Using Stock as the Numeraire Asset

Consider a risk-free money market account B_t following

$$dB_t = rB_t dt,$$

and a risky asset S_t under empirical measure $\mathbb{P}$ following

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{1}$$

where W_t is a standard Brownian motion under $\mathbb{P}$.

1 B_t as Numeraire (easy)

To move into the risk-neutral measure $\mathbb{Q}^*$ associated with the B_t numeraire, we let $X_t = \frac{S_t}{B_t}$, and use Itô's formula to derive the stochastic differential equation

$$dX_t = (\mu - r)X_t dt + \sigma X_t dW_t.$$

Under the risk-neutral measure $\mathbb{Q}^*$ associated with the numeraire B_t, the X_t process should be a martingale, so that

$$\begin{aligned} dX_t &= \sigma X_t \left(dW_t + \frac{\mu - r}{\sigma} dt \right) \\ &= \sigma X_t dW_t^*, \end{aligned}$$

where W_t^* is a standard Brownian motion under $\mathbb{Q}^*$. Hence

$$dW_t^* = dW_t + \frac{\mu - r}{\sigma} \qquad \Rightarrow \qquad dW_t = dW_t^* - \frac{\mu - r}{\sigma},$$

and substituting this back to Equation (1), we have the stock price model under $\mathbb{Q}^*$:

$$dS_t = rS_t dt + \sigma S_t dW_t^*.$$

The solution to this is

$$S_T = S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T^*},$$

and we can verify the martingale property by evaluating the expectation under the $\mathbb{Q}^*$ measure:

$$\begin{aligned} \mathbb{E}^* \left[\frac{S_T}{B_T} \right] &= \mathbb{E}^* \left[\frac{S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T^*}}{B_0 e^{rT}} \right] \\ &= \frac{S_0}{B_0} e^{-\frac{\sigma^2 T}{2}} \mathbb{E}^* \left[e^{\sigma W_T^*} \right] \\ &= \frac{S_0}{B_0} e^{-\frac{\sigma^2 T}{2}} e^{\frac{\sigma^2 T}{2}} = \frac{S_0}{B_0}. \end{aligned}$$

2 S_t as Numeraire

To move into the risk-neutral measure $\mathbb{Q}^S$ associated with the stock numeraire, we let $Y_t = \frac{B_t}{S_t}$, and use Itô's formula to derive the stochastic differential equation

$$dY_t = (r + \sigma^2 - \mu)Y_t dt - \sigma^2 Y_t dW_t.$$

This should be a martingale under $\mathbb{Q}^S$, so we write

$$\begin{aligned} dY_t &= -\sigma^2 Y_t \left(dW_t - \frac{r + \sigma^2 - \mu}{\sigma} dt \right) \\ &= -\sigma^2 Y_t dW_t^S, \end{aligned}$$

where W_t^S is a standard Brownian motion under $\mathbb{Q}^S$. Hence

$$dW_t^S = dW_t - \frac{r + \sigma^2 - \mu}{\sigma} dt \qquad \Rightarrow \qquad dW_t = dW_t^S + \frac{r + \sigma^2 - \mu}{\sigma} dt,$$

and substituting this back to Equation (1), we have the stock price model under $\mathbb{Q}^S$:

$$dS_t = (r + \sigma^2) S_t dt + \sigma S_t dW_t^S. \tag{2}$$

The solution to this is

$$S_T = S_0 e^{\left(r + \frac{\sigma^2}{2}\right)T + \sigma W_T^S},$$

and we can verify the martingale property by evaluating the expectation under the $\mathbb{Q}^S$ measure:

$$\begin{aligned} \mathbb{E}^S \left[\frac{B_T}{S_T} \right] &= \mathbb{E}^S \left[\frac{B_0 e^{rT}}{S_0 e^{\left(r + \frac{\sigma^2}{2}\right)T + \sigma W_T^S}} \right] \\ &= \frac{B_0}{S_0} e^{-\frac{\sigma^2 T}{2}} \mathbb{E}^S \left[e^{-\sigma W_T^S} \right] \\ &= \frac{B_0}{S_0} e^{-\frac{\sigma^2 T}{2}} e^{\frac{\sigma^2 T}{2}} = \frac{B_0}{S_0}. \end{aligned}$$

3 Adding Another Risky Asset

Now suppose there are two stocks, both of which we have already switched to the $\mathbb{Q}^*$ measure:

$$\begin{aligned} dS_t^1 &= r S_t^1 dt + \sigma_1 S_t^1 dW_t^{*,1} \\ dS_t^2 &= r S_t^2 dt + \sigma_2 S_t^2 dW_t^{*,2} \end{aligned}$$

where $dW_t^{*,1} \cdot dW_t^{*,2} = \rho\, dt$. Let dZ_t be an independent Brownian motion from $dW_t^{*,1}$, we substitute for $dW_t^{*,2} = \rho dW_t^{*,1} + \sqrt{1 - \rho^2} dZ_t$ to obtain:

$$\begin{aligned} dS_t^1 &= r S_t^1 dt + \sigma_1 S_t^1 dW_t^{*,1} \\ dS_t^2 &= r S_t^2 dt + \rho \sigma_2 S_t^2 dW_t^{*,1} + \sqrt{1 - \rho^2} \sigma_2 S_t^2 dZ_t \end{aligned}$$

Consider the first stock S^1: if we now want to use it as the numeraire by working under the $\mathbb{Q}^{S^1}$ measure, we need $\frac{B_t}{S_t^1}$ to be a martingale, then we will get Equation (2) in the previous section. Comparing between

$$\begin{aligned} \mathbb{Q}^* : &\qquad dS_t^1 = r S_t^1 dt + \sigma_1 S_t^1 dW_t^{*,1} \\ \mathbb{Q}^{S^1} : &\qquad dS_t^1 = (r + \sigma_1^2) S_t^1 dt + \sigma_1 S_t^1 dW_t^{S^1} \end{aligned}$$

we see that

$$dW_t^{S^1} = dW_t^{*,1} - \sigma_1 dt.$$

We can also verify this relationship from the Change of Numeraire Theorem. Note that to change the measure from $\mathbb{Q}^*$ to $\mathbb{Q}^{S^1}$, we need the Radon-Nikodyn derivative

$$\frac{d\mathbb{Q}^{S^1}}{d\mathbb{Q}^*} = \frac{S_T^1/S_0^1}{B_T/B_0} = \frac{\frac{S_0^1 e^{\left(r-\frac{\sigma_1^2}{2}\right)T+\sigma_1 W^{*,1}}}{S_0^1}}{\frac{B_0 e^{rT}}{B_0}} = e^{\sigma_1 W_T^{*,1} - \frac{\sigma_1^2 T}{2}}.$$

Girsanov's Theorem states that

> If W_t is a $\mathbb{P}$-Brownian, and $\mathbb{Q}$ is equivalent to $\mathbb{P}$, then
>
> $$\begin{aligned}\frac{d\mathbb{Q}}{d\mathbb{P}} &= \exp\left(-\kappa W_T - \frac{\kappa^2 T}{2}\right)\\ dW_t^Q &= dW_t + \kappa dt\end{aligned}$$
>
> And W_t^Q is a $\mathbb{Q}$-Brownian.

Comparing

$$\exp\left(\sigma_1 W_T^{*,1} - \frac{\sigma_1^2 T}{2}\right) \qquad \text{to} \qquad \exp\left(-\kappa W_T - \frac{\kappa^2 T}{2}\right),$$

we can see clearly that $\kappa = -\sigma_1$, so

$$dW_t^{S^1} = dW_t^{*,1} - \sigma_1 dt,$$

where $W_t^{S^1}$ is a $\mathbb{Q}^{S^1}$-Brownian. Substituting this back to the differential equation for dS_t^2, we have

$$\begin{aligned}dS_t^2 &= rS_t^2 dt + \rho\sigma_2 S_t^2 dW_t^{*,1} + \sqrt{1-\rho^2}\sigma_2 S_t^2 dZ_t\\ &= (r+\rho\sigma_1\sigma_2)S_t^2 dt + \rho\sigma_2 S_t^2 dW_t^{S^1} + \sqrt{1-\rho^2}\sigma_2 S_t^2 dZ_t\end{aligned}$$

The solution is given by

$$S_T^2 = S_0^2 \exp\left[\left(r+\rho\sigma_1\sigma_2 - \frac{\sigma_2^2}{2}\right)T + \rho\sigma_2 W_T^{S^1} + \sqrt{1-\rho^2}\sigma_2 Z_T\right].$$

We can verify the martingale relationship:

$$\begin{aligned}\mathbb{E}^{S^1}\left[\frac{S_T^2}{S_T^1}\right] &= \mathbb{E}^{S^1}\left[\frac{S_0^2\exp\left[\left(r+\rho\sigma_1\sigma_2-\frac{\sigma_2^2}{2}\right)T+\rho\sigma_2 W_T^{S^1}+\sqrt{1-\rho^2}\sigma_2 Z_T\right]}{S_0^1\exp\left[\left(r+\frac{\sigma_1^2}{2}\right)T+\sigma_1 W_T^{S^1}\right]}\right]\\
&= \frac{S_0^2}{S_0^1}\frac{\exp\left[\left(r+\rho\sigma_1\sigma_2-\frac{\sigma_2^2}{2}\right)T\right]}{\exp\left[\left(r+\frac{\sigma_1^2}{2}\right)T\right]}\mathbb{E}^{S^1}\left[\frac{\exp\left(\rho\sigma_2 W_T^{S^1}+\sqrt{1-\rho^2}\sigma_2 Z_T\right)}{\exp\left(\sigma_1 W_T^{S^1}\right)}\right]\\
&= \frac{S_0^2}{S_0^1}e^{\left(\rho\sigma_1\sigma_2-\frac{\sigma_2^2}{2}-\frac{\sigma_1^2}{2}\right)T}\mathbb{E}^{S^1}\left[e^{(\rho\sigma_2-\sigma_1)W_T^{S^1}+\sqrt{1-\rho^2}\sigma_2 Z_T}\right]\\
&= \frac{S_0^2}{S_0^1}e^{\left(\rho\sigma_1\sigma_2-\frac{\sigma_2^2}{2}-\frac{\sigma_1^2}{2}\right)T}e^{\frac{(\rho\sigma_2-\sigma_1)^2}{2}T+\frac{(1-\rho^2)\sigma_2^2}{2}T}\\
&= \frac{S_0^2}{S_0^1}e^{\left(\rho\sigma_1\sigma_2-\frac{\sigma_2^2}{2}-\frac{\sigma_1^2}{2}\right)T}e^{\frac{\rho^2\sigma_2^2-2\rho\sigma_1\sigma_2+\sigma_1^2}{2}T+\frac{(1-\rho^2)\sigma_2^2}{2}T} = \frac{S_0^2}{S_0^1}.\end{aligned}$$