Efficient Frontier

Wang Wei Mun

Lee Kong Chian School of Business
Singapore Management University

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Outline

Asset Allocation vs Asset Pricing

- Asset allocation (or portfolio choice) is theory of how investor allocates wealth amongst financial assets
- For simplicity, assume investor is "price taker" $\implies$ scale of investment is small enough to not affect prices
- Also for simplicity, assume perfect "frictionless" financial market with no taxes or transaction costs, etc.
- Asset allocation goes hand-in-hand with asset pricing: if we know how all investors choose to allocate their wealth, then we can find equilibrium prices to balance supply and demand
- Harry Markowitz pioneered modern concept of asset allocation with theory of mean-variance-efficient frontier in 1952

Investment Environment

- Financial market consists of $n \geq 2$ risky tradable assets with normal returns (and no riskless asset)
- Let $\mathbf{R} = (R_1, \ldots, R_n)'$ be $n \times 1$ vector of expected returns
- Let $\mathbf{V}$ be $n \times n$ covariance matrix of returns, which consists of variances on diagonal and covariances on off-diagonal
- $\mathbf{V}$ must be symmetric: $\mathbf{V}' = \mathbf{V}$, and positive definite: $\mathbf{z}'\mathbf{V}\mathbf{z} > 0$ for any non-zero $n \times 1$ vector $\mathbf{z}$
- Assume no redundant assets, so returns must be linearly independent and covariance matrix must be invertible:

$$\exists\ \mathbf{V}^{-1} \text{ such that } \mathbf{V}^{-1}\mathbf{V} = \mathbf{I}$$

- Then $\mathbf{V}^{-1}$ is also symmetric and positive definite

Portfolio Weights

- Let $\mathbf{w} = (w_1, \ldots, w_n)'$ be $n \times 1$ vector of portfolio weights, which represents proportion of investor's wealth allocated to each tradable financial asset
- No restriction on individual portfolio weights: positive weight indicates normal investment (or "long position") while negative weight indicates short-selling (or "short position")
- Only restriction is that portfolio weights must sum to one: $\mathbf{w}'\mathbf{e} = 1$, where $\mathbf{e} = (1, \ldots, 1)'$ is $n \times 1$ unit vector
- Investor can allocate more than available wealth into any individual financial asset, but cannot allocate more than available wealth in aggregate
- Then $\mathbf{w}'\mathbf{R}$ is expected return for investor's portfolio, and $\mathbf{w}'\mathbf{V}\mathbf{w} > 0$ is variance of return for investor's portfolio

Asset Allocation Problem – Part 1

- Investor's ultimate goal is to create "optimal" portfolio that maximises expected utility (of wealth)
- Requires knowledge of investor's utility function, which is difficult to observe in reality
- Since asset returns have normal distribution, investor's expected utility depends only on mean return and variance of return for investor's portfolio
- Risk-averse investor will always prefer higher mean return and lower variance of return
- Hence simpler problem is to identify all "efficient" or "frontier" portfolios that minimise risk for given mean return, which includes specific optimal portfolio for any risk-averse investor

Asset Allocation Problem – Part 2

- Portfolio weights for portfolio with mean return of R_p must satisfy two conditions:
  - Portfolio weights must sum to one: $\mathbf{w}'\mathbf{e} = 1$
  - Portfolio must have mean return of R_p: $\mathbf{w}'\mathbf{R} = R_p$
- Out of all portfolios that satisfy these two conditions, frontier portfolio has lowest variance of return: $\mathbf{w}'\mathbf{V}\mathbf{w}$
- In mathematical terms, finding portfolio weights for frontier portfolio is constrained minimisation problem with quadratic objective function and equality constraints
- Fortunately, this minimisation problem is guaranteed to have unique solution that is global minimum

Asset Allocation Problem – Part 3

- Solve constrained minimisation problem by adding Lagrange multipliers to objective function (or "Langragian") to ensure that equality constraints are satisfied:

$$\min_{\{\mathbf{w},\lambda,\gamma\}} \mathcal{L} = \mathbf{w}'\mathbf{V}\mathbf{w} + \lambda\left(R_p - \mathbf{w}'\mathbf{R}\right) + \gamma\left(1 - \mathbf{w}'\mathbf{e}\right)$$

- Here λ and γ are Lagrange multipliers for equality constraints on mean return and portfolio weights, respectively
- From economic perspective, Lagrange multiplier represents marginal cost ("shadow price") of relaxing constraint
- Alternative "dual problem" is to maximise mean return for specified variance of return, but more difficult to solve

Frontier Portfolios – Part 1

- Set partial derivative of Lagrangian to zero to get first-order optimality condition for portfolio weights of frontier portfolio:

$$\frac{\partial \mathcal{L}}{\partial \mathbf{w}} = \mathbf{V}\mathbf{w}^* - \lambda \mathbf{R} - \gamma \mathbf{e} = 0$$

- Use other first-order optimality conditions (for Lagrange multipliers) to confirm that equality constraints are satisfied:

$$\frac{\partial \mathcal{L}}{\partial \lambda} = R_p - \mathbf{w}'\mathbf{R} = 0 \quad \Longrightarrow \quad \mathbf{w}'\mathbf{R} = R_p$$
$$\frac{\partial \mathcal{L}}{\partial \gamma} = 1 - \mathbf{w}'\mathbf{e} = 0 \quad \Longrightarrow \quad \mathbf{w}'\mathbf{e} = 1$$

Frontier Portfolios – Part 2

- Pre-multiply first-order optimality condition for frontier portfolio weights by $\mathbf{V}^{-1}$ and rearrange:

$$\mathbf{V}\mathbf{w}^* - \lambda\mathbf{R} - \gamma\mathbf{e} = 0 \quad \Longrightarrow \quad \mathbf{w}^* = \lambda\mathbf{V}^{-1}\mathbf{R} + \gamma\mathbf{V}^{-1}\mathbf{e}$$

- Pre-multiply by $\mathbf{R}'$ and apply constraint for mean return:

$$\mathbf{R}'\mathbf{w}^* = \lambda\mathbf{R}'\mathbf{V}^{-1}\mathbf{R} + \gamma\mathbf{R}'\mathbf{V}^{-1}\mathbf{e} = R_p$$

- Pre-multiply by $\mathbf{e}'$ and apply constraint for portfolio weights:

$$\mathbf{e}'\mathbf{w}^* = \lambda\mathbf{e}'\mathbf{V}^{-1}\mathbf{R} + \gamma\mathbf{e}'\mathbf{V}^{-1}\mathbf{e} = 1$$

Frontier Portfolios – Part 3

- Solve simultaneous equations to find Lagrange multipliers:

$$\lambda = \frac{\delta R_p - \alpha}{\zeta\delta - \alpha^2}; \qquad \gamma = \frac{\zeta - \alpha R_p}{\zeta\delta - \alpha^2}$$

- Here α is scalar, while ζ and δ are strictly positive scalars:

$$\alpha = \mathbf{R}'\mathbf{V}^{-1}\mathbf{e}; \qquad \zeta = \mathbf{R}'\mathbf{V}^{-1}\mathbf{R}; \qquad \delta = \mathbf{e}'\mathbf{V}^{-1}\mathbf{e}$$

- Confirm that denominator is strictly positive:

$$(\alpha\mathbf{R} - \zeta\mathbf{e})'\,\mathbf{V}^{-1}\,(\alpha\mathbf{R} - \zeta\mathbf{e}) = \zeta\left(\zeta\delta - \alpha^2\right) > 0$$

Frontier Portfolios – Part 4

- Substitute for λ and γ to find portfolio weights for frontier portfolio with mean return of R_p:

$$\mathbf{w}^* = \left(\frac{\delta R_p - \alpha}{\zeta\delta - \alpha^2} \right) \mathbf{V}^{-1}\mathbf{R} + \left(\frac{\zeta - \alpha R_p}{\zeta\delta - \alpha^2} \right) \mathbf{V}^{-1}\mathbf{e}$$

- Rearrange to get linear relationship: $\mathbf{w}^* = \mathbf{a} + \mathbf{b}R_p$, where:

$$\mathbf{a} = \frac{\zeta\mathbf{V}^{-1}\mathbf{e} - \alpha\mathbf{V}^{-1}\mathbf{R}}{\zeta\delta - \alpha^2}; \qquad \mathbf{b} = \frac{\delta\mathbf{V}^{-1}\mathbf{R} - \alpha\mathbf{V}^{-1}\mathbf{e}}{\zeta\delta - \alpha^2}$$

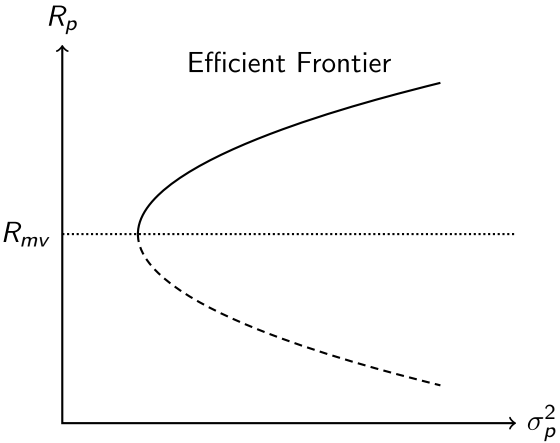
- Minimum-variance frontier consists of portfolios with lowest amount of risk, for different values of R_p

Efficient Frontier – Part 1

- Variance of return for frontier portfolio:

$$\begin{aligned}\sigma_p^2 = (\mathbf{w}^*)' \mathbf{V} \mathbf{w}^* &= (\mathbf{a} + \mathbf{b} R_p)' \mathbf{V} (\mathbf{a} + \mathbf{b} R_p) \\ &= \frac{\delta R_p^2 - 2\alpha R_p + \zeta}{\zeta\delta - \alpha^2} \\ &= \frac{1}{\delta} + \frac{\delta}{\zeta\delta - \alpha^2} (R_p - R_{mv})^2\end{aligned}$$

- $R_{mv} = \frac{\alpha}{\delta}$ is mean return for global minimum-variance portfolio
- Minimum-variance frontier is parabola when plotted with variance of return on y-axis and expected return on x-axis
- Standard practice to flip axes, as shown on next slide

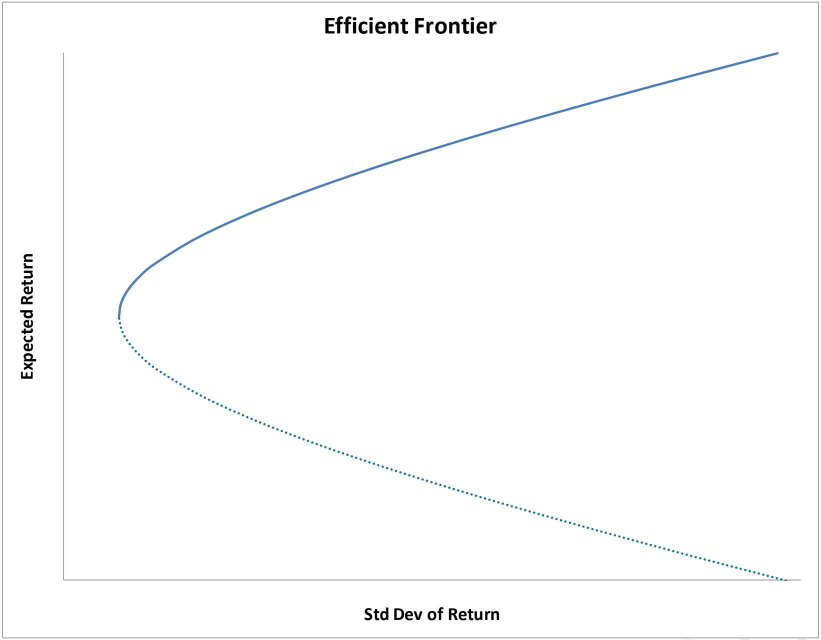

R_p
Efficient Frontier
R_{mv}
σ_p^2

Efficient Frontier – Part 2

- Top half of minimum-variance frontier (where $R_p \geq R_{mv}$) is known as efficient frontier, consisting of portfolios with highest mean return for given amount of risk
- If use standard deviation of return (instead of variance of return) on x-axis, then minimum-variance frontier is hyperbola with center at $(0, R_{mv})$ and asymptotes:

$$R_p = R_{mv} \pm \left(\zeta - \frac{\alpha^2}{\delta}\right)^{\frac{1}{2}} \sigma_p$$

- To maximise expected utility, investor will choose optimal portfolio where indifference curve is tangent to frontier


Efficient Frontier
Expected Return
Std Dev of Return

Portfolio Separation – Part 1

- Affine combination is linear combination with (positive or negative) coefficients that sum to one, so affine combination of two portfolios is also portfolio:

$$[\kappa \mathbf{w}_1 + (1-\kappa)\,\mathbf{w}_2]'\,\mathbf{e} = \kappa \mathbf{w}_1'\mathbf{e} + (1-\kappa)\,\mathbf{w}_2'\mathbf{e} = 1$$

- If p_1 and p_2 are both frontier portfolios, then affine combination of p_1 and p_2 is also frontier portfolio:

$$\begin{aligned}\kappa \mathbf{w}_1^* + (1-\kappa)\,\mathbf{w}_2^* &= \kappa\left(\mathbf{a} + \mathbf{b}R_{p_1}\right) + (1-\kappa)\left(\mathbf{a} + \mathbf{b}R_{p_2}\right)\\ &= \mathbf{a} + \mathbf{b}\left[\kappa R_{p_1} + (1-\kappa)\,R_{p_2}\right]\\ &= \mathbf{a} + \mathbf{b}R_{p_3}\end{aligned}$$

Portfolio Separation – Part 2

- Here p_3 is frontier portfolio with mean return of $R_{p_3} = \kappa R_{p_1} + (1-\kappa)\, R_{p_2}$
- No restriction on value of κ, so investor can generate entire minimum-variance frontier with different affine combinations of any two frontier portfolios
- Two-fund (or mutual fund) separation theorem: investor can construct optimal portfolio with appropriate affine combination of any two frontier portfolios
- In theory, more convenient since investor can generate efficient frontier and construct optimal portfolio without knowing **R** and **V** for individual risky assets

Orthogonal Frontier Portfolios – Part 1

- Covariance of return between two frontier portfolios:

$$\begin{aligned}(\mathbf{w}_1^*)' \mathbf{V} \mathbf{w}_2^* &= (\mathbf{a} + \mathbf{b} R_{p_1})' \mathbf{V} (\mathbf{a} + \mathbf{b} R_{p_2}) \\ &= \frac{1}{\delta} + \frac{\delta}{\zeta\delta - \alpha^2} (R_{p_1} - R_{mv})(R_{p_2} - R_{mv})\end{aligned}$$

- For given p_1, set covariance to zero to find mean return for p_2, which is frontier portfolio that is "orthogonal" to p_1:

$$R_{p_2} = R_{mv} - \frac{\zeta\delta - \alpha^2}{\delta^2 (R_{p_1} - R_{mv})}$$

- If p_1 is efficient, then p_2 must be "inefficient" (and vice versa)

Orthogonal Frontier Portfolios – Part 2

- Can measure slope at any point of minimum-variance frontier:

$$\frac{\partial R_p}{\partial \sigma_p} = \frac{\zeta\delta - \alpha^2}{\delta\left(R_p - R_{mv}\right)}\sigma_p$$

- Evaluate at (σ_{p_1}, R_{p_1}) to get slope of minimum-variance frontier at point corresponding to specific frontier portfolio p_1
- Hence equation for line (with unknown y-intercept of R_0) that is tangent to frontier at point corresponding to p_1:

$$R_p = R_0 + \left[\frac{\zeta\delta - \alpha^2}{\delta\left(R_{p_1} - R_{mv}\right)}\sigma_{p_1}\right]\sigma_p$$

Orthogonal Frontier Portfolios – Part 3

- Evaluate at (σ_{p_1}, R_{p_1}) and solve for y-intercept:

$$
\begin{aligned}
R_0 &= R_{p_1} - \frac{\zeta\delta - \alpha^2}{\delta\left(R_{p_1} - R_{mv}\right)}\sigma_{p_1}^2 \\
&= R_{p_1} - \frac{\zeta\delta - \alpha^2}{\delta\left(R_{p_1} - R_{mv}\right)}\left[\frac{1}{\delta} + \frac{\delta}{\zeta\delta - \alpha^2}\left(R_{p_1} - R_{mv}\right)^2\right] \\
&= R_{mv} - \frac{\zeta\delta - \alpha^2}{\delta^2\left(R_{p_1} - R_{mv}\right)} \\
&= R_{p_2}
\end{aligned}
$$

- Hence y-intercept for tangent line at p_1 also shows mean return for frontier portfolio that is orthogonal to p_1

Outline

Investment Environment with Riskless Asset

- Financial market consists of $n \geq 2$ risky assets (with normal returns) and riskless asset with risk-free rate of R_f
- Let $\mathbf{w}$ be vector of portfolio weights for risky assets, so that $1 - \mathbf{w}'\mathbf{e}$ is proportion of wealth invested in riskless asset
- If $\mathbf{w}'\mathbf{e} < 1$, then investor is lending money (to other investors, through bank) at risk-free rate
- If $\mathbf{w}'\mathbf{e} > 1$, then investor is borrowing money (from other investors, through bank) at risk-free rate
- Expected return for investor's portfolio:

$$\mathbf{w}'\mathbf{R} + (1 - \mathbf{w}'\mathbf{e})\, R_f = R_f + \mathbf{w}'\left(\mathbf{R} - R_f\mathbf{e}\right)$$

- Here $\mathbf{R} - R_f\mathbf{e}$ represents $n \times 1$ vector of risk premiums

Asset Allocation with Riskless Asset – Part 1

- Lagrangian for asset allocation problem:

$$\min_{\{\mathbf{w},\lambda\}} \mathcal{L} = \frac{1}{2}\mathbf{w}'\mathbf{V}\mathbf{w} + \lambda\left[R_p - R_f - \mathbf{w}'\left(\mathbf{R} - R_f\mathbf{e}\right)\right]$$

- Use optimality condition to find frontier portfolio weights:

$$\mathbf{V}\mathbf{w}^* - \lambda\left(\mathbf{R} - R_f\mathbf{e}\right) = 0 \implies \mathbf{w}^* = \lambda\mathbf{V}^{-1}\left(\mathbf{R} - R_f\mathbf{e}\right)$$

- Use expected return to solve for Lagrange multiplier:

$$\begin{aligned} R_p &= R_f + \lambda\left(\mathbf{R} - R_f\mathbf{e}\right)'\mathbf{V}^{-1}\left(\mathbf{R} - R_f\mathbf{e}\right) \\ &= R_f + \lambda\left(\zeta - 2\alpha R_f + \delta R_f^2\right) \implies \lambda = \frac{R_p - R_f}{\zeta - 2\alpha R_f + \delta R_f^2} \end{aligned}$$

Asset Allocation with Riskless Asset – Part 2

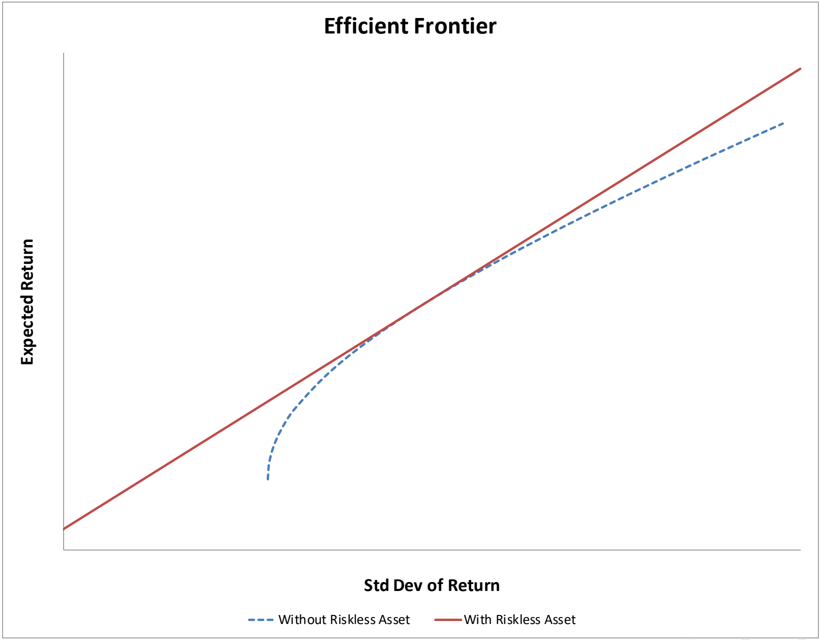
- Variance of return for portfolio on minimum-variance frontier:

$$\sigma_p^2 = (\mathbf{w}^*)' \mathbf{V}\mathbf{w}^* = \lambda^2 (\mathbf{R} - R_f \mathbf{e})' \mathbf{V}^{-1} (\mathbf{R} - R_f \mathbf{e})$$
$$= \frac{(R_p - R_f)^2}{\zeta - 2\alpha R_f + \delta R_f^2}$$

- Minimum-variance frontier in (σ_p, R_p)-space:

$$R_p = R_f \pm \left(\zeta - 2\alpha R_f + \delta R_f^2\right)^{\frac{1}{2}} \sigma_p$$

- Hence efficient frontier consists of straight line with y-intercept of R_f and positive slope of $\left(\zeta - 2\alpha R_f + \delta R_f^2\right)^{\frac{1}{2}}$


Efficient Frontier
Expected Return
Std Dev of Return
Without Riskless Asset
With Riskless Asset

Portfolio Separation with Riskless Asset – Part 1

- If $R_f < R_{mv} = \frac{\alpha}{\delta}$, then efficient frontier (with riskless asset) is tangent to top half of risky-asset-only frontier
- To verify this result, consider adding riskless asset to existing frontier generated by n risky assets
- Capital allocation line (CAL) is line in (σ_p, R_p)-space joining riskless asset to any risky portfolio
- CAL shows mean return and std dev of return for different affine combinations of risky portfolio and riskless asset
- Slope of CAL shows Sharpe ratio for all affine combinations
- "Tangency" portfolio is unique risky portfolio where CAL is tangent to existing frontier generated by n risky assets

Portfolio Separation with Riskless Asset – Part 2

- Tangency portfolio must have highest Sharpe ratio out of all possible risky portfolios
- Set y-intercept to R_f, and use previous result for equation of tangent line to get mean return for tangency portfolio:

$$R_{tg} = R_{mv} - \frac{\zeta\delta - \alpha^2}{\delta^2 (R_f - R_{mv})} = \frac{\alpha R_f - \zeta}{\delta R_f - \alpha}$$

- Hence risk premium for tangency portfolio:

$$R_{tg} - R_f = \frac{\alpha R_f - \zeta}{\delta R_f - \alpha} - R_f = -\frac{\zeta - 2\alpha R_f + \delta R_f^2}{\delta (R_f - R_{mv})}$$

Portfolio Separation with Riskless Asset – Part 3

- Variance of return for tangency portfolio:

$$\sigma_{tg}^2 = \frac{1}{\delta} + \frac{\delta\left(R_{tg} - R_{mv}\right)^2}{\zeta\delta - \alpha^2} = \frac{1}{\delta} + \frac{\zeta\delta - \alpha^2}{\delta^3\left(R_f - R_{mv}\right)^2}$$

$$= \frac{1}{\delta}\left[1 + \frac{\zeta\delta - \alpha^2}{\left(\delta R_f - \alpha\right)^2}\right] = \frac{\zeta - 2\alpha R_f + \delta R_f^2}{\delta^2\left(R_f - R_{mv}\right)^2}$$

- Choose positive square root to get std dev of return:

$$\sigma_{tg} = -\frac{\left(\zeta - 2\alpha R_f + \delta R_f^2\right)^{\frac{1}{2}}}{\delta\left(R_f - R_{mv}\right)}$$

Portfolio Separation with Riskless Asset – Part 4

- Hence slope of CAL and Sharpe ratio for tangency portfolio:

$$\frac{R_{tg} - R_f}{\sigma_{tg}} = \left[-\frac{\zeta - 2\alpha R_f + \delta R_f^2}{\delta (R_f - R_{mv})} \right] \left[-\frac{\delta (R_f - R_{mv})}{\left(\zeta - 2\alpha R_f + \delta R_f^2\right)^{\frac{1}{2}}} \right]$$
$$= \left(\zeta - 2\alpha R_f + \delta R_f^2\right)^{\frac{1}{2}}$$

- Confirms that CAL for tangency portfolio is efficient frontier (with riskless asset), since risk-averse investors prefer to hold risky portfolio with highest possible “reward-to-risk” ratio
- Opposite result for $R_f > R_{mv}$: tangency portfolio lies on bottom half of existing frontier and has lowest Sharpe ratio

Outline

CARA Utility: Economic Environment

- Financial market consists of $n \geq 2$ risky assets (with normal returns) and riskless asset with risk-free rate of R_f
- Let $\tilde{\mathbf{R}}$ be $n \times 1$ vector of asset returns, so portfolio return:

$$\tilde{R}_p = R_f + \mathbf{w}' \left(\tilde{\mathbf{R}} - R_f \mathbf{e} \right)$$

- Investor has constant absolute risk aversion, so let $b_r = bW_0$ be coefficient of relative risk aversion (at initial wealth)
- Hence investor's utility of (random) final wealth:

$$U\left(\tilde{W}\right) = -e^{-b\tilde{W}} = -e^{-b_r \frac{\tilde{W}}{W_0}} = -e^{-b_r \tilde{R}_p}$$

CARA Utility: Asset Allocation

- Asset returns have joint normal distribution, so utility of final wealth has lognormal distribution:

$$\begin{aligned} E\left[U\left(\tilde{W}\right)\right] &= E\left[-e^{-b_r \tilde{R}_p}\right] \\ &= -e^{-b_r[R_f + \mathbf{w}'(\mathbf{R} - R_f\mathbf{e})] + \frac{1}{2}b_r^2\mathbf{w}'\mathbf{V}\mathbf{w}} \end{aligned}$$

- Exponential function is monotonically increasing, so minimise (negative) exponent to maximise expected utility:

$$\max_{\mathbf{w}} \left\{\mathbf{w}'\left(\mathbf{R} - R_f\mathbf{e}\right) - \frac{1}{2}b_r\mathbf{w}'\mathbf{V}\mathbf{w}\right\}$$

CARA Utility: Optimal Portfolio

- Use optimality condition to find optimal portfolio weights:

$$\mathbf{R} - R_f\mathbf{e} - b_r\mathbf{V}\mathbf{w}^* = 0 \quad \Longrightarrow \quad \mathbf{w}^* = \frac{1}{b_r}\mathbf{V}^{-1}(\mathbf{R} - R_f\mathbf{e})$$

- Pre-multiply by $W_0\mathbf{e}'$ to find absolute (dollar) amount of wealth invested in risky assets:

$$W_0\mathbf{e}'\mathbf{w}^* = \frac{1}{b}(\alpha - \delta R_f)$$

- Notice that W_0 doesn't show up on RHS $\Longrightarrow$ all investors with same coefficient of absolute risk aversion will invest same (dollar) amount in risky assets, regardless of initial wealth