QF602 Derivatives

Lecture 7 - Smile Models 1

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The smile problem

Over the past three decades, quants have employed three broad strategies in an attempt to produce models whose BS implied volatilities are consistent with the smile.

1. The first strategy is to move away from traditional geometric Brownian motion for the evolution of the underlying asset.
2. The second directly models the movements of the Black Scholes implied volatility surface rather than the underlying asset.
3. The third, more pragmatically, avoids formal models of either the underlying asset or the implied volatility, and instead tries to construct heuristics for pricing and hedging.

The First Approach

- The first approach is the most fundamental, but also the most ambitious. It attempts to explicitly model the **stochastic evolution of the stock price** S via a more general process than geometric Brownian motion.
- The advantage of this approach is that arbitrage violations are more easily avoided, but finding a stochastic process that accurately describes the evolution of a stock price turns out to be very difficult. Such attempts typically involve more complex stochastic differential equations with additional stochastic variables, such as **realized volatility** or stock price **jumps**.
- The difficulty is to find the right complexity depending on the problem that we are trying to solve.
- This lecture focuses in the first approach.

The Second Approach

- The second approach directly models **the behavior of the Black Scholes implied volatility**.
- Traders automatically think about options in terms of implied volatility, which they observe every day as they make markets. For them it is natural to describe the dynamics of it. Implied volatility statistics are easily obtained and can be used to calibrate a model without too much difficulty, but this approach has its own problems.
- First, one has to be very careful in modeling the stochastic evolution of implied volatility directly, because changing implied volatility changes all option prices, and it is difficult to avoid violating the constraints imposed by the principle of no riskless arbitrage.

- Second, one must not forget the fact that implied volatility is a parameter of the Black Scholes model itself, which fails to describe option values correctly, and that we are therefore trying, perhaps illogically, to model the parameter of an inaccurate model.
- One of the examples of this approach is, *Consistent Variance Curve Models by Hans Buehler, Finance Stochastics, Vol. 10, No. 2, 2006*.

The Third Approach

- The third approach, which avoids formal models, is extremely flexible. Practitioners may value this flexibility, but without a solid theoretical foundation it becomes difficult to avoid inconsistencies that lead to arbitrage opportunities. A well-known and widely used example of this approach is the so-called vanna-volga model.
- See, *Vanna-Volga Methods Applied to FX Derivatives: From Theory to Market Practice, by Bossens et al. 2010*

The Road Map

- Bachelier and normal implied vol
- Shifted lognormal model
- Poisson process and Merton's jump diffusion
- Fourier method as a general pricing method for European option
- Heston stochastic volatility model
- SABR (Stochastic Alpha Beta Rho)

Bachelier model

- Bachelier model assumes the process S_t follows a normal SDE:

$$dS_t = \mu dt + \sigma^N dW_t$$

where μ is the drift and σ^N is the so-called normal volatility. The solution of the SDE is given as:

$$\begin{aligned} S_T &= S_0 + \int_0^T \mu dt + \int_0^T \sigma^N dW_t \\ &= S_0 + \mu T + \sigma^N \sqrt{T} x \end{aligned}$$

where x is a standard normal r.v. It turns out that there is a closed-form solution of the call option under the Bachelier model:

$$E_0\left[(S_T - K)^+\right] = (S_0 + \mu T - K)\Phi(d) + \sigma^N \sqrt{T}\phi(d)$$

where $\phi(.)$ is the probability density function of the standard normal distribution.

Proof of the Bachelier call option formula

- Let $d = \frac{S_0+\mu T-K}{\sigma^N\sqrt{T}}$

$$
\begin{aligned}
& E_0\left[(S_T-K)^+\right] \\
= & \int_{-\infty}^{\infty}\left(S_0+\mu T+\sigma^N\sqrt{T}x-K\right)^+\phi(x)dx \\
= & \int_{-d}^{\infty}\left(S_0+\mu T+\sigma^N\sqrt{T}x-K\right)\phi(x)dx \\
= & \left(S_0+\mu T-K\right)\Phi(d)+\sigma^N\sqrt{T}\int_{-d}^{\infty}x\phi(x)dx \\
= & \left(S_0+\mu T-K\right)\Phi(d)+\sigma^N\sqrt{T}\phi(d)
\end{aligned}
$$

- The integral in red can be computed as the following. Recall $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ and let $y = x^2$, therefore $dy = 2xdx$.

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}} \int_a^{\infty} x e^{-x^2/2} dx &= \frac{1}{\sqrt{2\pi}} \int_{\sqrt{a}}^{\infty} \frac{1}{2} e^{-y/2} dy \\
&= \frac{1}{\sqrt{2\pi}} \left[-e^{-y/2} \right]_{\sqrt{a}}^{\infty} \\
&= \frac{1}{\sqrt{2\pi}} \left[e^{-y/2} \right]_{\infty}^{\sqrt{a}} \\
&= \frac{1}{\sqrt{2\pi}} \left[e^{-x^2/2} \right]_{\infty}^{a} \\
&= \frac{1}{\sqrt{2\pi}} e^{-a^2/2} \\
&= \phi(a)
\end{aligned}$$

- Notice that ϕ is an even function, therefore $\phi(a) = \phi(-a)$.

Bachelier model and interest rate option

- Ever since interest rate became negative after the GFC (to be precise, forward interest rate), the Bachelier model becomes the standard for quoting interest rate option for G10 currencies. Some of the EM rates option are still quoted using Black implied vol but they are relatively small and less liquid market.
- Under the Bachelier model, the forward LIBOR F follows a Brownian motion with Bachelier implied volatility (aka Normal implied volatility) σ^N in the T_2 forward measure

$$dF_t = \sigma^N dW_t^{T_2}$$

- Forward LIBOR rate is defined as, for $t \leq T_1$

$$F_t \equiv F_t(T_1, T_2) = \frac{1}{T_2 - T_1}\left(\frac{Z_t(T_1) - Z_t(T_2)}{Z_t(T_2)}\right)$$

- The solution of the SDE can be computed as

$$F_T = F_0 + \sigma^N \sqrt{T}x, \quad x \sim \Phi(0, 1)$$

- The closed form formula for **undiscounted** call and put options in the Bachelier model:

$$BachCall = (F_0 - K)\Phi(d) + \sigma^N \sqrt{T}\phi(d)$$

$$BachPut = (K - F_0)\Phi(-d) + \sigma^N \sqrt{T}\phi(-d)$$

$$d = \frac{F_0 - K}{\sigma^N \sqrt{T}}$$

- Call option on forward LIBOR is called caplet. Put option on forward LIBOR is called floorlet.
- Interest rate option are quoted using absolute moneyness relative to the forward, i.e. the moneyness is defined as $K - F_0$. The unit is number of basis points.

- If we were to force to quote interest rate option using Black implied vol. We will observe the volatility smile is skewed.
- This can be partially understood by the tendency of interest rates to move normally rather than lognormally as rates get low. But why does it translate into Black volatility skew?
- Suppose that an interest rates evolves under Bachelier model. If you insist on viewing this as a Black model (i.e. plotting the Black implied vol vs moneyness), then you must write

$$dF_t = \sigma^N dW_t^{T_2} \equiv (\sigma F_t) dW_t^{T_2}$$

  where $\sigma = \sigma^N / F_t$ is the Black implied volatility and is convex in F_t.
- Assume σ^N is constant (i.e. forward LIBOR is truly normal), as F_t gets smaller, σ must increases to keep the ratio constant. This is the reason why Bachelier model would generate a negatively sloped Black implied vol skew.

Bachelier vs Black: Lognormal Implied Volatility

- Consider the case that, for option maturity $T = 1m$, an ATM call option on 3m LIBOR is traded in the market at price P.
- The forward is 2%, the strike K is 2%.
- Assuming one trader uses Black model, the other uses Bachelier model.
- Since there is only one option traded in the market, they can only calibrate to that price.
- If they completely trust their models and a client ask them to quote the options with strikes at 1.5% and 2.5%. How much will they quote?

- Let's denote the **undiscounted** Black call option as

$$BlackCall := F_0 N(d_1) - KN(d_2).$$

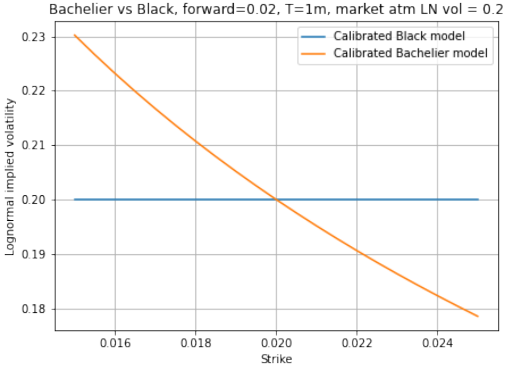
- For a trader who uses Black model, he calibrates his model and get a Black implied vol, aka lognormal implied vol or just lognormal vol.

$$\sigma \equiv \sigma^{LN} = BlackCall^{-1}(F_0, K, T = 1m, Price = P).$$

- For a trader who uses a Bachelier model, he calibrates his model and get a Bachelier implied vol, aka normal vol.

$$\sigma^N = BachCall^{-1}(F_0, K, T = 1m, Price = P).$$

- Assume the market is quoting using lognormal vol and the calibrated σ^{LN} is 20%. Since the Black model assumes constant lognormal vol for all strikes, if we use the calibrated Black model to price options for all other strikes, the corresponding implied vols will be flat at 20%.
- For the trader who uses Bachelier model, since he is calibrated to the ATM option price, his model must produce the same lognormal implied vol for ATM. That's where the blue line crosses the orange line.
- But for non-ATM options, the two models will produce different option prices.



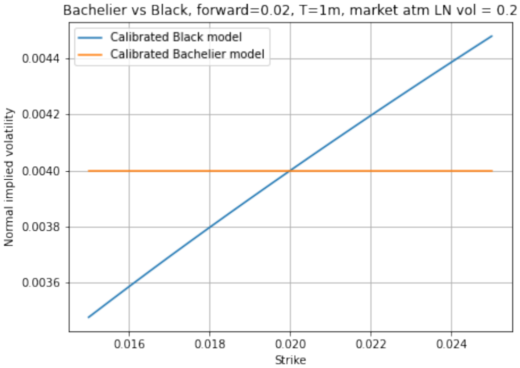
- For options with strike lower than the forward, the Bachelier model gives higher prices than Black.
- For options with strike higher than the forward, the Bachelier model gives lower prices than Black.

How to generate the Lognormal Ivol graphs

1. Calibrate a model (Black or Bachelier) to the market option price.
2. Use the calibrated model to price options with various strikes.
3. Convert those model generated option prices to **Lognormal** implied vols.

Bachelier vs Black: Normal Implied Volatility

- Lognormal implied volatility is not the only way to quote option prices. In fact, the default quoting method for G7 interest rate option is using normal implied volatility.
- The diagram below shows the same information as the previous diagram but in **normal** ivol instead of lognormal ivol.



How to generate the Normal Ivol graphs

1. Calibrate a model (Black or Bachelier) to the market option price.
2. Use the calibrated model to price options with various strikes.
3. Convert those model generated option prices to **Normal** implied vols.

Shifted Lognormal Model

- In 1983, Mark Rubinstein proposed the following SDE for option pricing

$$dF_t = \sigma^{SLN}(\alpha)(F_t + \alpha)dW_t$$

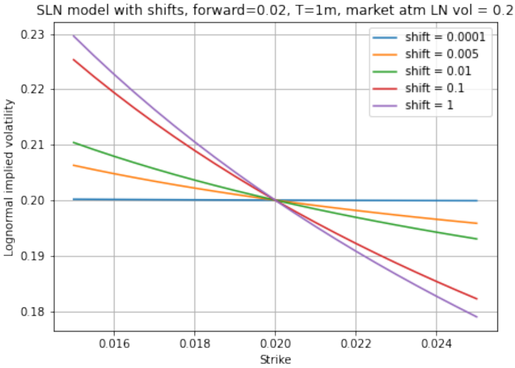
- Also known as displaced diffusion model.
- $\sigma^{SLN}(\alpha)$ is called the shifted lognormal vol with the shift α.
- When α is small, the SDE behaves like a Black model.
- When α is large, the SDE behaves like a Bachelier model.
- The call option price with strike K under shifted lognormal model can be formulated as a modified Black model

$$SLNCall(F_0, K, \sigma^{SLN}(\alpha), T) = BlackCall(F_0+\alpha, K+\alpha, \sigma^{SLN}(\alpha), T)$$

- The domain of the underlying in SLN is $(-\alpha, \infty)$. Given this model is highly tractable, if the underlying is not restricted to be positive, this is usually the first thing that a quant would try to extend beyond Black and Bachelier model.

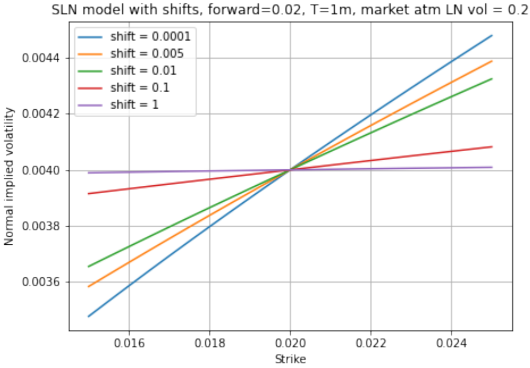
Shifted lognormal model - Lognormal Vol

- The diagram shows the lognormal vol skews that are generated with various shift sizes. We can see small shift generates a flat skew which corresponds to a Black model. The bigger the shift size, the closer to Bachelier model.



Shifted lognormal model - Normal Vol

- The diagram shows the normal vol skews with various shift size. The biggest shift size generates an almost flat normal skew which corresponds to the Bachelier model.



Merton's Jump Diffusion Model

- Merton (1976) introduced and analyzed one of the first models with both jump and diffusion terms for the pricing of derivative. It can be specified through the SDE in the risk-neutral measure

$$\frac{dS_t}{S_{t-}} = (r - q - \lambda m)dt + \sigma dW_t + dJ_t$$

- where S_{t-} stands for the left limit of S at time t, r is a risk-free rate, q is a divdend yield, σ is a diffusion volatility, W_t is a standard Brownian motion.
- The **compound Poisson process** J_t is given by

$$J_t = \sum_{i=1}^{N_t} (Y_i - 1)$$

- N_t is a **Poisson process** with constant intensity λ which is independent of W_t.
- Y_i is the amplitude of the multiplicative jump size which is lognormally distributed with $E[\ln Y_i] = a$, $Var[\ln Y_i] = b^2$ and $m = e^{a+b^2/2} - 1$.

Road Map for Merton's Jump Diffusion Model

The MJD model looks quite complicated at first, especially if this is the first time you come across Poisson process. We are going to tackle it piece-by-piece by answering the following questions:

1. How to add jump? **Answer**: Poisson process.
2. How to make a Poisson process martingale? **Answer**: compensated Poisson process.
3. Can we have a distribution for the jump size? **Answer**: compound Poisson process.
4. How to make a compound Poisson process martingale? **Answer**: Compensated compound Poisson process.
5. Can we use any distribution for the jump size? Merton suggested to use lognormal distribution for the multiplicative jump size so that there is a closed form solution for European options.

Poisson process

A stochastic process N_t is said to be a Poisson process having intensity $\lambda > 0$, if

- $N_0 = 0$. It starts counting from 0.
- The process has independent increments.
- The number of events in a time interval Δt is Poisson distributed with mean $\lambda \Delta t$. That is, for all $t \geq 0$

$$Prob\left(N_{t+\Delta t} - N_t = n\right) = e^{-\lambda \Delta t}\frac{(\lambda \Delta t)^n}{n!}, n = 0, 1, ...$$

- The expectation is given as $E[N_{t+\Delta t} - N_t] = \lambda \Delta t$.
- The variance is given as $Var[N_{t+\Delta t} - N_t] = \lambda \Delta t$.
- In our context, $N_{t+\Delta t} - N_t$ represents the number of the jumps happen in the time interval Δt.
- For a given time interval $[0, T]$, the probability that there is **no jump** (i.e. $n = 0$) is $e^{-\lambda T}$.

Exponential Distribution and Jump Arrival Time

- Consider a Poisson process and let's denote the time of the n-th jump by τ_n. The sequence $\{\tau_1, \tau_2, ...\tau_n\}$ is called the sequence of arrival times. For instance, if $\tau_1 = 5$ and $\tau_2 = 10$, then the first jump of the Poisson process would have occurred at time 5 and the second at time 15.
- We shall now determine the distribution of the τ_n. To do so, we first note that the event $\{\tau_1 > t\}$ take place if and only if there is no jumps occur in the interval $[0, t]$ and thus

$$Prob(\tau_1 > t) = Prob(N_t = 0) = e^{-\lambda t}$$

- Therefore, the probability that there is a jump between $[0, t]$ is

$$Prob(\tau_1 \leq t) = 1 - e^{-\lambda t}$$

We call τ_1 has an **exponential distribution** with the parameter λ.

- **Definition.** A continuous random variable τ is said to have an exponential distribution with parameter λ, $\lambda > 0$, if its probability density function is given by

$$\begin{aligned} f(t) &= \lambda e^{-\lambda t}, \; t \geq 0 \\ &= 0, \qquad\quad t < 0 \end{aligned}$$

or equivalently, if its cumulative distribution function is given by

$$\begin{aligned} F(t) = \int_{-\infty}^{t} f(y)dy &= 1 - e^{-\lambda t}, \; t \geq 0 \\ &= 0, \qquad\qquad t < 0 \end{aligned}$$

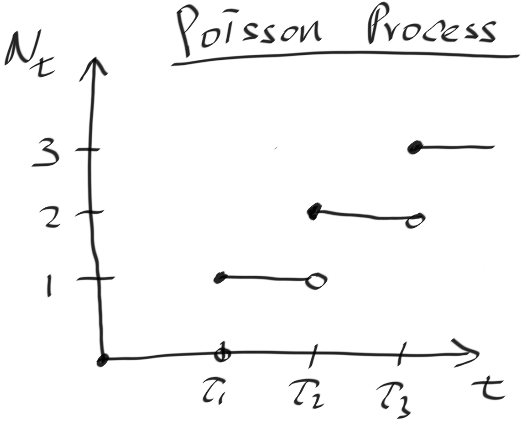
- Since Poisson process has independent increment, the distribution of $\tau_2 > t$ conditions on $\tau_1 = s$, for $t > s$, is

$$\begin{aligned} Prob(\tau_2 > t | \tau_1 = s) &= Prob(\tau_2 > t - s) \\ &= e^{-\lambda(t-s)} \end{aligned}$$

- It basically says at any point in time, the Poisson process probabilistically restarts itself.

A Sample Path of Poisson process

- τ_i are the jump arrival times and exponential distributed with the parameter λ. The jump size is always 1.



Compensated Poisson process

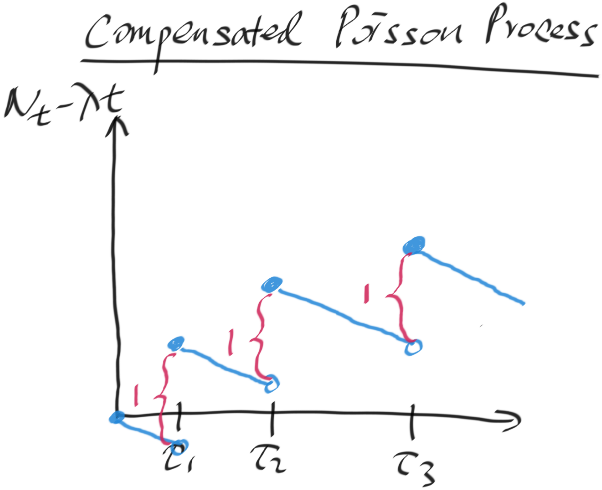
- Poisson process N_t is a non-decreasing process with a non-zero probability of upward jump. It clearly cannot be a martingale.
- We can show the process $N_t - \lambda t$ is a martingale and it is called the compensated Poisson process.
- Let $\bar{N}_T = N_T - \lambda T$. The conditional expectation can be computed as, for $t < T$

$$\begin{aligned} E_t[\bar{N}_T] &= E_t[N_T - \lambda T] \\ &= E_t[N_T - N_t + N_t - \lambda T - \lambda t + \lambda t] \\ &= E_t[N_T - N_t - \lambda T + \lambda t] + \bar{N}_t \\ &= E_t[N_T - N_t - \lambda(T - t)] + \bar{N}_t \\ &= E_t[N_{t+\Delta t} - N_t] - \lambda \Delta t + \bar{N}_t \\ &= \lambda \Delta t - \lambda \Delta t + \bar{N}_t \\ &= \bar{N}_t \end{aligned}$$

where $\Delta t = T - t$.

A Sample Path of Compensated Poisson process

- Compensated Poisson process is nothing more than a Poisson process plus a deterministic drift $-\lambda t$.



Compound Poisson Process

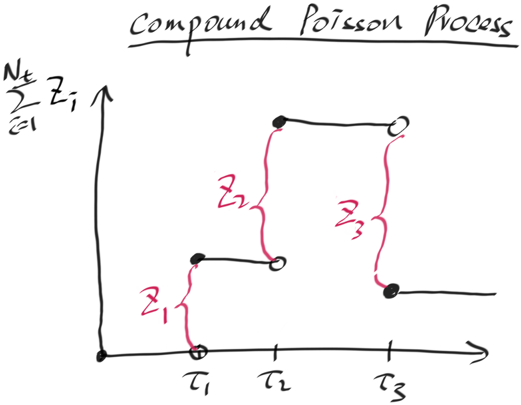
- A stochastic process J_t is said to be a compound Poisson process if it can be represented as

$$J_t = \sum_{i=1}^{N_t} Z_i$$

  where N_t is a Poisson process and Z_i is a family of independent and identically distributed random variables that is also independent of N_t.
- This is a generalization of Poisson process. If $Z_i = 1$, J_t is reduced to N_t.
- In MJD, Z_i is set as $Y_i - 1$ where Y_i is lognormally distributed.

A Sample Path of Compound Poisson process

- A generalization of Poisson process. The jump size is not a constant but instead, is a variable Z_i but not correlated with the jump arrival time τ_i.



Compensated Compound Poisson Process

- In MJD, the compound Poisson process is

$$J_t = \sum_{i=1}^{N_t} (Y_i - 1)$$

  what is the compensator to make J_t to be a martingale?

- A standard property of the Poisson process is that $N_t - \lambda t$ is a martingale, i.e. the compensator is $-\lambda t$. A generalization of this property is that

$$\sum_{i=1}^{N_t} h(Y_i) - \lambda E[h(Y)]t$$

  is a martingale for i.i.d Y, Y_i and any function h for which $E[h(Y)]$ is finite.

- In our case, $h(Y) = Y - 1$, and $E[\ln Y] = a$, $Var[\ln Y] = b^2$. Let $m = e^{a+b^2/2} - 1$. The compensator can be computed as

$$-\lambda E[Y - 1]t = -\lambda(e^{a+b^2/2} - 1)t = -\lambda mt$$

- Therefore, $dJ_t - \lambda mdt$ is a martingale as described in the MJD SDE.

S_t vs S_{t-} and Multiplicative Jump Size

- In the presence of jumps, S_t is potentially ambiguous: if it is possible for S to jump at t, we needs to specify whether S_t means the value of S just before or just after the jump.
- We define S_{t-} to be the value just before a jump (if it happens at t).
- Looking at the MJD SDE, we can see if there is a jump at time τ_i, the process S changes from $S_{\tau_i -}$ to S_{τ_i}. The change is the jump size at τ_i which is given as

$$\begin{aligned} S_{\tau_i} - S_{\tau_i -} = S_{\tau_i -} dJ_t &\equiv S_{\tau_i -}(J_{\tau_i} - J_{\tau_i -}) \\ &= S_{\tau_i -}(Y_i - 1) \end{aligned}$$

  hence

$$S_{\tau_i} = S_{\tau_i -} Y_i$$

- This shows that Y_i are the ratios of the asset price before and after a jump, in other words, the jumps are **multiplicative**. This also explains why we wrote $Y_i - 1$ rather than simply Y_i in the definition of J_t.

Solution of MJD SDE

- Our next task is to solve the MJD SDE in order to compute the closed form solution of European options.
- By restricting Y_i to be positive random variables, we ensure that S_t can never become negative. Since Y_i are independent of W_t, the solution of the MJD SDE is given by

$$S_t = S_0 e^{\left(\mu-\sigma^2/2\right)t+\sigma W_t} \prod_{i=1}^{N_t} Y_i$$

  where $\mu = r - q - \lambda m$.
- Recall that $\ln(Y_j) \sim \Phi(a, b^2)$. This means Y_j is lognormal $\sim LN(a, b^2)$. Products of lognormal variables are lognormal. For any fixed n, we have

$$\prod_{i=1}^{n} Y_i \sim LN(an, b^2 n).$$

  It follows that conditional on $N_t = n$, S_t has the distribution of

$$\begin{aligned} S_0 e^{\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma W_t} \prod_{i=1}^{n} Y_i &\sim S_0 \cdot LN\left(\left(\mu-\sigma^2/2\right)t, \sigma^2 t\right) \cdot LN(an, b^2 n) \\ &= LN\left(lnS_0 + (\mu-\sigma^2/2)t + an, \sigma^2 t + b^2 n\right) \end{aligned}$$

  using the independence of Y_i and W_t.

European Call Option Formula

- The spot is S_0, strike K, maturity T, diffusion vol σ, risk-free rate r, $\ln(Y_i) \sim \Phi(a, b^2)$, jump arrival rate is λ. The price of an European call option under MJD can be expressed as an infinite sum of risk-neutral expectation conditions on $N_T = n$.

$$e^{-rT} E_0\left[(S_T - K)^+\right] = e^{-rT} \sum_{n=0}^{\infty} \underbrace{e^{-\lambda T} \frac{(\lambda T)^n}{n!}}_{Prob(N_T=n)} E_0\left[(S_T - K)^+ | N_T = n\right]$$

- Before we compute the expectation in blue. Let's have a recap on computing an un-discounted call price when the underlying has lognormal distribution with mean $E_0[\ln S_T]$ and variance $Var_0[\ln S_T]$.
- The un-discounted call price can be expressed as:

$$E_0\left[(S_T - K)^+\right] = F\Phi(d_1) - K\Phi(d_2)$$

where

$$F = e^{E_0[\ln S_T] + \frac{1}{2} Var_0[\ln S_T]}$$

$$d_1 = \frac{\ln\left(\frac{F}{K}\right) + \frac{1}{2} Var_0[\ln S_T]}{\sqrt{Var_0[\ln S_T]}}, \quad d_2 = d_1 - \sqrt{Var_0[\ln S_T]}$$

- For the Black Scholes dynamics, $\frac{dS_t}{S_t} = (r-q)dt + \sigma dW_t$, we have

$$E_0[\ln S_T] = \ln S_0 + (r - q - \sigma^2/2)T$$

and variance

$$Var_0[\ln S_T] = \sigma^2 T$$

- Substitute to the formula and we recover the familiar Black Scholes formula for undiscounted call option price

$$F = e^{E_0[\ln S_T] + \frac{1}{2} Var_0[\ln S_T]} = S_0 e^{(r-q)T}$$

$$d_1 = \frac{\ln\left(\frac{F}{K}\right) + \frac{1}{2} Var_0[\ln S_T]}{\sqrt{Var_0[\ln S_T]}} = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - q + \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

- Now we return to compute the expectation in blue.
- Recall that S_T conditions on $N_T = n$ is lognormal with mean

$$E[\ln S_T] = lnS_0 + \left((r - q - \lambda m) - \frac{\sigma^2}{2}\right) T + an$$

and variance

$$Var[\ln S_T] = \sigma^2 T + b^2 n$$

- We define $\sigma_n := \sqrt{\sigma^2 + \frac{b^2 n}{T}}$, so $\sigma_n^2 T = Var[\ln S_T]$.
- The expectation in blue can be computed as

$$E_0\left[(S_T - K)^+ | N_T = n\right] = F_n \Phi(d_{1,n}) - K\Phi(d_{2,n})$$

where

$$\begin{aligned} F_n &= S_0 exp\left((r - q - \lambda m)T + an + \frac{b^2 n}{2}\right) \\ d_{1,n} &= \frac{\ln\left(\frac{F_n}{K}\right) + \frac{1}{2}\sigma_n^2 T}{\sigma_n \sqrt{T}}, \;\; d_{2,n} = d_{1,n} - \sigma_n \sqrt{T} \end{aligned}$$

Put them all together

- We are now in the position to compute the closed-form formula of European call option in MJD.
- Assume the spot is S_0, strike K, maturity T, diffusion vol σ, risk-free rate r, dividend yield q, $\ln(Y_i) \sim \Phi(a, b^2)$, jump arrival rate is λ, the European call option can be computed as

$$
\begin{aligned}
e^{-rT} E_0\left[(S_T - K)^+\right] &= e^{-rT} \sum_{n=0}^{\infty} e^{-\lambda T} \frac{(\lambda T)^n}{n!} E_0\left[(S_T - K)^+ | N_T = n\right] \\
&= e^{-rT} \sum_{n=0}^{\infty} e^{-\lambda T} \frac{(\lambda T)^n}{n!} \left(F_n \Phi(d_{1,n}) - K\Phi(d_{2,n})\right)
\end{aligned}
$$

where

$$
\begin{aligned}
F_n &= S_0 exp\left((r - q - \lambda m)T + an + \frac{b^2 n}{2}\right) \\
d_{1,n} &= \frac{\ln\left(\frac{F_n}{K}\right) + \frac{1}{2}\sigma_n^2 T}{\sigma_n \sqrt{T}}, \quad d_{2,n} = d_{1,n} - \sigma_n \sqrt{T} \\
m &= e^{a + b^2/2} - 1 \\
\sigma_n &= \sqrt{\sigma^2 + \frac{b^2 n}{T}}
\end{aligned}
$$

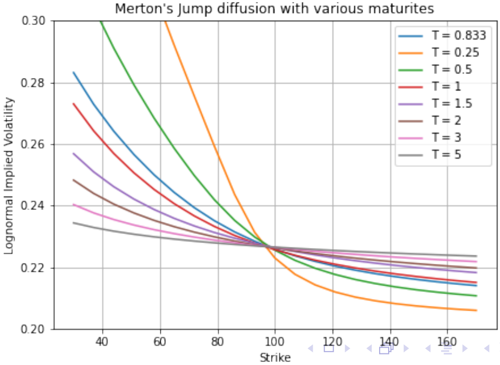
Python Code

The code below shows an implementation of the Merton's jump diffusion European call option formula

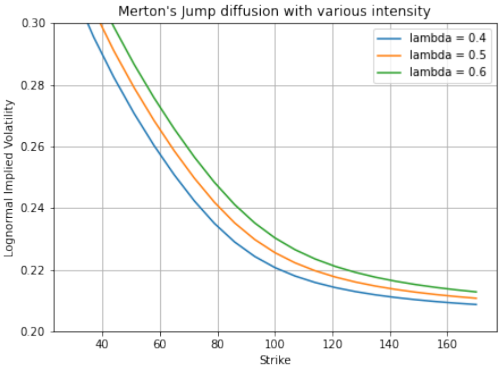
```
import numpy as np
import scipy.stats as st

def MertonJumpDiffusionCall(S0, K, r, q, vol, a, b, lam, T):
    #S0 - spot
    #K - strike
    #r - risk free rate, q - dividend yield
    #vol - diffusion vol
    #a - mean of the log jump size
    #b - vol of the log jump size
    #lam - jump intensity
    #T - maturity
    m = np.exp(a + b*b/2) - 1
    F_n = lambda n: S0 * np.exp( (r-q-lam*m)*T + a*n + b*b*n/2 )
    vol_n = lambda n: np.sqrt(vol*vol + b*b*n/T)
    d1_n = lambda n: (np.log(F_n(n)/K) + 0.5*vol_n(n)**2*T)/(vol_n(n)*np.sqrt(T))
    d2_n = lambda n: d1_n(n) - vol_n(n)*np.sqrt(T)
    value_n = lambda n: F_n(n) * st.norm.cdf(d1_n(n)) - K *st.norm.cdf(d2_n(n))
    valueExact = value_n(0.0)
    n_idx = range(1, 40)
    for n in n_idx:
        valueExact += np.power(lam * T, n)*value_n(n) / np.math.factorial(n)
    valueExact *= np.exp(-r*T) * np.exp(-lam * T)
    return valueExact
```

MJD example

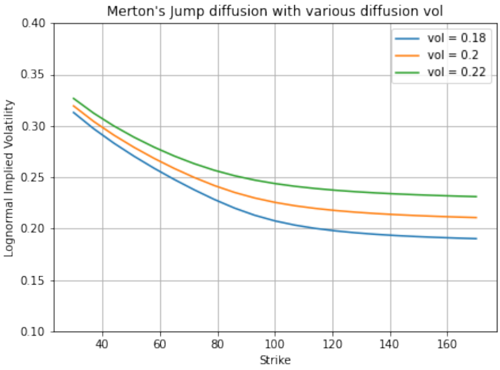
- Let $S_0 = 100$, $r = q = 0$, $\sigma = 0.2$, $\lambda = 0.5$, $a = -0.15$, $b = 0.05$. The values are taken from p.63 of Jim Gatheral's book, The Volatility Surface, A Practitioner's Guide.
- $a = -0.15$, this means the expected direction of the jump is going down.
- The diagram shows MJD lognormal skew with various maturities.



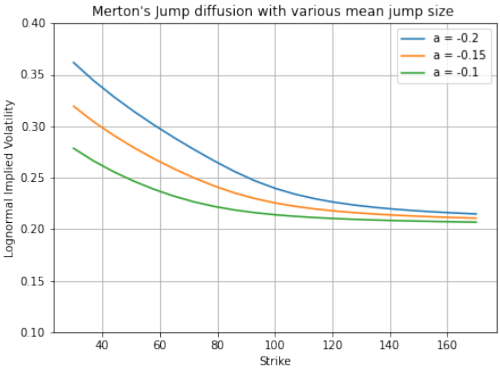
- The diagram shows MJD lognormal skew with various λ. The maturity is 0.5.



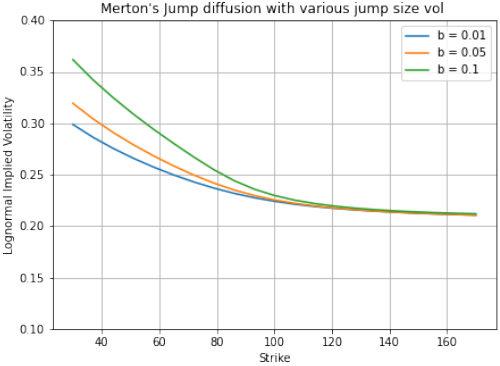
- The diagram shows MJD lognormal skew with various σ. The maturity is 0.5.



- The diagram shows MJD lognormal skew with various a. The maturity is 0.5.



- The diagram shows MJD lognormal skew with various b. The maturity is 0.5.



Fourier-Based Option Pricing

- Option pricing problem has so-far been presented as an expectation of the payoff under a pricing measure.
- Expectation is nothing more than an integration of the payoff against the probability density function of the underlying asset S_T. For example, an undiscounted call option price on S_T with strike K can be computed as

$$E_0[(S_T - K)^+] = \int_{-\infty}^{\infty} (S_T - K)^+ \phi(S_T) dS_T$$

  where $\phi(S_T)$ is the probability density function of S_T under a pricing measure.
- One of the main reasons that there is a closed-form solution for call option price in Black model is that lognormal density function can be expressed in closed form. In practice, we normally work in terms of $\ln S_T$ and the density function that we need in closed-form is normal distribution.

- However, many popular models that are used in the industry do not have closed-form density function, e.g Heston model. Fourier based method is one of the key technologies to allow us to have a semi-closed form solution for European option.
- A very interesting fact is that even if the random variable does not have close-form density function, the characteristic function (CF) of this random variable can be expressed in closed-form.
- It turns out if we can find the closed-form CF for $\ln S_T$, then we can have a semi-closed form formula for European call option. Semi-closed form means it involves one numerical integration.
- We are going to introduce some basic facts about Fourier transform and characteristic function so that we can formulate option price using them. This is by no means a complete treatment of the subject. This is a huge subject and there are many text books have written on it.

Fourier transform and characteristic function

- A function is **square integrable** if

$$\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty.$$

  A *necessary* condition for the above integral to be finite is

$$\lim_{x \to \pm\infty} f(x) = 0.$$

- Fourier transform of a square integrable function f is

$$\hat{f}(u) := \int_{-\infty}^{\infty} e^{iux} f(x) dx$$

  where $i = \sqrt{-1}$.

- To recover f from a given $\hat{f}$, we can use an inverse Fourier transform which is defined as

$$f(x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \hat{f}(u) du$$

- Characteristic function ψ of a random variable X with density ϕ is defined as
$$\psi(u) := \int_{-\infty}^{\infty} e^{iux} \phi(x) dx = E[e^{iuX}]$$
In other words, characteristic function is the Fourier transform of the density function.
- We can recover the density from a characteristic function by using inverse Fourier transform
$$\phi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \psi(u) du$$
- Any reasonable density function would be square integrable. In other words, the corresponding characteristic function exists.

Carr-Madan Formula for European Option

- Carr-Madan (1999) developed a semi closed-form formula to compute the price of a call option on S_T with strike K, maturity T

$$Call(K,T) = e^{-rT} E_0[(S_T - K)^+]$$

for a given characteristic function, ψ, of $\ln(S_T)$. The formula is

$$\begin{aligned} Call(K,T) &= \frac{e^{-\alpha k}}{\pi} \int_0^\infty Re\left(e^{-iuk}\hat{\psi}(u)\right) du \\ \hat{\psi}(u) &= \frac{e^{-rT}\psi(u-(\alpha+1)i)}{\alpha^2+\alpha-u^2+i(2\alpha+1)u} \end{aligned} \tag{1}$$

- where $k = ln(K)$, $\alpha > 0$ is the damping factor
- $Re(a+ib) = a$, i.e. the real part of the complex number.

Proof

- The price of an European call option with strike K and maturity T can be computed as a risk-neutral expectation

$$Call(K,T) = e^{-rT} E_0[(S_T - K)^+]$$

Let $s = \ln S_T$, $k = \ln K$, ϕ be the density function of s. We can express the call price as

$$Call(K,T) \equiv C_T(k) = e^{-rT} \int_k^{\infty} \left(e^s - e^k\right) \phi(s) ds$$

- For Black Scholes model, $\phi(s)$ is a normal density function with mean $\ln S_T + \left(r - q - \frac{1}{2}\sigma^2\right) T$ and variance $\sigma^2 T$.
- Note that $C_T(k)$ tends to S_0 as k tends to $-\infty$. This means $C_T(k)$ is **not square integrable** and implies the Fourier transform of $C_T(k)$ does not exist. To obtain a square integrable function, we define

$$c_T(k) := e^{\alpha k} C_T(k)$$

for $\alpha > 0$. In Schoutens et al. (2004), A perfect calibration! Now what? Wilmott Magazine: 66-78, the authors suggest $\alpha = 0.75$.

- The Fourier transform of $c_T(k)$ can be computed as

$$\begin{aligned}\hat{\psi}(u) &= \int_{-\infty}^{\infty} e^{iuk} c_T(k) dk \\ &= \int_{-\infty}^{\infty} e^{iuk} \left[e^{-rT} \int_{k}^{\infty} e^{\alpha k}(e^s - e^k)\phi(s) ds \right] dk \\ &= e^{-rT} \int_{-\infty}^{\infty} \phi(s) \left[\int_{-\infty}^{s} (e^s - e^k) e^{iuk} e^{\alpha k} dk \right] ds\end{aligned}$$

- Consider only the integral in the square bracket

$$\begin{aligned}\int_{-\infty}^{s} (e^s - e^k) e^{iuk} e^{\alpha k} dk &= e^s \int_{-\infty}^{s} e^{(\alpha+iu)k} dk - \int_{-\infty}^{s} e^{(\alpha+iu+1)k} dk \\ &= \frac{e^s}{\alpha+iu} \left[e^{(\alpha+iu)k} \right]_{-\infty}^{s} - \frac{1}{\alpha+iu+1} \left[e^{(\alpha+iu+1)k} \right]_{-\infty}^{s} \\ &= \frac{e^{(\alpha+iu+1)s}}{\alpha+iu} - \frac{e^{(\alpha+iu+1)s}}{\alpha+iu+1}\end{aligned}$$

- Then we have

$$\begin{aligned}\hat{\psi}(u) &= e^{-rT} \int_{-\infty}^{\infty} \phi(s) \left[\frac{e^{(\alpha+iu+1)s}}{\alpha+iu} - \frac{e^{(\alpha+iu+1)s}}{\alpha+iu+1} \right] ds \\ &= e^{-rT} \int_{-\infty}^{\infty} \phi(s) \left[\frac{e^{(\alpha+iu+1)s}}{(\alpha+iu)(\alpha+iu+1)} \right] ds\end{aligned}$$

- We can express

$$e^{(\alpha+iu+1)s} = e^{i(-i\alpha+u-i)s} = e^{i(u-(\alpha+1)i)s}$$

- We have

$$\begin{aligned}
\hat{\psi}(u) &= \frac{e^{-rT}}{(\alpha+iu)(\alpha+iu+1)} \underbrace{\int_{-\infty}^{\infty} \phi(s) \left[e^{i(u-(\alpha+1)i)s}\right] ds}_{\text{Characteristic function of } s} \\
&= \frac{e^{-rT}}{(\alpha+iu)(\alpha+iu+1)} \psi(u-(\alpha+1)i) \\
&= \frac{e^{-rT}\psi(u-(\alpha+1)i)}{\alpha^2+\alpha-u^2+i(2\alpha+1)u}
\end{aligned}$$

- Once we have $\hat{\psi}$, the European call option can be computed as inverse Fourier transform

$$C_T(k) = \frac{e^{-\alpha k}}{2\pi} \int_{-\infty}^{\infty} e^{-iuk} \hat{\psi}(u) du = \frac{e^{-\alpha k}}{\pi} \int_{0}^{\infty} Re\left(e^{-iuk}\hat{\psi}(u)\right) du$$

- The second equality holds because $C_T(k)$ is real, which implies that the function $\hat{\psi}(u)$ is odd in its imaginary part and even in its real part. **The proof is done**.

Characteristic functions of popular models

- The horribly looking formula in Equation (1) is very flexible. It is a general recipe to price European call option under any model as long as you can find the closed form characteristic function. This includes various jumps and stochastic volatility models.
- Therefore, the call option pricing problem boils down to computing the characteristic function of a model and some of the most popular ones are:

Black Scholes

- $F_0 = S_0 e^{(r-q)T}$

$$\frac{dS_t}{S_t} = (r-q)dt + \sigma dW_t$$

$$\psi^{BS}(u) = e^{iu\left(\ln F_0 - \frac{\sigma^2}{2}T\right) - \frac{\sigma^2}{2}u^2 T} \tag{2}$$

- See *Oosterlee and Grzelak, Mathematical Modeling and Computation in Finance, p143, Example 5.3.1* for the derivation.

Merton's Jump Diffusion

- Spot is S_0, maturity is T, diffusion vol is σ, risk-free rate is r, jump size with lognormal distribution $\ln(Y) \sim \Phi(a, b^2)$, jump arrival rate is λ.
- $m = E[Y] - 1 = e^{a+b^2/2} - 1$

$$\frac{dS_t}{S_{t-}} = (r-q)dt + \sigma dW_t + (dJ_t - \lambda m dt)$$

$$\psi^{MJD}(u) = e^{iu\left(lnF_0 - (\frac{\sigma^2}{2}+\lambda m)T\right) - \frac{\sigma^2}{2}u^2T + \lambda(e^{iua-u^2b^2/2}-1)T} \tag{3}$$

- Drift term: $iu\left(\ln F_0 - (\frac{\sigma^2}{2} + \lambda m)T\right)$
- Diffusion term: $-\frac{\sigma^2}{2}u^2T$
- Jump term: $\lambda(e^{iua-u^2b^2/2} - 1)T$
- See *Oosterlee and Grzelak, Mathematical Modeling and Computation in Finance, p133-135, Chapter 5.2.2* for the derivation.

When Fourier met Black and Scholes

- We are now going to use the Fourier method to price call option under the Black Scholes model. This is of course an overkill but also a good way to check if our implementation of the Fourier method is correct or not.
- The call option price under Black Scholes can be computed as

$$
\begin{aligned}
Call(K,T) &= \frac{e^{-\alpha k}}{\pi}\int_0^\infty Re\left(e^{-iuk}\hat{\psi}(u)\right)du \\
\hat{\psi}(u) &= \frac{e^{-rT}\psi^{BS}(u-(\alpha+1)i)}{\alpha^2+\alpha-u^2+i(2\alpha+1)u}
\end{aligned}
$$

- ψ^{BS} is given in Equation (2).

Python Code

The code below shows an implementation of the Carr-Madan formula with Black Scholes characteristic function.

```
import numpy as np
import scipy.integrate as integrate

# This is psi hat times exp(-iuk)
def carr_madan_integrand(cf, logk, u, alpha):
    top = cf(complex(u, -(alpha+1.0))) * np.exp(complex(0.0, -u*logk))
    bottom = complex(alpha*alpha + alpha - u*u, (2.0*alpha + 1.0)*u)
    result = top/bottom
    return result.real

# Black Scholes characteristic function
def black_cf(forward, vol, t, u):
    lnf_factor = complex(0, u*np.log(forward))
    drift_and_diffusion_term = complex(-0.5*vol*vol*u*u, -0.5*vol*vol*u)
    return np.exp(lnf_factor + drift_and_diffusion_term * t)

# The function to call to price a European call option
def carr_madan_black_call_option(forward, vol, t, k, r, alpha = 0.75):
    cf = lambda u: black_cf(forward, vol, t, u)
    integrand = lambda u: carr_madan_integrand(cf, np.log(k), u, alpha)
    # lower limit is 0.0, upper limit is infinity
    result, error = integrate.quad(integrand, 0.0, np.inf)
    return np.exp(-r*t) * np.exp(-alpha*np.log(k)) / np.pi * result
```

When Fourier met Merton

- Similarly, the call option price under MJD can be computed as

$$
\begin{aligned}
Call(K,T) &= \frac{e^{-\alpha k}}{\pi}\int_0^\infty Re\left(e^{-iuk}\hat{\psi}(u)\right)du \\
\hat{\psi}(u) &= \frac{e^{-rT}\psi^{MJD}(u-(\alpha+1)i)}{\alpha^2+\alpha-u^2+i(2\alpha+1)u}
\end{aligned}
$$

- ψ^{MJD} is given in Equation (3).

Python Code

The code below shows an implementation of the Carr-Madan formula with MJD characteristic function.

```
import numpy as np
import scipy.integrate as integrate

# Merton's jump diffusion charactersitic function
def MDJ_cf(forward, vol, a, b, lam, t, u):
    lnf_factor = complex(0, u*np.log(forward))
    jump_compensation = lam*(np.exp(a+b*b/2)-1)
    drift_and_diffusion_term = complex(-0.5*u*u*vol*vol, u*(-0.5*vol*vol - jump_compensation))
    jump_term = lam*(np.exp(complex(-0.5*b*b*u*u, u*a)) - 1)
    return np.exp(lnf_factor + (drift_and_diffusion_term + jump_term)*t )

# The function to call to price a European call option
# carr_madan_integrand is defined in the code with Black Scholes CF
def carr_madan_MJD_call_option(forward, vol, a, b, lam, t, k, r, alpha=0.75):
    cf = lambda u: MDJ_cf(forward, vol, a, b, lam, t, u)
    integrand = lambda u: carr_madan_integrand(cf, np.log(k), u, alpha)

    # lower limit is 0.0, upper limit is infinity
    result, error = integrate.quad(integrand, 0.0, np.inf)
    return np.exp(-r*t) * np.exp(-alpha*np.log(k)) / np.pi * result
```

Summary

- Bachelier model and normal implied vol.
- Shifted lognormal model as a generalization of Bachelier and Black model.
- What types of implied vol smile that different models generate.
- Poisson process as a counting process.
- Exponential jump arrival time.
- Derivation of European call option formula in Merton's jump diffusion model.
- Carr-Madan approach of European option pricing using Fourier method. This is because some models don't have closed form for density function but characteristic function.
- Python code for Fourier method for Black Scholes and Merton's jump diffusion models.