$dr_t = \mu\, dt + \sigma\, dW_t^*$

SINGAPORE MANAGEMENT UNIVERSITY

Session 8
Ho-Lee & Hull-White Models

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QF605 Fixed Income Securities

Note: Integrating W_t wrt t

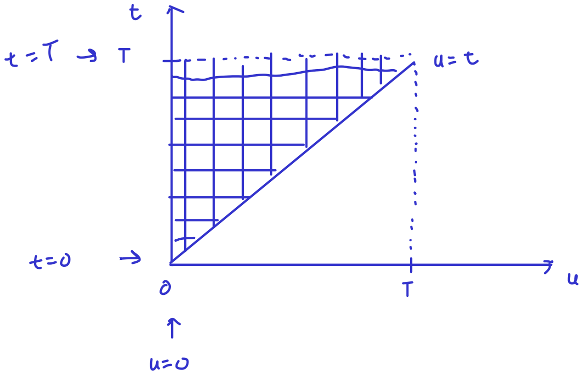
$$W_t = \int_0^t 1 \cdot dW_u$$

Consider the following integral:

$$\begin{aligned}\int_0^T W_t \, dt &= \int_0^T \int_0^t dW_u \, dt \\ &= \int_0^T \int_u^T dt \, dW_u \\ &= \int_0^T (T-u) \, dW_u \\ &= \lim_{N\to\infty} \sum_{i=1}^{N} (T-t_i)(W_{t_{i+1}} - W_{t_i})\end{aligned}$$

As a deterministic function $(T - t_i)$ weighted sum of independent Brownian increment, this integral must be normally distributed.

$\Rightarrow$ It remains to determine the mean and variance of the integral.



$$\int_{t=0}^{t=T} \int_{u=0}^{u=t} dW_u \, dt = \int_{u=0}^{u=T} \int_{t=u}^{t=T} dt \, dW_u$$

(stochastic) Fubini's Theorem

Note: Integrating W_t wrt t

The mean is given by

$$\mathbb{E}\left[\int_0^T (T-u)\, dW_u\right] = 0$$

since all stochastic integral has zero-mean, and

$$\begin{aligned} V\left[\int_0^T (T-u)\, dW_u\right] &= \mathbb{E}\left[\int_0^T (T-u)^2\, du\right] \\ &= \int_0^T \left(T^2 - 2uT + u^2\right) du \\ &= \left[T^2 u - u^2 T + \frac{u^3}{3}\right]_0^T = \frac{T^3}{3} \end{aligned}$$

where we have used Itô's Isometry.

Affine : $f(x) = a + bx$

Linear : $f(x) = bx$

Equilibrium Affine Models

Definition It can be shown that in any **equilibrium short rate model** (e.g. Vasicek, CIR), the zero coupon bond prices can be reconstructed as

$$e^{-R(t,T)(T-t)} = D(t,T) = e^{A(t,T) - r_t B(t,T)}$$

for some deterministic functions $A(t,T)$ and $B(t,T)$ of t and T only.

This implies that the **spot curve or zero rate curve** can be written as

$$R(t,T) = \frac{1}{T-t}\Big(-A(t,T) + r_t B(t,T)\Big)$$

for this class of model.

In this class of model, spot rates are affine functions of the short rate, and so this class is referred to as the class of **affine term structure models**.

Affine function is composed of a linear function plus a constant (translation).

No-Arbitrage Affine Models

Equilibrium: models

$dr_t = \mu dt + \sigma dW_t^*$

$dr_t = \kappa(\theta - r_t) dt + \sigma dW_t^*$

However, equilibrium models only have a few model parameters—there is no guarantee that we will be able to fit to the observed term structure.

Although it is possible to perform a least square optimization to match the observed discount factors as closely as possible, to prevent arbitrage, we must be able to fit exactly to liquid discount instruments.

Ho-Lee, and subsequently **Hull-White**, proposed to address this problem by letting the model parameters be deterministic function of time – this way, we can match any observed spot curve $R(t, T)$.

$\theta(t)$

Standard terminology for these models is that these are **no-arbitrage short rate models**:

$$\text{Ho-Lee:} \qquad dr_t = \theta(t)dt + \sigma dW_t^*.$$
$$\text{Hull-White:} \qquad dr_t = \kappa(\theta(t) - r_t)dt + \sigma dW_t^*.$$

The simplest no-arbitrage model is the Ho-Lee model: where we choose the deterministic function $\theta(t)$ to match the observed spot curve.

Ho-Lee Model

In the Ho-Lee interest rate model, the short rate follows:

$$dr_t = \theta(t)dt + \sigma dW_t^*,$$

where W_t^* is a Brownian motion under the measure $\mathbb{Q}^*$. To **fit the initial term structure**, we require that

$$\theta(T) = -\frac{\partial^2}{\partial T^2}\log D(0,T) + \sigma^2 T.$$

To show this, first write out the interest rate process by integrating both sides:

$$r_t = r_0 + \int_0^t \theta(s)\, ds + \int_0^t \sigma\, dW_s^*.$$

Next, integrate again to obtain an expression for the integrated rate:

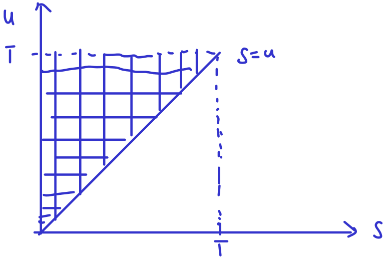
$$\begin{aligned}\int_0^T r_u\, du &= \int_0^T r_0\, du + \int_0^T \int_0^u \theta(s)\, ds\, du + \int_0^T \int_0^u \sigma\, dW_s^*\, du \\ &= r_0 T + \int_0^T \theta(s)(T-s)\, ds + \int_0^T \sigma(T-s)\, dW_s^*.\end{aligned}$$

Model: $dr_t = \theta(t)\,dt + \sigma\,dW_t^*$

$$\int_0^t dr_s = \int_0^t \theta(s)\,ds + \int_0^t \sigma\,dW_s^*$$

$$r_t = r_0 + \int_0^t \theta(s)\,ds + \int_0^t \sigma\,dW_s^*$$

$$\int_0^T r_u\,du = r_0\,(T-0) + \int_0^T \int_0^u \theta(s)\,ds\,du + \int_0^T \int_0^u \sigma\,dW_s^*\,du$$



$$= r_0 T + \int_{s=0}^{s=T} \int_{u=s}^{u=T} \theta(s)\, du\, ds + \int_{s=0}^{s=T} \int_{u=s}^{u=T} \sigma\, du\, dW_s^*$$

$$= r_0 T + \int_0^T \theta(s)(T-s)\, ds + \int_0^T \sigma (T-s)\, dW_s^*$$

Ho-Lee Model

$V\left[\int_0^T r_u du\right] = V\left[\int_0^T \sigma(T-s)\, dW_s^*\right] = \mathbb{E}\left[\left(\int_0^T \sigma(T-s)\, dW_s^*\right)^2\right]$

The mean of this stochastic integral is given by

Itô Isometry $= \int_0^T \sigma^2(T-s)^2 ds$

$$\mathbb{E}\left[\int_0^T r_u \, du\right] = r_0 T + \int_0^T \theta(s)(T-s)\, ds,$$

and the variance is given by

$$V\left[\int_0^T r_u \, du\right] = \int_0^T \sigma^2 (T-s)^2 \, ds = \frac{1}{3}\sigma^2 T^3,$$

where we have used **Itô Isometry**.

$X \sim N(\_\_, \_\_)$

$\mathbb{E}[e^{-X}] = e^{-\mu + \frac{\sigma^2}{2}}$

Therefore, the **zero-coupon discount bond can be reconstructed** as

$$D(0,T) = \mathbb{E}\left[e^{-\int_0^T r_u \, du}\right] = \exp\left[-r_0 T - \int_0^T \theta(s)(T-s)\, ds + \frac{1}{6}\sigma^2 T^3\right].$$

Since we can express $D(0,T)$ in the form of $e^{A(0,T) - r_0 B(0,T)}$, we see that <u>Ho-Lee is an affine model</u>.

$$\therefore \quad P(0,T) = e^{-R(0,T)\cdot(T-0)}$$

$$\therefore \Rightarrow \quad R(0,T) = \frac{1}{T-0}\left[r_0 T + \int_0^T \theta(s)(T-s)\,ds - \frac{1}{6}\sigma^2 T^3 \right]$$

Ho-Lee Model

Fitting the initial term structure

From here we can work out that

$$\begin{aligned}
\log D(0,T) &= -r_0 T - \int_0^T \theta(s)(T-s)\,ds + \frac{1}{6}\sigma^2 T^3 \\
\frac{\partial}{\partial T}\log D(0,T) &= -r_0 - \int_0^T \theta(s)\,ds + \frac{1}{2}\sigma^2 T^2 \\
\frac{\partial^2}{\partial T^2}\log D(0,T) &= -\theta(T) + \sigma^2 T \\
\Rightarrow \quad \theta(T) &= -\frac{\partial^2}{\partial T^2}\log D(0,T) + \sigma^2 T.
\end{aligned}$$

This allows Ho-Lee model to fit the initial term structure $D(0,T)$ observed in the market.

$$\log D(0,T) = -r_0 T - \int_0^T \theta(s)(T-s)\,ds + \frac{1}{6}\sigma^2 T^3$$

$$\frac{\partial}{\partial T}\log D(0,T) = -r_0 - \left[\theta(T)\cancelto{0}{(T-T)}\cdot\frac{dT}{dT} - \theta(0)\cancelto{0}{(T-0)\cdot\frac{d0}{dT}} + \int_0^T \theta(s)\,ds\right]$$

$$+ \frac{1}{2}\sigma^2 T^2$$

$$= -r_0 - \int_0^T \theta(s)\,ds + \frac{1}{2}\sigma^2 T^2$$

$$\frac{\partial^2}{\partial T^2}\log D(0,T) = 0 - \left[\theta(T)\cdot\frac{dT}{dT} - \theta(0)\cdot\frac{d0}{dT} + \int_0^T 0\,ds\right] + \sigma^2 T$$

$$= -\theta(T) + \sigma^2 T$$

$dr_t = \theta(t)\,dt + \sigma\,dW_t^*$

Ho-Lee Model

$dD(t,T) = \,?$

We have shown that Ho-Lee model allows us to reconstruct the discount factor

$$D(t,T) = e^{A(t,T) - r_t B(t,T)},$$

where

$$A(t,T) = -\int_t^T \theta(s)(T-s)\,ds + \frac{\sigma^2 (T-t)^3}{6},$$
$$B(t,T) = T - t.$$

What does Ho-Lee model tell us about the evolution of discount factors over time?

⇒ Note that the reconstructed discount factor is given as a function of time and short rate, i.e. $D(t,T) = f(t, r_t)$.

This means that we can use **Itô's formula** to derive the stochastic differential equation describing the evolution of the discount factors over time.

$P(t,T) = f(t, r_t)$

Ho-Lee Model

First, we work out the partial derivatives

$$f(t,x) = e^{A(t,T)-xB(t,T)}$$

$$f_t(t,x) = e^{A(t,T)-xB(t,T)}\left[\frac{\partial A(t,T)}{\partial t} - x\cdot\frac{\partial B(t,T)}{\partial t}\right]$$

$$f_x(t,x) = e^{A(t,T)-xB(t,T)}\Big[-B(t,T)\Big]$$

$$f_{xx}(t,x) = e^{A(t,T)-xB(t,T)}\Big[B(t,T)^2\Big],$$

where an application of **Leibniz's rule** yields

$$A(t,T) = -\int_t^T \theta(s)(T-s)\,ds + \frac{\sigma^2(T-t)^3}{6}$$

$$\frac{\partial A(t,T)}{\partial t} = \theta(t)(T-t) - \frac{\sigma^2(T-t)^2}{2}.$$

On the other hand, the time derivative for $B(t,T)$ is simply

$$\frac{\partial B(t,T)}{\partial t} = -1.$$

$$A(t,T) = -\int_t^T \theta(s)(T-s)\,ds + \frac{\sigma^2(T-t)^3}{6}$$

$$\frac{\partial A}{\partial t} = -\left[\cancelto{0}{\theta(T)(T-T)} \cdot \frac{dT}{dt} - \theta(t)(T-t)\cdot\frac{dt}{dt} + \cancelto{0}{\int_t^T 0\,ds} \right] + \frac{\sigma^2(T-t)^2}{2}\cdot(-1)$$

$$= \theta(t)(T-t) - \frac{\sigma^2(T-t)^2}{2}$$

$$B(t,T) = (T-t)$$

$$\frac{\partial B}{\partial t} = -1$$

Ho-Lee Model

Applying **Itô's formula**, we obtain the following stochastic differential equation:

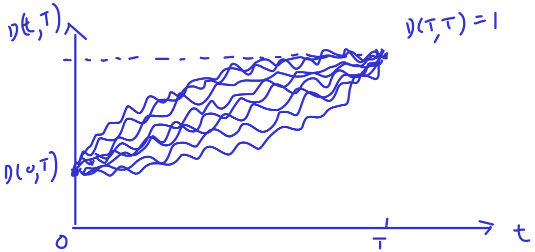
$$\begin{aligned}
dD(t,T) &= f_t(t,r_t)dt + f_x(t,r_t)dr_t + \frac{1}{2}f_{xx}(t,r_t)(dr_t)^2 \\
&= D(t,T)\left[\frac{\partial A(t,T)}{\partial t} - r_t \cdot \frac{\partial B(t,T)}{\partial t}\right]dt \\
&\qquad - D(t,T)(T-t)\Big(\theta(t)dt + \sigma dW_t^*\Big) \\
&\qquad + \frac{1}{2}D(t,T)(T-t)^2\sigma^2 dt \\
&= r_t D(t,T)dt - (T-t)\sigma D(t,T)dW_t^*.
\end{aligned}$$

$$dD(t,T) = D(t,T)\left[\cancel{\theta(t)(T-t)} - \cancel{\frac{\sigma^2 (T-t)^2}{2}} + r_t \right] dt$$

$$- D(t,T)(T-t)\left(\cancel{\theta(t)}\, dt + \sigma\, dW_t^* \right)$$

$$+ \frac{1}{2} D(t,T) \cdot \cancel{(T-t)^2}\; \sigma^2 dt \quad \leftarrow (dr_t)^2$$

$$dD(t,T) = r_t D(t,T)\, dt - \sigma \cdot (T-t) \cdot D(t,T)\, dW_t^*$$



$dr_t = \theta(t)dt + \sigma dW_t^*$

$\int_0^t \sigma dW_s^*$

Ho-Lee Binomial Tree

Integrating the Ho-Lee model from 0 to t, we obtain:

$x_i \begin{cases} +1 \\ -1 \end{cases}$

$$r_t = r_0 + \int_0^t \theta(s)\, ds + \sigma W_t^*$$

$\approx r_0 + \sum_{i=1}^{n-1} \theta_i \cdot \Delta t + \sum_{i=1}^{n-1} \sigma \cdot \sqrt{\Delta t} \cdot x_i$

Example Ho-Lee binomial tree

- Suppose we have an initial 1-year rate of $R = 5\%$ (with continuous compounding).
- We assume that the probability of a rate increase/decrease is $\frac{1}{2}$.
- At every node, we assume that the rate randomly increases by 1% or decreases by -1% (this is determined by the volatility of the short rate).
- Start at time $t = 0$. At time $s + 1$, we also add a deterministic amount $\sum_{u=0}^{u=s} \theta_u$ to the rate, in all nodes.
- We then choose the θ_u to ensure that we can match the observed spot rates $R(0, 2)$, $R(0, 3)$, $\cdots$.

Ho-Lee Binomial Tree

The 2-period binomial tree model looks as follows:

$\therefore D(0,1) = e^{-0.05 \cdot 1}$

r_0

$$r=5\% \quad D(0,2)=\ldots$$

$$\nearrow \quad r=6\%+\theta_0 \quad D(1,2)=e^{-(6\%+\theta_0)\cdot 1}$$

$$\searrow \quad r=4\%+\theta_0 \quad D(1,2)=e^{-(4\%+\theta_0)\cdot 1}$$

We can choose θ_0 to match the observed spot rate $R(0,2)$.

$$D(0,1) = e^{-0.05 \times 1} \Rightarrow r_0 = 5\%$$

we know

$$D(0,2y) = \mathbb{E}^*\left[D(0,1)\ D(1,2) \right]$$

$$= D(0,1)\ \mathbb{E}^*\left[D(1,2) \right]$$

$$= e^{-0.05} \times \left[\frac{1}{2} \times e^{-0.06-\theta_0} + \frac{1}{2} \times e^{-0.04-\theta_0} \right]$$

Solve for θ_0 = ______

Ho-Lee Binomial Tree

The 3-period binomial tree model looks as follows:

$$
\begin{tikzcd}[column sep=small]
 & & \begin{array}{c} r=7\%+\theta_0+\theta_1 \\ D(2,3)=e^{-(7\%+\theta_0+\theta_1)\cdot 1} \end{array} \\
 & \begin{array}{c} r=6\%+\theta_0 \\ D(1,3)=\dots \end{array} \arrow[ur] \arrow[dr] & \\
\begin{array}{c} r=5\% \\ D(0,3)=\dots \end{array} \arrow[ur] \arrow[dr] & & \begin{array}{c} r=5\%+\theta_0+\theta_1 \\ D(2,3)=e^{-(5\%+\theta_0+\theta_1)\cdot 1} \end{array} \\
 & \begin{array}{c} r=4\%+\theta_0 \\ D(1,3)=\dots \end{array} \arrow[ur] \arrow[dr] & \\
 & & \begin{array}{c} r=3\%+\theta_0+\theta_1 \\ D(2,3)=e^{-(3\%+\theta_0+\theta_1)\cdot 1} \end{array}
\end{tikzcd}
$$

We can then choose θ_1 to match the observed spot rate $R(0,3)$.

$$D(0,3) = \mathbb{E}^*\left[D(0,1) \cdot D(1,2) \cdot D(2,3) \right]$$

$$= D(0,1)\, \mathbb{E}^*\left[D(1,2)\, D(2,3) \right]$$

$$= D(0,1)\, \mathbb{E}^*\left[D(1,2)\, \mathbb{E}_1^*\left[D(2,3) \right] \right]$$

$$= e^{-0.05} \times \Big[\frac{1}{2} \cdot e^{-0.06-\theta_0} \cdot \left(\frac{1}{2} \cdot e^{-0.07-\theta_0-\theta_1} + \frac{1}{2} \cdot e^{-0.05-\theta_0-\theta_1} \right)$$

$$+ \frac{1}{2} e^{-0.04-\theta_0} \cdot \left(\frac{1}{2} \cdot e^{-0.05-\theta_0-\theta_1} + \frac{1}{2} e^{-0.03-\theta_0-\theta_1} \right) \Big]$$

Ho-Lee Binomial Tree

Example Consider the same Ho-Lee binomial tree given in the previous example. Suppose we observe the following in the interest rate market:

Instrument	Value
$D(0, 1y)$	0.95123
$D(0, 2y)$	0.90
$D(0, 3y)$	0.86

Determine the no-arbitrage value of θ_0 and θ_1.

ans.: $\theta_0 = 0.00556$, $\theta_1 = -0.01$.

No-Arbitrage Models

We should always use **no-arbitrage affine short rate models** because they allow us to fit the zero curve exactly.

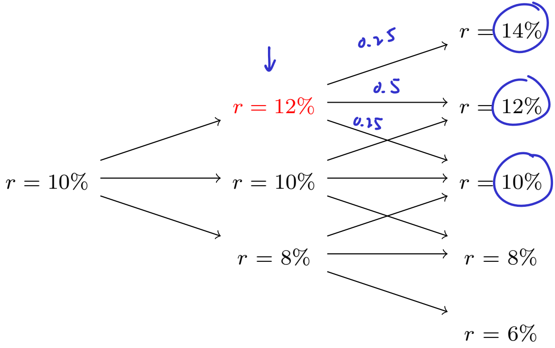
The notion is that if we are to hedge our exposure using bonds and swaps, our model must at least be able to price them correctly.

In practice, short rate models like Hull-White are more frequently implemented using **trinomial trees** (with 3 rather than 2 branches).

The extra branch makes it easier to capture features like **mean reversion**.

- ⇒ Time is discretized into steps of size Δt.
- ⇒ The underlying variable that evolves across the tree is the continuously compounded Δt-period short rate.
- ⇒ A key difference to binomial/trinomial tree models of the stock price is that discounting varies across branches.

Trinomial Tree



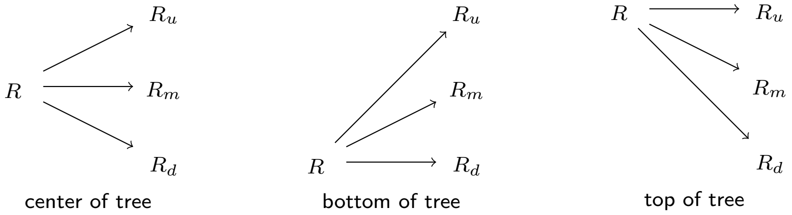
Suppose at the node indicated in red, the probabilities are $p_u = 0.25$, $p_m = 0.5$, and $p_d = 0.25$. Then at that node, a claim that pays off $100(r - 0.11)^+$ at the third date is worth $(\Delta t = 1)$

$$e^{-0.12 \cdot \Delta t}\Big(0.25 \times 3 + 0.5 \times 1 + 0.25 \times 0\Big) = 1.11$$

Hull-White Trinomial Tree

It is sometimes useful to use non-standard branching at the top and bottom of the tree.

Especially in models with mean reversion, this can improve the numerical stability of the procedure.



Hull-White Trinomial Tree

A trinomial tree with non-standard branching at the top and bottom: