QF602 Homework

XueYuanhuang

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Assignment 5: Due 1/3/24

Problem 1.

Step1:

$P(W_T \le x, M_T \ge y)$ for $x \le y$, $y > 0 \quad U = \{W_t = y, t \in [0,T]\}$

$= P(W_T \le x, U)$

$= P(W_T \le x|U)P(U)$

$= P(W_T \ge 2y - x|U)P(U)$

$= P(U|W_T \ge 2y - x)P(W_T \ge 2y - x)$

$= P(W_T \ge 2y - x)$

$= 1 - \Phi(\frac{2y-x}{\sqrt{T}})$

Step2:

$P(Z_T \le x, M_T^Z \ge y)$, where $Z_t = vt + \sigma W_t$.

Let $Z_t = \sigma B_t$, $B_t = \mu t + W_t$, $\mu = v/\sigma$ and event A $= \{Z_T \le x, M_T^Z \ge y\}$

Where W_t is a P Brownian motion, B_t is a Q Brownian motion.

$P(Z_T \le x, M_T^Z \ge y)$

$= E^P[1_A]$

$= E^Q[1_A \frac{dP}{dQ}]$

$= E^Q[1_A e^{\mu B_T - \frac{1}{2}\mu^2 T}]$

$= E^Q[1_A e^{\frac{\mu Z_T}{\sigma} - \frac{1}{2}\mu^2 T}]$

$= E^Q[1_{\{Z_T \geq 2y-x\}} e^{\frac{\mu(2y-Z_T)}{\sigma} - \frac{1}{2}\mu^2 T}]$

$= e^{\frac{2vy}{\sigma^2}} E^Q[1_{\{Z_T \geq 2y-x\}} e^{-\mu B_T - \frac{1}{2}\mu^2 T}]$

$\frac{dS_T}{dQ_T} = e^{-\mu B_T - \frac{1}{2}\mu^2 T}$, where $X_t = \mu t + B_t$ under S Brownian motion

$= e^{\frac{2vy}{\sigma^2}} E^S[1_{\{Z_T \geq 2y-x\}}]$

$= e^{\frac{2vy}{\sigma^2}} E^S[1_{\{\sigma B_T \geq 2y-x\}}]$

$= e^{\frac{2vy}{\sigma^2}} E^S[1_{\{\sigma X_T \geq 2y-x+vT\}}]$

$= e^{\frac{2vy}{\sigma^2}} [1 - \Phi(\frac{2y-x+vT}{\sigma\sqrt{T}})]$

Step3:

$V_0 = N_0 E^*[\frac{(K-S_T)^+}{N_T} 1_{M_T^S \geq H}]$

$= N_0 K E^*[\frac{1}{N_T} 1_{M_T^S \geq H}] - N_0 E^*[\frac{S_T}{N_T} 1_{M_T^S \geq H}]$

$N_0 K E^*[\frac{1}{N_T} 1_{M_T^S \geq H}]$, where $N_t = e^{rt}$,

$= e^{-rT} K E^*[1_{M_T^S \geq H}] = e^{-rT} K P(1_{M_T^S \geq H})$

Let $Z_t = ln\frac{S_t}{S_0}$, $x = ln\frac{K}{S_0}$, $y = ln\frac{H}{S_0}$, $v = \mu - \frac{1}{2}\sigma^2$, $\mu = r - q$

$= e^{-rT} K P(\mu_T^Z \geq y, Z_T \leq x)$

$= e^{-rT} K e^{\frac{2vy}{\sigma^2}} (1 - \Phi(\frac{2y-x+vT}{\sigma\sqrt{T}}))$

$= e^{-rT} K (\frac{H}{S_0})^{\frac{2\mu}{\sigma^2}-1} \Phi \frac{ln(\frac{S_0 K}{H^2} - (\mu - \frac{1}{2}\sigma^2)T)}{\sigma\sqrt{T}})$

$N_0 E^*[\frac{S_T}{N_T} 1_{M_T^S \geq H}]$, where $N_t = S_t e^{qt}$ under stock measure,

$$= S_0 e^{-qT} E^S[1_{M_T^S \geq H, S_T \leq K}] = S_0 e^{-qT} P(M_T^S \geq H, S_T \leq K)$$

Let $v = \mu + \frac{1}{2}\sigma^2$

$$= S_0 e^{-qT} (\frac{H}{S_0})^{\frac{2\mu}{\sigma^2}+1} \Phi(\frac{ln\frac{S_0 K}{H^2} - (\mu + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}})$$

Putting them together,

$$e^{-rT} K (\frac{H}{S_0})^{\frac{2\mu}{\sigma^2}-1} \Phi(\frac{ln\frac{S_0 K}{H^2} - (\mu - \frac{1}{2}\sigma^2)T)}{\sigma\sqrt{T}}) - S_0 e^{-qT} (\frac{H}{S_0})^{\frac{2\mu}{\sigma^2}+1} \Phi(\frac{ln\frac{S_0 K}{H^2} - (\mu + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}})$$