$c > S_0 e^{-qT}$

$-S_0 + C + (S_0 - S_0 e^{-qT}) > 0$

$t = T$

$S_T > K$

$(S_0 - S_0 e^{-qT}) e^{rT} + (-S_0 + C) e^{rT}$

$S_T < K$

$S_T + (S_0 - S_0 e^{-qT}) e^{rT}$

$P = C - e^{-rT}(F_0(T) - K)$

$F_0(T) = S_0 e^{(r-q)T}$

$C - P = e^{-rT}(F_0(T) - K)$

QF602 - Homework 3

$C_0 = e^{-rT}(F_0(T)\phi(d_1) - K\phi(d_2))$

$d_1 = \dfrac{\ln(F_0(T)/K) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$ $\quad d_2 = d_1 - \sigma\sqrt{T}$

Question 1

- What happens to the price of a vanilla call option as the volatility tends to infinity? How about put option? $S_0 e^{-qT}$ — $e^{-rT}K$
- What happens to the price of a vanilla call option as the volatility tends to 0? How about put option? $e^{-rT}(F_0(T) - K)^+$ $\quad e^{-rT}(K - F_0(T))^+$
- What are the upper and lower bounds of the price of a call and put option on a non-dividend paying stock?

$0 \le \text{call} \le S_0 e^{-qT}$

$0 \le \text{put} \le K Z_0(T)$

Question 2

$d_1 = \dfrac{\ln(F_0(T)/K) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$

- Black Scholes Vega is given as $e^{-qT}S_0\phi(d_1)\sqrt{T}$, can you find the strike that gives the maximum vega for a given maturity T? let $d_1 = 0$

$-\ln\dfrac{S_0 e^{(r-q)T}}{K} = \frac{1}{2}\sigma^2 T$

$e^{-\frac{1}{2}\sigma^2 T}$

$= \dfrac{S_0 e^{(r-q)T}}{K}$

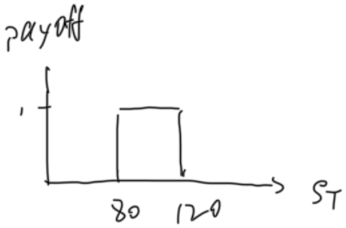
$K = S_0 e^{(r-q)T + \frac{1}{2}\sigma^2 T}$

Question 3

- Consider a digital option with a payoff at maturity $T = 1$

$$1_{L<S_T<U}$$

where $L = 80$ and $U = 120$ are the lower and upper barriers.

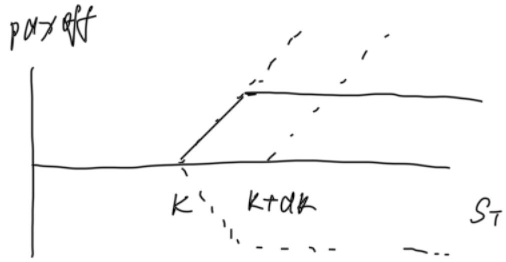


- Explain how to replicate the digital option using European options.
- Draw the Black Scholes delta profile of the digital option. Assume the implied vol is 0.2, risk free rate and dividend yield are 0.
- Draw the Black Scholes vega profile of the digital option.

Question 4 $\frac{\partial V}{\partial S_0}$ $\quad \frac{\partial Put_0}{\partial S_0} = x - e^{-qT}$

- If the delta of a call with maturity T and strike K is x, what is the delta of a put with the same maturity and strike?
- If the vega of a call with maturity T and strike K is y, what is the vega of put with the same maturity and strike? same $\quad \frac{\partial V}{\partial \sigma}$

$C - P = e^{-rT}[F_0(T) - K]$



call spread $c(K) - c(K+dK) \geq 0$

$F_0(T) = S_0 e^{(r-q)T}$

DCN_call : $e^{-rT} \cdot \Phi(d_2)$

put : $-d_2)$

$$d_1 = \frac{\ln(F_0(T)/K) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Delta $\frac{\partial V}{\partial S_0}$ $\quad e^{-rT}\phi(d_2)\frac{\partial d_2}{\partial S_0}$

$$\frac{\partial d_2}{\partial S_0} = \frac{K}{\sigma\sqrt{T}F_0(T)} \qquad \frac{\partial F_0(T)}{\partial S_0} = e^{(r-q)T}$$

Vega

$$S_T = S_0 e^{(r-q-\frac{1}{2}\sigma^2)T + \sigma\sqrt{T}x}$$

$$d < S_0 e^{(r-q-\frac{1}{2}\sigma^2)T + \sigma\sqrt{T}x}$$

$$x_1 > \frac{\ln\frac{d}{S_0} - (r-q-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$V_0 = e^{-rT}\int_{-\infty}^{\infty} \mathbb{1}_{d<S_T<U}\, \phi(x)\, dx$$

$$x_2 < \frac{\ln\frac{U}{S_0} - (r-q-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$= e^{-rT}\int_{x_1}^{x_2} \phi(x)\, dx$$

$$= e^{-rT}[\Phi(x_2) - \Phi(x_1)]$$

$$\frac{\partial V}{\partial S_0} = e^{-rT}\left[\phi(x_2) \cdot \frac{S_0}{U\sigma\sqrt{T}} \times\left(-\frac{d}{S_0^2}\right) - \phi(x_1)\frac{S_0}{d\sigma\sqrt{T}} \times\left(-\frac{d}{S_0^2}\right)\right]$$

$$= e^{-rT}\left[\phi(x_1)\frac{1}{S_0\sigma\sqrt{T}} - \phi(x_2)\frac{1}{S_0\sigma\sqrt{T}}\right]$$

$$\frac{\partial V}{\partial \sigma} = e^{-rT}\left[\phi(x_2)\cdot\left(-\frac{\ln\frac{U}{S_0} - (r-q)T}{\sigma^2\sqrt{T}} + \frac{1}{2}\sqrt{T}\right) - \phi(x_1)\left(-\frac{\ln\frac{d}{S_0} - (r-q)T}{\sigma^2\sqrt{T}} + \frac{1}{2}\sqrt{T}\right)\right]$$

$$= e^{-rT} \times \sqrt{T}\,[\phi(x_2) - \phi(x_1)] \times \left(\frac{1}{2} + \frac{(r-q)}{\sigma^2}\right)$$

$$+ e^{-rT}\left(\phi(x_1)\cdot\frac{\ln\frac{d}{S_0}}{\sigma^2\sqrt{T}} - \phi(x_2)\frac{\ln\frac{U}{S_0}}{\sigma^2\sqrt{T}}\right)$$