QF602 Derivatives

Lecture 9 - Structured Products

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What are Structure Products

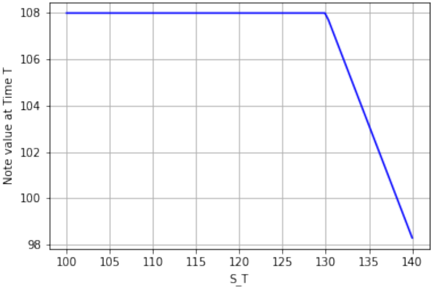
- Structured products are financial instruments whose performance or value is linked to that of an underlying asset, product, or index. These may include market indices, individual or baskets of stocks, bonds, and commodities, currencies, interest rates or a mix of these.
- In this course so far, we have covered forwards, futures, vanilla options, American options, barrier options, quanto and other exotic options. All of these can be regarded as building blocks of structured products.
- There are in general two forms of structured products, note form and swap form.

Note Form

- You are an investor who is bearish on a stock S and would like to sell an upside call option on **one unit** of S with maturity T to a bank.
- Since you short an option, there is a risk to the bank if you walk away when the option is in-the-money. There are a few ways to mitigate **the credit contingent market risk**. **Structured note** is one of the ways to achieve that.
- The bank would issue a structured note which has present value N_0 (usually at par, i.e. 100) for you to purchase. The value of the note at time T would be
$$N_0(1 + C_T) - (S_T - K)^+$$
where C_T is the coupon (in %) that you receive at time T by selling the option. The component $-(S_T - K)^+$ is the obligation for selling the option on one unit of S.
- Since you paid N_0 to the bank at time 0, unless $N_0 C_T - (S_T - K)^+ < -N_0$, the bank would have no risk for you to walk away in all circumstances. The whole term, $N_0 C_T - (S_T - K)^+$, is called a **structured coupon**.
- This is a common way for a bank to raise capital especially if the bank does not have a strong presence in the deposit taking business.

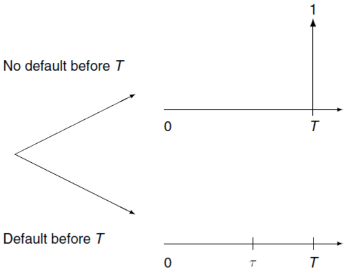
Example

- $S_0 = 100$, $K = 130$, $T = 1$. Assume the value of $C_T = 0.08$ and $N_0 = 100$.
- The graph shows the value of the note at maturity T for various S_T. The break-even spot price S_T for the investor is 138.
- One might wonder if the investor has missed out on the potential interest rate that one can earn if the 100 dollars is not used to purchase the structured note. In fact, bank's funding rate is one of the key factors to determine the value of C_T. If a bank has a funding need, it can offer higher funding rate (which means higher C_T) to attract investor to purchase its structure note assuming everything else being equal.



Modelling Credit Risk of the Bank

- Different banks would have different funding rates. One way to interpret that is that banks have different credit risks. Higher the credit risk, the market demands higher funding rate. We here present a way to model the credit risk.
- Let τ be the default arrive time. Consider a credit risky zero coupon bond issued by a bank with a face value of 1 which is due to mature at time T. Assume also that if there is a default, the recovered amount will be zero.
- The two possible payments are shown in the diagram below.



- We can write the present value of this risky zero coupon bond as the discounted expectation of the payoff at time T in the risk-neutral measure:

$$\hat{Z}_0(T) = E_0\left[e^{-\int_0^T r dt} 1_{\tau > T}\right]$$

where r is the risk free rate and we use the indicator function to capture the risk of default

$$\begin{aligned} 1_{\tau > T} &= 1 \ \textit{if} \ \ \tau > T \text{ and the bond survives} \\ &= 0 \ \textit{if} \ \ \tau \leq T \text{ and the bond defaults} \end{aligned}$$

- To price this bond, we need to know the (risk-neutral) survival probability $\mathbb{P}(\tau > T)$. If we further assume the default arrival time is independent to the risk free rate, we have

$$\begin{aligned} \hat{Z}_0(T) &= E_0\left[e^{-\int_0^T r dt} 1_{\tau > T}\right] \\ &= E_0\left[e^{-\int_0^T r dt}\right] E_0\left[1_{\tau > T}\right] \\ &= Z_0(T)\mathbb{P}(\tau > T) \end{aligned}$$

- The next question is: how do we model the default arrive time τ in order to compute $\mathbb{P}(\tau > T)$?

- A common approach is to model the default as the **first** arrival time of a Poisson process.
- Recall that for a Poisson process with intensity λ, the probability that there is **no jump** for a given time interval $[0, T]$ is $e^{-\lambda T}$. If there is no default (i.e. no jump) between $[0, T]$, this means the bond survives. In other words, we have

$$\mathbb{P}(\tau > T) = e^{-\lambda T}$$

- The risky zero coupon bond can be computed as

$$\hat{Z}_0(T) = Z_0(T)e^{-\lambda T}$$

- If we further assume r is constant, we have

$$\hat{Z}_0(T) = e^{-(r+\lambda)T}$$

We can interpret $r + \lambda$ as a credit adjusted discount rate or funding rate.

Example

Question

- A bank issues a zero coupon bond with maturity $T = 1$ with face value 100. The market price of the zero coupon bond is 95.5. The market risk free rate is 0.02. What is the implied default intensity λ? What is the risk neutral survival probability $\mathbb{P}(\tau > T)$? What is the expected default time?

Answer

- The PV of the risky zero coupon can be computed as:

$$\hat{Z}_0(T) = e^{-(r+\lambda)T}$$

The implied default intensity is

$$\lambda = -\frac{\ln(\hat{Z}_0(T))}{T} - r$$

Substitute the values and we have

$$\lambda = -\frac{\ln(95.5)}{1} - 0.02 = 0.026044$$

The survival probability is

$$\mathbb{P}(\tau > T) = e^{-0.026044 \times 1} = 0.97429$$

- If we model the default as the first jump of a Poisson process, the default arrival time τ is exponentially distributed with the parameter λ. The density function of τ is

$$f(t) = \lambda e^{-\lambda t}$$

for $t \geq 0$. We can calculate the expected default time using integration by parts

$$\begin{aligned}
E_0[\tau] &= \int_0^\infty t f(t) dt \\
&= \int_0^\infty t \lambda e^{-\lambda t} dt \\
&= -\left[t e^{-\lambda t} \right]_0^\infty + \int_0^\infty e^{-\lambda t} dt \\
&= 0 - \left[\frac{e^{-\lambda t}}{\lambda} \right]_0^\infty \\
&= \frac{1}{\lambda} = 38.4 \text{ years}
\end{aligned}$$

Example

Question

- An investor wants to buy a structured note with present value $N_0 = 100$ which has maturity $T = 1$ and receives the notional N_0 and the following structured coupon at maturity:

  $$N_0 C_T - (S_T - K)^+$$

  where S_T is the spot at T, K is the strike price and C_T is the coupon which pays at T. There are 2 banks competing to win the deal. Bank A has a better credit rating with a more aggressive trading desk. Bank B has a lower credit rating with a less aggressive trading desk. Here are the relevant market data

Bank	A	B
option premium paid at T	0.05	0.04
risk free rate r	0.02	0.02
default intensity λ	0.03	0.05

  Option premiums are expressed in percentage of N_0.
- Which bank's structured note offers a higher C_T?

Answer

- C_T is a function of the option premium paid at T and the funding rate. The funding rate is the risk free rate plus the default intensity.
- For bank A, its trading desk can show the bid for the option at 0.05 (i.e. the investor can sell at 0.05). But its funding desk is only willing to give $0.02 + 0.03 = 0.05$ funding rate. In other words, if you deposit $N_0 = 100$ to bank A, it will return you with $100 \times e^{(0.02+0.03)\times 1} = 105.13$ in one year time. This means the maximum value of C_T that bank A can offer is

$$0.05 + e^{(0.02+0.03)\times 1} - 1 = 0.1013$$

- For bank B, the maximum value of C_T it can offer is

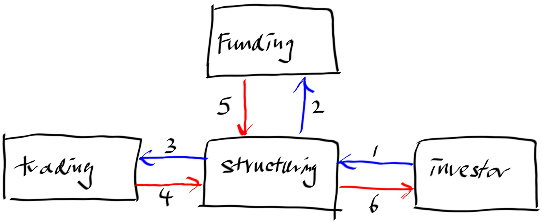
$$0.04 + e^{(0.02+0.05)\times 1} - 1 = 0.1125$$

- We can see that having a more sophisticated pricing model doesn't necessary mean winning the deal in terms of structured note pricing. The bank's funding rate is also an important factor.

Relationship of the desks

In the previous example, we can see pricing a structured note involves the trading desk and the funding desk. We are going to show a typical work flow of pricing a structured product on a trading floor. Let's start with a structured note that we used in the previous example for bank B.

1. Investor pays 100 to Bank B's structuring desk to purchase the structured note with maturity $T = 1$.
2. Structuring deposits 100 to the funding desk for 1 year at the continuous rate 0.07.
3. Structuring sells an option to the trading desk and receives the option premium which is worth $100 \times 0.04 = 4$ at T.
4. Structuring receives 4 from trading and pays $(S_T - K)^+$ to trading at T
5. Structuring receives 107.25 from the funding desk at T.
6. Structuring pays the investor $100 + 7.25 + 4 - (S_T - K)^+$ at T

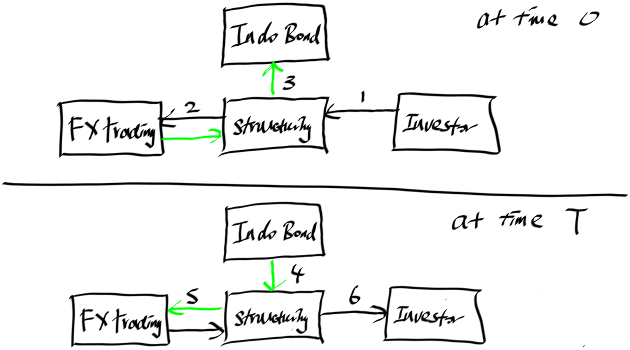


Credit Linked Note

- Assuming an investor would like to buy an Indonesian government bond denominated in IDR because it offers higher yield. Due to some restrictions, the investor does not have access to it directly.
- A bank can offer 2 ways for the investor to have an exposure to the bond.
- The first way is using the note form. The investor pays the bank in USD, the bank would change it to IDR using the prevailing FX rate to purchase the bond on behalf of the client. When the bond generates cash flow in IDR, the bank would convert the IDR to USD using the prevailing FX rate and pass it to the investor.
- The process is the similar to the equity structured note in the previous example except it doesn't involved the funding desk. The structuring desk simply collects the USD from the investor, convert it to IDR and purchase the bond on behalf of the investor.
- This product is called credit linked note (CLN) or funded CLN. It is called funded because the investor doesn't borrow any money from the bank in this transaction.

Here is the work flow of the funded CLN.

1. Investor pays 100 USD to structuring to purchase the CLN with maturity $T = 1$ denominated in USD.
2. Structuring converts 100 USD to IDR using the prevailing USDIDR spot rate with the FX trading desk.
3. Structuring purchases the Indo Govie denominated in IDR.
4. At maturity, structuring sells the Indo Govie and get back the IDR.
5. Structuring converts the IDR to USD using the prevailing USDIDR spot.
6. Structuring pays the investor the USD from selling the bond.



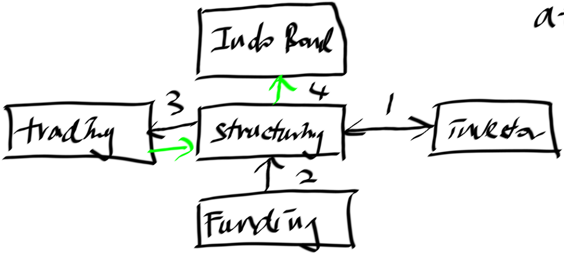
Swap Form

- The second way is to use a swap form. This is suitable for an investor who does not want to pay the USD upfront, instead, one would borrow the USD from the bank and pays the interest in USD instead. This structure is called unfunded total return swap (TRS).
- Since there is no collateral given to the bank, the bank is exposed to the risk that the investor would walk away. Therefore, the swap form is usually accessible to an investor who has an existing relationship or a legally enforceable claim to the investor in case one defaults on the payment.

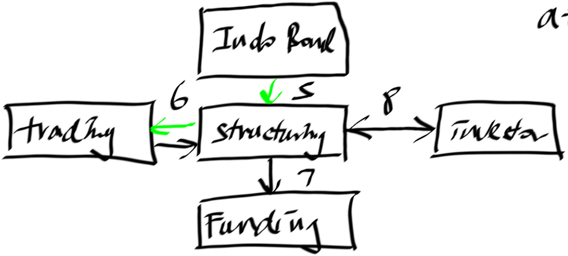
Here is the work flow of the unfunded TRS:

1. Investor enters a swap contract with structuring in which the investor pays the borrowing cost on 100 USD and receives the cash flow generates by the bond in USD. No cash flow is exchanged.
2. Structuring borrows 100 USD from the funding desk.
3. Structuring converts 100 USD to IDR using the prevailing USDIDR spot rate.
4. Structuring purchases the Indo Govie denominated in IDR.
5. At maturity, structuring sells the Indo Govie and get back the IDR.
6. Structuring converts the IDR to USD using the prevailing USDIDR spot rate.
7. Structuring returns 100 USD plus interest to the funding desk.
8. Structuring pays the USD from selling the bond and receives the interest on 100 USD from the investor.

at time 0



at time T



Target Redemption Forward

- A target redemption forward (TARF) is a foreign exchange product that allows the holder, usually a corporate, to buy or sell a currency at **an enhanced rate** for a number of expiry dates, with zero upfront premium. The product automatically expires if a **knock out target level** is reached. But if spot moves in the wrong direction, holders can be forced to trade regularly at **unfavourable rates** for the full life of the product.
- Perhaps the easiest way to explain is by a few real-world examples. There are many variations of TARF.

We are going to cover one of the variations called Discrete Pivot TARF on USDTWD. Here are some important features of the product:

- There are 5 Fixing and Delivery dates in this product.
- The settled amount for each period is a function of the corresponding USDTWD Fixing Rate only.
- The accumulated intrinsic count **will not have effect** on the knock out target. This means this TARF is just a series of 5 **European payoffs** with different Fixing and Delivery dates.

The term sheet can be found on eLearn which contains the full details.

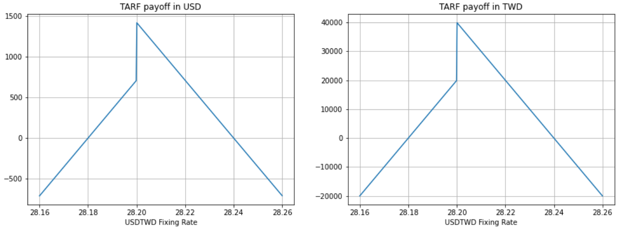
Example: Discrete Pivot TARF on USDTWD

- The table below shows the payoff from the holder's perspective of the product for a given sample path of USDTWD Fixing Rate.

No.	USDTWD Fixing Rate	Payoff in USD
1	28.25	-353.98
2	28.23	354.23
3	28.21	1063.45
4	28.19	354.74
5	28.17	-354.99

- Payoff number 1 is given as: $-\frac{(28.25-28.24)}{28.25} \times 1,000,000 = -353.98$
- Payoff number 2 is given as: $\frac{(28.24-28.23)}{28.23} \times 1,000,000 = 354.23$
- Payoff number 3 is given as: $\frac{(28.24-28.21)}{28.21} \times 1,000,000 = 1063.45$
- Payoff number 4 is given as: $\frac{(28.19-28.18)}{28.19} \times 1,000,000 = 354.74$
- Payoff number 5 is given as: $-\frac{(28.18-28.17)}{28.17} \times 1,000,000 = -354.99$

- Note that TWD is a restricted currency. In general, most USDTWD derivatives cannot be settled in TWD. Therefore, it is settled in USD and that's the reason why the settled amount is divided by USDTWD Fixing Rate to convert from TWD to USD.
- The LHS diagram is the TARF payoff in USD which is the one specified in the term sheet.
- The RHS diagram is the corresponding payoff in TWD (if it is allowed).
- The peak in the RHS is 1,000,000 x (28.24 - 28.2) = 40,000 TWD. The peak in the LHS is 40,000 / 28.2 $\approx$ 1418 USD.



Example: TARF with EKI (Exact) on USDKRW

- We are going to cover another variation which has accumulative feature. When the Cumulated Intrinsic Value (CIV) is greater than or equal to the **Target Intrinsic Value (TIV)**, the product automatically expires.

Here are some important features of the product:

- Intrinsic Value (IV) on the j-th Fixing Date is defined as:

$$IV_j = Max(\text{USDKRW Fixing Rate}_j - \text{Strike}, 0)$$

  if USDKRW Fixing Rate is larger than Strike.

- The CIV on the j-th Fixing Date is defined as

$$CIV_j = \sum_{i=1}^{j} IV_i$$

- If the knock out event hasn't occurred, the bank pays to the holder

$$\text{Notional}\left(\frac{\text{USDKRW Fixing Rate}_j - \text{Strike}}{\text{USDKRW Fixing Rate}_j}\right)$$

- If the knock out event has occurred on j-th Fixing Date, i.e. $CIV_j \geq TIV$, the bank will pay the holder

$$\text{Notional} \left(\frac{TIV - CIV_{j-1}}{\text{USDKRW Fixing Rate}_j} \right)$$

This feature is to ensure the bank don't pay the overshoot in the j-th Fixing.

- There is another important level called European Knock In (EKI). If the Fixing Rate is below EKI, the holder needs to pay the bank

$$\text{Notional} \left(\frac{\text{Strike} - \text{USDKRW Fixing Rate}_j}{\text{USDKRW Fixing Rate}_j} \right)$$

at the j-th Fixing Date. This means the holder effectively sold an European Knock In Put Option to the bank on each Fixing Date.

- The table below shows the holder's payoff of the product for a given sample path of USDKRW Fixing Rate. Strike is 1091, EKI level is 1071, TIV is 100 KRW per USD.
- Due to the "Exact" feature, the payoff at the 9th fixing is $\frac{38\times 200{,}000}{1150} = 6,609$ instead of $\frac{59\times 200{,}000}{1150} = 10,261$

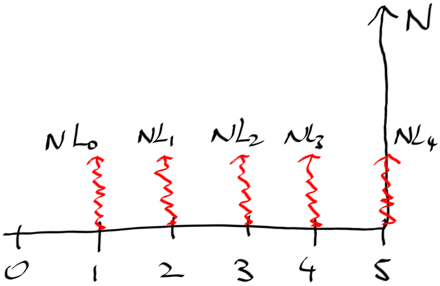
No.	USDKRW	IV	CIV	EKI payoff	Payoff in USD	Remark
1	1095	4	4	0	731	ITM
2	1080	0	4	0	0	OTM
3	1065	0	4	-26	-4,883	EKI
4	1050	0	4	-41	-7,810	EKI
5	1070	0	4	-21	-3,925	EKI
6	1090	0	4	0	0	OTM
7	1110	19	23	0	3,423	ITM
8	1130	39	62	0	6,903	ITM
9	1150	59	121	0	6,609	Target reached
10	1170	79	200	0		
11	1190	99	299	0		
12	1210	119	418	0		

Some discussion about the pricing of TARF

- For TARF that doesn't have accumulative feature, it is equivalent to a collection of European payoffs. This means the payoff can be priced by a model that is calibrated to an European option surface with the corresponding maturities.
- For TARF that has accumulative and "Exact" features, two models that are calibrated to the same option surface would produce different prices. This is because the product depends on the forward smile.
- To see this, let's consider the 9th Fixing in the example. CIV_8 is 62. TIV is 100. Strike is 1091. This means the target would be reached if the 9th Fixing is at or higher than $1091 + 38 = 1129$. With the "Exact" feature, the payoff for the 9th Fixing is therefore a call spread with strikes 1091 and 1129. This in turn is sensitive to the forward smile from the **start date** at the 8th Fixing Date to the **maturity** at the 9th Fixing Date.
- Note that 1129 is a function of CIV_8, this means it is path-dependent. For the two models to produce the same price of this product, they must produce the same forward smiles for all USDKRW levels at the **start Date** to all USDKRW levels at the **maturity**.

Floating Rate Note

- Floating Rate Note (FRN) pays the investor a series of coupons until the maturity. The coupons are a function of the interest rates determined at the corresponding fixing dates.
- The diagram below shows the cash flows of a FRN with 5y maturity. The red colour denotes the floating cash flow in a sense that it is determined in the future time. L_i denotes the floating interest rate L (stands for Libor) that is determined at the time i. N is the notional of the FRN which will be paid at maturity.



It seems one would need to compute the expectation of L_i for $i = 0, ..., 4$ in order to price the FRN. However, one can use a semi-static replication argument to price it. The replication strategy is as follows

- Deposit the notional N to a bank at time 0 with the rate L_0. One will receive NL_0 at time 1.
- Deposit N at time 1 with the rate L_1. One will receive NL_1 at time 2.
- Repeat until time 4.

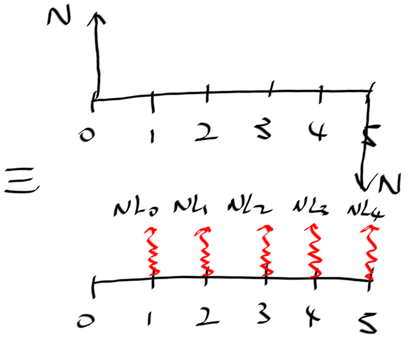
This would generate the same cash flows as the FRN in all interest rate scenarios. This implies the cost of both transactions must be the same. The cost of our replication strategy is N at time 0 and this means the FRN is priced at par at time 0. There are a few assumptions for this replication strategy

- The payment dates are the same as the fixing dates.
- The estimation curve and the discount curve are the same.

For our purpose, we assume both are true in our settings.

Using the same argument above, we can show the following two sets of cash flows are equivalent:

- Receive N at time 0 and pay N at time 5.
- Receive NL_i at time $i + 1$ for $i = 0, ..., 4$.



We will use this result later when discussing the pricing of Callable Fixed Rate Note.

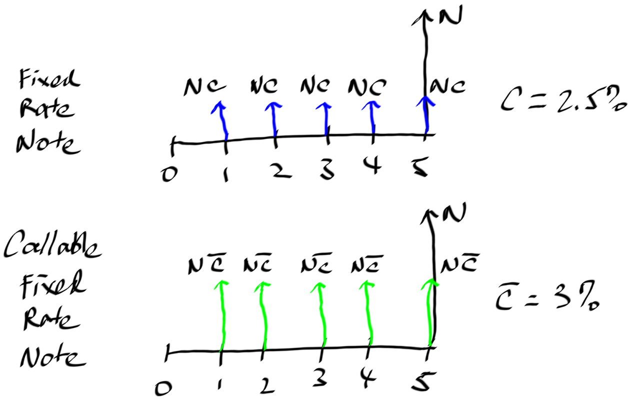
Callable Fixed Rate Note

- Callable Notes are securities with a "call" option that allow the issuer (i.e. the bank) to redeem the security prior to its maturity at par. The investor, in return, will receive an above-market interest rate.
- The issuer may call these securities when the current interest rate drops below the interest rate on the security.
- Callable Notes are beneficial to investors who believe the current interest rates will either remain the same or increase. If this were to happen, the issue will most likely not call the notes and the investor will enjoy higher interest rate payment (compare to the current interest rate) until maturity.
- Since the call option can only be exercised by the issuer, not the investor, Callable Notes can be risky for investors who are interested in steady income and predictable payment dates. This risk increases if current interest rates are expected to drop. Also, all payment on Callable Notes are subject to the creditworthiness of the issuer.

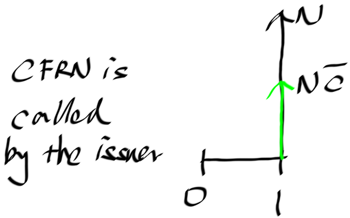
Example

- The current 5y swap rate with annual fixed payment frequency is 2.5%. Instead of receiving floating interest rates, investor can choose to receive 2.5% fixed coupon annually for 5y. We also assume the issuer default probability is 0. Therefore, a 5y Fixed Rate Note with annual fixed coupon from this issuer is priced at 2.5%.
- However, if the investor is willing to give the right to the issuer to call at par at some specific times in the future, the investor can earn higher interest rate, say, 3%. In other words, the option to cancel is worth 0.5% per annum.
- The call dates are usually the same as the coupon payment dates, except for the last payment date. In this cases, the call dates are 1y, 2y, 3y and 4y.

- The diagram below show the cash flows of the Fixed Rate Note and the Callable Fixed Rate Note.

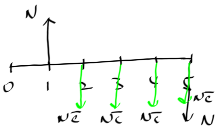


- When the note is called by the issuer at time 1, the cash flows of the Callable Fixed Rate Note would become the following



- The investor would still receive the 3% coupon at time 1 but he would also receive the notional N back.
- If the current interest rate is low, the issuer can borrow with lower cost and that's the reason the callable fixed rate note are likely to be called if the interest rate decreases.

This means the call action at time 1 by the issuer generates the following cash flows to offset the original cash flows of the Callable Fixed Rate Note.



By using the results before, the equivalent cash flows can be shown as below. The equivalent cash flows coincides with the receive float, pay fixed swaps **starts at time 2, ends at time 5**. This means the investor has sold an European swaption to the issuer that investor would receive float, pay fix at the strike $\bar{C}$ with the notional N.

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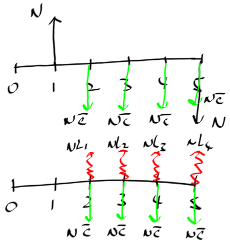

Bermudan Swaption

- However, this is not the end of the story for Callable Fixed Rate Note. Recall that the issuer can call the note **once** at time 1, 2, 3, or 4. The example above covers the case when the issuer call at time 1.
- The decision for the issuer to choose which time to call is similar to pricing an American option. Let's consider we are at time 4. At time 4, the issuer would call if the European swaption with the underlying swap **starts at time 4, ends at time 5** is ITM from the issuer point of view.
- At time 3, the decision is slightly more tricky. The issuer needs to compare the intrinsic value of the European swaption **starts at time 3, ends at time 5** against the value of European swapton **start at time 4, ends at time 5** at **time 3**. The collection of European swaptions with only one of them can be exercised is called Bermudan swaption. Note that the end dates of the European swaptions in the collection must be the same.
- At time 2, the issuer compares the intrinsic value of the European swaption **starts at time 2, ends at time 5** against the value of the Bermudan swaption **starts at 3, ends at 5** at time 2 and so on for time 1.
- In order to price a Bermudan swaption, one would need a term structure model like Hull-White which models the dynamics of the whole yield curve. This is beyond the scope of this course.

Summary

- Note form and swap form.
- Modelling credit risk using Poisson process.
- Relationship of different desks within an investment bank.
- Credit Linked Note.
- Total Return Swap.
- Target Redemption Forward.
- Float Rate Note.
- Callable Fixed Rate Note and Bermudan swaption.