Multi-Period Asset Pricing

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Multi-Period Setting – Part 1

- Consider investor with T-period planning horizon and time-separable utility of consumption:

$$V(C_0, \ldots, C_T) = E\left[\sum_{t=0}^{T} \delta^t U(C_t)\right]$$

- Here $\delta \in (0, 1)$ is subjective discount factor that reflects investor's rate of time preference, while U is strictly increasing and concave utility function
- Investor is endowed with initial wealth of W_0 and trades in n risky assets with (random) return of $R_{i,t+1}$ over time interval from t to $t + 1$, for $i = 1, \ldots, n$ and $t = 0, \ldots, T - 1$

Multi-Period Setting – Part 2

- Investor can rebalance portfolio at start of each time interval
- Investor allocates proportion $w_{i,t}$ of remaining wealth of $(W_t - C_t)$ to i'th asset at time t, subject to $\sum_{i=1}^{n} w_{i,t} = 1$:

$$W_{t+1} = (W_t - C_t) \sum_{i=1}^{n} w_{i,t} R_{i,t+1}$$

- Investor will choose consumption and asset allocation for each time period to maximise lifetime expected utility
- Optimal solution can be found using dynamic programming: solve static optimisation problem in last two time periods, and repeat for investor with $(T-1)$-period planning horizon, etc.

Asset Pricing Formula

- Intertemporal allocation condition for optimal consumption:

$$U'(C_t^*) = \delta E_t \left[U'\left(C_{t+1}^*\right) R_{t+1} \right]$$

- Here $E_t[\cdot]$ is expectation conditional on information at time t
- Divide through by $U'(C_t^*)$ to get asset pricing formula:

$$E_t \left[\delta \frac{U'\left(C_{t+1}^*\right)}{U'(C_t^*)} R_{t+1} \right] = E_t[M_{t+1} R_{t+1}] = 1$$

- Here $M_{t+1} = \delta U'\left(C_{t+1}^*\right) / U'(C_t^*)$ is investor's intertemporal marginal rate of substitution over time interval from time t to time $t+1$, which acts as (one-period) pricing kernel

Dividend Discount Model – Part 1

- Consider "long-lived" asset that has price of P_{t+i} and pays dividend of D_{t+i} at time $t+i$, for $i = 0, \ldots, T$
- Holding period return over first time interval:

$$R_{t+1} = \frac{D_{t+1} + P_{t+1}}{P_t}$$

- Use asset pricing formula to get price of long-lived asset at time t and time $t+1$:

$$P_t = E_t[M_{t+1}(D_{t+1} + P_{t+1})]$$
$$P_{t+1} = E_{t+1}[M_{t+2}(D_{t+2} + P_{t+2})]$$

Dividend Discount Model – Part 2

- Substitute for P_{t+1} and use law of iterated expectations:

$$\begin{aligned} P_t &= E_t[M_{t+1}D_{t+1} + M_{t+1}E_{t+1}[M_{t+2}\left(D_{t+2} + P_{t+2}\right)]] \\ &= E_t[M_{t+1}D_{t+1} + M_{t,t+2}\left(D_{t+2} + P_{t+2}\right)] \end{aligned}$$

- Here $M_{t,t+2} = M_{t+1}M_{t+2} = \delta^2 U'\left(C^*_{t+2}\right)/U'(C^*_t)$ is pricing kernel over time interval from time t to time $t+2$
- By extension, general pricing formula for long-lived asset:

$$P_t = E_t\left[\sum_{i=1}^{T} M_{t,t+i}D_{t+i} + M_{t,t+T}P_{t+T}\right]$$

Dividend Discount Model – Part 3

- Here $M_{t,t+i} = M_{t+1} \cdots M_{t+i} = \delta^i U'\left(C^*_{t+i}\right) / U'(C^*_t)$ is pricing kernel over time interval from time t to time $t+i$
- If investor has infinite lifetime, and long-lived asset has no fixed maturity date, then can take limit as $T \to \infty$:

$$P_t = E_t\left[\sum_{i=1}^{\infty} M_{t,t+i} D_{t+i}\right]$$

- Assumes no price "bubbles": $E_t[M_{t,t+T} P_{t+T}] \to 0$
- Interpret each term in infinite sum as price of individual "dividend claim" that delivers one single future dividend:

$$P_{i,t} = E_t[M_{t,t+i} D_{t+i}] \implies P_t = P_{1,t} + P_{2,t} + \cdots$$

Endowment Economy

- For simplicity, assume "endowment economy" where aggregate economic output grows randomly over time
- Investor who invests in market portfolio will receive share of aggregate economic output $\implies$ aggregate dividend (from market portfolio) is equivalent to aggregate economic output
- Investor immediately consumes any dividend that is received $\implies$ aggregate consumption must be equal to aggregate dividend in every time period: $\overline{C}_t = D_t$ for all $t = 0, 1, 2, \ldots$
- If financial market is "complete" and frictionless, then there will be unique pricing kernel that prices all assets
- Equivalent to economy where single representative investor consumes (per capita) aggregate consumption and invests in market portfolio to receive (per capita) aggregate dividend

Power Utility

- Suppose that representative investor has power utility:

$$U(C_t) = \frac{\overline{C}_t^{1-\gamma}}{1-\gamma} \implies M_{t+i} = \delta^i \left(\frac{\overline{C}_{t+i}}{\overline{C}_t} \right)^{-\gamma}$$

- Aggregate consumption is always equal to aggregate dividend, so price-dividend ratio for market portfolio:

$$\frac{P_t}{D_t} = E_t \left[\sum_{i=1}^{\infty} \delta^i \left(\frac{D_{t+i}}{D_t} \right)^{1-\gamma} \right]$$

- Must specify stochastic process for aggregate consumption (and dividend and output) to solve for price-dividend ratio

Lognormal Growth: Economic Environment

- Suppose that aggregate consumption evolves as lognormal random walk with drift:

$$\ln \overline{C}_{t+1} = \ln \overline{C}_t + \mu + \sigma\epsilon_{t+1}$$

- Here $\epsilon_t \sim N(0, 1)$ is independent and identically distributed (i.i.d.) random variable that captures random fluctuations
- Hence continuously compounded aggregate consumption growth rate has normal distribution over every time interval
- Then μ represents expected aggregate consumption growth rate, while σ represents volatility of economic fluctuations
- Let $\rho = -\ln\delta$ be investor's rate of time preference

Lognormal Growth: Market Portfolio – Part 1

- Dividend claim that delivers aggregate dividend at time $t+1$ has constant price-dividend ratio:

$$\frac{P_{1,t}}{D_t} = E_t\left[\delta\left(\frac{\overline{C}_{t+1}}{\overline{C}_t}\right)^{1-\gamma}\right] = E_t\left[\delta e^{(1-\gamma)(\mu+\sigma\epsilon_{t+1})}\right]$$
$$= e^{-\rho+(1-\gamma)\mu+\frac{1}{2}(1-\gamma)^2\sigma^2} = \theta$$

- Price-dividend ratio for dividend claim that delivers aggregate dividend at time $t+i$:

$$\frac{P_{i,t}}{D_t} = E_t\left[\delta^i\left(\frac{\overline{C}_{t+i}}{\overline{C}_t}\right)^{1-\gamma}\right] = E_t\left[\prod_{j=0}^{i-1}\delta\left(\frac{\overline{C}_{t+j+1}}{\overline{C}_{t+j}}\right)^{1-\gamma}\right]$$

Lognormal Growth: Market Portfolio – Part 2

- Consumption growth is i.i.d., so all dividend claims have constant price-dividend ratio:

$$\frac{P_{i,t}}{D_t} = \prod_{j=0}^{i-1} E_t\left[\delta\left(\frac{\overline{C}_{t+j+1}}{\overline{C}_{t+j}}\right)^{1-\gamma}\right] = \prod_{j=0}^{i-1} \frac{P_{1,t+j}}{D_{t+j}} = \theta^i$$

- Hence market portfolio will also have finite constant price-dividend ratio when $\theta < 1$:

$$\frac{P_t}{D_t} = \sum_{i=1}^{\infty} \frac{P_{i,t}}{D_t} = \sum_{i=1}^{\infty} \theta^i = \frac{\theta}{1-\theta}$$

Lognormal Growth: Market Portfolio – Part 3

- Market portfolio also has constant expected return:

$$
\begin{aligned}
E_t[R_{t+1}] &= E_t\left[\frac{D_{t+1}+P_{t+1}}{P_t}\right] = \frac{D_t}{P_t}E_t\left[\frac{D_{t+1}}{D_t}\left(1+\frac{P_{t+1}}{D_{t+1}}\right)\right] \\
&= \frac{1-\theta}{\theta}E_t\left[\frac{D_{t+1}}{D_t}\left(1+\frac{\theta}{1-\theta}\right)\right] = \frac{1}{\theta}E_t\left[\frac{\overline{C}_{t+1}}{\overline{C}_t}\right] \\
&= e^{\rho+\gamma\mu-\frac{1}{2}\gamma^2\sigma^2+\gamma\sigma^2}
\end{aligned}
$$

- First dividend claim has same mean return as market portfolio:

$$
E_t\left[\frac{D_{t+1}}{P_{1,t}}\right] = \frac{D_t}{P_{1,t}}E_t\left[\frac{D_{t+1}}{D_t}\right] = \frac{1}{\theta}E_t\left[\frac{\overline{C}_{t+1}}{\overline{C}_t}\right] = E_t[R_{t+1}]
$$

Lognormal Growth: Equity Premium

- Suppose there exists riskless asset that always delivers one unit of output in next time period:

$$P_{f,t} = E_t[M_{t+1}] \implies R_{f,t} = \frac{1}{P_{f,t}} = e^{\rho+\gamma\mu-\frac{1}{2}\gamma^2\sigma^2}$$

- Hence continuously compounded equity premium:

$$\ln E_t[R_{t+1}] - \ln R_{f,t} = \gamma\sigma^2$$

- Equity premium puzzle: annual consumption growth is very smooth (with $\sigma \approx 2\%$), so annual equity premium of only 4% even with $\gamma = 100$

Rare Disasters: Economic Environment

- Now suppose that aggregate consumption also contains i.i.d. random variable that represents effect of rare disaster:

$$\ln \overline{C}_{t+1} = \ln \overline{C}_t + \mu + \sigma \epsilon_{t+1} + \nu_{t+1},$$

$$\nu_t = \begin{cases} \ln \phi & \text{with probability of } \pi \\ 0 & \text{with probability of } 1 - \pi \end{cases}$$

- As before, use $\pi = 1.7\%$ and $\phi = 0.65$
- Price-dividend ratio for first dividend claim:

$$\frac{P_{1,t}}{D_t} = \theta E_t \left[e^{(1-\gamma)\nu_{t+1}} \right] = \theta \left\{ 1 + \pi \left(\phi^{1-\gamma} - 1 \right) \right\}$$

Rare Disasters: Equity Premium – Part 1

- Market portfolio has same mean return as first dividend claim:

$$E_t[R_{t+1}] = \frac{D_t}{P_{1,t}} E_t\left[\frac{\overline{C}_{t+1}}{\overline{C}_t}\right] = \frac{e^{\mu+\frac{1}{2}\sigma^2}\left\{1+\pi\left(\phi-1\right)\right\}}{\theta\left\{1+\pi\left(\phi^{1-\gamma}-1\right)\right\}}$$

- Can use $\ln(1+x) \approx x$ as long as γ is reasonably small:

$$\ln E_t[R_{t+1}] \approx \rho + \gamma\mu - \frac{1}{2}\gamma^2\sigma^2 + \gamma\sigma^2 - \pi\phi\left(\phi^{-\gamma}-1\right)$$

- Risk-free rate for riskless bond:

$$R_{f,t} = E_t[M_{t+1}]^{-1} = \left\{1+\pi\left(\phi^{-\gamma}-1\right)\right\}^{-1} e^{\rho+\gamma\mu-\frac{1}{2}\gamma^2\sigma^2}$$

Rare Disasters: Equity Premium – Part 2

- Can also use $\ln(1+x) \approx x$ as long as γ is reasonably small:

$$\ln R_{f,t} \approx \rho + \gamma\mu - \frac{1}{2}\gamma^2\sigma^2 - \pi\left(\phi^{-\gamma} - 1\right)$$

- Hence continuously compounded equity premium:

$$\ln E_t[R_{t+1}] - \ln R_f \approx \gamma\sigma^2 + \pi\,(1-\phi)\left(\phi^{-\gamma} - 1\right)$$

- Annual equity premium of around 7.5% for $\gamma = 6$, which represents reasonable degree of (relative) risk aversion:

$$6 \times 0.02^2 + 0.017 \times 0.35 \times \left(0.65^{-6} - 1\right) = 7.5\%$$