QF602 - Homework 4

Question 1

- Assume $r = 0$, $q^X = q^Y = q^Z = 0$; X, Y and Z are driven by the following dynamics in the risk neutral measure:

$$\frac{dX_t}{X_t} = \sigma^X dW_t^X, \ \frac{dY_t}{Y_t} = \sigma^Y dW_t^Y, \ \frac{dZ_t}{Z_t} = \sigma^Z dW_t^Z$$

where $E[dW_t^X dW_t^Y] = \rho dt, E[dW_t^X dW_t^Z] = 0, E[dW_t^Y dW_t^Z] = 0$.

Derive the pricing formula for the payoff which pays

$$\max(X_T/Y_T - Z_T, 0)$$

at time T.

Answer. Let $S = X/Y$, by Ito's lemma, we have

$$\begin{aligned}
dS_t &= \frac{\partial S}{\partial X} dX_t + \frac{\partial S}{\partial Y} dY_t + \frac{1}{2}\frac{\partial^2 S}{\partial Y^2} < dY_t > + \frac{\partial^2 S}{\partial X \partial Y} < dX_t, dY_t > \\
\frac{dS_t}{S_t} &= \sigma^X dW_t^X - \sigma^Y dW_t^Y + (\sigma^Y)^2 dt - \sigma^X \sigma^Y \rho dt \\
&= \Big((\sigma^Y)^2 - \sigma^X \sigma^Y \rho\Big) dt + \Big(\sigma^X dW_t^X - \sigma^Y dW_t^Y\Big) \\
&= -q^S dt + \sigma^S dW_t
\end{aligned}$$

where

$$q^S = -(\sigma^Y)^2 + \sigma^X \sigma^Y \rho$$

$$\sigma^S = \sqrt{(\sigma^X)^2 + (\sigma^Y)^2 - 2\sigma^X \sigma^Y \rho}$$

The next step is to apply the spread option closed form formula:

$$S_0 e^{-q^S T} \Phi(d_1) - Z_0 \Phi(d_2)$$

where

$$d_{1,2} = \frac{\ln(S_0/Z_0) + (-q^S \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$\sigma = \sqrt{(\sigma^S)^2 + (\sigma^Z)^2}$$

Question 2

- Assume X and Y are driven by the following dynamics in the risk neutral measure:

$$\frac{dX_t}{X_t} = (r - q^X)dt + \sigma^X dW_t^X$$
$$\frac{dY_t}{Y_t} = (r - q^Y)dt + \sigma^Y dW_t^Y$$

where $E[dW_t^X dW_t^Y] = \rho dt$.

Derive the pricing formula for the payoff which pays

$$\max(X_T, Y_T)$$

at time T.

Answer. The payoff can be expressed as

$$Y_T + \max(X_T - Y_T, 0)$$

The first term is a forward which has the value at time 0 is

$$e^{-rT} E_0^{\beta}[Y_T] = e^{-rT} Y_0 e^{(r-q^Y)T} = Y_0 e^{-q^Y T}$$

The second term is a spread option with $K = 0$, the value at time 0 is

$$X_0 e^{-q^X T}\Phi(d_1) - Y_0 e^{-q^Y T}\Phi(d_2)$$

where

$$d_{1,2} = \frac{\ln(X_0/Y_0) + (q^Y - q^X \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$
$$\sigma = \sqrt{(\sigma^X)^2 + (\sigma^Y)^2 - 2\sigma^X\sigma^Y\rho}$$

Put them all together the answer is

$$X_0 e^{-q^X T}\Phi(d_1) - Y_0 e^{-q^Y T}(\Phi(d_2) - 1)$$

Question 3

- Using the same setup as Question 2. Derive the pricing formula for the payoff which pays

$$\min(X_T, Y_T)$$

at time T.

Answer. The payoff can be expressed as

$$Y_T + \min(X_T - Y_T, 0) = Y_T - \max(Y_T - X_T, 0)$$

The first term has value $Y_0^{-q^Y T}$ as shown in Question 2. The second term is **short** a spread option with $K = 0$ with X and Y swap place, the value at time 0 is

$$-Y_0 e^{-q^Y T}\Phi(d_3) + X_0 e^{-q^X T}\Phi(d_4)$$

where

$$d_{3,4} = \frac{\ln(Y_0/X_0) + (q^X - q^Y \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$\sigma = \sqrt{(\sigma^X)^2 + (\sigma^Y)^2 - 2\sigma^X\sigma^Y\rho}$$

Put them all together and we have

$$Y_0 e^{-q^Y T}(1 - \Phi(d_3)) + X_0 e^{-q^X T}\Phi(d_4)$$

Question 4

- Assume you are a SGD investor. Let S be a SG stock, Y be a US stock, X be the USDSGD FX rate. The dynamics of the processes are given as:

$$\frac{dS_t}{S_t} = (r^d - q^S)dt + \sigma^S dW_t^S,$$

$$\frac{dY_t}{Y_t} = (r^f - q^Y)dt + \sigma^Y dW_t^Y,$$

$$\frac{dX_t}{X_t} = (r^d - r^f)dt + \sigma^X dW_t^X$$

- where dW^S and dW^X are BMs under the SGD risk neutral measure.
- dW^Y is a BM under the USD risk neutral measure.
- The correlations of the BMs are

$$E[dW_t^S dW_t^Y] = \rho_{SY}dt$$

$$E[dW_t^S dW_t^X] = \rho_{SX}dt$$

$$E[dW_t^Y dW_t^X] = \rho_{YX}dt$$

.

Derive the pricing formula for the payoff which pays

$$\max(S_T/S_0 - Y_T/Y_0, 0)$$

at time T from your perspective.

Answer.

- We first find the drift of Y in SGD risk neutral measure, which is

$$\frac{dY_t}{Y_t} = (r^f - q^Y - \sigma^Y \sigma^X \rho_{YX})dt + \sigma^Y d\bar{W}_t^Y,$$

  where $\bar{W}^Y$ is a BM under SGD risk neutral measure.

- Then we rewrite the payoff as

$$\max(S_T/S_0 - Y_T/Y_0, 0) = \max(\bar{S}_T - \bar{Y}_T, 0)$$

  where $\bar{S}_T = S_T/S_0$ and $\bar{Y}_T = Y_T/Y_0$, and note that S_0 and Y_0 are constants, so the SDE of $\bar{S}_T$ and $\bar{Y}_T$ are

$$\begin{aligned}
\frac{d\bar{S}_t}{\bar{S}_t} &= (r^d - q^S)dt + \sigma^S dW_t^S \\
\frac{d\bar{Y}_t}{\bar{Y}_t} &= (r^f - q^Y - \sigma^Y \sigma^X \rho_{YX})dt + \sigma^Y d\bar{W}_t^Y \\
&= (r^d - r^d + r^f - q^Y - \sigma^Y \sigma^X \rho_{YX})dt + \sigma^Y d\bar{W}_t^Y \\
&= (r^d - \bar{q}^Y)dt + \sigma^Y d\bar{W}_t^Y
\end{aligned}$$

  where

$$\bar{q}^Y = r^d - r^f + q^Y + \sigma^Y \sigma^X \rho_{YX}$$

- The next step is to apply the spread option formula with $K = 0$, we have

$$\bar{S}_0 e^{-q^S T}\Phi(d_1) - \bar{Y}_0 e^{-\bar{q}^Y T}\Phi(d_2)$$

$$d_{1,2} = \frac{\ln(\bar{S}_0/\bar{Y}_0) + (\bar{q}^Y - q^S \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$\sigma = \sqrt{(\sigma^S)^2 + (\sigma^Y)^2 - 2\sigma^S\sigma^Y\rho_{SY}}$$