QF602 - Homework 5

Question 1

- Assume the spot prices follows a lognormal process with risk free rate r, dividend yield q and volatility σ:

  $$\frac{dS_t}{S_t} = (r-q)dt + \sigma dW_t$$

  under the risk neutral measure. Derive the formula for up-and-in put option which has the payoff at maturity T:

  $$(K - S_T)^+ 1_{M_T^S \geq H}$$

  where K is the strike and $H > S_0$ is the upper barrier. M_T^S is the maximum of S_t between $[0,T]$.

  You need to show all the steps clearly to get the full mark.

  Answer.

  Step 1. Find $\mathbb{P}(W_T \leq x, M_T \geq y)$ for $x \leq y$, $y > 0$. Let the event that W_t touches y between $[0,T]$ be $U := \{W_t = y, t \in [0,T]\}$

  $$\begin{aligned}
  \mathbb{P}(W_T \leq x, M_T \geq y) &= \mathbb{P}(W_T \leq x, U) \\
  &= \mathbb{P}(W_T \leq x | U)\mathbb{P}(U) \\
  &= \mathbb{P}(W_T \geq 2y - x | U)\mathbb{P}(U) \\
  &= \underbrace{\mathbb{P}(U | W_T \geq 2y - x)}_{1}\mathbb{P}(W_T \geq 2y - x) \\
  &= \mathbb{P}(W_T \geq 2y - x) \\
  &= 1 - \Phi(2y - x)
  \end{aligned}$$

  Step 2. Find $\mathbb{P}(Z_T \leq x, M_T^Z \geq y)$, where $Z_t = \nu t + \sigma W_t$.

  We let $Z_t = \sigma B_t$, $B_t = \mu t + W_t$, $\mu = \nu/\sigma$ and event A be

  $$A = \{Z_T \leq x, M_T^Z \geq y\}$$

where W_t is a $\mathbb{P}$-Brownian motion, B_t is a $\mathbb{Q}$-Brownian motion.

$$\begin{aligned}
\mathbb{P}(Z_T \leq x, M_T^Z \geq y) &= E^{\mathbb{P}}[1_A] \\
&= E^{\mathbb{Q}}\left[1_A \frac{d\mathbb{P}_T}{d\mathbb{Q}_T}\right] \\
&= E^{\mathbb{Q}}\left[1_A e^{\mu B_T - \frac{1}{2}\mu^2 T}\right] \\
&= E^{\mathbb{Q}}\left[1_A e^{\frac{\mu Z_T}{\sigma} - \frac{1}{2}\mu^2 T}\right] \\
&= E^{\mathbb{Q}}\left[1_{\{Z_T \geq 2y-x\}} e^{\frac{\mu(2y-Z_T)}{\sigma} - \frac{1}{2}\mu^2 T}\right] \\
&= e^{\frac{2\nu y}{\sigma^2}} E^{\mathbb{Q}}\left[1_{\{Z_T \geq 2y-x\}} e^{-\mu B_T - \frac{1}{2}\mu^2 T}\right]
\end{aligned}$$

We can regard the exponential term as Radon-Nikodyn derivative

$$\frac{d\mathbb{S}_T}{d\mathbb{Q}_T} = e^{-\mu B_T - \frac{1}{2}\mu^2 T}$$

and $X_t = \mu t + B_t$ is a $\mathbb{S}$-Brownian motion.

$$\begin{aligned}
&= e^{\frac{2\nu y}{\sigma^2}} E^{\mathbb{S}}\left[1_{\{Z_T \geq 2y-x\}}\right] \\
&= e^{\frac{2\nu y}{\sigma^2}} E^{\mathbb{S}}\left[1_{\{\sigma B_T \geq 2y-x\}}\right] \\
&= e^{\frac{2\nu y}{\sigma^2}} E^{\mathbb{S}}\left[1_{\{\sigma X_T - \nu T \geq 2y-x\}}\right] \\
&= e^{\frac{2\nu y}{\sigma^2}} E^{\mathbb{S}}\left[1_{\{\sigma X_T \geq 2y-x+\nu T\}}\right]
\end{aligned}$$

Finally we have

$$\mathbb{P}(Z_T \leq x, M_T^Z \geq y) = e^{\frac{2\nu y}{\sigma^2}}\left(1 - \Phi\left(\frac{2y - x + \nu T}{\sigma\sqrt{T}}\right)\right)$$

Step 3. Compute the price of up-and-in put using the joint distribution.

The price of the up-and-in put can computed as under a measure induced by the numeraire N_t:

$$\begin{aligned}
V_0 &= N_0 E_0\left[\frac{(K - S_T)^+}{N_T} 1_{M_T^S \geq H}\right] \\
&= \underbrace{N_0 K E_0\left[\frac{1}{N_T} 1_{(M_T^S \geq H, S_T \leq K)}\right]}_{V_0^1} - \underbrace{N_0 E_0\left[\frac{S_T}{N_T} 1_{(M_T^S \geq H, S_T \leq K)}\right]}_{V_0^2}
\end{aligned}$$

For V_0^1, we choose $N_t = e^{rt}$ and we have

$$e^{-rT} K E_0\left[1_{(M_T^S \geq H, S_T \leq K)}\right] = e^{-rT} K \mathbb{P}\left(M_T^S \geq H, S_T \leq K\right)$$

We set $Z_t = \ln(S_t/S_0)$, $x = \ln(K/S_0)$, $y = \ln(H/S_0)$. In the risk neutral measure, $\nu = \mu - \frac{1}{2}\sigma^2$, $\mu = r - q$. We have

$$\begin{aligned}
&= e^{-rT} K \mathbb{P}\left(M_T^Z \geq y, Z_T \leq x\right) \\
&= e^{-rT} K e^{\frac{2\nu y}{\sigma^2}} \left(1 - \Phi\left(\frac{2y - x + \nu T}{\sigma\sqrt{T}}\right)\right) \\
&= e^{-rT} K \left(\frac{H}{S_0}\right)^{\frac{2\mu}{\sigma^2}-1} \left(1 - \Phi\left(\frac{\ln(\frac{H^2}{S_0 K}) + (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)\right) \\
&= e^{-rT} K \left(\frac{H}{S_0}\right)^{\frac{2\mu}{\sigma^2}-1} \Phi\left(\frac{\ln(\frac{S_0 K}{H^2}) - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)
\end{aligned}$$

For V_0^2, we choose $N_t = Se^{qt}$ and we have

$$S_0 e^{-qT} E_0\left[1_{(M_T^S \geq H, S_T \leq K)}\right] = S_0 e^{-qT} \mathbb{P}\left(M_T^S \geq H, S_T \leq K\right)$$

In the stock measure, $\nu = \mu + \frac{1}{2}\sigma^2$, we have

$$\begin{aligned}
&= S_0 e^{-qT} \mathbb{P}\left(M_T^S \geq H, S_T \leq K\right) \\
&= S_0 e^{-qT} e^{\frac{2\nu y}{\sigma^2}} \left(1 - \Phi\left(\frac{2y - x + \nu T}{\sigma\sqrt{T}}\right)\right) \\
&= S_0 e^{-qT} \left(\frac{H}{S_0}\right)^{\frac{2\mu}{\sigma^2}+1} \left(1 - \Phi\left(\frac{\ln(\frac{H^2}{S_0 K}) + (\mu + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)\right) \\
&= S_0 e^{-qT} \left(\frac{H}{S_0}\right)^{\frac{2\mu}{\sigma^2}+1} \Phi\left(\frac{\ln(\frac{S_0 K}{H^2}) - (\mu + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)
\end{aligned}$$

Put all together, the price of up-and-in put is

$$e^{-rT} K \left(\frac{H}{S_0}\right)^{\frac{2\mu}{\sigma^2}-1} \Phi\left(\frac{\ln(\frac{S_0 K}{H^2}) - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) - S_0 e^{-qT} \left(\frac{H}{S_0}\right)^{\frac{2\mu}{\sigma^2}+1} \Phi\left(\frac{\ln(\frac{S_0 K}{H^2}) - (\mu + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)$$