

QF602 - Homework 1

Question 1

- The initial forward price $F_0(T)$ is 105, $T = 1.5$. The spot price S_0 is trading at 100. Assume the dividend yield is 0, what is the implied risk free rate?

Answer.

$$\begin{aligned}F_0(T) &= S_0 e^{(r-d)T} \\ r - d &= \frac{1}{T} \ln \left(\frac{F_0(T)}{S_0} \right)\end{aligned}$$

Substitute the values and we get

$$r = \frac{1}{1.5} \ln \left(\frac{105}{100} \right) \approx 0.0325$$

- Now let's say the dividend yield is 2%, what is the implied risk free in this case?

Answer. With $d = 0.02$, we get

$$r = 0.02 + \frac{1}{1.5} \ln \left(\frac{105}{100} \right) \approx 0.0525$$

Question 2

- We have two forward prices with different maturities, 1 year and 2 years: $F_0(1) = 105$, $F_0(2) = 109$. Spot price S_0 is trading at 100. Assume the risk free rate is piece-wise constant, dividend yield is 0, what is the implied risk free rate between $T = 1$ and $T = 2$?

Answer. Since we assume piece-wise constant risk free rate, we can write

$$\begin{aligned}F_0(T_2) &= S_0 e^{(r_1-d_1)T_1 + (r_2-d_2)(T_2-T_1)} \\ F_0(T_2) &= F_0(T_1) e^{(r_2-d_2)(T_2-T_1)} \\ r_2 - d_2 &= \frac{1}{T_2 - T_1} \ln \left(\frac{F_0(T_2)}{F_0(T_1)} \right)\end{aligned}$$

where $T_1 = 1, T_2 = 2$. Substitute the values and we get

$$r_2 = \ln \left(\frac{109}{105} \right) \approx 0.0374$$

Question 3

- Spot S_0 is trading at 100. One year forward price $F_0(1)$ is 105. Risk free rate is 2%. Dividend yield is 1%. Is there an arbitrage opportunity? If yes, describe how would you monetize such opportunity and how much you would earn?
- What is the no-arbitrage forward price?

Answer. The no-arbitrage 1y forward price can be computed as

$$S_0 e^{(r-d)T} = 100e^{(0.02-0.01)1} \approx 101.01$$

To monetize, we **sell** one forward contract at 105. Then we borrow 100 at risk-free rate 2% to buy one stock at 100. Hold the stock for 1y. The stock will pay 1% dividend yield which earns us **1.01**. At maturity, we fulfill the forward contract by selling the stock at **105**. We pay back the 100 we borrowed with interest which costs us **102.02**. In this process we earn

$$105 - 102.02 + 1.01 = 3.99$$

Question 4

- You enter a long position of a forward contract with 1 year maturity on 100 shares of a stock. The spot S_0 is trading at 100. The risk free rate is 2% and dividend yield is 0%.
- One month later, the company releases an annual report and the spot is now trading at 120. Assume the risk free rate and dividend yield doesn't change, what is the PnL of your forward position?

Answer. The forward price at time 0 is

$$F_0(1) = 100e^{(0.02-0)1} \approx 102.02$$

One month later, the spot is now at 120, the 11 months forward price is now

$$F_{1/12}(1) = 120e^{(0.02-0) \times \frac{11}{12}} \approx 122.22$$

The PnL of a forward contract on ONE share is

$$\begin{aligned} (F_{1/12}(1) - F_0(1)) \times Z_{1/12}(1) &= (F_{1/12}(1) - F_0(1)) \times e^{-0.02 \times \frac{11}{12}} \\ &= (122.22 - 102.02) \times 0.9818 \\ &\approx 19.83 \end{aligned}$$

Therefore, the PnL of a forward contract on 100 shares is $19.83 \times 100 = 1983$

- What if the company decides to increase the dividend yield from 0% to 5%, what would be the PnL of your forward position?

Answer. The forward price at time 0 is still at

$$F_0(1) = 100e^{(0.02-0)1} \approx 102.02$$

One month later, the spot is now at 120, the 11 months forward price is now

$$F_{1/12}(1) = 120e^{(0.02-0.05) \times \frac{11}{12}} \approx 116.75$$

The PnL of a forward contract on ONE share is

$$\begin{aligned} (F_{1/12}(1) - F_0(1)) \times Z_{1/12}(1) &= (F_{1/12}(1) - F_0(1)) \times e^{-0.02 \times \frac{11}{12}} \\ &= (116.75 - 102.02) \times 0.9818 \\ &\approx 14.46 \end{aligned}$$

Therefore, the PnL of a forward contract on 100 shares is $14.46 \times 100 = 1446$

Question 5

- Explain why American call option on the underlying which pays dividend may be optimal to exercise early? Please give an example to illustrate that.

Answer. It may be optimal to early exercise an American call option if the underlying pays dividend and the dividend yield is high enough to compensate for the loss of interest on the strike price due to the early exercise. For example, assume risk free rate is 0 but the dividend is 10%. Early exercise would cause no interest loss and one would own the stock early to enjoy the dividend yield.

- How about the underlying that pays no dividend?

Answer. If there is no dividend, one should not early exercise an American call option. This is because one would pay the strike price to own a stock early but the stock doesn't generate any extra income. One could have waited until the maturity to exercise if $S_T > K$ or simply don't if $S_T \leq K$.

- How about American put option?

Answer. Without dividend, it may be optimal to exercise a deep ITM American put option. This is because the lower bound of the stock price is 0, therefore, the maximum upside is capped and it makes sense to early exercise in order to sell the stock at K and to earn risk free interest.

Question 6

- Explain why forward price and futures price might be different for the same underlying asset.

Answer. This is because forwards are traded OTC and futures are traded on exchange. Exchanges require each counterparty to maintain a margin account in order to mitigate the credit risk. Money in margin accounts earn interest and if the correlation between the interest rate and the underlying asset is non-zero, the forward and futures prices will be different.

- Describe a situation that a futures price could be **lower** than the forward price.

Answer. If the margin account interest rate is negatively correlated with the underlying, futures price will be lower than the forward price.