Expected Utility Theory

Wang Wei Mun

Lee Kong Chian School of Business
Singapore Management University

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Decision Theory

- Investors will inevitably be exposed to risk and uncertainty when making investment decisions
- Define "lottery" or "gamble" as any situation where decision-maker is faced with two or more possible outcomes $\Longrightarrow$ eventual realised outcome cannot be predicted in advance
- How to measure "attractiveness" (or utility) of lottery?
- Probability distribution shows probability of each possible outcome for discrete lottery, while probability density function provides same information for continuous lottery
- In most cases, can translate probability distribution into single (real) number that represents overall utility

Expected Utility

- Most common approach is based on expected utility:
  - Use appropriate (von Neumann–Morgenstern) utility function to measure utility for individual outcomes
  - Use probability distribution of utility outcomes to calculate (probabilistic) expectation of utility outcomes
- Investor prefers lottery with higher expected utility
- Utility function should be chosen to be consistent with observed behaviour of real-life investors
- Even so, expected utility automatically implies certain forms of behaviour that appear to be inconsistent with reality (regardless of choice of utility function)

Non-Satiation

- For simplicity, suppose that investors derive utility from existing wealth: $U(W)$
- Here W is investor's (non-random) level of existing wealth (which can be measured using dollars or some form of numéraire), while $U(\cdot)$ is appropriate vN-M utility function
- Non-satiation means that investor always prefers more wealth, so utility function must be strictly increasing in wealth:

$$U(W+\theta) > U(W) \quad \forall \quad \theta > 0$$

- If utility function is differentiable, then marginal utility must be strictly positive: $U'(W) > 0$

Marginal Utility

- Marginal utility determines amount of additional utility that investor receives from infinitesimal rise in existing wealth
- Investor's marginal utility should be decreasing in wealth:

$$U'(W+\theta) \leq U'(W) \quad \forall \quad \theta \geq 0$$

- Implies that investor's utility function should be concave:

$$\begin{aligned} &U(\kappa W_1 + (1-\kappa)\, W_2) \geq \\ &\kappa U(W_1) + (1-\kappa)\, U(W_2) \quad \forall \quad \kappa \in [0,1] \end{aligned}$$

- If utility function is twice-differentiable, then $U''(W) \leq 0$

Risk Aversion

- Suppose that lottery $\tilde{\epsilon}$ gives probability p of winning $\epsilon_+ > 0$ and probability $1 - p$ of losing $\epsilon_- < 0$
- Fair lottery has expected outcome of $p\epsilon_+ + (1 - p)\,\epsilon_- = 0$
- If utility function is concave, then utility of existing wealth (without participating in fair lottery) will exceed expected utility of potential wealth (after participating in fair lottery):

$$\begin{aligned} U(W) = U(p\,(W + \epsilon_+) + (1 - p)\,(W + \epsilon_-)) &\geq \\ E[U(W + \tilde{\epsilon})] = pU(W + \epsilon_+) + (1 - p)\,U(W + \epsilon_-) \end{aligned}$$

- Hence investor with concave utility function will be risk averse, in sense of being unwilling to accept fair lottery

(Insurance) Risk Premium

- Risk-averse investor will be willing to give up some wealth to avoid fair lottery, so define (absolute) risk premium of π_a:

$$U(W - \pi_a) = E[U(W + \tilde{\epsilon})]$$

- Suppose that utility function is twice-differentiable, and $\tilde{\epsilon}$ is "small" gamble (compared to investor's existing wealth)
- Convert utility functions to approximate Taylor polynomials:

$$\begin{aligned} U(W - \pi_a) &\approx U(W) - \pi_a U'(W) \\ E[U(W + \tilde{\epsilon})] &\approx E\left[U(W) + \tilde{\epsilon} U'(W) + \frac{1}{2}\tilde{\epsilon}^2 U''(W)\right] \\ &= U(W) + \frac{1}{2}\sigma_\epsilon^2 U''(W) \end{aligned}$$

Absolute Risk Aversion

- Here $\sigma_\epsilon^2 = E\left[\tilde{\epsilon}^2\right]$ is variance of fair lottery (with zero mean)
- If investor is never satiated, then marginal utility is strictly positive, so rearrange to obtain expression for risk premium:

$$\pi_a = -\frac{1}{2}\sigma_\epsilon^2 \frac{U''(W)}{U'(W)} = \frac{1}{2}\sigma_\epsilon^2 R_a(W)$$

- Here $R_a(W) = -\frac{U''(W)}{U'(W)}$ is coefficient of absolute risk aversion, which usually depends on investor's existing wealth
- If investor is risk averse, then utility function is concave, so $U''(W) \leq 0 \implies R_a(W) \geq 0 \implies \pi_a \geq 0$

Relative Risk Aversion

- Let $\tilde{\eta}$ be proportional lottery, where investor gambles on (small) proportion of existing wealth
- Define relative risk premium of π_r:

$$U(W - \pi_r W) = E[U(W + \tilde{\eta} W)]$$

- Apply approximate Taylor expansion to utility functions and rearrange to obtain expression for relative risk premium:

$$\pi_r = -\frac{1}{2}\sigma_\eta^2 W \frac{U''(W)}{U'(W)} = \frac{1}{2}\sigma_\eta^2 R_r(W)$$

- Here $R_r(W) = -W\frac{U''(W)}{U'(W)} = WR_a(W)$ is coefficient of relative risk aversion

Quadratic Utility

- Quadratic utility function:

$$U(W) = W - \frac{1}{2}bW^2, \qquad b > 0$$

- Marginal utility is $U'(W) = 1 - bW$, so utility function is increasing for $W \leq 1/b$ and decreasing otherwise
- Absolute risk aversion is increasing:

$$R_a(W) = \frac{b}{1 - bW} \implies \frac{dR_a(W)}{dW} = \frac{b^2}{(1 - bW)^2} > 0$$

Exponential Utility

- Exponential utility function:

$$U(W) = -e^{-bW}, \qquad b > 0$$

- Utility function is strictly increasing, and marginal utility is strictly positive: $U'(W) = be^{-bW} > 0$
- Utility function is strictly concave, and marginal utility is strictly decreasing: $U''(W) = -b^2 e^{-bW} < 0$
- Absolute risk aversion is constant: $R_a(W) = b$
- Relative risk aversion is increasing: $R_r(W) = bW$

Power Utility

- Power utility function:

$$U(W) = \frac{W^{1-\gamma}}{1-\gamma}, \quad \gamma > 0$$

- Utility function is strictly increasing, and marginal utility is strictly positive: $U'(W) = W^{-\gamma} > 0$
- Utility function is strictly concave, and marginal utility is strictly decreasing: $U''(W) = -\gamma W^{-(\gamma+1)} < 0$
- Absolute risk aversion is decreasing: $R_a(W) = \frac{\gamma}{W}$
- Relative risk aversion is constant: $R_r(W) = \gamma$
- Reduces to logarithmic utility for $\gamma = 1 : U(W) = \ln W$

Normal Returns – Part 1

- Suppose that investor has initial wealth of W_0
- Let $\tilde{R}$ be (one plus) random return on investor's portfolio, so investor's (random) final wealth is given by $\tilde{W} = W_0 \tilde{R}$
- If only one investor, then set $W_0 = 1$, so investor's expected utility only depends on portfolio return: $U\left(\tilde{W}\right) = U\left(\tilde{R}\right)$
- Let μ be mean portfolio return, and let σ^2 be variance of portfolio return, and apply Taylor expansion to utility function:

$$
\begin{aligned}
U\left(\tilde{R}\right) &= U(\mu) + U'(\mu)\left(\tilde{R} - \mu\right) + \frac{1}{2}U''(\mu)\left(\tilde{R} - \mu\right)^2 + \cdots \\
E\left[U\left(\tilde{R}\right)\right] &= U(\mu) + \frac{1}{2}\sigma^2 U''(\mu) + \cdots
\end{aligned}
$$

Normal Returns – Part 2

- If utility function is quadratic, then expected utility only depends on mean and variance of portfolio return
- Otherwise, expected utility also depends on higher moments (such as skewness and kurtosis), which is much less convenient
- Unless portfolio return has probability distribution that only depends on mean and variance, such as normal or lognormal
- Normal distribution is stable under addition, but is also unbounded from below, which implies unlimited liability
- By contrast, lognormal distribution is bounded from below, but is not stable under addition
- Assume that portfolio has normal returns: $\tilde{R} \sim N(\mu, \sigma^2)$

Normal Returns – Part 3

- Let $\tilde{z} = \frac{\tilde{R}-\mu}{\sigma}$ be standard normal variable, so expected utility of final wealth:

$$E\left[U\left(\tilde{R}\right)\right] = \int_{-\infty}^{\infty} U(\mu + z\sigma)\,\phi(z)\,dz$$

- Here $\phi(\cdot)$ is standard normal probability density function
- If investor is never satiated, then marginal utility is strictly positive, so expected utility will increase with expected return:

$$\frac{\partial}{\partial\mu} E\left[U\left(\tilde{R}\right)\right] = \int_{-\infty}^{\infty} U'(\mu + z\sigma)\,\phi(z)\,dz > 0$$

Normal Returns – Part 4

- If investor is risk averse, then higher standard deviation of return should produce lower expected utility:

$$\frac{\partial}{\partial\sigma} E\left[U\left(\tilde{R}\right)\right] = \int_{-\infty}^{\infty} U'(\mu + z\sigma)\, z\phi(z)\, dz < 0$$

- Split integral into two pieces, for $+$ve and $-$ve values:

$$\int_{-\infty}^{\infty} U'(\mu + z\sigma)\, z\phi(z)\, dz =$$
$$\int_{0}^{\infty} U'(\mu + z\sigma)\, z\phi(z)\, dz + \int_{-\infty}^{0} U'(\mu + y\sigma)\, y\phi(y)\, dy$$

Normal Returns – Part 5

- Let $y = -z$, so $dy = -dz$ and lower limit of integral changes from $y = -\infty$ to $z = \infty$, and also use $\phi(-z) = \phi(z)$:

$$\begin{aligned}\int_{-\infty}^{0} U'(\mu + y\sigma)\, y\phi(y)\, dy &= \int_{\infty}^{0} U'(\mu - z\sigma)\, z\phi(-z)\, dz \\ &= -\int_{0}^{\infty} U'(\mu - z\sigma)\, z\phi(z)\, dz\end{aligned}$$

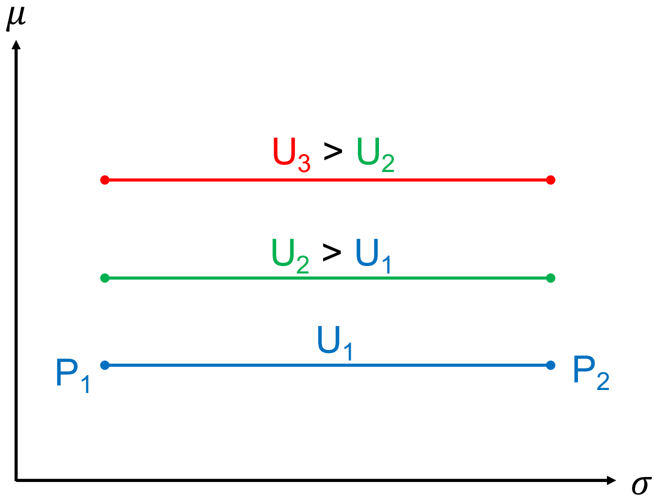
- If investor is risk averse, then marginal utility is decreasing:

$$U'(\mu + z\sigma) < U'(\mu - z\sigma) \ \text{ for } \ z > 0 \implies$$
$$\frac{\partial}{\partial\sigma} E\left[U\left(\tilde{R}\right)\right] = \int_{0}^{\infty} \left\{U'(\mu + z\sigma) - U'(\mu - z\sigma)\right\} z\phi(z)\, dz < 0$$

Indifference Curve – Part 1

- Indifference curve consists of portfolios with same expected utility, when plotted on graph with expected return on y-axis and standard deviation of return on x-axis: (σ, μ)-space
- Let P_1 and P_2 be two portfolios that lie on same indifference curve: $E\left[U\left(\tilde{R}_1\right)\right] = E\left[U\left(\tilde{R}_2\right)\right] = \overline{U}$
- If investor is risk-neutral, then P_1 and P_2 must have same expected return, but can have different std dev of return
- Hence indifference curves will be horizontal lines in (σ, μ)-space, with higher expected utility going north
- Notice that different indifference curves can never intersect

Indifference Curves for Risk-Neutral Investor



Indifference Curve – Part 2

- What happens to indifference curves if investor is risk averse?
- If $\mu_1 < \mu_2$, then $\sigma_1 < \sigma_2$ to give same expected utility for risk-averse investor, so P_2 lies northeast of P_1 in (σ, μ)-space
- Let P_3 be any convex combination of P_1 and $P_2 \implies \tilde{R}_3 = w\tilde{R}_1 + (1-w)\,\tilde{R}_2$, where $w \in [0,1]$, so expected return:

$$\mu_3 = w\mu_1 + (1-w)\,\mu_2$$

- Also let $\rho_{12} \in [-1,1]$ be correlation coefficient of return between P_1 and P_2, so variance of return for P_3:

$$\sigma_3^2 = w^2\sigma_1^2 + 2w\,(1-w)\,\rho_{12}\sigma_1\sigma_2 + (1-w)^2\,\sigma_2^2$$

Indifference Curve – Part 3

- Since $\rho_{12} \leq 1$, it follows that std dev of return for P_3 can be less than combined of std dev of returns for P_1 and P_2:

$$\sigma_3 \leq w\sigma_1 + (1-w)\,\sigma_2$$

- If $\rho_{12} = 1$, then P_3 lies on line joining P_1 and P_2
- Otherwise if $\rho_{12} < 1$, then P_3 must lie to left of line joining P_1 and P_2 in (σ, μ)-space

- But risk-averse investor must have concave utility function:

$$U\left(\tilde{R}_3\right) = U\left(w\tilde{R}_1 + (1-w)\,\tilde{R}_2\right) \geq wU\left(\tilde{R}_1\right) + (1-w)\,U\left(\tilde{R}_2\right)$$

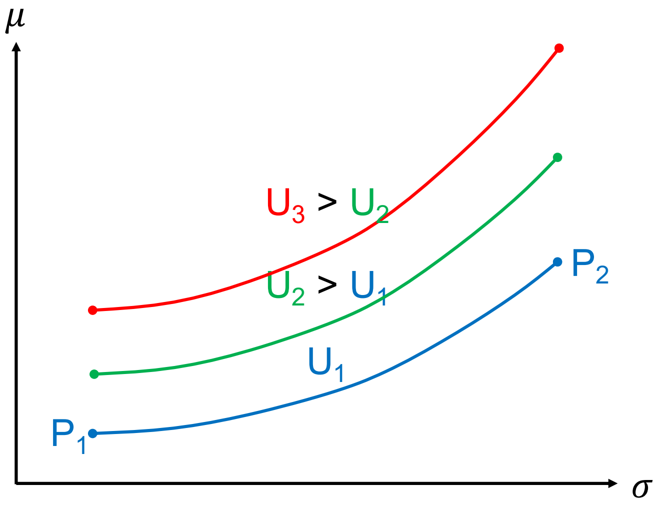
Indifference Curve – Part 4

- Hence P_3 offers higher expected utility than P_1 and P_2:

$$E\left[U\left(\tilde{R}_3\right)\right] \geq wE\left[U\left(\tilde{R}_1\right)\right] + (1-w)\,E\left[U\left(\tilde{R}_2\right)\right] = \overline{U}$$

- Indifference curve containing P_1 and P_2 cannot curve left, since it might end up containing (or overtaking) P_3
- Hence indifference curve must curve right $\implies$ risk-averse investor has convex indifference curves in (σ, μ)-space, with higher expected utility going north
- Indifference curve will become more convex, and also more "tilted", for higher levels of risk aversion (since P_2 must lie further north relative to P_1)

Indifference Curves for Risk-Averse Investor



Indifference Curve – Part 5

- Expected utility will rise when expected return rises, or when standard deviation of return falls
- To stay on indifference curve, effect of changes must cancel out, so that expected utility remains unchanged:

$$dE\left[U\left(\tilde{R}\right)\right] = \frac{\partial}{\partial\mu}E\left[U\left(\tilde{R}\right)\right]d\mu + \frac{\partial}{\partial\sigma}E\left[U\left(\tilde{R}\right)\right]d\sigma = 0$$

- Shows trade-off between risk and reward along indifference curve, and confirms that indifference curve has positive slope:

$$\frac{d\mu}{d\sigma} = -\frac{\partial}{\partial\sigma}E\left[U\left(\tilde{R}\right)\right]\Big/\frac{\partial}{\partial\mu}E\left[U\left(\tilde{R}\right)\right] > 0$$