QF602 Derivatives

Lecture 4 - Exotic Options in the Black Scholes World

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Chooser Options

- Chooser option is an agreement in which the holder has the right to choose at some future date T_c whether the option is to be a call or put option with a common exercise price K and remaining time to expiry $T - T_c$.
- The payoff at option maturity T is

$$V_T = (S_T - K)^+ \mathbb{I}_{\mathcal{A}} + (K - S_T)^+ \mathbb{I}_{\mathcal{A}^c}$$

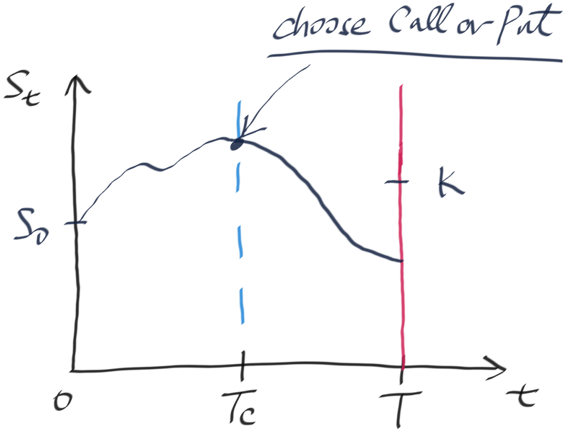
  where $\mathcal{A}$ is the event happens at T_c such that

$$Call_{T_c}(T - T_c, K) > Put_{T_c}(T - T_c, K)$$

  and $\mathcal{A}^c$ is the complement of $\mathcal{A}$.
- In other words, the holder chooses if he wants a call or a put with the remaining maturity $T - T_c$, **based on the information he has at T_c** . The value of the chooser option at T_c is therefore

$$V_{T_c} = \max\left(Call_{T_c}(T - T_c, K), Put_{T_c}(T - T_c, K)\right)$$

- A sample path and choosing a Call or a Put at T_c.



- Put call parity implies at time T_c

  $$Put_{T_c}(T - T_c, K) = Call_{T_c}(T - T_c, K) - Z_{T_c}(T)(F_{T_c}(T) - K)$$

  where $F_{T_c}(T) = S_{T_c} e^{(r-q)(T-T_c)}$, and thus

  $$\begin{aligned} V_{T_c} &= \mathbf{max}\Big(Call_{T_c}(T - T_c, K), \\ &\quad Call_{T_c}(T - T_c, K) - Z_{T_c}(T)(F_{T_c}(T) - K)\Big) \\ &= Call_{T_c}(T - T_c, K) + \mathbf{max}\Big(0, -Z_{T_c}(T)(F_{T_c}(T) - K)\Big) \\ &= Call_{T_c}(T - T_c, K) + \mathbf{max}\Big(0, Z_{T_c}(T)(K - F_{T_c}(T))\Big) \\ &= Call_{T_c}(T - T_c, K) + Z_{T_c}(T)(K - F_{T_c}(T))^+ \end{aligned}$$

- The last equality implies that the choose option is equivalent to the portfolio of a long call option and a long put option (with different expiry dates).

- Let's look at the call first. This is a plain vanilla European call option on spot S_T, strike K and maturity T. The value of the call at time 0 is

$$Call_0(T,K) = Z_0(T)(F_0(T)\Phi(d_1) - K\Phi(d_2))$$

where $d_{1,2} = \frac{\ln\left(\frac{F_0(T)}{K}\right) \pm \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$.
- Next we look at the put. It is a put option with the underlying $F_{T_c}(T)$ with strike K, maturity T_c and the notional is scaled by $Z_{T_c}(T)$.
- The forward price $F_t(T)$ has the dynamics under the T-forward measure

$$dF_t(T) = \sigma F_t(T) dW_t^T$$

- The numeraire asset is $Z_t(T)$ not $Z_t(T_c)$.

- The put payoff at time T_c is

$$Z_{T_c}(T)(K - F_{T_c}(T))^+$$

Note that the maturity of the option is at T_c not T.

- The value of the put at time 0 can be computed as

$$\begin{aligned} & Z_0(T)E_0^T\left[\frac{Z_{T_c}(T)(K - F_{T_c}(T))^+}{Z_{T_c}(T)}\right] \\ = \;& Z_0(T)E_0^T\left[(K - F_{T_c}(T))^+\right] \\ = \;& Z_0(T)(K\Phi(-\bar{d}_2) - F_0(T)\Phi(-\bar{d}_1)) \end{aligned}$$

where $\bar{d}_{1,2} = \frac{\ln\left(\frac{F_0(T)}{K}\right) \pm \frac{1}{2}\sigma^2 T_c}{\sigma\sqrt{T_c}}$

- Put them all together, the price of the chooser option with choose date T_c and maturity T with strike K is

$$Z_0(T)\Big(F_0(T)(\Phi(d_1) - \Phi(-\bar{d}_1)) + K(\Phi(-\bar{d}_2) - \Phi(d_2))\Big)$$

where $d_{1,2} = \frac{\ln\left(\frac{F_0(T)}{K}\right) \pm \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$

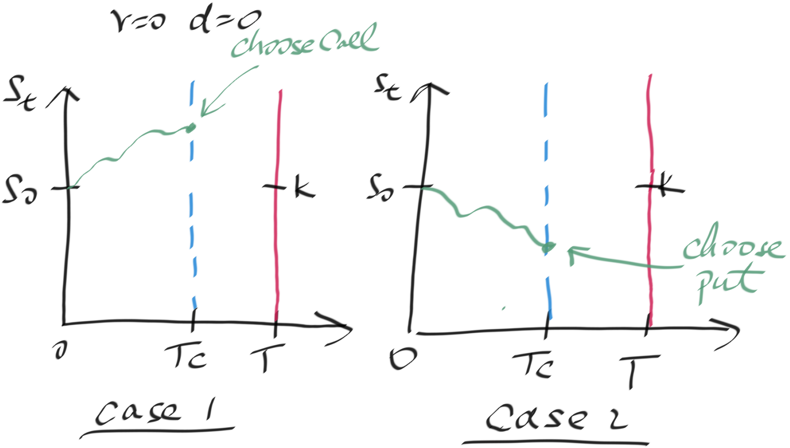
How to hedge a chooser option

- The portfolio (a long call and a long put) also suggests a semi-static way to hedge a chooser option. To illustrate the idea, we assume r and q are 0 for simplicity.
- Let's assume you **sold** a chooser option at time 0. In order to hedge, at time 0, you are going to buy (1) a call option that pays $(S_T - K)^+$ at T and (2) a put option that pays $(K - S_{T_c})^+$ at T_c.
- At T_c, there are 2 possibilities:
  1. $Call_{T_c}(T - T_c, K) \geq Put_{T_c}(T - T_c, K)$ when $S_{T_C} \geq K$
  2. $Call_{T_c}(T - T_c, K) < Put_{T_c}(T - T_c, K)$ when $S_{T_C} < K$
- In case 1, the holder will choose a call. Our hedge instrument (1) will hedge this exactly. On the other hand, hedge instrument (2) will be expired worthless.
- In case 2, the holder will choose a put. Our hedge instrument (2) will receive $(K - S_{T_c})$ and we still have a call that is worth $Call_{T_c}(T - T_c, K)$ which is not yet expired. Our aim now is to check if

$$(K - S_{T_c}) + Call_{T_c}(T - T_c, K) = Put_{T_c}(T - T_c, K)$$

  If so, we are perfectly hedge.
- By put call parity with $r = q = 0$, we can show that it is indeed the case.

- Illustrations of Case 1 and 2.



Spread option

- Spread option with strike K has the following payoff at maturity T

$$V_T = (X_T - Y_T - K)^+, \quad K \geq 0$$

- Assuming both X_T and Y_T are joint lognormal, there is no closed form pricing formula for $K \neq 0$. However, it can be expressed in terms of a numerical integration.
- Let

$$\frac{dX_t}{X_t} = (r - q^X)dt + \sigma^X dW_t^X$$

$$\frac{dY_t}{Y_t} = (r - q^Y)dt + \sigma^Y dW_t^Y$$

  where $E[dW_t^X dW_t^Y] = \rho dt$
- W^X and W^Y are BMs under the risk neutral measure, i.e. the numeraire asset is $\beta_t = e^{rt}$.

$K = 0$

- Let's consider the special case $K = 0$. There exists a closed form formula for the price of the spread option and is given as

$$X_0 e^{-q^X T} \Phi(d_1) - Y_0 e^{-q^Y T} \Phi(d_2)$$

where

$$d_{1,2} = \frac{\ln(X_0/Y_0) + (q^Y - q^X \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$\sigma = \sqrt{(\sigma^X)^2 + (\sigma^Y)^2 - 2\sigma^X \sigma^Y \rho}$$

- Note that the price of the spread option with $K = 0$ is independent of r. This is because we can replicate the option by just trading X and Y without the need to hold any cash, i.e. the interest rate is irrelevant.

A few results

- We need a few results before we can proceed with the derivation for the case $K = 0$.
- Let W_t^x and W_t^y be BMs with correlation ρ. We can express

$$\sigma^x dW_t^x - \sigma^y dW_t^y = \sigma dW_t$$

where $\sigma = \sqrt{(\sigma^x)^2 + (\sigma^y)^2 - 2\sigma^x\sigma^y\rho}$ and W_t is another BM which is uncorrelated with W_t^x and W_t^y.

We can show it by matching their first and the second moments.

- The first moment for both $\sigma^x dW_t^x - \sigma^y dW_t^y$ and σdW_t are zero.
- The second moment of $\sigma^x dW_t^x - \sigma^y dW_t^y$ can be computed as

$$\begin{aligned} & Var(\sigma^x dW_t^x) + Var(\sigma^y dW_t^y) - 2Covar(\sigma^x dW_t^x, \sigma^y dW_t^y) \\ = \quad & (\sigma^x)^2 dt + (\sigma^y)^2 dt - 2\sigma^x\sigma^y\rho dt \end{aligned}$$

- The second moment of σdW_t is $\sigma^2 dt$, so we get

$$\begin{aligned} \sigma^2 &= (\sigma^x)^2 + (\sigma^y)^2 - 2\sigma^x\sigma^y\rho \\ \sigma &= \sqrt{(\sigma^x)^2 + (\sigma^y)^2 - 2\sigma^x\sigma^y\rho}. \end{aligned}$$

Derivation for $K = 0$

- We can use the change of numeraire method to find the closed form formula.
- The price of the option can be computed as

$$V_0 = N_0 E_0 \left[\frac{(X_T - Y_T)^+}{N_T} \right] = N_0 E_0 \left[\frac{Y_T}{N_T} \left(\frac{X_T - Y_T}{Y_T} \right)^+ \right]$$

- If we pick $N_t = Y_t e^{q^Y t}$, then we have

$$V_0 = Y_0 e^{-q^Y T} E_0^Y \left[\left(\frac{X_T}{Y_T} - 1 \right)^+ \right]$$

- Note that if there is a dividend yield, we cannot just use Y_t as the numeraire as this does not include the dividend that pays out. We can think of $Y_t e^{q^Y t}$ as the total return of the stock with dividend reinvested continuously.
- The similar analogy is that if the risk-free rate $r \neq 0$, we cannot use 1 as the numeraire as this would not account for the interest that we receive. In order to account for the interest, we need to use a money market account e^{rt} as the numeraire.

- The ultimate goal is to find the SDE of $\frac{X}{Y}$ in the Y measure. Note that the ratio of two lognormals is still a lognormal, therefore, we can find the Black Scholes type formula for the spread option with $K = 0$.
- Let's first find out what is the SDE of Y in the Y measure, which we assume is lognormal with an unknown drift μ:

$$dY_t = \mu Y_t dt + \sigma^Y Y_t d\bar{W}_t^Y$$

- $\bar{W}_t^Y$ is a BM in the Y measure.
- By Girsanov's theorem for BM, we know that the volatility won't change even if the measure is changed, hence the volatility is still at σ^Y.
- By the martingale pricing formula, all numeraire rebased asset is a martingale under the measure induced by the numeraire:

$$\frac{V_0}{N_0} = E_0^N \left[\frac{V_T}{N_T} \right]$$

  This implies the drift of $\frac{V}{N}$ must be zero. [1] Using these facts, we can find μ as follows.

[1]This does not imply $\mu = 0$.

- Let $Z_t = \beta_t/(Y_t e^{q^Y t})$. Recall $d\beta_t = r\beta_t dt$.
- By Ito's lemma we have

$$\begin{aligned} dZ_t &= \frac{\partial Z}{\partial t}dt + \frac{\partial Z}{\partial \beta_t}d\beta_t + \frac{\partial Z}{\partial Y}dY_t + \frac{1}{2}\frac{\partial^2 Z}{\partial Y^2} < dY_t > \\ &= -q^Y Z_t dt + rZ_t dt - Z_t(\mu dt + \sigma^Y d\bar{W}_t^Y) + Z_t(\sigma^Y)^2 dt \end{aligned}$$

- Rearrange and we get

$$\frac{dZ_t}{Z_t} = \left(r - q^Y - \mu + (\sigma^Y)^2\right) dt - \sigma^Y d\bar{W}_t^Y$$

In order for the ratio Z_T to be a martingale, we must have

$$r - q^Y - \mu + (\sigma^Y)^2 = 0$$

therefore,

$$\mu = r - q^Y + (\sigma^Y)^2$$

- Follow the similar approach, we go to find the SDE of X in the Y measure. Let

$$dX_t = \kappa X_t dt + \sigma^X X_t d\bar{W}_t^X$$

with some unknown drift κ and $\bar{W}_t^X$ is a BM in Y measure.
- Let $Z_t = X_t e^{q^X t}/(Y_t e^{q^Y t})$.
- By Ito's lemma we have

$$\begin{aligned} dZ_t &= \frac{\partial Z}{\partial t}dt + \frac{\partial Z}{\partial X_t}dX_t + \frac{\partial Z}{\partial Y}dY_t + \frac{1}{2}\frac{\partial^2 Z}{\partial Y^2} < dY_t > + \frac{\partial^2 Z}{\partial X \partial Y} < dX_t, dY_t > \\ &= (q^X - q^Y)Z_t dt + Z_t(\kappa dt + \sigma^X d\bar{W}_t^X) - Z_t(\mu dt + \sigma^Y d\bar{W}_t^Y) \\ &\quad + Z_t(\sigma^Y)^2 dt - Z_t \sigma^X \sigma^Y \rho dt \end{aligned}$$

- Rearrange and we get

$$\frac{dZ_t}{Z_t} = \left(q^X - q^Y + \kappa - \mu + (\sigma^Y)^2 - \sigma^X\sigma^Y\rho\right)dt + \left(\sigma^X d\bar{W}_t^X - \sigma^Y d\bar{W}_t^Y\right)$$

In order for the ratio Z_T to be a martingale, we must have

$$q^X - q^Y + \kappa - \mu + (\sigma^Y)^2 - \sigma^X\sigma^Y\rho = 0$$

therefore,

$$\kappa = r - q^X + \sigma^X\sigma^Y\rho$$

- So far, we have derived the SDEs for X and Y with in Y measure:

$$\frac{dX_t}{X_t} = \left(r - q^X + \sigma^X \sigma^Y \rho\right) dt + \sigma^X d\bar{W}_t^X$$

$$\frac{dY_t}{Y_t} = \left(r - q^Y + (\sigma^Y)^2\right) dt + \sigma^Y d\bar{W}_t^Y$$

where $E[d\bar{W}_t^X d\bar{W}_t^Y] = \rho dt$

- We now going to compute the price of the spread option

$$V_0 = Y_0 e^{-q^Y T} E_0^Y \left[\left(\frac{X_T}{Y_T} - 1 \right)^+ \right]$$

By Ito's lemma, we can find

$$\frac{d(X_t/Y_t)}{(X_t/Y_t)} = (q^Y - q^X)dt + (\sigma^X d\bar{W}_t^X - \sigma^Y d\bar{W}_t^Y)$$

As shown previously, we can express a linear combination of two BMs as one single BM, so the SDE can be rewritten as

$$\frac{d(X_t/Y_t)}{(X_t/Y_t)} = (q^Y - q^X)dt + \sigma d\bar{W}_t$$

where $\sigma = \sqrt{(\sigma^X)^2 + (\sigma^Y)^2 - 2\sigma^X \sigma^Y \rho}$

- We can now regard the spread option as a call option on $\frac{X_T}{Y_T}$ with strike equals to 1. The expectation of $\frac{X_t}{Y_t}$ at maturity T is

$$E_0^Y\left[\frac{X_T}{Y_T}\right] = \frac{X_0}{Y_0}e^{(q^Y - q^X)T}$$

- Put it all together, we have

$$\begin{aligned} V_0 &= Y_0 e^{-q^Y T} E_0^Y\left[\left(\frac{X_T}{Y_T} - 1\right)^+\right] \\ &= Y_0 e^{-q^Y T}\left\{E_0^Y\left[\frac{X_T}{Y_T}\right]\Phi(d_1) - \Phi(d_2)\right\} \\ &= Y_0 e^{-q^Y T}\left\{\frac{X_0}{Y_0}e^{(q^Y - q^X)T}\Phi(d_1) - \Phi(d_2)\right\} \\ &= X_0 e^{-q^X T}\Phi(d_1) - Y_0 e^{-q^Y T}\Phi(d_2) \end{aligned}$$

where

$$d_{1,2} = \frac{\ln(X_0/Y_0) + (q^Y - q^X \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

The derivation is done.

Numerical Integration Formula For $K \neq 0$

- Let $F^X = X_0 e^{(r-q^X)T}$, $F^Y = Y_0 e^{(r-q^Y)T}$.
- The price of the spread option can be computed under the risk neutral measure:

$$
\begin{aligned}
V_0 &= e^{-rT} E_0^{\beta}\left[(X_T - Y_T - K)^+\right] \\
&= e^{-rT} E_0^{\beta}\left[(F^X e^{-\frac{1}{2}\sigma_X^2 T + \sigma_X \sqrt{T} x} - F^Y e^{-\frac{1}{2}\sigma_Y^2 T + \sigma_Y \sqrt{T} y} - K)^+\right]
\end{aligned}
$$

  where x and y are standard normal r.v. with correlation ρ.
- Two correlated normal r.v. (x, y) with correlation ρ can be expressed in terms of two independent normal r.v. (z_1, z_2):

$$
\begin{aligned}
x &= \rho z_1 + \sqrt{1-\rho^2} z_2 \\
y &= z_1
\end{aligned}
$$

- Use this result, we can rewrite the above equation as

$$
e^{-rT} E_0^{\beta}\left[(F^X e^{-\frac{1}{2}\sigma_X^2 T + \sigma_X \sqrt{T}(\rho z_1 + \sqrt{1-\rho^2} z_2)} - F^Y e^{-\frac{1}{2}\sigma_Y^2 T + \sigma_Y \sqrt{T} z_1} - K)^+\right]
$$

- Note that since z_1 and z_2 are independent, we can express the joint normal density as the product of two marginal densities

$$\int_{z_1}\int_{z_2}\phi(z_1,z_2)dz_1dz_2 = \int_{z_1}\int_{z_2}\phi(z_1)\phi(z_2)dz_1dz_2$$

- Rewrite the expectation in terms of two integrals.

$$e^{-rT}\int_{z_1}\phi(z_1)dz_1\int_{z_2}\Big(F^X e^{-\frac{1}{2}\sigma_X^2 T+\sigma_X\sqrt{T}(\rho z_1+\sqrt{1-\rho^2}z_2)} - F^Y e^{-\frac{1}{2}\sigma_Y^2 T+\sigma_Y\sqrt{T}z_1} - K\Big)^+\phi(z_2)dz_2$$

- Conditions on a value of z_1, we can perform the integration w.r.t z_2. We want to express the integration problem in the form such that we can express the solution in terms of the Black Scholes formula:

$$\int_z\Big(\bar{F}e^{-\frac{1}{2}\bar{\sigma}^2+\bar{\sigma}\sqrt{T}z} - \bar{K}\Big)^+\phi(z)dz = \bar{F}\Phi(\bar{d}_1) - \bar{K}\Phi(\bar{d}_2)$$

where $\bar{d}_{1,2} = \frac{\ln(\bar{F}/\bar{K})\pm\frac{1}{2}\bar{\sigma}^2 T}{\bar{\sigma}\sqrt{T}}$

- The z_2 integral can be expressed as

$$
\begin{aligned}
& \int_{z_2} \left(F^X e^{-\frac{1}{2}\sigma_X^2 T + \sigma_X \sqrt{T}(\rho z_1 + \sqrt{1-\rho^2} z_2)} - F^Y e^{-\frac{1}{2}\sigma_Y^2 T + \sigma_Y \sqrt{T} z_1} - K \right)^+ \phi(z_2) dz_2 \\
= \quad & \int_{z_2} \left(\bar{F} e^{-\frac{1}{2}\bar{\sigma}^2 + \bar{\sigma}\sqrt{T} z_2} - \bar{K} \right)^+ \phi(z_2) dz_2 \\
= \quad & \bar{F}\Phi(\bar{d}_1) - \bar{K}\Phi(\bar{d}_2)
\end{aligned}
$$

with

$$
\begin{aligned}
\bar{K} &= K + F^Y e^{-\frac{1}{2}\sigma_Y^2 T + \sigma_Y \sqrt{T} z_1} \\
\bar{\sigma} &= \sigma_X \sqrt{1-\rho^2} \\
\bar{F} &= F^X e^{-\frac{1}{2}\sigma_X^2 T + \sigma_X \sqrt{T} \rho z_1 + \frac{1}{2}\bar{\sigma}^2 T} \\
&= F^X e^{-\frac{1}{2}\sigma_X^2 \rho^2 T + \sigma_X \sqrt{T} \rho z_1}
\end{aligned}
$$

- Note that $\bar{K}$ and $\bar{F}$ are functions of z_1. Put them all together and we get

$$
V_0 = e^{-rT} \int_{z_1} \left(\bar{F}\Phi(\bar{d}_1) - \bar{K}\Phi(\bar{d}_2) \right) \phi(z_1) dz_1
$$

This integral cannot be computed analytically and needs to be evaluated using numerical integration.

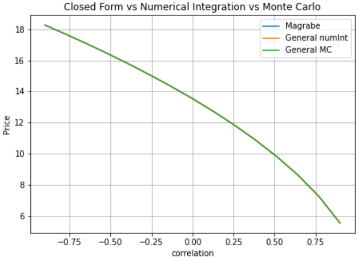
Python Code

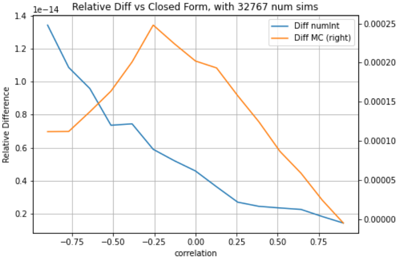
The code below shows an implementation of the closed form formula for $K = 0$ and the numerical integration approach for $K \geq 0$.

```
import numpy as np
from scipy.stats import norm
import scipy.integrate as integrate

# pays max(X(T) - Y(T), 0) at T
def Magrabe(x0, y0, qx, qy, volx, voly, rho, T):
    v = np.sqrt(volx*volx + voly*voly - 2*volx*voly*rho)
    d1 = (np.log(x0/y0) + (qy - qx + 0.5 * v*v)*T)/(v*np.sqrt(T))
    d2 = d1 - v*np.sqrt(T)
    return x0 * np.exp(-qx*T) * norm.cdf(d1) - y0 * np.exp(-qy*T) * norm.cdf(d2)

# pays max(X(T) - Y(T) - K, 0) at T
def general_spread_option(forwardX, forwardY, volx, voly, rho, T, K, DF):
    integrand = lambda z1: general_spread_option_integrand(forwardX, forwardY, volx, voly, rho, T, K, z1)
                             * norm.pdf(z1)
    undisc_spread_option, error = integrate.quad(integrand, -8, 8)
    return DF * undisc_spread_option

def general_spread_option_integrand(forwardX, forwardY, volx, voly, rho, T, K, z1):
    v = volx * np.sqrt(1.0 - rho*rho)
    K_ = K + forwardY*np.exp(-0.5*voly*voly*T+voly*np.sqrt(T)*z1)
    F_ = forwardX * np.exp(-0.5*volx*volx*rho*rho*T + volx*np.sqrt(T)*rho*z1)
    d1 = (np.log(F_/K_) + 0.5 * v*v*T)/(v*np.sqrt(T))
    d2 = d1 - v*np.sqrt(T)
    return F_ * norm.cdf(d1) - K_ * norm.cdf(d2)
```

Example 1, $K = 0$

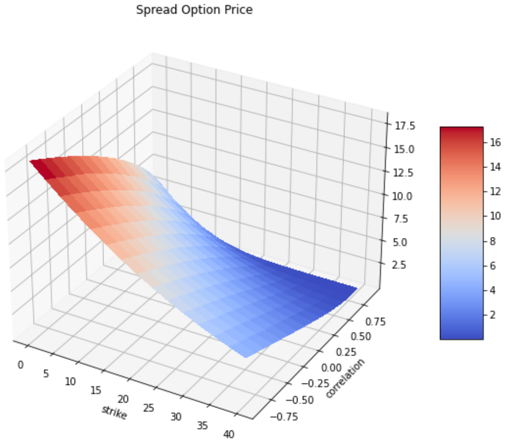
- Consider the following setup: $K = 0$, $T = 1$, $X_0 = 105$, $Y_0 = 100$, $r = 0.03$, $q^X = 0.1$, $q^Y = 0.05$, $\sigma^X = 0.2$, $\sigma^Y = 0.3$.
- The diagram on LHS shows the price of the spread option with various values of ρ using (1) Closed Form, (2) Numerical Integration, (3) Monte Carlo with 32767 simulations. RHS shows the relative error of (2) and (3) vs (1).
- We can see (2) is almost the same as (1) with relative errors below 10^{-13} whereas for (3), the errors are in the region of 10^{-4}.
- It is always good to have more than one way to implement the same pricing model for cross checking.





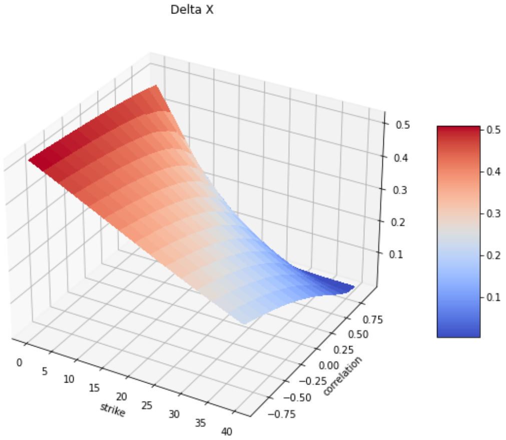
Example 2, $K \geq 0$

- Consider the same setup but now with the strike $K \geq 0$. The diagram shows the price of the spread option with various value of ρ and K using numerical integration.
- The line along $K = 0$ corresponds to the previous graph.
- We can see the spread option price increases as ρ decreases. This is because as ρ increases, the spread vol, $\sigma := \sqrt{(\sigma^X)^2 + (\sigma^Y)^2 - 2\sigma^X\sigma^Y\rho}$ decreases. This in turn reduces the price of the option.



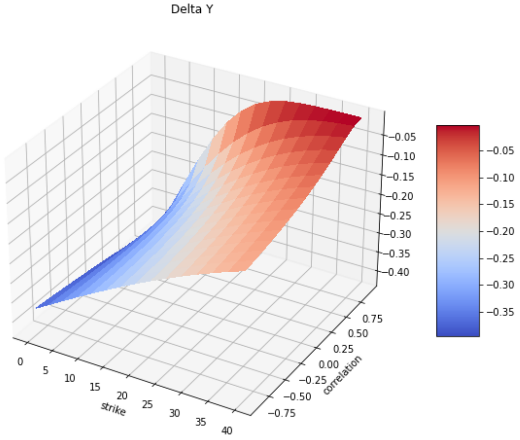
$\frac{\partial V_0}{\partial X_0}$

- Delta w.r.t X_0 is positive for all strikes and correlation.
- Delta decreases as strike increases.



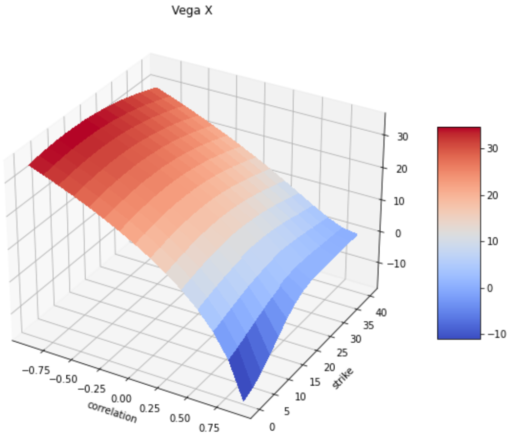
$\frac{\partial V_0}{\partial Y_0}$

- Delta w.r.t Y_0 is negative for all strikes and correlation.
- Delta decreases as strike decreases.



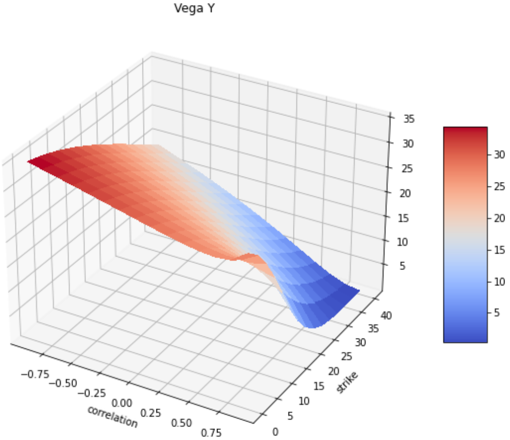
$\frac{\partial V_0}{\partial \sigma^X}$

- Vega w.r.t σ^X is positive for all strikes and correlation.
- Vega decreases as the correlation increases.



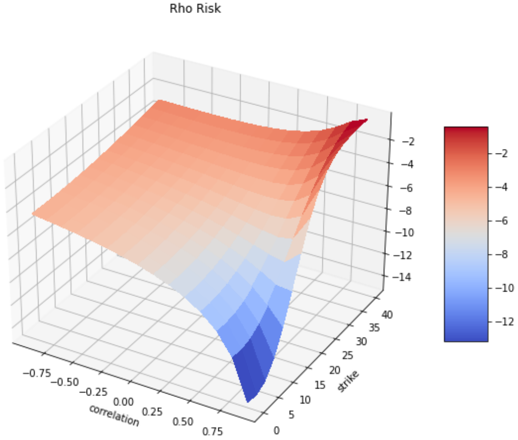
$\frac{\partial V_0}{\partial \sigma^Y}$

- Vega w.r.t σ^Y is positive for all strikes and correlation.
- Vega decreases as the correlation increases.



$\frac{\partial V_0}{\partial \rho}$

- The sensitivity to correlation is negative for all strikes and correlation. As increase in correlation reduces the spread volatility which in turn reduces the value of the spread option.



Spread Option with Time Dependent Volatilities

- Consider the case that we want to price a spread option that pays

$$(X_T - Y_T)^+$$

at time T. Assume there is a liquid European option market for X and Y with various maturities up to time T.

- In Lecture 2 we described how to calibrate a time dependent Black Scholes model to a term structure of ATM volatilities.
- In order to price a spread option on X_T and Y_T correctly, we must be able to have a model to price European options on X_T and Y_T individually. The natural extension is to allow for time dependent volatilities:

$$\frac{dX_t}{X_t} = (r - q^X)dt + \sigma_t^X dW_t^X$$

$$\frac{dY_t}{Y_t} = (r - q^Y)dt + \sigma_t^Y dW_t^Y$$

where $E[dW_t^X dW_t^Y] = \rho dt$

- For simplicity r, q^X and q^Y remains constant. The **instantaneous** correlation between the 2 BMs remains ρ. The term instantaneous refers to the correlation over a small period, $[t, t + dt]$, for all t.

Example

Consider the setup:

- $X_0 = 100,\ Y_0 = 100,\ r = 0,\ q^X = 0,\ q^Y = 0,\ \rho = -0.1,\ T = 2.$
- Implied volatilities for X: $\Sigma^X(1) = 0.2$, $\Sigma^X(2) = 0.18$
- Implied volatilities for Y: $\Sigma^Y(1) = 0.3$, $\Sigma^Y(2) = 0.25$.
- We assume σ_t^X and σ_t^Y are piece-wise constant.
- σ_1^X is σ_t^X between time 0 to time $t_1 = 1$, which can be computed as

$$\sigma_1^X = \Sigma^X(1) = 0.2$$

σ_2^X can be computed as

$$\sigma_2^X = \sqrt{\frac{\Sigma^X(2)^2 \times 2 - \Sigma^X(1)^2}{2-1}} = \sqrt{\frac{0.18^2 \times 2 - 0.2^2}{2-1}} = 0.15748$$

- Similarly, σ_1^Y and σ_2^Y can be computed as

$$\sigma_1^Y = \Sigma^Y(1) = 0.3$$

$$\sigma_2^Y = \sqrt{\frac{\Sigma^Y(2)^2 \times 2 - \Sigma^Y(1)^2}{2-1}} = \sqrt{\frac{0.25^2 \times 2 - 0.3^2}{2-1}} = 0.18708$$

Recall the spread option formula for $K = 0$:

$$X_0 e^{-q^X T}\Phi(d_1) - Y_0 e^{-q^Y T}\Phi(d_2)$$

where

$$d_{1,2} = \frac{\ln(X_0/Y_0) + (q^Y - q^X \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$\sigma = \sqrt{(\sigma^X)^2 + (\sigma^Y)^2 - 2\sigma^X\sigma^Y\rho}$$

- The question is: what values we should use for σ^X, σ^Y and $\sigma^X\sigma^Y\rho$ in the formula? There is a potential abuse of notations so I highlight the terms to green in the formula to indicate that they are different.

Answer

- $\sigma^X = \sqrt{\frac{1}{T}\int_0^T(\sigma_s^X)^2 ds} = \sqrt{\frac{1}{2}\left(0.2^2 \times 1 + 0.15748^2 \times (2-1)\right)} = 0.18$
- $\sigma^Y = \sqrt{\frac{1}{T}\int_0^T(\sigma_s^Y)^2 ds} = \sqrt{\frac{1}{2}\left(0.3^2 \times 1 + 0.18708^2 \times (2-1)\right)} = 0.25$
- $\sigma^X\sigma^Y\rho = \frac{1}{T}\int_0^T \sigma_s^X\sigma_s^Y\rho ds =$
  $\frac{1}{2}\left(0.2 \times 0.3 \times -0.1 \times 1 + 0.15748 \times 0.18708 \times -0.1 \times (2-1)\right) = -0.00447$
- Therefore, $\sigma = \sqrt{0.18^2 + 0.25^2 - 2 \times -0.00447} = 0.32225$
- Put all the values in the formula and we get the price equals 18.0296

The Impact on correlation due to Time Dependent Vols

- By now, it should be clear that correlation has a major role to determine the price of a spread option. But which correlation are we talking about?
- Let's define **terminal correlation** be:

$$\rho^{Term} := \frac{\int_0^T \sigma_s^X \sigma_s^Y \rho ds}{\sqrt{\int_0^T (\sigma_s^X)^2 ds}\sqrt{\int_0^T (\sigma_s^Y)^2 ds}}$$

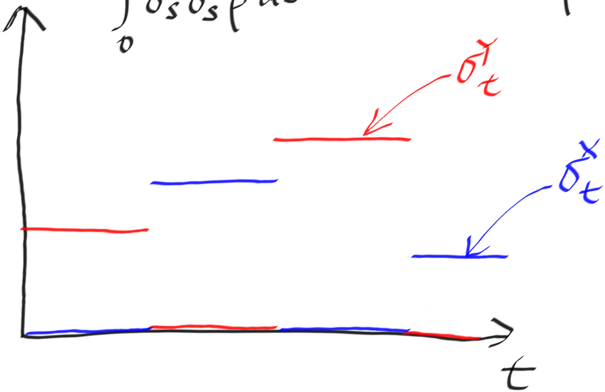
- The example in the previous slide shows that (assuming σ_t^X and σ_t^Y are calibrated to their respective European options) it is this term

$$\int_0^T \sigma_s^X \sigma_s^Y \rho ds$$

  which determines the spread option price rather than just the instantaneous correlation ρ.
- We can see that if σ_t^X and σ_t^Y are constants then $\rho^{Term} = \rho$. Also, ρ^{Term} is bounded from above by ρ.

- We can also show that even if $\rho \neq 0$, we can still have $\rho^{Term} = 0$.
- **Example.** We have 2 piece-wise constant vols, σ_t^X and σ_t^Y. If σ_t^X is exactly zero whenever σ_t^Y is non-zero, and vice versa. Then $\rho^{Term} = 0$ even if $\rho \neq 0$.

$$\int_0^T \sigma_s^X \sigma_s^Y \rho \, ds = 0 \text{ for all } \rho$$



Forward Starting Option and Forward Implied Volatility

- Forward starting option is just like an European option but the strike is not set until some time in the future.
- The payoff of a forward starting call option is given as:

$$(S_{T_2} - \alpha S_{T_1})^+$$

  where $T_1 < T_2$ and α is the parameter which determines the moneyness with $\alpha = 1$ means ATM.
- If we assume S is driven by the dynamics:

$$\frac{dS_t}{S_t} = (r - q)dt + \sigma dW_t$$

  where W_t is BM under risk neutral measure, then there is a closed form formula for the forward starting call option.

- By the martingale pricing formula:

$$V_0 = e^{-rT_2} E_0 \left[(S_{T_2} - \alpha S_{T_1})^+ \right]$$

- By the tower law of expectation, we have

$$V_0 = e^{-rT_2} E_0 \left[E_1 \left[(S_{T_2} - \alpha S_{T_1})^+ \right] \right] \tag{1}$$

- Let's concentrate on the expectation taken at T_1 and let $\tau := T_2 - T_1$:

$$\begin{aligned} E_1 \left[(S_{T_2} - \alpha S_{T_1})^+ \right] &= F_{T_1}(T_2)\Phi(d_1) - \alpha S_{T_1}\Phi(d_2) \\ &= S_{T_1} e^{(r-q)\tau}\Phi(d_1) - \alpha S_{T_1}\Phi(d_2) \\ &= S_{T_1} \left(e^{(r-q)\tau}\Phi(d_1) - \alpha\Phi(d_2) \right) \end{aligned} \tag{2}$$

where

$$d_{1,2} = \frac{\ln(F_{T_1}(T_2)/\alpha S_{T_1}) \pm \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}} = \frac{-\ln\alpha + (r - q \pm \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$$

- Now we substitute (2) into (1) and we have

$$V_0 = e^{-rT_2} E_0\left[S_{T_1}\left(e^{(r-q)\tau}\Phi(d_1) - \alpha\Phi(d_2)\right)\right]$$

Since the terms in the red bracket are non-random, we can take them outside of the expectation

$$\begin{aligned} &= e^{-rT_2} E_0\left[S_{T_1}\right]\left(e^{(r-q)\tau}\Phi(d_1) - \alpha\Phi(d_2)\right) \\ &= e^{-rT_2} F_0(T_1)\left(e^{(r-q)\tau}\Phi(d_1) - \alpha\Phi(d_2)\right) \\ &= e^{-rT_2}\left(F_0(T_2)\Phi(d_1) - \alpha F_0(T_1)\Phi(d_2)\right) \qquad (3) \end{aligned}$$

- Equation (3) is the closed form solution of forward starting option in Black Scholes model.
- If we are given a **price for a forward starting option**, we can invert Equation (3) in order to find σ in the similar spirit as finding the implied volatility from an European option. This is called the **forward implied volatility**.

Greeks

- In the Black Scholes model, unlike European options, forward starting options only has delta and vega but **no gamma**.
- The delta of forward staring options in the Black Scholes model is

$$\frac{\partial V_0}{\partial S_0} = e^{-rT_2}\left(e^{(r-q)T_2}\Phi(d_1) - e^{(r-q)T_1}\alpha\Phi(d_2)\right)$$

  which we can see that the delta is not a function of S_0, in other words, gamma is 0.
- However, once the strike has been fixed at T_1, i.e. $K = \alpha S_{T_1}$ then the forward starting option will become an ordinary European option, then the gamma is no longer 0.
- Vega of forward starting options is

$$\frac{\partial V_0}{\partial \sigma} = e^{-rT_2}F_0(T_2)\sqrt{\tau}\phi(d_1)$$

Proof

- The value of the forward starting option is

$$V_0 = e^{-rT_2}\Big(F_0(T_2)\Phi(d_1) - \alpha F_0(T_1)\Phi(d_2)\Big)$$

where $d_{1,2} = \frac{-\ln\alpha + (r-q\pm\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$. We can see only $F_0(T_2)$ and $F_0(T_1)$ are functions of S_0. Hence the delta is

$$\frac{\partial V_0}{\partial S_0} = e^{-rT_2}\Big(e^{(r-q)T_2} - \alpha e^{(r-q)T_1}N(d_2)\Big)$$

The proof is done.

- Vega is a bit tedious but straight forward to compute. Here are the steps:

$$\begin{aligned}
\frac{\partial V_0}{\partial \sigma} &= e^{-rT_2}\left(F_0(T_2)\phi(d_1)\frac{\partial d_1}{\partial \sigma} - \alpha F_0(T_1)\phi(d_2)\frac{\partial d_2}{\partial \sigma}\right)\\
\phi(d_2) &= \phi(d_1)e^{\sigma\sqrt{\tau}d_1 - \frac{1}{2}\sigma^2\tau} = \phi(d_1)\frac{e^{(r-q)\tau}}{\alpha}\\
\frac{\partial d_1}{\partial \sigma} &= \frac{\ln\alpha}{\sigma^2\sqrt{\tau}} + \frac{\sqrt{\tau}}{2}\\
\frac{\partial d_2}{\partial \sigma} &= \frac{\ln\alpha}{\sigma^2\sqrt{\tau}} - \frac{\sqrt{\tau}}{2}
\end{aligned}$$

- Collect all the terms and we get

$$\frac{\partial V_0}{\partial \sigma} = e^{-rT_2}\Big(F_0(T_2)\phi(d_1)\Big(\frac{\ln \alpha}{\sigma^2\sqrt{\tau}} + \frac{\sqrt{\tau}}{2}\Big) - \alpha F_0(T_1)\phi(d_1)\frac{e^{(r-q)\tau}}{\alpha}\Big(\frac{\ln \alpha}{\sigma^2\sqrt{\tau}} - \frac{\sqrt{\tau}}{2}\Big)\Big)$$

- It can be simplified as

$$\begin{aligned}
\frac{\partial V_0}{\partial \sigma} &= e^{-rT_2}\Big(F_0(T_2)\phi(d_1)\Big(\frac{\ln \alpha}{\sigma^2\sqrt{\tau}} + \frac{\sqrt{\tau}}{2}\Big) - F_0(T_2)\phi(d_1)\Big(\frac{\ln \alpha}{\sigma^2\sqrt{\tau}} - \frac{\sqrt{\tau}}{2}\Big)\Big) \\
&= e^{-rT_2}\Big(F_0(T_2)\phi(d_1)\Big(\frac{\sqrt{\tau}}{2} + \frac{\sqrt{\tau}}{2}\Big)\Big) \\
&= e^{-rT_2}F_0(T_2)\sqrt{\tau}\phi(d_1)
\end{aligned}$$

The proof is done.

Forward Starting Option with Time Dependent Volatilities

- Let's examine forward starting option if we have a time dependent Black Scholes model:
$$\frac{dS_t}{S_t} = (r - q)dt + \sigma_t dW_t$$

For simplicity, r and q remains constant. Consider the setup:

- $S_0 = 100,\ r = 0.05,\ q = 0.02.$
- Implied volatilities are $\Sigma(1) = 0.2,\ \Sigma(2) = 0.18$
- We assume σ_t is piece-wise constant.
- σ_1 is σ_t between time 0 to time $t_1 = 1$, which can be computed as
$$\sigma_1 = \Sigma(1) = 0.2$$
σ_2 is σ_t between $t_1 = 1$ to $t_2 = 2$ which can be computed as
$$\sigma_2 = \sqrt{\frac{\Sigma(2)^2 \times 2 - \Sigma(1)^2}{2-1}} = \sqrt{\frac{0.18^2 \times 2 - 0.2^2}{2-1}} = 0.15748$$
- What's the price of the forward starting option with payoff
$$(S_{T_2} - \alpha S_{T_1})^+$$
for $\alpha = 1.05,\ T_1 - 1,\ T_2 = 2.$

Answer

- Recall the Equation (3) for the pricing formula:

$$
\begin{aligned}
V_0 &= e^{-rT_2}\Big(F_0(T_2)\Phi(d_1) - \alpha F_0(T_1)\Phi(d_2)\Big) \\
d_{1,2} &= \frac{-\ln\alpha + (r - q \pm \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}
\end{aligned}
$$

  with $\tau = T_2 - T_1$.

- The question is: what value we should use for the forward starting volatility σ?
- We should use the volatility which covers the period $[T_1, T_2]$, for our case, this is $\sqrt{\frac{1}{\tau}\int_{T_1}^{T_2}\sigma_s^2 ds} = \sigma_2 = 0.15748$
- Put all values to the formula and we get 5.2191.

Another Example

- Using the same setup, what's the price of the forward starting option with the payoff
$$(S_{1.2} - \alpha S_{0.7})^+$$
for $\alpha = 1$.

Answer

- The forward starting volatility that we should use is

$$\begin{aligned}\sqrt{\frac{1}{1.2-0.7}\int_{0.7}^{1.2}\sigma_s^2 ds} &= \sqrt{\frac{1}{0.5}\Big(0.2^2 \times (1-0.7) + 0.15748^2 \times (1.2-1)\Big)} \\ &= 0.18417\end{aligned}$$

- Put all the values to the formula and we get 5.7909.
- These 2 examples show that forward starting options depends on the the term structure of the implied volatilities (i.e. $\Sigma(1)$ and $\Sigma(2)$). It also depends on the modelling assumption, in this case, piece-wise constant for σ_t.

FX rate in domestic risk neutral measure

- If we assume an FX rate X is lognormally distributed, under the **domestic risk neutral measure**, the SDE of X is

$$\frac{dX_t}{X_t} = (r^d - r^f)dt + \sigma dW_t$$

  where r^d and r^f are domestic and foreign risk free rates respectively. W_t is a BM under domestic risk neutral measure.

Proof

- We can apply the martingale pricing formula with the numeraire asset to be the domestic money market account $\beta^d(t) = e^{r^d t}$. Note that an FX rate itself is not a tradable asset but a foreign money market account denominated in the domestic currency is, i.e. $X_t \beta^f_t$.
- Let $\frac{dX_t}{X_t} = \mu dt + \sigma dW_t$ and μ is the unknown drift that we want to find.
- By the martingale pricing formula, we have

$$\begin{aligned} \frac{X_0 \beta^f_0}{\beta^d_0} &= E_0\left[\frac{X_t \beta^f_t}{\beta^d_t}\right] \\ X_0 &= e^{(-r^d + r^f)t} E_0\left[X_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t}\right] = X_0 e^{(-r^d + r^f + \mu)t} \end{aligned}$$

- To solve the equation, μ must be equal to $r^d - r^f$. The proof is done.

FX rate in foreign risk neutral measure

- If we assume an FX rate X is lognormally distributed, under the **foreign risk neutral measure**, the SDE of X is

$$\frac{dX_t}{X_t} = (r^d - r^f + \sigma^2)dt + \sigma d\bar{W}_t$$

where $\bar{W}_t$ is a BM under foreign risk neutral measure.

Proof

- Foreign investors see the domestic money market account denominated in foreign currency is a tradable asset, $\frac{\beta_t^d}{X_t}$. We then apply the martingale pricing formula with β_t^f as the numeraire.
- Let $\frac{dX_t}{X_t} = \mu dt + \sigma d\bar{W}_t$ and μ is the unknown drift that we want to find.

$$\begin{aligned} \frac{\beta_0^d}{\beta_0^f X_0} &= E_0\left[\frac{\beta_t^d}{\beta_t^f X_t}\right] \\ \frac{1}{X_0} &= e^{(r^d-r^f)t} E_0\left[\frac{1}{X_0} e^{\left(-\mu+\frac{1}{2}\sigma^2\right)t-\sigma W_t}\right] = \frac{1}{X_0} e^{(r^d-r^f-\mu+\sigma^2)t} \end{aligned}$$

- To solve the equation, μ must be equal to $r^d - r^f + \sigma^2$. The proof is done.

Quanto Option

- A quanto option is an option that the currency of the underlying is different to the currency of the option payoff. Also, the FX rate is **fixed** at the inception. Therefore, **the quanto option owner** does not have FX delta risk.
- For example, Google stock is an USD tradable asset but it is perfectly legit to write an option on the return of Google but the notional is specified in SGD.
- A **non-quanto** equity call on an USD tradable asset S with maturity T is defined as:

$$N_{USD}\left(\frac{S_T}{S_0}-k\right)^+$$

- N_{USD} is the notional in the same currency as S. k is the strike factor, e.g. 1 if at-the-money. This form is normally called the "fixed notional".
- Another less common form, "fixed units" is defined as

$$n(S_T-K)^+$$

- where $K=kS_0$ and $n=N_{USD}/S_0$ is the number of units. The two forms are equivalent.

- Using the fixed notional form, it is easier to see what a quanto option is. A **quanto** equity call can be defined as:

$$N_{SGD}\left(\frac{S_T}{S_0}-k\right)^{+}.$$

- N_{SGD} is the notional in SGD. Note that this is equivalent to a non-quanto equity call with the FX rate is fixed at inception at some FX rate $\bar{X}$

$$N_{USD}\bar{X}\left(\frac{S_T}{S_0}-k\right)^{+}.$$

- The key to understand quanto option pricing is to keep a firm grasp on what tradable quantities are. Suppose we are a SGD investor, our numeraire is SGD money market account. Google stock is denominated in USD. In order for us to buy the stock, we first need to convert our SGD cash into USD cash and then to buy the stock. This means our PnL of the trade at time t is

$$X_t S_t - X_0 S_0$$

  In other words, we have exposures in both USDSGD and Google.
- We can convert a USD tradable into a SGD tradable by multiplying with an FX rate but the question is: what's the stochastic process of S_t (not $X_t S_t$) in SGD risk neutral measure?

In order to price the quanto option, we need to find out the SDE of S_t in SGD risk neutral measure. Let's first identify what processes are involved.

- β_t^d denotes the SGD money market account with rate r^d.
- β_t^f denotes the USD money market account with rate r^f.
- X_t denotes the value of one USD in SGD at time t and is assumed to be lognormally distributed. W_t^X is a BM in SGD risk neutral measure.
- S_t denotes the value of the a US stock at time t and is assumed to be lognormally distributed and don't pay dividend. W_t^S is a BM in USD risk neutral measure.

The processes that we have are

$$\begin{aligned} d\beta_t^d &= r^d \beta_t^d dt \\ d\beta_t^f &= r^f \beta_t^f dt \\ \frac{dS_t}{S_t} &= r^f dt + \sigma_S dW_t^S \\ \frac{dX_t}{X_t} &= (r^d - r^f)dt + \sigma_X dW_t^X \end{aligned}$$

The correlation between W_t^S and W_t^X is ρ.

- Our aim is to find the drift of S_t in SGD risk neutral measure, i.e. β_t^d as the numeraire.
- Recall that any numeraire rebased (domestically) tradable asset must be a martingale under the measure induced by the numeraire. Consider the tradable asset is $X_t S_t$ and the numeraire is β_t^d, we have

$$\frac{X_t S_t}{\beta_t^d} = E_t^{\beta^d}\left[\frac{X_T S_T}{\beta_T^d}\right]$$

- Let $Y_t = \frac{X_t S_t}{\beta_t^d}$. The equation above implies the drift of Y_t under the SGD risk neutral measure is 0.
- By Ito's lemma, we can find the SDE of Y_t as

$$\begin{aligned}
dY_t &= \frac{\partial Y_t}{\partial t}dt + \frac{\partial Y_t}{\partial X_t}dX_t + \frac{\partial Y_t}{\partial S_t}dS_t + \frac{\partial^2 Y_t}{\partial X_t S_t} < dX_t, dS_t > \\
&= -r^d Y_t dt + \frac{S_t}{\beta_t^d}dX_t + \frac{X_t}{\beta_t^d}dS_t + \frac{1}{\beta_t^d} < dX_t, dS_t > \\
&= -r^d Y_t dt + Y_t\Big((r^d - r^f)dt + \sigma_X dW_t^X\Big) + Y_t\Big(\mu dt + \sigma_S d\bar{W}_t^S\Big) \\
&\quad + Y_t \sigma_X \sigma_S \rho dt
\end{aligned}$$

- where $d\bar{W}_t^S$ is a BM under SGD risk neutral measure and μ is the drift that we are trying to find when the measure is changed from USD risk neutral to SGD risk neutral.

- Collecting the terms, we have

$$\frac{dY_t}{Y_t} = -r^d dt + \left((r^d - r^f)dt + \sigma_X dW_t^X\right) + \left(\mu dt + \sigma_S d\bar{W}_t^S\right) + \sigma_X \sigma_S \rho dt$$

- In order for Y_t to be a martingale, the drift of Y_t must be zero. This implies

$$-r^d + (r^d - r^f) + \mu + \sigma_X \sigma_X \rho = 0$$

- Finally, we have

$$\mu = r^f - \sigma_X \sigma_S \rho$$

- The SDE of S under the SGD risk neutral measure is

$$\frac{dS_t}{S_t} = (r^f - \sigma_X \sigma_S \rho)dt + \sigma_S d\bar{W}_t^S$$

- The extra term $-\sigma_X \sigma_S \rho$ is called the quanto adjustment. We can see that if the correlation ρ is zero, the quanto adjustment is also zero.
- Another important point is that the SDE does not depend on the FX spot X_t but depends on the FX spot volatility σ_X.

- After we have found the drift of S in SGD risk neutral measure, we come back to our original problem: pricing a quanto option with the payoff at maturity T

$$N_{SGD}\left(\frac{S_T}{S_0}-k\right)^+$$

- Apply the martingale pricing formula, we have

$$\begin{aligned} V_0 &= N_{SGD}e^{-r^dT}E^{\beta^d}\left[\left(\frac{S_T}{S_0}-k\right)^+\right] \\ &= N_{SGD}e^{-r^dT}E^{\beta^d}\left[\left(e^{(r^f-\sigma_X\sigma_S\rho-\frac{1}{2}\sigma_S^2)T+\sigma_S\bar{W}_T^S}-k\right)^+\right] \end{aligned}$$

- We can apply the Black Scholes call option formula with forward $F_0(T)=e^{(r^f-\sigma_X\sigma_S\rho)T}$, strike equals k and volatility equals σ_S:

$$V_0 = N_{SGD}e^{-r^dT}\Big(F_0(T)\Phi(d_1)-k\Phi(d_2)\Big) \tag{4}$$

where $d_{1,2}=\frac{\ln(F_0(T)/k)\pm\frac{1}{2}\sigma_S^2T}{\sigma^S\sqrt{T}}$

Example

- Consider the following setup. Assume we are a SGD investor. S_t is the price of a US stock at time t. $S_0 = 100$, $r^d = 0.03$, $r^f = 0.01$, $\sigma_S = 0.4$, $\sigma_X = 0.1$, $\rho = -0.3$, $k = 1.1$, $T = 1$, $N_{SGD} = 200,000$
- The price of the Quanto call option is

$$V_0 = N_{SGD} e^{-r^d T} E_0 \left[\left(\frac{S_T}{S_0} - k \right)^+ \right]$$

- We can apply for quanto call optio pricing formula

$$V_0 = N_{SGD} e^{-r^d T} \Big(F_0(T)\Phi(d_1) - k\Phi(d_2) \Big)$$

where

$$F_0(T) = e^{(0.01 - 0.1 \times 0.4 \times -0.3)1} = 1.02224$$

$$d_{1,2} = \frac{\ln(F_0(T)/k) \pm \frac{1}{2}\sigma_S^2 T}{\sigma_S \sqrt{T}}$$

- The price is 12820.4 SGD.

Greeks for quanto opiton

Equation (4) shows that the quanto call option is a function of

- The initial stock price S_0.
- SGD risk free rate r^d through the discount factor $e^{-r^d T}$.
- USD risk free rate r^f through the forward $F_0(T)$.
- The stock implied volatility σ_S. We will discuss more about this in the next slide.
- The FX rate implied volatility σ_X through the quanto adjustment.
- The correlation ρ between the log returns of X_t and S_t through the quanto adjustment.
- An important omission is the dependency of the initial FX rate X_0.

Vega with respect to the stock implied volatility σ_S

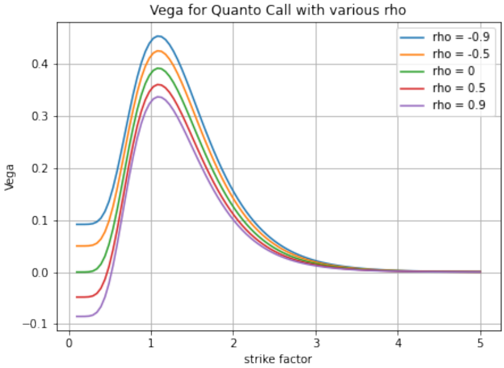
- Notice that the stock implied volatility σ_S appears twice in the Quanto call option formula: (1) in the quanto adjustment, (2) "the volatility" in the Black Scholes call option formula. Let's compute the vega w.r.t σ_S:

$$\begin{aligned}
\frac{\partial V_0}{\partial \sigma_S} &= N_{SGD} e^{-r^d T}\left(F_0(T)\frac{\partial \Phi(d_1)}{\partial \sigma_S} + \Phi(d_1)\frac{\partial F_0(T)}{\partial \sigma_S} - k\frac{\partial \Phi(d_2)}{\partial \sigma_S}\right)\\
&= N_{SGD} e^{-r^d T}\left(F_0(T)\phi(d_1)\frac{\partial d_1}{\partial \sigma_S} - \sigma_X \rho T F_0(T)\Phi(d_1) - k\phi(d_2)\frac{\partial d_2}{\partial \sigma_S}\right)\\
\phi(d_2) &= \phi(d_1)e^{\sigma_S\sqrt{T}d_1 - \frac{1}{2}\sigma_S^2 T} = \phi(d_1)\frac{F_0(T)}{k}\\
\frac{\partial d_1}{\partial \sigma_S} &= \frac{\ln(K/F_0(T))}{\sigma_S^2\sqrt{T}} + \frac{\sqrt{T}}{2} + \frac{\frac{\partial F_0(T)}{\partial \sigma_S}}{F_0(T)\sigma_S\sqrt{T}}\\
\frac{\partial d_2}{\partial \sigma_S} &= \frac{\ln(K/F_0(T))}{\sigma_S^2\sqrt{T}} - \frac{\sqrt{T}}{2} + \frac{\frac{\partial F_0(T)}{\partial \sigma_S}}{F_0(T)\sigma_S\sqrt{T}}
\end{aligned}$$

- Collecting all the terms and we get

$$\begin{aligned}
\frac{\partial V_0}{\partial \sigma_S} &= N_{SGD} e^{-r^d T}\left(F_0(T)\phi(d_1)\frac{\partial d_1}{\partial \sigma_S} - \sigma_X \rho T F_0(T)\Phi(d_1) - k\phi(d_2)\frac{\partial d_2}{\partial \sigma_S}\right)\\
&= N_{SGD} e^{-r^d T}\left(F_0(T)\phi(d_1)\sqrt{T} - \sigma_X \rho T F_0(T)\Phi(d_1)\right)\\
&= N_{SGD} e^{-r^d T} F_0(T)\left(\phi(d_1)\sqrt{T} - \sigma_X \rho T \Phi(d_1)\right) \qquad (5)
\end{aligned}$$

- Consider the following setup. Assume we are a SGD investor. S_t is the price of a US stock at time t. $S_0 = 100$, $r^d = 0.03$, $r^f = 0.01$, $\sigma_S = 0.4$, $\sigma_X = 0.1$, $T = 1$.
- The graph below shows $\frac{\partial V_0}{\partial \sigma_S}$ profile for various value of ρ.
- Notice that the vega of a quanto call option can be negative if $\rho > 0$ for deep ITM call option. This is due to the term $-\sigma_X \rho T \Phi(d_1)$ in Equation (5).
- The intuition is that for deep ITM call option, the call option becomes a linear product. The value of the quanto adjusted forward, $F_0(T) = e^{(r^f - \sigma_X \sigma_S \rho)T}$ dictates the value of the quanto call. We can see the if $\rho > 0$, as σ_S increases, $F_0(T)$ decreases, hence for deep ITM call option, the vega is negative.



How to Hedge a Quanto Forward

This is to illustrate why the correlation between the log return of S_t and X_t is involved in the quanto adjustment.

- For simplicity, assume $r^d = 0$ and $r^f = 0$ for all cases below. Therefore, all forward prices equals to the corresponding spot prices.
- A quanto equity forward on a US stock has the following payoff in SGD at maturity T

$$Q_T = N_{SGD}\left(\frac{S_T}{S_0} - k\right)$$

- Assume we **sold** a quanto forward at time 0 and we need to hedge against it until maturity T. We **long** a **non-quanto** equity forward to hedge the equity delta risk which has the following payoff at T

$$V_T = N_{USD}\left(\frac{S_T}{S_0} - k\right)$$

  where we choose $N_{USD} = N_{SGD}/X_0$, X_0 is the FX rate at 0.

Case 1

- Assume USDSGD FX rate, X is a constant at the level X_0, then $Q(T) = V(T)$ in all scenarios. We are fully hedged.

Case 2

- We now consider the case where X is stochastic but S is a constant at the level S_0, i.e. $S_T = S_0$. In this case, the only source of risk is FX. In order to hedge at time 0, we enter an FX forward contract to sell USD and buy SGD at T. The USD notional amount of the FX forward contract is

$$N_{FX} = N_{USD}\left(\frac{S_0}{S_0} - k\right)$$

- The amount is chosen to be the value for non-quanto equity forward contract (which is a constant in Case 2). The FX forward price is X_0 because both r^d and r^f are 0.
- At maturity T, the non-quanto forward payoff in **SGD** is (the USD amount can be converted to SGD using the prevailing FX rate at time T, X_T)

$$N_{USD}\left(\frac{S_T}{S_0} - k\right)X_T = N_{USD}\left(\frac{S_0}{S_0} - k\right)X_T$$

- The FX forward payoff in SGD is

$$-N_{FX}(X_T - X_0) = -N_{USD}\left(\frac{S_0}{S_0} - k\right)(X_T - X_0)$$

- The quanto equity payoff in SGD is

$$-N_{SGD}\left(\frac{S_0}{S_0} - k\right)$$

- Recall $N_{SGD} = N_{USD} X_0$. Put them all together and the PnL that we have at time T is

$$
\begin{aligned}
& N_{USD}\left(\frac{S_0}{S_0} - k\right) X_T - N_{USD}\left(\frac{S_0}{S_0} - k\right)(X_T - X_0) - N_{SGD}\left(\frac{S_0}{S_0} - k\right) \\
= \quad & N_{USD}\left(\frac{S_0}{S_0} - k\right)(X_T - (X_T - X_0) - X_0) \\
= \quad & 0
\end{aligned}
$$

- We can see that the value of the non-quanto forward and the FX forward replicates the quanto equity payoff for all X_T.

A numerical example for case 2

- Consider the following setup: $S_0 = 100$, $X_0 = 1.4$, $N_{SGD} = 1.4$, $k = 0.9$.
- At time 0, (1) we sold a quanto equity forward with k. (2) We long a non-quanto equity forward with $N_{USD} = 1$. (3) We also enter an FX forward that sell USD buy SGD at the forward rate 1.4.
- At maturity T, $S_T = S_0 = 100$, USDSGD drops to $X_T = 1.2$ (USD depreciates). The three instruments have the following payoff from our perspective:
- Non-quanto forward: $N_{USD}\left(\frac{S_0}{S_0} - k\right) X_T = 1(1 - 0.9)1.2 = 0.12SGD$
- FX forward:
  $-N_{USD}\left(\frac{S_0}{S_0} - k\right)(X_T - X_0) = -1(1 - 0.9)(1.2 - 1.4) = 0.02SGD$
- Quanto forward: $-N_{SGD}\left(\frac{S_0}{S_0} - k\right) = -1.4(1 - 0.9) = -0.14SGD$
- Our PnL is zero. In fact, we are fully hedged in all scenarios of X_T

Case 3

- We now consider the case where both X and S are stochastic. At time 0, the notional of non-quanto equity forward is N_{USD}. We also enter into an FX forward contract with FX forward price X_0 and USD notional

$$N_{FX} = N_{USD} E_0^{\beta}\left[\left(\frac{S_T}{S_0} - k\right)\right] = N_{USD}\left(\frac{F_0(T)}{S_0} - k\right)$$

- At time T, both variables have changed to X_T and S_T respectively. The three instruments have the following value in SGD at T:
- Non-quanto forward: $N_{USD}\left(\frac{S_T}{S_0} - k\right) X_T$
- FX forward: $-N_{USD}\left(\frac{F_0(T)}{S_0} - k\right)(X_T - X_0)$
- Quanto forward: $-N_{SGD}\left(\frac{S_T}{S_0} - k\right)$
- Put them all together and the PnL that we have at time T is

$$\begin{aligned}
& N_{USD}\left(\frac{S_T}{S_0} - k\right) X_T - N_{USD}\left(\frac{F_0(T)}{S_0} - k\right)(X_T - X_0) - N_{SGD}\left(\frac{S_T}{S_0} - k\right) \\
= \quad & N_{USD}\left(\frac{S_T}{S_0} - k\right)(X_T - X_0) - N_{USD}\left(\frac{F_0(T)}{S_0} - k\right)(X_T - X_0) \\
= \quad & N_{USD}(X_T - X_0)\left(\left(\frac{S_T}{S_0} - k\right) - \left(\frac{F_0(T)}{S_0} - k\right)\right) \\
= \quad & N_{USD}(X_T - X_0)\left(\frac{S_T - F_0(T)}{S_0}\right)
\end{aligned}$$

A numerical example for Case 3

- Consider the following setup: $S_0 = 100$, $F_0(T) = 100$, $X_0 = 1.4$, $N_{SGD} = 1.4$, $k = 0.9$.
- At time 0, (1) we sold a quanto equity forward with k. (2) We long a non-quanto equity forward with $N_{USD} = 1$. (3) We also enter an FX forward that sell USD buy SGD at the forward rate 1.4.
- At maturity T, $S_T = 85$ and $X_T = 1.2$. The three instruments have the following payoff from our perspective:
- Non-quanto forward: $N_{USD}\left(\frac{S_T}{S_0} - k\right) X_T = 1(0.85 - 0.9)1.2 = -0.06SGD$
- FX forward:
  $-N_{USD}\left(\frac{F_0(T)}{S_0} - k\right)(X_T - X_0) = -1(1 - 0.9)(1.2 - 1.4) = 0.02SGD$
- Quanto forward: $-N_{SGD}\left(\frac{S_T}{S_0} - k\right) = -1.4(0.85 - 0.9) = 0.07SGD$
- Our PnL is 0.03 SGD.

- The PnL formula $N_{USD}(X_T - X_0)\left(\frac{S_T - F_0(T)}{S_0}\right)$ shows that if X_T and S_T has positive correlation, i.e. tends to move in the same direction, then we will have positive PnL.
- This means the buyer of the quanto forward should get compensated for this "biased" and this is reflected in the quanto adjustment of the forward:
$$F_0(T)^{quanto} = F_0(T)e^{-\sigma_x \sigma_S \rho T}$$
- We showed that if $\rho > 0$, the seller of the quanto forward has a biased for positive PnL by the hedging using non-quanto forward and FX forward. The quanto adjustment will adjust the forward price to be lower to compensate for that.

Summary

- Chooser option. Semi-static hedging using a call and a put.
- Spread option. Closed form solution when $K = 0$. Numerical Integration for $K \neq 0$.
- Terminal correlation vs instantaneous correlation.
- Forward staring option. No Gamma and the dependency of time dependent volatility.
- FX rates drift in domestic and foreign risk neutral measure.
- Quanto option. How to hedge a quanto payoff and the intuition of the quanto adjustment.