QF602 Homework

XueYuanhuang

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Assignment 4: Due 2/23/24

Problem 1.

$$S = \frac{X}{Y}$$

$$dS_t = \frac{1}{Y_t}dX_t - \frac{X_t}{Y_t^2}dY_t - \frac{1}{Y_t^2}dX_t dY_t + \frac{X_t}{Y_t^3}(dY_t)^2$$

$$dS_t = \frac{X_t}{Y_t}\sigma^X dW_t^X - \frac{X_t}{Y_t}\sigma^Y dW_t^Y - \frac{X_t}{Y_t}\sigma^X \sigma^Y \rho dt + \frac{X_t}{Y_t}(\sigma^Y)^2 dt$$

$$\frac{dS_t}{S_t} = \sigma^X dW_t^X - \sigma^Y dW_t^Y - \sigma^X \sigma^Y \rho dt + (\sigma^Y)^2 dt$$

$$\frac{dS_t}{S_t} = ((\sigma^Y)^2 - \sigma^X \sigma^Y \rho)dt + (\sigma^X dW_t^X - \sigma^Y dW_t^Y)$$

$$\frac{dS_t}{S_t} = -q^S dt + \sigma^S dW_t^S$$

$$q^S = -((\sigma^Y)^2 - \sigma^X \sigma^Y \rho), \quad \sigma^S = \sqrt{(\sigma^X)^2 + (\sigma^Y)^2 - 2\sigma^X \sigma^Y \rho}$$

As,

$$V_T = (X_T - Y_T)^+$$

$$\frac{dX_t}{X_t} = (r - q^X)dt + \sigma^X dW_t^X$$

$$\frac{dY_t}{Y_t} = (r - q^Y)dt + \sigma^Y dW_t^Y$$

where $E[dW_t^X dW_t^Y] = \rho dt$

$$V_0 = X_0 e^{-q^X T}\Phi(d_1) - Y_0 e^{-q^Y T}\Phi(d_2)$$

where

$$d_{1,2} = \frac{ln(X_0/Y_0) + (q^Y - q^X \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$\sigma = \sqrt{(\sigma^X)^2 + (\sigma^Y)^2 - 2\sigma^X\sigma^Y\rho}$$

Implement this formular into the result,

$$V_0 = S_0 e^{-q^S T}\Phi(d_1) - Z_0\Phi(d_2)$$

where

$$d_{1,2} = \frac{ln(S_0/Z_0) + (-q^S \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$\sigma = \sqrt{(\sigma^S)^2 + (\sigma^Z)^2}$$

Problem 2.

$$max(X_T, Y_T) = Y_T + max(X_T - Y_T, 0)$$

First term at t = 0 is

$$e^{-rT}E^*[Y_T] = e^{-rT}Y_0 e^{(r-q^Y)T} = Y_0 e^{-q^Y T}$$

Second term at t = 0 is

$$X_0 e^{-q^X T}\Phi(d_1) - Y_0 e^{-q^Y T}\Phi(d_2)$$

$$d_{1,2} = \frac{ln(X_0/Y_0) + (q^Y - q^X \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$\sigma = \sqrt{(\sigma^X)^2 + (\sigma^Y)^2 - 2\sigma^X\sigma^Y\rho}$$

$V_0 =$

$$X_0 e^{-q^X T}\Phi(d_1) - Y_0 e^{-q^Y T}(\Phi(d_2) - 1)$$

Problem 3.

$$min(X_T, Y_T) = min(X_T - Y_T, 0) + Y_T = Y_T - max(Y_T - X_T, 0)$$

First term at t = 0 is

$$Y_0 e^{-q^Y T}$$

Second term at t = 0 is

$$-Y_0 e^{-q^Y T}\Phi(d_1) + X_0 e^{-q^X T}\Phi(d_2)$$

$$d_{1,2} = \frac{ln(Y_0/X_0) + (q^X - q^Y \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$
$$\sigma = \sqrt{(\sigma^X)^2 + (\sigma^Y)^2 - 2\sigma^X\sigma^Y\rho}$$

$$V_0 = Y_0 e^{-q^Y T}(1 - \Phi(d_1)) + X_0 e^{-q^X T}\Phi(d_2)$$

Problem 4.

$$max(S_T/S_0 - Y_T/Y_0, 0) = max(\bar{S_T} - \bar{Y_T}, 0), \quad \bar{S_T} = S_T/S_0 \quad \bar{Y_T} = Y_T/Y_0$$

Under the SGD risk neutral measure,

$$\frac{dY_t}{Y_t} = (r^f - q^Y - \sigma^Y\sigma^X\rho_{YX})dt + \sigma^Y d\bar{W}_t^Y$$

As S_0, Y_0 is constant,

$$\frac{d\bar{S}_t}{\bar{S}_t} = (r^d - q^S)dt + \sigma^S dW_t^S$$
$$\frac{d\bar{Y}_t}{\bar{Y}_t} = (r^f - q^Y - \sigma^Y\sigma^X\rho_{YX})dt + \sigma^Y d\bar{W}_t^Y$$
$$\frac{d\bar{Y}_t}{\bar{Y}_t} = (r^d - \bar{q}^Y)dt + \sigma^Y d\bar{W}_t^Y, \quad \bar{q}^Y = r^d - r^f + q^Y + \sigma^Y\sigma^X\rho_{YX}$$

$$V_0 = \bar{S}_0 e^{-q^S T}\Phi(d_1) - \bar{Y}_0 e^{-\bar{q}^Y T}\Phi(d_2)$$
$$d_{1,2} = \frac{ln(\bar{S}_0/\bar{Y}_0) + (\bar{q}^Y - q^S + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$
$$\sigma = \sqrt{(\sigma^S)^2 + (\sigma^Y)^2 - 2\sigma^S\sigma^Y\rho_{SY}}$$