# QF602 Derivatives

## Lecture 2 - Black Scholes, Bachelier and Volatility Smile

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# Brownian Motion

- The basic building block in math finance.
- A standard Brownian Motion on $[0, T]$, is a stochastic process $W_t, 0 \leq t \leq T$ with the following properties:
  - BM starts at 0. $W_0 = 0$.
  - Continuous path. The mapping, $t \to W_t$ is, with probability 1, a continuous function on $[0, T]$.
  - Independent increment. The increments $W_{t_k} - W_{t_{k-1}}$ for all $k$ are independent for any $k$. All you care is the distance between $t_k$ and $t_{k-1}$.
  - Normally distributed increment. The Brownian increment has the following distribution: $W_t - W_s \sim N(0, t - s)$ for any $0 \leq s \leq t \leq T$.
- What is the covariance of $W_1$ and $W_2$?

# Brownian Motion with Drift

- For a constant $\mu$ and $\sigma > 0$, we call a process $X_t$ a Brownian motion with drift $\mu$ and volatility $\sigma$ if

$$\frac{X_t - \mu t}{\sigma}$$

is a standard Brownian motion.

- We can construct $X$ from a standard Brownian motion $W$ by setting

$$X_t = \mu t + \sigma W_t.$$

- It follows that $X_t \sim N(\mu t, \sigma^2 t)$.
- Moreover, $X$ solves the stochastic differential equation (SDE)

$$dX_t = \mu dt + \sigma dW_t$$

- The assumption that $X_0 = 0$ is a natural normalization, but we may construct a Brownian motion with parameters $\mu$ and $\sigma$ and initial value $x$ by simply adding $x$ to each $X_t$.

# Geometric Brownian Motion

- A stochastic process $S_t$ is a geometric Brownian motion (GBM) if $\ln S_t$ is a Brownian motion with initial value $\ln S_0$.
- In other words, a GBM is an exponential Brownian motion.
- GBM is the most fundamental model of the value of a financial asset.
- In 1900, Louis Bachelier developed a model of stock prices that in retrospect we described as Brownian motion. The mathematics of Brownian motion had not yet been developed.
- In 1960s, Paul Samuelson pioneered the use of GBM as model in finance.

# Geometric Brownian Motion

- Brownian motion can take negative values, an undesirable feature in a model of the price of a stock.
- GBM is always positive because the exponential function takes only positive values.
- The dynamics of a GBM can be specified by the following SDE

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

- To find the solution of the SDE, we set $f(S_t) = \ln S_t$, by Ito's lemma, we have

$$df(S_t) = f^{'}(S_t)dS_t + \frac{1}{2}f^{''}(S_t) < dS_t >$$

where, $f^{'}(S) = \frac{1}{S}$, $f^{''}(S) = -\frac{1}{S^2}$, $< dS >= \sigma^2 S^2 dt$.

# Geometric Brownian Motion

- Substitute in all the terms, we have

$$d \ln S_t = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t.$$

- Express in the integral form, we have

$$\begin{aligned} \ln S_T &= \ln S_0 + \int_0^T \left(\mu - \frac{1}{2}\sigma^2\right) du + \int_0^T \sigma dW_u \\ &= \ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right) T + \sigma\sqrt{T}x, \end{aligned}$$

where $x \sim \Phi(0, 1)$. The solution is given as

$$S_T = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}x}$$

## Geometric Brownian Motion

- What is the expectation of $S_T$?
- Let's start with something simpler. What is the expectation of the exponential of a normal random variable $x$?

$$E\left[e^{ax}\right] = e^{\frac{1}{2}a^2}$$

where $a$ is an arbitrary number.

- This is the moment generating function of $x \sim \Phi(0,1)$.
- All other terms in $S_T$ are non-random, so

$$\begin{aligned} E_0[S_T] &= S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)T} E_0\left[e^{\sigma\sqrt{T}x}\right] \\ &= S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)T} e^{\frac{1}{2}\sigma^2 T} \\ &= S_0 e^{\mu T} \end{aligned}$$

# Black Scholes

Key assumptions:

- Volatility is constant over time.
- Underlying is traded continuously and is log-normally distributed.
- One can always short sell.
- No transaction costs.
- One can sell any fraction of a share.
- One can borrow and lend cash at a constant risk free rate.
- Stock pays a constant dividend yield.

# Risk neutral pricing and replication

- The fundamental assumption behind risk-neutral pricing is to use a replicating portfolio of assets with known prices to remove any risk.
- In the Black Scholes world, options are considered to be redundant in the sense that one can replicate the payoff of an European option on stock using the stock itself and risk-free bonds.
- Since options can be replicated and their theoretical values do not depend upon investors' risk preferences.
- The idea of replication is one of the most important contributions by Black and Scholes.

# Martingale pricing formula

- Let $V_t$ be a tradable asset price and $N_t$ be a strictly **positive** tradable asset, for $t < T$, we have

$$\frac{V_t}{N_t} = E_t^N \left[ \frac{V_T}{N_T} \right]$$

- The subscript $t$ denotes the expectation is taken at time $t$.
- The superscript $N$ denotes the expectation is taken under the measure induced by the numeraire asset $N$.
- Note that not all tradeable asset can be a numeraire, for example, an interest rate swap cannot be a numeraire as its value can be negative.
- If $N_t$ is chosen to be the money market account, $e^{rt}$, the measure associated with it is called the risk neutral measure.

# Dividend Paying Stock

- According to the martingale pricing formula, discounted stock prices are martingales under the risk neutral measure. This is the case provided *the stock pays no dividend*.
- The key feature of the risk neutral measure is that it causes discounted portfolio values, i.e. $\frac{V_T}{N_T}$, to be martingales. In order for the discounted value of a portfolio that invests in a dividend paying stock to be a martingale, the discounted value of the stock **with the dividends reinvested** must be a martingale, but the discounted price itself is not a martingale.
- One can think of the stock with the dividends reinvested from time 0 to time $T$ as the **total return** of the stock over the period, i.e. $\mathbf{G}_T = S_T e^{qT}$, where $q$ is the dividend yield.
- In other words, if $q \neq 0$, $\frac{G_T}{N_T}$ is a martingale but not $\frac{S_T}{N_T}$.

# Black Scholes Formula for European Option

- Let $S_0$ be the spot price at time 0.
- $\sigma$ be the volatility of the log return of the underlying.
- $r$ and $q$ be the interest rate and dividend yield respectively.
- Forward price at time 0 with maturity $T$ is

$$F_0(T) = S_0 e^{(r-q)T}$$

- The price of an European call option is given by

$$Call_0 = e^{-rT}(F_0(T)\Phi(d_1) - K\Phi(d_2))$$

$$d_1 = \frac{ln(F_0(T)/K) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, d_2 = d_1 - \sigma\sqrt{T}$$

$$\Phi(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-u^2/2} du$$

# Proof of Black Scholes Call Formula

- Under the risk neutral measure, assume the stock price follows $dS_t = (r-q)S_t dt + \sigma S_t dW_t$. By the martingale pricing formula, we have

$$\begin{aligned}
V_0 &= E_0^{\beta}\left[\frac{(S_T-K)^+}{e^{rT}}\right] \\
&= e^{-rT}\left[\int_{-\infty}^{\infty}\left(S_0 e^{(r-q-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}x}-K\right)^+\phi(x)dx\right] \\
&= e^{-rT}\left[\int_{\bar{x}}^{\infty}\left(S_0 e^{(r-q-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}x}-K\right)\phi(x)dx\right] \\
&= e^{-rT}\left[S_0 e^{(r-q-\frac{1}{2}\sigma^2)T}\int_{\bar{x}}^{\infty} e^{\sigma\sqrt{T}x}\phi(x)dx - K\int_{\bar{x}}^{\infty}\phi(x)dx\right]
\end{aligned}$$

where $\bar{x} = \frac{\ln(K/S_0)-(r-q-0.5\sigma^2 T)}{\sigma\sqrt{T}}$

$$K\int_{\bar{x}}^{\infty}\phi(x)dx = K(1-\Phi(\bar{x})) = K\Phi(-\bar{x})$$

$$
\begin{aligned}
\int_{\bar{x}}^{\infty} e^{\sigma\sqrt{T}x}\phi(x)dx &= \frac{1}{\sqrt{2\pi}}\int_{\bar{x}}^{\infty} e^{\sigma\sqrt{T}x}e^{-x^2/2}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{\bar{x}}^{\infty} e^{-\frac{1}{2}\left(x^2-2\sigma\sqrt{T}x\right)}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{\bar{x}}^{\infty} e^{-\frac{1}{2}\left((x-\sigma\sqrt{T})^2-\sigma^2 T\right)}dx \\
&= \frac{e^{\frac{1}{2}\sigma^2 T}}{\sqrt{2\pi}}\int_{\bar{x}}^{\infty} e^{-\frac{1}{2}(x-\sigma\sqrt{T})^2}dx \\
&= e^{\frac{1}{2}\sigma^2 T}\Phi(-(\bar{x}-\sigma\sqrt{T}))
\end{aligned}
$$

Substitute all the terms and we get

$$
\begin{aligned}
V_0 &= e^{-rT}\left[S_0 e^{(r-q-\frac{1}{2}\sigma^2)T} e^{\frac{1}{2}\sigma^2 T}\Phi(-\bar{x}+\sigma\sqrt{T}) - K\Phi(-\bar{x})\right] \\
&= e^{-rT}\left[S_0 e^{(r-q)T}\Phi(-\bar{x}+\sigma\sqrt{T}) - K\Phi(-\bar{x})\right]
\end{aligned}
$$

where

$$
-\bar{x}+\sigma\sqrt{T} = \frac{\ln(S_0/K)+(r-q+0.5\sigma^2 T)}{\sigma\sqrt{T}}, -\bar{x} = \frac{\ln(S_0/K)+(r-q-0.5\sigma^2 T)}{\sigma\sqrt{T}}
$$

The proof is done.

- The price of an European put option is given by

$$Put_0 = e^{-rT}(K\Phi(-d_2) - F_0(T)\Phi(-d_1))$$

- We can show that Black Scholes Call minus Black Scholes Put becomes

$$e^{-rT}(F_0(T) - K).$$

- Put Call parity works!!!!!
- In fact, this is a model-free result and must be satisfied by any non-arbitrage models.
- See the paper which appears to be the first one in modern academic literature, *Stoll, Hans R. (December 1969). "The Relationship Between Put and Call Option Prices". Journal of Finance. 24 (5): 801–824*

# Another formulation - BS in T-forward measure

- Instead of starting from the SDE of the spot $S$, we can start from the SDE of the forward price $F_t(T)$ at time $t$ which matures at $T$.
- Note that $F_T(T) = S_T$, this is because at maturity, the forward price must be the same as the underlying spot.
- Recall that the forward price can be written as $F_t(T) = \frac{S_t e^{-q(T-t)}}{Z_t(T)}$. [1] The SDE of the forward price is a martingale under the T-forward measure:

$$dF_t(T) = \sigma F_t(T) dW_t^T$$

  where $W^T$ is a standard BM in the T-forward measure.
- The numeraire asset is the zero coupon bond $Z_t(T)$ matures at time $T$.

[1] This is because the investor wants to buy exactly one stock of $S$ at the maturity $T$. Taking into account of the dividend reinvestment, the number of unit that she must acquire at time $t$ is $S_t e^{-q(T-t)}$.

# Another formulation - BS in T-forward measure

- Apply the martingale pricing formula, we have

$$V_0 = N_0 E_0^T \left[ \frac{(F_T(T) - K)^+}{N_T} \right] = Z_0(T) E_0^T \left[ \frac{(S_T - K)^+}{Z_T(T)} \right]$$

- Note that the superscript is changed to $T$ to denote this is T-forward measure, not risk-neutral measure.
- $N_T = Z_T(T) = 1$, so we have

$$V_0 = Z_0(T) E_0^T \left[ (S_T - K)^+ \right].$$

- This is equivalent to the result that we have before. This is the so-called the Black 76 formulation.
- This formulation is more general as we have not assumed interest rate is constant or even deterministic at all. All we need is a zero coupon bond price at time $t$ with maturity $T$.

# Proof of BS Call Formula in T-forward measure

- Under the T-forward measure and by the martingale pricing formula, we have

$$
\begin{aligned}
V_0 &= Z_0(T)E_0^T\left[(F_T(T)-K)^+\right] \\
&= Z_0(T)\left[\int_{-\infty}^{\infty}\left(F_0(T)e^{-\frac{1}{2}\sigma^2T+\sigma\sqrt{T}x}-K\right)^+\phi(x)dx\right] \\
&= Z_0(T)\left[\int_{\bar{x}}^{\infty}\left(F_0(T)e^{-\frac{1}{2}\sigma^2T+\sigma\sqrt{T}x}-K\right)^+\phi(x)dx\right] \\
&= Z_0(T)\left[F_0(T)e^{-\frac{1}{2}\sigma^2T}\int_{\bar{x}}^{\infty}e^{\sigma\sqrt{T}x}\phi(x)dx-K\int_{\bar{x}}^{\infty}\phi(x)dx\right] \\
&= Z_0(T)\left[F_0(T)e^{-\frac{1}{2}\sigma^2T}\Phi(-\bar{x}+\sigma\sqrt{T})-K\Phi(-\bar{x})\right]
\end{aligned}
$$

where $\bar{x}=\frac{\ln(K/F_0(T))+0.5\sigma^2T}{\sigma\sqrt{T}}$ and

$$
-\bar{x}+\sigma\sqrt{T}=\frac{\ln(F_0(T)/K)+0.5\sigma^2T}{\sigma\sqrt{T}},\quad -\bar{x}=\frac{\ln(F_0(T)/K)-0.5\sigma^2T}{\sigma\sqrt{T}}
$$

The proof is done.

# Stochastic interest rate in Black Scholes

- If the short rate $r_t$ is stochastic, the money market account is now a stochastic process

$$\beta_t = e^{\int_0^t r_s ds}$$

The zero coupon bond at time $t$ matures at $T$ can be expressed as

$$Z_t(T) = E_t^{\beta}\left[e^{-\int_t^T r_s ds}\right].$$

- **Proof.** By martingale pricing formula, the value of the zero coupon bond at time t in the risk neutral measure is (recall $Z_T(T) = 1$):

$$Z_t(T) = \beta_t E_t^{\beta}\left[\frac{1}{\beta_T}\right] = e^{\int_0^t r_s ds} E_t^{\beta}\left[\frac{1}{e^{\int_0^T r_s ds}}\right] = E_t^{\beta}\left[e^{-\int_t^T r_s ds}\right]$$

The proof is done.

- The call option under the risk neutral measure can be computed as (Note: $\beta_0 = 1$)

$$Call_0 = \beta_0 E_t^{\beta}\left[\frac{(S_T - K)^+}{\beta_T}\right] = E_t^{\beta}\left[e^{-\int_0^T r_s ds}(S_T - K)^+\right]$$

- If we further assume $r_t$ and $S_t$ are independent then

$$\begin{aligned} Call_0 &= E_t^{\beta}\left[e^{-\int_0^T r_s ds}(S_T - K)^+\right] \\ &= E_t^{\beta}\left[e^{-\int_0^T r_s ds}\right] E_t^{\beta}\left[(S_T - K)^+\right] \\ &= Z_0(T) E_t^{\beta}\left[(S_T - K)^+\right] \\ &= Z_0(T)\left(F_0(T)\Phi(d_1) - K\Phi(d_2)\right) \end{aligned}$$

- We basically end up with the Black Scholes call option formula using the T-forward measure. In other words, the T-forward measure approach covers the special case where stochastic interest rate is uncorrelated with the Spot price.

# Difference between $\beta_t$ and $Z_t(T)$

- A common question is: when should we choose $\beta_t$ or $Z_t(T)$ as the numeraire?
- There is a fundamental difference between the two numeraires: $\beta_t$ has no maturity, $Z_t(T)$ matures at $T$. In other words, $Z_t(T)$ only exists up to $t \leq T$.
- If we are only interested in the pricing problem up to maturity $T$, then $Z_t(T)$ could be a good choice.
- If interest rate is stochastic, i.e. $\int_0^t r_s ds$ is a random variable. The money market account

$$\beta_t = e^{\int_0^t r_s ds}$$

  is path-dependent. On the other hand, $Z_t(T)$ is a non-path-dependent variable. This makes a crucial difference if one decides to choose which numerical method to use when pricing exotic options.

# Greeks - a Brief Introduction

- The buying and selling a derivative creates a position with various sources of risk, some of which may be unwanted.
- Hedging is the act of reducing these risks by engaging in financial transactions that counterbalance these risks.
- When a bank sells a derivative to a client, it should understand all the risks associated with the product and hedge its accordingly.
- Once a sale is done, the product is added to an existing portfolio and the position must be risk managed.
- In order to see where the risks lie, the trader will need to know the sensitivity of the derivative's price to the various parameters that impact its value.
- The sensitivities of the option are often known as Greeks.

- Note that Greeks are defined as the derivative of the price with respect to some model parameter/input. In other words, they are model dependent.
- Delta is defined as $\frac{\partial Call_0}{\partial S_0}$, first order sensitivity to the current spot price. In plain English: how much does the price of the call change if the stock moves up by one dollar?
- E.g. delta is 0.5, this means if the stock moves up by one dollar, the call will increase by 0.5 dollar.
- Vega is the sensitivity of the option price to a movement in the implied volatility $\sigma$ of the underlying asset which is defined as $\frac{\partial Call_0}{\partial \sigma}$. This says how much does the price of the call change is the implied vol moves up by 1%.
- We will cover Greeks in more details later.

# Implied volatility

- The volatility parameter is different from all the other parameters in the BS formula. Some parameters, the strike and the maturity are set by terms of the contract. Some can be observed in the market like the current riskless rate and the underlying price. However, the volatility is the *future* volatility of the stock which is unknown.
- Assume for the moment that we have an estimate for future vol. Insert this number, along with other parameters (spot, strike, maturity, risk free rate) into the BS option formula, produces an option price.
- If the BS model is true then the option price produced by the BS option formula would be consistent with the observed market price.
- Conversely, if there is a liquid option market, we can take an option's market price and then calculate the *market implied future volatility* that makes the BS model price agree with the market price.

- If BS model was true then the implied volatility smile (implied vol plot against the strike for a given maturity) would be flat (Why?). However, a typical volatility smile (depends on the asset class) would look like


- This means that we cannot use a single BS model (i.e. a single implied vol) to price all the options simultaneously.

# How to compute implied volatility

- In the Black Scholes call option formula, volatility $\sigma$ (more precisely $\sigma(K,T)$) is one of the input parameters:

$$CallPrice_0 = BSCall(F_0(T), K, r, \sigma(K,T), T).$$

- Implied volatility is defined as the value of $\sigma(K,T)$ of such that the Black Schloes call option formula produces the same price as the observable option price:

$$\sigma(K,T) = BSCall^{-1}(F_0(T), K, r, T, CallPrice_0).$$

- "Implied volatility is the wrong number to put into the wrong formula to get the right price of plain-vanilla options"
- One would need to invert the Black Scholes call option formula subject to the implied volatility.

- Recall the Black Scholes call option formula

$$BSCall(\sigma) = Z_0(T)(F_0(T)N(d_1) - KN(d_2))$$

$$d_1 = \frac{\ln(F_0(T)/K) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, d_2 = d_1 - \sigma\sqrt{T}$$

$$N(a) = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{a} e^{-u^2/2du}$$

- There is no known closed form solution to the implied volatility by inverting the Black Scholes formula.
- One needs to use numerical methods.

# Root searching algorithm

- Problem statement: given a function $f(x)$, can you find $x$ such that $f(x) = 0$.
- If the function $f$ is differentiable with respect to $x$ then we can use Newton Raphson's method (NR), $x_0$ is the initial guess,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

- A toy problem: can you find $x$ such that $x^2 = 7$?
- We set the function $f(x) = x^2 - 7$, $f^{'}(x) = 2x$.
- Initial guess $x_0 = 2.5$, $f(x_0) = 2.5^2 - 7 = -0.75$, $f^{'}(x_0) = 5$.
- Apply NR, we have $x_1 = 2.5 - \frac{-0.75}{5} = 2.5$.
- It converges at the third iteration, $x = 2.6457513111$
- What is the convergence rate? Does it always converge?

| iteration | x | f(x) | f'(x) |
|---|---|---|---|
| 0 | 2.5000000000 | -0.7500000000 | 5.0000000000 |
| 1 | 2.6500000000 | 0.0225000000 | 5.3000000000 |
| 2 | 2.6457547170 | 0.0000180224 | 5.2915094340 |
| 3 | 2.6457513111 | 0.0000000000 | 5.2915026221 |
| 4 | 2.6457513111 | 0.0000000000 | 5.2915026221 |
| 5 | 2.6457513111 | 0.0000000000 | 5.2915026221 |
| 6 | 2.6457513111 | 0.0000000000 | 5.2915026221 |
| 7 | 2.6457513111 | 0.0000000000 | 5.2915026221 |
| 8 | 2.6457513111 | 0.0000000000 | 5.2915026221 |

# Implied vol using NR

- To find implied volatility, one can set the function to be

$$f(\sigma) = BSCall(\sigma) - price$$

$$f^{'}(\sigma) = vega(\sigma)$$

- where *price* is the observable call option price.
- Have a sensible initial guess for $\sigma_0$, say 20%.
- Then we are ready to use the NR iteration formula

$$\sigma_{i+1} = \sigma_i - \frac{BSCall(\sigma_i) - price}{vega(\sigma_i)}$$

# Example

- The observable option price is 21.9078.
- $S_0 = 100, K = 80, r = q = 0, T = 1.$
- It converges after 3 iterations.

| S | 100 | | iteration | ivol | f(ivol) | f'(ivol) |
|---|---|---|---|---|---|---|
| K | 80 | | 0 | 0.20000000 | -0.72191 | 19.05342 |
| input vol | 0.23456 | | 1 | 0.23788877 | 0.07550 | 22.82026 |
| r | 0 | | 2 | 0.23458037 | 0.00046 | 22.54126 |
| q | 0 | | 3 | 0.23456000 | 0.00000 | 22.53951 |
| ttm | 1 | | 4 | 0.23456000 | 0.00000 | 22.53951 |
| | | | 5 | 0.23456000 | 0.00000 | 22.53951 |
| call price | 21.9078 | | 6 | 0.23456000 | 0.00000 | 22.53951 |

Note that for deep out or deep in the money options, the NR method will fail. One of the reasons is that the vega is too small. If you are going to implement this at work, refer to the works by Peter Jaeckel on this topic (see, http://www.jaeckel.org/).

- The non-flat smiles tell us that actual option markets are more complicated than we have been assuming, and that their prices violate the BS model.
- In other words, the world is not lognormal.
- Nevertheless, traders everywhere use implied BS volatilities to quote option prices. This is not because traders believe the world is truly lognormal. The BS implied volatility is just treated as a tool to quote option prices.
- This is similar for you to quote your condo in SGD per square foot rather than in just SGD. With SGD per square foot, it is easier to compare the prices of the condos in two different regions in Singapore.

# Implied volatility surface

- If you are an option market maker dealing in a variety of strikes and maturities, it is useful to describe how implied volatility for a particular underlier varies with both strike and time to expiration. The relationship of these three variables defines a surface.

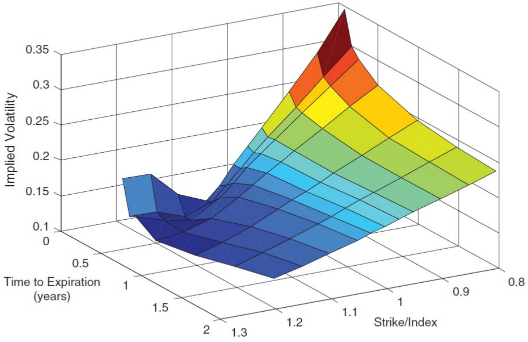



Figure: SP 500 Volatility Surface

- There are discrete number of strikes and maturities for observable option prices. In order to model the whole vol surface, one would need to interpolate from discrete observations to a continuous surface without violating no-arbitrage constraints.
- You will be surprised how easy it is to create arbitrage within the vol surface if you use simple inter(extra)polation scheme like linear or cubic spline.
- For example, see Roger Lee (2004), The Moment Formula for Implied Volatility at Extreme Strikes, on how one should extrapolate the volatility smile.
- We will come back to the inter(extra)polation problem when we discuss model beyond Black Scholes in the later lectures.

# How to graph the smile

- Though the most logical way to specify the strike price in an option contract is in dollars (or euro, or yen, etc.), it is difficult to compare implied volatilities using dollar strikes when the underliers have very different prices.
- For example, if the implied volatility of a \$120 strike three-month call on XOM is 18% and the implied volatility of a \$600 strike three-month call on GOOG is 15%, how can we meaningfully compare these values?
- One solution is to use relative strike prices, i.e. to quote the implied vol as a function of the option moneyness.
- Using moneyness $K/S_t$ or forward moneyness $K/F_t$ allows us to compare the values of different strikes at the same expiration. While moneyness is useful for comparing options on different underliers with the same time to expiration, it is less useful for comparing options with different times to expiration or significantly different volatilities.

- We can normalize implied volatilities across strikes and maturities by comparing them as a function of delta. Note that delta is a function of strike, spot (or forward), maturity and implied volatility.
- In FX option market, traders quote options using delta rather strike or moneyness. E.g. A 25 delta EUR call USD put option has delta = 0.25. This approach has several attractive features:
  - The x-axis of the plot is standardized for a vanilla call, delta always varies between 0 and 1, and for a put between 0 and -1.
  - The delta for a given implied volatility immediately indicates the amount of underlying you need to hedge the option in the BS model.

# Smile in different option markets

- Different types of securities (equities, FX rates, bonds, etc.) have smiles with different characteristic shapes. In each case, these differences hint at the difference between our idealized geometric Brownian motion with constant volatility and the actual behavior of these securities in markets, differences that need to be accounted for if we are going to value options accurately.
- We discuss an overview, focusing on the general shape of the smiles in this lecture.

# Equity Indexes

- Prior to the crash of 1987, the volatility smile in equity index markets was almost flat in the strike dimension, consistent with the BS model, though there was often a dependence on the time to expiration.
- Technically, the non-flat term structure is also a violation of the BS model's assumption of constant volatility, but a non-flat term structure can easily be reconciled with BS by allowing forward volatilities to vary with time.
- Since the crash of 1987, in almost all equity index option markets around the world, BS implied volatilities have exhibited a persistent and dramatically skewed structure in the strike dimension that cannot be reconciled with the BS model.
- Here are some of the most salient characteristics of the equity index smile.

- The most noteworthy feature of every index volatility smile is its negative slope as a function of strike. Implied volatilities often, but not always, reach a minimum near the at-the-money strike and then increase slightly for higher strike prices.
- The negative skew is partially due to an asymmetry in the way equity indexes move: Large negative returns are much more frequent than large positive returns. The SP 500 has experienced 20% downward moves in one day, but never 20% upward moves; there are no 'up crashes'. Since crashes are difficult for option market makers to hedge, their likelihood tends to elevate the relative cost of far out-of-the-money puts. There is also a demand component that contributes to the negative skew. Investors who own equities may want to hedge against large losses. For them, buying out-of-the-money puts is a form of insurance for which they are willing to pay a premium.

- The negative skew or slope with respect to the strike is generally steeper for short maturities.

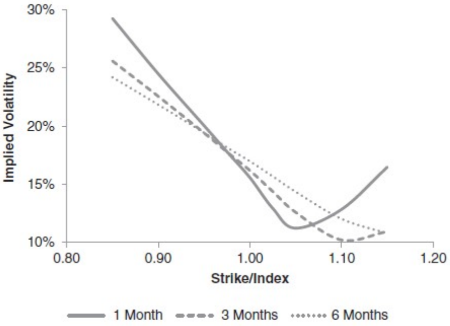
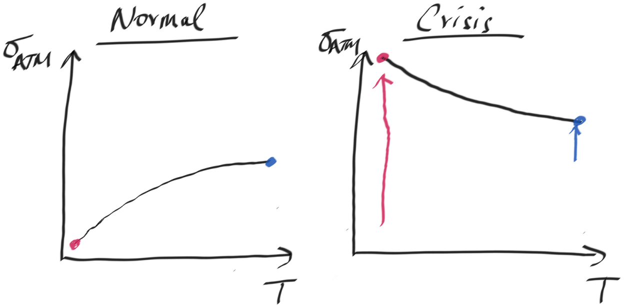



Figure: SP 500 vol smiles

- Unlike the strike structure, which almost always has a negative slope, the term structure of the volatility surface can slope up or down. Its shape is heavily influenced by the market's expectation of future volatility. During a crisis—and a crisis is always characterized by high volatility—the term structure is likely to be downward sloping. The high short-term volatility and lower long-term volatility reflect market participants' belief that uncertainty in the near term will eventually give way to more typical volatility.



- For equity indexes, implied volatility and index returns are negatively correlated. Equity index markets tend to drift up with low volatility, but crash down. When an index moves down sharply, realized volatility by definition increases, and this leads to a fearful increase in implied volatility.

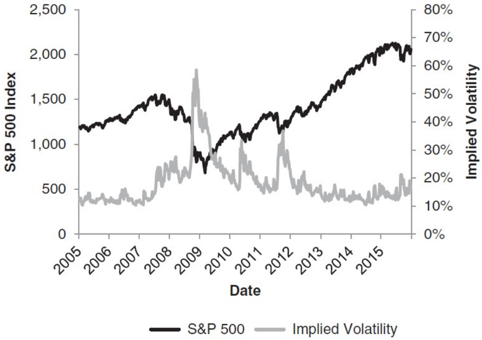



Figure: SP 500 level and 3m ATM implied vol

# Single Stocks

- Single-stock smiles tend to be more symmetric than index smiles.
- Unlike equity indexes, which tend to crash down but move up slowly, single stock prices can move sharply up or down. For example, if a quarterly earnings announcement for a company is much better or worse than expected, the company's stock price can move up or down significantly.
- Because single stocks can experience both large positive and large negative shocks, far out-of-the-money options in both directions are more likely to generate large payoffs. At the same time, hedging far out-of-the-money options is difficult. Because of this, market makers demand an extra premium over at-the-money options, a premium that corresponds to higher implied volatility for both extremely high and low strike prices.

# Single Stocks vs Equity Indexes

- How is it that the smile of individual stocks is more symmetric than the smile of the index, which is composed of individual stocks?
- Stock-specific shocks tend to be relatively uncorrelated across stocks. Apple (AAPL) might release positive news on the same day that Exxon Mobil (XOM) releases negative news.
- Large economic shocks that impact all companies, though, are more likely to be negative. Because of this, if we consider only large negative returns, stocks appear to have a higher correlation than if we consider only positive returns. As a result, the returns of equity indexes tend to be more negatively skewed than we would expect on the basis of the skewness of the individual index components.

# Foreign Exchange

- The smiles for foreign exchange (FX) options can be index-like or single-stock-like. They tend to be roughly symmetric for "equally powerful" currencies, less so for "unequal" ones.
- This can be understood in part by the perceived likelihood of the exchange rate to move up or down.
- The currencies of large developed countries or regions (e.g., USD, EUR, JPY) tend to be relatively stable, with the exchange rates between these currencies typically just as likely to move up as down.
- There is also symmetry on the demand side: There are investors for whom a move down in the dollar is painful, but there are investors for whom a move down in the yen (i.e., up in the dollar) is equally painful. Hence, for equally powerful currencies, smiles tend resemble a symmetric smile.

# Foreign Exchange

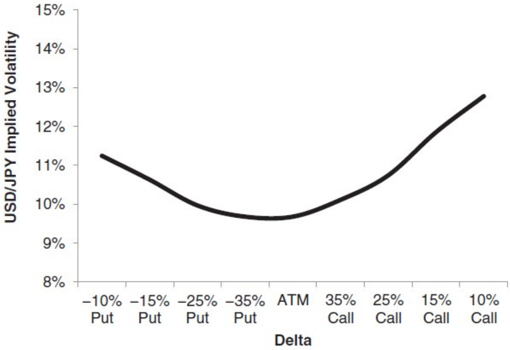



Figure: USDJPY vol smile

# Foreign Exchange

- Emerging market economies, on the other hand, tend to be less stable, and their currencies are much more likely, periodically, to fall dramatically, rather than rise, relative to major currencies. The Asian financial crisis in 1997 is a good example. During that crisis, exchange rates versus the U.S. dollar for several emerging Asian currencies fell by more than 30%, and in some cases by more than 80%. Smiles for exchange rates between emerging market currencies and major currencies therefore tend to resemble an index smile. The USD/MXN smile is an example.

# Foreign Exchange

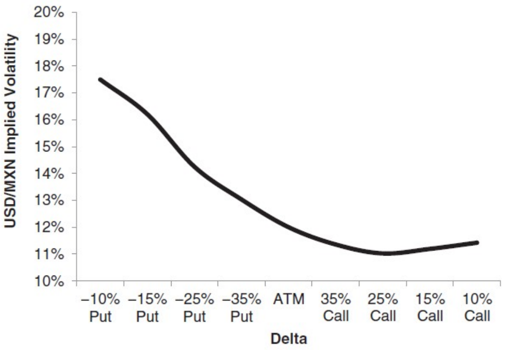



Figure: USDMXN vol smile

# Time-dependent Black Scholes

- Turns out that it is straight forward to extend the Black Scholes model to incorporate time-dependent drift and volatility. The time-dependent BS SDE is

$$\frac{dS_t}{S_t} = r_t dt + \sigma_t dW_t$$

- By Ito's lemma, we have

$$d \ln S_t = \left( r_t - \frac{1}{2}\sigma_t^2 \right) dt + \sigma_t dW_t$$

- The integral form is

$$\begin{aligned} \ln S_T &= \ln S_0 + \int_0^T \left( r_u - \frac{1}{2}\sigma_u^2 \right) du + \int_0^T \sigma_u dW_u \\ &= \ln S_0 + \int_0^T r_u du - \frac{1}{2}\int_0^T \sigma_u^2 du + \left( \sqrt{\int_0^T \sigma_u^2 du} \right) x, \end{aligned}$$

where $x \sim \Phi(0,1)$

- In time-dependent BS, we allow the risk-free rate to be time dependent. This is much more realistic than assuming constant risk-free rate between 0 to $T$. In practice, we further assume $r_t$ to be piece-wise constant, i.e. the risk-free rate between $t_1$ and $t_2$ is a constant value. We denote the value of the risk free rate between $t_i$ and $t_{i+1}$ as $r_{i+1}$. The time integral becomes

$$\int_0^T r_u du = \sum_{i=0}^{N-1} r_{i+1}(t_{i+1} - t_i)$$

  with $t_0 = 0, t_N = T$.
- The values of $r_t$ are calibrated to the current yield curve which is expressed in terms of discount factors, $Z_0(t)$.
- Assuming $Z_0(t)$ are given for all $t$. The calibration procedure for $r_t$ is
  1. Set $r_1 = \frac{-\ln(Z_0(t_1))}{t_1}$
  2. Set $r_i = \frac{-\ln(Z_0(t_i)/Z_0(t_{i-1}))}{t_i - t_{i-1}}$ for $i \geq 2$

- We also assume the volatility of $S$, i.e. $\sigma_t$, to be piece-wise constant. The values of $\sigma_t$ can be calibrated to the ATM implied volatility observed in the market for various maturities. We denote the value of $\sigma_t$ between $t_i$ and $t_{i+1}$ as $\sigma_{i+1}$. The time integral will then become:

$$\int_0^T \sigma_u^2 du = \sum_{i=0}^{N-1} \sigma_{i+1}^2 (t_{i+1} - t_i)$$

  with $t_0 = 0, t_N = T$.
- Since the market implied vols are quoted using BS model, the volatility calibration of the time-dependent BS model is straight forward. This is because the log of stock return in the BS model is normally distributed; and the fact that a normal variate plus another normal variate is also a normal variate.

- Recall the variance of the sum of the 2 normal variates $x$ and $y$ is

$$var(x+y) = var(x) + var(y) + 2covar(x,y).$$

  Let $x$ be the log return of $S$ between $t_1$ and $t_2$, and $y$ be the log return between $t_2$ and $t_3$. The variance of the log return between $t_1$ to $t_3$ can be computed as the sum of the variance of $x$ and $y$ because the correlation between the two non-overlapping period returns is 0.
- This observation allows us to have a straight forward calibration routine for time-dependent BS model.
- We denote the market observed ATM implied volatility with maturity $T$ as $\Sigma(T)$. The calibration procedure for $\sigma_t$ is
  1. Set $\sigma_1 = \Sigma(t_1)$
  2. Set $\sigma_i = \sqrt{\frac{\Sigma^2(t_i)t_i - \Sigma^2(t_{i-1})t_{i-1}}{t_i - t_{i-1}}}$ for $i \geq 2$

- We can see that the market implied variance (not volatility) $\Sigma^2(T)$ can be interpreted as the time-average of the instantaneous variance $\sigma_t^2$ over the period 0 to $T$:

$$
\begin{aligned}
\Sigma^2(T)T &= \int_0^T \sigma_u^2 du \\
\Sigma^2(T) &= \frac{1}{T}\int_0^T \sigma_u^2 du \\
&= \frac{1}{T}\sum_{i=0}^{N-1} \sigma_{i+1}^2 (t_{i+1} - t_i)
\end{aligned}
$$

  with $t_0 = 0$ and $t_N = T$.
- In a nutshell, the advantage of having time-dependency in the BS model is that we can now have one consistent model which is calibrated to a term structure of implied volatility with various maturities.

# Example

**Question**

- Assume $S_0 = 100$, dividend yield is 0 and we have a yield curve, $Z_0(1) = 0.99, Z_0(2) = 0.97$; a term-structure of implied vol $\Sigma(1) = 0.2, \Sigma(2) = 0.18$. Can you calibrate a time-dependent BS model to those market data? In other words, find $r_t$ and $\sigma_t$ such that we can reprice those options.

**Answer**

- Let's calibrate $r_t$ first. We assume $r_t$ is piecewise constant.
- $r_1$ is the short rate between time 0 to time $t_1 = 1$. It can be computed as

$$r_1 = -\frac{ln(0.99)}{1} = 0.01005$$

- $r_2$ can be computed as

$$r_2 = -\frac{ln(0.97/0.99)}{(2-1)} = 0.02041$$

## Sanity Check

- Discount factor with deterministic short rate $r_t$ can be computed as

$$Z_0(T) = e^{-\int_0^T r_s ds}$$

If $r_t$ is piecewise constant then the discount factors can be computed as

$$Z_0(1) = e^{-\int_0^1 r_s ds} = e^{-r_1 \times 1} = 0.99$$

$$Z_0(2) = e^{-\left(\int_0^1 r_s ds + \int_1^2 r_s ds\right)} = e^{-(r_1 \times 1 + r_2 \times (2-1))} = 0.97$$

**Answer**

- Next we calibrate $\sigma_t$. We assume $\sigma_t$ is piecewise constant.
- $\sigma_1$ is the model volatility between time 0 to time $t_1 = 1$. It can be computed as

$$\sigma_1 = \Sigma(1) = 0.2$$

- $\sigma_2$ can be computed as

$$\sigma_2 = \sqrt{\frac{\Sigma^2(2) \times 2 - \Sigma^2(1)}{2-1}} = \sqrt{\frac{0.18^2 \times 2 - 0.2^2}{2-1}} = 0.15748$$

## Sanity Check

- Implied vol with deterministic vol $\sigma_t$ can be computed as

$$\Sigma(T) = \sqrt{\frac{1}{T}\int_0^T \sigma_u^2 du}$$

If $\sigma_t$ is piecewise constant then the implied vol can be computed as

$$\Sigma(1) = \sqrt{\frac{1}{1}\int_0^1 \sigma_u^2 du} = \sqrt{\frac{1}{1}\left({\color{red}\sigma_1}^2 \times 1\right)} = 0.2$$

$$\Sigma(2) = \sqrt{\frac{1}{2}\int_0^2 \sigma_u^2 du} = \sqrt{\frac{1}{2}\left({\color{red}\sigma_1}^2 \times 1 + {\color{blue}\sigma_2}^2 \times (2-1)\right)} = 0.18$$

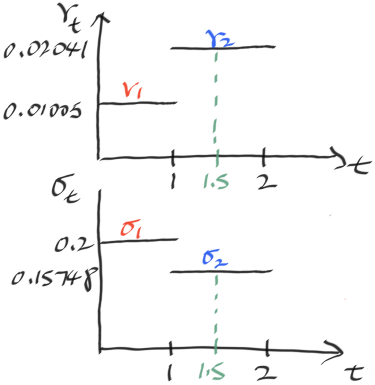
# Another Example

**Question**

- Using the previous calibrated time-dependent BS model. What is the price of the European call option with strike 100 and maturity 1.5 years?

**Answer**

- Let's represent our calibration results, i.e. $r_t$ and $\sigma_t$, in graphs.



## Answer

- Discount factor with 1.5 year maturity can be computed as

$$Z_0(1.5) = e^{-\int_0^{1.5} r_s ds} = e^{-(0.01005\times 1+0.02041\times(1.5-1))} = 0.979949$$

- The implied vol with 1.5 year maturity can be computed as

$$\begin{aligned}\Sigma(1.5) &= \sqrt{\frac{1}{1.5}\int_0^{1.5}\sigma_u^2 du} \\ &= \sqrt{\frac{1}{1.5}(0.2^2\times 1 + 0.15748^2\times(1.5-1))} = 0.186905\end{aligned}$$

- The forward price $F_0(1.5)$ is

$$F_0(1.5) = S_0/Z_0(1.5) = 100/0.979949 = 102.0461$$

- The call option with $K = 100$ and maturity 1.5 is

$$Z_0(1.5)\left(F_0(1.5)\Phi(d_1) - K\Phi(d_2)\right) = \mathbf{10.586}$$

where $d_1 = \frac{\ln(F_0(1.5)/K)+\frac{1}{2}\Sigma(1.5)^2\times 1.5}{\Sigma(1.5)\sqrt{1.5}}$ , $d_2 = d_1 - \Sigma(1.5)\sqrt{1.5}$

# Consequences of the smile for trading

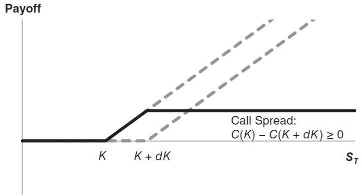
- The assumed dynamics of the underlier in the BS model is inconsistent with the existence of the smile, and this discrepancy manifests itself in both hedging and pricing.
- For very liquid options, where an option price is taken from the market and then used to generate an implied BS vol that is then used to quote the price, the fact that the model is wrong isn't a major problem. The model is merely a quoting convention. The model would matter if you wanted to generate your own idea of fair option values and then arbitrage them against market prices.
- The model becomes critical for vanilla options, even liquid ones, when you want to **hedge** them, because even if the option price is known, the option's hedge ratio is model-dependent.

# Consequences of the smile for trading

- If you don't get the hedge ratio right, you cannot replicate the option and recoup its value accurately.
- The model is also critical if you want to trade illiquid exotic options, whose prices are not obtainable from a listed market. In that case, you have no choice but to use a model to estimate both the price and a hedge ratio.

# No-arbitrage bounds on the smile - Call spread

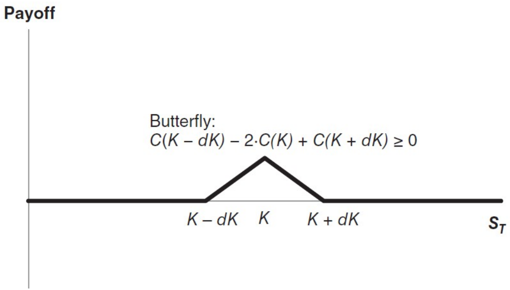
- There are no-arbitrage constraints on option prices as a function of strike. The following diagram shows the payoff at expiration of a European call spread consisting of a long position in a call with strike $K$, and a short position in a call with a higher strike $(K + dK)$, both with the same expiration.




- No matter what values we choose for $K$ and $dK$, the call spread will always have a non-negative payoff and therefore, by the principle of no riskless arbitrage, must have a non-negative value at all times prior to expiration. This means that a call with a higher strike cannot be worth more than a call with a lower strike.

# No-arbitrage bounds on the smile - Butterfly

- We can also find constraints on European call prices in terms of Butterfly. To see why this is the case, imagine a butterfly spread constructed from calls whose payoff at expiration is shown in the diagram below.




- Since the payoff of the butterfly is always greater than or equal to zero, by the principle of no riskless arbitrage, the current value of the butterfly must also be greater than or equal to zero.

# No-arbitrage bounds on the smile - Calendar Spread

- Calendar spread constraint is defined as:

$$Call_0(K, T_1) \leq Call_0(K, T_2)$$

  for $T_1 \leq T_2$.
- In other words, for the same underlying and the same strike, the call option with the longer maturity would be worth as much as the shorter maturity one.

# Example

### Question

- The table below shows the **call** option prices with various maturities and strikes. Assuming $r = 0$, $q = 0$, what are the ranges of $x, y$ such that there is no arbitrage?

| **Maturity/Strike** | **90** | **100** | **110** |
|---|---|---|---|
| **1** | x | 6 | 4 |
| **2** | 12 | 7 | y |

# Example

**Answer**

- We first use the butterfly constraint to find the range of $y$:

$$12 - 2 \times 7 + y \geq 0 \rightarrow y \geq 2$$

- We can further narrow the range by using calendar spread. The call with strike 110 and 1y maturity is 4, this yields $y \geq 4$
- Finally, the call with strike 110 cannot worth more than the call with strike 100 with the same maturity, this yields $7 \geq y \geq 4$
- Then we use the butterfly and calendar spread constraint to find the range of $x$:

$$x - 2 \times 6 + 4 \geq 0 \rightarrow x \geq 8$$

- Using calendar spread constraint, we can narrow the range to be $12 \geq x \geq 8$

# Summary

- Basic properties of Brownian Motion.
- Geometric Brownian Motion and the solution of the SDE.
- Moment generating function of a normal random variable.
- Black Scholes in risk neutral and T-forward measure
- what is implied volatility and how to compute it.
- what is implied volatility surface.
- Time dependent Black Scholes model.
- Some arbitrage bound on the smile.