# State Prices

Wang Wei Mun

Lee Kong Chian School of Business
Singapore Management University

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# Economic Environment

- Financial market consists of $n$ risky assets with random returns
- Financial market has $k \geq 2$ "states of nature", where each state corresponds to unique set of outcomes for asset returns
- Let $\pi_s > 0$ be probability for state $s$, where $\sum_{s=1}^{k} \pi_s = 1$
- Let $X_{si}$ be payoff (or liquidation value) for one share of $i$'th asset in state $s$, and let $\mathbf{X}$ be $k \times n$ matrix that shows payoffs for one share of each asset, in each possible state of nature:

$$\mathbf{X} = \begin{bmatrix} X_{11} & \cdots & X_{1n} \\ \vdots & \ddots & \vdots \\ X_{k1} & \cdots & X_{kn} \end{bmatrix}$$

# Complete Market

- Financial market is complete if $n \geq k$ and $\mathbf{X}$ has $k$ linearly independent columns and rows $\implies$ $\mathbf{X}$ has rank $k$
- If $n > k$, then can form $k$ portfolios with linearly independent payoffs $\implies$ assume that $n = k$, so that $\mathbf{X}$ is invertible
- Let $\mathbf{Y} = [Y_1, \ldots, Y_k]'$ be any $k \times 1$ vector of desired payoffs in each possible state of nature
- Let $\mathbf{N} = [N_1, \ldots, N_k]'$ be $k \times 1$ vector of required shares in each asset, in order to create portfolio that delivers $\mathbf{Y}$:

$$\mathbf{Y} = \mathbf{X}\mathbf{N} \implies \mathbf{N} = \mathbf{X}^{-1}\mathbf{Y}$$

- Hence if market is complete, then can always create appropriate portfolio to deliver any set of desired payoffs

# State Prices

- Let $\mathbf{P} = [P_1, \ldots, P_k]'$ be $k \times 1$ vector of initial price for one share of each asset
- Assuming no arbitrage, portfolio that delivers desired payoffs of $\mathbf{Y}$ must have initial price of $P_Y = \mathbf{P}'\mathbf{N} = \mathbf{P}'\mathbf{X}^{-1}\mathbf{Y}$
- Let $\mathbf{e}_s$ be elementary security (also known as primitive security or Arrow–Debreu security) that delivers payoff of one in state $s$, and zero in all other states
- Initial price of elementary security is known as state price, which represents present value of receiving one unit of future consumption in given state of nature:

$$p_s = \mathbf{P}'\mathbf{X}^{-1}\mathbf{e}_s \qquad \forall \qquad s = 1, \ldots, k$$

# Pricing Kernel

- There exists unique set of state prices in complete market
- Investors who are non-satiated will always be willing to pay for more consumption, so state prices must be strictly positive
- Assuming no arbitrage, initial price of portfolio that delivers desired payoffs of $\mathbf{Y}$ can be expressed in terms of state prices:

$$P_Y = \sum_{s=1}^{k} p_s Y_s = \sum_{s=1}^{k} \pi_s \left(\frac{p_s}{\pi_s}\right) Y_s = \sum_{s=1}^{k} \pi_s M_s Y_s = E\left[\tilde{M}\tilde{Y}\right]$$

- Hence there exists unique pricing kernel in complete market, which must have value of $M_s = p_s/\pi_s > 0$ in state $s$

# Risk-Neutral Probabilities – Part 1

- Price of riskless asset with payoff of one in every state:

$$P_f = \sum_{s=1}^{k} p_s = \sum_{s=1}^{k} \pi_s M_s = E\left[\tilde{M}\right] = \frac{1}{R_f}$$

- Define $\widehat{\pi}_s = R_f p_s > 0$ for $s = 1, \ldots, k$, and interpret as set of (adjusted) state probabilities since $\sum_{s=1}^{k} \widehat{\pi}_s = 1$
- Then initial price of portfolio that delivers payoffs given by **Y**:

$$P_Y = \sum_{s=1}^{k} p_s Y_s = \frac{1}{R_f} \sum_{s=1}^{k} \widehat{\pi}_s Y_s = \frac{1}{R_f} \widehat{E}\left[\tilde{Y}\right]$$

# Risk-Neutral Probabilities – Part 2

- Here $\widehat{E}[\cdot]$ is expectation under probability distribution of $\widehat{\pi}$
- Then all portfolios have same expected return under probability distribution of $\widehat{\pi}$, equal to risk-free rate:

$$R_Y = \frac{1}{P_Y}\widehat{E}\left[\tilde{Y}\right] = R_f$$

- Interpret $\widehat{\pi}$ as risk-neutral probability distribution, for which pricing kernel is non-random: $\widehat{M}_s = R_f^{-1}$ for all $s$
- Hence expected payoffs under risk-neutral probability distribution must be discounted by risk-free rate
- Then $\pi$ represents physical probability distribution

# Risk-Neutral Probabilities – Part 3

- Notice that $\widehat{\pi}$ puts more (less) weight on states where pricing kernel is larger (smaller) than average, compared to $\pi$:

$$\widehat{\pi}_s = R_f p_s = R_f M_s \pi_s = \left( \frac{M_s}{E\left[\tilde{M}\right]} \right) \pi_s$$

- Hence $\widehat{\pi}$ puts more weight on "bad" states (where consumption is low and marginal utility is high), and less weight on "good" states, compared to $\pi$
- Interpret $\widehat{\pi}$ as risk-adjusted probability distribution, in order to eliminate risk premium and induce risk-neutral behavior

# Binomial Model – Part 1

- Consider "binomial" model with two states of nature
- Risky stock has initial price of $S$, which can either rise to $uS$ or drop to $dS$, where $u > d$
- Riskless bond has initial price of $P_f = R_f^{-1}$, where $u > R_f > d$
- Vector of initial prices and matrix of final payoffs:

$$\mathbf{P} = \begin{bmatrix} S \\ P_f \end{bmatrix}, \qquad \mathbf{X} = \begin{bmatrix} uS & 1 \\ dS & 1 \end{bmatrix}$$

- Vector of state prices:

$$\begin{bmatrix} p_u & p_d \end{bmatrix} = \mathbf{P}'\mathbf{X}^{-1} = \begin{bmatrix} \dfrac{1 - dP_f}{u - d} & \dfrac{uP_f - 1}{u - d} \end{bmatrix}$$

# Binomial Model – Part 2

- Vector of risk-neutral probabilities:

$$\left[ \begin{array}{cc} \hat{\pi}_u & \hat{\pi}_d \end{array} \right] = R_f \left[ \begin{array}{cc} p_u & p_d \end{array} \right] = \left[ \begin{array}{cc} \frac{R_f - d}{u - d} & \frac{u - R_f}{u - d} \end{array} \right]$$

- Pricing formula for portfolio that delivers $Y_u$ and $Y_d$:

$$P_Y = p_u Y_u + p_d Y_d = \frac{1}{R_f} \left( \hat{\pi}_u Y_u + \hat{\pi}_d Y_d \right)$$

- Binomial model is often used for option-pricing
- Not very realistic with just one time period, but becomes more realistic when extended to multiple time periods

# Example: Binomial Model – Part 1

- Stock has initial price of 6 and final payoff of 10 or 5
- Riskless bond has risk-free rate of 1.05
- Vector of initial prices and matrix of final payoffs:

$$\mathbf{P} = \begin{bmatrix} 6 \\ \frac{1}{1.05} \end{bmatrix}, \qquad \mathbf{X} = \begin{bmatrix} 10 & 1 \\ 5 & 1 \end{bmatrix}$$

- Vector of state prices:

$$\begin{bmatrix} p_u & p_d \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 6 & \frac{1}{1.05} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -5 & 10 \end{bmatrix} = \begin{bmatrix} 0.248 & 0.705 \end{bmatrix}$$

# Example: Binomial Model – Part 2

- Vector of risk-neutral probabilities:

$$\left[\begin{array}{cc} \hat{\pi}_u & \hat{\pi}_d \end{array}\right] = 1.05 \times \left[\begin{array}{cc} 0.248 & 0.705 \end{array}\right] = \left[\begin{array}{cc} 0.26 & 0.74 \end{array}\right]$$

- Alternatively, using stock returns of $u = \frac{5}{3}$ and $d = \frac{5}{6}$, and risk-free rate of $R_f = 1.05$:

$$\hat{\pi}_u = \frac{1.05 - \frac{5}{6}}{\frac{5}{3} - \frac{5}{6}} = \frac{26}{120} \times \frac{6}{5} = 0.26$$

$$\hat{\pi}_d = \frac{\frac{5}{3} - 1.05}{\frac{5}{3} - \frac{5}{6}} = \frac{37}{60} \times \frac{6}{5} = 0.74$$