QF605 Supplementary Note
Derivation of CMS Replication Formula

In this note, we show how to arrive at the CMS replication formula based on the risk-neutral density of the forward swap rate. Recall that the second-order partial derivatives of the payer and receiver IRR-settled swaptions yield the expression for the risk-neutral density:

$$f(K) = \begin{cases} \frac{1}{D(0,T)}\frac{1}{\text{IRR}(K)} \times \frac{\partial^2 V^{\text{pay}}(K)}{\partial K^2} & \text{when} \quad K > S_{n,N}(0), \\ \frac{1}{D(0,T)}\frac{1}{\text{IRR}(K)} \times \frac{\partial^2 V^{\text{rec}}(K)}{\partial K^2} & \text{when} \quad K < S_{n,N}(0). \end{cases}$$

Suppose we wish to pay a generic function g of the forward swap rate S, i.e. $g(S)$. Based on the static replication approach, let $F = S_{n,N}(0)$ be the expansion point, and $h(K) = \frac{g(K)}{\text{IRR}(K)}$, the value of this contract can be written as:

$$\begin{aligned}
V_0 &= D(0,T)\mathbb{E}[g(S)] \\
&= D(0,T)\int_0^\infty g(K)f(K)dK \\
&= D(0,T)\int_0^\infty g(K)\frac{1}{D(0,T)}\frac{1}{\text{IRR}(K)} \times \frac{\partial^2 V(K)}{\partial K^2}dK \\
&= \int_0^F h(K)\frac{\partial^2 V^{\text{rec}}(K)}{\partial K^2}dK + \int_F^\infty h(K)\frac{\partial^2 V^{\text{pay}}(K)}{\partial K^2}dK \\
&= \left[h(K)\frac{\partial V^{\text{rec}}(K)}{\partial K}\right]_0^F - \int_0^F h'(K)\frac{\partial V^{\text{rec}}(K)}{\partial K}dK \\
&\quad + \left[h(K)\frac{\partial V^{\text{pay}}(K)}{\partial K}\right]_F^\infty - \int_F^\infty h'(K)\frac{\partial V^{\text{pay}}(K)}{\partial K}dK \\
&= h(F)\frac{\partial V^{\text{rec}}(F)}{\partial K} - \cancelto{0}{h(0)\frac{\partial V^{\text{rec}}(0)}{\partial K}} - [h'(K)V^{\text{rec}}(K)]_0^F + \int_0^F h''(K)V^{\text{rec}}(K)dK \\
&\quad + \cancelto{0}{h(\infty)\frac{\partial V^{\text{pay}}(\infty)}{\partial K}} - h(F)\frac{\partial V^{\text{pay}}(F)}{\partial K} - [h'(K)V^{\text{pay}}(K)]_F^\infty + \int_F^\infty h''(K)V^{\text{pay}}(K)dK \\
&= h'(F)\frac{\partial V^{\text{rec}}(F)}{\partial K} - h'(F)V^{\text{rec}}(F) + \cancelto{0}{h'(0)V^{\text{rec}}(0)} + \int_0^F h''(K)V^{\text{rec}}(K)dK \\
&\quad - h(F)\frac{\partial V^{\text{pay}}(F)}{\partial K} - \cancelto{0}{h'(\infty)V^{\text{pay}}(\infty)} + h'(F)V^{\text{pay}}(F) + \int_F^\infty h''(K)V^{\text{pay}}(K)dK \\
&= -h(F)\left[\frac{\partial V^{\text{pay}}(F)}{\partial K} - \frac{\partial V^{\text{rec}}(F)}{\partial K}\right] + h'(F)[V^{\text{pay}}(F) - V^{\text{rec}}(F)] \\
&\quad + \int_0^F h''(K)V^{\text{rec}}(K)dK + \int_F^\infty h''(K)V^{\text{pay}}(K)dK
\end{aligned}$$

The put-call parity relationship for IRR-settled swaptions is given by

$$\begin{aligned} V^{\text{pay}}(K) - V^{\text{rec}}(K) &= D(0,T)\mathbb{E}[\text{IRR}(S)(S-K)^+] - D(0,T)\mathbb{E}[\text{IRR}(S)(K-S)^+] \\ &= D(0,T)\text{IRR}(S)(S-K). \end{aligned}$$

When $K = F = S_{n,N}$, the ATM payer and receiver swaptions are worth the same amount, i.e. $V^{\text{pay}}(F) - V^{\text{rec}}(F) = 0$. Also, the first order derivative of the put-call parity relationship with respect to strike (K) yields:

$$\frac{\partial V^{\text{pay}}(K)}{\partial K} - \frac{\partial V^{\text{rec}}(K)}{\partial K} = -D(0,T)\text{IRR}(S)$$

Substituting this back into the derivation on the previous page, we obtain

$$\begin{aligned} V_0 = &-h(F)\left[\frac{\partial V^{\text{pay}}(F)}{\partial K} - \frac{\partial V^{\text{rec}}(F)}{\partial K}\right] + h'(F)[V^{\text{pay}}(F) - V^{\text{rec}}(F)] \\ &+ \int_0^F h''(K)V^{\text{rec}}(K)dK + \int_F^\infty h''(K)V^{\text{pay}}(K)dK \\ = &\ D(0,T)h(F)\text{IRR(F)} + h'(F)[V^{\text{pay}}(F) - V^{\text{rec}}(F)] + \int_0^F h''(K)V^{\text{rec}}(K)dK + \int_F^\infty h''(K)V^{\text{pay}}(K)dK \\ = &\ D(0,T)g(F) + h'(F)[V^{\text{pay}}(F) - V^{\text{rec}}(F)] + \int_0^F h''(K)V^{\text{rec}}(K)dK + \int_F^\infty h''(K)V^{\text{pay}}(K)dK. \end{aligned}$$

This is the static-replication formula in the course material.

For example, for CMS rate, the payoff is $g(F) = F$, and recognizing that $V^{\text{pay}}(F) - V^{\text{rec}}(F) = 0$, we have the following CMS replication formula:

$$V_0 = D(0,T)F + \int_0^F h''(K)V^{\text{rec}}(K)dK + \int_F^\infty h''(K)V^{\text{pay}}(K)dK.$$