QF 604
ECONOMETRICS OF FINANCIAL MARKETS

LECTURE 6

Professor Lim Kian Guan

SMU Master of Science in
Quantitative Finance 2023-2024 T2

LECTURE OUTLINE

01 METHOD OF MOMENTS

02 CONSUMPTION-BASED ASSEST PRICING

03 GENERALIZED METHOD OF MOMENTS

04 TESTING EULER EQUATION

05 INTEREST RATE MODELS

01 Method of Moments

Suppose: X_t is stationary ergodic, i.e., sample moments involving X_t converge over time

$g(X_t, \theta)$ is a continuous bounded function in X_t and θ.

$\mathrm{E}[g(X_t, \theta)] = 0$ where θ is unique.

A Taylor series expansion of $g(X_t, \theta)$ about an estimate $\hat{\theta}$ gives:

$$g(X_t, \theta) = g(X_t, \hat{\theta}) + (\theta - \hat{\theta})g'(X_t, \hat{\theta}) + \frac{(\theta - \hat{\theta})^2}{2!} g''(X_t, \hat{\theta}) + \cdots$$

Taking the sample average,

$$\frac{1}{T}\sum_{t=1}^{T} g(X_t, \theta) = \frac{1}{T}\sum_{t=1}^{T} g(X_t, \hat{\theta}) + (\theta - \hat{\theta})\frac{1}{T}\sum_{t=1}^{T} g'(X_t, \hat{\theta}) + \frac{(\theta-\hat{\theta})^2}{2!}\frac{1}{T}\sum_{t=1}^{T} g''(X_t, \hat{\theta}) + \cdots$$

As $T \uparrow \infty$, LHS = 0. Hence RHS = 0. The only way RHS = 0 is to put $\frac{1}{T}\sum_{t=1}^{T} g(X_t, \hat{\theta}) = 0$ and for $\hat{\theta} \xrightarrow{p} \theta$. This $\hat{\theta}$ is a method of moments estimator of θ.

01 Method of Moments – Examples

- Suppose the demand for goods at time t is y_t. Demand quantity is given by:

$$y_t = d_0 + d_1 P_t + \varepsilon_t$$

where P_t is price of the goods, ε_t is a disturbance term with zero mean and a constant variance. Typically, $d_0 > 0$ and $d_1 < 0$.

- The supply quantity is given by

$$y_t = s_0 + s_1 P_t + \eta_t$$

where η_t is a disturbance term with zero mean and a constant variance. $Cov(\eta_t, \varepsilon_t)$ needs not be zero. Typically, $s_0 > 0$ and $s_1 > 0$.

- Solve the equilibrium price from the demand and supply equations to obtain

$$P_t = \frac{s_0 - d_0}{d_1 - s_1} + \frac{\eta_t - \varepsilon_t}{d_1 - s_1}$$

- $Cov(P_t, \eta_t) = (d_1 - s_1)^{-1}(Var(\eta_t) - Cov(\eta_t, \varepsilon_t)) \neq 0$

The non-zero correlation implies that price is an endogenous variable (as it also impacts on the disturbances) - called endogeneity bias.

Black dots: the equilibrium price and quantity at each t are observed price and quantity.

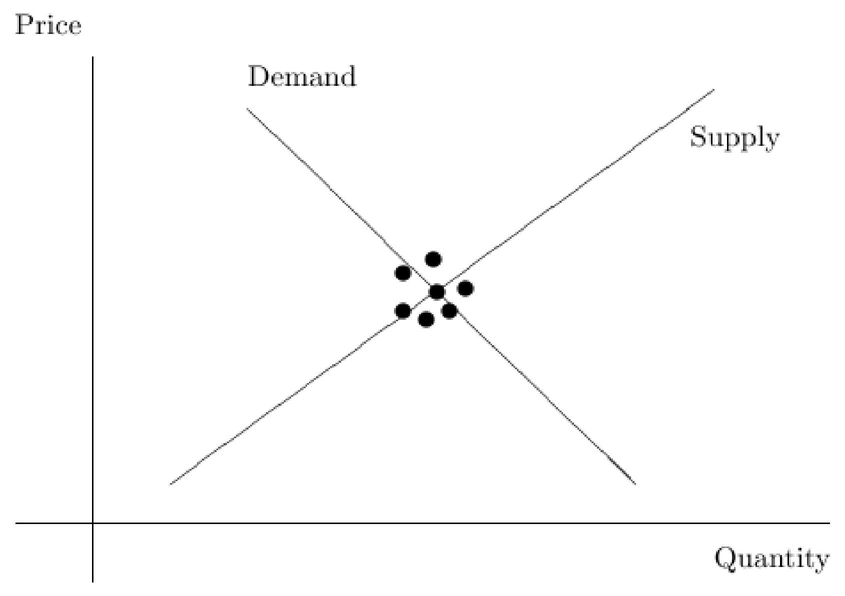

Figure 8.1: Equilibrium Price and Quantity with Stochastic Demand and Supply

01 Method of Moments – Examples

- Suppose the demand for goods at time t is given by:

$$y_t = d_0 + d_1 P_t + d_2 Z_t + \varepsilon_t$$

Z_t: an exogenous variable to both demand and supply.
The supply equation remains the same.
$Cov(Z_t, \varepsilon_t) = 0$ and $Cov(Z_t, \eta_t) = 0$.

- Due to additional effect of random Z_t, these black dots now scatter along the supply curves (shifted by random η_t, given ε_t).
- The estimation of the supply curve

$$y_t = s_0 + s_1 P_t + \eta_t$$

is as follows. The exogeneity of Z_t gives

$$E(Z_t(y_t - s_0 - s_1 P_t)) = 0$$

And $\quad E(y_t - s_0 - s_1 P_t) = 0.$

These two moment conditions help to identify s_0 and s_1 for estimation.

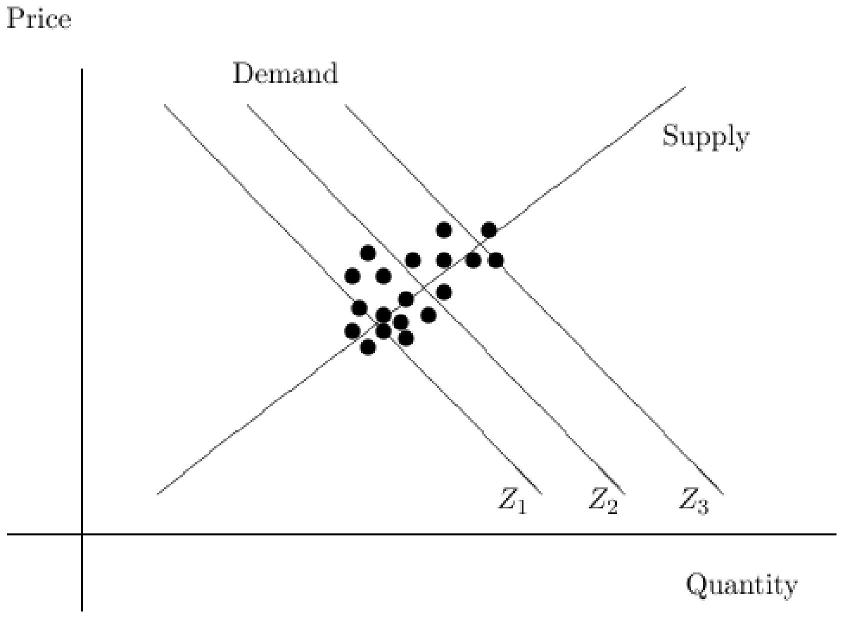

Figure 8.2: Equilibrium Price and Quantity with Stochastic Demand and Supply but only Demand driven by Exogenous Variable X

01 Method of Moments – Examples

- The two associated sample moments of the orthogonality conditions are

$$\frac{1}{N}\sum_{t=1}^{N} Z_t\left(y_t - s_0 - s_1 P_t\right) \quad \text{and} \quad \frac{1}{N}\sum_{t=1}^{N}\left(y_t - s_0 - s_1 P_t\right).$$

- Assuming further that Z_t and η_t are stationary ergodic (over time), then these sample moments converge to zeros at least in probability. The set of two sample moment conditions can also be written in matrix form as:

$$\frac{1}{N}\left(Z^T Y - Z^T X B\right) \xrightarrow{p} 0$$

where $Z_{N\times 2}$ consists of a first column of ones and a second column of $\{Z_t\}$, $X_{N\times 2}$ consists of a first column of ones and a second column of $\{P_t\}$, and $Y_{N\times 1}$ consists of a column of $\{Y_t\}$. $B = (s_0, s_1)^T$.

- By putting $\frac{1}{N}\left(Z^T Y - Z^T X B\right) = 0$, we solve for the method of moments estimator $\hat{B}$.

$$\hat{B} = \left(\frac{1}{N} Z^T X\right)^{-1}\left(\frac{1}{N} Z^T Y\right).$$

This is identical with the instrumental variables (IV) estimator discussed in Chapter 6. It is consistent.

01 Two-Stage Least Sqs(2SLS)

- Another approach in the use of instrument is the **two-stage least-squares (2SLS)** method.

  Suppose the price in the supply curve P_t is explained by an exogenous variable Z_t that has zero correlations with ε_t and η_t. Instead of using Z_t directly as above in the IV regression, we perform a linear regression of P_t on Z_t in the first stage to obtain fitted $\hat{P}_t$.

- In the new set of demand and supply equations,

$$P_t = c_0 + c_1 Z_t + \xi_t$$
$$c_0 = (s_1 - d_1)^{-1}(d_0 - s_0)$$
$$c_1 = (s_1 - d_1)^{-1} d_2$$
$$\xi_t = (s_1 - d_1)^{-1}(\varepsilon_t - \eta_t)$$

- The estimated $\hat{P}_t = \hat{c}_0 + \hat{c}_1 Z_t$

  Therefore, $Cov(\hat{P}_t, \eta_t) = \hat{c}_1 Cov(Z_t, \eta_t) = 0$ for any supply residual noise

  But $Cov(P_t, \hat{P}_t) = Cov(c_1 Z_t + \xi_t, \hat{c}_1 Z_t) \neq 0$

- $\hat{P}_t$ is a reasonable instrument in place of P_t for the second-stage regression in the supply equation:

$$y_t = s_0' + s_1' \hat{P}_t + \eta_t'$$

02 Consumption-Based Asset Pricing Models

- The intertemporal CAPM is based on optimizing lifetime consumption. Particularly, the continuous time modeling using Wiener processes as risk innovations makes only the instantaneous means and variances matter in the pricing.
- Suppose we use another approach in intertemporal asset pricing with discrete time and without specifying the form of the returns distributions except that they have finite moments.
- This latter pricing approach can subsume other approaches using more stringent utility or distributional assumptions.

Advantages:

(1) not having to specify (hence no mis-specification) the cdf of the returns distributions

(2) not having to specify the utility function U

(3) allowing a nonlinear GMM econometric method to simultaneously do estimation and testing of the pricing model.

Disadvantages:

Assuming a single representative agent and asymptotic instead of finite sample statistical results.

02 Consumption-Based Asset Pricing Models

The representative individual's problem is

$$\max_{\{C_t, w_t\}} E_0 \left[\sum_{t=0}^{\infty} \rho^t U(C_t) \right]$$

$$s.t. \quad W_{t+1} = (W_t - C_t) \sum_i w_i R_i \,,$$

$$\sum_i w_i = 1, \forall t$$

where we assume stationary return distribution of all stock returns R_i and W_t is wealth at time t, with initial condition as W_0.

Optimal asset weights $w_t = (w_1, w_2, \dots, w_N)$ where N is the number of securities. Returns are independent of consumption decisions. $\lim_{t \to \infty} U(C_t) < \infty$.

02 Consumption-Based Asset Pricing Models

Solving the representative agent's optimal consumption-investment problem:

$$U'(C_t) = \rho E_t[U'(C_{t+1})R_i] \qquad (8.8)$$

See **Proof** on pages 233 – 235 of book.

By putting $R_i = \frac{P_{t+1}^i}{P_t^i}$ where P_t^i is stock i's price at time t, then $P_t^i\, U'(C_t) = \rho E_t\left[U'(C_{t+1})P_{t+1}^i\right]$. This last equation is called the stochastic Euler equation and is a necessary but not sufficient condition for any solution to a rational asset pricing equilibrium. We may write the stochastic Euler equation as

$$E_t[M_{t+1}R_{t+1}] = 1 \qquad (8.9)$$

where $M_{t+1} = \rho\frac{U'(C_{t+1})}{U'(C_t)}$ is sometimes called the price kernel or stochastic discount factor (as it discounts the future stock price to the current price) and letting R_{t+1} denote the return to any traded security at time $t + 1$.

02 Consumption-Based Asset Pricing Models

Eq. (8.9) $\Rightarrow E(R_{t+1})E(M_{t+1}) + \mathrm{Cov}(M_{t+1}, R_{t+1}) = 1$

As the risk-free asset with return R_f also satisfies Eq. (8.9), then

$R_f E(M_{t+1}) = 1$ or $E(M_{t+1}) = 1/R_f$.

$$E(R_{t+1}) - R_f = -\frac{Cov(M_{t+1}, R_{t+1})}{E(M_{t+1})} \qquad (8.10)$$

Or,

$$E(R_{t+1}) = R_f + \frac{Cov(-M_{t+1}, R_{t+1})}{Var(M_{t+1})} \times \frac{Var(M_{t+1})}{E(M_{t+1})} \qquad (8.11)$$

Define $\beta_C = \frac{Cov(-M_{t+1}, R_{t+1})}{Var(M_{t+1})}$, and $\lambda_M = \frac{Var(M_{t+1})}{E(M_{t+1})}$ in Eq. (8.11), then

β_C: a consumption beta specific to the asset with return R_{t+1}

λ_M: a market premium (> 0) or the price of common risk to all assets

02 Consumption-Based Asset Pricing Models

Eq. (8.10) can be re-written as: $E(R_{t+1}) - R_f = -\rho_{MR}\sigma_M\sigma_R / E(M_{t+1})$

where ρ_{MR} is the correlation coefficient between M_{t+1} and R_{t+1} . σ_M, σ_R denote the respective standard deviations. Then

$$\frac{E(R_{t+1} - R_f)}{\sigma_R} = -\rho_{MR}\frac{\sigma_M}{E(M_{t+1})} \leq \frac{\sigma_M}{E(M_{t+1})} = R_f\sigma_M \tag{8.12}$$

LHS: the Sharpe ratio or performance of any stock (including the market index) showing its equity premium (or excess return over risk-free rate) divided by its return volatility.

RHS is called the Hansen-Jagannathan bound, which provides a theoretical upper bound to the Sharpe ratio or to the equity premium.

03 Generalized Method of Moments

GMM: an econometric method to perform estimation of parameters and to test the plausibility of a model that is based on expectations or moment conditions derived from the theoretical model itself.

Assume:

(1) A stationary stochastic process $\{X_t\}_{t=1,2,\ldots}$ where X_t is vector of random variables at time *t*.

(2) A finite sample from the stochastic vector process as $\{x_1, x_2, \ldots, x_T\}$ of sample size *T*, where each x_t is a vector of realised values at *t* that are observable.

The model is derived based on a set of *K* number of moment conditions:

$$E[f_1(X_t, \theta)] = 0$$

$$E[f_2(X_t, \theta)] = 0$$

$$\vdots$$

$$E[f_K(X_t, \theta)] = 0 \qquad (8.13)$$

where $f_j(.)$ is in general a smooth nonlinear function. θ is an m dimension unique vector of unknown parameters, m < *K*.

03 Generalized Method of Moments

Find $\hat{\theta}$ such that the corresponding sample (empirical) moments are close to zero

$$\frac{1}{T}\sum_{t=1}^{T} f_1\left(x_t, \hat{\theta}\right) \approx 0$$

$$\frac{1}{T}\sum_{t=1}^{T} f_2\left(x_t, \hat{\theta}\right) \approx 0$$

$$\vdots$$

$$\frac{1}{T}\sum_{t=1}^{T} f_K\left(x_t, \hat{\theta}\right) \approx 0$$

The Law of Large Numbers suggests that as $T \uparrow \infty$, these sample moments would converge to $E[f_j(X_t, \hat{\theta})]$

03 Generalized Method of Moments

Let all the observable values of sample size T be $Y_T \equiv \{x_t\}_{t=1,2,\ldots,T}$. Let

$$g(Y_T, \hat{\theta}) \equiv \begin{pmatrix} \frac{1}{T}\sum_{t=1}^{T} f_1(x_t, \hat{\theta}) \\ \frac{1}{T}\sum_{t=1}^{T} f_2(x_t, \hat{\theta}) \\ \vdots \\ \frac{1}{T}\sum_{t=1}^{T} f_K(x_t, \hat{\theta}) \end{pmatrix}$$

be a $K \times 1$ vector of sampling moments. Suppose W_T is a $K \times K$ symmetric positive definite weighting matrix which may be a function of data Y_T. The GMM estimator is found by minimising the scalar function:

$$\min_{\hat{\theta}} \Gamma(\hat{\theta}) \equiv g(Y_T, \hat{\theta})^T W_T(Y_T, \theta) g(Y_T, \hat{\theta}). \tag{8.15}$$

Note that if $W_T(.\,,.)$ is any arbitrary symmetric positive definite matrix, then estimators $\hat{\theta}$ will still be consistent though not asymptotically efficient. However, an optimal weighting matrix function $W_T(.\,,.)$ can be found so that the estimators $\hat{\theta}$ will be asymptotically efficient.

03 Generalized Method of Moments

- Let vector function $F_{K\times 1}(X_t,\theta) = \ (f_1(X_t,\theta), f_2(X_t,\theta), \ldots, f_K(X_t,\theta))^T$.

  Then $g(Y_T,\theta) = \frac{1}{T}\sum_{t=1}^{T} F(x_t,\theta)$.

- Let $\sum_0 = \sum_{j=-N}^{N} E\left[F(X_t,\theta)F\left(X_{t-j},\theta\right)^T\right]$ be a $K\times K$ covariance matrix that is the sum of contemporaneous covariance matrix $E[F(X_t,\theta)F(X_t,\theta)^T]$ and $2N$ number of serial covariance matrices $E\left[F(X_t,\theta)F\left(X_{t-j},\theta\right)^T\right]$.

- Note that the covariance matrix of random vector $g(Y_T,\theta)$

  $= T^{-1}\sum_0$

  for an asymptotically large T.

- By the Central Limit Theorem, as T $\uparrow \infty$, $Tg(Y_T,\theta)^T \sum_0^{-1} g(Y_T,\theta) \rightarrow \chi_k^2$.

03 Generalized Method of Moments

The minimisation of Eq.(8.15) gives vector first-order condition:

$$2\frac{\partial g(Y_T,\hat{\theta})^T}{\partial\hat{\theta}}W_T(Y_T,\theta)g(Y_T,\hat{\theta}) = 0_{m\times 1}$$

conditional on a given weighting matrix $W_T(Y_T,\theta)$. In principle, the above m equations in the vector equation can be solved, i.e.

$$\frac{\partial g(Y_T,\hat{\theta})^T}{\partial\hat{\theta}}W_T(Y_T,\theta)g(Y_T,\hat{\theta}) = 0_{m\times 1}, \tag{8.16}$$

to obtain the m estimates in $m \times 1$ vector $\hat{\theta}$. In this first step, $W_T(Y_T,\theta)$ can be initially selected as $I_{K\times K}$ The solution $\hat{\theta}_1$ is consistent but not efficient. The consistent estimates $\hat{\theta}_1$ in this first step are then employed to find the optimal weighting matrix $W_T^*(Y_T,\hat{\theta}_1)$.

03 Generalized Method of Moments

- Let $g'(Y_T, \hat{\theta}_1) = \frac{1}{T}\sum_{t=1}^{T} \frac{\partial F(x_t, \hat{\theta}_1)}{\partial \theta}$. Let a consistent estimator of Σ_0 be, for T much larger than N:

$$\begin{aligned}\hat{\Sigma}_0(\hat{\theta}_1) &= \frac{1}{T}\sum_{t=1}^{T} F(x_t, \hat{\theta}_1) F(x_t, \hat{\theta}_1)^T \\ &+ \sum_{j=1}^{N}\left(\frac{1}{T}\sum_{t=j+1}^{T} F(x_t, \hat{\theta}_1) F(x_{t-j}, \hat{\theta}_1)^T\right) \\ &+ \sum_{j=1}^{N}\left(\frac{1}{T}\sum_{t=1}^{T-j} F(x_t, \hat{\theta}_1) F(x_{t+j}, \hat{\theta}_1)^T\right)^T .\end{aligned} \tag{8.17}$$

- Then employ $W_T^* = \hat{\Sigma}_0(\hat{\theta}_1)^{-1}$ as the optimal weighting matrix in Eq. (8.15) and minimize the function again in the second step to obtain the efficient and consistent GMM estimator $\hat{\theta}^*$.

- The minimized function in Eq. (8.15) is now

$$\Gamma_T(\hat{\theta}^*) \equiv g(Y_T, \hat{\theta}^*)^T \hat{\Sigma}_0(\hat{\theta}_1)^{-1} g(Y_T, \hat{\theta}^*),$$

where the first-order condition is satisfied: $\frac{\partial g(Y_T, \hat{\theta}^*)^T}{\partial \hat{\theta}} \hat{\Sigma}_0(\hat{\theta}_1)^{-1} g(Y_T, \hat{\theta}^*) = 0_{m\times 1}$.

03 Generalized Method of Moments

- As T $\uparrow \infty$, $\hat{\theta}_1, \hat{\theta}^* \rightarrow \theta$, $\hat{\sum}_0(\hat{\theta}_1) \rightarrow \sum_0$, and

$$T\,\Gamma_T(\hat{\theta}^*) \equiv T g\left(Y_T, \hat{\theta}^*\right)^T \hat{\sum}_0\left(\hat{\theta}_1\right)^{-1} g\left(Y_T, \hat{\theta}^*\right) \rightarrow \chi^2_{k-m} \qquad (8.19)$$

For the above equation:

LHS: called the *J*-statistic, $J_T = T\,\Gamma_T$.

This is the **GMM test statistic** and is asymptotically chi-square of *K* - *m* degrees of freedom and not *K* degrees of freedom (when the population parameter θ is in the arguments instead). Number of overidentifying restrictions is *K* - *m*.

- Eq. (8.19) is the test statistic measuring the sampling moment deviations from the means imposed by the theoretical restrictions in Eq. (8.13). If the test statistic is too large and exceeds the critical boundaries of the chi-square random variable, then the moment conditions of Eq. (8.13) and thus the theoretical restrictions would be rejected.

03 Generalized Method of Moments

- In estimating $\sum_0$ in Eq. (8.17), the estimator $\hat{\sum}_0\,(\hat{\theta}_1)$is consistent.

  However, in finite sample, this computed matrix may not be positive semi-definite.

  This may result in Eq. (8.19) being negative.

- Newey-West HACC (heteroskedastic and autocorrelation consistent covariance) matrix estimator provides for a positive semi-definite covariance estimator of $\sum_0$ that can be used. This takes the form:

$$\hat{\sum}_0(\hat{\theta}_1) = \frac{1}{T}\sum_{t=1}^{T} F(X_t, \hat{\theta}_1)F(X_t, \hat{\theta}_1)^T + \sum_{j=1}^{N}\left(1 - \frac{j}{N+1}\right)\left(\hat{\Omega}_j + \hat{\Omega}_j^T\right)$$

where $\hat{\Omega}_j = \left(\frac{1}{T}\sum_{t=j+1}^{T} F(X_t, \hat{\theta}_1)F(X_{t-j}, \hat{\theta}_1)^T\right)$

03 Generalized Method of Moments

- Besides obtaining the efficient GMM estimators $\hat{\theta}^*$ and the asymptotic test statistic, the GMM method also allows for testing if the GMM estimates are significantly different from θ.

  Asymptotically, for large T

$$\sqrt{T}\left(\hat{\theta}^* - \theta\right) \rightarrow N(0, V) \qquad (8.20)$$

  where $V = \left[\frac{\partial_g g\left(Y_T, \hat{\theta}^*\right)^T}{\partial \hat{\theta}} \widehat{\Sigma}_0^{-1} \frac{\partial_g\left(\hat{\theta}^*\right)}{\partial \theta^T}\right]^{-1}$. So, $\hat{\theta}^* \sim N(\theta, V/T)$

  See **Proofs** of asymptotic test statistic and estimator distribution on pages 239 – 241.

- For finding enough moment conditions (restrictions) in Eq. (8.19), if the theoretical model prescribes $E_{t-1}[f_j(X_t, \theta)] = 0$, additional moment restrictions can be imposed using the iterated expectation theorem and market rational expectations (where feasible). If X_{t-1} is observed at *t*-1, we can use it as an instrument to form $E_{t-1}[f_j(X_t, \theta)\, X_{t-1}] = 0$.
- Moment restrictions are also called orthogonality conditions because the instruments are orthogonal to the original argument.

04 Testing the Euler Equation

- Hansen and Singleton tested the Euler condition: $E_t\left[\rho\, R_{t+1}\frac{U'(C_{t+1})}{U'(C_t)}\right] = 1$, where $R_{t+1} = P_{t+1}/P_t$ is the return over [t, t+1].
- Assume the utility function: $U(C_t) = C_t^{\gamma}/\gamma$ for $\gamma < 1$. Relative risk aversion parameter : $-\,C_t \times U''(C_t)\;/\;U'(C_t) = 1 - \gamma > 0$
- Euler equation is: $E_t\left[\rho\, R_{t+1} Q_{t+1}^{\gamma-1}\right] = 1$ (8.21)

  where $Q_{t+1} \equiv C_{t+1}/C_t$ is the per capita consumption ratio, as $U'(C_t) = C_t^{\gamma-1}$.
- Employ the lagged values of R_t's or Q_t's as instruments, then form three moment restrictions:
  (1) $\mathrm{E}\left[\rho\, R_{t+1} Q_{t+1}^{\delta} - 1\right] = 0$
  (2) $\mathrm{E}\left[\rho\, R_{t+1} R_t Q_{t+1}^{\delta} - R_t\right] = 0$
  (3) $\mathrm{E}\left[\rho\, R_{t+1} R_{t-1} Q_{t+1}^{\delta} - R_{t-1}\right] = 0$ where $\delta = \gamma - 1$

04 Testing the Euler Equation

Table 8.1: GMM Test of the Euler Equation under Constant Relative Risk Aversion. 37 quarterly observations are used. 2000 - 2009. Standard errors and covariance are estimated using Newey-West HACC. Convergence is obtained after 13 iterations. Instruments are one-period are two-period lagged market returns.

	Coefficient	Std. Error	t-Statistic	Prob.
$\hat{\rho}$	0.9980	0.0236	42.30	0.0000
$\hat{\delta}$	-1.3615	0.5149	-2.64	0.0122
Mean dependent var	0.000000	S.D. dependent var		0.000000
S.E. of regression	0.091516	Sum squared resid		0.293129
Durbin-Watson stat	1.897556	J-statistic		4.115583
Instrument rank	3	Prob(J-statistic)		0.042490

Employs Quarterly US consumption data 2000 – 2009 on a per capita basis

- From Table 8.1: time discount factor estimate $\hat{\rho}$ is 0.998.

  Relative risk aversion coefficient estimate is $1 - \gamma = -\delta = 1.3615$.
- Both are significantly different from zero at p-values of less than 2%. The J-statistic $\sim \chi^2$ with one degree of freedom is 4.12 with a p-value of 0.0425.
- Therefore, the moment restrictions (1), (2), and (3) implied by the model and rational expectations are not rejected at the 1% significance level, though it would be rejected at the 5% level.

04 Testing the Euler Equation

- Another test of Eq. (8.21): $E_t\left[\rho R_{t+1}\left(\frac{C_{t+1}}{C_t}\right)^{\gamma-1}\right]=1$ is based on Brown and Gibbons(1985). The Brown and Gibbons study employed an assumption that aggregate consumption each period is a constant fraction of aggregate wealth, i.e., $C_t = kW_t$ where W_t is aggregate wealth of all individuals in the market.

$$C_t = kW_t = k[(W_{t-1} - kW_{t-1})(1 + r_{mt})] = (1-k)C_{t-1}(1 + r_{mt})$$

where r_{mt} is the market portfolio return rate over [*t*-1, *t*].

- Express Eq. (8.21) as

$$E_{t-1}[\rho(1 + r_t)(1-k)^{\gamma-1}(1 + r_{mt})^{\gamma-1}] = 1 \tag{8.22}$$

where r_t is the return rate of any stock or portfolio (investable asset) over [*t*-1, *t*].

- Derive from Eq. (8.22):

$$E_{t-1}[(1 + r_{mt})^{\gamma}] = \rho^{-1}(1-k)^{1-\gamma}$$

and

$$E_{t-1}\left[(1 + r_{mt})^{\gamma-1}(1 + r_{ft})\right] = \rho^{-1}(1-k)^{1-\gamma}$$

04 Testing the Euler Equation

- Since a one-period treasury bill with maturity at t has a market price at $t-1$, the risk-free rate r_{ft} over $[t-1, t]$ is known at $t-1$, i.e. r_{ft}

  Unlike other risky asset returns, r_t is in the information set at $t-1$.

  Dividing $E_{t-1}[(1+r_{mt})^{\gamma}] = E_{t-1}\left[(1+r_{mt})^{\gamma-1}(1+r_{ft})\right]$ by $(1+r_{ft})^{\gamma}$:

$$E_{t-1}\left[(X_t - 1)X_t^{\gamma-1}\right] = 0 \qquad (8.23)$$

  where $X_t = (1+r_{mt})/(1+r_{ft})$, assuming X_t is stationary ergodic. Ergodic means that its sample moments will converge to population moments when the series is long enough.

- The sample moment corresponding to Eq. (8.23) is

$$\frac{1}{T}\sum_{t=1}^{T}(X_t - 1)X_t^{\gamma-1} = 0$$

  for asymptotically large sample size T, and relative risk aversion parameter $1-\gamma > 0$.

05 Test of Interest Rate Models

The short rate is the spot interest rate when the term (tenure) goes to zero.

Let the short rate be r. This is the instantaneous spot rate.

An example of a diffusion process of the short rate is:

$$dr_t = (\alpha - \beta r_t)dt + \sigma r_t^{\lambda} dW_t^P \tag{11.5}$$

where, α, β, and λ are constants.

In the literature on interest rate modelling, many different models were applied to study interest rate dynamics and the associated bond prices.

Many of the models are subsumed under the class represented by Eq. (11.5)

(1) Dothan model: $\alpha = \beta = 0$ and $\lambda = 1$
(2) Brennan-Schwarz model: just $\lambda = 1$
(3) Vasicek model: just $\lambda = 0$
(4) Cox-Ross-Ingersoll model: when $\lambda = 1/2$

05 Test of Interest Rate Models

- Some solutions of the short-rate model can be intractable. If the time-interval between each short-rate observation (e.g., a day) is small, a discretized approximation can be used.
- For Eq. (11.5): $\Delta r_t = (\alpha - \beta r_{t-1})\, h + e_t$

  where $h \equiv 1/365$, $\Delta r_t \equiv r_t - r_{t-1}$, $e_t = \sigma r_{t-1}^{\lambda}\, \Delta W_t^P$, $\Delta W_t^P \sim N(0, h)$ under physical/empirical measure.

  $\alpha, \beta, \sigma, \lambda$ are constants to be estimated. The discretized short-rate model can be simplified as:

  $$\Delta r_t = b_0 + b_1\, r_{t-1} + e_t$$

  where $\alpha = b_0/h$, $\beta = -b_1/h$. Let $b_2 = \sigma^2$ and $b_3 = 2\lambda$. r_t is a risk-free spot rate over short horizon h.
- The parameter estimation can be performed using 4 conditional moment conditions.

  (1) $E_{t-1}(\text{error1}) \equiv E_{t-1}(\Delta r_t - b_0 - b_1\, r_{t-1}) = 0$

  (2) $E_{t-1}(\text{error2}) \equiv E_{t-1}(\, [\Delta r_t - b_0 - b_1\, r_{t-1}]^2 - b_2 h\, r_{t-1}^{b3}\,) = 0$

  (3) $E_{t-1}(\text{error3}) \equiv E_{t-1}(\, [\Delta r_t - b_0 - b_1\, r_{t-1}]\, r_{t-1}\,) = 0$

  (4) $E_{t-1}(\text{error4}) \equiv E_{t-1}(\, [[\Delta r_t - b_0 - b_1\, r_{t-1}]^2 - b_2 h\, r_{t-1}^{b3}]\, r_{t-1}) = 0$

  For over-identifying restriction, we can add: (5) $E_{t-1}(\, [\Delta r_t - b_0 - b_1\, r_{t-1}]\, r_{t-1}^2\,) = 0$

05 Test of Interest Rate Models

Taking iterated expectations, the conditional moment conditions lead to unconditional moment conditions allowing sampling mean to represent them.

(1) $E(\text{error1}) \equiv E(\Delta r_t - b_0 - b_1 r_{t-1}) = 0$

(2) $E(\text{error2}) \equiv E([\Delta r_t - b_0 - b_1 r_{t-1}]^2 - b_2 h\, r_{t-1}{}^{b3}) = 0$

(3) $E(\text{error3}) \equiv E([\Delta r_t - b_0 - b_1 r_{t-1}]\, r_{t-1}) = 0$

(4) $E(\text{error4}) \equiv E([[\Delta r_t - b_0 - b_1 r_{t-1}]^2 - b_2 h\, r_{t-1}{}^{b3}]\, r_{t-1}) = 0$

(5) $E(\text{error5}) \equiv E([\Delta r_t - b_0 - b_1 r_{t-1}]\, r_{t-1}{}^2) = 0$

The Generalized Method of Moments (assuming reasonably large sample size, but not concerned about endogeneity or autocorrelation or other heteroskedasticity issues) can then be applied using (1) – (4) for estimation, and (1) – (5) for estimation and testing with test-statistic χ_1^2.

Practice Exercise (not graded)

+Interest_Rate_Model.ipynb

Interest_Rate.csv