# QF605 Additional Examples
# Session 5: Constant Maturity Swap Payoffs

## 1 Questions

1. Suppose the LIBOR rate follows the stochastic differential equation

$$dL_i(t) = \sigma_i L_i(t) dW_t^{i+1},$$

Consider a contract paying $L_i(T_i) + L_i^2(T_i)$ at time $T_{i+1}$. Derive the valuation formula for this contract.

2. Suppose the swap rate follows the stochastic differential equation

$$dS_{n,N}(t) = \sigma_{n,N} S_{n,N}(t) dW^{n+1,N}.$$

Determine the valuation formula of a pvbp-or-nothing digital swaption that pays

$$V_{n,N}^{dig}(T) = P_{n+1,N}(T) \mathbb{1}_{S_{n,N}(T)>K}.$$

3. The valuation formulae of an *IRR-settled* payer and receiver swaptions are given by

$$\begin{cases} V^{pay}(t) = D(t,T) \displaystyle\int_K^\infty \text{IRR}(S)(S-K)\ f(S)\ dS \\ V^{rec}(t) = D(t,T) \displaystyle\int_0^K \text{IRR}(S)(K-S)\ f(S)\ dS \end{cases}$$

where $D(t,T)$ is a discount factor, $S$ is the swap rate at time $T$, $K$ is the strike of the swaption, $f(S)$ is the risk-neutral probability density function, and $\text{IRR}(S)$ is the cash-settled annuity function. Following the Breeden-Litzenberger approach, derive an expression of the risk-neutral probability density function $f(\cdot)$.

*Note: IRR$(S)$ is a function of the terminal swap rate only. You can write its derivatives as IRR$'(S)$, IRR$''(S)$ and so on.*

4. Following up on the previous question, use the Carr-Madan approach to carry out integration-by-parts twice, and derive the static replication formula for a European payoff on the constant maturity swap rate $g(S)$ paid at time $T$. Use this formula to obtain the replication formula for the payoff $g(S) = S$.

## 2 Suggested Solutions

1. Given the stochastic differential equation

$$dL_i(t) = \sigma_i L_i(t) dW_t^{i+1},$$

where $W_t^{i+1}$ is a Brownian motion under the risk-neutral measure associated to the zero-coupon discount bond $D(t, T_{i+1}) = D_{i+1}(t)$. The solution is given by

$$L_i(T) = L_i(0) e^{-\frac{1}{2}\sigma_i^2 T + \sigma_i W_T^{i+1}}.$$

A contract paying $L_i(T_i) + L_i^2(T_i)$ at time $T_{i+1}$ can be valued as follow:

$$\begin{aligned} V_0 &= D_{i+1}(0)\mathbb{E}^{i+1}[L_i(T_i) + L_i^2(T_i)] \\ &= D_{i+1}(0)\left[L_i(0) + L_i^2(0)e^{\sigma_i^2 T}\right]. \quad \triangleleft \end{aligned}$$

2. We write

$$\begin{aligned} \frac{V_{n,N}^{dig}(0)}{P_{n+1,N}(0)} &= \mathbb{E}^{n+1,N}\left[\frac{V_{n,N}^{dig}(T)}{P_{n+1,N}(T)}\right] \\ V_{n,N}^{dig}(0) &= P_{n+1,N}(0)\mathbb{E}^{n+1,N}\left[\mathbb{1}_{S_{n,N}(T)>K}\right] \\ &= P_{n+1,N}(0)\Phi\left(\frac{\log\left(\frac{S_{n,N}(0)}{K}\right) - \frac{1}{2}\sigma_{n,N}^2 T}{\sigma_{n,N}\sqrt{T}}\right). \quad \triangleleft \end{aligned}$$

3. An IRR-settled payer swaption is priced by

$$\begin{aligned} V(K) &= D(t,T)\mathbb{E}^T[\text{IRR}(S)(S-K)^+] \\ &= D(t,T)\int_K^\infty \text{IRR}(S)(S-K)\, f(S)\, dS, \end{aligned}$$

where $f(S)$ denotes the probability density function of the terminal swap rate under T-measure. Differentiating, we obtain

$$\begin{aligned} \frac{\partial V}{\partial K} &= -D(t,T)\int_K^\infty \text{IRR}(S) f(S) dS \\ \frac{\partial^2 V}{\partial K^2} &= D(t,T)\text{IRR}(K) f(K) \\ \Rightarrow \quad f(K) &= \frac{1}{D(t,T)\text{IRR}(K)}\frac{\partial^2 V(K)}{\partial K^2}. \quad \triangleleft \end{aligned}$$

Doing the same for both IRR-settled payer and receiver swaptions, we obtain

$$f(K) = \begin{cases} \frac{1}{D(0,T)}\frac{1}{\text{IRR}(K)} \times \frac{\partial^2 V^{\text{pay}}(K)}{\partial K^2} & \text{when} \quad K > S_{n,N}(0), \\ \frac{1}{D(0,T)}\frac{1}{\text{IRR}(K)} \times \frac{\partial^2 V^{\text{rec}}(K)}{\partial K^2} & \text{when} \quad K < S_{n,N}(0). \end{cases}$$

4. Suppose we wish to pay a generic function $g$ of the forward swap rate $S$, i.e. $g(S)$. Based on the static replication approach, let $F = S_{n,N}(0)$ be the expansion point, and $h(K) = \frac{g(K)}{\text{IRR}(K)}$, the value of this contract can be written as:

$$
\begin{aligned}
V_0 &= D(0,T)\mathbb{E}[g(S)] \\
&= D(0,T)\int_0^\infty g(K)f(K)dK \\
&= D(0,T)\int_0^\infty g(K)\frac{1}{D(0,T)}\frac{1}{\text{IRR}(K)} \times \frac{\partial^2 V(K)}{\partial K^2}dK \\
&= \int_0^F h(K)\frac{\partial^2 V^{\text{rec}}(K)}{\partial K^2}dK + \int_F^\infty h(K)\frac{\partial^2 V^{\text{pay}}(K)}{\partial K^2}dK \\
&= \left[h(K)\frac{\partial V^{\text{rec}}(K)}{\partial K}\right]_0^F - \int_0^F h'(K)\frac{\partial V^{\text{rec}}(K)}{\partial K}dK \\
&\quad + \left[h(K)\frac{\partial V^{\text{pay}}(K)}{\partial K}\right]_F^\infty - \int_F^\infty h'(K)\frac{\partial V^{\text{pay}}(K)}{\partial K}dK \\
&= h(F)\frac{\partial V^{\text{rec}}(F)}{\partial K} - \cancelto{0}{h(0)\frac{\partial V^{\text{rec}}(0)}{\partial K}} - [h'(K)V^{\text{rec}}(K)]_0^F + \int_0^F h''(K)V^{\text{rec}}(K)dK \\
&\quad + \cancelto{0}{h(\infty)\frac{\partial V^{\text{pay}}(\infty)}{\partial K}} - h(F)\frac{\partial V^{\text{pay}}(F)}{\partial K} - [h'(K)V^{\text{pay}}(K)]_F^\infty + \int_F^\infty h''(K)V^{\text{pay}}(K)dK \\
&= h(F)\frac{\partial V^{\text{rec}}(F)}{\partial K} - h'(F)V^{\text{rec}}(F) + \cancelto{0}{h'(0)V^{\text{rec}}(0)} + \int_0^F h''(K)V^{\text{rec}}(K)dK \\
&\quad - h(F)\frac{\partial V^{\text{pay}}(F)}{\partial K} - \cancelto{0}{h'(\infty)V^{\text{pay}}(\infty)} + h'(F)V^{\text{pay}}(F) + \int_F^\infty h''(K)V^{\text{pay}}(K)dK \\
&= -h(F)\left[\frac{\partial V^{\text{pay}}(F)}{\partial K} - \frac{\partial V^{\text{rec}}(F)}{\partial K}\right] + h'(F)[V^{\text{pay}}(F) - V^{\text{rec}}(F)] \\
&\quad + \int_0^F h''(K)V^{\text{rec}}(K)dK + \int_F^\infty h''(K)V^{\text{pay}}(K)dK
\end{aligned}
$$

The put-call parity relationship for IRR-settled swaptions is given by

$$
\begin{aligned}
V^{\text{pay}}(K) - V^{\text{rec}}(K) &= D(0,T)\mathbb{E}[\text{IRR}(S)(S-K)^+] - D(0,T)\mathbb{E}[\text{IRR}(S)(K-S)^+] \\
&= D(0,T)\text{IRR}(S)(S-K).
\end{aligned}
$$

When $K = F = S_{n,N}$, the ATM payer and receiver swaptions are worth the same amount, i.e. $V^{\text{pay}}(F) - V^{\text{rec}}(F) = 0$. Also, the first order derivative of the put-call parity relationship with respect to strike ($K$) yields:

$$
\frac{\partial V^{\text{pay}}(K)}{\partial K} - \frac{\partial V^{\text{rec}}(K)}{\partial K} = -D(0,T)\text{IRR}(S)
$$

Substituting this back into the derivation on the previous page, we obtain

$$
\begin{aligned}
V_0 &= -h(F)\left[\frac{\partial V^{\text{pay}}(F)}{\partial K} - \frac{\partial V^{\text{rec}}(F)}{\partial K}\right] + h'(F)[V^{\text{pay}}(F) - V^{\text{rec}}(F)] \\
&\qquad + \int_0^F h''(K)V^{\text{rec}}(K)dK + \int_F^\infty h''(K)V^{\text{pay}}(K)dK \\
&= D(0,T)h(F)\text{IRR(F)} + h'(F)[V^{\text{pay}}(F) - V^{\text{rec}}(F)] + \int_0^F h''(K)V^{\text{rec}}(K)dK + \int_F^\infty h''(K)V^{\text{pay}}(K)dK \\
&= D(0,T)g(F) + h'(F)[V^{\text{pay}}(F) - V^{\text{rec}}(F)] + \int_0^F h''(K)V^{\text{rec}}(K)dK + \int_F^\infty h''(K)V^{\text{pay}}(K)dK.
\end{aligned}
$$

This is the static-replication formula in the course material.

For example, for CMS rate, the payoff is $g(F) = F$, and recognizing that $V^{\text{pay}}(F) - V^{\text{rec}}(F) = 0$, we have the following CMS replication formula:

$$
V_0 = D(0,T)F + \int_0^F h''(K)V^{\text{rec}}(K)dK + \int_F^\infty h''(K)V^{\text{pay}}(K)dK.
$$