QF605 Fixed-Income Securities
Assignment 2, Due Date: 14-Feb-2024

1. Suppose the swap rate follows the stochastic differential equation

$$dS_{n,N}(t) = \sigma_{n,N} S_{n,N}(0) dW^{n+1,N},$$

derive a valuation formula for a payer swaption in the Black normal model

$$V^{pay}_{n,N}(0) = P_{n+1,N} \mathbb{E}^{n+1,N}[(S_{n,N}(T) - K)^+].$$

2. Consider a displaced-diffusion LIBOR model, defined as

$$dL_i(t) = \sigma_i \big[\beta L_i(t) + (1-\beta) L_i(0)\big] dW^{i+1}_t,$$

Evaluate the following expectations under the risk-neutral measure $\mathbb{Q}^{i+1}$, associated with the zero-coupon bond $D_{i+1}(t)$:

(a) $\mathbb{E}^{i+1}[L_i(T_i)]$

(b) $\mathbb{E}^{i+1}[(L_i(T_i) - K)^+]$

3. Write down the expectation of a receiver swaption payoff maturing at T and struck at K. Show that we cannot evaluate the expectation under $\mathbb{Q}^*$, the risk-neutral measure associated with the risk-free money market account numeraire $B_t = B_0 e^{\int_0^t r_u \, du}$, but by changing the measure to $\mathbb{Q}^{n+1,N}$, the risk-neutral measure associated with the present value of a basis point (PVBP) numeraire $P_{n+1,N}(t) = \sum_{i=n+1}^{N} \Delta_{i-1} D_i(t)$, we can derive an analytical expression for the receiver swaption.

1. $dS_{n,N}(t) = \sigma_{n,N} S_{n,N}(0)\, dW^{n+1,N}$ $\qquad \bar{x} > \dfrac{K - S_{n,N}(0)}{\sigma_{n,N} S_{n,N}(0)\sqrt{T}} = d_1$

$$S_{n,N}(T) = S_{n,N}(0) + \sigma_{n,N} S_{n,N}(0) W^{n+1,N}$$

$$V^{pay}_{n,N}(0) = P_{n+1,N} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \left(S_{n,N}(0) + \sigma_{n,N} S_{n,N}(0)\sqrt{T}\, x - K \right)^{+} e^{-\frac{x^2}{2}}\, dx$$

$$= P_{n+1,N} \frac{1}{\sqrt{2\pi}} \int_{\bar{x}}^{\infty} \left(S_{n,N}(0) + \sigma_{n,N} S_{n,N}(0)\sqrt{T}\, x - K \right) e^{-\frac{x^2}{2}}\, dx$$

$$= P_{n+1,N} (S_{n,N}(0) - K)\, \Phi(-d_1) - P_{n+1,N} \frac{1}{\sqrt{2\pi}} \int_{\bar{x}}^{\infty} \sigma_{n,N} S_{n,N}(0) \sqrt{T}\, e^{-\frac{x^2}{2}}\, d\cdot\frac{x^2}{2}$$

$$= P_{n+1,N} (S_{n,N}(0) - K)\, \Phi(-d_1) + P_{n+1,N}\, \sigma_{n,N} S_{n,N}(0) \sqrt{T}\, \varphi(d_1)$$

2. $L_i(T) = \dfrac{L_i(0)}{\beta} e^{-\frac{\beta^2 \sigma_i^2 T}{2} + \beta \sigma_i W_T^{i+1}}$ $\qquad d_2 = d_1 - \beta \sigma_i \sqrt{T_i}$

$$d_1 = x > \frac{\ln \dfrac{(1-\beta) L_i(0) + K\beta}{L_i(0)} + \dfrac{\beta^2 \sigma_i^2 T_i}{2}}{\beta \sigma_i \sqrt{T_i}}$$

(a) $E^{i+1}[L_i(T_i)] = L_i(0)$

(b) $$E^{i+1}[(L_i(T_i) - K)^+] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{L_i(0)}{\beta} e^{-\frac{\beta^2 \sigma_i^2 T_i}{2} + \beta \sigma_i \sqrt{T_i}\, x} - \frac{1-\beta}{\beta} L_i(0) - K \right)^{+} e^{-\frac{x^2}{2}}\, dx$$

$$= \frac{L_i(0)}{\beta} \Phi(-d_2) - \left(\frac{1-\beta}{\beta} L_i(0) + K \right) \Phi(-d_1)$$

3.

$$\frac{V^{rec}_{n,N}(0)}{B_0} = E^{*}\left[\frac{V^{rec}_{n,N}(T_n)}{B_{T_n}} \right]$$

$$V^{rec}_{n,N}(0) = E^{*}\left[\frac{B_0 V^{rec}_{n,N}(T_n)}{B_0 e^{\int_0^{T_n} r_u du}} \right] = E^{*}\left[V^{rec}_{n,N}(T_n) \frac{P_{n+1,N}(T_n) / P_{n+1,N}(0)}{B_{T_n}/B_0} \right] \frac{P_{n+1,N}(0)}{P_{n+1,N}(T_n)}$$

$$= E^{n+1,N}\left[P_{n+1,N}(T) (K - S_{n,N}(T))^{+} \right] \frac{P_{n+1,N}(0)}{P_{n+1,N}(T_n)}$$

$$= P_{n+1,N}(0)\, E^{n+1,N}\left[(K - S_{n,N}(T))^{+} \right]$$