SINGAPORE MANAGEMENT UNIVERSITY

Session 6
Change of Numeraire Theorem
& Convexity Correction

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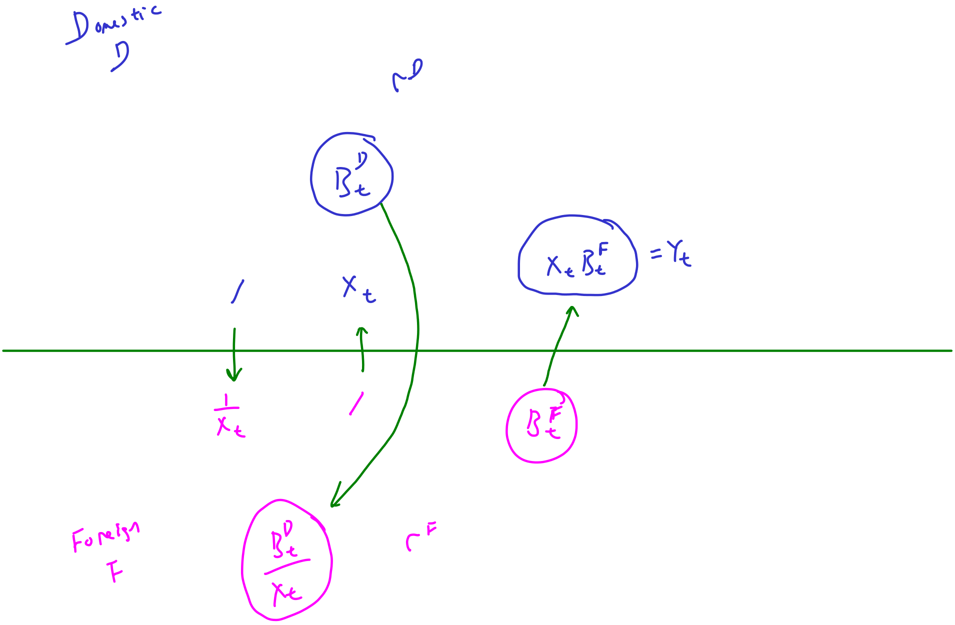
QF605 Fixed Income Securities

FX Rate Process

- While the previous example of using a stock process as a numeraire appears rather stylised, it is no longer so when we look at the foreign exchange (FX) market.
- FX products can be calculated based on either the **domestic** or the **foreign** currency denomination.
- In an arbitrage-free market, both values must be the same.
- Consider the following international economy: in the domestic market D there is a money-market account B_t^D, earning a risk-free interest rate of r^D; in the foreign market F there is the corresponding money-market account B_t^F and interest rate r^F.

Let the FX rate follows a **geometric Brownian motion**, then the three price processes can be written as

$$\begin{aligned} dB_t^F &= r^F B_t^F dt \\ dX_t &= \mu X_t dt + \sigma X_t dW_t \\ dB_t^D &= r^D B_t^D dt. \end{aligned}$$


Domestic
D
r^D
B_t^D
1
X_t
$X_t B_t^F = Y_t$
$\frac{1}{X_t}$
1
B_t^F
Foreign
F
$\frac{B_t^D}{X_t}$
r^F

$$dB_t^F = r^F B_t^F \, dt$$

$$\mathbb{P}: \quad dX_t = \mu X_t \, dt + \sigma X_t \, dW_t$$

$$dB_t^D = r^D B_t^D \, dt$$

$$f(b, x) = bx$$

$$f_b = x$$

$$f_x = b \, , \quad f_{xx} = 0$$

Let $Y_t = B_t^F X_t = f(B_t^F, X_t)$

Itô's:

$$dY_t = f_b(B_t^F, X_t) \cdot dB_t^F + f_x(B_t^F, X_t) \, dX_t + \frac{1}{2} f_{xx}(B_t^F, X_t)(dX_t)^2$$

$$= X_t \, r^F B_t^F \, dt + B_t^F \left(\mu X_t \, dt + \sigma X_t \, dW_t \right) + 0$$

$$\mathbb{P}: \quad dY_t = (r^F + \mu) Y_t \, dt + \sigma Y_t \, dW_t$$

$$Z_t = \frac{Y_t}{B_t^D} = g(B_t^D, Y_t)$$

$$g(b,y) = \frac{y}{b}$$

$$g_b = -\frac{y}{b^2}$$

$$g_y = \frac{1}{b}, \quad g_{yy} = 0$$

Itô's:

$$dZ_t = g_b(B_t^D, Y_t)\, dB_t^D + g_y(B_t^D, Y_t)\, dY_t + \frac{1}{2} g_{yy}(B_t^D, Y_t)\,(dY_t)^2$$

$$= -\frac{Y_t}{(B_t^D)^2} \cdot r^D B_t^D\, dt + \frac{1}{B_t^D} \cdot \left[(r^F + \mu) Y_t\, dt + \sigma Y_t\, dW_t \right] + 0$$

$$= -r^D Z_t\, dt + (r^F + \mu) Z_t\, dt + \sigma Z_t\, dW_t$$

$$dZ_t = (r^F - r^D + \mu) Z_t\, dt + \sigma Z_t\, dW_t$$

$\mathbb{P}$:

$$dZ_t = \sigma Z_t \left(dW_t + \frac{r^F - r^D + \mu}{\sigma} dt \right)$$

$\mathbb{Q}^D$

$$dZ_t = \sigma Z_t \, dW_t^D$$

$$dW_t^D = dW_t + \frac{r^F - r^D + \mu}{\sigma} dt$$

$\mathbb{P}$:

$$dX_t = \mu X_t \, dt + \sigma X_t \, dW_t$$

$\mathbb{Q}^D$:

$$dX_t = \mu X_t \, dt + \sigma X_t \left(dW_t^D - \frac{r^F - r^D + \mu}{\sigma} dt \right)$$

$$dX_t = (r^D - r^F) X_t \, dt + \sigma X_t \, dW_t^D$$

$$X_T = X_0 e^{(r^D - r^F - \frac{\sigma^2}{2})T + \sigma W_T^D}$$

$$\mathbb{E}^{Q^D}[X_T] = X_0 e^{(r^D - r^F - \frac{\sigma^2}{2})T} \mathbb{E}^{Q^D}\left[e^{\sigma W_T^D}\right]$$

$$= X_0 e^{(r^D - r^F - \frac{\sigma^2}{2})T} \cdot e^{\frac{\sigma^2 T}{2}}$$

$$= X_0 e^{(r^D - r^F)T} = X_0 \cdot \frac{e^{r^D T}}{e^{r^F T}}$$

$$FX_T = FX_0 \cdot \frac{(1+r^D)^T}{(1+r^F)^T}$$

FX Rate Process

From a domestic market point of view, there are two **marketed assets**: the domestic money-market account B_t^D and the value of the foreign money-market account in domestic terms, given by $B_t^F X_t$. Applying **Itô's formula** to the function $Y_t = f(B_t^F, X_t) = B_t^F X_t$, we obtain

$$dY_t = (r^F + \mu)Y_t dt + \sigma Y_t dW_t.$$

In the domestic market, we can use the domestic money-market account as a numeraire. Defining the process $Z_t = g(B_t^D, Y_t) = \frac{Y_t}{B_t^D} = \frac{B_t^F X_t}{B_t^D}$, we again apply Itô's formula to obtain

$$dZ_t = (r^F - r^D + \mu)Z_t dt + \sigma Z_t dW_t.$$

Applying Girsanov's theorem with $\kappa = \frac{r^F - r^D + \mu}{\sigma}$ to obtain the equivalent martingale measure $\mathbb{Q}^D$, under which the process $Z_t = \frac{B_t^F X_t}{B_t^D}$ is a martingale. We see that under this $\mathbb{Q}^D$ measure, the exchange rate process follows

$$dX_t = (r^D - r^F)X_t dt + \sigma X_t dW_t^D.$$

FX Rate Process

Taking the perspective of the **foreign market**, we have two marketed assets: B_t^F and $\frac{B_t^D}{X_t}$. Using B_t^F as a numeraire, we can derive the process $\frac{B_t^D}{X_t B_t^F}$ as follow

$$d\left(\frac{B_t^D}{X_t B_t^F}\right) = (r^D - r^F - \mu + \sigma^2)\left(\frac{B_t^D}{X_t B_t^F}\right)dt - \sigma\left(\frac{B_t^D}{X_t B_t^F}\right)dW_t.$$

Applying Girsanov's theorem with $\kappa = -\frac{r^D - r^F - \mu}{\sigma} - \sigma$, we obtain the unique equivalent martingale measure $\mathbb{Q}^F$. Under this measure, the exchange rate $\frac{1}{X_t}$ follows the process

$$d\frac{1}{X_t} = (r^F - r^D)\frac{1}{X_t}dt - \sigma\frac{1}{X_t}dW_t^F.$$

In the foreign economy, the trader will see exactly the same price for an FX option.

Single-currency Change of Numeraire Theorem

CMS Quanto

Products requiring **convexity correction** are characterised by the fact that the underlying assets are paid at the wrong time or in the wrong denomination.

In this case, the forwards need to be adjusted to reflect the fact that they are paid incorrectly. This adjustment is known in the market as convexity correction.

We are already familiar with the **single-currency change of numeraire theorem**, which states that in an arbitrage-free economy, we have

$$\mathbb{E}^N[H_T] = \mathbb{E}^M\left[H_T \frac{N_T/N_0}{M_T/M_0}\right],$$

where

$$\frac{d\mathbb{Q}^N}{d\mathbb{Q}^M} = \frac{N_T/N_0}{M_T/M_0}.$$

Many of the products we are interested in will be multi-currency products. We can extend the single-currency theorem to include multi-currency economies.

$\mathbb{Q}^*$: $$\frac{V_0}{B_0} = \mathbb{E}^*\left[\frac{V_T}{B_T}\right] \Rightarrow V_0 = B_0\, \mathbb{E}^*\left[\frac{V_T}{B_T}\right]$$

unique

$\mathbb{Q}^T$: $$V_0 = D(0,T)\, \mathbb{E}^T\left[\frac{V_T}{D(T,T)}\right]$$

$$D(0,T)\, \mathbb{E}^T\left[\frac{V_T}{\cancel{D(T,T)}}\right] = B_0\, \mathbb{E}^*\left[\frac{V_T}{B_T}\right]$$

$$\mathbb{E}^T\left[V_T\right] = \mathbb{E}^*\left[V_T \cdot \frac{\overset{D(T,T)}{1}/D(0,T)}{B_T/B_0}\right]$$

$$= \mathbb{E}^*\left[V_T \cdot \frac{d\mathbb{Q}^T}{d\mathbb{Q}^*}\right]$$

Domestic

D

Numeraire → N^D

$\mathbb{Q}^{N,D}$: $\frac{Z_0^D}{N_0^D} = \mathbb{E}^{N,D}\left[\frac{Z_T^D}{N_T^D}\right]$ Z^D

$\mathbb{Q}^{XM,D}$: $\frac{\cancel{X_0} Z_0^F}{\cancel{X_0} M_0^F} = \mathbb{E}^{XM,D}\left[\frac{\cancel{X_T} Z_T^F}{\cancel{X_T} M_T^F}\right]$

$X_t M^F$ $X_t Z_t^F$

X_t /

Foreign

F

/ $\frac{1}{X_t}$

$\mathbb{Q}^{M,F}$: $\frac{Z_0^F}{M_0^F} = \mathbb{E}^{M,F}\left[\frac{Z_T^F}{M_T^F}\right]$

Z^F

$$\boxed{\mathbb{Q}^{MF} \equiv \mathbb{Q}^{XM,D}}$$

Numeraire → M^F

Multi-currency Change of Numeraire Theorem

We can put convexity correction on a firm mathematical basis by showing that it can be interpreted as the side-effect of a change of numeraire.

This means that we will show how convexity correction can be understood mathematically as the expected value of an underlying asset under a different probability measure than its own martingale measure.

Suppose we have a domestic economy D and a foreign economy F, together with the exchange rate X_t that expresses the value at time t of one unit of foreign currency in terms of domestic currency.

This implies that for a numeraire N^D in the domestic economy there exists a unique martingale measure $\mathbb{Q}^{N,D}$ such that all N^D rebased traded assets in the domestic economy become martingales.

Note that we have 2 types of traded assets in the domestic economy:

- the domestic assets Z^D
- the domestic value of the foreign assets which are given by $X \cdot Z^F$

Multi-currency Change of Numeraire Theorem

Let us now consider 2 numeraires, one in the domestic economy N^D and one in the foreign economy M^F.

As the exchange rate is strictly positive, the domestic value of the foreign numeraire in domestic denomination $X \cdot M^F$ is a valid domestic numeraire.

Hence, there exists a unique martingale measure $\mathbb{Q}^{XM,D}$ such that all XM^F rebased traded assets in the domestic economy become martingales.

What is the relation between the probability measure $\mathbb{Q}^{XM,D}$ in the domestic economy and $\mathbb{Q}^{M,F}$ in the foreign economy? All $X \cdot M^F$ rebased traded assets in the domestic economy are martingales under $\mathbb{Q}^{XM,D}$.

This would mean that the domestic value of the foreign traded assets

$$\frac{X \cdot Z^F}{X \cdot M^F} = \frac{Z^F}{M^F}$$

are martingales also. This implies that $\mathbb{Q}^{XM,D}$ and $\mathbb{Q}^{M,F}$ are the same probability measure. So under the measure $\mathbb{Q}^{M,F}$ all $X \cdot M^F$ rebased traded assets are martingales in the domestic economy and all M^F rebased traded assets are martingales in the foreign economy.

Multi-currency Change of Numeraire Theorem

From the domestic economy perspective we have N^D and $X \cdot M^F$ as domestic numeraires. Hence, we can apply the single-currency change of numeraire theorem which yields

$$d\mathbb{Q}^{XM,D} = \frac{d\mathbb{Q}^{N,D}}{d\mathbb{Q}^{M,F}} = \frac{\frac{N_T^D}{N_0^D}}{\frac{X_T M_T^F}{X_0 M_0^F}}$$

From the foreign economy perspective, we have $\frac{1}{X} \cdot N^D$ and M^F as foreign numeraires. Hence, we can apply the single currency change of numeraire theorem to this case as well which yields

$$d\mathbb{Q}^{\frac{N}{X},F} = \frac{d\mathbb{Q}^{N,D}}{d\mathbb{Q}^{M,F}} = \frac{\frac{\frac{1}{X_T} N_T^D}{\frac{1}{X_0} N_0^D}}{\frac{M_T^F}{M_0^F}}.$$

Note that the two expressions are identical with simple rearrangement. This is the **multi-currency change of numeraire theorem**. Note also that the single-currency change of numeraire theorem can be seen as a special case of the multi-currency theorem with $X = 1$ and $F = D$.

Lehman Brother's 100 million dollar formula

$dL_i(t) = \sigma_i \cdot L_i(t)\, dW^{i+1}(t)$

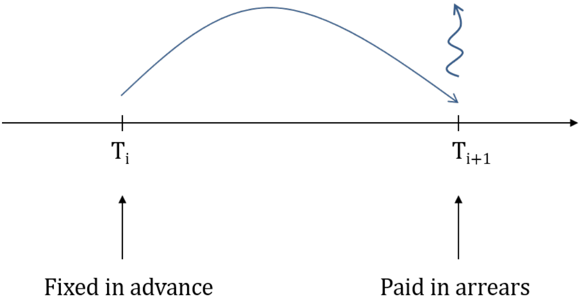
Convexity correction

$$\frac{d\mathbb{Q}^i}{d\mathbb{Q}^{i+1}} = \frac{D_i(T_i)/D_i(0)}{D_{i+1}(T_i)/D_{i+1}(0)}$$

- Discovered the right formula
- Gained long exposures to LIBOR-in-Arrears
- "80 cents on the dollar"

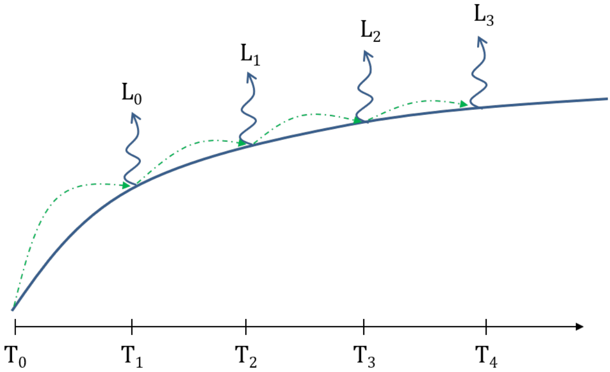
$$\mathbb{E}^i[L_i(T_i)] = \frac{L_i(0) + \Delta_i L_i(0)^2 e^{\sigma^2 T_i}}{1 + \Delta_i L_i(0)}$$

In a standard LIBOR product, the interest rate is fixed in advance, paid in arrears.



The rate you receive at time T_{i+1} is fixed at T_i.

Floating leg of a standard interest rate swaps:



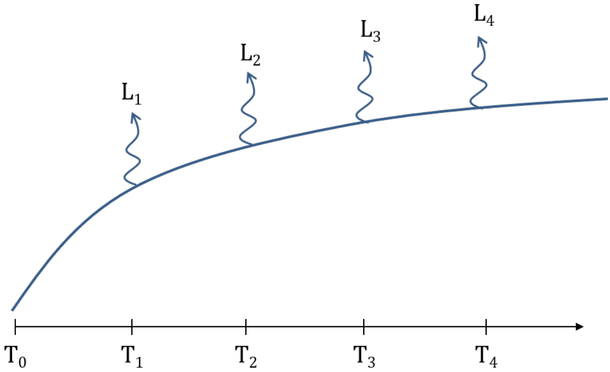
4% 8%

6%

3m rate = 4%

6m rate = 6%

Floating leg of an in-arrears interest rate swaps:



In-arrears Products: Historical Context

In the early 90s (1992-1993), the yield curve was extremely steep, prompting the development of LIBOR-in-arrears products.

⇒ In contrast to standard IR products, the floating leg of in-arrears products is fixed and paid at the same time.

⇒ This is meant to be the rate for the next period for standard products.

If the yield curve is upward sloping, then the expected coupons of the in-arrears floating leg will be higher.

⇒ Long payer swap with in-arrears floating leg to capitalize.

On the other hand, if you think that LIBOR will not rise as fast as the curve predicts, you can long a receiver in-arrears swap to capitalize on your view.

In-arrears Products: Historical Context

issue fixed-coupon bonds &

Generally speaking, corporate clients or financial institutions hold receiver swaps.

If the yield curve is steep, the rate they receive on the fixed leg of the in-arrears product is higher – more attractive.

⇒ Their view is that the in-arrears rates are overestimated, the realized rates will be lower.

Similarly, if the view is that LIBOR will be falling, a receiver swap with an in-arrears floating leg is also more beneficial.

⇒ The rate will be lower at the end of each period than at the beginning.

LIBOR-in-Arrears

In a standard vanilla LIBOR payment, the LIBOR interest rate L_i is observed at time T_i and paid at the end of the accrual period at time T_{i+1}, as $\Delta_i L_i(T_i)$, where Δ_i denotes the daycount or accrual fraction.

The forward LIBOR rate $L_i(t)$ is defined as

$$L_i(t) = \frac{D_i(t) - D_{i+1}(t)}{\Delta_i D_{i+1}(t)},$$

where D_i denotes the discount factor maturing at time T_i.

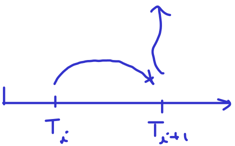
Hence if we choose D_{i+1} as the numeraire then under the associated martingale measure $\mathbb{Q}^{i+1}$, the forward LIBOR rate L_i is a martingale.

In a **LIBOR-in-Arrears** contract, the interest payment at time T_i is based on $L_i(T_i)$. Hence, the LIBOR contract is fixed only at the end of the interest rate period, i.e. fixed in arrears. We want to evaluate

$$V^{\text{LIA}}(0) = D_i(0)\mathbb{E}^i[L_i(T_i)],$$

where L_i is not a martingale under the measure $\mathbb{Q}^i$.

Vanilla Libor



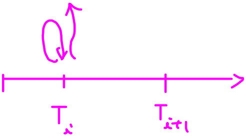
$$dL_i(t) = \sigma_i L_i(t)\, dW^{i+1}(t)$$

$$\frac{V_0}{D(0,T_{i+1})} = \mathbb{E}^{i+1}\left[\frac{V_{T_{i+1}}}{\cancel{D(T_{i+1},T_{i+1})}^{\,1}}\right]$$

$$V_0 = D(0,T_{i+1})\, \mathbb{E}^{i+1}\left[L_i(T_{i+1})\right]$$

$$= D(0,T_{i+1})\, L_i(0)$$

LIA



$$\frac{V_0}{D(0,T_i)} = \mathbb{E}^{i}\left[\frac{V_{T_i}}{\cancel{D(T_i,T_i)}^{\,1}}\right]$$

$$V_0 = D(0,T_i)\, \mathbb{E}^{i}\left[L_i(T_i)\right]$$

Markov approximation : $V_0 \simeq D(0,T_i)\, L_i(0)$

$$\mathbb{E}^{i}\left[L_i(T) \right] = \mathbb{E}^{i+1}\left[L_i(T) \cdot \frac{d\mathbb{Q}^{i}}{d\mathbb{Q}^{i+1}} \right]$$

$$= \mathbb{E}^{i+1}\left[L_i(T) \cdot \frac{D_i(T) / D_i(0)}{D_{i+1}(T) / D_{i+1}(0)} \right]$$

$$= \mathbb{E}^{i+1}\left[L_i(T) \cdot \frac{D_i(T) / D_{i+1}(T)}{D_i(0) / D_{i+1}(0)} \right]$$

$$L_i(t) = \frac{1}{\Delta_i} \cdot \frac{D_i(t) - D_{i+1}(t)}{D_{i+1}(t)}$$

$$1 + \Delta_i L_i(t) = \frac{D_i(t)}{D_{i+1}(t)}$$

$$dL_i(t) = \sigma_i L_i(t)\, dW^{i+1}(t)$$

$$L_i(T) = L_i(0)\, e^{-\frac{\sigma_i^2 T}{2} + \sigma_i W^{i+1}(T)}$$

$$\therefore \quad \mathbb{E}^{i}\left[L_i(T)\right] = \mathbb{E}^{i+1}\left[L_i(T) \cdot \frac{1 + \Delta_i L_i(T)}{1 + \Delta_i L_i(0)}\right]$$

$$= \frac{1}{1 + \Delta_i L_i(0)} \, \mathbb{E}^{i+1}\left[L_i(T) + \Delta_i L_i(T)^2\right]$$

$$= \frac{1}{1 + \Delta_i L_i(0)} \, \mathbb{E}^{i+1}\left[L_i(0) e^{-\frac{\sigma_i^2 T}{2} + \sigma_i W^{i+1}(T)} + \Delta_i L_i(0)^2 e^{-\sigma_i^2 T + 2\sigma_i W^{i+1}(T)}\right]$$

$$= \frac{1}{1 + \Delta_i L_i(0)} \left[L_i(0) \cdot 1 + \Delta_i L_i(0)^2 e^{-\sigma_i^2 T + 2\sigma_i^2 T}\right]$$

$$= \frac{1}{1 + \Delta_i L_i(0)} \left[L_i(0) + \Delta_i L_i(0)^2 e^{\sigma_i^2 T}\right]$$

$$= L_i(0) \cdot \underbrace{\frac{1 + \Delta_i L_i(0) e^{\sigma_i^2 T}}{1 + \Delta_i L_i(0)}}_{> 1}$$

LIBOR-in-Arrears

To calculate the convexity correction, we proceed as follow. We write down the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}^i}{d\mathbb{Q}^{i+1}} = \frac{D_i(T_i)/D_i(0)}{D_{i+1}(T_i)/D_{i+1}(0)}.$$

Using the definition of LIBOR rate, it simplifies to

$$\frac{d\mathbb{Q}^i}{d\mathbb{Q}^{i+1}} = \frac{1+\Delta_i L_i(T_i)}{1+\Delta_i L_i(0)}.$$

Using this expression, we can now evaluate the expectation as

$$\begin{aligned}
\mathbb{E}^i[L_i(T_i)] &= \mathbb{E}^{i+1}\left[\frac{d\mathbb{Q}^i}{d\mathbb{Q}^{i+1}} L_i(T_i)\right] \\
&= \frac{1}{1+\Delta_i L_i(0)} \mathbb{E}^{i+1}\left[L_i(T_i)(1+\Delta L_i(T_i))\right] \\
&= \frac{\mathbb{E}^{i+1}[L_i(T_i)] + \Delta_i \mathbb{E}^{i+1}[L_i(T_i)^2]}{1+\Delta_i L_i(0)}.
\end{aligned}$$

LIBOR-in-Arrears

Recall our **LIBOR Market Model** for the forward Libor rate:

$$dL_i(t) = \sigma_i L_i(t) dW^{i+1}.$$

The solution is given by

$$L_i(T) = L_i(0) \exp\left[-\frac{1}{2}\sigma_i^2 T + \sigma_i W^{i+1}\right].$$

Substituting back to our expectation, we obtain

$$\begin{aligned}\mathbb{E}^i[L_i(T_i)] &= \frac{\mathbb{E}^{i+1}[L_i(T_i)] + \Delta_i \mathbb{E}^{i+1}[L_i(T_i)^2]}{1 + \Delta_i L_i(0)} \\ &= \frac{L_i(0) + \Delta_i L_i(0)^2 e^{\sigma_i^2 T_i}}{1 + \Delta_i L_i(0)}.\end{aligned}$$

Cholesky Decomposition of Brownian Motions

Given two stochastic processes that are correlated with a correlation of ρ, it is often beneficial to be able to decompose them and express them as independent Brownian motions. Consider two correlated SDEs

$$
\begin{aligned}
dX_t &= \mu_X X_t dt + \sigma_X X_t dW_t^X \\
dY_t &= \mu_Y Y_t dt + \sigma_Y Y_t dW_t^Y,
\end{aligned}
$$

where $dW_t^X dW_t^Y = \rho\, dt$. We can decompose the two correlated Brownian processes as follow:

$$
\begin{aligned}
dW_t^X &= \alpha_{11} dZ_t^{(1)} \\
dW_t^Y &= \alpha_{12} dZ_t^{(1)} + \alpha_{22} dZ_t^{(2)},
\end{aligned}
$$

where $dZ_t^{(1)} dZ_t^{(2)} = 0$, and

$$
\alpha_{11} = 1, \quad \alpha_{12} = \rho, \quad \alpha_{22} = \sqrt{1 - \alpha_{12}^2}.
$$

Moment Matching

$$V\left[W_t^X\right] = V\left[\alpha_{11} Z_t^{(1)}\right]$$

$$t = \alpha_{11}^2 \cdot t \quad \Rightarrow \quad \alpha_{11} = 1$$

$$Cov\left(W_t^X, W_t^Y\right) = Cov\left(Z_t^{(1)}, \alpha_{12} Z_t^{(1)} + \alpha_{22} Z_t^{(2)}\right)$$

$$\rho t = \alpha_{12} \cdot t + 0$$

$$V\left[W_t^Y\right] = V\left[\rho Z_t^{(1)} + \alpha_{22} Z_t^{(2)}\right]$$

$$t = \rho^2 t + \alpha_{22}^2 \cdot t$$

$$1 = \rho^2 + \alpha_{22}^2$$

Cholesky Decomposition of Brownian Motions

This is because of moment matching. First, the variance relationship is

$$V\left[dW_t^X\right] = V\left[\alpha_{11} dZ_t^{(1)}\right]$$
$$dt = \alpha_{11}^2 dt \quad \Rightarrow \quad \alpha_{11} = 1.$$

Next, the covariance relationship is

$$\mathrm{Cov}\left[dW_t^X,\ dW_t^Y\right] = \mathrm{Cov}\left[\alpha_{11} dZ_t^{(1)},\ \alpha_{12} dZ_t^{(1)} + \alpha_{22} dZ_t^{(2)}\right]$$

$$\rho\, dt = \mathrm{Cov}\left[\alpha_{11} dZ_t^{(1)},\ \alpha_{12} dZ_t^{(1)}\right] + \cancelto{0}{\mathrm{Cov}\left[\alpha_{11} dZ_t^{(1)}, \alpha_{22} dZ_t^{(2)}\right]}$$
$$= \alpha_{11}\alpha_{12}\, dt \quad \Rightarrow \quad \alpha_{12} = \rho.$$

And finally,

$$V\left[dW_t^Y\right] = V\left[\alpha_{12} dZ_t^{(1)} + \alpha_{22} dZ_t^{(2)}\right]$$
$$dt = \alpha_{12}^2\, dt + \alpha_{22}^2\, dt \quad \Rightarrow \quad \alpha_{22} = \sqrt{1-\rho^2}.$$

Forward Exchange Rate Process

$F_t = S_t e^{r(T-t)}$

Earlier, we mentioned that from the domestic investor's perspective, the spot exchange rate follows

$$dX_t = (r^D - r^F)X_t dt + \sigma_X X_t dW_t^D,$$

while from the foreign investor's perspective, the spot FX rate follows

$$d\frac{1}{X_t} = (r^F - r^D)\frac{1}{X_t}dt + \sigma_X \frac{1}{X_t}dW_t^F.$$

We have also covered that the forward exchange rate can be written as

$$\begin{aligned}\mathbb{E}^D[X_T] &= \mathbb{E}^D\left[X_t e^{\left(r^D - r^F - \frac{\sigma_X^2}{2}\right)(T-t) + \sigma_X(W_T^D - W_t^D)}\right]\\ &= X_t e^{(r^D - r^F)(T-t)}.\end{aligned}$$

Let $F_t = X_t e^{(r^D - r^F)(T-t)}$ denote the forward exchange rate process (maturity at T), we can use Itô's formula to show that

$$dF_t = \sigma_X F_t dW_t^D.$$

Forward Exchange Rate Process

Let $D^D(t,T)$ denote the LIBOR discount factor in the domestic economy, and $D^F(t,T)$ denote the LIBOR discount factor in the foreign economy. Let us express the forward exchange rate F_t (maturing at T) as:

$$F_t = X_t \cdot \frac{D^F(t,T)}{D^D(t,T)}$$

By the same argument, we also have

$$\mathbb{E}^F\left[\frac{1}{X_T}\right] = \frac{1}{X_t} e^{(r^F - r^D)(T-t)}$$

and the forward exchange rate (maturity at T) from the foreign investor's perspective as

$$d\frac{1}{F_t} = \sigma_X \frac{1}{F_t} dW_t^F.$$

Therefore, we express it as

$$\frac{1}{F_t} = \frac{1}{X_t} \cdot \frac{D^D(t,T)}{D^F(t,T)}$$

Pricing Quanto LIBOR

In a quanto LIBOR contract, a foreign LIBOR rate L_i^F is observed at T_i and is paid in domestic denomination at time T_{i+1}. Suppose the Brownian motions W_t^{i+1} and W_t^F have a correlation of ρ.

To price this contract, we apply our multi-currency convexity correction formula

$$
\begin{aligned}
\mathbb{E}^{i+1,D}\left[L_i^F(T)\right] &= \mathbb{E}^{i+1,F}\left[L_i^F(T)\cdot\frac{d\mathbb{Q}^{i+1,D}}{d\mathbb{Q}^{i+1,F}}\right] \\
&= \mathbb{E}^{i+1,F}\left[L_i^F(T)\cdot\frac{\frac{D_{i+1}^D(T_{i+1})}{D_{i+1}^D(0)}}{\frac{X_T D_{i+1}^F(T_{i+1})}{X_0 D_{i+1}^F(0)}}\right] \\
&= \mathbb{E}^{i+1,F}\left[L_i^F(T)\cdot\frac{\frac{D_{i+1}^D(T_{i+1})}{X_T D_{i+1}^F(T_{i+1})}}{\frac{D_{i+1}^D(0)}{X_0 D_{i+1}^F(0)}}\right] \\
&= \mathbb{E}^{i+1,F}\left[L_i^F(T)\cdot\frac{\frac{1}{F_{T_{i+1}}}}{\frac{1}{F_0}}\right]
\end{aligned}
$$

Pricing Quanto LIBOR

Substituting for $L_i^F(T)$ and the forward exchange rate, the expectation to evaluate becomes

$$\mathbb{E}^{i+1,F}\left[L_i^F(0)e^{-\frac{\sigma_i^2 T}{2}+\sigma_i W^{i+1}}\cdot\frac{1}{F_0}e^{-\frac{\sigma_X^2 T}{2}+\sigma_X W^F}\right]\times F_0.$$

Next, we apply Cholesky decomposition:

$$\begin{aligned} W^{i+1} &:\longrightarrow Z_1 \\ W^F &:\longrightarrow \rho Z_1 + \sqrt{1-\rho^2}Z_2 \end{aligned}$$

where $Z_1 \perp Z_2$.

Finally, the convexity corrected foreign LIBOR rate paid in domestic denomination is given by

$$\tilde{L}_i^F(T) = L_i^F(0)e^{\rho\sigma_X\sigma_i T}.$$