QF602 Derivatives

Lecture 5 - Barrier Options

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Barrier Options

- European call option has the payoff $(S_T - K)^+$ at maturity T.
- Barrier call option is similar to European call option with an extra feature, the barrier.
- There are 2 flavours of barrier, knock-in and knock out. For **knock-in**, if the barrier level **has never been touched** in the life time of the option, i.e. between 0 to T, the payoff will be 0 otherwise $(S_T - K)^+$. On the other hand, for **knock-out**, if the barrier level **has been touched**, the payoff will be 0 otherwise $(S_T - K)^+$.
- If we have a portfolio of two options, one knock-in and one knock-out, this is equivalent to holding an European option with no barrier, since one but not both will always pay. Therefore we have the replication formula

$$knock - in + knock - out = European$$

This is the so-called **In-Out Parity**.

- We can have any combination of up/down, in/out and call/put that we like. The most interesting combinations are those for which **the barrier is in-the-money**. For example, an up-and-out call option with barrier at 120 and strike at 100, has an interesting payoff profile. Below 100 the value is zero, from 100 to 120, the value increases in a straight line to 20, and then above 120 it drops immediately to zero.
- Why buy a barrier option?
- For the purpose of hedging they are not particularly useful. However, they are cheaper than European options, so if a speculator has a very strong view on how an asset price will move then he can make more money by purchasing an option which express those views precisely.
- For example, if the speculator believes that the asset will greatly increase in value and will not go below 90 in the mean time, he could purchase a down-and-out call with barrier at 90 saving a little on the option's premium.

- A technical issue with continuous barrier options is the question of how to agree what it means for the asset price to cross the barrier.
- Most asset prices are observable discretely even though some asset classes like FX are traded continuously from New York Sunday afternoon to Friday afternoon. Having the infrastructure to monitor all asset prices continuously can be expensive.
- For this reason, truly continuous barrier options are rare. Instead, the option price is generally sampled on a daily basis. The option is therefore really a discrete barrier option; however the continuous barrier is a very good approximation to the daily sampled barrier.

- The first option that we focus on is a **down-and-in call option**. The price of an option will be computed as its discounted expectation. To compute this expectation, we will need to know the probability that the barrier will be breached and the distribution of the final value of spot given that barrier has been breached. To compute these what we really need is the **joint distribution** of the minimum and the terminal value for a Brownian motion with drift.
- To derive the closed form formula for down-and-in option, we start with the using the **reflection principle** to compute the joint distribution for Brownian motion **with no drift**. Then we use Girsanov's theorem to compute the joint distribution for Brownian motion **with drift**. Then we finally use the joint distribution to compute the price of a down-and-in option.

Reflection Principle

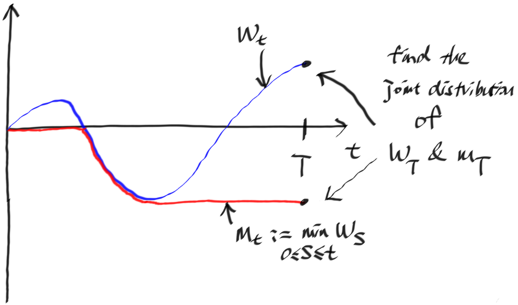
- Let W_t be a Brownian motion. Let m_T denotes the minimum value of W_t over the interval $[0, T]$

$$m_T := \min_{0 \le s \le T} W_s$$

- We want to compute the joint probability

$$\mathbb{P}(W_T \ge x, m_T \le y)$$

for $x \ge y$ and $y < 0$.



- To compute the joint probability, it is helpful to consider the event that the Brownian motion touches the barrier at some time before T,

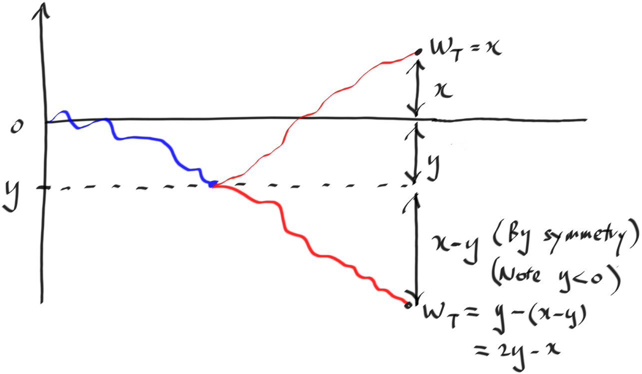
$$\mathcal{U} = \{W_t = y, t \in [0, T]\}.$$

- Touching the barrier at some point is exactly equivalent to the minimum being at or below the barrier. Then we have, by the product rule,

$$\begin{aligned} \mathbb{P}(W_T \geq x, m_T \leq y) &= \mathbb{P}(W_T \geq x, \mathcal{U}) \\ &= \mathbb{P}(W_T \geq x | \mathcal{U}) \mathbb{P}(\mathcal{U}) \end{aligned}$$

- The reflection principle will allow us to calculate $\mathbb{P}(W_T \geq x | \mathcal{U})$. The condition on $\mathcal{U}$ tells us to consider only paths that touch the barrier. Let's suppose this happens at time t, and consider the path from t onwards. We are interested in a path that then moves back upwards and ends up with $W_T > x$.

- By the symmetry of Brownian motion, the path that is the exact reflection has identical probability so that $\mathbb{P}(W_T \geq x|\mathcal{U}) = \mathbb{P}(W_T \leq 2y - x|\mathcal{U})$. This is illustrated in the diagram below



- Then our joint probability becomes

$$\begin{aligned}\mathbb{P}(W_T \geq x, m_T \leq y) &= \mathbb{P}(W_T \leq 2y - x|\mathcal{U})P(\mathcal{U}) \\ &= \mathbb{P}(W_T \leq 2y - x, \mathcal{U}) \\ &= \mathbb{P}(\mathcal{U}|W_T \leq 2y - x)\mathbb{P}(W_T \leq 2y - x)\end{aligned}$$

- But since $2y - x$ is the reflection of x in the barrier, it is certainly lower than the barrier and therefore the conditional probability of touching the barrier is $\mathbb{P}(\mathcal{U}|W_T \leq 2y - x) = 1$. This leaves us with a delightfully simple result

$$\mathbb{P}(W_T \geq x, m_T \leq y) = \mathbb{P}(W_T \leq 2y - x)$$

- A problem depending on the full path of the Brownian motion has been reduced to a problem depending on the terminal value of a new Brownian motion constructed from the original and its reflection in the barrier.

Joint Distribution for Brownian motion with drift and its minimum

- We have derived the joint distribution of the minimum and terminal value for a Brownian motion **with no drift**. We now apply Girsanov's theorem to derive the joint distribution for a Brownian motion **with drift**.
- Let W_t be a Brownian motion. Let $Y_t = \sigma W_t$, and m_t^Y be the minimum of Y_t up to time t. We then have for $y < 0$ and $x \geq y$ that

$$\mathbb{P}(Y_t \geq x, m_t^Y \leq y) = \mathbb{P}(Y_t \leq 2y - x) = \Phi\left(\frac{2y - x}{\sigma\sqrt{t}}\right)$$

  This follows from the result for Brownian motion, as the volatility term makes no real difference as the derivation is based on the symmetry of Brownian motion.
- We wish to prove a similar result for a Brownian motion with drift. Let $Z_t = \nu t + \sigma W_t$ and m_t^Z denotes its minimum up to time t. **Our main result** is

$$\mathbb{P}(Z_t \geq x, m_t^Z \leq y) = e^{\frac{2\nu y}{\sigma^2}} \Phi\left(\frac{2y - x + \nu t}{\sigma\sqrt{t}}\right)$$

  for $y < 0$ and $x \geq y$

Proof

- Let's recall the **Girsanov's theorem** for Brownian motion: Let W_t be a Brownian motion under probability measure $\mathbb{P}$. Consider the process $B_t = \mu t + W_t$. Define the equivalent measure $\mathbb{Q}$ by

  $$\frac{d\mathbb{Q}_t}{d\mathbb{P}_t} = e^{-\mu W_t - \frac{1}{2}\mu^2 t}$$

  which is the Radon-Nikodym derivative to change from $\mathbb{Q}$ to $\mathbb{P}$. B_t is a $\mathbb{Q}$-Brownian motion and

  $$\frac{d\mathbb{P}_t}{d\mathbb{Q}_t} = e^{\mu B_t - \frac{1}{2}\mu^2 t}$$

  is the Radon-Nikodym derivative to change back from $\mathbb{P}$ to $\mathbb{Q}$.
- We denote expectation under the $\mathbb{P}$ and $\mathbb{Q}$ measure by $E^{\mathbb{P}}$ and $E^{\mathbb{Q}}$ respectively.

- Let an event A be

$$A = \{Z_t \geq x, m_t^Z \leq y\}$$

and let $Z_t = \sigma B_t$, $B_t = \mu t + W_t$ and $\mu = \frac{\nu}{\sigma}$. W_t is a Brownian motion under the risk-neutral measure and let's denote the measure as $\mathbb{P}$ in this proof.

- The probability of event A can be computed as

$$\begin{aligned}
\mathbb{P}(A) &= E^{\mathbb{P}}[1_A] \\
&= E^{\mathbb{P}}\left[1_{\{Z_t \geq x, m_t^Z \leq y\}}\right] \\
&= E^{\mathbb{Q}}\left[1_{\{Z_t \geq x, m_t^Z \leq y\}} \frac{d\mathbb{P}}{d\mathbb{Q}}\right]
\end{aligned}$$

- Under the $\mathbb{Q}$ measure, B_t is Brownian motion with the Radon-Nikodym derivative

$$\frac{d\mathbb{P}_t}{d\mathbb{Q}_t} = e^{\mu B_t - \frac{1}{2}\mu^2 t}$$

- Substitute the term and we have

$$\begin{aligned}
&= E^{\mathbb{Q}}\left[1_{\{Z_t \geq x, m_t^Z \leq y\}} e^{\mu B_t - \frac{1}{2}\mu^2 t}\right] \\
&= E^{\mathbb{Q}}\left[1_{\{Z_t \geq x, m_t^Z \leq y\}} e^{\frac{\mu}{\sigma} Z_t - \frac{1}{2}\mu^2 t}\right]
\end{aligned}$$

- Under the measure $\mathbb{Q}$ and using the reflection principle, we can replace the joint distribution with a marginal distribution, and replace Z_t with $2y - Z_t$ [1]

$$
\begin{aligned}
&= E^{\mathbb{Q}}\left[1_{\{Z_t \leq 2y-x\}} e^{\frac{\mu}{\sigma}(2y-Z_t)-\frac{1}{2}\mu^2 t}\right] \\
&= e^{\frac{2\nu y}{\sigma^2}} E^{\mathbb{Q}}\left[1_{\{Z_t \leq 2y-x\}} e^{-\frac{\mu}{\sigma}Z_t-\frac{1}{2}\mu^2 t}\right] \\
&= e^{\frac{2\nu y}{\sigma^2}} E^{\mathbb{Q}}\left[1_{\{Z_t \leq 2y-x\}} e^{-\mu B_t-\frac{1}{2}\mu^2 t}\right]
\end{aligned}
$$

- We can regard the exponential term as the Radon-Nikodym derivative which changes from measure $\mathbb{S}$ to measure $\mathbb{Q}$

$$
\frac{d\mathbb{S}_t}{d\mathbb{Q}_t} = e^{-\mu B_t - \frac{1}{2}\mu^2 t}
$$

$X_t = \mu t + B_t$ is a $\mathbb{S}$-Brownian motion.

[1] This is because

$$
\begin{aligned}
P(Z_t \geq x, m_t^Z \leq y) &= P(Z_t \leq 2y - x, m_t^Z \leq y) \\
&= P(2y - Z_t \geq x, m_t^Z \leq y)
\end{aligned}
$$

- We finally have

$$
\begin{aligned}
\mathbb{P}(Z_t \geq x, m_t^Z \leq y) &= e^{\frac{2\nu y}{\sigma^2}} E^{\mathbb{Q}}\left[1_{\{Z_t \leq 2y-x\}} e^{-\mu B_t - \frac{1}{2}\mu^2 t}\right] \\
&= e^{\frac{2\nu y}{\sigma^2}} E^{\mathbb{S}}\left[1_{\{Z_t \leq 2y-x\}}\right] \\
&= e^{\frac{2\nu y}{\sigma^2}} E^{\mathbb{S}}\left[1_{\{\sigma B_t \leq 2y-x\}}\right] \\
&= e^{\frac{2\nu y}{\sigma^2}} E^{\mathbb{S}}\left[1_{\{\sigma X_t - \nu t \leq 2y-x\}}\right] \\
&= e^{\frac{2\nu y}{\sigma^2}} E^{\mathbb{S}}\left[1_{\{\sigma X_t \leq 2y-x+\nu t\}}\right] \\
&= e^{\frac{2\nu y}{\sigma^2}} \Phi\left(\frac{2y - x + \nu t}{\sigma\sqrt{t}}\right)
\end{aligned}
$$

- The proof is done.

Down-and-In Call Option

- We want to price a down-and-in call option with a lower barrier at H and strike K with the payoff at T, for $K > H$ and $S_0 > H$
$$(S_T - K)^+ 1_{m_T^S \le H}$$
under the Black Scholes model. The spot S_t follows the log-normal dynamics
$$\ln S_t = \ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t, \mu = r - q$$
- The price of the barrier option V_0 can be computed as under the measure induced by the numeraire N_t:
$$\begin{aligned} V_0 &= N_0 E_0 \left[\frac{(S_T - K)^+ 1_{m_T^S \le H}}{N_T}\right] \\ &= \underbrace{N_0 E_0 \left[\frac{S_T 1_{(S_T \ge K, m_T^S \le H)}}{N_T}\right]}_{V_0^1} - \underbrace{K N_0 E_0 \left[\frac{1_{(S_T \ge K, m_T^S \le H)}}{N_T}\right]}_{V_0^2} \end{aligned}$$
- In the previous slides, we derived the joint distribution
$$\mathbb{P}(Z_t \ge x, m_t^Z \le y)$$
where $Z_t = \nu t + \sigma W_t$.

- Let's compute V_0^2 first. We set $Z_t = \ln(S_t/S_0)$, $x = \ln(K/S_0)$, $y = \ln(H/S_0)$, such that

$$\mathbb{P}(Z_T \geq x, m_T^Z \leq y) = \mathbb{P}(S_T \geq K, m_T^S \leq H)$$

- We then choose $N_t = \beta_t = e^{rt}$ and the drift becomes $\nu = \mu - \frac{1}{2}\sigma^2$.

$$\begin{aligned}
V_0^2 &= Ke^{-rT} E_0^{\beta}\left[1_{(S_T \geq K, m_T^S \leq H)}\right] \\
&= Ke^{-rT}\mathbb{P}(Z_T \geq x, m_T^Z \leq y) \\
&= Ke^{-rT} e^{\frac{2\nu y}{\sigma^2}} \Phi\left(\frac{2y - x + \nu T}{\sigma\sqrt{T}}\right) \\
&= Ke^{-rT} e^{\frac{2(\mu - \frac{1}{2}\sigma^2)\ln(H/S_0)}{\sigma^2}} \Phi\left(\frac{2\ln(H/S_0) - \ln(K/S_0) + (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \\
&= Ke^{-rT}\left(\frac{H}{S_0}\right)^{\left(\frac{2\mu}{\sigma^2} - 1\right)} \Phi\left(\frac{\ln(H^2/(KS_0)) + (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)
\end{aligned}$$

- Next, we compute V_0^1. It can be computed under the stock measure, i.e. $N_t = S_t e^{qt}$. In this measure, $\nu = \mu + \frac{1}{2}\sigma^2$ and we have

$$
\begin{aligned}
V_0^1 &= S_0 e^{-qT} E_0^S \left[1_{(S_T \geq K, m_T^S \leq H)} \right] \\
&= S_0 e^{-qT} \mathbb{P}(Z_T \geq x, m_T^Z \leq y) \\
&= S_0 e^{-qT} e^{\frac{2\nu y}{\sigma^2}} \Phi\left(\frac{2y - x + \nu T}{\sigma\sqrt{T}} \right) \\
&= S_0 e^{-qT} e^{\frac{2(\mu + \frac{1}{2}\sigma^2)\ln(H/S_0)}{\sigma^2}} \Phi\left(\frac{2\ln(H/S_0) - \ln(K/S_0) + (\mu + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right) \\
&= S_0 e^{-qT} \left(\frac{H}{S_0} \right)^{\left(\frac{2\mu}{\sigma^2} + 1\right)} \Phi\left(\frac{\ln(H^2/(KS_0)) + (\mu + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right)
\end{aligned}
$$

- Putting the results together and we have the closed form formula:

$$
DIC(H, K, T) = S_0 e^{-qT} \left(\frac{H}{S_0} \right)^{\left(\frac{2\mu}{\sigma^2} + 1\right)} \Phi(Q_+) - K e^{-rT} \left(\frac{H}{S_0} \right)^{\left(\frac{2\mu}{\sigma^2} - 1\right)} \Phi(Q_-)
$$

where

$$
Q_{\pm} = \frac{\ln(H^2/(KS_0)) + (\mu \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}
$$

Down-and-Out Call Option

- If we have the closed form formula for down-and-in call option, the closed form formula for down-and-out call option with the same barrier H, strike K and maturity T can be easily computed using the in-out parity:

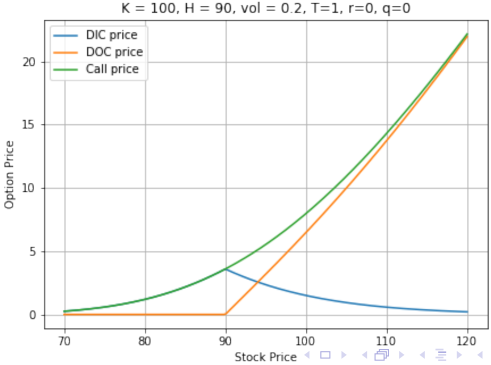
$$DOC(H,K,T) = Call(K,T) - DIC(H,K,T)$$

- The greeks of down-and-out call can also be computed in the similar fashion, for example, delta of down-and-out call:

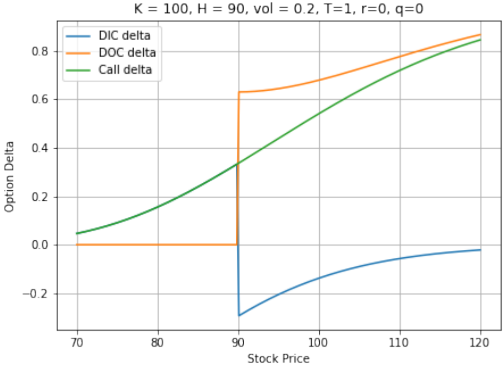
$$\frac{\partial DOC(H,K,T)}{\partial S_0} = \frac{\partial Call(K,T)}{\partial S_0} - \frac{\partial DIC(H,K,T)}{\partial S_0}$$

Risk Analysis for Down-and-Out Option

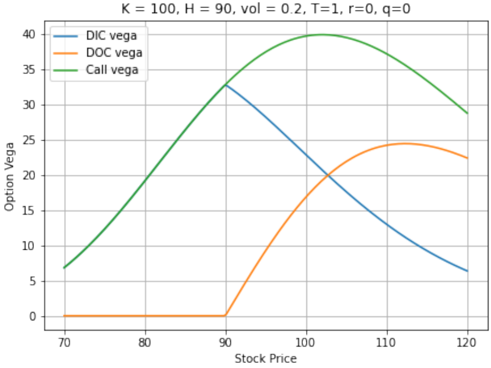
- Let's consider the case where strike $K = 100$, down side barrier $H = 90$, volatility $\sigma = 0.2$, time to maturity $T = 1$, rate $r = 0$ and dividend yield $q = 0$.
- The orange line is the **price** of down-and-out call option (DOC) for various stock price at time 0, S_0. The price of DOC is 0 when $S_0 \leq H$ because it has already been knocked out. The green line is the European call price. The blue line is the DIC price, it can be computed as the difference between the green and orange lines by the in-out parity. For $S_0 \leq H$, DIC is the same as European call because it has already been knocked-in.
- DIC price decreases as S_0 increases. This is because the probability being knocked in is lower as S_0 increases. We can deduce the delta of DIC is negative.



- The graph below shows the **deltas**. For $S_0 \leq H$, the delta of DOC call is 0. The delta profile changes dramatically from 0 to around 0.6 at the barrier $H = 90$. A discontinuity in delta is known as **delta gap**, and its presence presents a significant delta-hedging challenge.
- In practice, spot does not always move around slowly and smoothly. We cannot assume that we will always be able to close out our delta hedge at a spot rate exactly equal to the barrier level. We must take the delta gap into account when calculating option values. This is typically done using techniques known as **barrier over-hedging** method which will be discussed later.



- The graph below shows the **vegas**. The DIC vega is peaked at the barrier but the DOC vega is peaked away from the barrier, in this case, around 110. On one hand, the payoff $(S_T - K)^+$ benefits from the increases of volatility, i.e. vega is positive. On the other hand, volatility increases will increase the probability of being knocked out. These two opposite forces generate the DOC vega profile is shown as the orange line in the graph.



Downside One Touch Option

- There are many variants of one touch option. We study a variant which pays 1 **at expiry** if at any point during the life of the option, a given lower barrier H has been breached. The **downside one touch** payoff at the maturity T is

$$1_{m_T^S \leq H}$$

The value of the option can be computed by taking the expectation in the risk neutral measure:

$$e^{-rT} E_0^{\beta}\left[1_{m_T^S \leq H}\right] = e^{-rT}\mathbb{P}\left(m_T^S \leq H\right)$$

- In order to price one touch option, we need to find $\mathbb{P}\left(m_T^S \leq H\right)$.
- For $y \leq 0$ and $x \geq y$, we have the identity:

$$\mathbb{P}(m_t^Z \leq y) = \mathbb{P}(Z_t \geq x, m_t^Z \leq y) + \mathbb{P}(Z_t \leq x, m_t^Z \leq y)$$

- $\mathbb{P}(m_t^Z \leq y)$ can be found by setting $x = y$, since it is not a function of x. The event that the minimum is less than y and the terminal value is less than y is the same as the event that the terminal value is less than y. We therefore have

$$\begin{aligned}
\mathbb{P}(m_t^Z \leq y) &= \mathbb{P}(Z_t \geq y, m_t^Z \leq y) + \underbrace{\mathbb{P}(Z_t \leq y, m_t^Z \leq y)}_{\mathbb{P}(Z_t \leq y)} \\
&= e^{\frac{2\nu y}{\sigma^2}} \Phi\left(\frac{2y - y + \nu t}{\sigma\sqrt{t}}\right) + \Phi\left(\frac{y - \nu t}{\sigma\sqrt{t}}\right)
\end{aligned}$$

- We can set $Z_t = \ln(S_t/S_0)$, $y = \ln(H/S_0)$, $\nu = \mu - \frac{1}{2}\sigma^2$, such that

$$
\begin{aligned}
& \mathbb{P}\left(m_T^S \leq H\right) \\
= \quad & \mathbb{P}\left(m_T^Z \leq y\right) \\
= \quad & e^{\frac{2\nu y}{\sigma^2}} \Phi\left(\frac{y+\nu T}{\sigma\sqrt{T}}\right) + \Phi\left(\frac{y-\nu T}{\sigma\sqrt{T}}\right) \\
= \quad & \left(\frac{H}{S_0}\right)^{\frac{2\mu}{\sigma^2}-1} \Phi\left(\frac{\ln(\frac{H}{S_0}) + (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) + \Phi\left(\frac{\ln(\frac{H}{S_0}) - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)
\end{aligned}
$$

- The value of the downside one touch option with barrier H is

$$
\begin{aligned}
DownOneTouch(H,T) \quad = \quad & e^{-rT}\mathbb{P}\left(m_T^S \leq H\right) \\
= \quad & e^{-rT}\left(\frac{H}{S_0}\right)^{\frac{2\mu}{\sigma^2}-1} \Phi\left(\frac{\ln(\frac{H}{S_0}) + (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \\
& + e^{-rT}\Phi\left(\frac{\ln(\frac{H}{S_0}) - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)
\end{aligned}
$$

Downside No Touch Option

- Downside no touch option pays 1 at expiry if at any point during the life of the option, a given lower barrier H **has never been breached**.
- The downside no touch payoff at maturity T is

  $$1_{m_T^S > H}$$

  The value of the option can be computed under the risk neutral measure:

  $$\begin{aligned} DownNoTouch(H,T) &= e^{-rT} E_0^{\beta}\left[1_{m_T^S > H}\right] \\ &= e^{-rT}\mathbb{P}\left(m_T^S > H\right) \\ &= e^{-rT}\left(1 - \mathbb{P}\left(m_T^S \leq H\right)\right) \\ &= e^{-rT} - DownOneTouch(H,T) \end{aligned}$$

Upside One Touch Option

- We have concentrated on studying the distribution of the minimum of Brownian motion which is relevant when studying down-and-in and downside one touch options. If we wish to price **upside one touch** option, we will need similar theorems for the maximum. Let M_T^Z denotes the maximum over the interval $[0, T]$

$$M_T^Z := \max_{0 \le t \le T} Z_t$$

- The upside one touch payoff with upper barrier H at the maturity T is

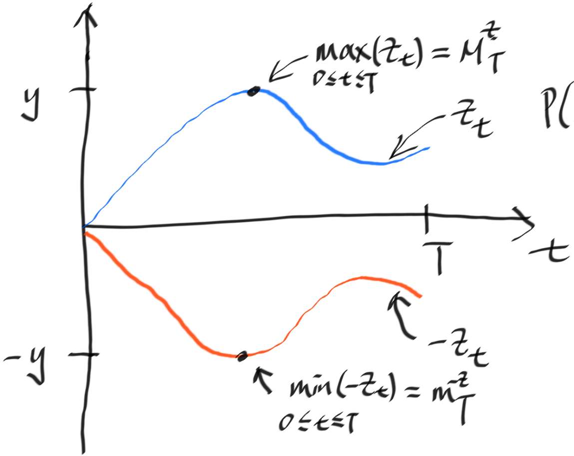
$$1_{M_T^S \ge H}$$

The value of the option can be computed by taking the risk neutral expectation:

$$e^{-rT} E_0^{\beta} \left[1_{M_T^S \ge H} \right] = e^{-rT} \mathbb{P}(M_T^S \ge H)$$

- Fortunately, the fact that the negative of a Brownian motion with drift is a also a Brownian motion with drift, it means that the distribution of the maximum can be deduced from the distribution of the minimum. We can write

$$M_t^Z = \max_{0 \le t \le T} (\sigma W_t + \nu t) = - \min_{0 \le t \le T} (-\sigma W_t - \nu t) = - \min_{0 \le t \le T} (-Z_t) = -m_T^{-Z}$$



$$P(M_T^Z \geq y) = P(m_T^{-Z} \leq -y)$$

- Note that $-Z_T$ has the drift $-\nu T$ and volatility σ because $-W_T$ is just another Brownian motion. We have

$$\begin{aligned}
\mathbb{P}(M_T^Z \geq y) &= \mathbb{P}(m_T^{-Z} \leq -y) \\
&= e^{\frac{2(-\nu)(-y)}{\sigma^2}} \Phi\left(\frac{-y + (-\nu)T}{\sigma\sqrt{T}}\right) + \Phi\left(\frac{-y - (-\nu)T}{\sigma\sqrt{T}}\right) \\
&= e^{\frac{2\nu y}{\sigma^2}} \Phi\left(\frac{-y - \nu T}{\sigma\sqrt{T}}\right) + \Phi\left(\frac{-y + \nu T}{\sigma\sqrt{T}}\right)
\end{aligned}$$

- By setting $Z_T = \ln(S_T/S_0)$, $y = \ln(H/S_0)$, $\nu = \mu - \frac{1}{2}\sigma^2$, such that the value of upside one touch can be computed as

$$\begin{aligned}
UpOneTouch(H, T) &= e^{-rT}\mathbb{P}(M_T^S \geq H) \\
&= e^{-rT}\mathbb{P}(M_T^Z \geq y) \\
&= e^{-rT}\left(e^{\frac{2\nu y}{\sigma^2}} \Phi\left(\frac{-y - \nu T}{\sigma\sqrt{T}}\right) + \Phi\left(\frac{-y + \nu T}{\sigma\sqrt{T}}\right)\right) \\
&= e^{-rT}\left(\frac{H}{S_0}\right)^{\frac{2\mu}{\sigma^2} - 1} \Phi\left(\frac{\ln(\frac{S_0}{H}) - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \\
&\quad + e^{-rT}\Phi\left(\frac{\ln(\frac{S_0}{H}) + (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)
\end{aligned}$$

Upside No Touch Option

- Upside no touch option pays 1 at expiry if at any point during the life of the option, a given higher barrier H **has never been breached**.
- The upside no touch payoff at maturity T is

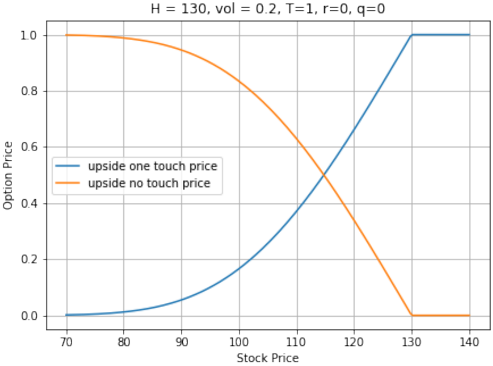
$$1_{M_T^S < H}$$

The value of the option can be computed under the risk neutral measure:

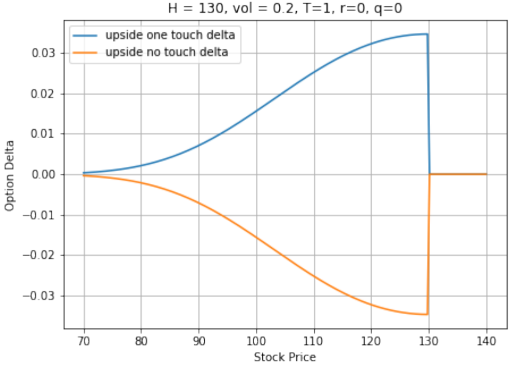
$$\begin{aligned} UpNoTouch(H, T) &= e^{-rT} E_0^{\beta}\left[1_{M_T^S < H}\right] \\ &= e^{-rT}\mathbb{P}\left(M_T^S < H\right) \\ &= e^{-rT}\left(1 - \mathbb{P}\left(M_T^S \geq H\right)\right) \\ &= e^{-rT} - UpOneTouch(H, T) \end{aligned}$$

Risk Analysis for Upside No Touch

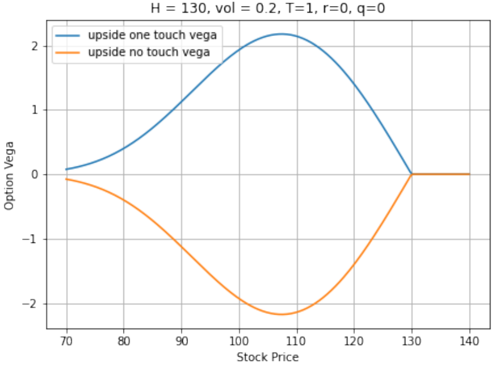
- Let's consider the case where the upside barrier $H = 130$, volatility $\sigma = 0.2$, time to maturity $T = 1$, rate $r = 0$, dividend yield $q = 0$.
- The price of upside no touch (UNT) decreases from 1 to 0 as S_0 approaches the barrier H. On the other hand, the price of upside one touch (UOT) increases from 0 to 1 as S_0 approaches the barrier H.



- The graph below shows the deltas. UOT delta is always positive as increasing the S_0 would only increases the probability of touching the upper barrier.



- The graph below shows the vegas. UOT vega is always positive as increase the volatility would only increase the probability of touching the upper barrier.



Up-and-In Call Option

- For down-and-in call option, the barrier is located in region that the call is OTM. We now consider the case that the knocked in barrier is in the region of the call is ITM, such that $S_0 < K < H$. When the barrier is in the ITM region, they are called **reversed** barrier option.
- The payoff of the up-and-in call at maturity T is

$$(S_T - K)^+ 1_{M_T^S \geq H}$$

- To compute the value of the option, we need to find the joint distribution

$$\mathbb{P}(Z_T \geq x, M_T^Z \geq y)$$

  where $Z_t = \nu t + \sigma W_t$ for $y \geq x \geq 0$, which is given as

$$e^{\frac{2\nu y}{\sigma^2}} \Phi\left(\frac{-y - \nu t}{\sigma\sqrt{t}}\right) + \Phi\left(\frac{-y + \nu t}{\sigma\sqrt{t}}\right) - e^{\frac{2\nu y}{\sigma^2}} \Phi\left(\frac{-2y + x - \nu t}{\sigma\sqrt{t}}\right)$$

- We are going to use the formula above as it is to compute the price of up-and-in call first, then we come back to prove the formula.
- The price of the up-and-in call can be computed as an expectation in the measure induced by a numeraire N_t:

$$
\begin{aligned}
UIC(H,K,T) &= N_0 E_0\left[\frac{(S_T-K)^+}{N_T} 1_{M_T^S \geq H}\right] \\
&= N_0 E_0\left[\frac{(S_T-K)}{N_T} 1_{(S_T\geq K, M_T^S \geq H)}\right] \\
&= N_0 E_0\left[\frac{S_T}{N_T} 1_{(S_T\geq K, M_T^S \geq H)}\right] - N_0 E_0\left[\frac{K}{N_T} 1_{(S_T\geq K, M_T^S \geq H)}\right] \\
&= \underbrace{S_0 e^{-qT} E_0^S\left[1_{(S_T\geq K, M_T^S \geq H)}\right]}_{V_0^1} - \underbrace{K e^{-rT} E_0^\beta\left[1_{(S_T\geq K, M_T^S \geq H)}\right]}_{V_0^2}
\end{aligned}
$$

- For V_0^1, we choose $N_t = S_t e^{qt}$.
- Tor V_0^2, we choose $N_t = \beta_t = e^{rt}$.

- The first term V_0^1 can be computed as

$$
\begin{aligned}
V_0^1 &= S_0 e^{-qT} \mathbb{P}(Z_T \geq x, M_T^Z \geq y) \\
&= S_0 e^{-qT} \left(e^{\frac{2\nu y}{\sigma^2}} \Phi\left(\frac{-y-\nu T}{\sigma\sqrt{T}} \right) + \Phi\left(\frac{-y+\nu T}{\sigma\sqrt{T}} \right) \right. \\
&\quad \left. - e^{\frac{2\nu y}{\sigma^2}} \Phi\left(\frac{-2y+x-\nu T}{\sigma\sqrt{T}} \right) \right)
\end{aligned}
$$

with $Z_t = \ln(S_t/S_0)$, $x = \ln(K/S_0)$, $y = \ln(H/S_0)$, $\nu = \mu + \frac{1}{2}\sigma^2$.

- Substitute all the terms and we have

$$
\begin{aligned}
V_0^1 = S_0 e^{-qT} \Bigg(& \left(\frac{H}{S_0} \right)^{\frac{2\mu}{\sigma^2}+1} \Phi\left(\frac{\ln(\frac{S_0}{H}) - (\mu + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right) \\
& + \Phi\left(\frac{\ln(\frac{S_0}{H}) + (\mu + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right) \\
& - \left(\frac{H}{S_0} \right)^{\frac{2\mu}{\sigma^2}+1} \Phi\left(\frac{\ln(\frac{S_0 K}{H^2}) - (\mu + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right) \Bigg)
\end{aligned}
$$

- The second term V_0^2 can be computed as

$$\begin{aligned} V_0^2 &= Ke^{-rT}\mathbb{P}(Z_T \geq x, M_T^Z \geq y) \\ &= Ke^{-rT}\left(e^{\frac{2\nu y}{\sigma^2}}\Phi\left(\frac{-y-\nu T}{\sigma\sqrt{T}}\right) + \Phi\left(\frac{-y+\nu T}{\sigma\sqrt{T}}\right) \right. \\ & \qquad \left. - e^{\frac{2\nu y}{\sigma^2}}\Phi\left(\frac{-2y+x-\nu T}{\sigma\sqrt{T}}\right)\right) \end{aligned}$$

with $Z_t = \ln(S_t/S_0)$, $x = \ln(K/S_0)$, $y = \ln(H/S_0)$, $\nu = \mu - \frac{1}{2}\sigma^2$.

- Substitute all the terms and we have

$$\begin{aligned} V_0^2 = Ke^{-rT}\Bigg(\left(\frac{H}{S_0}\right)^{\frac{2\mu}{\sigma^2}-1} &\Phi\left(\frac{\ln(\frac{S_0}{H}) - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \\ &+ \Phi\left(\frac{\ln(\frac{S_0}{H}) + (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \\ &- \left(\frac{H}{S_0}\right)^{\frac{2\mu}{\sigma^2}-1}\Phi\left(\frac{\ln(\frac{S_0 K}{H^2}) - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)\Bigg) \end{aligned}$$

- Collecting all the terms, the price of a up-and-in call option is

$$
\begin{aligned}
UIC(H,K,T) = S_0 e^{-qT} \Bigg(& \left(\frac{H}{S_0}\right)^{\frac{2\mu}{\sigma^2}+1} \Phi\left(\frac{\ln(\frac{S_0}{H}) - (\mu + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \\
& + \Phi\left(\frac{\ln(\frac{S_0}{H}) + (\mu + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \\
& - \left(\frac{H}{S_0}\right)^{\frac{2\mu}{\sigma^2}+1} \Phi\left(\frac{\ln(\frac{S_0 K}{H^2}) - (\mu + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)\Bigg) \\
- K e^{-rT} \Bigg(& \left(\frac{H}{S_0}\right)^{\frac{2\mu}{\sigma^2}-1} \Phi\left(\frac{\ln(\frac{S_0}{H}) - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \\
& + \Phi\left(\frac{\ln(\frac{S_0}{H}) + (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \\
& - \left(\frac{H}{S_0}\right)^{\frac{2\mu}{\sigma^2}-1} \Phi\left(\frac{\ln(\frac{S_0 K}{H^2}) - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)\Bigg)
\end{aligned}
$$

Proof of $\mathbb{P}(Z_T \geq x, M_T^Z \geq y)$

- We are going to prove the joint distribution by using the similar procedure, i.e. using reflection principle to find the joint distribution for Brownian motion with no drift and then apply the Girsanov's theorem to find the final result.
- Our task is to find the joint distribution $\mathbb{P}(Z_T \geq x, M_T^Z \geq y)$ for $y \geq x \geq 0$. $Z_t = \nu t + \sigma W_t$. W_t is a $\mathbb{P}$-Brownian motion. This can be computed as

$$\mathbb{P}(M_T^Z \geq y) = \mathbb{P}(Z_T \geq x, M_T^Z \geq y) + \mathbb{P}(Z_T \leq x, M_T^Z \geq y)$$

- When pricing upside one touch option, we have already computed

$$\mathbb{P}(M_T^Z \geq y) = e^{\frac{2\nu y}{\sigma^2}} \Phi\left(\frac{-y - \nu T}{\sigma\sqrt{T}}\right) + \Phi\left(\frac{-y + \nu T}{\sigma\sqrt{T}}\right)$$

- $\mathbb{P}(Z_T \leq x, M_T^Z \geq y)$ can be computed using the reflection principle. We first consider the joint distribution for W_T:

$$\begin{aligned}
\mathbb{P}(W_T \leq x, M_T \geq y) &= \mathbb{P}(W_T \leq x | M_T \geq y)\mathbb{P}(M_T \geq y) \\
&= \mathbb{P}(W_T \geq 2y - x | M_T \geq y)\mathbb{P}(M_T \geq y) \\
&= \mathbb{P}(M_T \geq y | W_T \geq 2y - x)\mathbb{P}(W_T \geq 2y - x) \\
&= \mathbb{P}(W_T \geq 2y - x)
\end{aligned}$$

- Let an event A be
$$A = \{Z_t \leq x, M_t^Z \geq y\}$$
and let $Z_t = \sigma B_t$, $B_t = \mu t + W_t$ and $\mu = \frac{\nu}{\sigma}$.
- The probability of event A can be computed as
$$\begin{aligned}
\mathbb{P}(A) &= E^{\mathbb{P}}[1_A] \\
&= E^{\mathbb{P}}\left[1_{\{Z_t \leq x, M_t^Z \geq y\}}\right] \\
&= E^{\mathbb{Q}}\left[1_{\{Z_t \leq x, M_t^Z \geq y\}} \frac{d\mathbb{P}}{d\mathbb{Q}}\right]
\end{aligned}$$
- Under the $\mathbb{Q}$ measure, B_t is Brownian motion with the Radon-Nikodym derivative
$$\frac{d\mathbb{P}_t}{d\mathbb{Q}_t} = e^{\mu B_t - \frac{1}{2}\mu^2 t}$$
- Substitute the term and we have
$$\begin{aligned}
&= E^{\mathbb{Q}}\left[1_{\{Z_t \leq x, M_t^Z \geq y\}} e^{\mu B_t - \frac{1}{2}\mu^2 t}\right] \\
&= E^{\mathbb{Q}}\left[1_{\{Z_t \leq x, M_t^Z \geq y\}} e^{\frac{\mu}{\sigma} Z_t - \frac{1}{2}\mu^2 t}\right]
\end{aligned}$$

- Under the measure $\mathbb{Q}$ and using the reflection principle, we can replace the joint distribution with a marginal distribution, and replace Z_t with $2y - Z_t$ [2]

$$
\begin{aligned}
&= E^{\mathbb{Q}}\left[1_{\{Z_t \geq 2y-x\}} e^{\frac{\mu}{\sigma}(2y-Z_t)-\frac{1}{2}\mu^2 t}\right] \\
&= e^{\frac{2\nu y}{\sigma^2}} E^{\mathbb{Q}}\left[1_{\{Z_t \geq 2y-x\}} e^{-\frac{\mu}{\sigma}Z_t-\frac{1}{2}\mu^2 t}\right] \\
&= e^{\frac{2\nu y}{\sigma^2}} E^{\mathbb{Q}}\left[1_{\{Z_t \geq 2y-x\}} e^{-\mu B_t-\frac{1}{2}\mu^2 t}\right]
\end{aligned}
$$

- We can regard the exponential term as the Radon-Nikodym derivative which changes from measure $\mathbb{S}$ to measure $\mathbb{Q}$

$$
\frac{d\mathbb{S}_t}{d\mathbb{Q}_t} = e^{-\mu B_t - \frac{1}{2}\mu^2 t}
$$

$X_t = \mu t + B_t$ is a $\mathbb{S}$-Brownian motion.

---

[2] This is because if Z_t touches the level y anywhere then the terminal distribution of $2y - Z_t$ is equal to that of Z_t.

- We have

$$
\begin{aligned}
\mathbb{P}(Z_t \leq x, M_t^Z \geq y) &= e^{\frac{2\nu y}{\sigma^2}} E^{\mathbb{Q}}\left[1_{\{Z_t \geq 2y - x\}} e^{-\mu B_t - \frac{1}{2}\mu^2 t}\right] \\
&= e^{\frac{2\nu y}{\sigma^2}} E^{\mathbb{S}}\left[1_{\{Z_t \geq 2y - x\}}\right] \\
&= e^{\frac{2\nu y}{\sigma^2}} E^{\mathbb{S}}\left[1_{\{\sigma B_t \geq 2y - x\}}\right] \\
&= e^{\frac{2\nu y}{\sigma^2}} E^{\mathbb{S}}\left[1_{\{\sigma X_t - \nu t \geq 2y - x\}}\right] \\
&= e^{\frac{2\nu y}{\sigma^2}} E^{\mathbb{S}}\left[1_{\{\sigma X_t \geq 2y - x + \nu t\}}\right] \\
&= e^{\frac{2\nu y}{\sigma^2}} \left(1 - \Phi\left(\frac{2y - x + \nu t}{\sigma\sqrt{t}}\right)\right) \\
&= e^{\frac{2\nu y}{\sigma^2}} \Phi\left(\frac{-2y + x - \nu t}{\sigma\sqrt{t}}\right)
\end{aligned}
$$

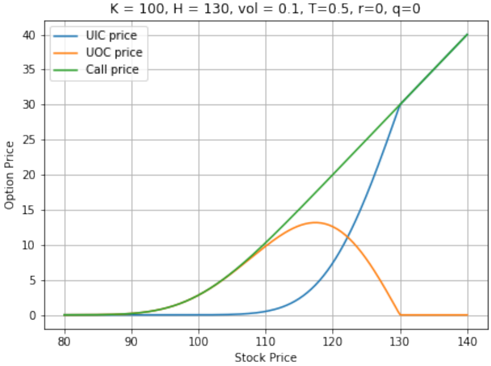
- Collecting all the terms and finally we have

$$
\begin{aligned}
& \mathbb{P}(Z_T \geq x, M_T^Z \geq y) \\
= \; & \mathbb{P}(M_T^Z \geq y) - \mathbb{P}(Z_T \leq x, M_T^Z \geq y) \\
= \; & e^{\frac{2\nu y}{\sigma^2}} \Phi\left(\frac{-y - \nu T}{\sigma\sqrt{T}}\right) + \Phi\left(\frac{-y + \nu T}{\sigma\sqrt{T}}\right) - e^{\frac{2\nu y}{\sigma^2}} \Phi\left(\frac{-2y + x - \nu T}{\sigma\sqrt{T}}\right)
\end{aligned}
$$

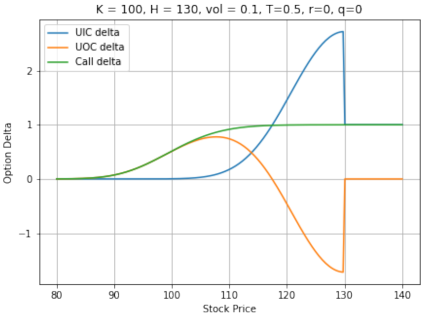
- The proof is done.

Risk Analysis of Up-and-Out Option

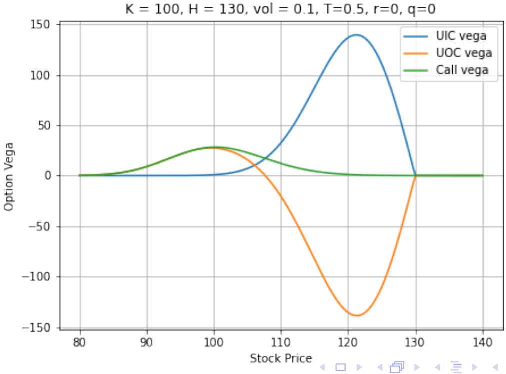
- Let's consider the case where strike $K = 100$, upside barrier $H = 130$, $T = 0.5$, $r = 0$, $q = 0$.
- The up-and-out call (UOC) price is not a monotonic increasing function of S_0. This is because there are two countering effects on the UOC price when S_0 increases. (1) the value of the payoff $(S_T - K)^+$ increases, (2) the probability of getting knocked out at H also increases.
- On the other hand, up-and-in call (UIC) price always increases as S_0 increases.



- UOC delta changes sign somewhere between the strike K and the upper barrier H. The intuition is that when H is far away from S_0, the probability of getting knocked out is small. Hence the increase in S_0 contributes mostly to increasing the value of the payoff $(S_T - K)^+$, just like a standard European call.
- When S_0 is close to the barrier, the probability of being knocked out is large. Also, this is the region where the payoff $(S_T - K)^+$ is the largest. In an extreme case, where S_0 is just slightly below H, any increases in S_0 would make the UOC value to be 0. This is the reason why we have a large negative delta when S_0 is close to H.
- Another interesting observation is that UIC has delta large than 1. The intuition is similar to UOC. In an extreme case, where S_0 is just slightly below H, any increases in S_0 would make the UIC value to be a deep ITM European call.

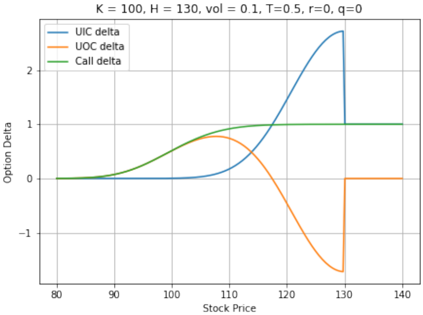


- Vega of European option and UIC is always positive. UOC vega changes sign somewhere between K and H.
- When S_0 is far away from H, European Call and UOC behaves similarly. However, when S_0 approaches H, the vega of UOC becomes negative. This means the value of UOC decreases as volatility increases.
- Consider an extreme cases that we have zero volatility and S_0 is just slightly below H. This means the UOC is deep ITM. Zero volatility means the underlying spot would move along the forward curve. Assume rate and dividend yield are 0, forward curve is equal to S_0. This means the UOC would not be knocked out before the maturity. Any increase in volatility would certainly decrease the value of the UOC as there is no more up side in terms of the payoff but only increases the probability of being knocked out.



Barrier over-hedging for UOC

- The fair value that comes out of the Black Scholes model is valid only in the circumstances that the hedging strategy specified by the model is employed. It assumes the spot price moves continuously and we can delta hedge accordingly. In practice, we can never trade continuously but as long as the delta does not gap, we can maintain the portfolio delta neutral. For barrier option, this is not the case.
- In the previous slides, we showed how the delta profile of UOC can show discontinuities across the barrier level. The delta profile in the diagram below is from **the buyer's perspective**.



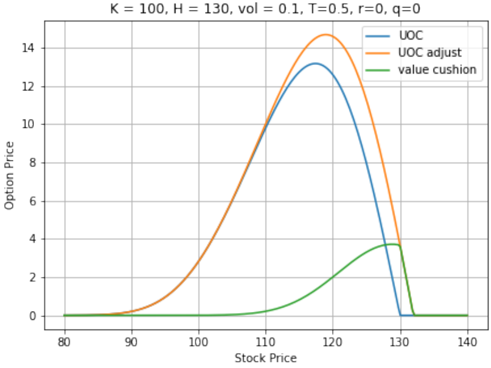
- The argument in this section is from **the market maker's perspective** as most of the buyers of UOC do not delta hedge. Assume the market maker sold the UOC and delta hedge. The current spot is 120. This means the short position of UOC has delta about $+0.5$. The market maker needs to sell 0.5 of the notional of the stock to maintain delta neutral.
- Then, the spot moves higher to 129, delta of short UOC is about $+1.7$. The market maker needs to sell an extra 1.2 of the stock.
- Finally, the spot breaches the barrier $H = 130$ and moves to 132. The UOC is knocked out. The market maker earns the fully premium and doesn't need to pay the buyer anything. However, he has the delta position to unwind: he needs to buy back -1.7 of stock that he sold. If the spot moves slowly towards 130 and he can manage to close his delta position at 130. It would be perfect. But since the spot moves fast from 129 to 132. He can only buy back at 132 which means he lost an extra \$2 per stock that he sold.
- It is therefore clear that, when we are making the price, we must apply some method to account for such an unfavourable scenario. These types of methods for achieving this go under the name of **barrier over-hedging**.

- Let's suppose that we need to make an offer price for the above UOC. As a first step towards over-hedging, a typical technique is to value the contract with an **adjusted barrier level**, chosen so that the contract is worth more to the buyer than the contract with the true barrier level.
- This means we would adjust the barrier higher (say, from 130 to 132) and price the option using the same market data and model. Let's denote the price of UOC with barrier H as $UOC(H)$. We have

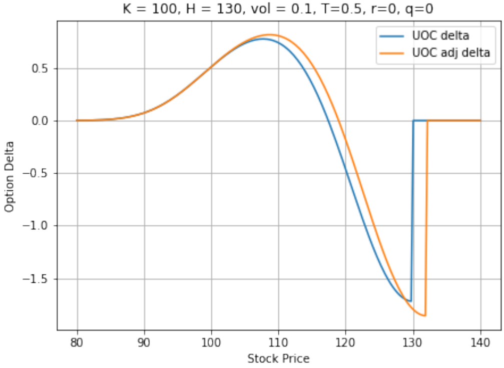
  $$UOC(130) < UOC(132)$$

  The adjustment would be based on the type of the underlying and how aggressive the firm is. Slower the market and more aggressive the firm is, the smaller the barrier adjustment.
- Once we have entered a trade that has been priced in this way, we do two things: (1) risk manage the adjusted barrier option, and (2) recalculate the valuation cushion.
- Operationally, this is typically achieved by having one trading book which contains the adjusted-barrier options, and another trading book which contains the true (client-confirmed) counterparts, back-to-back with internally booked reverse positions in the adjusted-barrier options. The former book is the main one that is risk-managed. The latter book is not risk-managed, but its value is monitored, consisting entirely as it does of all the value cushions.

- The diagram below shows the prices of UOC(130) and UOC(132). The value cushion is the difference between them.
- Note that the value cushion is not static but is a function of time, spot and volatility.



- The diagram shows the delta of UOC(130) and UOC(132). The major benefit of risk managing using the delta of UOC(132) is that the delta gapping at 130 (which is the true contract) disappears, or I should say, shifts to 132 instead. Also, since we make the offer price using UOC(132), the delta of UOC(132) is a more appropriate choice of hedging.
- An astute reader may ask, this would not mitigate the issue if the spot is moved from 129 to 140. That's why the adjustment is a function of how fast the market is. If the market is volatile, we might need to adjust the barrier to 140. But this would make our offer price too high and may not close the deal.



Discretely Monitored Barrier Options

- An alternative to the continuous barrier is a discretely monitored barrier for which the spot level is measured only at intervals (say once per day). The price is higher than a contract with continuous barrier at the same level since it is possible for the spot to breach the barrier for a short period in between monitoring instants
- Broadie, Glasserman, and Kou (1997) provide a simple but high quality approximation for the price by moving the barrier away a little to account for this effect, and then pricing as a continuous barrier.
- The correction shifts the barrier away by a factor of $\exp(\beta\sigma\sqrt{\delta t})$, where $\beta = 0.5826$, σ is the volatility and δt is the time between monitoring instants.

Summary

- In-out parity.
- Reflection principle to compute the joint distribution for Brownian motion with no drift and its minimum.
- Girsanov's theorem to compute the joint distribution for Brownian motion with drift and its minimum.
- Down-and-in Call. Down-and-out Call.
- Down side one touch. Down side no touch.
- Up side on touch. Up side no touch.
- Up-and-in Call. Up-and-out Call.
- Barrier over-hedging.
- Discretely monitor barrier option.