Efficient Frontier Revisited

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Outline

Black–Litterman Model

std err $= \frac{20\%}{\sqrt{100}} = 2\%$

67% (6.6%, 10.6%)

95% (4.6%, 12.6%)

risk premium 8.6%

std 20%

- Biggest econometric issue with constructing efficient frontier is difficulty of estimating mean return
- Previously assumed that investors know exact mean return for all tradable assets, but not true in reality
- Sample mean has standard error of $\sigma/\sqrt{m}$, where σ is standard deviation of return and m is number of observations
- Returns are volatile and data is limited, so estimate of mean return tends to have large standard error
- Related issue is that small change in vector of mean returns can produce large change in location of efficient frontier
- Fischer Black and Robert Litterman developed model (at Goldman Sachs) to overcome issues with efficient frontier

Prior Distribution – Part 1

- Assume that excess returns for n tradable risky assets has joint normal distribution with covariance matrix of $\mathbf{\Sigma}$:

$n \times 1$ $\quad$ $n \times n$ diagonal

$$\tilde{\mathbf{R}}_{\mathbf{e}} \sim N(\tilde{\mu}, \mathbf{\Sigma})$$

a random variable. $\quad$ true

- Here $\tilde{\mu}$ is $n \times 1$ vector of (unobservable) population risk premiums, which has independent normal distribution with covariance matrix of $\mathbf{\Sigma}_{\mu}$:

$z_\mu \sim N(0,1)$

$\hat{\mu} = \pi + \sigma_\mu \cdot z_\mu$

$\sigma_\mu = \tau \cdot \sigma$

$$\tilde{\mu} \sim N(\pi, \mathbf{\Sigma}_{\mu})$$

$\hat{R} = \hat{\mu} + \sigma \cdot \tilde{z}$

$= (\pi + \sigma_\mu \cdot \hat{z}_\mu) + \sigma \cdot \tilde{z}$

$= \pi + \sigma(1+\tau) \cdot \tilde{z}$

- Then π is $n \times 1$ vector of (observable) sample risk premiums, which provides noisy estimate of population risk premiums

Prior Distribution – Part 2

- For simplicity, assume that $\boldsymbol{\Sigma}_\mu = \tau \boldsymbol{\Sigma}$, where τ is constant
- In practice, often set $\tau = 1/m$, where m is number of data points used to estimate $\boldsymbol{\Sigma}$
- Reflects standard error of sample mean, when used as estimate of population mean
- So joint normal distribution for excess returns, expressed in terms of sample risk premiums:

$\frac{1}{m}$

$$\tilde{\mathbf{R}}_\mathbf{e} \sim N\left(\boldsymbol{\pi}, (1+\tau)\,\boldsymbol{\Sigma}\right)$$

- Sampling error increases effective volatility of asset returns

Implied Risk Premiums

assumption → ① all CARA

- Assume that investor has constant absolute risk aversion, and that investor's optimal choice is ② to invest in market portfolio
- Use observed weights of risky assets in market portfolio to determine implied sample risk premiums

indirect estimate R−Rf. br V.

$$\pi = \lambda \boldsymbol{\Sigma} \mathbf{w}_m$$

*

$\lambda = b_r = b \cdot W_0$

- Here λ is coefficient of relative risk aversion (based on initial wealth) for investor with constant absolute risk aversion
- Calibrate λ using market risk premium or Sharpe ratio:

sharpe ratio

$$\lambda = \frac{\mathbf{w}_m' \pi}{\mathbf{w}_m' \boldsymbol{\Sigma} \mathbf{w}_m} = \frac{R_m - R_f}{\sigma_m^2} = \frac{S_m}{\sigma_m}$$

Investor Views

~~mean~~ ?

- Black–Litterman model also incorporates investor's "views" on (absolute or relative) expected returns of risky assets
- Suppose that investor has $k \geq 1$ views on expected returns
- Let $\mathbf{P}$ be $\overset{\text{views}}{k} \times \overset{\text{assets}}{n}$ vector of asset weights corresponding to investor's views, and let $\mathbf{Q}$ be $k \times 1$ vector of expected returns corresponding to investor's views
- Also let $\boldsymbol{\Omega}$ be $k \times k$ covariance matrix based on confidence of investor's views
- For simplicity, assume that $\boldsymbol{\Omega}$ is diagonal matrix, so if investor is equally confident in all views, then $\boldsymbol{\Omega}$ will be identity matrix

$$\Omega = \begin{bmatrix} 1 & 0 & & \\ 0 & 1 & & \\ & & \ddots & 0 \\ & & 0 & 1 \end{bmatrix}$$

Example: Investor Views

- Suppose that investible universe consists of three risky assets (or asset classes, or mutual funds)
- Investor expects first risky asset to earn return of 5% per year
- Investor expects that second risky asset to outperform third risky asset by 100 basis points (i.e., 1%) per year

~~P·R = Q~~

absolute view

- Then $\underset{2\times 3}{\mathbf{P}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ and $\mathbf{Q} = \begin{bmatrix} 0.05 \\ 0.01 \end{bmatrix}$

relative view

single asset.

- First row of **P** corresponds to absolute view (where weights sum to one), while second row of **P** corresponds to relative view (where weights sum to zero)

related to difference of ~~diffs~~ different assets 2&3 diff)

Posterior Distribution

- Bayes' theorem is used to update probability of given hypothesis H when new evidence E is observed:

evidence

$$\Pr(H|E) = \frac{\Pr(E|H)}{\Pr(E)}\Pr(H)$$

- Black–Litterman model uses Bayes' theorem to incorporate investor's views into distribution of excess returns
- Conditional on investor's views, excess returns have normal distribution of $\boxed{N(\hat{\pi}, \mathbf{M})}$ where:

excess return updated incorporate P & Q.

?

$$\hat{\pi} = \pi + \tau\mathbf{\Sigma}\mathbf{P}'\left(\tau\mathbf{P}\mathbf{\Sigma}\mathbf{P}' + \mathbf{\Omega}\right)^{-1}(\mathbf{Q} - \mathbf{P}\pi),$$

$$\mathbf{M} = \mathbf{\Sigma} + \left(\frac{1}{\tau}\mathbf{\Sigma}^{-1} + \mathbf{P}'\mathbf{\Omega}^{-1}\mathbf{P}\right)^{-1}$$

Outline

1. Black–Litterman
2. Skewness

Skewed Returns – Part 1

- Markowitz efficient frontier ignores higher moments of return distribution, such as skewness and kurtosis
- Appropriate if investors have quadratic utility, or if returns have normal distribution, but neither assumption is realistic
- Incorporating higher moments directly into investor's objective function would be very complicated and messy
- Alternative is to use measure of downside risk (such as below-target semi-variance) in place of variance
- If using below-target semi-variance, then tangency portfolio will maximise Sortino ratio instead than Sharpe ratio
- If returns have normal distribution, then maximising Sortino 相同 ratio is equivalent to maximising Sharpe ratio

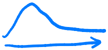
Sortino ratio $= \frac{E(R_i - R_f)}{\sqrt{SV(R_i, R_t)}}$

Skewed Returns – Part 2

- Skewness (coefficient) of return is usually defined as:

$$\text{Skew}\left(\tilde{R}\right) = E\left(\frac{\tilde{R}-\mu}{\sigma}\right)^3 = \frac{\mu_3}{\sigma^3}$$

- Here μ is expected return and σ is std dev of return, while $\mu_3 = E\left(\tilde{R}-\mu\right)^3$ is third central moment of return



左偏

- Risk-averse investor is willing to pay to receive positive 右部出现极端值故而拉长尾部 skewness (such as participating in actual lotteries)

右偏

- Risk-averse investor is also willing to pay to avoid negative skewness (such as insuring against disasters)

Skewed Returns – Part 3

- If return distribution is asymmetric, then risk-averse investor faces trade-off between maximising skewness of return or minimising variance of return, for given mean return
- Trade-off between skewness and variance can be adjusted by using lower partial moment in place of semi-variance:

investor 可能受到的最小损失

$$\mathrm{LPM}\left(\tilde{R}_i; \tilde{R}_t, \kappa\right) = E\left[\min\left\{\tilde{R}_i - \tilde{R}_t,\ 0\right\}^{\kappa}\right]$$

- Here $\kappa = 1$ corresponds to risk neutrality, while $\kappa > 1$ corresponds to (increasing) risk aversion and $\kappa < 1$ corresponds to (increasing) risk affinity

喜好

Three-Moment CAPM – Part 1

return is asymatric

undisarable.

$\beta_i \uparrow$ more risky.

high risk premium

$r_i < 0$, skew → negative

- Alan Kraus and Robert Litzenberger extended CAPM to account for skewness risk:

risk premium

variance skewness

$r_i \uparrow$, more skew

less risky

more disarable.

$$E\left(\tilde{R}_i\right) - R_f = \beta_i \pi_1 + \gamma_i \pi_2$$

skewness risk.

- Here π_1 is market variance risk premium and π_2 is market skewness risk premium, while γ_i is normalised coskewness coefficient for i'th asset:

OLS regression

covariance = $E[(X-\bar{X})(Y-\bar{Y})]$.

$$\gamma_i = \frac{\tau_{imm}}{\tau_m^3} = \frac{E\left[\left(\tilde{R}_i - \mu_i\right)\left(\tilde{R}_m - \mu_m\right)^2\right]}{E\left(\tilde{R}_m - \mu_m\right)^3}$$

coskewness

Three-Moment CAPM – Part 2

- Using empirical data from U.S. financial markets, Kraus and Litzenberger found that $\pi_1 > 0$: i.e., investors demand economic compensation for taking on variance risk
- Kraus and Litzenberger also found that $\pi_2 < 0$: i.e., investors demand economic compensation for taking on negative skewness, but are willing to give up economic compensation for taking on positive skewness
- Three-moment CAPM is not widely used in finance industry, where economists and practitioners prefer to use multi-factor model such as Fama–French more, three-factor model
- Three-moment CAPM is widely used in insurance industry, where accounting for skewness risk is particular important