Convexity



Session 1: Bond Market and Bond Risk Management Tee Chyng Wen

QF605 Fixed Income Securities



Compounding

Remember that time is money.

— Benjamin Franklin (1706–1790)

Gentlemen prefer bonds.

— Andrew Mellon (1855-1937)



The convention used in the market is to treat all quoted interests as <u>annualized</u>, i.e. the total amount of interest paid for a whole year.

Simple/Linear Compounding Investing \$1 at the rate R for a period Δ yields

$$\mathsf{Return} = 1 + \Delta \cdot R$$

Discrete Compounding Interest is compounded at discrete frequency:

$$\mathsf{Return} = \left(1 + \frac{R}{m}\right)^{m \times n}$$

where m is the number of payments per year and n is the total number of years.

Continuous Compounding Interest is continuously compounded:

$$\mathsf{Return} = e^{R \cdot T}$$

where T is the number of years. Continuous compounding is often easier to deal with mathematically.

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We can also turn this question around and ask how much money we need today to compound to \$1 at the maturity date.

This brings in the concept of **discount factor**: D(t,T), denoting the discount factor that discounts from T to back t, in other words, how much money we need at t to compound to \$1 at T.

Depending on the compounding convention, we have

$$D(0, \Delta) = \frac{1}{1 + \Delta \cdot R}$$

$$D(0, n) = \frac{1}{\left(1 + \frac{R}{m}\right)^{m \times n}}$$

$$D(0, T) = e^{-R \cdot T}$$

- \Rightarrow Interest rate is generally positive (R > 0), so discount factors are generally less than 1.
- ⇒ Liquidity Preference: people generally prefer to receiver money earlier rather than later, hence D(0,T) should be decreasing in T.

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Compound interest is the <u>interest on interest</u>, hence the frequency at which the interest is being compounded is important.

Suppose we invest for one year at an interest rate of 8%. Different compounding frequency leads to different investment outcome.

Compound. freq.	$\frac{r}{m}$	$m \times N$	<i>FV</i> of \$100
Annual	8%/1	1×1	$100 \times (1 + 0.08)^1 = 108$
Semi-annual	8%/2	2×1	$100 \times (1 + 0.04)^2 = 108.16$
Quarterly	$8\%/_{4}$	4×1	$100 \times (1 + 0.02)^4 = 108.24$
Monthly	8%/12	12×1	$100 \times (1 + 0.00667)^{12} = 108.3$
Continuous	8%	1	$100 \times e^{0.08 \times 1} = 108.3287$

Money makes money. And the money that money makes, makes money.

— Benjamin Franklin (1706–1790)

Compound interest is the eighth wonder of the world.

— Attributed to Albert Einstein (1879–1955)



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Effective Annual Rate

There is a distinction between <u>stated</u> annual interest rate and **effective annual** rate (EAR) — the interest actually earned over a year.

$$r_{EAR} = \left(1 + \frac{r_S}{m}\right)^m - 1.$$

In our example, \$1 investment that earns 8.16% compounded annually gives the same FV as a \$1 investment that earns 8% compounded semiannually.

For an 8% stated annual interest rate with semi-annual compounding, the EAR is 8.16%

Special Case: **Bond Equivalent Yield** is conventionally used in the U.S. fixed-income market, and restates the yield in terms of <u>semi-annual basis</u>.

$$r_{BEY} = \left[\left(1 + \frac{r_S}{m} \right)^{\frac{m}{2}} - 1 \right] \times 2$$

Example The effective annual yield on a fixed-income instrument is 9%. What is the yield on a bond equivalent basis? (ans. 8.81%)

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Side Note: Exponential Constant

Bond Market

An account starts with \$1 and pays 100% per year. If the interest is credited once, at the end of the year, the value of the account at year end will be \$2. What happens if the interest is computed and credited more frequently during the year?



Jacob Bernoulli (1655–1705)

As the payment frequency increases, the interest paid out needs to be divided by the number of payment, but the interest earned is multiplied by the same amount. Taking the limit yields:

$$\lim_{n \to \infty} \left(1 + \frac{1}{n} \right)^n = e$$

Day Count Conventions

Calculating Δ is important because it tells us how much interest we have accrued over a given period.

Three very common ways of calculating this are

- Act/Act: Read "actual/actual", simply count the number of calendar days.
- 30/360: Read "thirty three-sixty", assumes that there are 30 days in a month and 360 days in a year.
- Act/360: Read "actual three-sixty", each months has the right number of days, but there are only 360 days in a year.

Business Day Convention If payment date falls on a non-business day, then we need to move the date using business day convention:

- Actual: on the actual day, regardless.
- Following(/Previous): rolled onto the following business day.
- Modified Following(/Previous): rolled only the following business day, except if it is the next calendar month, in this case roll backwards.

10/40

Day Count Conventions

Example A simple interest rate contract whereby you are paid $\Delta \times 4.5\%$ (Δ is the day count fraction) gives you the optionality to choose one of the following day count convention:

• 30/360

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- Act/360
- Act/365
- Act/Act

The 4.5% rate accrues from 1-Jan-2024 to 30-June-2024. Which would you choose and why?

Day Count Conventions

Example Everything else being equal, should a 1m expiry call option on a non-dividend paying stock be more valuable on

(A) 15 Jan

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(B) 15 Feb



Basic Instruments in the Bond Market

- Zero-coupon Bonds: These are pure discount instruments making a single payment on the maturity date T.
 - ⇒ The final payment is commonly referred to as principal, face value, or notional.
 - ⇒ They tend to have very short maturities (e.g. treasury bills or commercial papers).
- Coupon Bonds In addition to the final notional, these also pay a <u>periodic</u> fixed coupon over the life of the bond.
 - \Rightarrow Tends to have longer maturity.
 - ⇒ US convention: zero-coupon bonds are called <u>bills</u>, bonds with maturity 2-10 years are called <u>notes</u>, bonds with maturities longer than 10 year are called <u>bonds</u>.
- Floating Rate Bonds Coupon payments are not fixed but depend on some benchmark interest rate (floating).
 - $\Rightarrow\,$ E.g. quarterly compounded 3m benchmark rate plus 10 basis points.

Zero-coupon Bonds

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The importance of the zero-coupon bonds is that we can write any deterministic set of cashflows as a linear multiple of such bonds.

Bond Market

For instance, suppose we agreed to lend a company a principal N at T=1 and the company is to pay a fixed six-monthly annualized rate of 10% for 5 years. and then at T=6 return the principal.

We can write this transaction in terms of zero-coupon bonds:

$$-N \cdot D(0,1) + \sum_{i=1}^{10} 0.05 \cdot N \cdot D(0,1+0.5i) + N \cdot D(0,6)$$

The convenient property is that—upon substituting the values of the zero-coupon bonds—we have a present value for the entire transaction.

If this number is 0, the loan is at fair value. This technique for valuing trades is very standard and is called present-valuing (or PV).

Zero rate (or spot rate) is the yield-to-maturity of a zero-coupon bond. From the price D(t,T), we can calculate the continuously compounded **spot rate** R(t,T) that is set at t and pays at T.

Continuously compounded zero rate

$$D(t,T) = e^{-R(t,T)(T-t)} \quad \Rightarrow \quad R(t,T) = -\frac{\log D(t,T)}{T-t}.$$

By no-arbitrage, the zero rate should be the interest rate earned on any investment at time t and pays at $T\Rightarrow$ useful for discounting.

Discretely compounded zero rate

$$D(t,T) = \frac{1}{\left(1 + \frac{R(t,T)}{m}\right)^{(T-t)\times m}} \quad \Rightarrow \quad R(t,T) = m\left[D(t,T)^{-\frac{1}{m(T-t)}} - 1\right]$$

- \Rightarrow If R(t,T) is specified for all values of T, then we have a zero rate curve.
- $\Rightarrow R(t,T)$ is the interest rate implicit in a discount factor/zero-coupon bond D(t,T).

For coupon bonds, **yield-to-maturity (YtM)** is a more convenient interest rate measure. The YtM is defined as the interest rate such that if all the different cash flows are discounted at this rate (y), then the resulting net present value is equal to the current price of the bond. Assuming continuous compounding:

$$B = \sum_{i=1}^{N} c_i e^{-y \cdot T_i}$$

Here c_i is the cash flow paid at time T_i .

The price of the bond can be regarded as a function of the yield. Note that the price of the bond B(y) is a decreasing function of the yield y:

$$\frac{\partial B}{\partial y} = B'(y) = -\sum_{i=1}^{N} T_i c_i e^{-yT_i} < 0.$$

Therefore, if the yield of the bond goes up, then the price of the bond goes down, and vice versa.

The yield-to-maturity or internal rate of return or redemption yield is the <u>discount rate</u> that makes the market price of a bond equal to the discounted value of its future cashflows.

Although continuous compounding is elegant, in practice we use <u>discrete</u> compounding to compute bond prices.

Compare the formula with zero rates and formula with the yield-to-maturity:

$$B = \sum_{i=1}^{N} \frac{c_i}{\left(1 + \frac{y}{m}\right)^{m \times T_i}} = \sum_{i=1}^{N} \frac{c_i}{\left(1 + \frac{R_i}{m}\right)^{m \times T_i}}$$

Notice that the <u>yield</u> is a blend or a kind of <u>average of the different zero rates</u> associated with the cashflows.

In other words, the yield must be between the highest and lowest zero rates.



Bond Market

Compounding

Consider a 1.5y semi-annual coupon bond with a coupon of 8.5%. Compare the two formulas for the bond:

$$1.043066 = \frac{0.0425}{\left(1 + \frac{0.0554}{2}\right)^1} + \frac{0.0425}{\left(1 + \frac{0.0545}{2}\right)^2} + \frac{1.0425}{\left(1 + \frac{0.0547}{2}\right)^3}$$
$$1.043066 = \frac{0.0425}{\left(1 + \frac{0.054704}{2}\right)^1} + \frac{0.0425}{\left(1 + \frac{0.054704}{2}\right)^2} + \frac{1.0425}{\left(1 + \frac{0.054704}{2}\right)^3}$$

- \Rightarrow Using the zero rates 5.54%, 5.45%, and 5.47%, the bond price is 1.043066 per dollar par value.
- \Rightarrow The implied yield of 5.4704% is a kind of average of the discount rates 5.54%, 5.45%, and 5.47%,
- ⇒ The YtM is useful because it provides a way of converting a price into something resembling an interest rate. This sometimes makes it easier to compare how cheap or expensive a bond is.

<u>Par Yield</u> is the coupon rate that makes the value of the bond equal to its **face** value. In other words, par yield is the value of the coupon rate such that

$$B = \mathsf{Face} \; \mathsf{Value}$$

Bonds are typically issued at par.

Under discrete compounding, a bond will trade at par whenever its yield-to-maturity is equal to the coupon rate.

Example Consider the following annual coupon bond:

$$B = \sum_{i=1}^{N} \frac{c_i}{(1+y)^i}.$$

Show that when the yield-to-maturity y is equal to its coupon rate (c=100y), then B=100.

Bootstrapping a Bond Curve

Example Suppose we have 3 coupon bonds (annual coupon):

Bond	Maturity	Coupon	Price
A	1y	5	101
B	2y	6.5	102
C	3y	7	102.5

What can we say about D(0,1), D(0,2), and D(0,3) and the continuously compounded zero rates R(0,1), R(0,2), R(0,3)?

ans.:
$$D(0,1) = 0.9619$$
, $D(0,2) = 0.899$, $D(0,3) = 0.8362$

Bootstrapping a Bond Curve

Compounding

Example Following the previous question, suppose we have another coupon bond D that pays a coupon of 3 annually, with 2 years left to maturity.

- **1** What should be the no-arbitrage price for this bond?
- 2 Suppose this bond trades at \$94. Form an arbitrage.

ans.:
$$B_D = 95.483$$
. $w_A = 0.0313$, $w_B = -0.967$.

Bootstrapping a Bond Curve

Example Consider the following bonds from the same issuer:

Maturity (years)	Coupon	Price
0.25	0	97.5
0.50	0	94.9
1.00	0	90.0
1.50	8 (semi-annual)	96.0
2.00	12 (semi-annual)	101.6

Bootstrap the continuously compounded zero rate curve.

ans.:
$$D(0, 1.5) = 0.85196$$
, $D(0, 2) = 0.8056$.

On-the-run treasuries are the <u>most recently issued</u> US Treasury bonds of a particular maturity. Because on-the-run issues are the <u>most liquid</u>, they typically trade at a slight <u>premium</u> and therefore yield a little less than their **off-the-run** counterparts.

Long-Term Capital Management (LTCM) successfully exploited this price differential through an arbitrage strategy that involves selling on-the-run treasuries and buying off-the-run treasuries between 1994–1998.

The idea is that this disparity in pricing should not persist. At the same time, there was an economic justification underpinning the disparity.

Convergence Arbitrage

Compounding

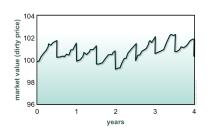
- Trade made between two different securities that tend to become more similar over time. Because they differ from one another, there is daily mark-to-market risk.
- But over the longer term the prices will tend to move closer and closer to each other because it will become increasingly obvious that the two securities are similar.

Clean Price vs. Dirty Price

The price of bonds quoted in the market are clean prices. That is, they are quoted without any accrued interest. The accrued interest is the amount of interest that has built up since the last coupon payment.

The actual payment is called the **dirty price** and is the sum of the quoted clean price and the accrued interest.

The cash price of a coupon bond falls by the amount of the coupon just after the coupon has been paid, this means that the cash price have a "saw" pattern.

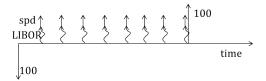


Clean Price = Dirty Price - Al

$${\rm AI} = c \times \frac{\rm days~since~last~coupon~date}{\rm days~between~coupon~dates}$$

Floating Rate Notes (FRN)

- FRN is a bond that pays a coupon linked to a variable interest rate index - typically LIBOR (or Euribor for EUR).
- FRN eliminates most of the interest rate sensitivity, making it almost a pure credit play - the price action of a FRN is driven mostly by the changes in the market-perceived credit quality of the note issuer.
- While the senior short-term floaters of AA-rated banks pay a coupon close to LIBOR and trade at a price close to par, in the credit markets, many floaters are issued by corporates with much lower credit ratings.
- Many AA-rated banks also issue FRN that are subordinate in the capital structure.



- Either way, investors require a higher yield to compensate them for the increased credit risk. At the same time, the coupons of the bonds must be discounted at a higher interest rate than LIBOR to account for this higher credit risk.
- Therefore, in order to issue the note at par, the coupon on the FRN must be set at a fixed <u>spread over LIBOR</u>, this is known as the **par floater** spread.
- ullet Example: 2y FRN with semi-annual payment paying LIBOR + 50bps.
- FRN has a much lower interest rate sensitivity than fixed-rate bonds. If LIBOR moves up, the increment in coupons will be offset by the higher discount rate and vice versa.
- On coupon dates, whether the price of a FRN is above or below par is determined by its par floater spread. If it is above the fixed spread in the coupon, then it will trade below par, and vice versa.

Floating Rate Notes (FRN)

- Between coupon dates, the LIBOR component of the impending coupon payment is fixed and its value is known today.
- However, it is discounted with LIBOR + spread, giving us interest rate exposure for this single cashflow - this is known as reset risk. It is usually small and drops to 0 as we approach the payment date.
- A large proportion of the FRN is issued by banks to satisfy their bank capital requirements.
- A large number of corporate and emerging market bonds are issued in FRN format.
- FRN is a way for credit investor to buy bond and take exposure to credit without taking exposure to interest rate movements.
- Most bonds are fixed rate, and so incorporate a significant interest rate sensitivity. We can turn them into pure credit exposure using asset swaps.

Floating Rate Notes (FRN)

By no-arbitrage, just after an interest payment has been made, the price of a floating rate bond must be equal to par if credit outlook remains constant.

Consider the following trading strategy, which requires an initial amount of cash equal to par:

- ⇒ Invest the cash at the floating rate, until the date of the next coupon payment on the floating rate bond.
- ⇒ When you receive the cash and interest, reinvest the cash amount at the new floating rate until the next coupon payment date.
- ⇒ At maturity of the floating rate bond, you receive the cash equal to par, plus the last interest payment.

Since the payoff of this strategy is equal to the payoff of buying the floating rate bond, the price of the bond must be equal to par.

Just before a coupon is paid, the price of the bond must be equal to par plus the value of the coupon.



Bond Duration

Compounding

Duration and **convexity** are two of the most important parameters to estimate when investing in a bond, other than its yield.

 \Rightarrow Bond price is a decreasing and convex function of the bond yield.

The most frequently used bond duration measure in practice is **modified** duration.

The **modified duration** of a bond is the rate of change of the price of the bond with respect to the yield of the bond, normalized by the price of the bond, and with opposite sign, i.e.

$$D = -\frac{1}{B} \frac{\partial B}{\partial y}.$$

The price B of the bond is considered a function of the yield y and with cash flows c_i at time t_i .

Modified duration measures the rate of return of the bond price for small changes in the bond yield

Bond Duration

Compounding

Under continuous compounding, we have

$$B = \sum_{i=1}^{n} c_i e^{-yt_i} \qquad \Rightarrow \qquad \frac{\partial B}{\partial y} = -\sum_{i=1}^{n} t_i c_i e^{-yt_i},$$

and it follows that the modified duration D of the bond is

$$D = \frac{1}{B} \sum_{i=1}^{n} t_i c_i e^{-yt_i}.$$

From here we see that this is just a $\underline{\text{"time weighted"}}$ average of the of the cashflows' NPV.

Example Suppose the bond yield is 4% and continuously compounded, what is the modified duration of a zero coupon bond paying maturing at the end of the 2^{nd} year. Ans.: 2

Example A 2y coupon bond is trading at 102. The continuously compounded yield is 5.2756%, and the coupon is 6.5 paid annually. Calculate the modified duration of this bond. Ans.: 1.9396

Bond Duration

Compounding

The Macaulay Duration of a bond is defined as the weighted averaged of the cash flow times, with weights equal to the value of the corresponding cash flow discounted with respect to the yield of the bond, i.e.

$$D_{\mathsf{Mac}} = \frac{1}{B} \sum_{i=1}^{n} t_i c_i \mathsf{Discount}(t_i, y),$$

where Discount (t_i, y) is the discount factor corresponding to time t_i , computed with respect to the yield of the bond.

Based on the definition, we see that if the bond yield is continuously compounded, then the Macaulay duration is exactly equal to the modified duration.

If the bond yield is discretely compounded (as is the case in practice), the modified duration is:

$$D = \frac{D_{\mathsf{Mac}}}{1 + \frac{y}{m}}.$$

- Macaulay duration estimates the point in time when the future value of the bond would remain unchanged for small parallel changes in the zero rate curve.
- Macaulay duration is defined as the "time weighted" average of the cashflows NPV.
- For continuous compounding, the discount factors are Discount $(t_i, y) = e^{-yt_i}$, and it follows that $D_{Mac} = D$.
 - ⇒ In other words, the Macaulay duration and the modified duration of a bond have the same value if interest is compounded continuously.
 - ⇒ This is not the case if interest is compounded discretely, when the modified duration of a bond is smaller than its Macaulay duration.



Using Modified Duration

Compounding

Let Δy be the change in the yield of the bond, and let

$$\Delta B = B(y + \Delta y) - B(y)$$

be the corresponding change in the price of the bond. The <u>discretized</u> version of the modified duration formula becomes

$$\begin{split} D &\approx -\frac{1}{B} \cdot \frac{B(y + \Delta y) - B(y)}{\Delta y} = -\frac{\Delta B}{B \cdot \Delta y} \\ &\Rightarrow \quad \frac{\Delta B}{B} \approx -D\Delta y \end{split}$$

In other words, the return $\frac{\Delta B}{B}$ of the bond can be approximated by the duration of the bond multiplied by the parallel shift in the yield curve, with opposite sign.

For very small changes in the yield of the bond, this approximation formula is accurate. For larger changes, convexity is used to better capture the effect of changes in the bond yield.

Bond Convexity

Compounding

The **Convexity** C of a bond with price B and yield y is defined as

$$C = \frac{1}{B} \frac{\partial^2 B}{\partial y^2}.$$

Under continuous compounding, we can see that

$$C = \frac{1}{B} \sum_{i=1}^{n} t_i^2 c_i e^{-yt_i}.$$

Let D and C be the modified duration and the convexity of a bond with yield yand value B = B(y). Then

$$\frac{\Delta B}{B} \approx -D\Delta y + \frac{1}{2}C \cdot (\Delta y)^2,$$

where $\Delta B = B(y + \Delta y) - B(y)$.

Bond Convexity

Compounding

Note that regardless of whether the yield goes up or down, a bond with higher convexity provides a better return when the bond yield moves.

Note also that an approximate value for the percentage return $\frac{\Delta B}{B}$ of a long bond position can be computed using this formula without requiring specific knowledge of the cash flows of the bond.

A practical note about bond trading—for two bonds with the same duration, the bond with higher convexity provides a higher return for small changes in the yield curve.

⇒ Regardless of whether the yield goes up or down, bond with higher convexity provides a higher return, and is the better investment, all other things being considered equal.



Bond Convexity

Compounding

Example A 2y coupon bond is trading at 102. The continuously compounded yield is 5.2756%, and the coupon is 6.5 paid annually.

- ① Calculate the convexity of this bond. ans.: 3.8187
- **2** Suppose the yield goes up by 2 basis points, calculate the return $\frac{\Delta B}{B}$ of this bond. ans.: -0.037%
- **3** Compare the bond return to the approximation we obtain using modified duration and convexity. ans.: -0.038%



Dollar Duration

Compounding

Modified duration and convexity are not well suited for analyzing bond portfolios since they are non-additive.

⇒ The modified duration and the convexity of a portfolio made of positions in different bonds are not equal to the sum of the modified durations or of the convexities, respectively, of the bond positions.

Dollar duration and **dollar convexity** are additive and can be used to measure the sensitivity of bond portfolios with respect to parallel changes in the zero rate curves.

Dollar Duration of a bond is defined as

$$D_{\$} = -\frac{\partial B}{\partial y},$$

and measures the sensitivity of the bond price with respect to small changes of the bond yield. It is easy to see that $D_{\$} = B \times D$.

Dollar Convexity

Compounding

Dollar Convexity of a bond is defined as

$$C_{\$} = \frac{\partial^2 B}{\partial y^2},$$

and measures the sensitivity of the dollar duration of a bond with respect to small changes of the bond yield. It is easy to see that $C_{\$} = B \times C$.

Note that the change in dollar amount of a bond is related to dollar duration and dollar convexity as

$$\Delta B \approx -D_{\$} \Delta y + \frac{C_{\$}}{2} \cdot (\Delta y)^2$$

Duration vs Convexity

Compounding

The duration of a bond is a measure of how long on average the holder of the bond has to wait before receiving cash payments.

- A zero-coupon bond that lasts n years has a duration of n years.
- However, a coupon-bearing bond lasting n years has a duration of less than n years, because the holder receives some of the cash payments prior to year n.
- The duration is therefore a weighted average of the times when payments are made.

The convexity of a bond portfolio tends to be greatest when the portfolio provides payments evenly over a long period of time. It is least when the payments are concentrated around one particular point in time.

 By matching convexity as well as duration, a company can make itself immune to relatively large parallel shifts in the zero curve. However, it is still exposed to nonparallel shifts.



 $\underline{DV01}$ stands for "dollar value of one basis point", and is often used instead of dollar duration when quoting the risk associated with a bond position or with a bond portfolio.

The DV01 of a bond measures the change in the value of the bond for a decrease of one basis point (i.e. 0.01% or 0.0001) in the yield of the bond:

$$DV01 = \frac{D_{\$}}{10,000}.$$

The DV01 of a bond is always positive, since a decrease in the bond yield results in an increase in the value of the bond:

$$\Delta B \approx DV01$$
 for $\Delta y = -0.0001$.

DV01 vs Modified Duration

- ⇒ DV01 is useful in a hedging context when you want to offset price movement between your portfolio with a hedging instrument.
- ⇒ Modified Duration is useful when comparing how sensitive different bonds are to yields.

DV01

Compounding

Example A 2y coupon bond is trading at 102. The continuously compounded yield is 5.2756%, and the coupon is 6.5 paid annually. Calculate the DV01 of this bond.

ans.: 0.0198

