1. (a) i. If we try to express the forward Libors using interpolated Libor discount factors, we will get

$$\begin{aligned}\Delta \times 2.5\% \times \Big[D_o(0,0.5) + D_o(0,1) + D_o(0,1.5) + D_o(0,2)\Big] \\ = D_o(0,0.5)\Delta L(0,0,5) + D_o(0,1)\Delta L(0.5,1) \\ + D_o(0,1.5)\Delta L(1,1.5) + D_o(0,2y)\Delta L(1.5,2)\end{aligned}$$

where $\Delta = 0.5$ is the day count fraction, $D_o$ is the OIS discount factor, and

$$L(1,1.5) = \frac{1}{\Delta}\frac{\tilde{D}(0,1) - \tilde{D}(0,1.5)}{\tilde{D}(0,1.5)} \quad \text{and} \quad L(1.5,2) = \frac{1}{\Delta}\frac{\tilde{D}(0,1.5) - \tilde{D}(0,2)}{\tilde{D}(0,2)}$$

where $\tilde{D}$ denote Libor discount factor. This ends up being a quadratic equation, which is cumbersome to solve. A better way is to use the Libor discount factors to work out what the $1.5y$ par swap rate should be

$$\text{IRS } 1.5y = \frac{1 - \tilde{D}(0,1.5)}{\Delta \times \Big(\tilde{D}(0,0.5) + \tilde{D}(0,1) + \tilde{D}(0,1.5)\Big)} = 2.39\%$$

Hence

$$\begin{aligned}2.39\% \cdot \Delta \cdot \Big[D_o(0,0.5) + D_o(0,1) + D_o(0,1.5)\Big] \\ = D_o(0,0.5)\Delta L(0,0.5) + D_o(0,1)\Delta L(0.5,1) + D_o(0,1.5)\Delta L(1,1.5)\end{aligned}$$

In both collateralized and uncollateralized markets, we have $L(1y6m, 2y) \approx 2.8\%$.

ii. Collateralized: 0.0138, uncollateralized: 0.0133

iii. $D_o(1y,2y) \approx 0.992$ $D(1y,2y) \approx 0.9726$

2. (a) i. If 1 unit of foreign currency is equal to $FX \times 1$ unit of domestic currency, then

$$FX_T = FX_0 \frac{D_f(0,T)}{D_d(0,T)}$$

ii. 0.983.

(b) 102.47

3. (a) Given the stochastic differential equation followed by the short rate $r_t$, we obtain:

$$\begin{aligned}dr_t &= \mu\, dt + \sigma\, dW_t^* \\ r_s &= r_t + \mu \cdot (s-t) + \sigma \cdot (W_s^* - W_t^*) \\ \int_t^T r_u\, du &= r_t \cdot (T-t) + \mu \int_t^T (u-t)\, du + \sigma \int_t^T (W_u^* - W_t^*)\, du \\ &= r_t \cdot (T-t) + \mu \left[\frac{u^2}{2} - tu\right]_t^T + \sigma \int_t^T \int_t^u dW_s\, du \\ &= r_t \cdot (T-t) + \frac{\mu}{2} \cdot (T-t)^2 + \sigma \int_t^T \int_s^T du\, dW_s \\ &= r_t \cdot (T-t) + \frac{\mu}{2} \cdot (T-t)^2 + \sigma \int_t^T (T-s)\, dW_s\end{aligned}$$

From this we can work out that

$$\mathbb{E}^*\left[\int_t^T r_u\ du\right] = r_t \cdot (T-r) + \frac{\mu}{2}\cdot(T-t)^2,$$

and

$$\begin{aligned} V\left[\int_t^T r_u\ du\right] &= V\left[\sigma \int_t^T (T-s)\ dW_s\right] \\ &= \sigma^2 \int_t^T (T-s)^2\ ds \\ &= \frac{\sigma^2}{3}(T-t)^3. \end{aligned}$$

Hence, we have the distribution

$$-\int_t^T r_u\ du \sim N\left(-r_t\cdot(T-r) - \frac{\mu}{2}\cdot(T-t)^2,\ \frac{\sigma^2}{3}(T-t)^3\right).$$

We can therefore express the discount factor as

$$D(t,T) = \mathbb{E}^*\left[e^{-\int_t^T r_u\ du}\right] = e^{-r_t\cdot(T-r)-\frac{\mu}{2}\cdot(T-t)^2+\frac{1}{6}(T-t)^3}.$$

(b) Since we can also express the discount factor as

$$D(t,T) = e^{-R(t,T)(T-t)},$$

we can identify that

$$R(t,T) = r_t + \frac{\mu}{2}\cdot(T-t) - \frac{\sigma^2}{6}\cdot(T-t)^2,$$

hence this is an affine short rate model.

4. (a) Under the Black-Scholes model of

$$dS_t = rS_t dt + \sigma S_t dW_t^*,$$

we obtain

$$S_T = S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma W_T^*},$$

and hence

$$\begin{aligned} \log\left(\frac{\alpha S_T}{\beta}\right) &= \log\left(\frac{\alpha}{\beta}\right) + \log(S_T) \\ &= \log\left(\frac{\alpha}{\beta}\right) + \log(S_0) + \left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T^*. \end{aligned}$$

Under Black-Scholes model, the contract can be priced as follows:

$$V_0 = e^{-rT}\mathbb{E}^*\left[\log\left(\frac{\alpha S_T}{\beta}\right)\right] = e^{-rT}\left[\log\left(\frac{\alpha}{\beta}\right) + \log(S_0) + \left(r-\frac{\sigma^2}{2}\right)T\right].$$

(b) To apply static replication approach, first we let

$$h(K) = \log\left(\frac{\alpha K}{\beta}\right),$$

and the first and second derivatives are given by

$$h'(K) = \frac{1}{K}, \qquad h''(K) = -\frac{1}{K^2}.$$

Choosing $F = S_0 e^{rT}$ as the expansion point, the Breeden-Litzenberger formula is given by

$$\begin{aligned} V_0 &= e^{-rT} h(F) + \int_0^F h''(K)P(K)\,dK + \int_F^\infty h''(K)C(K)\,dK \\ &= e^{-rT}\log\left(\frac{\alpha S_0 e^{rT}}{\beta}\right) - \int_0^F \frac{P(K)}{K^2}\,dK - \int_F^\infty \frac{C(K)}{K^2}\,dK \\ &= e^{-rT}\left[\log\left(\frac{\alpha}{\beta}\right) + \log(S_0) + rT\right] - \int_0^F \frac{P(K)}{K^2}\,dK - \int_F^\infty \frac{C(K)}{K^2}\,dK. \end{aligned}$$

5. Taking the perspective of the foreign investor, we note that $\frac{B_t^D}{X_t}$ is a foreign tradable. We use Itô's formula to derive the stochastic differential equation for this tradable:

$$dY_t = (r^D + \mu)Y_t dt + \sigma Y_t dW_t$$

Next, note that $Z_t = \frac{Y_t}{B_t^F}$ is a ratio of foreign tradable, and must be a martingale. Writing down the stochastic differential equation, we have

$$\begin{aligned} dZ_t &= (r^D - r^F + \mu)Z_t dt + \sigma Z_t dW_t \\ &= \sigma Z_t\left(dW_t + \frac{r^D - r^F + \mu}{\sigma}dt\right) \\ &= \sigma Z_t dW_t^F \end{aligned}$$

So

$$dW_t = dW_t^F - \frac{r^D - r^F + \mu}{\sigma}dt,$$

substituting to the FX process, we have

$$\begin{aligned} d\frac{1}{X_t} &= \mu\frac{1}{X_t}dt + \sigma\frac{1}{X_t}dW_t \\ &= \mu\frac{1}{X_t}dt + \sigma\frac{1}{X_t}\left(dW_t^F - \frac{r^D - r^F + \mu}{\sigma}dt\right) \\ &= (r^F - r^D)\frac{1}{X_t}dt + \sigma\frac{1}{X_t}dW_t^F. \end{aligned}$$

6. (a)

$$dL_i(t) = \sigma_i L_i(t) dW_t^{i+1}, \quad L_i(t) = \frac{1}{\Delta_i}\frac{D_i(t) - D_{i+1}(t)}{D_{i+1}(t)}.$$

(b) $D_{i+1}$, zero coupon bond maturing at $T_{i+1}$.

(c)

$$\begin{aligned}
D_{i+1}(0)\mathbb{E}^{i+1}\left[100 \times \mathbb{1}_{L_i(T)>5\%}\right] &= D_{i+1}(0) \cdot \frac{100}{\sqrt{2\pi}} \int_{x^*}^{\infty} e^{-\frac{x^2}{2}}\, dx \\
&= D_{i+1}(0) \cdot 100 \cdot \Phi\left(\frac{\log \frac{L_i(0)}{5\%} - \frac{\sigma_i^2 T}{2}}{\sigma_i \sqrt{T}}\right)
\end{aligned}$$

7. We use multi-currency change of numeraire theorem and Cholesky decomposition to evaluate the following expectation:

$$\begin{aligned}
\mathbb{E}^{i+1,D}\left[L_i^F(T)\right] &= \mathbb{E}^{i+1,F}\left[L_i^F(T) \cdot \frac{d\mathbb{Q}^{i+1,D}}{d\mathbb{Q}^{i+1,F}}\right] \\
&= \mathbb{E}^{i+1,F}\left[L_i^F(T) \cdot \frac{\frac{D_{i+1}^D(T_{i+1})}{D_{i+1}^D(0)}}{\frac{X_T D_{i+1}^F(T_{i+1})}{X_0 D_{i+1}^F(0)}}\right] \\
&= \mathbb{E}^{i+1,F}\left[L_i^F(T) \cdot \frac{\frac{D_{i+1}^D(T_{i+1})}{X_T D_{i+1}^F(T_{i+1})}}{\frac{D_{i+1}^D(0)}{X_0 D_{i+1}^F(0)}}\right] \\
&= \mathbb{E}^{i+1,F}\left[L_i^F(T) \cdot \frac{\frac{1}{F_{T_{i+1}}}}{\frac{1}{F_0}}\right] \\
&= \mathbb{E}^{i+1,F}\left[L_i^F(0)e^{-\frac{\sigma_i^2 T}{2}+\sigma_i W^{i+1}} \cdot \frac{1}{F_0} e^{-\frac{\sigma_X^2 T}{2}+\sigma_X W^F}\right] \times F_0 \\
&= \mathbb{E}^{i+1,F}\left[L_i^F(0)e^{-\frac{\sigma_i^2 T}{2}+\sigma_i W^{i+1}} \cdot e^{-\frac{\sigma_X^2 T}{2}+\sigma_X W^F}\right] \\
&= L_i^F(0)e^{-\frac{\sigma_i^2 T}{2}} \cdot e^{-\frac{\sigma_X^2 T}{2}} \mathbb{E}^{i+1,F}\left[e^{\sigma_i W^{i+1}} \cdot e^{\sigma_X W^F}\right] \\
&= L_i^F(0)e^{-\frac{\sigma_i^2 T}{2}} \cdot e^{-\frac{\sigma_X^2 T}{2}} \mathbb{E}^{i+1,F}\left[e^{\sigma_i Z_1} \cdot e^{\sigma_X\left(\rho Z_1+\sqrt{1-\rho^2} Z_2\right)}\right] = L_i^F(0)e^{\sigma_i \sigma_X \rho T}
\end{aligned}$$

8. (a)

$$\begin{aligned}
dL_i(t) &= \sigma_i L_i(t) dW_t^{i+1} \\
L_i(T) &= L_i(0)e^{-\frac{\sigma_i^2 T}{2}+\sigma_i W_T^{i+1}} \\
L_i(T)^2 &= L_i(0)^2 e^{-\sigma_i^2 T+2\sigma_i W_T^{i+1}}
\end{aligned}$$

So

$$\begin{aligned}
V_0 &= D_{i+1}(0)\mathbb{E}^{i+1}\left[L_i(0)^2 e^{-\sigma_i^2 T+2\sigma_i W_T^{i+1}}\right] \\
&= D_{i+1}(0) L_i(0)^2 e^{\sigma_i^2 T}
\end{aligned}$$

(b) Choosing the expansion point at the forward LIBOR rate $F = L_i(0)$, we obtain

$$V_0 = D(0,T)L_i(0)^2 + \int_0^{L_i(0)} 2\, V^f(K)\, dK + \int_{L_i(0)}^{\infty} 2\, V^c(K)\, dK$$

9. Integrating the stochastic differential equation from $t$ to $s$, we obtain

$$r_s = r_t + \int_t^s \theta(u)\, du + \int_t^s \sigma\, dW_u^*$$

Integrating both sides from $t$ to $T$, we obtain

$$\begin{aligned}\int_t^T r_s\, ds &= r_t(T-t) + \int_t^T \int_t^s \theta(u)\, du\, ds + \int_t^T \int_t^s \sigma\, dW_u^*\, ds \\ &= r_t(T-t) + \int_t^T \int_u^T \theta(u)\, ds\, du + \int_t^T \int_u^T \sigma\, ds\, dW_u^* \\ &= r_t(T-t) + \int_t^T \theta(u)(T-u)\, du + \int_t^T \sigma(T-u)\, dW_u^*.\end{aligned}$$

So

$$\begin{aligned}\mathbb{E}^*\left[\int_t^T r_s\, ds\right] &= r_t(T-t) + \int_t^T \theta(u)(T-u)\, du \\ V\left[\int_t^T r_s\, ds\right] &= \frac{\sigma^2(T-t)^3}{3}\end{aligned}$$

So we have

$$D(t,T) = \mathbb{E}^*\left[e^{-\int_t^T r_u\, du}\right] = \exp\left[-r_t(T-t) - \int_t^T \theta(u)(T-u)\, du + \frac{\sigma^2(T-t)^3}{6}\right].$$

We can therefore express the zero coupon bond as

$$D(t,T) = \mathbb{E}^*\left[e^{-\int_t^T r_u\, du}\right] = e^{A(t,T) - r_t B(t,T)},$$

where

$$A(t,T) = \int_t^T \theta(u)(u-T)\, du + \frac{\sigma^2(T-t)^3}{6}, \qquad B(t,T) = T-t.$$

10. (a) We note that

$$\mathbb{E}^*[dr_t] = \kappa(\theta - r_t)dt,$$

so if short rate is lower than the long run mean (i.e. $\theta > r_t$), the drift term is positive, and $dr_t$ is expected to drift upward, while if short rate is higher than the long run mean (i.e. $\theta < r_t$), the drift term is negative, and $dr_t$ is expected to drift downward. When $\theta = r_t$, the drift term is zero, and so $dr_t$ is expected to remain at the same level. $\kappa$ is a positive multiplier and controls the mean reversion speed. $\triangleleft$

(b) Consider the function $e^{\kappa t} r_t$, the total derivative is given by

$$\begin{aligned}d(e^{\kappa t} r_t) &= \kappa e^{\kappa t} r_t dt + e^{\kappa t} dr_t \\ &= \kappa\theta e^{\kappa t} dt + \sigma e^{\kappa t} dW_t.\end{aligned}$$

Integrating both sides from $0$ to $t$, we can obtain a solution to the stochastic differential equation

$$\begin{aligned}\int_0^t d(e^{\kappa u} r_u) &= \int_0^t \kappa\theta e^{\kappa u} du + \int_0^t \sigma e^{\kappa u} dW_u \\ r_t &= r_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}) + \sigma \int_0^t e^{\kappa(u-t)} dW_u.\end{aligned}$$

Taking expectation on both sides gives us the mean

$$\mathbb{E}[r_t] = r_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}).$$

Applying Itô's Isometry theorem, we can evaluate the variance of the short rate

$$\begin{aligned} V[r_t] &= \mathbb{E}\left[\left(\sigma \int_0^t e^{\kappa(u-t)} dW_u\right)^2\right] \\ &= \mathbb{E}\left[\sigma^2 \int_0^t e^{2\kappa(u-t)} du\right] = \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa t}\right). \end{aligned}$$

The distribution of $r_t$ is therefore given by

$$r_t \sim N\left(r_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}), \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa t})\right).$$

So we have

$$\begin{aligned} \lim_{t\to\infty} \mathbb{E}^*[r_t] &= \lim_{t\to\infty} \left(r_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t})\right) = \theta, \quad \lhd \\ \lim_{t\to\infty} V[r_t] &= \lim_{t\to\infty} \left(\frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa t})\right) = \frac{\sigma^2}{2\kappa}. \quad \lhd \end{aligned}$$