QF602 Derivatives

Lecture 6 - Option Strategies and Static Replication

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Option Strategies

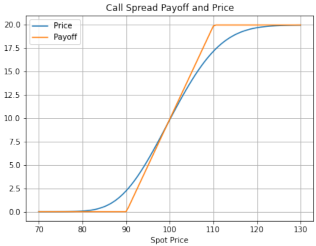
- We are going to cover some of the most common option strategies using European call and put options and their risk profiles.

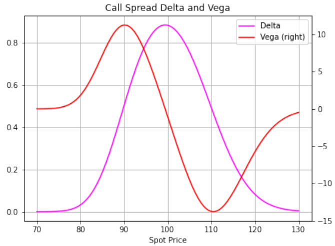
	long call	long put	short call	short put
call spread	y		y	
put spread		y		y
straddle	y	y		
strangle	y	y		
bullish risk reversal	y			y
bearish risk reversal		y	y	
call ratio	y		y	
put ratio		y		y

- There are many reasons to trade options.
- (1) Directional view of spot.
- (2) Directional view of volatility.
- (3) Carry.
- (4) Hedging.

Call Spread

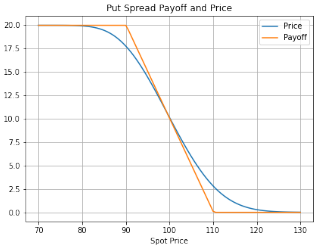
- $T = 0.1, r = 0.02, q = 0.03, \sigma = 0.2, K1 = 90, K2 = 110$. The payoff for call spread at T is
$$(S_T - K1)^+ - (S_T - K2)^+$$
- Bullish on spot. Long vega closer to the lower strike. Short vega closer to the upper strike.
- One cannot lose more than the premium paid but the upside gain is capped.

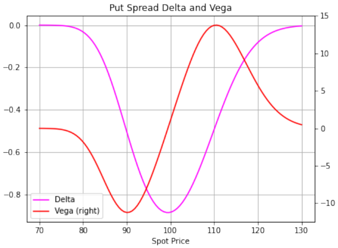




Put Spread

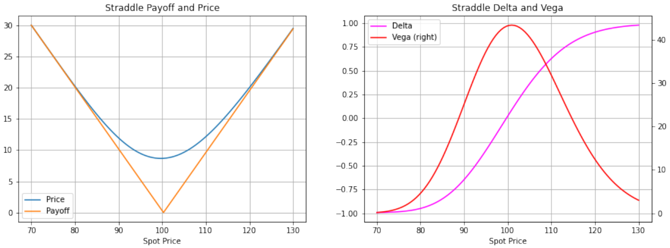
- $T = 0.1, r = 0.02, q = 0.03, \sigma = 0.2, K1 = 90, K2 = 110$. The payoff for put spread at T is
$$(K2 - S_T)^+ - (K1 - S_T)^+$$
- Bearish on spot. Long vega closer to the upper strike. Short vega closer to the lower strike.
- One cannot lose more than the premium paid but the upside gain is capped.





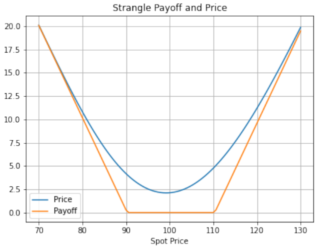
Straddle

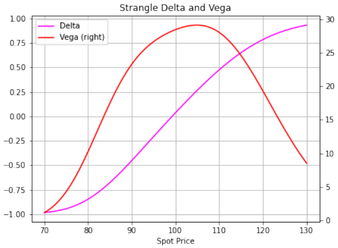
- $T = 0.3, r = 0.02, q = 0.03, \sigma = 0.2, K = 100$. The payoff for straddle at T is
$$(S_T - K)^+ + (K - S_T)^+$$
- A combination of two ATM options. Generally neutral on spot. Bullish on volatility. When spot is moved away from K, vega drops and delta increases/decreases.
- One cannot lose more than the premium paid and the upside is not capped.



Strangle

- $T = 0.3, r = 0.02, q = 0.03, \sigma = 0.2, K1 = 90, K2 = 110$. The payoff for strangle at T is
$$(S_T - K2)^+ + (K1 - S_T)^+$$
- A cheaper version of straddle which is a combination of two OTM options. Generally neutral on spot. Bullish on volatility.
- One cannot lose more than the premium paid and the upside is not capped.



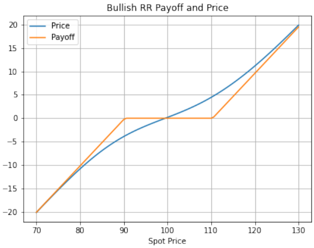


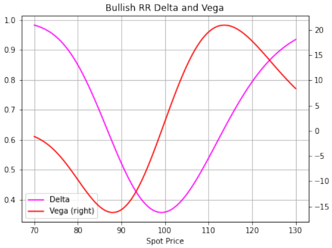
Bullish Risk Reversal

- $T = 0.3, r = 0.02, q = 0.03, \sigma = 0.2, K1 = 90, K2 = 110$. The payoff for bullish risk reversal at T is

$$(S_T - K2)^+ - (K1 - S_T)^+$$

- Bullish on spot. Long vega when the spot is closer to the upper strike. Short vega when the spot is closer to the lower strike.
- One can lose more than the premium paid and the upside is not capped.
- It is a generalized version of long forward contract.



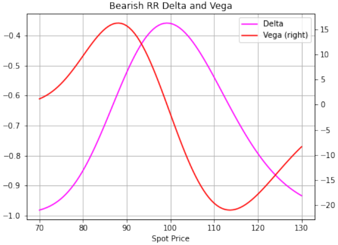


Bearish Risk Reversal

- $T = 0.3, r = 0.02, q = 0.03, \sigma = 0.2, K1 = 90, K2 = 110$. The payoff for bearish risk reversal at T is

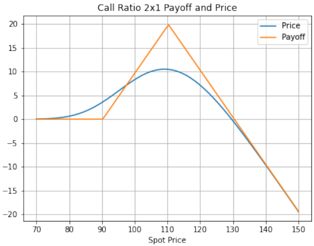
$$(K1 - S_T)^+ - (S_T - K2)^+$$

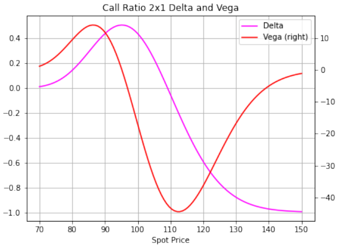
- Bearish on spot. Long vega when the spot is closer to the lower strike. Short vega when the spot is closer to the upper strike.
- One can lose more than the premium paid and the upside is not capped.
- It is generalized version of short forward contract.



Call Ratio

- $T = 0.3, r = 0.02, q = 0.03, \sigma = 0.2, K1 = 90, K2 = 110$. The payoff for call ratio at T is
$$(S_T - K1)^+ - 2(S_T - K2)^+$$
- Bullish on spot but up to the upper strike. Long vega when the spot is closer to the lower strike. Short vega when the spot is closer to the upper strike.
- Cheaper than call spread but one can lose more than the premium paid.



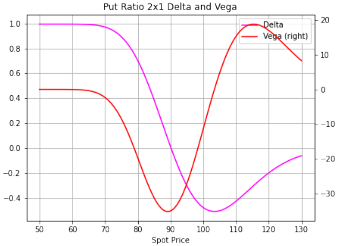


Put Ratio

- $T = 0.3, r = 0.02, q = 0.03, \sigma = 0.2, K1 = 90, K2 = 110$. The payoff for call spread at T is

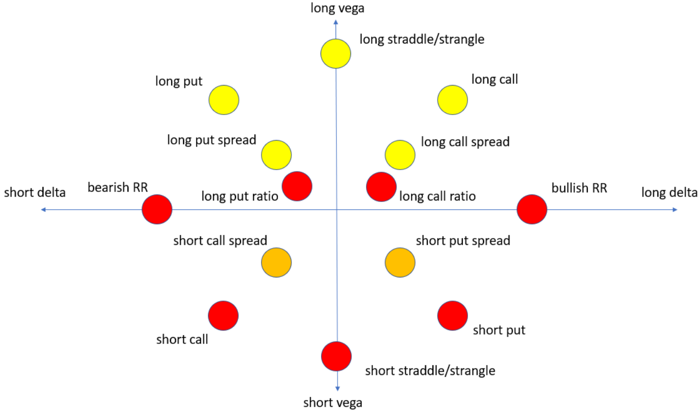
$$(K2 - S_T)^+ - 2(K1 - S_T)^+$$

- Bearish on spot but up to the lower strike. Long vega when the spot is closer to the upper strike. Short vega when the spot is closer to the lower strike.
- Cheaper than put spread but one can lose more than the premium paid.



Riskiness of option strategies

- This is a general assessment of the riskiness at inception of the trade. One would usually enter to the trade when the option is OTM.
- Yellow. Low risk. Fixed premium, short theta/carry and long volatility.
- Orange. Medium risk. Long theta/carry, short volatility, capped downside exposure.
- Red. High risk. Long theta/carry, short volatility, uncapped downside exposure.



Pricing general European-style payoff

- We here present a basic but powerful, technique which was introduced by Breeden and Litzenberger (1978) for pricing European-style options with implied volatility smile. It is based on the observation that the risk-neutral density for the underlying asset can be derived directly from market quotes of European call and put options.
- The result states that the value of a payoff depends on S_T which pays $V_T \equiv V_T(S_T)$ at maturity T can be computed as

$$\begin{aligned} V_0 &= e^{-rT} E_0^{\beta}[V_T] \\ &= e^{-rT} V_T(F_0(T)) + \int_0^{F_0(T)} Put(K,T) \frac{\partial^2 V_T(K)}{\partial K^2} dK \\ &\quad + \int_{F_0(T)}^{\infty} Call(K,T) \frac{\partial^2 V_T(K)}{\partial K^2} dK \end{aligned}$$

- where $F_0(T)$ is the forward price at maturity T. $Put(K,T)$ and $Call(K,T)$ are values of put and call option prices with strike K and maturity T.
- Assuming V_T is twice differential-able with respect to S_T.
- Note that this framework is related to the marginal density of S_T, we cannot price any payoff function which depends on multiple maturity times. As a consequence, path-dependent options cannot be priced in this framework.

Proof

- We first show that the density of S_T, $\phi(S_T)$, can be obtained from call and put option prices. The call option price can be computed as:

$$Call(K,T) = e^{-rT}\int_K^{\infty}(S_T - K)\phi(S_T)dS_T$$

Differentiate the call price with respect to strike K

$$\frac{\partial Call(K,T)}{\partial K} = e^{-rT}\frac{\partial}{\partial K}\int_K^{\infty}(S_T - K)\phi(S_T)dS_T$$

By the Leibniz integral rule

$$= -e^{-rT}\int_K^{\infty}\phi(S_T)dS_T$$

The second derivative with respect to K is

$$\frac{\partial^2 Call(K,T)}{\partial K^2} = -e^{-rT}\frac{\partial}{\partial K}\int_K^{\infty}\phi(S_T)dS_T = e^{-rT}\phi(K)$$

The implied density can thus be obtained from the call option as:

$$\phi(K) = e^{rT}\frac{\partial^2 Call(K,T)}{\partial K^2} \tag{1}$$

- By the put call parity, the density can also be obtained from the put option prices as:

$$\begin{aligned}\phi(K) &= e^{rT}\frac{\partial^2 Call(K,T)}{\partial K^2} \\ &= e^{rT}\frac{\partial^2}{\partial K^2}\Big(Put(K,T)+Z_0(T)(F_0(T)-K)\Big) \\ &= e^{rT}\frac{\partial^2 Put(K,T)}{\partial K^2} \qquad (2)\end{aligned}$$

- With two possibilities for implying the density for S_T, via calls or puts, we should use the most liquid traded ones. It is usually that OTM options are more liquid than ITM options. To determine the market implied density for S_T, one should preferably choose the OTM calls and puts.

- Let $V_T \equiv V_T(S_T)$ be a general payoff that depends only on S_T. Today's value can be expressed as a discounted risk neutral expectation:

$$V_0 = e^{-rT} \int_0^{\infty} V_T \phi(S_T) dS_T$$

We can split the integral into two parts:

$$= e^{-rT} \left(\int_0^{F_0(T)} V_T \phi(S_T) dS_T + e^{-rT} \int_{F_0(T)}^{\infty} V_T \phi(S_T) dS_T \right)$$

Substitute Equations (1) and (2) to the integrals. Note that e^{-rT} cancels out.

$$= \underbrace{\int_0^{F_0(T)} V_T \frac{\partial^2 Put(S_T, T)}{\partial S_T^2} dS_T}_{A(S_T)} + \underbrace{\int_{F_0(T)}^{\infty} V_T \frac{\partial^2 Call(S_T, T)}{\partial S_T^2} dS_T}_{B(S_T)}$$

- Let's change the variable S_T and rename to y for ease of notation:

$$= \int_0^{F_0(T)} \underbrace{V_T \frac{\partial^2 Put(y, T)}{\partial y^2} dy}_{A(y)} + \int_{F_0(T)}^{\infty} \underbrace{V_T \frac{\partial^2 Call(y, T)}{\partial y^2} dy}_{B(y)}$$

- Let's look at $A(y)$ first. By integration by parts, we get

$$\int_0^{F_0(T)} A(y)dy = \left[V_T(y)\frac{\partial Put(y,T)}{\partial y}\right]_{y=0}^{y=F_0(T)} - \int_0^{F_0(T)} \frac{\partial V_T(y)}{\partial y}\frac{\partial Put(y,T)}{\partial y}dy$$

Apply again integration by parts to the last integral, we get

$$\int_0^{F_0(T)} A(y)dy = \underbrace{\left[V_T(y)\frac{\partial Put(y,T)}{\partial y}\right]_{y=0}^{y=F_0(T)}}_{A_1} - \underbrace{\left[\frac{\partial V_T(y)}{\partial y}Put(y,T)\right]_{y=0}^{y=F_0(T)}}_{A_2} + \int_0^{F_0(T)} \frac{\partial^2 V_T(y)}{\partial y^2}Put(y,T)dy$$

- Similarly, by integration by parts twice to $B(y)$, we get

$$\int_0^{F_0(T)} B(y)dy = \underbrace{\left[V_T(y)\frac{\partial Call(y,T)}{\partial y}\right]_{y=F_0(T)}^{y=\infty}}_{B_1} - \underbrace{\left[\frac{\partial V_T(y)}{\partial y}Call(y,T)\right]_{y=F_0(T)}^{y=\infty}}_{B_2} + \int_{F_0(T)}^{\infty} \frac{\partial^2 V_T(y)}{\partial y^2}Call(y,T)dy$$

- The value of the general payoff can be expressed as

$$V_0 = A_1 - A_2 + B_1 - B_2 + \int_0^{F_0(T)} \frac{\partial^2 V_T(y)}{\partial y^2} Put(y,T)dy + \int_{F_0(T)}^{\infty} \frac{\partial^2 V_T(y)}{\partial y^2} Call(y,T)dy$$

- Let's look at $-A_2 - B_2$. As a put option has value zero for strike $y = 0$, and a call option has value zero for strike $y \to \infty$, we get

$$-A_2 - B_2 = -\left.\frac{\partial V_T(y)}{\partial y}\right|_{y=F_0(T)} Put(F_0(T),T) + \left.\frac{\partial V_T(y)}{\partial y}\right|_{y=F_0(T)} Call(F_0(T),T)$$

By the put call parity, for strike equals the forward $F_0(T)$, call and put has the same price, therefore

$$-A_2 - B_2 = 0$$

- Next, we look at $A_1 + B_1$.

$$A_1 + B_1 = \left[V_T(y) \frac{\partial Put(y,T)}{\partial y} \right]_{y=0}^{y=F_0(T)} + \left[V_T(y) \frac{\partial Call(y,T)}{\partial y} \right]_{y=F_0(T)}^{y=\infty}$$

- Differentiation of the put-call parity with respect to strike y, gives us

$$\frac{\partial Call(y,T)}{\partial y} + e^{-rT} = \frac{\partial Put(y,T)}{\partial y}$$

Substitute and we get

$$\begin{aligned} A_1 + B_1 &= \left[V_T(y) \left(\frac{\partial Call(y,T)}{\partial y} + e^{-rT} \right) \right]_{y=0}^{y=F_0(T)} + \left[V_T(y) \frac{\partial Call(y,T)}{\partial y} \right]_{y=F_0(T)}^{y=\infty} \\ &= \left[V_T(y) \frac{\partial Call(y,T)}{\partial y} \right]_{y=0}^{y=\infty} + e^{-rT} \Big(V_T(F_0(T)) - V_T(0) \Big) \end{aligned}$$

- Recall

$$\frac{\partial Call(y,T)}{\partial y} = -e^{-rT} \int_y^\infty \phi(S_T) dS_T = -e^{-rT} \left(1 - \Phi(y)\right)$$

- Substitute and we get

$$\left[V_T(y)\frac{\partial Call(y,T)}{\partial y}\right]_{y=0}^{y=\infty} = -e^{-rT}\Big(V_T(\infty)\big(1-\Phi(\infty)\big) - V_T(0)\big(1-\Phi(0)\big)\Big)$$

Since $S_T > 0$, thus $\Phi(0) = 0$. We also assume for $y \to \infty$, $\Phi(y) \to 1$ faster than $V_T(y) \to \infty$, we get

$$\left[V_T(y)\frac{\partial Call(y,T)}{\partial y}\right]_{y=0}^{y=\infty} = e^{-rT}V_T(0)$$

Therefore,

$$A_1 + B_1 = e^{-rT}V_T(0) + e^{-rT}\Big(V_T(F_0(T)) - V_T(0)\Big) = e^{-rT}V_T(F_0(T))$$

Put it all together, we get

$$\begin{aligned} V_0 = e^{-rT}V_T(F_0(T)) + \int_0^{F_0(T)} \frac{\partial^2 V_T(y)}{\partial y^2} Put(y,T)dy \\ + \int_{F_0(T)}^{\infty} \frac{\partial^2 V_T(y)}{\partial y^2} Call(y,T)dy \end{aligned}$$

The proof is done.

Example - Square Payoff

- Consider $V_T = S_T^2$. The value of the payoff can be computed using the B-L formula:

$$V_0 = e^{-rT} F_0(T)^2 + 2 \int_0^{F_0(T)} Put(y, T) dy + 2 \int_{F_0(T)}^{\infty} Call(y, T) dy$$

The integrals can be evaluated numerically. The upper limit of the call integral can be approximated as

$$F_0(T) e^{\kappa \sigma \sqrt{T}}$$

where κ is an integer which represents the number of standard deviation away from the mean. The higher the number, the more accurate the approximation. σ is the ATM implied vol.

- If we assume S_t follows lognormal dynamics:

$$dS_t = (r - q) S_t dt + \sigma S_t dW_t$$

We can compute V_0 analytically as

$$\begin{aligned}
V_0 &= e^{-rT} E_0[V_T] \\
&= e^{-rT} E_0[S_T^2] \\
&= e^{-rT} E_0\left[\left(S_0 e^{(r-q-0.5\sigma^2)T + \sigma W_T}\right)^2\right] \\
&= e^{-rT} S_0^2 e^{2(r-q)T - \sigma^2 T} E_0\left[e^{2\sigma W_T}\right] \\
&= S_0^2 e^{(r-2q+\sigma^2)T}
\end{aligned}$$

- Let $S_0 = 10$, $r = 0.02$, $q = 0.05$, $\sigma = 0.4$, $T = 1.5$. The table below shows the accuracy of the numerical integration against the analytical solution with various κ.

κ	V_0
1	108.15601273732547
2	112.07566800063283
3	112.70281029498769
4	112.74817164300457
5	112.74957520050991
6	112.74959316774772
Analytic	112.74968515793758

Python Code

The code below shows an implementation of B-L formula for the square payoff using the numerical integration approach with various κ.

```python
import numpy as np
import scipy.integrate as integrate

# BlackPut(f, k, T, vol) is Black put option with f-forward, k-strike, T-maturity, vol-implied vol
def numerical_integration_sq(S0, r, q, T, vol, SD):
    DF = np.exp(-r*T); DivF = np.exp(-q*T)
    f = S0*DivF/DF
    maxS = f * np.exp(vol * SD * np.sqrt(T))
    forward_part = f * f
    integrand_put = lambda k: 2.0 * BlackPut(f, k, T, vol)
    put_part, error = integrate.quad(integrand_put, 0, f)
    integrand_call = lambda k: 2.0 * BlackCall(f, k, T, vol)
    call_part, error = integrate.quad(integrand_call, f, maxS)
    return DF * (forward_part + put_part + call_part)

# driver routine
q = 0.05; r = 0.02; T = 1.5; S0 = 10; vol = 0.4
analytical_res = S0*S0*np.exp( r*T -2*q*T + vol*vol*T)
kappas = np.linspace(1, 6, 6)
numIntResults = [numerical_integration_sq(S0, r, q, T, vol, sd) for sd in kappas]
```

Variance Swap

- A variance swap is a forward contract that pays the difference between the realized variance and a predefined strike price K at maturity T:

$$V_T = RV(0, T) - K$$

where $RV(0, T)$ is the annualized realized variance of S_t between 0 and T, $0 = t_0 < t_1 < ... < t_m = T$

$$RV(0, T) := \frac{252}{m} \sum_{i=1}^{m} \left(\ln \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2 \tag{3}$$

252 represents the number of business days in a given year and the term 252/m annualizes the realized variance.

- With deterministic interest rate r, the contract value at 0 is given by

$$V_0 = e^{-rT} E_0^{\beta} \left[RV(0, T) - K \right]$$

In order for V_0 to be zero, the strike K is

$$K = E_0^{\beta} \left[RV(0, T) \right]$$

which is the risk-neutral expectation at time 0 of the realized variance of S_t between 0 to T.

- The limit of the log-term in (3), as the time grid gets finer, i.e. $\Delta t = t_i - t_{i-1} \to 0$,

$$\ln \frac{S_{t_i}}{S_{t_{i-1}}} = \ln S_{t_i} - \ln S_{t_{i-1}} \xrightarrow{\Delta t \to 0} d \ln S_t$$

  If we assume S_t is governed by the following diffusion dynamics (note that σ_t can be stochastic)

$$\frac{dS_t}{S_t} = (r - q)\, dt + \sigma_t dW_t \tag{4}$$

  By Ito's lemma,

$$d \ln S_t = \left(r - q - \frac{1}{2} \sigma_t^2 \right) dt + \sigma_t dW_t \tag{5}$$

  Recall $dt^2 = 0$, $dtdW_t = 0$ and $\left(dW_t\right)^2 = dt$, this yields

$$\left(d \ln S_t\right)^2 = \sigma_t^2 dt$$

  and thus

$$\int_0^T \left(d \ln S_t\right)^2 = \int_0^T \sigma_t^2 dt$$

- In the continuous case, as $\Delta t \to 0$, the term $252/m$ is modeled by $1/T$ and $RV(0,T)$ can be written as

$$RV(0,T) \to \frac{1}{T}\int_0^T (d\ln S_t)^2 = \frac{1}{T}\int_0^T \sigma_t^2 dt$$

- The strike K can now be computed as

$$K = E_0^\beta\left[\frac{1}{T}\int_0^T \sigma_t^2 dt\right]$$

- With (4) and (5), we have $\frac{dS_t}{S_t} - d\ln S_t = \frac{1}{2}\sigma_t^2 dt$. Substitue and we get

$$\begin{aligned} K &= \frac{2}{T}E_0^\beta\left[\int_0^T\left(\frac{dS_t}{S_t} - d\ln S_t\right)\right] \\ &= \frac{2}{T}E_0^\beta\left[\int_0^T \frac{dS_t}{S_t}\right] - \frac{2}{T}E_0^\beta\left[\ln\frac{S_T}{S_0}\right] \\ &= -\frac{2}{T}\left(E_0^\beta\left[\ln\frac{S_T}{S_0}\right] - (r-q)T\right) = -\frac{2}{T}E_0^\beta\left[\ln\frac{S_T}{F_0(T)}\right] \end{aligned}$$

- Note that the only source of randomness in the expectation is S_T. This suggests we can apply the B-L formula to value variance swap.

- Recall the B-L formula:

$$V_0 = e^{-rT} E_0^{\beta}[V_T] = e^{-rT} V_T(F_0(T)) + \int_0^{F_0(T)} Put(K,T) \frac{\partial^2 V_T(K)}{\partial K^2} dK + \int_{F_0(T)}^{\infty} Call(K,T) \frac{\partial^2 V_T(K)}{\partial K^2} dK$$

- Our problem is

$$K = -\frac{2}{T} E_0^{\beta}\left[\ln \frac{S_T}{F_0(T)} \right]$$

Apply the B-L formula, we get

$$K = -\frac{2}{T}\left(V_T(F_0(T)) + e^{rT} \int_0^{F_0(T)} Put(K,T) \frac{\partial^2 V_T(K)}{\partial K^2} dK + e^{rT} \int_{F_0(T)}^{\infty} Call(K,T) \frac{\partial^2 V_T(K)}{\partial K^2} dK \right)$$

- The 2nd order derivaive is $\frac{\partial^2 V_T(S_T)}{\partial S_T^2} = -\frac{1}{S_T^2}$
- Substitute and we get

$$K = -\frac{2}{T}\left(0 - e^{rT} \int_0^{F_0(T)} \frac{Put(K,T)}{K^2} dK - e^{rT} \int_{F_0(T)}^{\infty} \frac{Call(K,T)}{K^2} dK \right)$$

- Put it altogether and we finally have

$$K = \frac{2}{T} e^{rT} \left(\int_0^{F_0(T)} \frac{Put(K,T)}{K^2} dK + \int_{F_0(T)}^{\infty} \frac{Call(K,T)}{K^2} dK \right) \tag{6}$$

This can be approximated by

$$\frac{2}{T} e^{rT} \left(\sum_{i=1}^{N-1} \frac{Put(K_i,T)}{K_i^2} \Delta K_i + \sum_{i=N}^{M-1} \frac{Call(K_i,T)}{K_i^2} \Delta K_i \right) \tag{7}$$

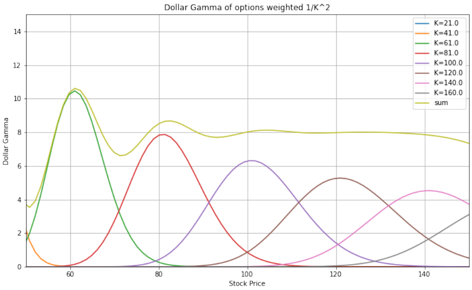
where $\Delta K_i = \frac{K_{i+1} - K_{i-1}}{2}$, K_0 and K_M are the lowest and highest strike available respectively, $K_N = F_0(T)$.

- The formula above shows that the fair value of the variance swap can be computed using a portfolio of puts and calls with the same maturity T which is the final observation date of the realized variance. This means that once we know the volatility smile with maturity T, we can price the variance swap with the realized variance is observed from time 0 to T.
- This is quite remarkable as variance swap is clearly a highly path-dependent payoff but it can be valued as a portfolio of non-path-dependent European options.
- The only assumption here is the path of S_t is continuous. The volatility σ_t can be stochastic. In other words, it is almost a model-free result.

Greeks for Variance Swap

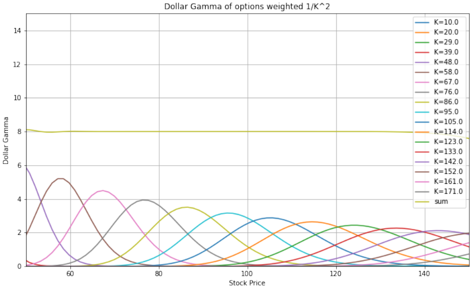
- Since the expectation of the realized variance can be computed using a portfolio of European options. The greeks of the variance swap can be expressed as a portfolio of the greeks of European options.
- Consider the case where the volatility smile is flat at 0.2, $T = 0.25$, $r = 0$, $q = 0$. The diagram below shows the **dollar gamma** of variance swap for various S_0. We use 8 options in the replication. Dollar gamma is defined as

$$\text{dollar gamma} = \Gamma S_0^2$$



- The golden line is the dollar gamma of the variance swap which is the sum of the dollar gamma of each option with a different strike K.

- If we use 18 options in the replication, the variance swap dollar gamma profile becomes a flat line

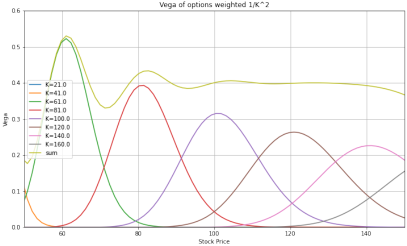


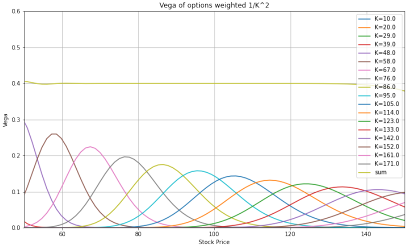
- Recall that, in Lecture 3, if we long an European option with implied vol Σ and then we delta hedge, the net infinitesimal profit or loss after time dt is:

$$Profit = \frac{1}{2}\Gamma S_t^2(\sigma^2 - \Sigma^2)dt$$

where σ is the annualized realized vol between t and $t + dt$, Γ is the gamma of the delta hedged portfolio.
- The gamma of an European option is peaked near the strike and tends to zero away from it. This would reduce the profit of the gamma scalping if the spot is away from the strike.
- The dollar gamma ΓS_t^2 of the variance swap is constant, this means the profit is the same no matter the level of S_t and is only a function of $\sigma^2 - \Sigma^2$.

- We get the similar results for the vega profile of the variance swap. The first diagram uses 8 options and the second diagram uses 18 options.





VIX

- The VIX is an index which gives a measure for the implied volatility of the SP500. Its value is based on options on the SP500 for which the time to maturity ranges from 23 to 37 days. An average of the implied volatility is calculated for 30 days options on the SP500. The VIX index is defined by the Chicago Board Options Exchange (CBOE) in their white paper as

$$VIX^2 = \frac{2}{T}\sum_{i=1}^{M}\frac{\Delta K_i}{K_i^2}e^{rT}Q(K_i) - \frac{1}{T}\left[\frac{F_0(T)}{K_f} - 1\right]^2$$

  where $Q(K_i)$ represents the price of OTM call and put options with strike K_i, and K_f being the hightest strike below the forward price $F_0(T)$. $\Delta K_i = \frac{K_{i+1} - K_{i-1}}{2}$.
- The formula for VIX looks remarkably similar to variance swap with an extra term. The **spot** VIX index is essentially a discrete strike approximation of the variance swap on the SP500 index with the assumption that the underlying is a diffusion process.
- We are going to prove the VIX formula.

Proof

- We start with the Equation (6) which provides a direct link between variance swaps and implied distributions. For some strike $K_F < F_0(T)$, we can rewrite Equation (6) as follows:

$$K = \frac{2}{T}e^{rT}\left[\int_0^{K_F}\frac{Put(K,T)}{K^2}dK + \int_{K_F}^{\infty}\frac{Call(K,T)}{K^2}dK\right] + \frac{2}{T}e^{rT}\int_{K_F}^{F_0(T)}\frac{Put(K,T)-Call(K,T)}{K^2}dK \quad (8)$$

By the put call parity, we have $Put(K,T) - Call(K,T) = e^{-rT}(K - F_0(T))$. The last expression in (8) can be expressed as

$$\begin{aligned}
& \frac{2}{T}e^{rT}\int_{K_F}^{F_0(T)}\frac{Put(K,T)-Call(K,T)}{K^2}dK \\
= \quad & \frac{2}{T}\int_{K_F}^{F_0(T)}\frac{K-F_0(T)}{K^2}dK \\
= \quad & \frac{2}{T}\left[\ln F_0(T) - \ln K_F - F_0(T)\left(-\frac{1}{F_0(T)}+\frac{1}{K_F}\right)\right] \\
= \quad & \frac{2}{T}\left[\ln\frac{F_0(T)}{K_F} + \left(1-\frac{F_0(T)}{K_F}\right)\right]
\end{aligned}$$

- Application of a Taylor series expansion to the logarithm, ignoring terms higher than second-order, and we get

$$\ln \frac{F_0(T)}{K_F} \approx \left(\frac{F_0(T)}{K_F} - 1 \right) - \frac{1}{2} \left(\frac{F_0(T)}{K_F} - 1 \right)^2$$

Substitute and we get

$$\frac{2}{T} e^{rT} \int_{K_F}^{F_0(T)} \frac{Put(K,T) - Call(K,T)}{K^2} dK \approx -\frac{1}{T} \left(\frac{F_0(T)}{K_F} - 1 \right)^2$$

and thus Equation (8) reads

$$K \approx \frac{2}{T} e^{rT} \left[\int_0^{K_F} \frac{Put(K,T)}{K^2} dK + \int_{K_F}^{\infty} \frac{Call(K,T)}{K^2} dK \right] - \frac{1}{T} \left(\frac{F_0(T)}{K_F} - 1 \right)^2$$

Discretizing the integrals and we get

$$\frac{2}{T} \sum_{i=1}^{M} \frac{\Delta K_i}{K_i^2} e^{rT} Q(K_i) - \frac{1}{T} \left[\frac{F_0(T)}{K_f} - 1 \right]^2$$

where $Q(K_i)$ is the OTM call and put option with strike K_i. The proof is done.

Weak static replication

- Path-dependent options, such as barrier options, have payoffs that depend on the path of the underlying price up to expiration. Traditionally, one uses dynamic hedging to hedge such options. But there are some practical difficulties with dynamic hedging.
- First, it is impossible to continuously rebalance the weights of a portfolio, so traders must adjust at discrete intervals. This causes small replication errors that compound over the life of the option.
- Second, there are transaction costs associated with rebalancing the portfolio. These costs grow with the frequency of balancing and can overwhelm the potential profit margin of the option. As a result, traders have to compromise between the accuracy of replication and the cost associated with more frequent rebalancing.

- In this section we describe a method of replication that bypasses some of these difficulties. Given an exotic option, we show how to construct a portfolio of European options, with static time-independent weights, which replicates the value of the exotic option for a specified range of future times and market levels. This portfolio is known as a weak static replicating portfolio.
- Weak static replication relies on matching the boundary payoffs of the replication portfolio to those of the exotic option. When the boundary comes into play **only** at the expiration of the option, the match can be made perfect, as in the sense of Breeden-Litzenberger.
- When the boundary comes into play at earlier times, for example, up-and-out barrier option, the match typically involves the value on the boundary of other non-expiring options.

- These values depend on the model being used to value them. That model could be Black Scholes, or something that perhaps works better. Either way, the value and composition of the replicating portfolio will depend on the model. The more closely the model resembles the true dynamics of the underlying, the better the static replicating portfolio will perform.
- This form of replication is called weak because the matching is model dependent.
- To illustrate weak static replication, we will use up and out barrier option. The underlying dynamics is assumed to be Black Scholes so that we have a closed form solution for the barrier option. The closed form solution serves as the benchmark for checking the accuracy of the replicating portfolios.

Replicating Up-And-Out Call with European Options

- Consider the following up-and-out call option. $S_0 = 100$, $H = 120$, $K = 100$, $T = 1$, $r = 0.05$, $q = 0.03$, $\sigma = 0.2$.
- Under the Black Scholes assumptions, the closed form price for the up-and-out call option is 1.10732.
- We are going to use 3 European call options to replicate the barrier option:

Option	Strike	Maturity	Quantity (w)
C_0	100	1	1
C_1	120	1	-2.94235
C_2	120	0.5	1.03796

- The quantities are chosen such that the portfolio of European options matches the value of the barrier option at the maturities and at the barrier.
- The quantity of C_0 is chosen to match the value of the barrier option at the maturities; we choose 1 for simplicity. The rest of the quantities are chosen to match the value at the barrier at $t = 0$ and $t = 0.5$.

- Let $C(S_t, K, t, T)$ be the value of a call option at time t with maturity T, strike K and the spot price at t is S_t.
- Quantity of C_1 is chosen to ensure the value of the portfolio is 0 when the stock hit the barrier $H = 120$ at **t=0.5**. It can be computed as:

$$\begin{aligned} C_0(S_t = H, K, t, T) + w_1 C_1(S_t = H, H, t, T) &= 0 \\ C_0(120, 100, 0.5, 1) + w_1 C_1(120, 120, 0.5, 1) &= 0 \\ w_1 = -\frac{C_0(120, 100, 0.5, 1)}{C_1(120, 120, 0.5, 1)} & \end{aligned}$$

- Under the Black Scholes dynamics with $\sigma = 0.2$, we have

$$w_1 = -2.94235$$

- Note that the volatility to be used should be the forward implied vols from time 0.5 to 1 with the corresponding moneyness. In this example, under the non-time-dependent Black Scholes dynamics, they are all equal to 0.2.

- Our next task is to find the quantity of C_2. It is chosen to ensure the value of the portfolio is 0 when the stock hit the barrier $H = 120$ at **t=0**.
- Note that the option C_2 must have maturity less than or equal to 0.5 or this will affect the value of w_1. The quantity of C_2 can be computed as

$$C_0(120, 100, 0, 1) + w_1 C_1(120, 120, 0, 1) + w_2 C_2(120, 120, 0, 0.5) = 0$$

- Under the Black Scholes dynamics with $\sigma = 0.2$, we have

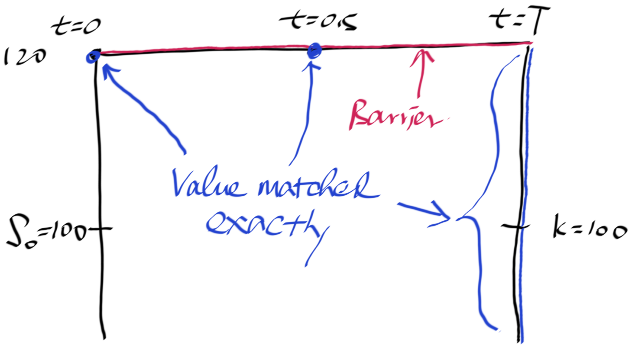
$$w_2 = 1.03796$$

- After getting the quantity for each option, we can compute the value of the option portfolio which replicates the up-and-out barrier option at $t = 0$. Recall that $S_0 = 100$, the value of the portfolio is:

$$C_0(100, 100, 0, 1) + w_1 C_1(100, 120, 0, 1) + w_2 C_2(100, 120, 0, 0.5) = 2.22986$$

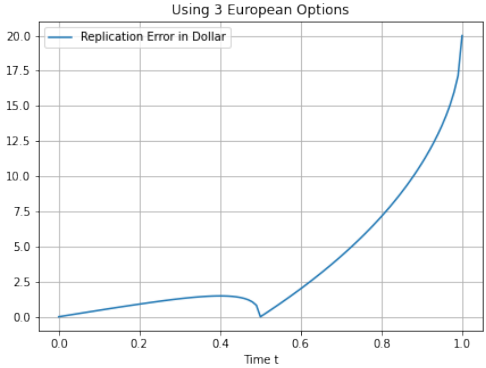
- The reason why this is quite different to 1.10732 is because we only ensure the value of the portfolio is 0 at barrier at $t = 0$ and $t = 0.5$.

- The diagram below shows the regions where the replication error is zero when using 3 European options.



Replication Errors with 3 European Options

- The diagram below shows the replication error **along the barrier** at various t using 3 European options. Our construction only ensure the replication error is 0 at $t = 0$ and $t = 0.5$.
- Note that the value of the replicating portfolio matches the barrier option at all points at maturity except when $S_T = 120$, $T = 1$.



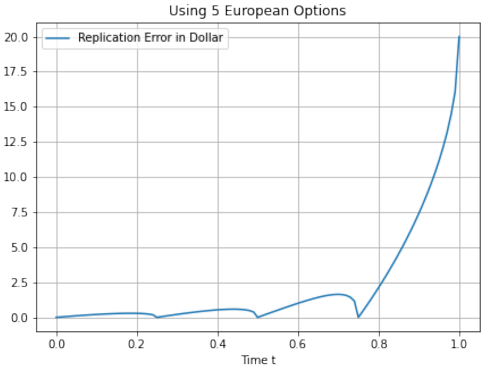
Replicating using More European Options

- The table shows the values of the replicating portfolios when using more European options.

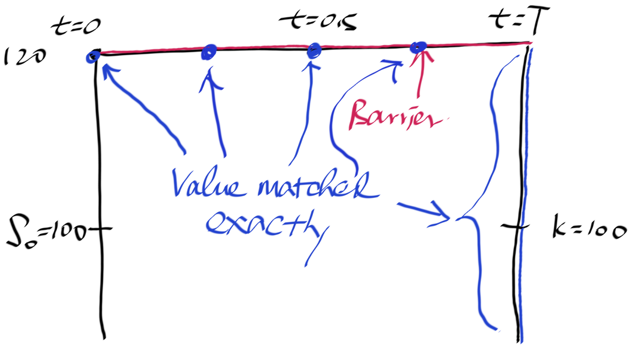
Number of Options	**Value of Portfolio at t=0**
3	2.22986
5	1.68177
12	1.31606
20	1.22784
30	1.18616
40	1.16590
100	1.13037
Closed form	1.10732

Replication Errors with 5 European Options

- The diagram below shows the replication error **along the barrier** at various t using 5 European options.

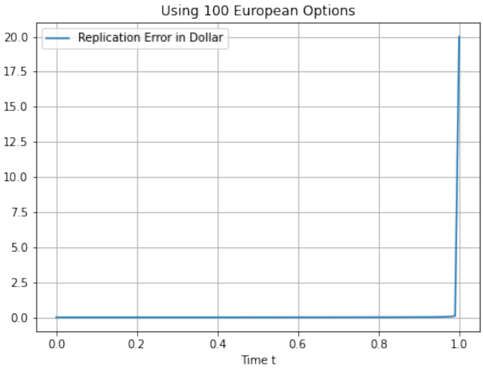


- The diagram below shows the regions where the replication error is zero when using 5 European options.



Replication Errors with 100 European Options

- The diagram below shows the replication error **along the barrier** at various t using 100 European options.



Unwind if S_t Hits the Barrier

- If the stock hits the barrier, the value of the actual up-and-out call is not only zero at that moment, but zero for all times thereafter.
- However, the value of the replicating portfolio would continue to change as the stock price continues to evolve.
- Therefore, if the stock hits the barrier, we must immediately liquidate the replicating portfolio.
- Of course, this strategy brings with it the risk that we might not be able to liquidate the portfolio close to the model price. Furthermore, if the stock price were to move discontinuously across the barrier, we would not be able to liquidate the portfolio at the right moment, which would further decrease the accuracy of the weak static replication strategy.

Summary

- Various option strategies using European options.
- Bredden-Litzenberger - any European payoff that is twice differentiable with respect to the underlying can be replicated by a portfolio of European options.
- Variance swap and VIX.
- Weak static replication of up and out barrier options.