



Session 7

Short Rate Models and Term Structure

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QF605 Fixed Income Securities

Term Structure Models

The **Market Models** and static replication method can handle the pricing of derivatives with **European payoffs**, such as caps, floors and European swaptions.

However, they are not able to handle derivatives with **path-dependent payoffs**, e.g. Bermudan or American option.

To value path-dependent products, we need a model of how the whole **term structure** (not just a single forward rate or bond) evolves.

One set of models specifies dynamics for the short rate under the risk-neutral measure. This then determines prices of zero coupon bonds, and hence, the entire term structure:

$$\mathbb{E}_t^* \left[e^{-\int_t^T r_u du} \right] = D(t, T) = e^{-R(t, T)(T-t)}$$

Term Structure Models

A typical **short rate model** will take the following form:

$$dr_t = \mu_t dt + \sigma_t dW_t^*$$

We begin by considering how two different features of the short rate model affect the spot curve that you obtain from the model:

- ① the drift in the short rate (under \mathbb{Q}^*)
- ② the volatility of the short rate (under \mathbb{Q}^*)

1. Drift: Suppose a (simplistic) short rate model specifies

$$dr_t = \mu dt,$$

where μ is a constant. The short rate grows linearly over time, and is deterministic. We could also write this as

$$\mu = \frac{dr_t}{dt}.$$

Drift in Short Rate Models

Example We consider a discrete approximation of positive μ . Suppose the initial $3m$ rate (with continuous compounding) is 5%. The next $3m$ rates will be 5.1%, 5.2%, 5.3%, \dots and so on.

$$D(0, 6m) = e^{-(0.05+0.051) \cdot 0.25} \quad \Rightarrow \quad R(0, 6m) = 0.0505$$

$$D(0, 9m) = e^{-(0.05+0.051+0.052) \cdot 0.25} \quad \Rightarrow \quad R(0, 9m) = 0.051$$

$$D(0, 12m) = e^{-(0.05+0.051+0.052+0.053) \cdot 0.25} \quad \Rightarrow \quad R(0, 12m) = 0.0515.$$

Based on the calculation, we conclude that the term structure is upward sloping.

If μ is negative, then the term structure will be downward sloping.

Drift in Short Rate Models

Mathematically, we proceed as follows:

- First, we integrate the short rate SDE from 0 to t to obtain an expression for the short rate process:

$$r_t = r_0 + \mu t.$$

- Next, we integrate the short rate process to obtain:

$$\int_t^T r_u du = r_0(T-t) + \frac{1}{2}\mu(T^2 - t^2) = r_t(T-t) + \frac{1}{2}\mu(T-t)^2$$

- We can now reconstruct the discount factor as

$$D(t, T) = \mathbb{E}_t^* \left[e^{-\int_t^T r_u du} \right] = e^{-r_t(T-t) - \frac{1}{2}\mu(T-t)^2}.$$

- Therefore, the spot curve in this stylized (simplified) model is given by

$$R(t, T) = -\frac{1}{T-t} \log D(t, T) = \frac{1}{2}\mu(T-t) + r_t.$$

Clearly, if $\mu > 0$, the spot curve is upward sloping, and if $\mu < 0$, the spot curve is downward sloping.

Volatility in Short Rate Models

2. Volatility: Suppose a (simplistic) short rate model specifies

$$dr_t = \sigma dW_t^*$$

where σ is a constant, and W_t^* is a Brownian motion under \mathbb{Q}^* .

The short rate follows a random walk without drift under \mathbb{Q} , where σ affects the variance of the “error term”.

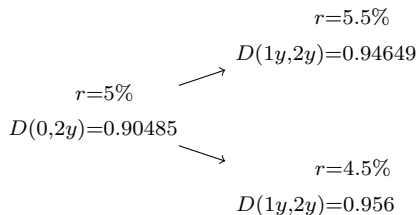
In discrete term, we have

$$r_{t+\Delta t} \approx r_t + \Delta r_t = r_t + \sigma \Delta W_t^*$$

Volatility in Short Rate Models

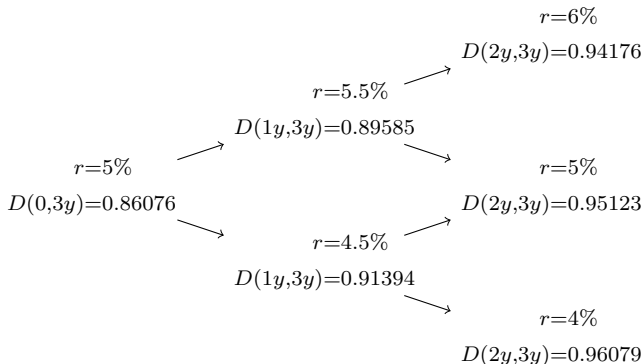
Example We consider a discrete approximation of this short rate model with small σ . Suppose the initial 1-year rate (with continuous compounding) is 5%. The following 1-year rates are describe by a tree, where each period the short rate can move up or down by 0.5%. The risk-neutral probability of an up/down move is always 0.5.

A 2-period tree looks as follows:



Volatility in Short Rate Models

A 3-period tree looks as follows:



From the zero coupon bond prices, we work out the spot rates:

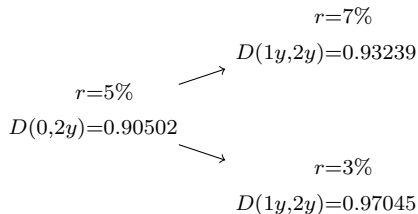
$$R(0, 1y) = 5\%, \quad D(0, 2y) = 0.90485 \quad \Rightarrow \quad R(0, 2y) = 4.9994\%$$

$$D(0, 3y) = 0.86076 \quad \Rightarrow \quad R(0, 3y) = 4.9979\%$$

Volatility in Short Rate Models

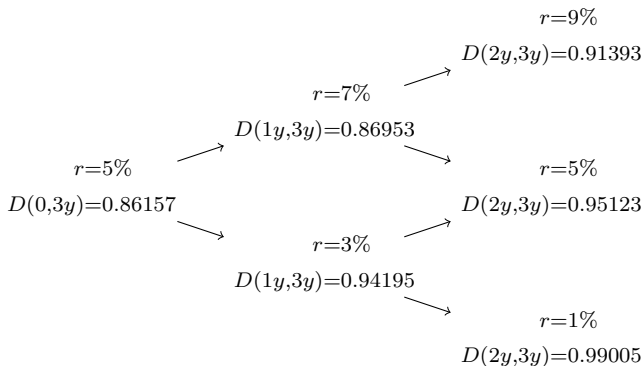
Example We now consider a discrete approximation of the short rate model with large σ . Suppose the initial 1-year rate (with continuous compounding) is 5%. The following 1-year rates are described by a tree, where each period the short rate can move up or down by 2%, and the risk-neutral probability of an up/down move is always $\frac{1}{2}$.

A 2-period tree looks as follows:



Volatility in Short Rate Models

A 3-period tree looks as follows:



From the zero coupon bond prices, we work out the spot rates:

$$R(0, 1y) = 5\%, D(0, 2y) = 0.90502 \Rightarrow R(0, 2y) = 4.99\%$$

$$D(0, 3y) = 0.86157 \quad \Rightarrow \quad R(0, 3y) = 4.9667\%$$

Volatility in Short Rate Models

Main Conclusions

- 1 Volatility of the short rate by itself produces a slightly downward sloping spot curve.
- 2 The higher the volatility, the more negative the slope of the spot curve.
- 3 This is a consequence of Jensen's inequality and the fact that $f(x) = e^{-x}$ and $f(x) = \frac{1}{1+x}$ are convex in x .

Jensen's inequality states that

$$\mathbb{E}[f(X)] \geq f(\mathbb{E}[X]) \text{ if } f \text{ is convex}$$

$$\mathbb{E}[f(X)] \leq f(\mathbb{E}[X]) \text{ if } f \text{ is concave}$$

Volatility in Short Rate Models

Mathematically, we proceed as follows:

- First, integrate the SDE from 0 to t to obtain the short rate process:

$$r_t = r_0 + \sigma W_t^*, \quad \text{where } r_t \sim N(r_0, \sigma^2 t)$$

- Next we integrate the short rate process to obtain:

$$\int_t^T r_u \, du = r_0(T - t) + \sigma \int_t^T W_u^* \, du = r_t(T - t) + \sigma \int_t^T (W_u^* - W_t^*) \, du.$$

Recall that in the previous term, we have demonstrated that by applying Itô's formula to the function $X_t = f(t, W_t) = tW_t$, we can write

$$\int_0^T W_u \, du = \int_0^T (T - u) \, dW_u,$$

so that this integral is normally distributed, with mean and variance:

$$\mathbb{E} \left[\int_0^T W_u \, du \right] = 0, \quad V \left[\int_0^T W_u \, du \right] = \frac{T^3}{3}.$$

Volatility in Short Rate Models

- Applying this results to our integrated short rate process, we note that

$$\mathbb{E} \left[\int_t^T r_u du \right] = r_t(T - t)$$
$$V \left[\int_t^T r_u du \right] = V \left[\sigma \int_t^T (W_u^* - W_t^*) du \right] = \frac{\sigma^2(T - t)^3}{3},$$

and hence

$$\int_t^T r_u du \sim N \left(r_t(T - t), \frac{\sigma^2}{3}(T - t)^3 \right).$$

- We can now reconstruct the discount factor as

$$D(t, T) = \mathbb{E}_t^* \left[e^{-\int_t^T r_u du} \right].$$

Volatility in Short Rate Models

- We know how to evaluate the expectation of a lognormal random variable. If $X \sim N(\mu, \sigma^2)$, then

$$\mathbb{E} \left[e^{\theta X} \right] = e^{\mu\theta + \frac{1}{2}\sigma^2\theta^2}.$$

- Using this, we have

$$D(t, T) = \mathbb{E}_t^* \left[e^{-\int_t^T r_u du} \right] = e^{-r_t(T-t) + \frac{\sigma^2}{6}(T-t)^3}.$$

- Finally, we can express the zero rate $R(t, T)$ as follows:

$$R(t, T) = -\frac{1}{T-t} \log D(t, T) = r_t - \frac{\sigma^2}{6}(T-t)^2.$$

- The further we look ahead (larger $T-t$), the larger the accumulated uncertainty, and hence the lower the corresponding spot rate. Also, the higher σ , the lower all spot rates.

Vasicek Model

The Vasicek model for interest rate is a classic short rate model:

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t^*$$

Here, κ is the mean reversion coefficient, θ is the long run mean of the short rate, and σ is the volatility of the short rate. Vasicek model is mean reverting.

Applying Itô's formula to $f(r_t, t) = r_t e^{\kappa t}$, we can show that

$$r_t = r_0 e^{-\kappa t} + \theta (1 - e^{-\kappa t}) + \sigma \int_0^t e^{\kappa(u-T)} dW_u^*$$

We conclude that r_t is normally distributed, with a mean of

$$\mathbb{E}[r_t] = r_0 e^{-\kappa t} + \theta (1 - e^{-\kappa t})$$

and a variance of

$$V[r_t] = \frac{\sigma^2}{2\kappa} (1 - e^{-2\kappa t}).$$

Vasicek Model

Once again, we can now write the integrated short rate process under Vasicek model as

$$\int_t^T r_u du \sim N \left(\mathbb{E} \left[\int_t^T r_u du \right], V \left[\int_t^T r_u du \right] \right).$$

This in turn allows us to reconstruct the discount factor as follows:

$$D(t, T) = \mathbb{E} \left[e^{-\int_t^T r_u du} \right].$$

Vasicek Model

Let $R(t, T)$ denote the zero rate covering the period $[t, T]$, so that

$$D(t, T) = e^{-R(t, T)(T-t)}.$$

After some algebra (see Session 7 Additional Examples Q2), we find that we can write

$$D(t, T) = e^{A(t, T) - B(t, T)r_t},$$

or (equivalently)

$$R(t, T) = \frac{1}{T-t} \left[-A(t, T) + B(t, T)r_t \right]$$

where

$$B(t, T) = \frac{1}{\kappa} \left(1 - e^{-\kappa(T-t)} \right)$$
$$A(t, T) = \frac{[B(t, T) - (T-t)](\kappa^2\theta - \frac{1}{2}\sigma^2)}{\kappa^2} - \frac{\sigma^2 B(t, T)^2}{4\kappa}$$

Cox-Ingersoll-Ross Model

In any model in which the short rate is normally distributed (including the Vasicek model), there is always a non-zero probability that the short rate is negative.

An alternative model will be the Cox-Ingersoll-Ross (CIR) model:

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t^*$$

However, r_t is non-centrally χ^2 -distributed in the CIR model.