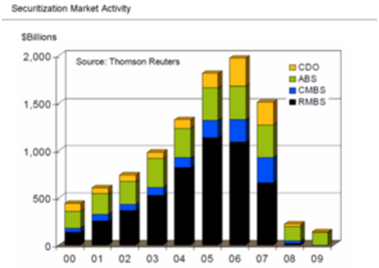
OTC

Background of Collateralization

- First sign on sub-prime lending issues began in mid 2007.
- Large amount of derivative contracts between banks and institutional investors.
- Banks began to worry about **counterparty's credit quality** (e.g. major investment banks vs. smaller financial institutions).
- Does **LIBOR discounting** still mean anything?
- Unique derivative price or counterparty specific valuation?



Importance of Collateralization

- Trades to be constantly **collateralized** to avoid financial loss in events of default.
- Portfolios are marked-to-market daily and **collaterals** posted.
- Either cleared through a **central clearing house** (e.g. London Clearing House LCH) or managed by operations at each banks.
- Interest will be paid on the posted collateral according to mutual agreement.
- **Credit Support Annex** (CSA) agreements are signed between counterparties to stipulate details of collateralizations.

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Issues of Collateralization Discounting

- What can be posted as collateral?
- What is the interest to be paid on collateral?
- Should collateralization details enter the picture of derivative valuation?
- How wrong is LIBOR (uncollateralized) discounting?
- Does martingale valuation framework still work?

Collateralization

- Collateralization requires us to formulate a theory of swap valuation in the presence of **bilateral mark-to-market** (MTM).
- MTM requires that counterparties post collateral in the amount of the current MTM value of the contract.
- This generates an important departure from the traditional theory, which assumes that all cashflows exchanged between counterparties occur on the periodic swap dates.
- Since collateral is generally costly to post, these payments induce economic costs (benefits) to the payer (receiver).
- Given that these credit enhancements are part of the swap contract, they must be accounted for in valuation (**Credit Support Annex** to the **ISDA Master Swap Agreement**).

Collateralization

- An important feature of bilateral collateralization is that interest on collateral is often rebated.

- It is important to note that the posting of collateral, regardless of what or how it is posted, entails a cost and, for the other counterparty, a benefit.

- To appreciate this, we just need to observe that:
  - The receiver of collateral reduces or eliminates any losses conditional on **default**.
  - Collateral receivers, when allowed, typically reuse or **rehypothecate** the collateral for other purposes.
  - Even when interest is rebated, there is often a cost to posting collateral as the interest rebated is typically less than the payer's funding costs.
  - Most market participants borrow short term at rates higher than LIBOR, which generates an additional cost.

Matching Collateral & Payment Currency

Guideline 1: If a given portfolio has cashflows paid in a single currency, and if the collateral is paid in the same currency, and if the portfolio can be statically replicated by fixed payments. Then the portfolio should be discounted at the interest rate index paid on the collateral.

Reason: Consider the case where we receive 1 unit of currency in one day. Our aim is to solve for the present value of the cashflow – the collateral call amount – so that the next day's payment are net flat, i.e. there's no credit risk. The mechanics of the collateral are that

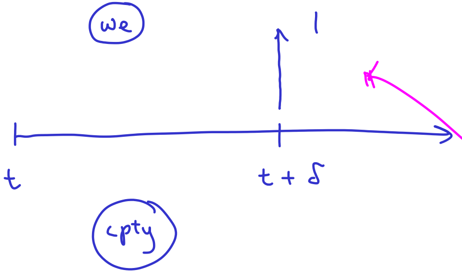
- We ask for the PV today: $D(t, t+\delta)$.
- On the next day, we pay: $D(t, t+\delta)[1+\delta f(t, t+\delta)]$.

In order for there to be no credit risk, the amount received on the next day must be equal to the amount of collateral we hold, i.e.

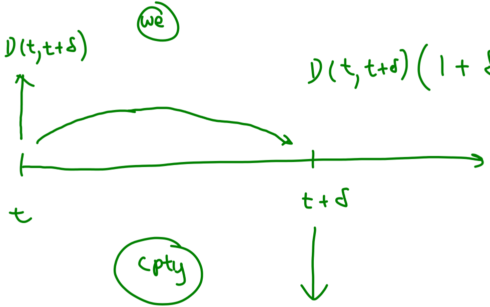
$$1 = D(t, t+\delta)[1+\delta f(t, t+\delta)] \quad \Rightarrow \quad D(t, t+\delta) = \frac{1}{1+\delta f(t, t+\delta)}.$$

Trade

$\delta = \frac{1}{360}$



Collateral



$D(t, t+\delta)(1 + \delta \cdot f) = 1$

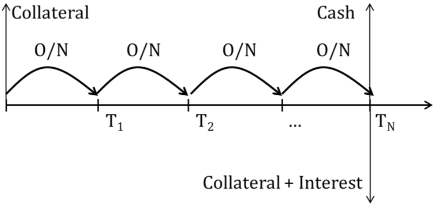
Matching Collateral & Payment Currency

The overnight discount factor is therefore based on the collateral interest rate.

Suppose now that the receivable is more than a day away, at time T in the future. Applying the above relationship recursively, we have

$$D(t,T) = \prod_{i=1}^{\frac{T-t}{\delta}} \frac{1}{1+\delta f(t+(i-1)\delta, t+i\delta)}$$

This formula is nothing more than the expression of term discount factors via the realised daily compound of overnight rates — the overnight index swap (OIS) market allows us to hedge the forwards at time t.



$$\prod_{i=1}^{6m}(1+\delta\cdot f_i) = 1+\Delta\cdot S$$

Matching Collateral & Payment Currency

Example You are expecting to receive a cashflow of \$250,000 in $6m$'s time. We see the following quotes in the market:

Instrument	Quote
$6m$ OIS	1.05%
$6m$ LIBOR	2.1%

6m

1.05%

Using 30/360 day count convention, calculate the PV of this cashflow

1. if the trade is collateralized.
2. if the trade is uncollateralized.

ans.: (1) $PV = 248,694$, (2) $PV = 247,402$

$$\frac{250,000}{1+0.5\cdot 0.0105} \qquad \frac{250,000}{1+0.5\cdot 0.021}$$

Different Collateral & Payment Currency

This is an important case – common collateral currencies are USD, EUR, GBP, JPY. For trades not denominated in one of those currencies, there is almost certainly going to be a mismatch between the payment currency and the collateral currency.

Guideline 2: If a given portfolio has cashflows paid in a single currency x, and the collateral paid in another currency y, and the portfolio can be statically replicated by fixed payments. Then the portfolio should be discounted at

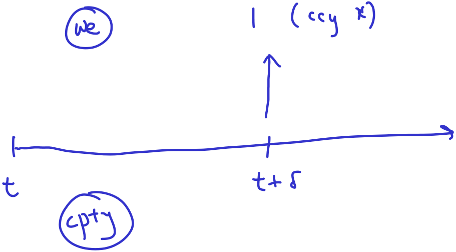
$$D_{x,y}(t,T) = D_y(t,T)\frac{FX_{x,y}(t,T)}{FX_{x,y}(t,t)}.$$

Reason: Again considering the case of single fixed receivable cashflow:

- The currency x's PV is $D_{x,y}(t,t+\delta)$.
- We convert to currency y using the spot FX rate – this is the amount we require in currency y: $FX_{x,y}(t,t)D_{x,y}(t,t+\delta)$.

$\boxed{\text{Trade}}$

ccy x



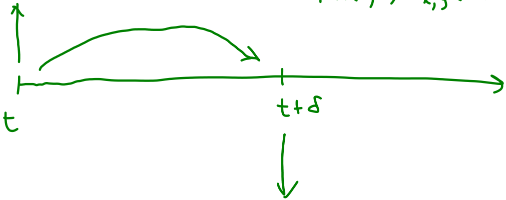
$$1 \cdot FX(t, t+\delta)$$

$$\|$$

$\boxed{\text{Collateral}}$

ccy y

$$FX(t,t) \cdot D_{x,y}(t, t+\delta)$$



$$FX(t,t) \cdot D_{x,y}(t, t+\delta) \cdot (1 + \delta f_y)$$

Different Collateral & Payment Currency

We put on an overnight FX forward hedge, so that we can assume that the $1d$ forward rate $f(t, t+\delta)$ is realised – this can be done at zero cost.

On the next day, we pay (note that we have accrued interest in y)

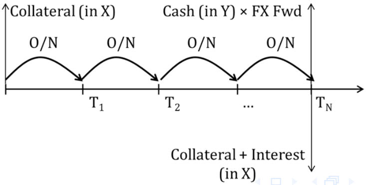
$$FX_{x,y}(t,t)D_{x,y}(t,t+\delta)[1+\delta f_y(t,t+\delta)]$$

Also on the next day, the counterparty pays one unit of currency x. In currency y this is worth $FX_{x,y}(t,t+\delta)$ as a result of our FX hedge.

As before, we equate the 2 payments and solve for the cross-currency discount factor

$$D_{x,y}(t,t+\delta) = \frac{1}{1+\delta f_y(t,t+\delta)} \frac{FX_{x,y}(t,t+\delta)}{FX_{x,y}(t,t)} = D_y(t,t+\delta)\frac{FX_{x,y}(t,t+\delta)}{FX_{x,y}(t,t)}.$$

$$\frac{1}{1+\delta \cdot f_y}$$



Different Collateral & Payment Currency

To extend to the case where the receivable is paid at time T in the future, we note that

- We are solving for the currency x present value $D_{x,y}(t,T)$.
- We convert to currency y using the spot rate (this is the amount we require in y): $FX_{x,y}(t,t)D_{x,y}(t,T)$.
- We put on an FX forward hedge to time T, so that at time T, when the counterparty pays 1 unit of currency x, we have, in currency y, thanks to our FX hedge: $FX_{x,y}(t,T)$.
- Now consider the full period, we pay at time T:

$$FX_{x,y}(t,t)D_{x,y}(t,T)\prod_{i=1}^{\frac{T-t}{\delta}}[1+\delta f_y(t+(i-1)\delta,t+i\delta)].$$

Different Collateral & Payment Currency

We can enter into a currency y overnight index swap (OIS) such that we are effectively paying

$$\frac{FX_{x,y}(t,t)D_{x,y}(t,T)}{D_y(t,T)}.$$

Equating pay and receive leg, and solving, we obtain

$$\begin{aligned} FX_{x,y}(t,T) &= \frac{FX_{x,y}(t,t)D_{x,y}(t,T)}{D_y(t,T)} \\ \Rightarrow \quad D_{x,y}(t,T) &= \frac{FX_{x,y}(t,T)D_y(t,T)}{FX_{x,y}(t,t)}. \end{aligned}$$

IRS Example

$D_0(0,1y) = \frac{1}{\left(1+\frac{0.005}{360}\right)^{360}}$, $D_0(0,2y) = \frac{1}{\left(1+\frac{0.005}{360}\right)^{720}}$

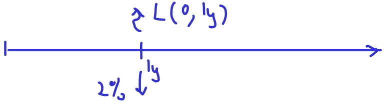
Example Consider the following market quotes of *collateralized* interest rate swaps (annual payment):

Maturity	Instrument	Rate
$1y$	IRS	2%
$2y$	IRS	2.5%

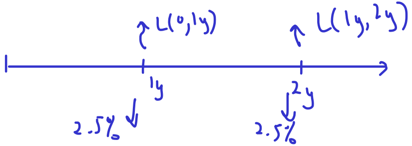
Determine the forward LIBOR rate $L(1y, 2y)$. Assume that collateral are posted in cash of the same denomination of the swap and that the daily overnight compounding rate $f(t_{i-1}, t_i)$ is flat at 0.5%. Use 30/360 day count convention.

ans.: $L(0, 1y) = 2\%$, $L(1y, 2y) = 3\%$.

1y IRS



2y IRS



$$PV_{flt} \neq 1 - \tilde{D}(0,2y)$$

$$\neq 1 - D_0(0,2y)$$

$$PV_{flt} = PV_{fix}$$

$$D_0(0,1y)\cdot \overset{\checkmark}{L(0,1y)} + D_0(0,2y)\cdot \overset{?}{L(1y,2y)} = D_0(0,1y)\cdot 2.5\% + D_0(0,2y)\cdot 2.5\%$$

$$L(0,1y) = \frac{1 - \tilde{D}(0,1y)}{\tilde{D}(0,1y)}, \quad L(1y,2y) = \frac{\tilde{D}(0,1y) - \tilde{D}(0,2y)}{\tilde{D}(0,2y)}$$

Liquidity Value Adjustment

LVA = OIS PV − Libor PV
DF ✓ DF ✗

- Liquidity Value Adjustment (LVA) is the present value of the difference between the risk free rate vs. the collateral rate paid/rebated on the collateral received/posted.

  ✓ = ✗ + LVA

- This corresponds to the cashflow generated daily due to the mark-to-market process between 2 counterparties with mutual collateralization agreement.

- From the perspective of the trading desk, as pricing and valuation architecture for collateral agreements has been rolled out, it is no longer meaningful to calculate LVA as a separate adjustment.

- All discounting should takes collateral agreement into account (instead of assuming risk free discounting and implementing a correction step later).

SINGAPORE MANAGEMENT UNIVERSITY

LMM

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Session 4
LIBOR and Swap Market Models

Tee Chyng Wen

QF605 Fixed Income Securities

About Market Models

> *These models postulate a geometric Brownian motion for the market rates under consideration, such that the Black (1976) formula is recovered for the price of an European option on the market rate.*
>
> *The Black formula is the market standard for calculating prices of European-style interest rate options.*
>
> — Antoon Pelsser

Martingales

Under the **risk-neutral valuation framework**, let V_t denote the value of a security at time t, we write

$$V_0 = e^{-rT}\mathbb{E}^*[V_T].$$

The expectation is taken under the risk-neutral measure associated with the risk-free bond numeraire.

This is valid because the asset ratio is a **martingale**

$$\frac{V_0}{B_0} = \mathbb{E}^*\left[\frac{V_T}{B_T}\right].$$

Under the risk-neutral measure, the best estimate based on the information at time t of the value of the discounted asset price at time T is the discounted asset price at time t:

$$\begin{aligned} M_t &= \mathbb{E}_t^*[M_T], \quad T > t \\ \therefore\ M_0 &= \mathbb{E}^*[M_T] \end{aligned}$$

$$dB_t = r B_t \, dt \qquad r \in \mathbb{R}$$

$$B_T = B_0 e^{rT}$$

$$dB_t = r_t B_t \, dt$$

$$\frac{dB_t}{B_t} = r_t \, dt$$

$$\int_t^T \frac{dB_u}{B_u} = \int_t^T r_u \, du$$

$$\Rightarrow B_T = B_t e^{\int_t^T r_u \, du}$$

$$\frac{V_t}{B_t} = \mathbb{E}^*\left[\frac{V_T}{B_T}\right]$$

$$V_t = \mathbb{E}^*\left[\frac{B_t V_T}{B_T}\right]$$

Zero-Coupon Bond as Numeraire

Suppose the interest rate r is not a constant but a function of time (i.e. r_t), then under martingale pricing, we can value a financial contract V_t under the risk-neutral measure associated to the risk-free money market account numeraire as

$$V_t = \mathbb{E}^* \left[e^{-\int_t^T r_u du} V_T \right].$$

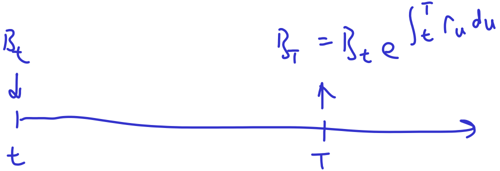
The expectation is evaluated under the probability measure $\mathbb{Q}^*$, which is associated to the **money market account numeraire** B_t.

Instead of using the value of the money market account B_t as a numeraire, the prices of **discount bonds** $D(t,T)$ **can also be used as a numeraire**.

A very convenient choice is to use the discount bond with maturity T as numeraire (co-inciding with the payoff time of the contract). A zero-coupon discount bond is given by

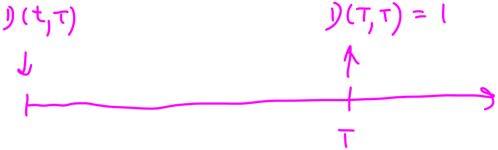
$$D(t,T) = \mathbb{E}^* \left[e^{-\int_t^T r_u du} \right].$$

Money market account



$\mathbb{Q}^*$: $\dfrac{V_t}{B_t} = \mathbb{E}^*\left[\dfrac{V_T}{B_T}\right]$

Zero coupon bond



$\mathbb{Q}^T$: $\dfrac{V_t}{D(t,T)} = \mathbb{E}^T\left[\dfrac{V_T}{D(T,T)}\right] = \mathbb{E}^T\left[V_T\right]$

$$\mathbb{Q}^* : \qquad \frac{D(t,T)}{B_t} = \mathbb{E}^*\left[\frac{\cancelto{1}{D(T,T)}}{B_T} \right]$$

$$D(t,T) = \mathbb{E}^*\left[\frac{B_t}{B_T} \right] = \mathbb{E}^*\left[e^{-\int_t^T r_u \, du} \right]$$

Zero-Coupon Bond as Numeraire

If we denote the probability measure associated to the numeraire $D(t, T)$ by $\mathbb{Q}^T$, we can apply the **change of numeraire theorem** to obtain

$\mathbb{Q}^T$:
$$\frac{V_t}{D(t,T)} = \mathbb{E}^T\left[\frac{V_T}{D(T,T)}\right].$$

However, at time T the price of the discount bond $D(T,T) = 1$, and so

$$V_t = D(t,T)\mathbb{E}^T\left[V_T\right]. \quad = \mathbb{E}^*\left[e^{-\int_t^T r_u du}\right] \mathbb{E}^T\left[V_T\right]$$

In words, by changing the measure from $\mathbb{Q}^*$ to $\mathbb{Q}^T$, we have managed to express the expectation of the discounted payoff as a discounted expectation of the payoff.

$\Rightarrow$ We have therefore eliminated the problem of the correlation between the discounting term and the payoff term.

LIBOR Market Model

In the LIBOR market, we can choose to lend (deposit) capital and earn the LIBOR rate, which is the rate for unsecured borrowing and lending between banks.

If you lend into the LIBOR market for a period of length Δ, you earn $1 + \Delta \cdot L$ one period later, where L denote the LIBOR rate you invested in.

Let $D(t, T)$ denote the value at time t of a discount bond which pays 1 at maturity T, the LIBOR rate and discount factor is related by

$$1 = (1 + \Delta \cdot L) \cdot D(0, \Delta).$$

Suppose we are at time t, and we commit into a forward LIBOR rate for the period $[T_i, T_{i+1}]$. We have the following relation

$$\begin{aligned} D(t, T_i) &= (1 + \Delta_i L_t(T_i, T_{i+1}))D(t, T_{i+1}) \\ \Rightarrow \quad L_t(T_i, T_{i+1}) &= \frac{1}{\Delta_i} \frac{D(t, T_i) - D(t, T_{i+1})}{D(t, T_{i+1})}. \end{aligned}$$

$$\frac{S_t}{B_t} = \mathbb{E}^*\left[\frac{S_T}{B_T}\right]$$

$$\mathbb{Q}^* : \quad dS_t = r S_t \, dt + \sigma S_t \, dW_t^*$$

LIBOR Market Model

$dL_i(t) = \underline{\ ?\ }$ $\mathbb{Q}^T$

In most markets, only one specific LIBOR tenor is liquidly traded. In Singapore, this will be the 6m SIBOR or SOR rate.

In other words, for all practical purposes, $[T_i, T_{i+1}]$ are not arbitrary. Let us denote $L_i(t) = L_t(T_i, T_{i+1})$ and $D_i(t) = D(t, T_i)$. Now consider the process

$$\Delta_i L_i(t) = \frac{D_i(t) - D_{i+1}(t)}{D_{i+1}(t)}.$$

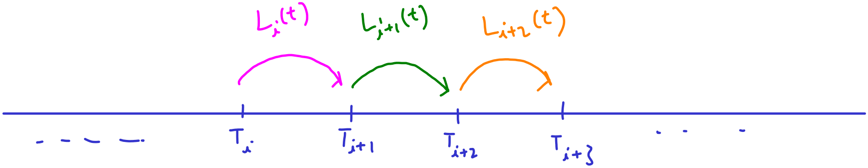
This is a **ratio of marketed assets**. If we take the discount bond $D_{i+1}(t)$ as numeraire, then under the martingale measure $\mathbb{Q}^{i+1}$ associated with the numeraire $D_{i+1}(t)$, the process $\Delta_i L_i(t)$ must be a martingale.

Since Δ_i is a constant, the process $L_i(t)$ must be a martingale under $\mathbb{Q}^{i+1}$. This gives rise to the **LIBOR Market Model (LMM)**

$dF_t = \sigma F_t\, dW_t \quad \Rightarrow \quad F_t = F_0\, e^{-\frac{1}{2}\sigma^2 t + \sigma W_t}$

$$dL_i(t) = \sigma_i L_i(t) dW^{i+1}(t) \quad \Rightarrow \quad L_i(t) = L_i(0) \exp\left[-\frac{1}{2}\sigma_i^2 t + \sigma_i W^{i+1}(t)\right],$$

where W^{i+1} is a Brownian motion under $\mathbb{Q}^{i+1}$.



$$Q^{i+1} : \quad dL_i(t) = \sigma_i L_i(t)\, dW^{i+1}(t)$$

$$Q^{i+2} : \quad dL_{i+1}(t) = \sigma_{i+1} L_{i+1}(t)\, dW^{i+2}(t)$$

$$Q^{i+3} : \quad dL_{i+2}(t) = \sigma_{i+2} L_{i+2}(t)\, dW^{i+3}(t)$$

$$\vdots$$