



Session 4

LIBOR and Swap Market Models

Tee Chyng Wen

QF605 Fixed Income Securities

About Market Models

These models postulate a geometric Brownian motion for the market rates under consideration, such that the Black (1976) formula is recovered for the price of an European option on the market rate.

The Black formula is the market standard for calculating prices of European-style interest rate options.

— Antoon Pelsser

Martingales

Under the **risk-neutral valuation framework**, let V_t denote the value of a security at time t , we write

$$V_0 = e^{-rT} \mathbb{E}^*[V_T].$$

The expectation is taken under the risk-neutral measure associated with the risk-free bond numeraire.

This is valid because the asset ratio is a **martingale**

$$\frac{V_0}{B_0} = \mathbb{E}^* \left[\frac{V_T}{B_T} \right].$$

Under the risk-neutral measure, the best estimate based on the information at time t of the value of the discounted asset price at time T is the discounted asset price at time t :

$$\begin{aligned} M_t &= \mathbb{E}_t^*[M_T], \quad T > t \\ \therefore M_0 &= \mathbb{E}^*[M_T] \end{aligned}$$

Zero-Coupon Bond as Numeraire

Suppose the interest rate r is not a constant but a function of time (i.e. r_t), then under martingale pricing, we can value a financial contract V_t under the risk-neutral measure associated to the risk-free money market account numeraire as

$$V_t = \mathbb{E}^* \left[e^{-\int_t^T r_u du} V_T \right].$$

The expectation is evaluated under the probability measure \mathbb{Q}^* , which is associated to the **money market account numeraire** B_t .

Instead of using the value of the money market account B_t as a numeraire, the prices of **discount bonds** $D(t, T)$ **can also be used as a numeraire**.

A very convenient choice is to use the discount bond with maturity T as numeraire (co-inciding with the payoff time of the contract). A zero-coupon discount bond is given by

$$D(t, T) = \mathbb{E}^* \left[e^{-\int_t^T r_u du} \right].$$

Zero-Coupon Bond as Numeraire

If we denote the probability measure associated to the numeraire $D(t, T)$ by \mathbb{Q}^T , we can apply the **change of numeraire theorem** to obtain

$$\frac{V_t}{D(t, T)} = \mathbb{E}^T \left[\frac{V_T}{D(T, T)} \right].$$

However, at time T the price of the discount bond $D(T, T) = 1$, and so

$$V_t = D(t, T) \mathbb{E}^T [V_T].$$

In words, by changing the measure from \mathbb{Q}^* to \mathbb{Q}^T , we have managed to express the expectation of the discounted payoff as a discounted expectation of the payoff.

⇒ We have therefore eliminated the problem of the correlation between the discounting term and the payoff term.

LIBOR Market Model

In the LIBOR market, we can choose to lend (deposit) capital and earn the LIBOR rate, which is the rate for unsecured borrowing and lending between banks.

If you lend into the LIBOR market for a period of length Δ , you earn $1 + \Delta \cdot L$ one period later, where L denote the LIBOR rate you invested in.

Let $D(t, T)$ denote the value at time t of a discount bond which pays 1 at maturity T , the LIBOR rate and discount factor is related by

$$1 = (1 + \Delta \cdot L) \cdot D(0, \Delta).$$

Suppose we are at time t , and we commit into a forward LIBOR rate for the period $[T_i, T_{i+1}]$. We have the following relation

$$\begin{aligned} D(t, T_i) &= (1 + \Delta_i L_t(T_i, T_{i+1})) D(t, T_{i+1}) \\ \Rightarrow L_t(T_i, T_{i+1}) &= \frac{1}{\Delta_i} \frac{D(t, T_i) - D(t, T_{i+1})}{D(t, T_{i+1})}. \end{aligned}$$

LIBOR Market Model

In most markets, only one specific LIBOR tenor is liquidly traded. In Singapore, this will be the 6m SIBOR or SOR rate.

In other words, for all practical purposes, $[T_i, T_{i+1}]$ are not arbitrary. Let us denote $L_i(t) = L_t(T_i, T_{i+1})$ and $D_i(t) = D(t, T_i)$. Now consider the process

$$\Delta_i L_i(t) = \frac{D_i(t) - D_{i+1}(t)}{D_{i+1}(t)}.$$

This is a **ratio of marketed assets**. If we take the discount bond $D_{i+1}(t)$ as numeraire, then under the martingale measure \mathbb{Q}^{i+1} associated with the numeraire $D_{i+1}(t)$, the process $\Delta_i L_i(t)$ must be a martingale.

Since Δ_i is a constant, the process $L_i(t)$ must be a martingale under \mathbb{Q}^{i+1} . This gives rise to the **LIBOR Market Model (LMM)**

$$dL_i(t) = \sigma_i L_i(t) dW^{i+1}(t) \Rightarrow L_i(t) = L_i(0) \exp \left[-\frac{1}{2} \sigma_i^2 t + \sigma_i W^{i+1}(t) \right],$$

where W^{i+1} is a Brownian motion under \mathbb{Q}^{i+1} .

Pricing a Caplet

The payoff of a **caplet** C_i at time T_{i+1} is given by

$$C_i(T_{i+1}) = \Delta_i(L_i(T_i) - K)^+.$$

Choosing D_{i+1} as a numeraire and working under the associated martingale measure \mathbb{Q}^{i+1} , we know that

$$\begin{aligned}\frac{C_i(0)}{D_{i+1}(0)} &= \mathbb{E}^{i+1} \left[\frac{C_i(T_{i+1})}{D_{i+1}(T_{i+1})} \right] \\ \Rightarrow C_i(0) &= D_{i+1}(0) \Delta_i \mathbb{E}^{i+1} [(L_i(T_i) - K)^+].\end{aligned}$$

The remaining steps required to derive a formula for a caplet price is identical to how we would handle a vanilla European option.

Pricing a Caplet

The LIBOR rate follows the stochastic differential equation

$$dL_i(t) = \sigma_i L_i(t) dW^{i+1}(t),$$

where $W^{i+1}(t)$ is a standard Brownian motion under the risk-neutral measure \mathbb{Q}^{i+1} associated with the numeraire $D_{i+1}(t)$. The solution is given by

$$L_i(T) = L_i(0) e^{-\frac{1}{2}\sigma_i^2 T + \sigma_i W^{i+1}(T)}.$$

Evaluating the expectation, we obtain

$$\begin{aligned} C_i(0) &= D_{i+1}(0) \Delta_i \mathbb{E}^{i+1}[(L_i(T_i) - K)^+] \\ &= D_{i+1}(0) \Delta_i [L_i(0) \Phi(d_1) - K \Phi(d_2)], \end{aligned}$$

where

$$d_1 = \frac{\log \frac{L_i(0)}{K} + \frac{1}{2}\sigma_i^2 T}{\sigma_i \sqrt{T}}, \quad d_2 = d_1 - \sigma_i \sqrt{T}.$$

Swap Market Model

Let us denote the **par swap rate** for the $[T_n, T_N]$ swap as $S_{n,N}$:

$$S_{n,N}(t) = \frac{D_n(t) - D_N(t)}{\sum_{i=n+1}^N \Delta_{i-1} D_i(t)}.$$

The term in the denominator is also called the **present value of a basis point** (PVBP)

$$P_{n+1,N}(t) = \sum_{i=n+1}^N \Delta_{i-1} D_i(t).$$

Note that a one-period swap rate $S_{i,i+1}$ is equal to the LIBOR rate. We can now write the value of a payer and receiver swap as

$$\begin{aligned}\text{Payer Swap} &= P_{n+1,N}(t)(S_{n,N}(t) - K) \\ \text{Receiver Swap} &= P_{n+1,N}(t)(K - S_{n,N}(t))\end{aligned}$$

Pricing a Swaption

The PVBP is a portfolio of traded assets and has strictly positive value. It can therefore be used as a numeraire.

If we use $P_{n+1,N}(t)$ as a numeraire, then under the measure $\mathbb{Q}^{n+1,N}$ associated to the numeraire $P_{n+1,N}(t)$, all $P_{n+1,N}$ rebased values must be martingales in an arbitrage-free world.

In particular, the par swap rate $S_{n,N}$ must be a martingale under $\mathbb{Q}^{n+1,N}$. The swap market model makes the assumption that $S_{n,N}$ is a lognormal martingale under $\mathbb{Q}^{n+1,N}$. We write down the process

$$dS_{n,N}(t) = \sigma_{n,N} S_{n,N}(t) dW^{n+1,N}(t),$$

where $W^{n+1,N}(t)$ is a Brownian motion under $\mathbb{Q}^{n+1,N}$.

A **swaption** (short for swap option) gives the right to enter at time T_n into a swap with fixed rate K . A **receiver swaption** gives the right to enter into a receiver swap, and a **payer swaption** gives the right to enter into a payer swap.

Pricing a Swaption

Swaptions are often denoted as $T_n \times (T_N - T_n)$, where T_n is the option expiry date (and also the start of the underlying swap), and $T_N - T_n$ is the tenor of the underlying swap.

The payoff of a payer swaption is given by

$$[P_{n+1,N}(T)(S_{n,N}(T) - K)]^+.$$

Using $P_{n+1,N}$ as a numeraire, we can value the payer swaption under the measure $\mathbb{Q}^{n+1,N}$

$$\begin{aligned} \frac{V_{n,N}^{\text{payer}}(0)}{P_{n+1,N}(0)} &= \mathbb{E}^{n+1,N} \left[\frac{V_{n,N}^{\text{payer}}(T_n)}{P_{n+1,N}(T_n)} \right] \\ \Rightarrow V_{n,N}^{\text{payer}}(0) &= P_{n+1,N}(0) \mathbb{E}^{n+1,N} [(S_{n,N}(T) - K)^+]. \end{aligned}$$

The remaining steps required to derive a formula for a swaption is identical to how we would handle a vanilla European option.

Pricing a Swaption

The swap rate follows the stochastic differential equation

$$dS_{n,N}(t) = \sigma_{n,N} S_{n,N}(t) dW^{n+1,N}(t),$$

where $W^{n+1,N}(t)$ is a Brownian motion under $\mathbb{Q}^{n+1,N}$. The solution is given by

$$S_{n,N}(T) = S_{n,N}(0) e^{-\frac{1}{2}\sigma_{n,N}^2 T + \sigma_{n,N} W^{n+1,N}(T)}.$$

Evaluating the expectation, we obtain

$$\begin{aligned} V_{n,N}^{payer}(0) &= P_{n+1,N}(0) \mathbb{E}^{n+1,N}[(S_{n,N}(T) - K)^+] \\ &= P_{n+1,N}(0) [S_{n,N}(0) \Phi(d_1) - K \Phi(d_2)], \end{aligned}$$

where

$$d_1 = \frac{\log \frac{S_{n,N}(0)}{K} + \frac{1}{2}\sigma_{n,N}^2 T}{\sigma_{n,N} \sqrt{T}}, \quad d_2 = d_1 - \sigma_{n,N} \sqrt{T}. \quad \triangleleft$$

Swaption Vols – ATM Vols

91) Asset		92) Actions		93) Settings		Interest Rate Volatility Cube						
USD	USD BVOL Cube (Default)		Bid	Date	10/29/15							
1) Analyze Cube		2) Market Data										
Swap Curve	(23) US Dollar				Index Tenor	3M	<input checked="" type="checkbox"/> Show Vol		Black			
View	Strike		ATM		Discounting	IBOR	<input type="checkbox"/> Show Strikes					
Table		Charts										
Expiry	1Yr	2Yr	3Yr	4Yr	5Yr	6Yr	7Yr	8Yr	9Yr	10Yr	12Yr	15Yr
1Mo	66.10	62.21	55.17	49.96	46.57	42.67	40.01	37.95	36.32	34.93	33.42	31.65
3Mo	70.23	63.36	57.30	52.53	48.96	44.96	42.20	40.23	38.56	37.12	35.62	33.86
6Mo	64.49	59.82	55.44	50.78	47.56	44.22	41.86	40.16	38.75	37.52	36.11	34.49
9Mo	61.24	56.74	52.60	48.62	45.77	43.00	40.94	39.50	38.25	37.09	35.75	34.21
1Yr	58.49	54.16	50.10	46.94	44.14	41.76	39.93	38.66	37.61	36.68	35.47	34.08
2Yr	52.60	48.17	44.87	42.30	40.20	38.71	37.53	36.65	35.84	35.19	34.21	32.99
3Yr	47.94	44.17	41.45	39.51	37.94	36.84	35.88	35.12	34.48	33.94	33.11	32.00
4Yr	43.43	40.52	38.55	37.09	35.81	34.97	34.30	33.72	33.27	32.92	32.17	31.12
5Yr	39.96	37.89	36.69	35.71	34.78	34.01	33.41	32.99	32.61	32.31	31.59	30.56
6Yr	37.44	36.00	35.03	34.21	33.40	32.80	32.32	31.97	31.66	31.40	30.73	29.77
7Yr	35.22	34.29	33.51	32.84	32.18	31.75	31.38	31.06	30.80	30.57	29.95	29.04
8Yr	33.81	32.87	32.23	31.74	31.25	30.91	30.60	30.33	30.09	29.90	29.32	28.48
9Yr	32.43	31.53	31.13	30.79	30.45	30.18	29.89	29.67	29.46	29.29	28.77	27.95
10Yr	31.21	30.41	30.19	29.98	29.75	29.52	29.26	29.07	28.88	28.72	28.24	27.45
12Yr	30.02	29.26	28.90	28.89	28.88	28.62	28.37	28.20	28.16	28.12	27.62	26.86
15Yr	28.25	27.56	27.43	27.33	27.36	27.26	27.19	27.20	27.23	27.23	26.76	25.99

Swaption ATM Vols

	1y	2y	3y	4y	5y	10y	15y	20y	25y	30y
1m 3m 6m	GAMMA									
1y 2y . . 15y 20y 30y	VEGA									
	1y	2y	3y	4y	5y	10y	15y	20y	25y	30y
1m 3m 6m	TOP LEFT							TOP RIGHT		
1y 2y 3y 5y 10y 20y 30y				INTERMEDIATES						
								BOTTOM RIGHT		

Swaption ATM Vols

77 Settings 98 Output 200 Show in Launchpad Page 1/2 ICAP Global Menu

EUR Cash IRR Cal Day EUR Phys (LCH) Cal Day EUR Cash IRR Bus Day EUR Phys (LCH) Bus Day GBP Calendar D...

ICAP EUR Swaption - BP Vol OIS Ph MSG Contributor 10:56:45

ICAP Global Menu -> ICAP EMEA -> Interest Rate Options -> IR Options - Digital -> Swaption Normal Vols -> EUR Phys (LCH)...

Zoom - + 100%

ICAP - ATM Swaptions

Term	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y
1) 1M	12.70	13.60	15.90	18.40	20.60	22.60	24.80	26.50	28.00
2) 2M	13.30	14.20	16.70	19.40	21.90	24.20	26.40	28.30	29.90
3) 3M	14.10	15.00	17.20	20.00	22.50	25.10	27.60	29.50	31.10
4) 6M	14.60	16.10	18.90	22.20	24.70	27.20	29.40	31.60	33.50
5) 9M	15.80	17.70	20.50	23.60	26.20	28.60	30.90	32.80	34.70
6) 1Y	16.80	19.00	22.10	24.90	27.50	30.00	32.20	34.10	35.80
7) 18Y	18.80	21.90	24.90	27.40	29.90	32.40	34.40	36.30	38.00
8) 2Y	21.70	24.50	27.80	30.40	32.40	34.40	36.60	38.20	39.90
9) 3Y	28.00	30.50	33.10	35.00	36.80	38.60	40.30	41.80	43.10
10) 4Y	33.40	35.50	37.50	39.10	40.40	41.60	43.00	44.20	45.50
11) 5Y	37.80	39.70	41.10	42.20	43.30	44.40	45.40	46.40	47.40
12) 6Y	41.90	43.00	43.90	45.20	45.70	46.70	47.50	48.40	49.10
13) 7Y	44.70	45.60	46.30	47.10	47.70	48.40	49.00	49.60	50.20
14) 10Y	49.70	49.80	50.30	50.80	51.10	51.40	51.70	51.80	52.10
15) 12Y	51.30	50.90	51.00	51.40	51.60	52.00	52.30	52.50	52.50
16) 15Y	51.70	51.50	51.90	52.00	52.00	52.10	52.20	52.10	52.40
17) 20Y	51.20	51.10	51.40	51.30	51.30	51.50	51.30	51.20	51.20
18) 25Y	50.30	50.30	50.50	50.40	50.30	50.10	49.90	49.30	49.30
19) 30Y	49.30	49.40	49.60	49.60	49.70	49.30	48.60	48.00	47.40

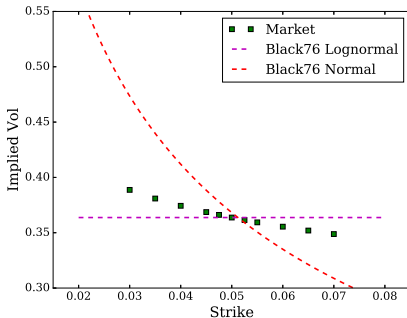
Suggested Functions FED See central bank info for the US GOVY See a government's richest/cheapest bond

Swaption Vols – Smile/Skew

Global Swaption Skews										Last Update	08:
Tullett Prebon											
SMKR412 (c) 2020 Tullett Prebon Information 16-Dec-2020 08:47 LDN											
EUR Swaption Volatility Smile based on Spot Premium and IBOR curve											
OPTION/ TENOR	(Normal Volatility)									ATM STRIKE	
	-200	-100	-50	-25	ATM	25	50	100	200		
1Y1Y	51.9	36.2	24.4	18.5	16.8	22.4	29.1	42.1	65.6	-0.57	
3M2Y	74.2	48.9	31.4	21.3	15.0	25.3	36.2	56.1	91.8	-0.54	
2Y2Y	46.5	34.5	26.7	24.1	24.4	27.9	32.5	42.4	61.5	-0.47	
1Y5Y	57.5	42.2	32.0	27.4	26.9	30.8	36.6	48.7	71.7	-0.43	
5Y5Y	46.0	42.4	41.3	41.7	42.4	43.4	44.7	48.0	56.0	-0.08	
3M10Y	88.3	61.5	43.7	35.3	32.2	39.6	50.1	70.9	109.3	-0.26	
1Y10Y	66.0	50.7	41.0	37.6	36.8	39.3	43.8	54.8	77.0	-0.21	
2Y10Y	58.4	48.7	42.9	41.2	40.8	41.9	44.1	50.4	65.0	-0.13	
5Y10Y	52.5	49.2	47.6	47.2	47.4	47.9	48.7	51.0	57.5	0.087	
10Y10Y	52.4	51.9	51.7	51.7	52.3	52.9	53.4	54.9	59.1	0.236	
15Y15Y	49.9	49.3	49.0	49.1	49.7	50.4	50.8	51.9	55.0	0.010	
10Y20Y	51.9	49.9	48.9	48.7	49.3	49.9	50.2	51.3	55.1	0.073	
5Y30Y	54.3	50.0	48.5	48.1	48.2	48.5	49.1	50.8	56.6	-0.00	
	-200	-100	-50	-25	ATM	25	50	100	200		

Swaption Vol Calibration

Suppose the implied volatility across strike for a given swaption maturity and tenor is given by the green markers in the following figure:



The at-the-money volatility is 0.36, and the forward swap rate is 0.05.

Extension to the Black Model

An immediate and straightforward extension is the Black Normal model:

$$dS_{n,N}(t) = \sigma_{n,N} dW^{n+1,N}(t).$$

This is an arithmetic Brownian motion.

If the implied volatility skew we observed in the market is between normal and lognormal, then we can make use of the displaced-diffusion (shifted lognormal) model:

$$dS_{n,N}(t) = \sigma_{n,N} [\beta S_{n,N}(t) + (1 - \beta) S_{n,N}(0)] dW^{n+1,N}(t).$$

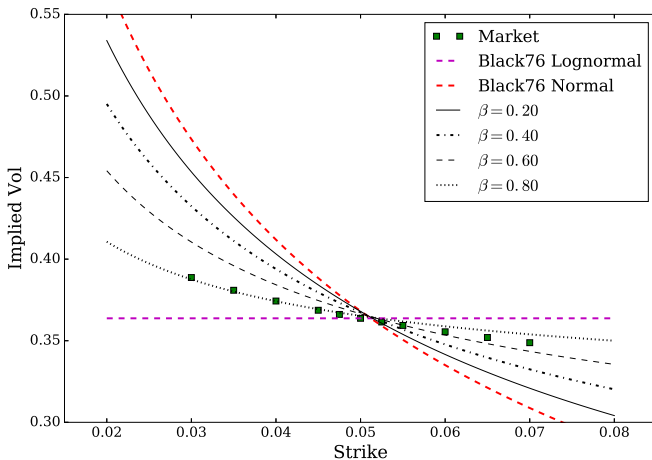
Recall that the solution is given by

$$S_{n,N}(T) = \frac{S_{n,N}(0)}{\beta} e^{\sigma_{n,N} \beta W^{n+1,N}(T) - \frac{\sigma_{n,N}^2 \beta^2 T}{2}} - \frac{1 - \beta}{\beta} S_{n,N}(0)$$

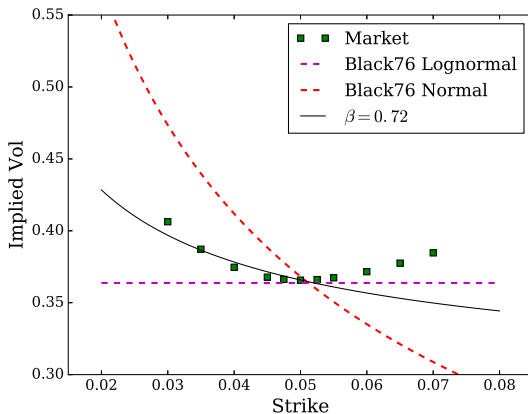
The swaption price under the displaced-diffusion model is

$$V_{n,N}(0) = P_{n+1,N}(0) \text{Black} \left(\frac{S_{n,N}(0)}{\beta}, K + \frac{1 - \beta}{\beta} S_{n,N}(0), \sigma \beta, T \right)$$

Swaption Vol Calibration – Displaced Diffusion

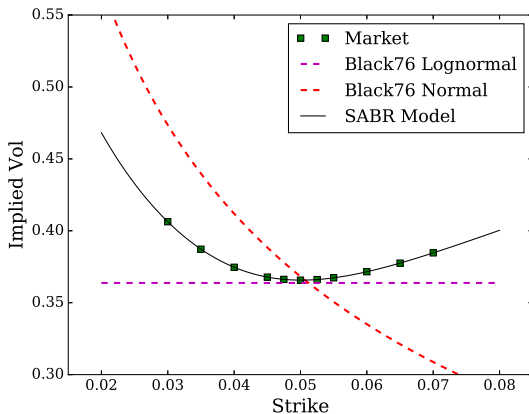


SABR Model



Displaced-diffusion model can only fit to implied volatility skew – there will be mismatch if the implied volatility surface also exhibit “smile” characteristic.

SABR Model



SABR model is able to fit both skew and smile in the implied volatility surface – this is the standard volatility model used in fixed-income market.