QF602 Derivatives

Lecture 1 - Introduction to Derivatives

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What are derivatives?

- Financial instruments that “derive” cash flows from prices of other assets.
- Typically, an agreement between two parties to exchange cash flows in the future.
- Someone once says: "if you can write down the payoff on a piece of paper, I can create a corresponding derivative."
- Cash equity and bond are examples of what is not a derivative.
- Derivatives can be used to hedge against unfavorable price changes, or speculate on favorable price changes.
- Hedging means mitigate or reduce risks.

Types of Underlying Assets

- Physical assets such as agricultural products, precious metals, or fossil fuels.
- Financial assets such as stocks or bonds (or even other derivatives!).
- Intangible assets such as electricity or weather.
- Derivatives can also be written on an FX rate, an interest rate or longevity (q-forward).
- In this course, we restrict ourselves to underlying asset which is tradable. This implies that we can hedge the delta risk induced by the derivative.
- Some are settled by delivery of the underlying asset, some are cash settled.

Types of Derivatives

- Some derivatives represent obligations. Both parties are required to follow through on transactions, irrespective of gain or loss. For example, a forward contract (or simply forward) on USDJPY with 6 months maturity (aka the value date in FX).
- If you long (buy) a 6 month forward contract on USDJPY at 105 with 1mio USD notional, you will receive 1mio USD and sell 105mio JPY at the value date.
- Other derivatives represent options. One party has the right but not the obligation to buy or sell an asset. The other party receives a fee to sell that right. For example, an European option on SP500 index with 1 month maturity.
- Some jurisdiction has capital control and the likes of non-deliverable forwards (NDF) can provide a means to hedge the exposure. E.g. USDIDR NDF.

Time Value of Money

- \$100 to be paid 1 year from now (Future Value).
- 5% interest rate per annum, denoted as r.
- Discount Factor (Zero coupon bond) at time t with maturity T is defined as $Z_t(T) = e^{-r(T-t)}$
- Present value (PV) can be computed as PV = Future Value x Discount Factor.
- $Z_0(T) = e^{-rT} = 0.9512$, where $T = 1, r = 5\%$.
- We assume continuous compounding in this course unless otherwise stated. This is merely a convention of how to represent Discount Factor.
- In practice, discount factors are derived from prices of market traded instruments, e.g deposit rates, OIS swaps, interest rate swaps, FX swaps, cross-currency swaps. For our purpose, we assume they are given in this course.

Forward contract

- The most fundamental instrument as it determines the expectation of the underlying asset.
- A forward contract is an agreement between 2 parties to buy or sell an asset at maturity.
- No cash flows exchange until maturity. (What happen if one party walks away before the maturity?)
- The price agreed to buy or sell the asset at maturity is called the strike price, K.
- The value of the forward contract V at t, where $0 \leq t \leq T$, is computed as:

$$V_t = (F_t(T) - K) \times Z_t(T).$$

- The forward contract that has 0 value at inception has the strike price equals to the forward price at time 0, i.e. $V_0 = 0$, if $F_0(T) = K$.

Forward price

- S_t denotes the price of a stock at time t.
- $F_t(T)$ denotes the forward price at time t with maturity T, $t \leq T$. Note that $F_T(T) = S_T$.
- Consider a stock that pays no dividends and is worth 50, $S_0 = 50$.
- 6 month interest rate is 6% per annum, $r = 6\%$.
- Forward price at time t can be computed as:

$$F_t(T) = S_t e^{r(T-t)} = S_t / Z_t(T)$$

To compute a 6 month forward price, we can set $t = 0$, $T = 6/12$ and we get

$$F_0(6/12) = 50 \times e^{6\% \times 6/12} = 51.52.$$

- Forward price of a stock can be viewed as equal to the spot price plus the cost of carrying it.
- We can compute the forward price by using a hedging argument:
- Let $S_0 = 50$, $r = 6\%$, a customer long a forward contract on one stock against a bank. The maturity is 6 months. How will the bank hedge such a position?
- At maturity, the bank will need to deliver one stock to the customer. To hedge such a position, it will first borrow 50 at $t = 0$ and buy one stock. Then the bank will hold it for 6 months.
- 6 months later, the bank will deliver the stock to the customer and receive $F_0(1/2)$ from the customer. The bank will repay $50 \times e^{6\%/2} = 51.52$ to the lender.
- If $F_0(1/2) \neq 51.52$, then there is an arbitrage.

- So what if the stock provides an additional income to the stockholder?
- If the stock pays dividend, the cost of carry decreases.
- Dividend yield, $q = 1\%$ for 6 month period.
- The 6 month forward price now becomes

$$50 \times e^{(6\%-1\%)\times 6/12} = 51.27$$

- Similar hedging argument can be applied here. When the bank bought one stock at $t = 0$ using 50 and the stock pays dividend. It provides an income to the bank.
- This will decrease the cost of carry for the stock so the forward price should be lower.

PnL of forward contract position

Question. You have just taken a long position in a 1.5 year forward contract on 10 shares of a stock. The stock is currently trading at \$100 per share. The risk-free rate and dividend yield are 5% and 1% per year respectively.

- Calculate the 1.5 year no-arbitrage forward price.
- It is now 6 months since you entered into the forward contract. The stock price has risen to \$103 per share. The risk-free rate and dividend yield remain the same. Calculate the value of your position in the forward contract.

Answer.

- The forward price with 1.5 years maturity can be computed as:

$$F_0(1.5) = 100e^{(5\%-1\%)1.5} = 106.18$$

- In 6 months time, the forward price with 1 year maturity is

$$F_{0.5}(1.5) = 103e^{(5\%-1\%)1.0} = 107.20$$

The PnL of the forward contract position for one share at time $t = 0.5$ is

$$\begin{aligned} V_t &= (F_t(T) - K) \times Z_t(T) \\ &= (F_{0.5}(1.5) - F_0(1.5))e^{-5\%} \\ &= (107.20 - 106.18) \times 0.9512 = 0.97 \end{aligned}$$

- We have 10 shares and the PnL is 9.7.

Foreign Exchange (FX) Forward

- FX forward is a contract that two parties exchange cash flows at maturity in two difference currencies with a pre-agreed rate at 0.
- USD interest rate is 2% and JPY interest rate is 1%. USDJPY FX spot is 100. USDJPY 1 month forward price can be computed as:

$$100 \times e^{(1\%-2\%)\times 1/12} = 99.92$$

- USD is the foreign (or base) currency and JPY is the domestic (or quote) currency. It has nothing to do with where you live or transcation takes place. They are market conventions.
- The market conventions for the rankings are EUR,GBP,AUD,NZD,USD,CAD,CHF,JPY.
- Except for EUR,GBP,AUD,NZD, all other major currencies are quoted as USDXXX.

- One can also use the hedging argument to price an FX forward:
- A customer long 1mio USDJPY forward contract against a bank with 1 month maturity. This means the bank will need to sell 1mio USD and buy $F_0(1/12)$ mio of JPY from the customer.
- To hedge such position, the bank will first buy 1mio USD sell 100 mio JPY at $t = 0$. This is equivalent to long 1mio USDJPY FX spot. And the bank hold the positions for 1 month.
- Since the bank long USD, it will receive 2% interest for 1 month in USD. On the flip side, it will pay 1% interest for 1 month in JPY.
- One can regard JPY interest as the "domestic" interest rate and USD interest as the dividend yield. Hence, this is the reason why the second currency (JPY of USDJPY) is called the domestic currency.

- A rule of thumb is to think the higher interest currency (USD) will depreciate against the lower interest currency (JPY). The current spot is 100, the FX forward must be lower than 100 or there will be an arbitrage.
- JPY is expected to be strengthened. This is a consequence of interest rate parity.
- In practice, we implied the non-USD interest rate from the FX forward market. This is because the FX market is much more liquid than the non-USD money market.
- However, there is only one FX forward but 2 interest rates per maturity. In order to sovle this, the market uses USD-OIS (Overnight index swap) rate as the interest rate for USD when computing FX forward.
- Given an USD interest rate, the JPY interest rate can now be implied from the the market traded FX forward rate.

PnL for USDJPY FX forward position

Question

- Today is 19 Apr 2021, you **short** 100mio JPY of USDJPY (short USD, long JPY) valued at 30 Apr 2021, i.e. in 11 calendar days. The forward price is 110. The JPY rate is 1%.
- One day later, the forward price is 109.45. Assume the JPY interest rate is the same, what is the PnL of your position?

Answer

- One way is to work out the USD notional that you **short** at 30 Apr 2021, which is $100 mio JPY / 110 = 909,091 USD$, then we apply the formula:

$$-909,091(109.45 - 110)e^{-1\% \frac{10}{365}} = 499,863 JPY$$

FX swaps

- An FX swap is a simultaneous purchase and sale of one currency for another with two different value dates.
- The value dates are the dates upon which delivery of the currencies. For FX spot, the value date is usually T+2 (important exceptions are USDCAD and USDPHP). For FX forward, it is at some forward date.
- FX swap = FX spot (or forward) + FX forward
- The notionals of the near-leg and far-leg can be different. It is called uneven FX swap.
- FX forward is quoted in terms of forward points which are the difference between the forward price $F_t(T)$ and the spot price S_t:

$$F_t(T) - S_t = S_t \left(e^{(r_d - r_f)(T-t)} - 1 \right)$$

- where r_d and r_f are domestic and foreign interest rates respectively. When interest rate differential gets larger, the absolute value of the forward point increases.

Bloomberg screen - USDJPY FX Forward points

- USDJPY FX spot bid (SP) = 103.58.
- 1m USDJPY forward price bid (1M) = 103.58 - 0.0332 = 103.5468
- Overnight forward points (ON) bid = -0.300
- Tomorrow (Tom) next forward points (TN) bid = -0.123

FX forward price for value today and value tom

- If you buy USDJPY value spot at 103.59, this means you are going to receive 1 USD and pay 103.59 JPY at T+2. T is the date for today and T+2 must be a good business day for both US and Japan.
- For example, today is 1 July 2021 (Thur), 5 July (Mon) is a US holiday. Value spot date is 6 July (Tue). In FX, value date does not mean trade date, it means the date for the cash flow settlement.
- Using the market data above, what's the price to buy USDJPY value TODAY and TOM?
- Value TOM offer is 103.59123 = 103.59 - (-0.00123) = spot offer - T/N bid.
- Value TODAY offer is 103.59423 = 103.59123 - (-0.003) = TOM offer - O/N bid.

What time does the value date T change?

- Now we know that long a USDJPY FX spot is equivalent to long USD and short JPY simultaneously at a value date $T+2$. The natural question is, what time does T change in the day?
- It turns out that if you long USDJPY at NY time 4.59PM and hold the positions for 1 minute, you will earn the full day interest in USD and pay the fully day interest in JPY.
- The roll time is NY time 5PM, except for NZD. The roll time for NZD is Wellington time 7AM which is earlier than NY time 5PM.

Non-Deliverable Forward

- It is a forward FX contract where the profit or loss on the trade is settled in USD. The PnL on the trade is calculated using the Fixing Rate published on the Fixing Date agreed in the contract.
- The payoff of an NDF is

$$N_{USD}(S_T - K)/S_T$$

where S_T is the Fixing Rate for the settlement date $T + 2$, K is the strike that is agreed at the inception of the contract. N_{USD} is the USD Notional.

- E.g. if $N_{USD} = \$1mio$, $K = 16000$ and $S_T = 16100$ then the party who long the contract receives

$$1,000,000 \times (16100 - 16000)/16100 = \$6211.18$$

- Many restricted Asian currencies have active NDF markets, e.g. USDINR, USDKRW, USDIDR and USDTWD. The fixing rate methodology, fixing source, fixing published time can be found in the corresponding central bank websites.

- The most liquid tenor is $T = 1$ month.
- In a deliverable spot or FX forward transaction if the value date of the transaction rolls down to today, we simply hold the long currency and run an overdraft on the short currency. However, in an NDF transaction, the contract effectively closes out on the fixing date.
- If we traded a 1 month NDF and we did not close the position, then we hedge the position the next day using 1 month NDF, we will create a mismatch in cashflows.
- Complication comes in when trading cross NDF, e.g. SGDINR. It requires fixing for both USDINR and USDSGD.
- Market usually uses the ABS fixing for USDSGD which is computed using the volume weighted average of the USDSGD spot transaction on the qualified platforms (brokers, EBS and Reuters) between 10.30am to 11.00am on the fixing date.

Futures contract

- A futures contract is an exchange-traded contract in which the holder has the obligation to buy/sell an asset on a future date at a market-determined price called the futures price.
- The contract specifications like quantity, time and place of delivery are determined by the exchange.
- The assets can be a commodity, a stock, an index, etc. . .
- Futures is considered safer than forward since the counter-party risk is almost totally eliminated.
- Futures contract are marked-to-market on a daily basis.
- Every buyer/seller must maintain a certain amount with an account at the exchange.
- If the price movement is against you, the balance of your account reduces. If it drops below the maintenance margin then you will need to top up to the initial margin.

Example: Mark-to-Market

- Futures price: \$100
- Initial margin: \$7 per contract
- Maintenance margin: \$4 per contract
- Investor A buys 500 contracts, while investor B sells (shorts) 500 contracts
- Initial margin: $500 \times \$7 = \$3,500$
- Maintenance margin: $500 \times \$4 = \$2,000$
- Also see the link for more details about initial and maintenance margin.

https://www.cmegroup.com/education/courses/introduction-to-futures/margin-know-what-is-needed.html

Example: Mark-to-Market

Trading Day	Settlement Price
1	\$99
2	\$97
3	\$98
4	\$95

- Settlement price is the price used for mark-to-market.

Investor A's Margin Account.

- Day 1
  - PnL $= (\$99 - \$100) \times 500 = -\$500$
  - Balance $= \$3500 - \$500 = \$3000$
- Day 2
  - PnL $= (\$97 - \$99) \times 500 = -\$1000$
  - Balance $= \$3000 - \$1000 = \$2000$
- Day 3
  - PnL $= (\$98 - \$97) \times 500 = \$500$
  - Balance $= \$2000 + \$500 = \$2500$
- Day 4
  - PnL $= (\$95 - \$98) \times 500 = -\$1500$
  - Balance $= \$2500 - \$1500 = \$1000$
  - Receives a margin call to deposit \$2500 to bring the balance up to initial margin.
- Total PnL $= \$1000 - \$3500 = -\$2500$

Investor B's Margin Account.

- Day 1
  - PnL $= (\$100 - \$99) \times 500 = \$500$
  - Balance $= \$3500 + \$500 = \$4000$
- Day 2
  - PnL $= (\$99 - \$97) \times 500 = \$1000$
  - Balance $= \$4000 + \$1000 = \$5000$
- Day 3
  - PnL $= (\$97 - \$98) \times 500 = -\$500$
  - Balance $= \$5000 - \$500 = \$4500$
- Day 4
  - PnL $= (\$98 - \$95) \times 500 = \$1500$
  - Balance $= \$4500 + \$1500 = \$6000$
- Total PnL $= \$6000 - \$3500 = \$2500$

Futures Price vs Forward Price

- A futures price may be different from a forward price when a margin account earns interest.
- If risk-free interest rate is constant, then the futures price will be the same as the forward price.
- If risk-free interest tends to move in the same direction as the futures price (i.e. positively correlated), then the long party tends to benefit at expense of the short party.
- To compensate, the futures price will be higher than the corresponding forward price.
- Conversely, if risk-free interest tends to move in the opposite direction as the futures price (i.e. negatively correlated), then the short party tends to benefit at expense of the long party.
- To compensate, the futures price will be lower than the corresponding forward price.

Common futures contracts

- The most liquid futures contracts in the world are futures on SP500 and US Treasuries.
- SP 500 E-mini - https://www.cmegroup.com/trading/why-futures/welcome-to-e-mini-s-and-p-500-futures.html

E-mini S&P 500 futures | View Full Contract Specs

Contract Size	$50 x S&P 500 Index
Minimum Tick	0.25 index points
Dollar Value of One Tick	$12.50 U.S. Dollars
Product Symbol	ES
Trading Hours	Sunday - Friday 5:00p.m. - 4:00p.m. CT with a trading halt from 3:15p.m. - 3:30p.m. CT; Daily Maintenance period Monday - Thursday 4:00p.m. - 5:00p.m.
Contract Months	Nearest five months in the quarterly cycle (Mar, Jun, Sep, Dec)
Options Available	Quarterly, Monthly, Weekly (Monday, Wednesday, Friday)

Common futures contracts

- 10Y US Treasuries - https://www.cmegroup.com/trading/why-futures/welcome-to-us-treasury-futures.html

	2-Year Note	5-Year Note	10-Year Note	Ultra 10	T-Bond	Ultra T-Bond
Contract Size	$200,000	$100,000	$100,000	$100,000	$100,000	$100,000
Deliverable Maturities	1 3/4 to 2 years	4 1/6 to 5 1/4 years	6 1/2 to 10 years	9 5/12 to 10 Years	15 years up to 25 years	25 years to 30 years
Product Symbol	ZT	ZF	ZN	TN	ZB	UB
Contract Months	Quarterly: March, June, September and December					
Trading Hours	Electronic: 5:00p.m. - 4:00p.m., Sunday - Friday (Central Time)					
Minimum Tick	1/8 of 1/32 of 1 point	1/4 of 1/32 of 1 point	1/2 of 1/32 of 1 point	1/2 of 1/32 of 1 point	1/32 of 1 point	1/32 of 1 point
Dollar Value of One Tick	$7.8125	$7.8125	$15.625	$15.625	$31.25	$31.25
Options Available	Quarterly, Serial, Weekly (Fridays & Wednesdays)					

Where can we trade derivatives?

- Some derivatives are traded on an exchange, such as Chicago Mercantile Exchange (CME).
- Exchange-traded derivatives standardize terms and conditions, high liquidity, and price transparency. For example, FX futures contract.
- Clearing house operated by exchange serves as "central counter-party" to ensure that all transactions are completed.

Where can we trade derivatives?

- Other derivatives, known as "over-the-counter" (OTC) derivatives, are direct bilateral agreements with no central counter-party. For example, FX forward contract.
- Generally, one party will be major financial institution acting as "market maker".
- OTC Derivatives have more flexibility for terms and conditions, but less price transparency and no protection from credit risk in general.
- After the Great Financial Crisis, there are more regulations and cost of capital requirement to trade OTC derivatives compare to exchange-traded derivatives.

Exchange-Traded vs OTC

Exchange-Traded

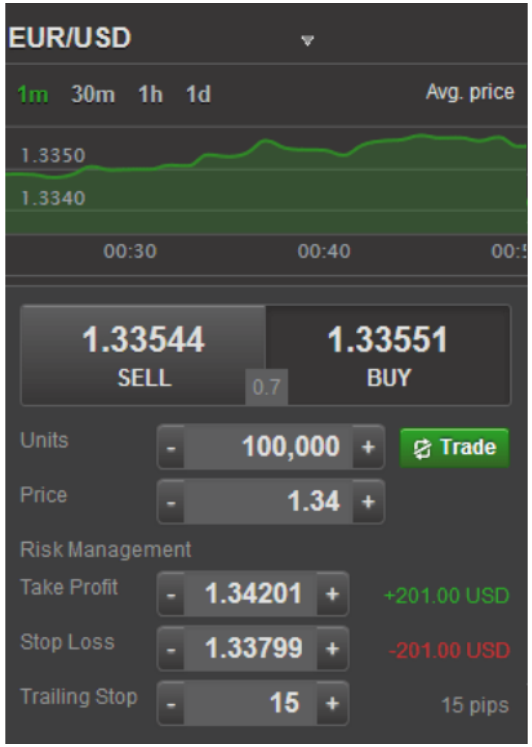
- Central counter-party eliminates almost all credit (default) risk.
- Standardized products and pricing transparency.

OTC

- Both parties subject to credit (default) risk.
- Flexible products but pricing opacity.
- More regulations and cost of capital. E.g. xVA, SIMM.

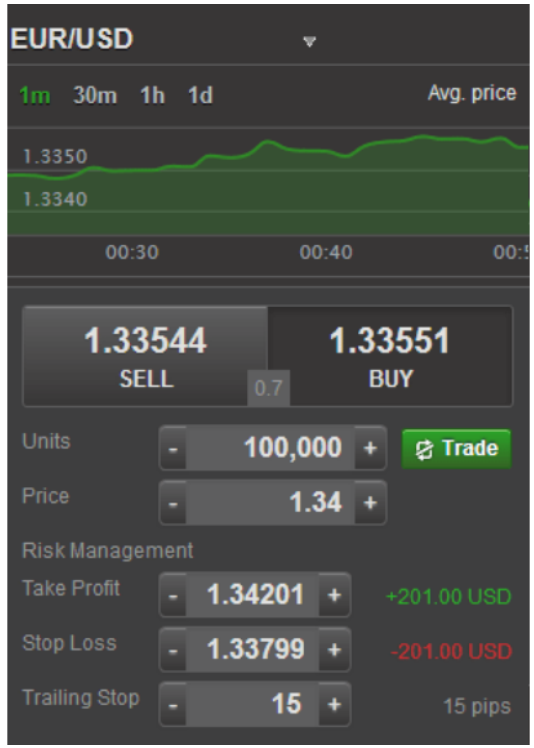
Market Marking

- OTC derivatives generally require a market marker as a counterparty. So what is market marking?
- If you want to long 100k EURUSD, what would you do?
- You find an FX broker, open and account, put some money into the account and then trade.
- Most of them will have a mobile app looks like below.



Some FX jargon

- For EURUSD, the 2nd digit after the dot is called "the big figure"/handle (i.e. 3)
- The 4th digit after the dot is called one pip (i.e. 4)
- 100 pips = 1 big figure
- The last digit is 1/10 of a pip.



- If you want to buy 100K EURUSD now, the price is 1.33551.
- But if you want to sell 100K EURUSD now, the price is 1.33544.
- Note that if you buy and sell simultaneously then you will guarantee to lose $100k \times (1.33544 - 1.33551) = \$7$.
- Who earns that \$7? Market makers.
- The \$7 is the bid-offer spread.

- Market makers are there to provide the liquidity to trade even in extreme market conditions, like the SNB un-pegged CHF against EUR on 15 Jan 2015.
- On return, they charge a spread to provide that service.

- EURCHF dropped to 0.85 from 1.2 in a matter of minutes.

- A few brokers went bust because of that event.

City forex broker and West Ham sponsor goes bust as clients suffer huge losses from Swiss currency bombshell

- **Alpari UK applies for insolvency after clients rack up huge losses**
- **Swiss central bank shocked currency markets by removing euro peg**
- **Positions held by forex traders went haywire as Swiss franc soared 13-15% against euro, sterling and dollar**

By ADRIAN LOWERY FOR THISISMONEY.CO.UK
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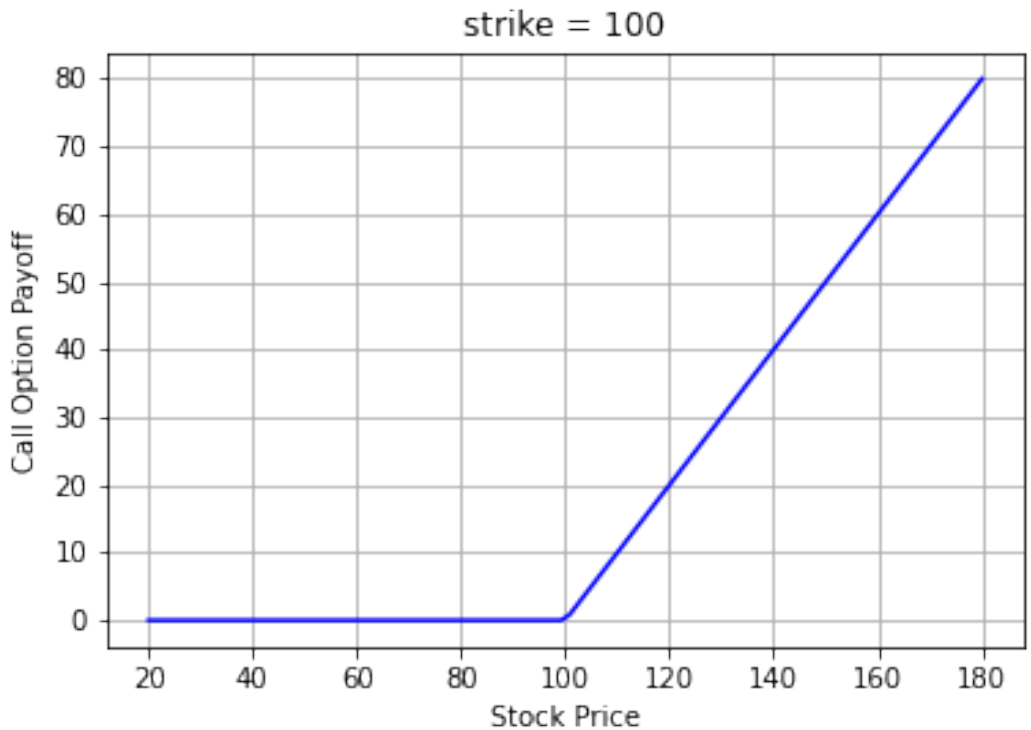
Evolution of the FX market

- 25 Years Ago
  - If you are a bank, you trade on: Reuters, EBS or voice broker
  - If you are a client: you call a bank
- 15 Years Ago
  - Banks: Reuters, EBS or voice broker
  - Client: the first offerings from ECN and Single Bank Platforms
- Now
  - Banks: Multiple venues of which Reuters and EBS are no longer dominant. Often all viewed on an internal aggregator.
  - Client: More ECN and platform choices than they can possibly use.
- The FX spot market marking business becomes highly automated. Forwards, NDF, swaps and options will go to the same direction. This requires different skill sets and strategies for banks and its employees.
- An interesting series of videos by DB, https://www.youtube.com/watch?v=lWbkhsgct7I

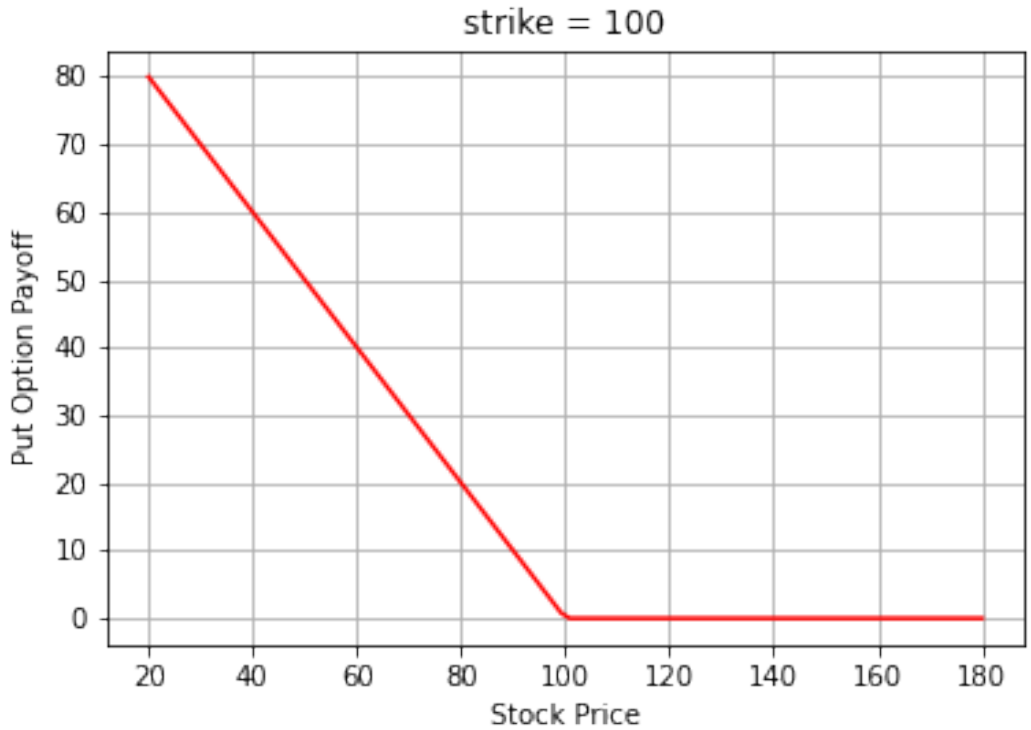
Call Option

- Call option is a right for the holder (or long party) to buy underlying asset from the writer (or short party).
- At maturity T, the payoff is $max(S_T - K, 0)$, where S_T is the stock at time T and K is the strike price.
- One can think of a call as the right to receive 1 stock and pay K in cash at maturity T.



Put Option

- Put option is a right for the holder to sell underlying asset to the writer.
- At maturity T, the payoff is $max(K - S_T, 0)$.
- One can think of a put as the right to sell 1 stock and receive K in cash at maturity T.



Option Contract

- The long party makes payment of option premium to short party. The payment can be upfront (i.e. at the trade date) or any time before the maturity (the price will be different).
- European-style option may only exercised at maturity.
- American-style option may be exercised at any time up to the maturity.
- Bermudan-style option may be exercised once at a fixed period of times up to the maturity.
- Option may be exchange-traded or OTC. Note that CME E-mini SP 500 Options (exchanged trade option) is American-style.
- Exchange-traded options tend to be more standardized and liquid then OTC options.

Example: Option on IBM

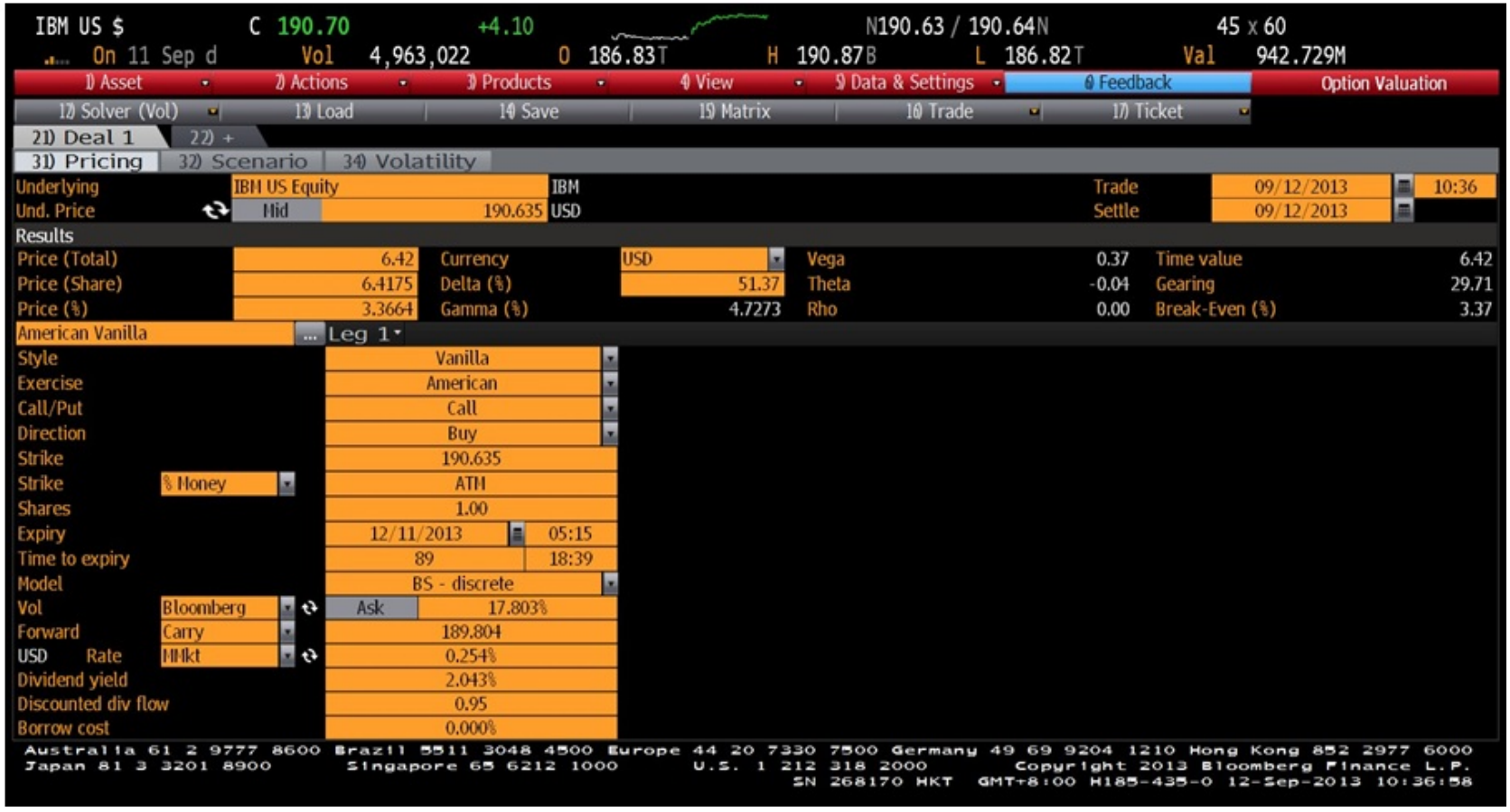


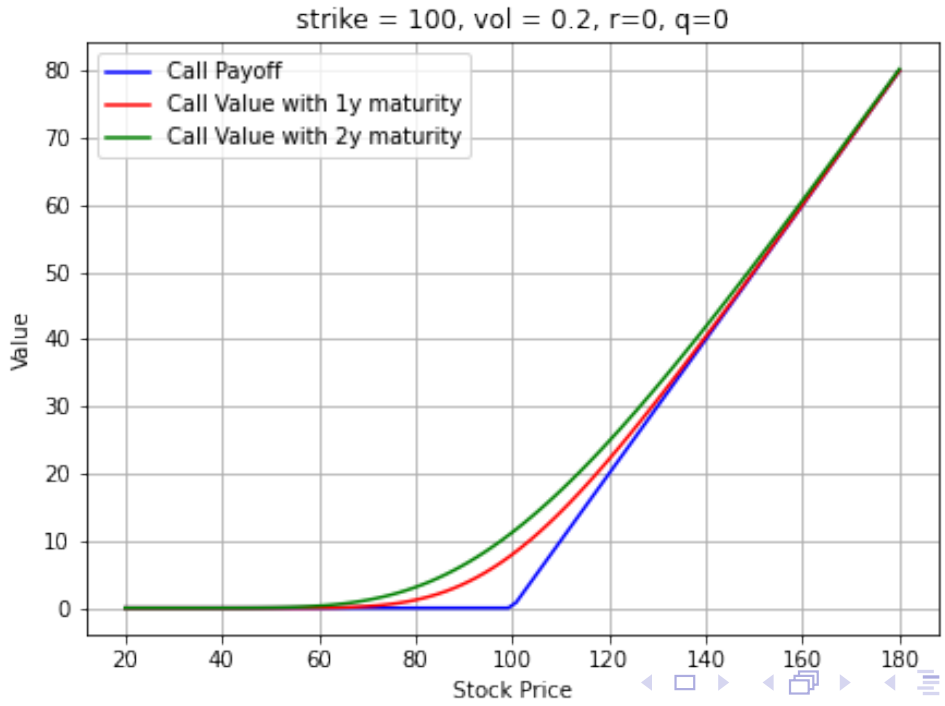
Figure: Bloomberg option pricer

Moneyness

- Let S_t be the spot price of an underlying asset at time t.
- Let K be the exercise price.
- Moneyness is defined as $\frac{K}{S_t}$.
- Option is in-the-money when exercise is profitable: $S_t > K$ for call and $S_t < K$ for put.
- Option is out-of-the-money when exercise is not profitable: $S_t < K$ for call and $S_t > K$ for put.
- Option is at-the-money when $S_t = K$.
- Note that this is not the only definition of moneyness.
- Moneyness can be set with respective to forward in some market like interest rates option. If $K = F_t(T)$, this is called at-the-money forward.
- For FX option, ATM strike is the strike such that a straddle has no delta. See Chapter 3.5 in Foreign Exchange Option Pricing: A Practitioner's Guide by Iain Clark for many more types of moneyness in FX.

Option Value

- Option Value = Intrinsic Value + Time Value.
- Intrinsic value is the payoff from immediate exercise: $max(S_t - K, 0)$ for call option and $max(K - S_t, 0)$ for put option.
- Time value is the value of not exercising since intrinsic value may increase if we don't exercise. It is a function of the volatility.



Put-Call Parity

- Let c_t and p_t be values of European call and put at time t respectively, with the same underlying S, strike price K and maturity T.
- Combination of long call and short put at the same strike delivers the same payoff as long a forward contract:

$$c_t - p_t = (F_t(T) - K)Z_t(T)$$

.

Put-Call Parity

It is easier to see the relationship at maturity, i.e. $t = T$:

$$max(S_T - K, 0) - max(K - S_T, 0) = S_T - K$$

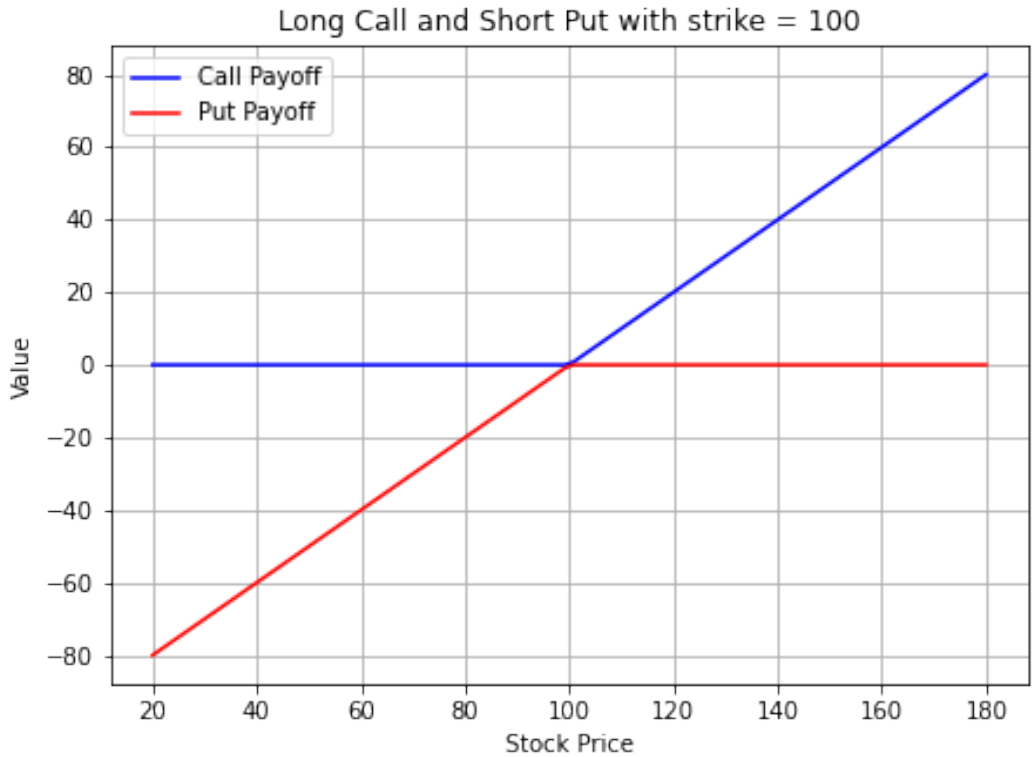


Figure: The payoff for long call and short put with the same strike.

American option

- European option can only be exercised at the maturity.
- American option can be exercised at any time before the maturity.
- Let C_t, P_t be an American call and put at time t.
- It is easy to see that $C_t \geq c_t$ and $P_t \geq p_t$ if all other terms are the same because American option has more choice than European option.
- Is there a situation that $C_t = c_t$ and $P_t = p_t$?

American call option

- Without dividends, we never exercise an American call early.
- Exercise early requires paying the strike price early, hence loses the time value of money because he doesn't receive interest on this cash amount.
- On the other hand, he would receive future dividends for holding the stock.
- If dividend yield is higher than the interest rate then it MAY BE optimal to exercise.
- And of course it also depends on where the spot is compare to the strike price.

American put option

- Without dividends, it can be optimal to exercise an American put early.
- Consider an American put with $K = 100$ on a stock with S_t very very close to 0.
- S_t cannot go any lower and the put option pays $max(K - S_t, 0) = 100$ is the maximum that one can earn for holding this put option.
- Exercise now, the option holder gets 100 today.
- Exercise later, the option holder gets 100 later.

Binomial Model for American option

- Assume the asset price S can go up to uS and down to dS during a time period Δt. Risk free rate is denoted as r. We have the following one period model:

$$uSq + dS(1-q) = Se^{r\Delta t}$$

Rearrange and we get the risk neutral up probability

$$q = \frac{e^{r\Delta t} - d}{u - d}$$

- We will refer to a state in the tree as a node and label nodes (i, j), where i indicates the number of time steps from time zero and j indicates the number of upward movements the asset price has made since time zero.

- We are free to choose u and d. However, for numerical efficiency, a popular choice is $u = 1/d$. This will yield a recombining binomial tree. We can then use the remaining free variable u to calibrate to the volatility.
- The asset price at node (i, j) is $S_{i,j} = Su^j d^{i-j}$ and the option price will be $P_{i,j}$.

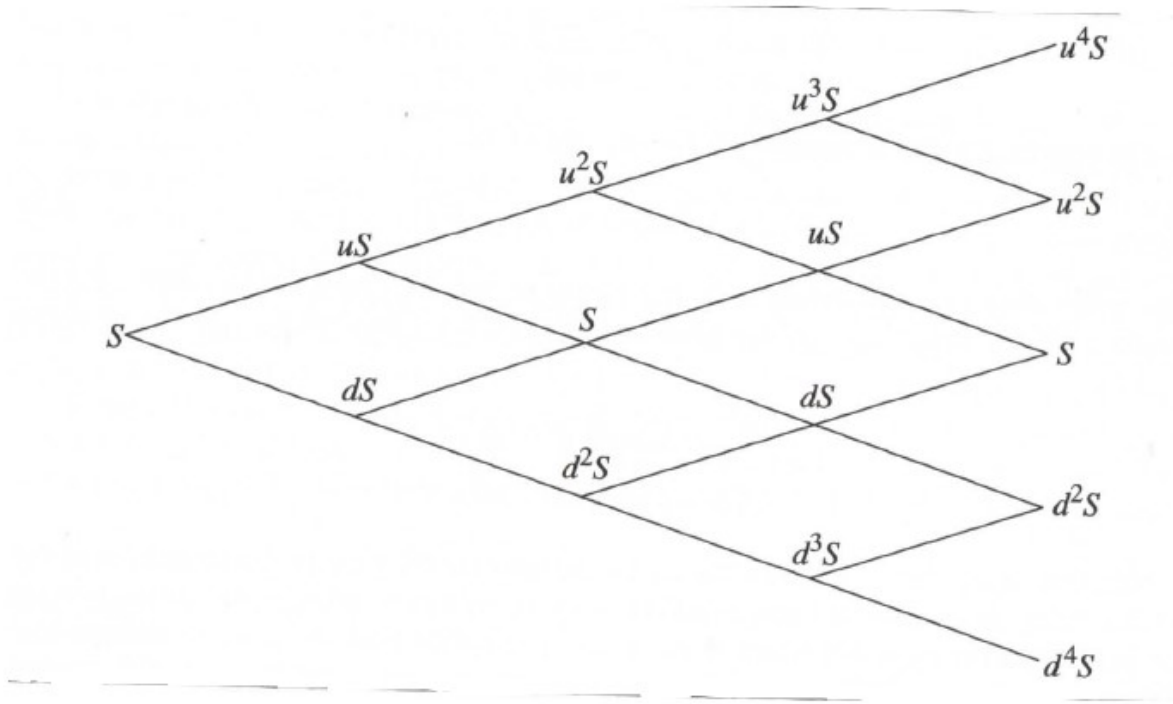


Figure: Four steps binomial tree

- We assume N time steps in total, where the N-th time step corresponds to the maturity date of the option.
- The value of the put option at the maturity is known:

$$P_{N,j} = max(K - S_{N,j}, 0)$$

- The holding value of the put option at time $i < N$ can be computed by backward induction:

$$H_{i,j} = e^{-r\Delta t}(qP_{i+1,j+1} + (1-q)P_{i+1,j})$$

- An American Put option at node (i, j) is the maximum value of the holding value and the immediate exercise value

$$P_{i,j} = max(H_{i,j}, K - S_{i,j})$$

- The value of the American Put option is given as $P_{0,0}$.

Time Option

- FX time option is a special type of forward contract that allows the buyer to enter into a spot FX transaction in a range of dates.
- A typical user of FX time option is corporate which has foreign currency receivables and the corporate wants to have the flexibility to decide when to hedge those foreign currency exposures.
- E.g. Today is 7 Sep 2020. A client wants to sell 10mio USD buy SGD between the 9 Sep 2020 to 4 Jan 2021.
- The holder (client) of this time option would have the right to sell 10mio USD and buy SGD at some rate at anytime between the period. However, the holder must exercise at the expiry, i.e. it just works like an FX forward but with an early exercise feature. When it is exercised, it becomes a FX spot transaction which the value date is normally T+2.

Time Option

The top diagram shows the cash flows of the time option at inception, $T = 0$. The bottom diagram shows the cash flows of the time option when it is exercised after inception at some time $t > 0$. At time t, when exercised, the time option becomes an FX spot with the agreed price, in this case, sell USDSGD at 1.35.

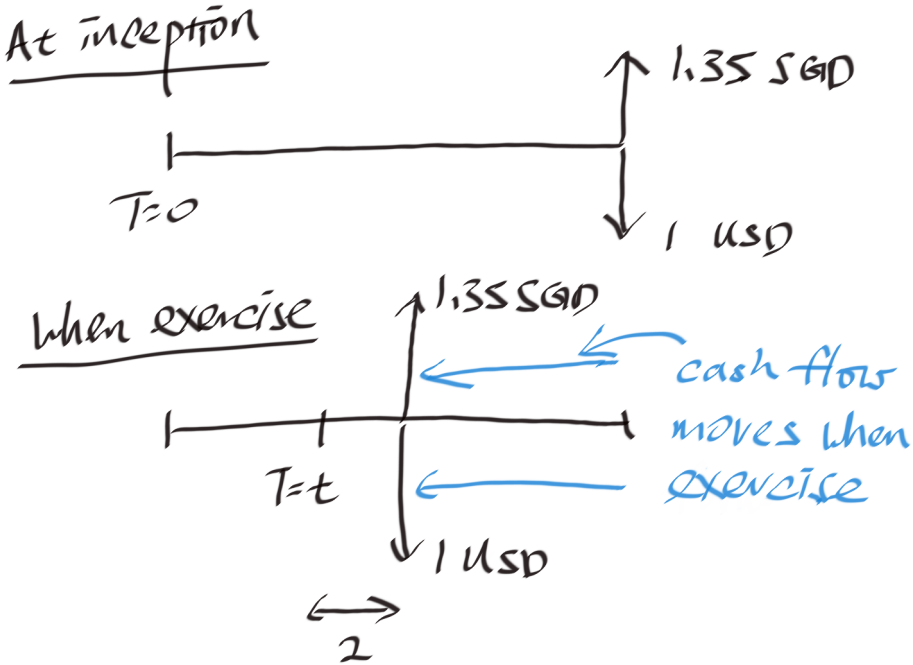


Figure: Cash flows of the time option when exercised.

- How would you price such a product if you are the seller? In other words, what rate would you agree to such a contract?
- If the time option holder buy USD sell SGD, i.e. she is lifting your offer side. You price it using the highest forward point in that period.
- Since you cannot predict when the holder will exercise the time option, i.e. converting it to a FX spot contract, you just price it assuming the worse case scenario.

- If USDSGD forward points is positive, e.g. USDSGD spot is 1.35 and USDSGD 1y forward is 1.36. The client purchases a time option to buy 10mio USD sell SGD between a period from spot to 1y. You would price it at 1.36.
- If the client early exercise the time option immedidately, she will buy 10mio USD and sell 13.6mio SGD value spot. The transaction would effectively cost him 100k SGD as he could have bought 10mio USDSGD at 1.35 instead.
- If the client decides to hold until maturity, then she will buy 10mio USDSGD at 1.36 in 1y time. This is the same as a standard FX forward.
- On the other hand, If the client purchases a time option to sell 10mio USD buy SGD between a period from spot to 1y. You would price it at 1.35.

- The motivation of buying time option is a function of the corporate needs, the level of the spot and the shape of the forward curve.
- To hedge such product, one would normally first hedge the spot risk using FX spot. For example, if client purchases a time option to buy USDSGD from spot to 1y, one would first sell USDSGD spot. This creates a sell/buy USD (or buy/sell SGD), spot to 1y, FX swap position.

Summary

- Time value of money. Cost of carry.
- Forward price and forward contract.
- Compute the PnL of a forward contract position.
- FX forward, FX swap and NDF.
- Futures contract and compute the PnL of a futures contract position.
- Futures price vs Forward price.
- Market making, exchange traded, OTC and credit.
- European call and put option. Moneyness and Put-Call Parity.
- American option vs European option.
- Time option.