QF602 - Homework 2

Question 1

- Suppose that Tesla is trading at \$750 per share. What does it cost to construct a derivative security that pays \$100 when Tesla hits \$1000 for the first time? Note that this security has no maturity. Assume no dividend, zero interest rate, you can trade any amount of Tesla share and no transaction cost.

  Answer. \$75 by using the replication argument. 0.1 shares of Tesla will worth \$100 when a Tesla share is trading at \$1000. The cost of buying 0.1 Tesla shares right now is \$75, therefore, the derivative must cost \$75.

Question 2

- The table below shows the put option prices for various maturities and strikes for the same underlying. Assume the interest rate is 0. What is the range of x and y such that there is no arbitrage?

Maturity/Strike	**90**	**100**	**110**
1	5	x	10
2	y	9	12

  Answer. Let's start with maturity 1. Using butterfly constraint, we have

$$5 - 2x + 10 \geq 0 \rightarrow x \leq 7.5$$

  We also know that put at 100 must worth more than put at 90, hence $5 \leq x \leq 7.5$. Then we apply butterfly constraint to maturity 2, we have

$$y - 2 \times 9 + 12 \geq 0 \rightarrow y \geq 6$$

  We also know that put at 100 must worth more than put at 90 for maturity 2, hence $9 \geq y \geq 6$.

Question 3

- Assume $S_0 = 100$, dividend yield is 0% and risk free rate is 0%; a term-structure of implied vol $\Sigma(1) = 0.3, \Sigma(2) = 0.2$, the strikes are 100. Is there any arbitrage in the market? What's the minimum value of $\Sigma(2)$ such that there is no arbitrage?

Answer. Yes, there is a calendar spread arbitrage as the call with maturity 1 has price = 11.9235 and call with maturity 2 has price = 11.2463. In order to have no arbitrage, the minimum price for call with maturity 2 needs to be at 11.9235. By inverting the Black Scholes call option formula, we have $\Sigma(2) = 0.21213$.
Another way to look at the problem is to calibrate a time-dependent Black Scholes model to this market data, we will get

$$\sigma_1 = \Sigma(1) = 0.3$$

$$\begin{aligned}
\sigma_2 &= \sqrt{\frac{(\Sigma(2)^2 \times 2 - \Sigma(1)^2 \times 1)}{(2-1)}} \\
&= \sqrt{\frac{(0.2^2 \times 2 - 0.3^2 \times 1)}{(2-1)}} \\
&= \sqrt{\frac{-0.01}{(2-1)}}
\end{aligned}$$

As shown above, the market data implies σ_2^2 is negative which is clearly not correct.