QF602 Derivatives

Lecture 3 - Greeks

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The Greeks

- In the previous lecture, we briefly covered delta and vega. We now come back to discuss more about the Greeks or sensitives of the option prices to the various parameters.
- Consider an European call and put option, each with the same strike, K, and maturity T. Assuming a continuous dividend yield, q, then put-call parity states

$$Call_0 - Put_0 = e^{-rT}(F_0(T) - K)$$

  where, $F_0(T) = S_0 e^{(r-q)T}$
- Put-call parity is useful for calculating Greeks. For example, it implies that Vega and Gamma for European call and put options are the same.
- Note that all Greeks are model dependent. This means that if you have two models that give you the same Call option price, their Greeks may be different.

Delta

- **Definition**: The delta of an option is the sensitivity of the option price to a change in the price of the underlying security, i.e. S_0.
- The delta of a European call option in the Black model is

$$\frac{\partial Call_0}{\partial S_0} = e^{-qT} N(d_1)$$

$$d_1 = \frac{\ln(F_0(T)/K) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

- By put-call parity, we have

$$\frac{\partial Put_0}{\partial S_0} = \frac{\partial Call_0}{\partial S_0} - e^{-qT}$$

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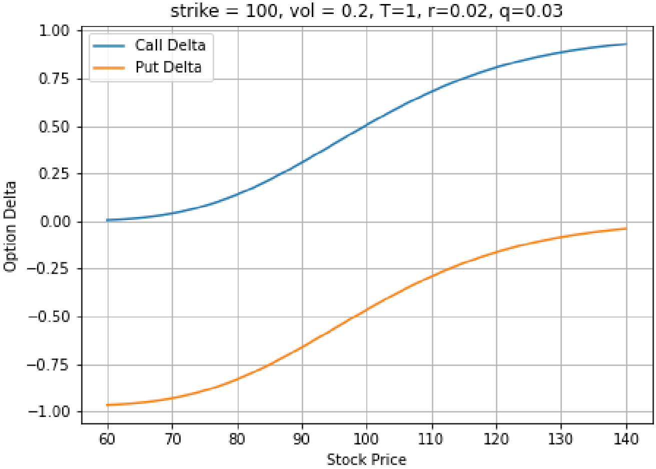
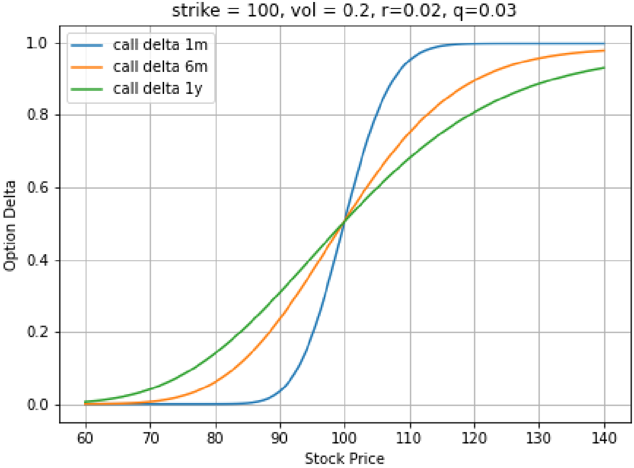
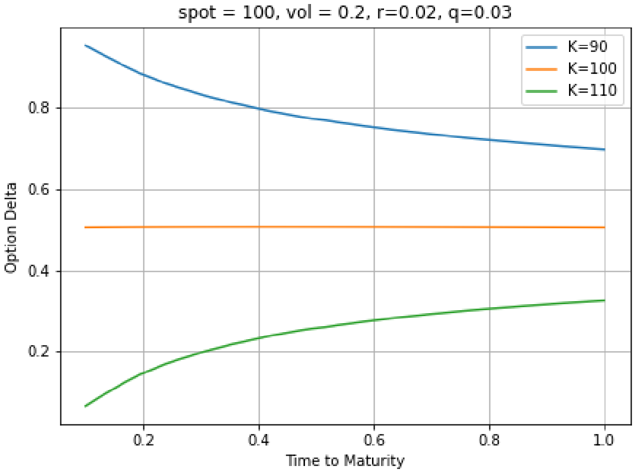
- The figure below shows the delta for a call and a put option, as a function of the underlying stock price S_0.



- The figure below shows the delta for a call option as a function of the underlying stock price for three different maturities, 1 month, 6 months and 1 year.



- The figure below shows the delta of a call option as a function of maturity for three options of different moneyness.



Premium adjusted delta

- The standard Black Scholes delta has an intuitive interpretation from a hedging point of view. It denotes the number of stocks an option seller has to buy to be hedged with respect to spot movements.
- Foreign exchange options have a nominal amount, such as 1,000,000 EUR. In this particular case, an option buyer of a EUR call USD put option with a strike of $K = 1.35$ will receive 1,000,000 EUR and pay 1,350,000 USD at maturity. The option value is by default measured in domestic currency units (i.e. in USD). In this example, the dollar value of the option is, say, 102,400 USD.
- A delta of 0.6 would imply that buying $60\% = 600,000$ EUR of the foreign notional is appropriate to hedge a sold option position.

- The option was sold for 102,400 USD and 600,000 EUR were bought for the hedge.
- One can reduce the quantity of the hedge by converting the 102,400 USD received as the premium to a EUR amount at a given rate of EURUSD at time 0, say, $S_0 = 1.39$.
- This USD premium is equivalent to 73,669 EUR. Consequently, the final hedge quantity will be 526,331 EUR which is the original delta quantity of 600,000 reduced by the received premium measured in EUR.
- This procedure implies adjusting the delta by the premium such that approximately 52.6% of the notional needs to be bought for the hedge (instead of 60%). The quantity 52.6% can also be viewed as a delta and is called the premium-adjusted spot delta.

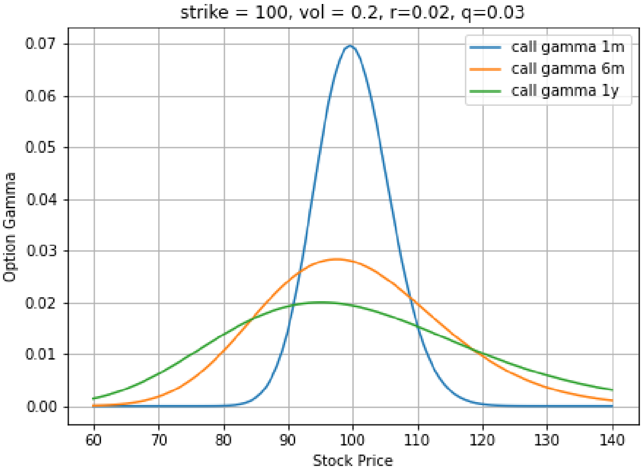
- For stock option, premium adjusted delta can be considered if the premium of the option is quoted in number of stocks.
- For example, a call option on a stock with $S_0 = 100$ USD, standard BS delta 50%, premium is 8 USD.
- In order to be delta neutral, the seller of the call option would need to buy 50 USD worth of stock.
- However, if the premium is paid in a fraction of stock, i.e. $\frac{8}{100}$ of a stock. Then the seller of the option would only need to buy 42 USD worth of stock in order to be delta neutral.
- This is because the premium that is received in this case (i.e. the stock) is not risk-free anymore.
- Similar for FX option on a currency pair like USDJPY which the premium is usually paid in USD (of course one can demand to be paid in JPY but this is not the standard).

Gamma

- **Definition**: The gamma of an option is the sensitivity of the option's delta to a change in the price of the underlying security. The gamma of a call option in the Black model is

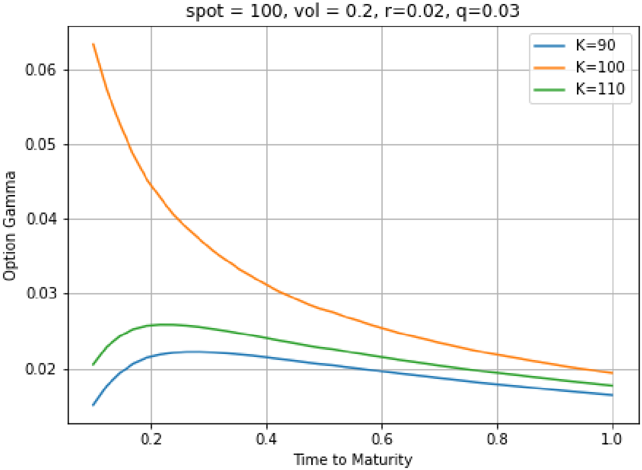
$$\frac{\partial^2 Call_0}{\partial S_0^2} = e^{-qT} \frac{\phi(d_1)}{\sigma S_0 \sqrt{T}}$$

- The figure below shows the gamma of an European option as a function of stock price for three different maturities.



- Note that by put-call parity, the gamma of European call and put option with the same strike are equal.
- Gamma is always positive due to option convexity. Traders who are long gamma can make money by gamma scalping. Gamma scalping is the process of regularly re-balancing your options portfolio to be delta-neutral. However, you must pay for this long gamma position up front with the option premium. We will cover this in details later this lecture.

- The figure below shows gamma as a function of maturity.

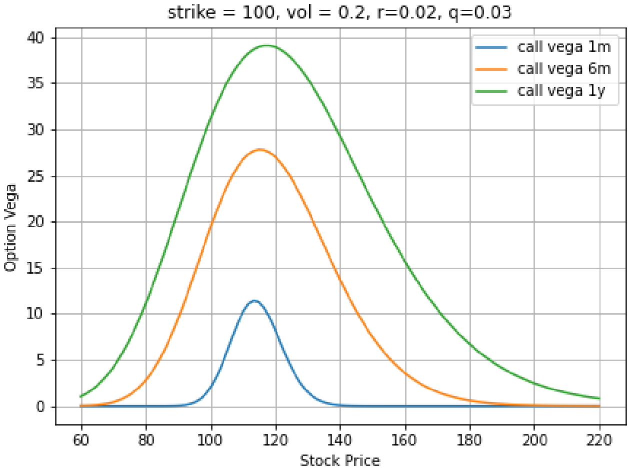


Vega

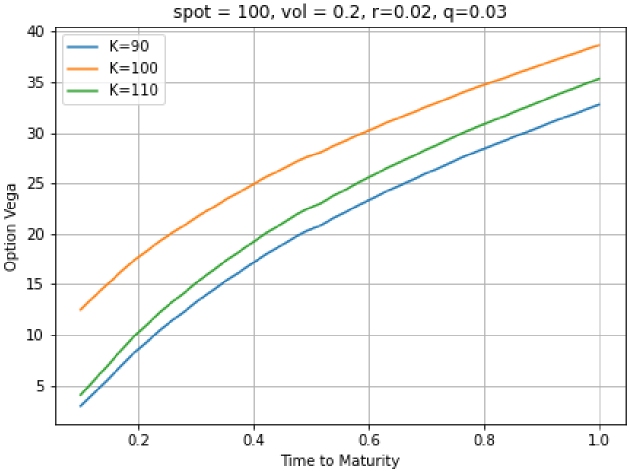
- **Definition**: The vega of an option is the sensitivity of the option price to a change in implied volatility. The Black vega of a call option is

$$\frac{\partial Call_0}{\partial \sigma} = e^{-qT} S_0 \sqrt{T} \phi(d_1)$$

- The figure below shows vega as a function of the underlying stock price.



- By put-call parity, the vega of a call option equals the vega of a put option with the same strike.
- In the figure below, the vega decreases to 0 as time-to-maturity goes to 0.



Delta with non-flat implied vol surface

- Let $\sigma(S_t, K, T)$ be the implied vol with the current spot S_t, option strike K and option maturity T.
- The delta we mentioned so far is the standard Black delta corresponding to a volatility surface that is sticky-strike.
- Sticky-strike vol surface assumes that as the spot moves (from S_t to S_{t+1}), the implied vol of the option with strike K and maturity T does not move. In other words,

  $$\sigma(S_t, K, T) = \sigma(S_{t+1}, K, T)$$

  This is consistent with the Black model which assumes constant volatility.

- If the implied vol of the option does move if the spot changes then the delta would have an additional term, the delta of the call option becomes

$$\frac{\partial Call_0}{\partial S_0} + \frac{\partial Call_0}{\partial \sigma}\frac{\partial \sigma}{\partial S_0}$$

- The intuition is that if the implied vol changes, this means the Black Scholes assumption is violated. The Black delta needs to be modified in order to be accurate.
- The first term is the standard Black delta. The second term is the product of the Black vega, $\frac{\partial, Call_0}{\partial \sigma}$ and the sensitive of the implied vol w.r.t S_0, $\frac{\partial \sigma}{\partial S_0}$.
- If the vol surface is flat, we have $\frac{\partial \sigma}{\partial S_0} = 0$. Then the modified delta reduces to the standard Black delta.

- In practice, we don't need to use the modified Black delta formula to compute the hedge ratio. This can be easily done by using finite difference method as this will automatically capture the impact of the non-flat surface:

$$Delta \approx \frac{Call_0(S_t + \epsilon) - Call_0(S_t - \epsilon)}{2\epsilon}$$

  ϵ is a small change in spot.
- Assume you have an implied vol engine which will return an implied vol if you give the current spot, the strike and the maturity of an option, i.e. the function $\sigma(S_t, K, T)$.
- The key is to make sure you get the correct implied vol when computing the option.
- We need to get the implied vol $\sigma(S_t + \epsilon, K, T)$ to compute the call price $Call_0(S_t + \epsilon)$. and $\sigma(S_t - \epsilon, K, T)$ to compute the call price $Call_0(S_t - \epsilon)$.

Sticky-moneyness and implied vol dynamics

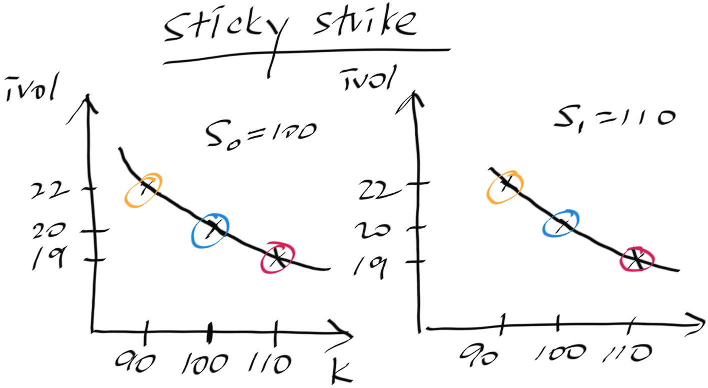
- While sticky-strike assumes the implied vol with a particular strike does not change. Sticky-moneyness assumes that the implied vol with a particular moneyness K/S_t does not change. This means

$$\sigma(S_t, K, T) = \sigma(S_{t+1}, \bar{K}, T)$$

  where $\bar{K} = \frac{S_{t+1}K}{S_t}$
- For example, if $S_t = 100$, $K = 100$ and $\sigma(S_t, K, T) = 0.2$. At time $t+1$, the spot moves up and becomes $S_{t+1} = 110$. Sticky-moneyness tells you that the implied vol at strike 110 is 0.2.
- In the real world, it is neither 100% sticky-strike nor sticky-moneyness. That's why it is very important to have an implied vol engine which is not just be able to reproduce the current market observed implied vol but also have a reasonable implied vol dynamics. Detail discussion of this topic is beyond the scope of this class.

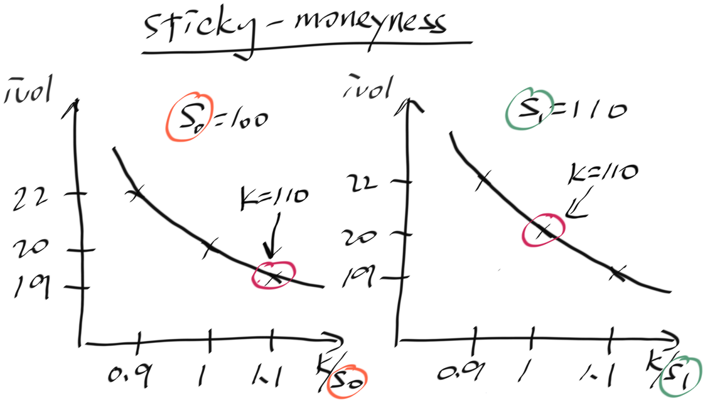
Sticky Strike example

- The spot moves from $S_0 = 100$ to $S_1 = 110$. We can see that the ivols for all 3 options are unchanged.



Sticky Moneyness example

- The spot moves from $S_0 = 100$ to $S_1 = 110$. The option with $K = 110$, the implied vol increases from 19% to 20%. This effect is what we called sticky moneyness.



Theta

- **Definition**: The theta of an option is the sensitivity of the option price to a negative change in maturity.
- The Black theta of a call option is

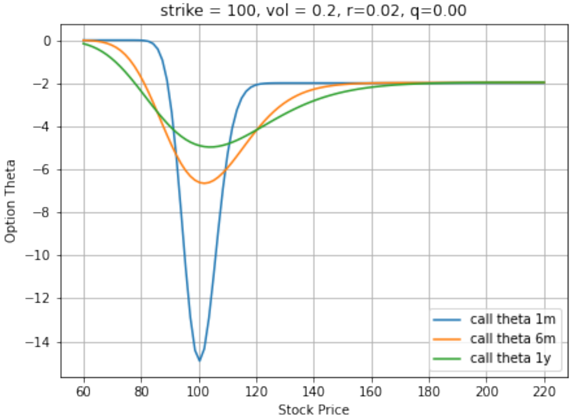
$$\frac{\partial Call_0}{\partial t} = -e^{-qT} S_0 \phi(d_1) \frac{\sigma}{2\sqrt{T}} + q e^{-qT} S_0 N(d_1) - rKe^{-rT} N(d_2)$$

- The Black theta of a put option is

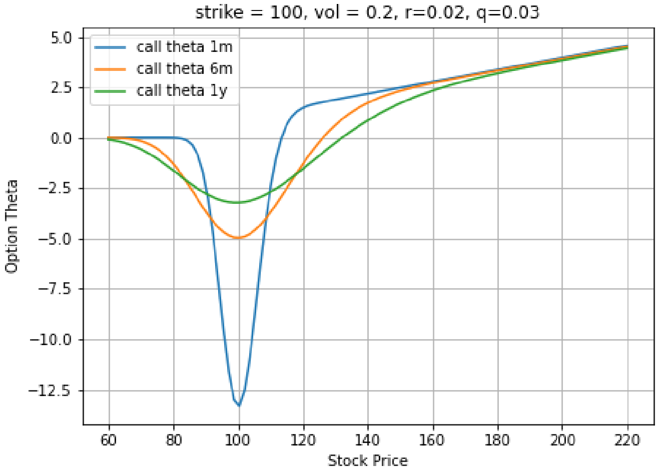
$$\frac{\partial Put_0}{\partial t} = -e^{-qT} S_0 \phi(d_1) \frac{\sigma}{2\sqrt{T}} - q e^{-qT} S_0 N(-d_1) + rKe^{-rT} N(-d_2)$$

Theta

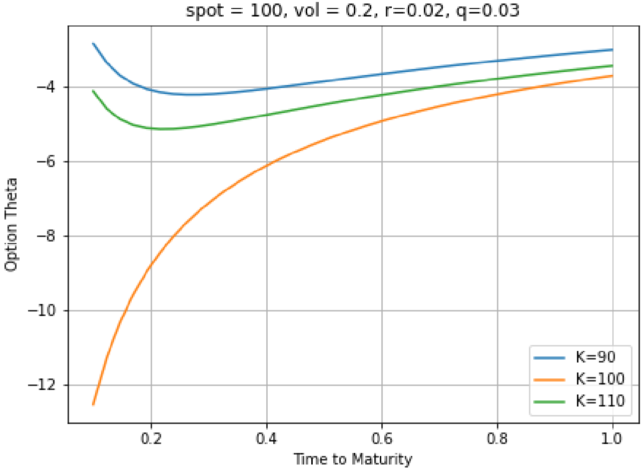
- The figure below shows theta for three call options of different maturity as a function of the underlying stock price.
- When there is no dividend, the theta for a call option is always negative.



- When there is a dividend, the theta for a call option can be positive for deep-in-the-money call option.



- The figure below plots the theta for three call options of different moneyness as a function of maturity. Note that the ATM option has the most negative theta and this gets more negative as maturity goes to 0. Can you explain why?

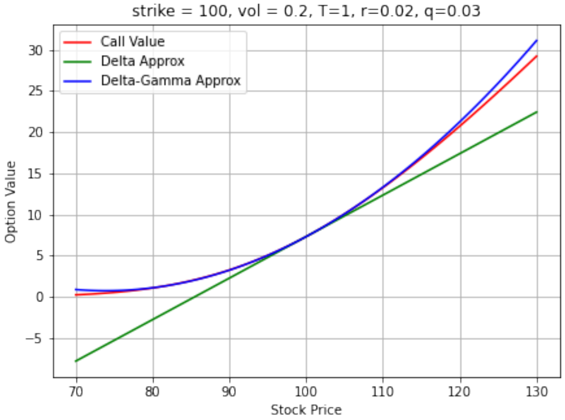


Delta-Gamma Approximations to Option Prices

- One of the applications of Greeks is to estimate the value of the portfolio of options when the underlying market has moved without having to recompute the portfolio.
- Assume at time 0, we have a portfolio of options which has value $V(S_0)$, delta δ and gamma γ. Also assume that this portfolio takes a very long time compute the value.
- Can we find out the value of the portfolio if the stock price is changed by ΔS without having to recompute the whole portfolio?
- An application of Taylor's Theorem says the value of the portfolio if the stock price is $S_0 + \Delta S$ can be approximated as

$$V(S_0 + \Delta S) \approx V(S_0) + \Delta S \times \delta + \frac{1}{2}(\Delta S)^2 \Gamma$$

- The figure below shows the case when the portfolio only contains a call option with $S_0 = 100$, $K = 100$ and $T = 1$, i.e. $V(S_0) = Call_0(S_0)$.
- The red line shows the true value for the call option for various ΔS. The green and blue lines are the approximated values of the portfolio using delta only and delta-gamma approximation.



- We can extend the concept to include the changes in implied volatility. The Taylor series approximation formula will be a little bit more complex:

$$
\begin{aligned}
V(S_0 + \Delta S, \sigma + \Delta\sigma) &\approx V(S_0) + \Delta S \times \delta + \Delta\sigma \times vega \\
&+ \frac{1}{2}(\Delta S)^2 \Gamma \\
&+ \Delta S \times \Delta\sigma \times \frac{\partial^2 V}{\partial\sigma\partial S_0}
\end{aligned}
$$

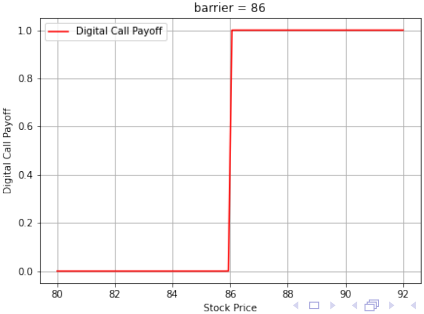
- The first row is the first order approximation for the changes in stock and implied vol. The second row is the gamma approximation. The third row is to cover the simultaneous changes in both stock and implied vol, $\frac{\partial^2 V}{\partial\sigma\partial S_0}$ is called Vanna.
- A common interpretation of Vanna is how much does the delta change if the implied vol is changed.

Digital Option Payoff

- So far, all the Greeks analysis are done using European call and put options.
- Let's look at the Greeks for digital call option.
- Recall the payoff of the digital call option:

$$1_{S(T)>H}$$

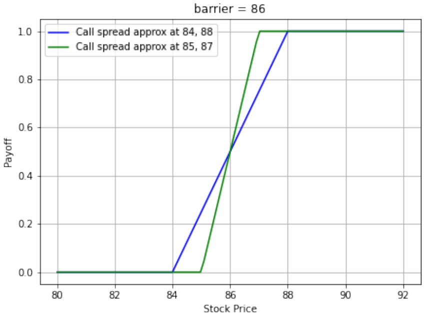
- 1 is the indicator function and H is the barrier.
- The graph below is the payoff diagram with $H = 86$.



Call Spread Payoff

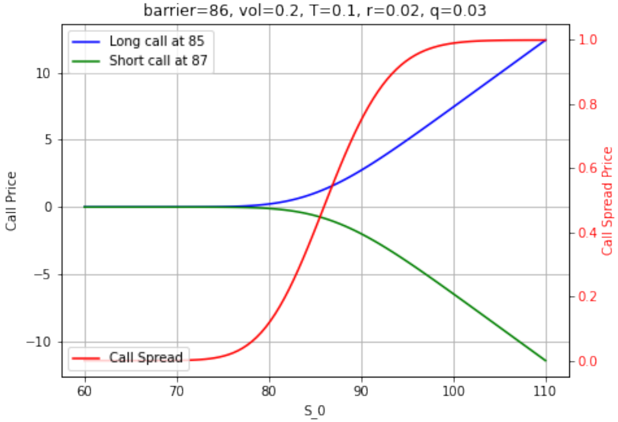
- We can use a call spread as an approx. to a digital call.
- Digital call option can be thought as a limit of a call spread:

$$
\begin{aligned}
Digital\_Call(H) &= \lim_{\epsilon \to 0} \frac{Call(H-\epsilon) - Call(H+\epsilon)}{2\epsilon} \\
&= -\frac{\partial Call}{\partial K}\bigg|_{K=H}
\end{aligned}
$$



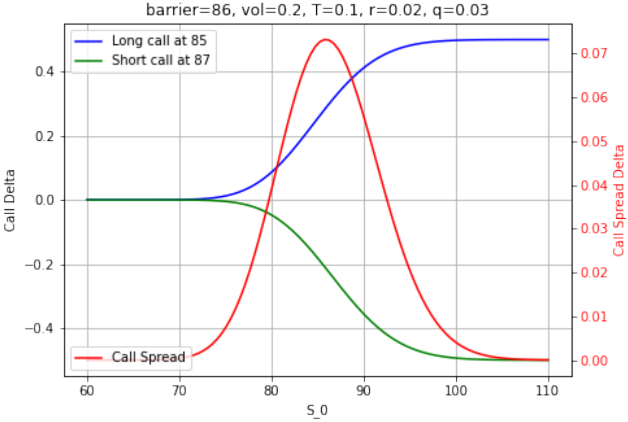
Call Spread Price

- Let $H = 86$ and $\epsilon = 1$.
- The graph below shows the price profile of a call spread and the call options with $r = 0.02, q = 0.03,\ vol = 0.2,\ T = 0.1$



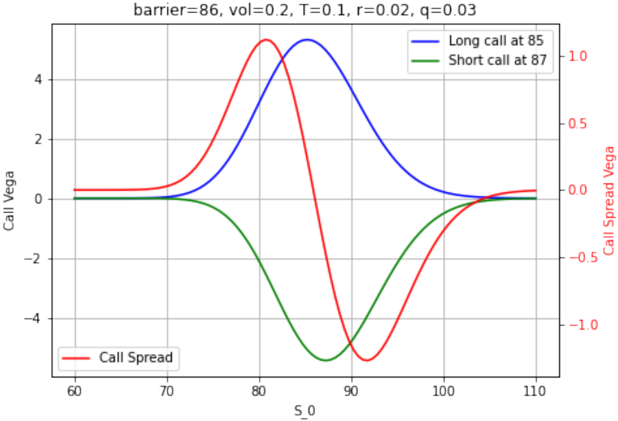
Call Spread Delta

- The graph below shows the delta profile of a call spread and the call options with $r = 0.02, q = 0.03,\ vol = 0.2,\ T = 0.1$



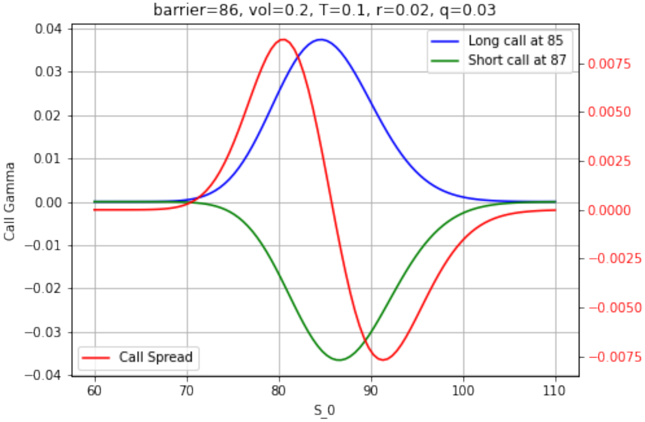
Call Spread Vega

- The graph below shows the vega profile of a call spread and the call options with $r = 0.02, q = 0.03,\ vol = 0.2,\ T = 0.1$



Call Spread Gamma

- The graph below shows the gamma profile of a call spread and the call options with $r = 0.02, q = 0.03,\ vol = 0.2,\ T = 0.1$



The Relationship between Delta, Theta and Gamma

- The BS PDE states that any derivative security with price P_t must satisfy

$$\frac{\partial P_t}{\partial t} + (r-q)S_t \frac{\partial P_t}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 P_t}{\partial S_t^2} = rP_t.$$

- Let θ, δ, Γ be theta, delta and gamma respectively, we get

$$\theta + (r-q)S_t\delta + \frac{1}{2}\sigma^2 S_t^2 \Gamma = rP_t.$$

- The equation holds in general for any portfolio of securities. If the portfolio in question is delta-hedged so that the portfolio $\delta = 0$ then we get

$$\theta + \frac{1}{2}\sigma^2 S_t^2 \Gamma = rP_t.$$

- There are 3 terms in the delta-hedged portfolio.
- The first term is theta. Recall that theta represents the time decay if one long an option, so in general, long option means $\theta < 0$.
- The second term is gamma multiplied with the half of the variance of the absolute change of the stock, dS_t. All the items in the second term is positive and hence the second is positive.
- If the risk free rate $r = 0$, we can see that

$$\theta = -\frac{1}{2}\sigma^2 S_t^2 \Gamma$$

Any gain due to gamma must be offset by the losses due to theta. In other words, the convexity (gamma) is paid for by the slowly bleeding in value which is quantified by theta. We will now explore what I meant by this statement.

Trading the Volatility - Example

- Consider the case that one has bought a call option at strike 100, spot at 100, maturity is 1 year, rate and dividend are 0.
- For simplicity, assume the implied vol of the option is 10% and remains as such until the maturity.
- The table below shows the the spot price movements, the corresponding option prices and deltas (using ivol at 10%).
- Assume one is always delta hedged as time goes by, what is the PnL at the end of the period?
- TTM stands for time to maturity.
- Notice that spot at TTM=0.96 is the same as inception

TTM	spot	option price	delta
1	100.0	3.99	0.52
0.99	104.6	6.77	0.69
0.98	99.0	3.45	0.48
0.97	103.6	6.03	0.66
0.96	100.0	3.91	0.52

- At inception, one would short 0.52 stocks @ 100.
- At TTM=0.99, spot moves up to 104.6, delta of the call increases to 0.69, one would need to sell more stocks, i.e. short another 0.17 stocks @ 104.6.
- At TTM=0.98, spot moves down to 99, delta drops, one needs to buy the stocks to remain delta hedged, i.e. long 0.21 @ 99.
- Similar for TTM=0.97 and 0.96. Total PnL = 1.57.

TTM	spot	option price	option delta	option pnl	hedge pnl	total pnl
1	100	3.99	0.52	0.00	0.00	0.00
0.99	104.6	6.77	0.69	2.78	-2.39	0.39
0.98	99	3.45	0.48	-3.32	3.88	0.56
0.97	103.6	6.05	0.66	2.60	-2.20	0.40
0.96	100	3.91	0.52	-2.15	2.37	0.22
						1.57

Higher realized volatility

- Consider the similar case but the spot realized volatility is higher.
- In order to have a fair comparison, note that the implied volatility for the option is the same, i.e. 10%.
- Notice the spot at TTM=0.96 is still the same as inception.
- Since the realized volatility is higher, the Total PnL is also larger at 2.1 instead of 1.57.

TTM	spot	option price	option delta	option pnl	hedge pnl	total pnl
1	100	3.99	0.52	0.00	0.00	0.00
0.99	108	9.30	0.79	5.31	-4.16	1.15
0.98	104	6.34	0.67	-2.96	3.18	0.22
0.97	98	2.97	0.44	-3.37	4.03	0.66
0.96	100	3.91	0.52	0.94	-0.88	0.06
						2.10

- In the previous 2 cases, the spot ends up with the same price as the beginning. This means the option intrinsic value is the same but the time value is lower.
- We show that by the delta hedging, if the realized vol is higher than the implied vol, we can actually make money by simply doing delta hedging.
- How about if the spot price just drifting up or drifting down?

Drifting up

- Implied vol of the option is still the same at 10%.
- All settings are the same except the spot just drifting up by 1 every period.
- The PnL is very close to zero.

TTM	spot	option price	option delta	option pnl	hedge pnl	total pnl
1	100	3.99	0.52	0.00	0.00	0.0000
0.99	101	4.51	0.56	0.52	-0.52	-0.0002
0.98	102	5.07	0.60	0.56	-0.56	-0.0004
0.97	103	5.66	0.64	0.60	-0.60	-0.0007
0.96	104	6.30	0.67	0.64	-0.64	-0.0009
						-0.0021

Drifting down

- Implied vol of the option is still the same at 10%.
- All settings are the same except the spot just drifting down by 1 every period.
- The PnL is very close to zero.
- These examples tell us that even if we delta hedge, the PnL is uncertain and is highly path dependent.

TTM	spot	option price	option delta	option pnl	hedge pnl	total pnl
1	100	3.99	0.52	0.00	0.00	0.0000
0.99	99	3.47	0.48	-0.52	0.52	0.0002
0.98	98	2.99	0.44	-0.48	0.48	0.0008
0.97	97	2.55	0.40	-0.44	0.44	0.0013
0.96	96	2.16	0.36	-0.40	0.40	0.0018
						0.0041

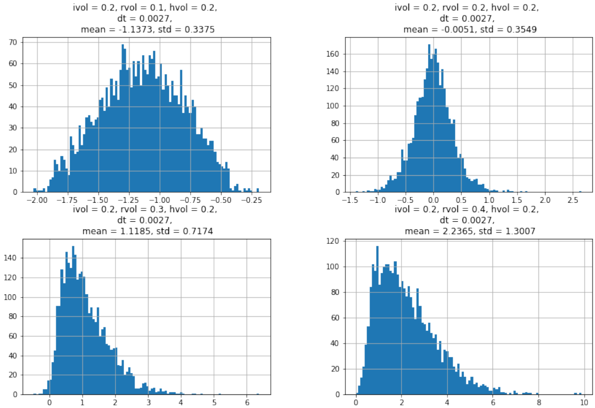
Delta Hedging Simulations - Hedge using Implied Vol

- Since the PnL is path dependent, we are going to simulate many sample paths and compute the PnL for each simulation. We are then going to plot a histogram of those PnLs and see how it looks like.
- Consider $S_0 = 100$, $K = 100$, $r = 0$, implied vol $= 0.2$, time step $dt = 1/365$, maturity $T = 30/365$. In other words, we have 30 time steps to reach the maturity of the call option.
- We use the **implied vol to compute the delta**.
- We simulate 3000 paths each with the realized volatility $\sigma = [0.1, 0.2, 0.3, 0.4]$ using the Black Scholes dynamics:

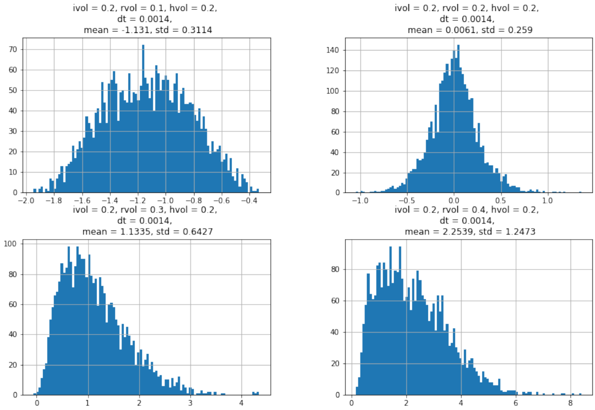
$$lnS_{t+1} = lnS_t + \left(r - \frac{\sigma^2}{2}\right) dt + \sigma dW_t$$

- Using those simulated paths, we are going to evaluate the total PnL for each path. Then we plot a histogram of the total PnL for each realized volatility σ.

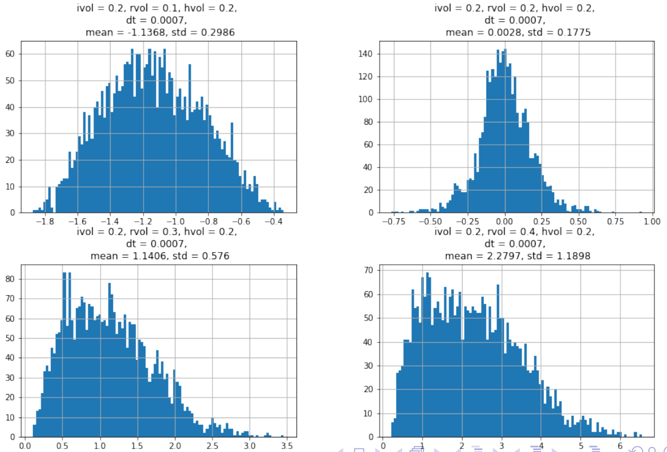
- The graphs below shows the histogram of the total PnL using 3000 paths with different σ (labelled as rvol in the title of each graph). The x-axis are the total PnL for each simulation.
- Two observations: (1) Higher the realized vol, on average, higher the PnL. (2) Higher the realized vol, higher the SD of the PnL.



- How about we make the hedging frequency higher, $dt = 1/730$, i.e. re-balance twice a day instead of once a day.
- The graphs below shows the histogram of the total PnL using 3000 paths with different σ with $dt = 1/730$
- We can see the SD reduces but the mean is more or less unchanged.



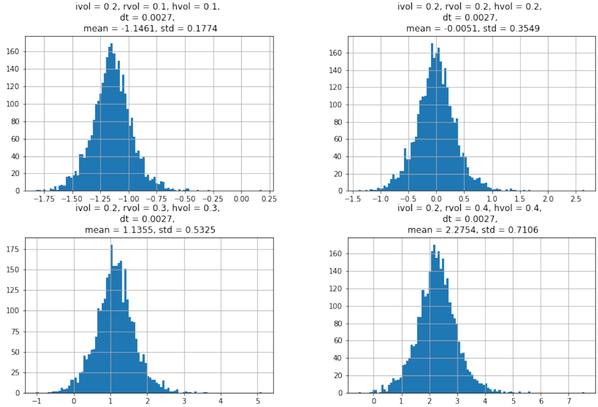
- For $dt = 1/1460$, we can see the SD reduces even further. This shows that the SD is due to the discrete hedging.
- But we are living in a discrete world and cannot hedge continuously. Is there anything we can do to reduce the SD further?



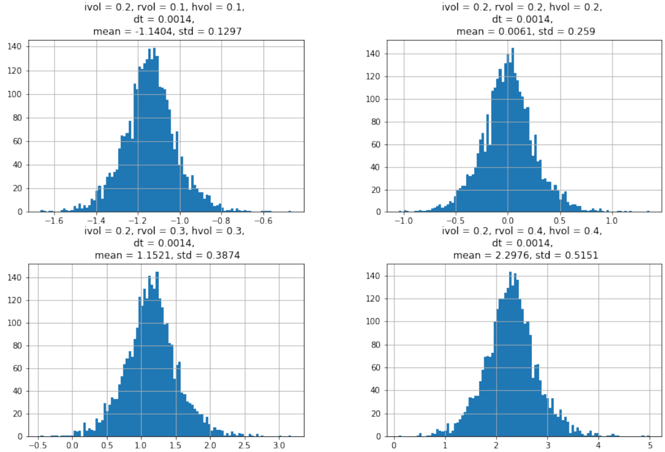
Hedge with Realized Vol

- Let's assume we have a very good volatility estimator and instead of computing the delta using the implied vol 0.2, we use the realized volatility to compute the delta and hedge accordingly. What will happen to the distribution of the total PnL?
- Since we generate the paths using the Black Scholes dynamics with a specific realized volatility σ, we can play "God" here by hedging using σ instead of 0.2.
- For example, if we generate the paths using $\sigma = 0.3$, then we compute the deltas with the Black Scholes delta formula and using 0.3 as the volatility.
- But remember, the option is still valued at the implied vol = 0.2.

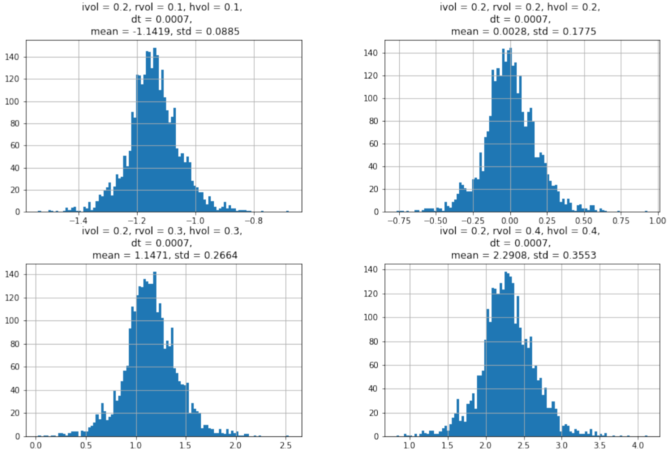
- The figure below shows the total PnL histogram where we still use implied vol to price the option but using realized vols to compute the delta and hedge. $dt = 1/365$.
- We can see the mean is similar as before but SD is lower.



- The figure below shows the total PnL histogram that we hedge using realized vol with $dt = 1/730$.



- The figure below shows the total PnL histogram that we hedge using realized vol with $dt = 1/1460$.



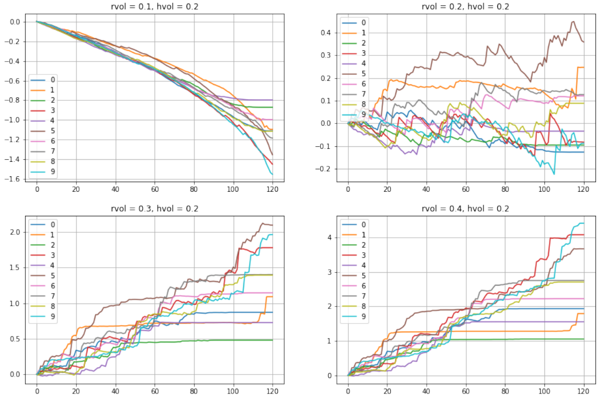
- The table below summarizes the results and it shows hedging with the realized vol significantly reduces the SD of the Total PnL instead of using implied vol.

dt	realized vol	hedge vol	mean	sd
1/1460	0.1	0.2	-1.1368	0.2986
1/1460	0.2	0.2	0.0028	0.1775
1/1460	0.3	0.2	1.1406	0.576
1/1460	0.4	0.2	2.2797	1.1898
1/1460	0.1	0.1	-1.1419	0.0885
1/1460	0.2	0.2	0.0028	0.1775
1/1460	0.3	0.3	1.1471	0.2664
1/1460	0.4	0.4	2.2908	0.3553

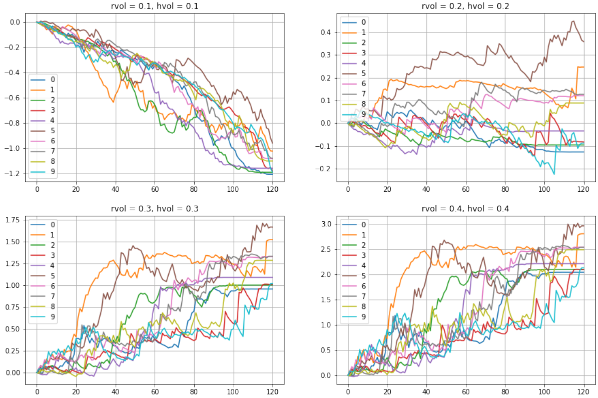
So which vol shall we use?

- We show that if we use the realized vol to delta hedge **until the maturity** of the option, the SD of the total PnL at **maturity** will be smaller than using the implied vol. Does it mean we should always use realized vol?
- There are 2 problems, (1) we do not know what the realized vol is going to be. (2) We have not considered the PnL **before we get to the option maturity**.
- To answer the second question, let's repeat the same simulation exercise. Instead of just showing the total PnL at maturity, we show the total PnL of every time step. For ease on the eyes, we just show 10 paths instead of 3000 with $dt = 1/1460$.

- The figure below shows the total PnL for each path when we hedge using implied vol at 0.2. We can see the paths are relatively smooth.



- The figure below shows the total PnL for each path when we hedge using the corresponding realized vol. We can see the paths are less smooth compare to the one we use implied vol to hedge.



Final remark

- This is assuming you are on the right side of the trade, i.e. buy when the realized vol is going to higher than implied, vice-versa.
- If you hedge using implied vol: (1) you will get smooth PnL until the maturity. (2) No need to estimate the realized vol. But (3) You don't know how much money you will make, only that it is positive.
- If you hedge using realized vol: (1) you will get volatile PnL until the maturity. (2) You need to have a accurate forecast of the realized vol. (3) The SD of the total PnL at maturity is reduce, so you have more certain of how much money you will make at the maturity.
- Most of the treatment of this topic in this lecture follows, *Ahmad and Wilmott (2005) Which Free Lunch Would You Like Today, Sir?: Delta Hedging, Volatility Arbitrage and Optimal Portfolios*. See there for a more detail discussion and derivation of the results.

Gamma squeeze

- This is a phenomenon when option market makers have sold a lot of deep OTM options and the underlying price moves towards the strikes in a short period of time. Option market makers need to hedge their delta exposures and causing the market move against them even more.
- Assume the current spot price is 100. Option market makers sold many call options around 500 strike. OTM call options have almost no delta.
- The spot suddenly moves up to 400 and the call options now have non-negligible deltas. Since option market makers short the call options, they have negative delta positions. In other words, they need to buy the underlying to be delta hedged. This delta re-balancing action causes the spot moves even higher and so on.

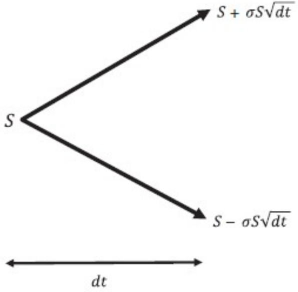
Summary

- Delta and premium adjusted delta. How does delta change with spot and time.
- Gamma, Vega and Theta.
- Delta with non-flat implied vol surface. Compute delta by finite difference to capture the smile effect.
- Sticky strike and sticky moneyness.
- Delta gamma approximation.
- Digital option. Digital call can be approximated by call spread.
- Digital option price, delta, gamma and vega profile.
- Relationship between delta, theta and gamma.
- Delta hedging option as trading the volatility.
- What vol should we use to compute the delta when hedging an option? What are the pros and cons for each choice?

Appendix: Dynamic Replication and Convexity

- Options theory is based on the insight that, in an idealized and simplified world, options are not an independent asset. Because of this, we can use dynamic replication, using simpler securities to mimic the payoff of options. How closely the actual world matches the hypothetical simplified one determines how well the theory works in practice.
- To begin with, for simplicity, assume that the expected rate of return of a stock is zero. An investor who is long the stock makes money if it goes up, and loses money if it goes down. The profit and loss (PnL) is linear in the price of the stock.

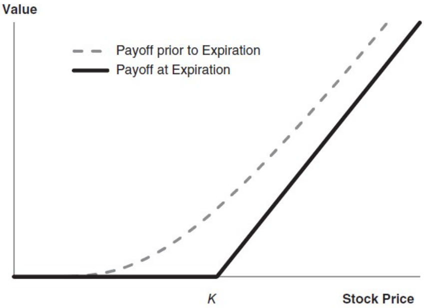
- The figure below shows a binomial model for a share of stock with current price S and volatility σ.



Binomial Model of Underlying Stock Price, $\mu = 0$

- The change in the value of the stock over dt is $dS = \pm\sigma S\sqrt{dt}$, so that $dS^2 = \sigma^2 S^2 dt$, irrespective of whether the stock moves up or down.

- The solid line displays the payoff of a vanilla call option at expiration, and the dashed line represents its value at some earlier time, both plotted as a function of the underlying stock price. The graph of the payoff is kinked, and the value at an earlier time is more smoothly curved. The option increases more in value if the stock moves above the strike than if it moves the same amount below the strike. Convexity is a valuable quality in a security, and the fundamental question of options valuation is: What should you pay for convexity?



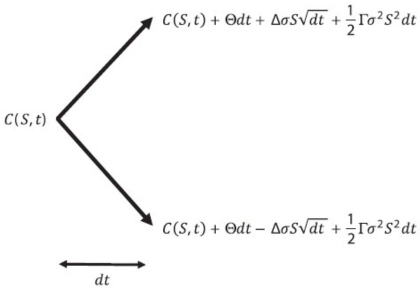
The Payoff of a Vanilla Call Option at Expiration

- We can answer this by using the principle of replication and the law of one price, as originally discovered by Black and Scholes, and Merton.
- We can specify the change in the price $C(S, t)$ of a vanilla call when the underlying stock, whose price is S at time t, changes by a small amount dS during time dt, by using a Taylor series expansion of the call price:

$$C(S + dS, t + dt) = C(S, t) + \frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial S} dS + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} dS^2 + \ldots$$

- We have terminated the Taylor series at the dS^2 term because the size of the squared change in S in our binomial model is proportional to dt. For small dt, terms involving dt^2 or $dSdt$ and any higher order terms will be extremely small and considered negligible.

- How would the value of a call option change in our binomial model when the underlying price changes as in the figure in p60?
- Let $\Theta = \frac{\partial C}{\partial t}$, $\Delta = \frac{\partial C}{\partial S}$, $\Gamma = \frac{\partial^2 C}{\partial S^2}$. We use the equation in p62 to calculate the corresponding change in the value of the call due to the stock price changes

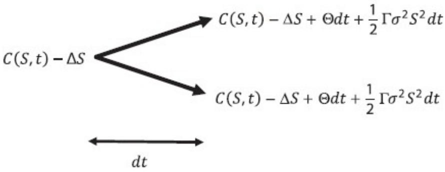


Binomial Model of the Value of a Call Option, $\mu = 0$

- Except for the $\pm\Delta\sigma S\sqrt{dt}$ terms, the payoffs are the same whether the stock moves up or down. If we could somehow eliminate this Δ term, we would have a guaranteed (i.e., riskless) payoff an instant later, and, based on the law of one price, we know that all riskless payoffs should earn the riskless rate of return. Requiring that the return on this instantaneously riskless portfolio be equal to the riskless rate would then lead to the BS option pricing formula.
- In the binomial framework, in order to cancel out the $\pm\Delta\sigma S\sqrt{dt}$ terms that distinguish the up-payoff from the down-payoff, we need to short Δ shares of the underlying stock S. The binomial evolution of the long-call/short-stock portfolio, which is called a delta-hedged portfolio.
- Since the delta-hedged portfolio has the same value whether the stock moves up or down, it is riskless.

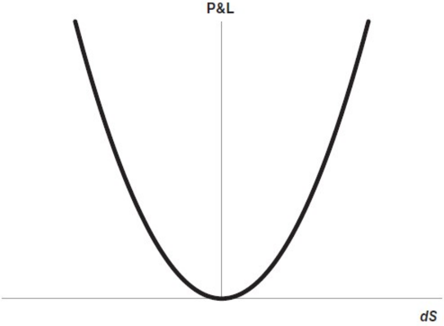
- The initial value of the delta-hedged portfolio is $V(S,t) = C(S,t) - \Delta S$. Recall $dS^2 = \sigma^2 S^2 dt$. The figure below shows the change in the value of the hedged portfolio is given by

$$
\begin{aligned}
dV(S,t) &= \Theta dt + \frac{1}{2}\Gamma dS^2 \\
&= \Theta dt + \frac{1}{2}\Gamma \sigma^2 S^2 dt
\end{aligned}
$$



Delta-Hedged Call Option, $\mu = 0$

- The second term in the equation is quadratic in dS. It is much smaller than the linear change in value of V, proportional to dS, which has been removed by the delta hedge. If Γ is positive, then we say that the option position displays positive convexity or is convex in dS. To get the benefit of pure curvature, you must delta-hedge away the linear part of the change in the call option's value due to dS, which would otherwise swamp the small but significant change, proportional to dS^2, that arises from the curvature.



P&L with Positive Convexity

What Should You Pay for Convexity?

- In our binomial model, the delta-hedged option position is riskless over an infinitesimal time dt, and should therefore, according to the law of one price, earn the riskless rate of return. If we continue with the additional assumption, convenient but not necessary, that the riskless rate is zero, then our delta-hedged position should earn zero profit, so there should be no change in the value of the position after a time dt passes.
- The equation for $dV(S,t)$ is

$$dV(S,t) = \Theta dt + \frac{1}{2}\Gamma\sigma^2 S^2 dt = 0$$

- For a long option position, when rates are zero, the amount Θdt that the option loses from time decay must be precisely offset by the gain $\frac{1}{2}\Gamma\sigma^2 S^2 dt$ that results from convexity as the stock price moves by $\pm\sigma S\sqrt{dt}$.

Hedging an Option Means Betting on Volatility

- We denote the **implied vol** by Σ, which in the framework of our model can be regarded as the market's anticipated value for future **realized volatility**, σ, which is unknown.
- If the **realized volatility** σ turns out to be different from what we expected, then the stock will move either more or less than we anticipated.
- If σ turns out to be greater than Σ the convex delta-hedged option position $V = C - \Delta S$ will increase in value more than anticipated, no matter which direction the stock moves. Similarly if σ is lower than anticipated, the hedged position will appreciate less.

- We can quantify the gain made from convexity and the loss from time decay for a long option position. Replacing σ with Σ to account for the fact that we anticipate a volatility of Σ we have

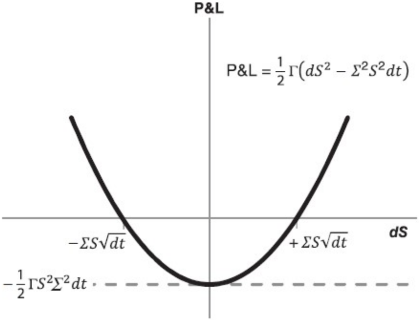
$$\Theta + \frac{1}{2}\Gamma\Sigma^2 S^2 = 0$$

- The amount we expect to lose due to time decay during time dt is $\frac{1}{2}\Gamma\Sigma^2 S^2 dt$. The gain from convexity, if the stock moves an amount $dS = \pm\sigma S\sqrt{dt}$ with a realized volatility σ is $\frac{1}{2}\Gamma\sigma^2 S^2 dt$
- The net infinitesimal profit or loss after time dt is then the difference between these two quantities:

$$Profit = \frac{1}{2}\Gamma S^2(\sigma^2 - \Sigma^2)dt$$

- The term ΓS^2 is called **dollar gamma** which we will discuss in more detail in Lecture 6 in the context of variance swap.

- The figure below illustrates how the P and L of the hedged position varies with the realized move dS in the stock price.
- When we delta-hedge a long option position, we are effectively making a bet on volatility. To profit, we need the realized volatility to be greater than the implied volatility. A short position profits when the opposite holds.



P&L from Implied versus Realized Volatility