Stochastic Integrals

Since the differentiation formulation does not work, let us try the integration formulation by expressing the stock price process as follow:

$$S_T = S_0 + \underbrace{\int_0^T \mu(u, S_u)\, du}_{\text{Riemann integral}} + \underbrace{\int_0^T \sigma(u, S_u)\, dW_u}_{\text{stochastic integral}}.$$

Note that on the right hand side, the first integral is a classic **Riemann integral**, and we know how to manage it. Recall the definition of a Riemann integral

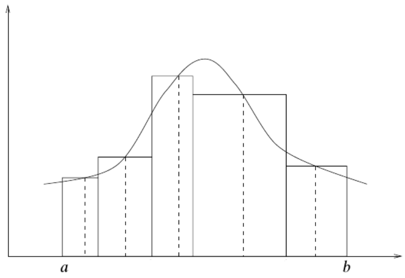
Riemann Integration

Let f be a regular function and P_n be a partition of the interval $[0, T]$, given by $\{t_0 = 0, t_1, t_2, \ldots, t_n = T\}$, then f is Riemann integrable if the following limit converges

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i) \times (t_i - t_{i-1}), \quad x_i \in [t_{i-1}, t_i].$$

Definition of Riemann Integrals

We can define and visualise a Riemann integral as the area under the curve. Consider a function $f : \mathbb{R} \to \mathbb{R}$ which we would like to integral over the interval $[a, b]$.



Partitioning the interval into

$$\{t_0 = a,\ t_1,\ t_2,\ t_3,\ t_4,\ t_5 = b\}$$

We can approximate the area as

$$S = \sum_{i=1}^{5} f(x_i)(t_i - t_{i-1}),$$

where

$$x_i \in [t_{i-1}, t_i].$$

The integral is therefore defined as $n \to \infty$.

Definition of Stochastic Integrals

What about the second integral? Let us define it using the same approach as the Riemann integral before. With the same notations, we define the **stochastic integral** as

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(t_{i-1})\times(W_{t_i}-W_{t_{i-1}}).$$

This limit exists in an appropriate sense. With this definition for stochastic integral, we note that

- the result of a stochastic integral is a random variable, as opposed to what we get from Riemann integral.
- in Riemann integral of a function $f(x_i)$, x_i can be any point in the interval $[t_{i-1}, t_i]$, whereas in stochastic integral, x_i must be taken at the left side of each interval (t_{i-1}). This is due to the **previsibility** condition.
- **Previsibility**: the value of $f(x_i)$ is only known at the beginning of the interval. Taking $x_i \neq t_{i-1}$ lead to different results.

Properties of Stochastic Integrals

Consider I_T defined as the stochastic integral:

$$I_T = \int_0^T f(u, W_u)\, dW_u.$$

Below are the key properties of stochastic integrals I_T:

1. $\mathbb{E}[I_T] = 0$
2. $\mathbb{E}[I_T^2] = \mathbb{E}\left[\left(\int_0^T f(u, W_u)\, dW_u\right)^2\right] = \mathbb{E}\left[\int_0^T f(u, W_u)^2\, du\right]$
3. If f is a deterministic function, then $I_T \sim N\left(0, \int_0^T f(u)^2\, du\right)$.
4. $\mathbb{E}\left[\int_0^T f(u)\, dW_u \times \int_0^T g(s)\, dW_s\right] = \mathbb{E}\left[\int_0^T f(u)g(u)\, du\right]$
5. **Itô's Isometry theorem** states that $\mathbb{E}\left[\left(\int_0^T X_t\, dW_t\right)^2\right] = \mathbb{E}\left[\int_0^T X_t^2\, dt\right]$.

Properties of Stochastic Integrals: Proof of (1)

Recall that our first principle definition of the stochastic integrals is given by

$$\int_0^T f(t, W_t)\, dW_t = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_{i-1}, W_{t_{i-1}}) \times (W_{t_i} - W_{t_{i-1}})$$

Note that Brownian motion's increments are independent, so $(W_{t_i} - W_{t_{i-1}})$ is independent from $W_{t_{i-1}}$. Hence, we have

$$\begin{aligned}
\mathbb{E}\left[\int_0^T f(t, W_t)\, dW_t\right] &= \mathbb{E}\left[\lim_{n\to\infty} \sum_{i=1}^{n} f(t_{i-1}, W_{t_{i-1}}) \times (W_{t_i} - W_{t_{i-1}})\right] \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \mathbb{E}\left[f(t_{i-1}, W_{t_{i-1}}) \times (W_{t_i} - W_{t_{i-1}})\right] \\
&= \lim_{n\to\infty} \sum_{i=i}^{n} \mathbb{E}\left[f(t_{i-1}, W_{t_{i-1}})\right] \times \mathbb{E}\left[(W_{t_i} - W_{t_{i-1}})\right] \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \mathbb{E}\left[f(t_{i-1}, W_{t_{i-1}})\right] \times 0 = 0
\end{aligned}$$

Properties of Stochastic Integrals: Proof of (2)

Let $\Delta W_{t_{i-1}} = W_{t_i} - W_{t_{i-1}}$ and $\Delta W_{t_{j-1}} = W_{t_j} - W_{t_{j-1}}$, note that

$$\mathbb{E}\Big[f(t_{i-1}, W_{t_{i-1}}) \cdot f(t_{j-1}, W_{t_{j-1}}) \cdot \Delta W_{t_{i-1}} \cdot \Delta W_{t_{j-1}}\Big] = \begin{cases} \mathbb{E}[f(t_{i-1}, W_{t_{i-1}})^2] \times (t_i - t_{i-1}) & i = j \\ 0 & i \neq j \end{cases}$$

We have

$$\begin{aligned} \mathbb{E}\left[\left(\int_0^T f(t, W_t)\, dW_t\right)^2\right] &= \mathbb{E}\left[\int_0^T f(s, W_s)\, dW_s \times \int_0^T f(u, W_u)\, dW_u\right] \\ &= \lim_{n\to\infty} \sum_{i=1,j=1}^{n} \mathbb{E}\Big[f(t_{i-1}, W_{t_{i-1}}) \cdot f(t_{j-1}, W_{t_{j-1}}) \cdot \Delta W_{t_{i-1}} \cdot \Delta W_{t_{j-1}}\Big] \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \mathbb{E}[f(t_{i-1}, W_{t_{i-1}})^2] \times (t_i - t_{i-1}) \\ &= \mathbb{E}\left[\int_0^T f(t, W_t)^2\, dt\right] \end{aligned}$$

Stochastic Integrals

Example Consider the stochastic integral

$$I_T = \int_0^T W_t \, dW_t,$$

determine its mean $\mathbb{E}[I_T]$ and variance $V[I_T]$.

Quadratic Variation

Strong Law of Large Numbers

Kolmogorov's strong law states that the average of a sequence of independent random variables having a common distribution will with probability 1 converge to the mean of that distribution:

$$n \to \infty \quad \Rightarrow \quad \mathbb{P}\left(\frac{X_1 + X_2 + X_3 + \cdots + X_n}{n} = \mu\right) = 1.$$

For example, if $X_1, X_2, \ldots$ is a sequence of independent binomial random variables taking values $+1$ or -1 with equal probability, then the Strong Law states that

$$\lim_{n\to\infty} \frac{X_1 + X_2 + \cdots + X_n}{n} = 0.$$

We can use the Law of Large Numbers to argue that $(\Delta W_t)^2$ and Δt are of the same order, leading to the Box calculus rule of $dW_t \cdot dW_t = dt$.

Quadratic Variation

Suppose W_t is a Brownian motion, and we have the time partition $\{t_0, t_1, t_2, \ldots, t_n\}$, where $t_0 = 0$ and $t_n = T$. Consider any two timestamps t_i and t_{i-1}, note that

$$\mathbb{E}\left[(W_{t_i} - W_{t_{i-1}})^2\right] = t_i - t_{i-1}$$

Let us define $\Delta W_t = W_{t_i} - W_{t_{i-1}}$, and $\Delta t = t_i - t_{i-1}$. We have

$$\mathbb{E}[(\Delta W_t)^2] = \Delta t$$

Also note that a quick rearrangement yields

$$\mathbb{E}\left[\frac{(\Delta W_t)^2}{\Delta t}\right] = 1.$$

The Law of Large Numbers asserts that

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\frac{(\Delta W_t)^2}{\Delta t}}{n} = 1.$$

Quadratic Variation

Now we write,

$$(\Delta W_t)^2 = \Delta t \cdot \frac{(\Delta W_t)^2}{\Delta t} = \frac{T}{n} \cdot \frac{(\Delta W_t)^2}{\Delta t} = T \cdot \frac{\frac{(\Delta W_t)^2}{\Delta t}}{n}$$

Summing up and sending the limit of $n \to \infty$, we have

$$\lim_{n\to\infty} \sum_{i=1}^{n} (\Delta W_t)^2 = T \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\frac{(\Delta W_t)^2}{\Delta t}}{n} = T.$$

Compare this with the expression

$$\lim_{n\to\infty} \sum_{i=1}^{n} \Delta t = T,$$

and using the differential notation, we arrive at the rule of

$$(dW_t)^2 = dt.$$

Note that this does not mean that every single $(\Delta W_t)^2$ is exactly equal to Δt. But when summed over a time interval, the Law of Large Numbers asserts that it converges to the mean.

Box Calculus Rules

From the discussion above, we see that $(\Delta W_t)^2$ is in the order of Δt, and hence ΔW_t is in the order of $\sqrt{\Delta t}$. Under the limit of $\Delta t \to 0$, we use differential notation. This yields

$$\Delta t \to dt$$
$$(\Delta t)^2 \to (dt)^2 = 0$$
$$\Delta W_t \to dW_t$$
$$(\Delta W_t)^2 \to dW_t^2 = dt$$
$$(\Delta t \cdot \Delta W_t) \to dt \cdot dW_t = 0$$

These rules are based on the formalism of **Box Calculus**:

$\cdot$	dt	dW_t
dt	0	0
dW_t	0	dt

So for example, we can write:

$$dX_t = \mu\, dt + \sigma\, dW_t \quad \Rightarrow \quad (dX_t)^2 = \sigma^2 dt.$$

The History...

Brown, a botanist, discovered the motion of pollen particles in water in 1827.

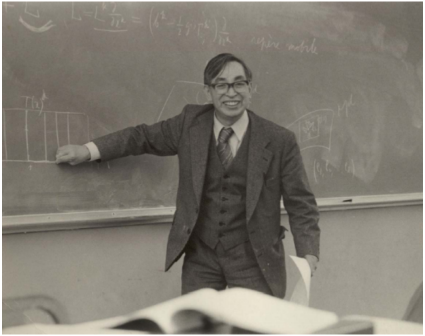
At the beginning of the 20^{th} *century, Brownian motion was studied by Einstein, Perrin and other physicists.*

In 1923, against this scientific background, Wiener defined probability measures in path spaces, and used the concept of Lebesgue integrals to lay the mathematical foundations of stochastic analysis.

In 1942, Dr. Itô began to reconstruct from scratch the concept of stochastic integrals, and its associated theory of analysis. He created the theory of stochastic differential equations, which describe motion due to random events.

Kiyosi Itô (1915–2008)

> *...I finally devised stochastic differential equations, after painstaking solitary endeavours.*
>
> — Kiyosi Itô (1915–2008)



Why Itô's Formula?

From a theoretical viewpoint:

- Now that we have defined the stochastic integral, we want to be able to manipulate it without coming back to the definition.
- General rules similar to ordinary calculus chain rules, product rules etc. will be very handy.

From a practical viewpoint:

- We have now defined the stock price process, and knowing that an option price is a function of that random process,
- We want to study and understand the infinitesimal evolution of the option price process as well.

The purpose of computing is insight, not numbers.

— Richard Hamming (1915–1998)

There is no doubt that the field of quantitative finance has been thoroughly transformed by the basic insights provided by Ito's calculus, both on a conceptual and on a computational level.

— Hans Föllmer (b. 1941)

Formulation of Itô's Formula

Under ordinary calculus, **Taylor expansion** yields

$$f(t+\Delta t) = f(t) + f'(t)\Delta t + \frac{1}{2!}f''(t)(\Delta t)^2 + \frac{1}{3!}f'''(t)(\Delta t)^3 + \cdots$$

If we let $y = f(t)$, a simple rearrangement yields

$$\begin{aligned} f(t+\Delta t) - f(t) &= f'(t)\Delta t + \frac{1}{2!}f''(t)(\Delta t)^2 + \frac{1}{3!}f'''(t)(\Delta t)^3 + \cdots \\ \Delta y &= f'(t)\Delta t + \frac{1}{2!}f''(t)(\Delta t)^2 + \frac{1}{3!}f'''(t)(\Delta t)^3 + \cdots \end{aligned}$$

Sending the limit of $\Delta t \to 0$, we obtain

$$\frac{dy}{dt} = f'(t).$$

Formulation of Itô's Formula

Now consider the case for stochastic calculus. We have

$$f(W_t + \Delta W_t) = f(W_t) + f'(W_t)\Delta W_t + \frac{1}{2!}f''(W_t)(\Delta W_t)^2 + \cdots$$

If we let $Y_t = f(W_t)$, a simple rearrangement now yields

$$\begin{aligned} f(W_t + \Delta W_t) - f(W_t) &= f'(W_t)\Delta W_t + \frac{1}{2!}f''(W_t)(\Delta W_t)^2 + \cdots \\ \Delta Y_t &= f'(W_t)\Delta W_t + \frac{1}{2!}f''(W_t)(\Delta W_t)^2 + \cdots \end{aligned}$$

Now when we send the limit of $\Delta t \to 0$, we arrive at

$$dY_t = f'(W_t)dW_t + \underbrace{\frac{1}{2}f''(W_t)dt}_{\text{extra term}}$$

Formulation of Itô's Formula

Suppose now we model the stock price discrete dynamics as

$$\Delta S_t = \mu \, \Delta t + \sigma \, \Delta W_t,$$

and we let $X_t = f(S_t)$, we have

$$f(S_t + \Delta S_t) = f(S_t) + f'(S_t)\Delta S_t + \frac{1}{2!} f''(S_t)(\Delta S_t)^2 + \mathcal{O}\big((\Delta S_t)^3\big).$$

The next step is obviously to send the limit of $\Delta t \to 0$. To that end, we must first analyse the expression

$$(\Delta S_t)^2 = \mu^2 \cdot (\Delta t)^2 + 2 \cdot \mu \cdot \sigma \cdot (\Delta t)(\Delta W_t) + \sigma^2 \cdot (\Delta W_t)^2.$$

Under the limit $\Delta t \to 0$, we obtain

$$\begin{aligned} dX_t &= f'(S_t) dS_t + \frac{1}{2} f''(S_t) \sigma^2 dt \\ &= \left(\mu f'(S_t) + \frac{\sigma^2}{2} f''(S_t) \right) dt + \sigma f'(S_t) \, dW_t. \end{aligned}$$

Itô's Formula

This leads us to the famous **Itô's formula** (sometimes known as **Itô's lemma**):

Itô's Formula (Function of a Stochastic Process)

If X_t is a stochastic process satisfying

$$dX_t = \mu_t \, dt + \sigma_t \, dW_t,$$

and the function $f : \mathbb{R} \to \mathbb{R}, f, f', f''$ are continuous, then $Y_t = f(X_t)$ is also a stochastic process and is given by

$$dY_t = \left(\mu_t f'(X_t) + \frac{1}{2}\sigma_t^2 f''(X_t) \right) dt + \sigma_t f'(X_t) dW_t.$$

Example Suppose $dX_t = \mu \, dt + \sigma \, dW_t$, and $Y_t = X_t^2$. Derive the stochastic differential equation for dY_t.

Itô's Formula

More generally, Itô's Formula also allows us to write down the stochastic differential equation of a function of stochastic processes and time:

Itô's Formula (Function of a Stochastic Process & Time)

If X_t is a stochastic process satisfying

$$dX_t = \mu_t \, dt + \sigma_t \, dW_t,$$

and the function $g : \mathbb{R}^2 \to \mathbb{R}, \; g(t,x), \; g_t, \; g_x, \; g_{xx}$ are continuous, then $Y_t = g(t, X_t)$ is also a stochastic process and is given by

$$dY_t = \left[g_t(t,X_t) + \mu_t g_x(t,X_t) + \frac{1}{2}\sigma_t^2 g_{xx}(t,X_t) \right] dt + \sigma_t g_x(t,X_t) dW_t.$$

Example Suppose $dX_t = \mu \, dt + \sigma \, dW_t$, and $Y_t = e^{X_t + t}$. Derive the stochastic differential equation for dY_t.

Itô's Formula: Applications

Example Show that

$$\int_0^T W_t \, dW_t = \frac{W_T^2}{2} - \frac{T}{2}$$

by applying Itô formula to $X_t = f(W_t) = W_t^2$.

Itô's Formula: Applications

Example Apply Itô formula to the function $X_t = f(t, W_t) = tW_t$, and show that

$$\int_0^T W_t \, dt = TW_T - \int_0^T t \, dW_t = \int_0^T (T - t) \, dW_t.$$

Use this to show that

$$V\left[\int_0^T W_t \, dt\right] = \frac{T^3}{3}.$$