

Stochastic Discount Factor

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Consumption Choice Model

- Investor receives utility from regular consumption of goods and services, which is financed by investor's existing wealth
- Consider static (or “one-period”) model in which investor only consumes at start and end of single time period
- Investor starts with initial wealth of W_0 and immediately consumes C_0 , which leaves remaining wealth of $(W_0 - C_0)$
- Investor can invest remaining wealth in any of n risky assets: i 'th asset has initial price of P_i and (random) final value of \tilde{X}_i
 \implies (random) return of $\tilde{R}_i = \tilde{X}_i / P_i$
- One of the risky assets may in fact be riskless bond with (non-random) risk-free rate of R_f

Portfolio Choice & Budget Constraint

- Suppose that investor invests proportion w_i of remaining wealth into i 'th asset, subject to constraint: $\sum_{i=1}^n w_i = 1$
- Investor's final wealth depends on realised portfolio return:

$$\tilde{W}_1 = (W_0 - C_0) \sum_{i=1}^n w_i \tilde{R}_i$$

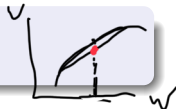
- No further opportunity for consumption after end of time period, so investor optimally chooses to consume final wealth:

$$\tilde{C}_1 = (W_0 - C_0) \sum_{i=1}^n w_i \tilde{R}_i$$

Investor's Utility

- Investor's overall utility of consumption will depend on both initial and final consumption: $V(C_0, \tilde{C}_1)$ *↗ durable good*
- For simplicity, assume that investor has time-separable utility of consumption \Rightarrow investor's utility of consumption at given point of time is not affected by past or future consumption:

$$V(C_0, \tilde{C}_1) = U(C_0) + \delta E[U(\tilde{C}_1)]$$



- Here $\delta \in (0, 1)$ *or 0.99* is **subjective discount factor** that reflects investor's rate of time preference (i.e., impatience)
- Assume that $U(\cdot)$ is strictly increasing and concave \Rightarrow investor will be non-satiated and risk averse

$$U[E\tilde{C}_1] \quad \text{X}$$

Consumption and Portfolio Choice Problem

$$\tilde{C}_1 = (w_0 - c_0) \sum w_i \tilde{R}_i$$

- At start of time period, investor chooses initial consumption of C_0 and portfolio weights of w_i (for investment portfolio) so as to maximise overall utility, subject to relevant constraints:

$$\max_{C_0, \{w_i\}} \mathcal{L} = \left\{ U(C_0) + \delta E[U(\tilde{C}_1)] + \lambda \left(1 - \sum_{i=1}^n w_i \right) \right\}$$

- First-order optimality condition for initial consumption, after applying chain rule since \tilde{C}_1 is function of C_0 :

$$\frac{\partial \mathcal{L}}{\partial C_0} = 0 \implies U'(C_0^*) = \delta E \left[U'(\tilde{C}_1^*) \sum_{i=1}^n w_i^* \tilde{R}_i \right]$$

Optimal Asset Allocation – Part 1

$$\tilde{c}_i = (w_0 - c_0) \sum w_i \tilde{R}_i \quad \frac{\partial U(\tilde{c}_1)}{\partial w_i} = U'(\tilde{c}_1) \cdot \tilde{R}_i - (w_0 - c_0)$$

- First-order optimality conditions for portfolio weights, after applying chain rule since \tilde{C}_1 is function of w_i :

$$\frac{\partial \mathcal{L}}{\partial w_i} = 0 \implies \delta E \left[U'(\tilde{C}_1^*) \tilde{R}_i \right] = \frac{\lambda}{W_0 - C_0^*} \quad \forall \quad i = 1, \dots, n$$

- All assets must have same expected marginal-utility-weighted return, based on marginal utility of optimal final consumption:

$$\underline{E \left[U'(\tilde{C}_1^*) \tilde{R}_i \right]} = E \left[U'(\tilde{C}_1^*) \tilde{R}_j \right] \quad \forall \quad i, j$$

what's the meaning of this

Optimal Asset Allocation – Part 2

- Additional dollar invested in i 'th asset produces return of \tilde{R}_i , which provides additional utility of $U'(\tilde{C}_1) \tilde{R}_i$ when consumed
- If i 'th asset provides higher expected marginal-utility-weighted return, then investor will shift investment into i 'th asset
- More investment in i 'th asset leads to higher correlation between \tilde{R}_i and $\tilde{C}_1 \implies \tilde{C}_1$ tends to higher and $U'(\tilde{C}_1)$ tends to be lower when \tilde{R}_i is above average, and vice versa
- Hence expected marginal-utility-weighted return for i 'th asset will fall since utility function is concave $\implies U''(\cdot) < 0$
- Investor will shift investment across risky assets until all assets have same expected marginal-utility-weighted return

Intertemporal Allocation

- Use equality of expected marginal-utility-weighted returns to simplify optimality condition for initial consumption:

$$U'(C_0^*) = \sum_{i=1}^n w_i^* \left(\delta E \left[U'(\tilde{C}_1^*) \tilde{R}_i \right] \right) = \delta E \left[U'(\tilde{C}_1^*) \tilde{R}_i \right]$$

- LHS represents marginal utility from one unit of initial consumption, while RHS represents discounted expected marginal utility from \tilde{R}_i units of final consumption
- Hence investor will shift between initial consumption and investment in final consumption to equalise marginal benefit
- Applies to all assets, as well as any combination of assets

Asset Pricing Formula

- Rearrange to get asset pricing formula:

$$E \left[\delta \frac{U'(\tilde{C}_1^*)}{U'(C_0^*)} \tilde{R}_i \right] = 1 \implies P_i = E \left[\delta \frac{U'(\tilde{C}_1^*)}{U'(C_0^*)} \tilde{X}_i \right]$$

- Here $\tilde{M} = \delta U'(\tilde{C}_1^*) / U'(C_0^*) > 0$ represents investor's **intertemporal marginal rate of substitution (IMRS)**
- Hence investor's IMRS acts as **pricing kernel** (or **stochastic discount factor**) that relates initial price to final value:

$$E \left[\tilde{M} \tilde{R}_i \right] = 1 \implies P_i = E \left[\tilde{M} \tilde{X}_i \right]$$

Consumption CAPM – Part 1

- Assume that riskless bond exists:

$$E[\tilde{M}R_f] = 1 \implies E[\tilde{M}] = R_f^{-1} > 0$$

- Expand expectation of product in asset pricing formula:

$$E[\tilde{M}\tilde{R}_i] = E[\tilde{M}]E[\tilde{R}_i] + \text{Cov}[\tilde{M}, \tilde{R}_i] = 1$$

- Rearrange to get pricing formula for **Consumption CAPM**:

$$E[\tilde{R}_i] - R_f = -\frac{\text{Cov}[\tilde{M}, \tilde{R}_i]}{E[\tilde{M}]} = -\frac{\text{Cov}[U'(\tilde{C}_1^*), \tilde{R}_i]}{E[U'(\tilde{C}_1^*)]}$$

Consumption CAPM – Part 2

- Suppose that \tilde{R}_i has negative correlation with $U'(\tilde{C}_1^*)$
- Implies that asset return tends to be high when marginal utility of final consumption is low, and vice versa
- Hence investor is likely to receive more consumption when consumption is less valuable, and vice versa
- Asset has undesirable payoff characteristics, so investor will demand large risk premium for holding this “risky” asset
- Conversely, if \tilde{R}_i has positive correlation with $U'(\tilde{C}_1^*)$, then investing in asset provides insurance against low consumption, so investor is willing to accept negative risk premium

Consumption CAPM \rightarrow CAPM

- Suppose that investor's optimal portfolio is affine combination of market portfolio and riskless asset
- Market return will have perfect negative correlation with investor's marginal utility of final consumption, and pricing kernel will be linear function of market return
- Market risk becomes only source of systematic risk, so Consumption CAPM will give same pricing formula as CAPM
- Requires that investor has quadratic utility of consumption, or that all risky assets have normal returns
- Hence investor will become satiated, and then investor's marginal utility will drop below zero, as consumption rises

Volatility Bound – Part 1

- Let $\mu_M = E[\tilde{M}] = R_f^{-1}$ and $\text{Cov}[\tilde{M}, \tilde{R}_i] = \rho\sigma_M\sigma_i$, where ρ is correlation coefficient between \tilde{M} and \tilde{R}_i
- Apply to pricing formula for Consumption CAPM:

$$E[\tilde{R}_i] - R_f = -\frac{\rho\sigma_M\sigma_i}{\mu_M} \implies \frac{E[\tilde{R}_i] - R_f}{\sigma_i} = -\rho\frac{\sigma_M}{\mu_M}$$

- Use $\rho \in [-1, 1]$ to get **Hansen–Jagannathan (H–J) bound**:

$$\left| \frac{E[\tilde{R}_i] - R_f}{\sigma_i} \right| \leq \frac{\sigma_M}{\mu_M}$$

Volatility Bound – Part 2

- LHS of H–J bound is Sharpe ratio of any risky asset, while RHS of H–J bound is “volatility ratio” for pricing kernel
- But H–J bound also applies to any portfolio, since pricing formula for Consumption CAPM applies to any portfolio
- Hence volatility ratio of pricing kernel cannot be less than highest Sharpe ratio out of all possible portfolios
- Annual risk premium of around 7% and annual standard deviation of around 17% for U.S. stock market returns \implies Sharpe ratio of around 0.4, so pricing kernel is highly volatile
- Pricing kernel has lower limit of zero but no upper limit \implies probability distribution should be heavily skewed on right side

Power Utility – Part 1

- Consider investor with power utility of consumption:

$$U(C) = \frac{C^{1-\gamma}}{1-\gamma} \implies$$
$$\tilde{M} = \delta \left(\frac{\tilde{C}_1^*}{C_0^*} \right)^{-\gamma} = \delta \exp \left[-\gamma \ln \left(\frac{\tilde{C}_1^*}{C_0^*} \right) \right]$$

- Suppose that optimal consumption growth has lognormal distribution with mean of μ_c and variance of σ_c^2 :

$$\ln \left(\frac{\tilde{C}_1^*}{C_0^*} \right) = \mu_c + \sigma_c \tilde{z}, \quad \tilde{z} \sim N(0, 1)$$

Power Utility – Part 2

- Apply result for variance of pricing kernel to volatility ratio:

$$\begin{aligned}\text{Var}[\tilde{M}] &= E[\tilde{M}^2] - E[\tilde{M}]^2 \\ \implies \sigma_M^2 &= \mu_{M^2} - \mu_M^2 \\ \implies \frac{\sigma_M}{\mu_M} &= \left(\frac{\mu_{M^2}}{\mu_M^2} - 1 \right)^{\frac{1}{2}}\end{aligned}$$

- Apply results for lognormal random variable:

$$\begin{aligned}\mu_M &= \delta E\left[e^{-\gamma(\mu_c + \sigma_c \tilde{z})}\right] = \delta e^{-\gamma\mu_c + \frac{1}{2}\gamma^2\sigma_c^2} = \eta \\ \mu_{M^2} &= \delta E\left[e^{-2\gamma(\mu_c + \sigma_c \tilde{z})}\right] = \delta e^{-2\gamma\mu_c + 2\gamma^2\sigma_c^2} = \eta^2 e^{\gamma^2\sigma_c^2}\end{aligned}$$

Power Utility – Part 3

- Substitute for μ_M and μ_{M^2} in equation for volatility ratio of pricing kernel, and apply $e^x \approx 1 + x$ for small values of x :

$$\frac{\sigma_M}{\mu_M} = \left(\frac{\mu_{M^2}}{\mu_M^2} - 1 \right)^{\frac{1}{2}} = \left(e^{\gamma^2 \sigma_c^2} - 1 \right)^{\frac{1}{2}} \approx \gamma \sigma_c$$

- Now substitute into result for H–J bound:

$$\frac{\sigma_M}{\mu_M} \approx \gamma \sigma_c \geq \left| \frac{E[\tilde{R}_i] - R_f}{\sigma_i} \right|$$

Equity Premium Puzzle

- Sharpe ratio of around 0.4 for U.S. stock market
- $\sigma_c \approx 2\%$ based on real annual per capita consumption for post-war U.S. economy (i.e., after World War II)
- Hence investor with power utility who consumes real per capita consumption must have $\gamma \gtrsim 20$, which is generally considered as unreasonably high degree of relative risk aversion
- **Equity premium puzzle:** investor with time-separable power utility of consumption and lognormal consumption growth must have unreasonably high degree of relative risk aversion
- Either investors don't have power utility of consumption, or consumption growth doesn't have lognormal distribution

Skewness Bound

- Investor's optimal consumption growth has lognormal distribution \implies small amount of negative (left) skewness
- So for investor with power utility of consumption, distribution for pricing kernel will have positive (right) skewness that increases with investor's (relative) risk aversion
- Empirical evidence suggests that probability distribution for pricing kernel should have large amount of positive skewness
- Hence investor must also have high degree of relative risk aversion to satisfy "skewness bound" for pricing kernel
- But what if empirical data on post-war consumption understates volatility and skewness of consumption growth?

Rare Disasters – Part 1

- Now suppose that optimal consumption growth also contains random variable that represents effect of rare disasters:

$$\ln \left(\frac{\tilde{C}_1^*}{C_0^*} \right) = \mu_c + \sigma_c \tilde{Z} + \tilde{\nu},$$
$$\tilde{\nu} = \begin{cases} \ln \phi & \text{with probability of } \pi \\ 0 & \text{with probability of } 1 - \pi \end{cases}$$

- Here $\pi \in [0, 1]$ is probability that rare disaster occurs
- Then $1 - \phi$ is fraction of optimal consumption that is lost in event of disaster, where $\phi \in (0, 1)$
- For simplicity, assume that $\tilde{\nu}$ is independent of \tilde{Z}

Rare Disasters – Part 2

- Disasters are events that result in great economic disruption, such as Great Depression, World War I, and World War II
- Other examples are outbreak of infectious and deadly viral pandemic, or asteroid striking Earth in densely populated area
- Historical data usually covers time periods without disasters, which makes consumption growth appear less volatile
- Moreover, excluding disasters severely understates amount of negative (left) skewness in consumption growth
- Robert Barro conducted survey of major disasters of 20th century, and concluded that $\pi = 1.7\%$ and $\phi = 0.65$ for real annual per capita consumption growth

Rare Disasters – Part 3

- Apply results for lognormal random variable:

$$\begin{aligned}\mu_M &= \eta E[e^{-\gamma \tilde{\nu}}] = \eta \{1 + \pi (\phi^{-\gamma} - 1)\} \\ \mu_{M^2} &= \eta^2 e^{\gamma^2 \sigma_c^2} E[e^{-2\gamma \tilde{\nu}}] = \eta^2 e^{\gamma^2 \sigma_c^2} \{1 + \pi (\phi^{-2\gamma} - 1)\}\end{aligned}$$

- Can also apply $1 + x \approx e^x$ as long as γ is reasonably small:

$$\begin{aligned}\mu_M &\approx \eta e^{\pi(\phi^{-\gamma}-1)} \\ \mu_{M^2} &\approx \eta^2 e^{\gamma^2 \sigma_c^2 + \pi(\phi^{-2\gamma}-1)} \\ \Rightarrow \frac{\sigma_M}{\mu_M} &= \left(\frac{\mu_{M^2}}{\mu_M^2} - 1 \right)^{\frac{1}{2}} \approx \left(e^{\gamma^2 \sigma_c^2 + \pi(\phi^{-\gamma}-1)^2} - 1 \right)^{\frac{1}{2}}\end{aligned}$$

Rare Disasters – Part 4

- If γ is reasonably small, then $\gamma^2 \sigma_c^2 \approx 0$, so apply $e^x \approx 1 + x$ to remaining terms:

$$\frac{\sigma_M}{\mu_M} \approx \left(e^{\pi(\phi^{-\gamma}-1)^2} - 1 \right)^{\frac{1}{2}} \approx \sqrt{\pi} (\phi^{-\gamma} - 1)$$

- No equity premium puzzle since $\gamma \gtrsim 3.3$ (based on Sharpe ratio of U.S. stock market), which represents acceptable degree of relative risk aversion:

$$\gamma = 3.3 \implies \frac{\sigma_M}{\mu_M} \approx \sqrt{0.017} (0.65^{-3.3} - 1) = 0.41$$