State Prices

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Economic Environment

- Financial market consists of n risky assets with random returns
- Financial market has $k \geq 2$ "states of nature", where each state corresponds to unique set of outcomes for asset returns
- Let $\pi_s > 0$ be probability for state s, where $\sum_{s=1}^{k} \pi_s = 1$
- Let X_{si} be payoff (or liquidation value) for one share of i'th asset in state s, and let $\mathbf{X}$ be $k \times n$ matrix that shows payoffs for one share of each asset, in each possible state of nature:

$$\mathbf{X} = \begin{bmatrix} X_{11} & \cdots & X_{1n} \\ \vdots & \ddots & \vdots \\ X_{k1} & \cdots & X_{kn} \end{bmatrix}$$

Complete Market

$Y = \begin{bmatrix} 20 \\ 10 \end{bmatrix} \quad X = \begin{bmatrix} 10 & 8 \\ 8 & 6 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$

- Financial market is complete if $n \geq k$ and $\mathbf{X}$ has k linearly independent columns and rows $\implies$ $\mathbf{X}$ has rank k
- If $n > k$, then can form k portfolios with linearly independent payoffs $\implies$ assume that $n = k$, so that $\mathbf{X}$ is invertible
- Let $\mathbf{Y} = [Y_1, \ldots, Y_k]'$ be any $k \times 1$ vector of desired payoffs in each possible state of nature
- Let $\mathbf{N} = [N_1, \ldots, N_k]'$ be $k \times 1$ vector of required shares in each asset, in order to create portfolio that delivers $\mathbf{Y}$:

$$\underset{k\times 1}{\mathbf{Y}} = \underset{k\times k}{\mathbf{X}}\underset{k\times 1}{\mathbf{N}} \implies \mathbf{N} = \mathbf{X}^{-1}\mathbf{Y}$$

- Hence if market is complete, then can always create appropriate portfolio to deliver any set of desired payoffs

 $Y = \begin{bmatrix} 20 \\ 10 \end{bmatrix}$ $X = \begin{bmatrix} 10 & 8 \\ 8 & 6 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$ $P = \begin{bmatrix} 9 \\ 7 \end{bmatrix}$

- Let $\mathbf{P} = [P_1, \ldots, P_k]'$ be $k \times 1$ vector of initial price for one share of each asset $P_Y = \begin{bmatrix} 9 & 7 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$
- Assuming no arbitrage, portfolio that delivers desired payoffs of $\mathbf{Y}$ must have initial price of $P_Y = \mathbf{P}'\mathbf{N} = \mathbf{P}'\mathbf{X}^{-1}\mathbf{Y}$
- Let $\mathbf{e}_s$ be elementary security (also known as primitive security or Arrow–Debreu security) that delivers payoff of one in state s, and zero in all other states $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
- Initial price of elementary security is known as state price, which represents present value of receiving one unit of future consumption in given state of nature: $\begin{bmatrix} 9 & 7 \end{bmatrix} \begin{bmatrix} 10 & 8 \\ 8 & 6 \end{bmatrix}^{-1} e_s$

$$p_s = \mathbf{P}'\mathbf{X}^{-1}\mathbf{e}_s \qquad \forall \qquad s = 1, \ldots, k$$

Pricing Kernel

- There exists unique set of state prices in complete market
- Investors who are non-satiated will always be willing to pay for more consumption, so state prices must be strictly positive
- Assuming no arbitrage, initial price of portfolio that delivers desired payoffs of **Y** can be expressed in terms of state prices:

initial total price — given s state of nature desired payoff

$$P_Y = \sum_{s=1}^{k} p_s Y_s = \sum_{s=1}^{k} \pi_s \left(\frac{p_s}{\pi_s} \right) Y_s = \sum_{s=1}^{k} \pi_s [M_s Y_s] = E\left[\tilde{M} \tilde{Y} \right]$$

given s state of nature 1 unit of payoff's price

- Hence there exists unique pricing kernel in complete market, which must have value of $M_s = p_s / \pi_s > 0$ in state s

probability of state s

$\frac{1}{R_f}$ $\quad R_f = 1 + r_f$

- P~~rice of riskless~~ asset with payoff of one in every state:

to get 1 in future

initial price

$$P_f = \sum_{s=1}^{k} p_s = \sum_{s=1}^{k} \pi_s M_s = E\left[\tilde{M}\right] = \frac{1}{R_f} \quad \checkmark$$

future value of p_s

- Define $\widehat{\pi}_s = R_f p_s > 0$ for $s = 1, \ldots, k$, and interpret as set of (adjusted) state probabilities since $\sum_{s=1}^{k} \widehat{\pi}_s = 1$
- Then initial price of portfolio that delivers payoffs given by $\mathbf{Y}$:

Expected payoff

$$P_Y = \sum_{s=1}^{k} p_s Y_s = \frac{1}{R_f} \boxed{\sum_{s=1}^{k} \widehat{\pi}_s Y_s} = \frac{1}{R_f} \boxed{\widehat{E}\left[\tilde{Y}\right]}$$

Risk-Neutral Probabilities – Part 2

- Here $\widehat{E}[\cdot]$ is expectation under probability distribution of $\widehat{\pi}$
- Then all portfolios have same expected return under probability distribution of $\widehat{\pi}$, equal to risk-free rate:

$$R_Y = \frac{1}{P_Y}\widehat{E}\left[\tilde{Y}\right] = R_f$$

- Interpret $\widehat{\pi}$ as risk-neutral probability distribution, for which pricing kernel is non-random: $\widehat{M}_s = R_f^{-1}$ for all s
- Hence expected payoffs under risk-neutral probability distribution must be discounted by risk-free rate
- Then π represents physical probability distribution

Risk-Neutral Probabilities – Part 3

- Notice that $\widehat{\pi}$ puts more (less) weight on states where pricing kernel is larger (smaller) than average, compared to π:

$$\widehat{\pi}_s = R_f p_s = R_f M_s \pi_s = \left(\frac{M_s}{E\left[\tilde{M}\right]} \right) \pi_s$$

- Hence $\widehat{\pi}$ puts more weight on "bad" states (where consumption is low and marginal utility is high), and less weight on "good" states, compared to π
- Interpret $\widehat{\pi}$ as risk-adjusted probability distribution, in order to eliminate risk premium and induce risk-neutral behavior

Binomial Model – Part 1

- Consider "binomial" model with two states of nature
- Risky stock has initial price of S, which can either rise to uS or drop to dS, where $u > d$
- Riskless bond has initial price of $P_f = R_f^{-1}$, where $u > R_f > d$
- Vector of initial prices and matrix of final payoffs:

$$\mathbf{P} = \begin{bmatrix} S \\ P_f \end{bmatrix}, \qquad \mathbf{X} = \begin{bmatrix} uS & 1 \\ dS & 1 \end{bmatrix}$$

- Vector of state prices:

$$\begin{bmatrix} p_u & p_d \end{bmatrix} = \mathbf{P}'\mathbf{X}^{-1} = \begin{bmatrix} \dfrac{1 - dP_f}{u - d} & \dfrac{uP_f - 1}{u - d} \end{bmatrix}$$

- Vector of risk-neutral probabilities:

$$\begin{bmatrix} \hat{\pi}_u & \hat{\pi}_d \end{bmatrix} = R_f \begin{bmatrix} p_u & p_d \end{bmatrix} = \begin{bmatrix} \dfrac{R_f - d}{u - d} & \dfrac{u - R_f}{u - d} \end{bmatrix}$$

- Pricing formula for portfolio that delivers Y_u and Y_d:

$$P_Y = p_u Y_u + p_d Y_d = \frac{1}{R_f} \left(\hat{\pi}_u Y_u + \hat{\pi}_d Y_d \right)$$

- Binomial model is often used for option-pricing
- Not very realistic with just one time period, but becomes more realistic when extended to multiple time periods

Example: Binomial Model – Part 1

$p = \begin{bmatrix} 6 \\ \frac{1}{1.05} \end{bmatrix} \quad x = \begin{bmatrix} 10 & 1 \\ 5 & 1 \end{bmatrix}$

- Stock has initial price of 6 and final payoff of 10 or 5
- Riskless bond has risk-free rate of 1.05
- Vector of initial prices and matrix of final payoffs:

$\begin{bmatrix} S & P_f \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -dS & uS \end{bmatrix}$

$\begin{bmatrix} S - dSP_f & uSP_f - S \end{bmatrix}$

$$\mathbf{P} = \begin{bmatrix} 6 \\ \frac{1}{1.05} \end{bmatrix}, \qquad \mathbf{X} = \begin{bmatrix} 10 & 1 \\ 5 & 1 \end{bmatrix}$$

$\frac{1}{S(u-d)} \begin{bmatrix} S(1-dP_f) & S(uP_f - 1) \end{bmatrix}$

$\begin{bmatrix} 6 & \frac{1}{1.05} \end{bmatrix} \begin{bmatrix} 10 & 1 \\ 5 & 1 \end{bmatrix}^{-1}$

- Vector of state prices:

$6 \times \left(\frac{10}{6} - \frac{5}{6}\right) = 5 \qquad \begin{bmatrix} 1 & -1 \\ -5 & 10 \end{bmatrix}$

$$\begin{bmatrix} p_u & p_d \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 6 & \frac{1}{1.05} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -5 & 10 \end{bmatrix} = \begin{bmatrix} 0.248 & 0.705 \end{bmatrix}$$

Example: Binomial Model – Part 2

- Vector of risk-neutral probabilities:

$$\left[\begin{array}{cc} \hat{\pi}_u & \hat{\pi}_d \end{array}\right] = 1.05 \times \left[\begin{array}{cc} 0.248 & 0.705 \end{array}\right] = \left[\begin{array}{cc} 0.26 & 0.74 \end{array}\right]$$

- Alternatively, using stock returns of $u = \frac{5}{3}$ and $d = \frac{5}{6}$, and risk-free rate of $R_f = 1.05$:

$$\hat{\pi}_u = \frac{1.05 - \frac{5}{6}}{\frac{5}{3} - \frac{5}{6}} = \frac{26}{120} \times \frac{6}{5} = 0.26$$
$$\hat{\pi}_d = \frac{\frac{5}{3} - 1.05}{\frac{5}{3} - \frac{5}{6}} = \frac{37}{60} \times \frac{6}{5} = 0.74$$