QF605 Additional Examples
Session 6: Change of Numeraire Theorem
& Convexity Correction

1 Questions

1. We have established in our course material that the domestic investor is seeing an FX process of

$$dX_t = (r^D - r^F)X_t dt + \sigma X_t dW_t^D.$$

   (a) What is the price of an FX forward from the domestic investor's perspective? You should be able to derive the interest rate parity relationship ($\mathbb{E}^D[X_T] = X_0 e^{(r^D - r^F)T}$) from this stochastic differential equation.

   (b) Show that the foreign investor will see the following SDEs

$$\begin{cases} d\dfrac{1}{X_t} = (r^F - r^D + \sigma^2)\dfrac{1}{X_t}dt - \sigma\dfrac{1}{X_t}dW_t^D \\ d\dfrac{1}{X_t} = (r^F - r^D)\dfrac{1}{X_t}dt - \sigma\dfrac{1}{X_t}dW_t^F \end{cases}$$

   (c) Derive the FX forward price from the foreign investor's perspective and show that its expectation (i.e. $\mathbb{E}^F\left[\frac{1}{X_T}\right]$) is consistent with the forward price obtained by the domestic investor.

2. In a LIBOR-in-arrear contract, the LIBOR rate L_i is observed at time T_i and paid immediately at time T_i.

   (a) Using LIBOR market model, evaluate the following expectation by performing single-currency convexity correction

$$\mathbb{E}^i[L_i(T_i)]$$

   (b) Derive the valuation formula for a LIBOR-in-arrear caplet paying

$$(L_i(T_i) - K)^+,$$

   observed and paid at T_i.

3. (a) A contract pays

$$\Delta_i \times L_i(T)$$

   at $T = T_{i+1}$. Derive a valuation formula for this contract using LIBOR market model.

   (b) A contract pays

$$\Delta_i \times L_i(T)$$

   at $T = T_i$. Derive a valuation formula for this contract using LIBOR market model.

4. Let L_i^D be a forward LIBOR rate in the domestic economy, observed at time T_i and paid at T_{i+1}. It follows the LIBOR market model in the domestic economy with a volatility of σ_i. There is also a forward foreign exchange process following

$$dF_t = \sigma_X F_t dW_t^D$$

from the domestic investor's perspective. Suppose the Brownian motion of the exchange rate process W_t^D and the LIBOR market model W_t^{i+1} is correlated with ρ, evaluate the following expectation (foreign investor's perspective)

$$\mathbb{E}^{i+1,F}\left[L_i^D(T)\right]$$

5. Given three correlated Brownian motions W_t^f, W_t^g and W_t^h with correlations

$$dW_t^f dW_t^g = \rho_{fg}dt, \quad dW_t^g dW_t^h = \rho_{gh}dt, \quad dW_t^f dW_t^h = \rho_{fh}dt,$$

determine the coefficients α_{11}, α_{12}, α_{22}, α_{13}, α_{23} and α_{33} for the Cholesky decomposition

$$\begin{aligned} dW_t^f &= \alpha_{11}dZ_t^1 \\ dW_t^g &= \alpha_{12}dZ_t^1 + \alpha_{22}dZ_t^2 \\ dW_t^h &= \alpha_{13}dZ_t^1 + \alpha_{23}dZ_t^2 + \alpha_{33}dZ_t^3, \end{aligned}$$

where Z_t^1, Z_t^2 and Z_t^3 are three mutually independent Brownian motions.

6. **Discussion** *Siegel's Exchange Rate Paradox*
Let us start by restricting ourselves to the simplified discrete FX market involving the SGD and USD economies, where the spot FX rate is $FX_0 \approx 1.25$. Using a one-step binomial model, if we have $u = \frac{6}{5}$, $d = \frac{5}{6}$, risk-neutral probabilities of $p^* = q^* = 0.5$, what is the expected forward exchange rate?

$$1.25 \begin{array}{l} \nearrow \; 1.25 \times \frac{6}{5} \\ \searrow \; 1.25 \times \frac{5}{6} \end{array}$$

From the perspective of a Singapore-based investor, we will see

$$\mathbb{E}[FX_T] = \frac{1}{2} \times \frac{6}{5} \times 1.25 + \frac{1}{2} \times \frac{5}{6} \times 1.25 = 1.423.$$

From the perspective of a US-based investor, we will see

$$\mathbb{E}\left[\frac{1}{FX_T}\right] = \frac{1}{2} \times \frac{5}{6} \times \frac{1}{1.25} + \frac{1}{2} \times \frac{6}{5} \times \frac{1}{1.25} = 0.726.$$

Note that $\frac{1}{0.726} = 1.377 \neq 1.423$. So apparently we have a paradox here, since the same binomial FX model gives rise to two different expectations, depending on which economy we use as denomination.

In the continuous time model, this paradox is equivalent to saying that if we see

$$dX_t = (r^D - r^F)X_t dt + \sigma X_t dW_t^D$$

in the Singapore economy, then we ought to see the following in the US economy

$$d\frac{1}{X_t} = (r^F - r^D)\frac{1}{X_t}dt - \sigma\frac{1}{X_t}dW_t^D.$$

This is obviously incorrect since the first SDE is based the martingale measure with the SGD money market account as numeraire. For a US investor, the SGD money market account is not a tradable, and hence cannot be used as a numeraire. Applying Itô's formula, we obtain the SDE for the process $\frac{1}{X_t}$ as follow

$$d\frac{1}{X_t} = (r^F - r^D + \sigma^2)\frac{1}{X_t}dt - \sigma\frac{1}{X_t}dW_t^D.$$

As we've shown in the previous section, we need to use the USD money market account as a numeraire when we are taking the US investor's perspective. In this case, we obtain

$$d\frac{1}{X_t} = (r^F - r^D)\frac{1}{X_t}dt - \sigma\frac{1}{X_t}dW_t^F,$$

which resolves the paradox.

2 Suggested Solutions

1. (a) First apply Itô's formula to solve the exchange rate process.

$$
\begin{aligned}
dX_t &= (r^D - r^F)X_t dt + \sigma X_t dW_t^D \\
d\log(X_t) &= \left(r^D - r^F - \frac{\sigma^2}{2}\right) dt + \sigma dW_t^D \\
X_T &= X_0 e^{\left(r^D - r^F - \frac{\sigma^2}{2}\right)T + \sigma W_T^D}
\end{aligned}
$$

Next, we take the expectation under the risk-neutral measure associated to the domestic risk-free bond to obtain

$$
\mathbb{E}^D[X_T] = X_0 e^{(r^D - r^F)T}. \quad \triangleleft
$$

This is the interest rate parity relationship describing the price of an FX forward contract from the domestic investor's perspective.

(b) Starting with

$$
dX_t = (r^D - r^F)X_t dt + \sigma X_t dW_t^D,
$$

define $Y_t = f(X_t) = \frac{1}{X_t}$, we apply Itô's formula to derive

$$
\begin{aligned}
dY_t &= f'(X_t)dX_t + \frac{1}{2}f''(X_t)(dX_t)^2 \\
d\frac{1}{X_t} &= (r^F - r^D + \sigma^2)\frac{1}{X_t}dt - \sigma \frac{1}{X_t} dW_t^D \quad \triangleleft
\end{aligned}
$$

On the other hand, starting with B_t^D and

$$
dX_t = \mu X_t dt + \sigma X_t dW_t,
$$

we note that $\frac{B_t^D}{X_t}$ is a foreign tradable. Let $Y_t = \frac{B_t^D}{X_t} = f(B_t^D, X_t)$, we again use Itô's formula to derive

$$
dY_t = (r^D + \sigma^2 - \mu)Y_t dt - \sigma Y_t dW_t.
$$

Next, note that $Z_t = \frac{Y_t}{B_t^F} = g(Y_t, B_t^F)$ is a ratio of foreign tradable assets, we use Itô's formula to derive

$$
\begin{aligned}
dZ_t &= (r^D - r^F + \sigma^2 - \mu)Z_t dt - \sigma Z_t dW_t \\
&= -\sigma Z_t \left(dW_t - \frac{r^D - r^F + \sigma^2 - \mu}{\sigma} dt \right) \\
&= -\sigma Z_t dW_t^F
\end{aligned}
$$

Substituting

$$
dW_t = dW_t^F + \frac{r^D - r^F + \sigma^2 - \mu}{\sigma} dt
$$

to

$$
dX_t = \mu X_t dt + \sigma X_t dW_t,
$$

we get

$$dX_t = (r^D - r^F + \sigma^2)X_t dt + \sigma X_t dW_t^F.$$

Finally, apply Itô's formula to obtain the stochastic differential equation for $\frac{1}{X_t}$ as

$$d\frac{1}{X_t} = (r^F - r^D)\frac{1}{X_t}dt - \sigma\frac{1}{X_t}dW_t^F. \quad \triangleleft$$

(c) Solving the stochastic differential equation, we get

$$\begin{aligned} d\log\left(\frac{1}{X_t}\right) &= \left(r^F - r^D - \frac{\sigma^2}{2}\right)dt - \sigma dW_t^F \\ \frac{1}{X_T} &= \frac{1}{X_0}e^{\left(r^F - r^D - \frac{\sigma^2}{2}\right)T - \sigma W_T} \end{aligned}$$

Taking expectation under the risk-neutral measure associated with the foreign bond numeraire, we obtain

$$\mathbb{E}\left[\frac{1}{X_T}\right] = \frac{1}{X_0}e^{(r^F - r^D)T}. \quad \triangleleft$$

2. (a) Under the LIBOR market model, the forward LIBOR $L_i(t)$ is a martingale under the risk-neutral measure associated to the numeraire $D_{i+1}(t)$:

$$L_i(T) = L_i(0)e^{-\frac{\sigma_i^2 T}{2} + \sigma_i W_T^{i+1}}.$$

Using Radon-Nikodym derivative to change the measure, we obtain

$$\begin{aligned} \mathbb{E}^i[L_i(T)] &= \mathbb{E}^{i+1}\left[\frac{d\mathbb{Q}^i}{d\mathbb{Q}^{i+1}}L_i(T)\right] \\ &= \mathbb{E}^{i+1}\left[\frac{D_i(T)/D_i(0)}{D_{i+1}(T)/D_{i+1}(0)}L_i(T)\right] \\ &= \frac{1}{1+\Delta_i L_i(0)}\mathbb{E}^{i+1}\Big[(1+\Delta_i L_i(T))\cdot L_i(T)\Big] \\ &= \frac{1}{1+\Delta_i L_i(0)}\Big[L_i(0) + \Delta_i L_i(0)^2 e^{\sigma_i^2 T}\Big] \\ &= L_i(0)\cdot\frac{1+\Delta_i L_i(0)e^{\sigma_i^2 T}}{1+\Delta_i L_i(0)} \quad \triangleleft \end{aligned}$$

(b) The LIBOR-in-arrear caplet contract can be valued as

$$
\begin{aligned}
V_0 &= D_i(0)\mathbb{E}^i[(L_i(T)-K)^+] \\
&= D_i(0)\mathbb{E}^{i+1}\left[\frac{d\mathbb{Q}^i}{d\mathbb{Q}^{i+1}}(L_i(T)-K)^+\right] \\
&= D_i(0)\mathbb{E}^{i+1}\left[\frac{D_i(T)/D_i(0)}{D_{i+1}(T)/D_{i+1}(0)}(L_i(T)-K)^+\right] \\
&= D_{i+1}(0)\mathbb{E}^{i+1}\Big[(1+\Delta_i L_i(T))\cdot(L_i(T)-K)^+\Big] \\
&= D_{i+1}(0)\Big\{\mathbb{E}^{i+1}[(L_i(T)-K)^+] + \Delta_i\mathbb{E}^{i+1}[L_i(T)(L_i(T)-K)^+]\Big\} \\
&= D_{i+1}(0)\left[L_i(0)\Phi\left(\frac{\log\frac{L_i(0)}{K}+\frac{\sigma_i^2 T}{2}}{\sigma_i\sqrt{T}}\right) - K\Phi\left(\frac{\log\frac{L_i(0)}{K}-\frac{\sigma_i^2 T}{2}}{\sigma_i\sqrt{T}}\right)\right] \\
&\quad + \Delta_i D_{i+1}(0)\left[L_i(0)^2 e^{\sigma_i^2 T}\Phi\left(-x^*+2\sigma_i\sqrt{T}\right) - L_i(0)K\Phi\left(-x^*+\sigma_i\sqrt{T}\right)\right] \\
&= D_{i+1}(0)\left[L_i(0)\Phi\left(\frac{\log\frac{L_i(0)}{K}+\frac{\sigma_i^2 T}{2}}{\sigma_i\sqrt{T}}\right) - K\Phi\left(\frac{\log\frac{L_i(0)}{K}-\frac{\sigma_i^2 T}{2}}{\sigma_i\sqrt{T}}\right)\right] \\
&\quad + \Delta_i D_{i+1}(0)\left[L_i(0)^2 e^{\sigma_i^2 T}\Phi\left(\frac{\log\frac{L_i(0)}{K}+\frac{3\sigma_i^2 T}{2}}{\sigma_i\sqrt{T}}\right)\right. \\
&\qquad \left. - L_i(0)K\Phi\left(\frac{\log\frac{L_i(0)}{K}+\frac{\sigma_i^2 T}{2}}{\sigma_i\sqrt{T}}\right)\right] \quad \triangleleft
\end{aligned}
$$

3. (a) The LIBOR market model is defined as

$$dL_i(t) = \sigma_i L_i(t) dW^{i+1}(t),$$

where $W^{i+1}(t)$ is a standard Brownian motion under the risk neutral measure $\mathbb{Q}^{i+1}$, associated to the zero coupon discount bond $D_{i+1}(t) = D(t, T_{i+1})$. The solution to the LIBOR market model is given by

$$L_i(T) = L_i(0)e^{-\frac{\sigma_i^2 T}{2}+\sigma_i W_T^{i+1}}.$$

Let V_t denote the value of the financial contract at time t. Under the martingale measure, we have

$$
\begin{aligned}
\frac{V_0}{D_{i+1}(0)} &= \mathbb{E}^{i+1}\left[\frac{V_T}{D_{i+1}(T)}\right] \\
V_0 &= D(0,T_{i+1})\mathbb{E}^{i+1}\left[\Delta_i L_i(T)\right] \\
&= D(0,T_{i+1})\Delta_i\mathbb{E}^{i+1}\left[L_i(0)e^{-\frac{\sigma_i^2 T}{2}+\sigma_i W_T^{i+1}}\right] \\
&= D(0,T_{i+1})\Delta_i L_i(0)e^{-\frac{\sigma_i^2 T}{2}+\frac{\sigma_i^2 T}{2}} \\
&= D(0,T_{i+1})\Delta_i L_i(0) \quad \triangleleft
\end{aligned}
$$

(b) We value the contract as follows:

$$
\begin{aligned}
V_0 &= D(0,T_i)\mathbb{E}^i\left[\Delta_i \times L_i(T_i)\right] \\
&= D(0,T_i)\mathbb{E}^{i+1}\left[\frac{d\mathbb{Q}^i}{d\mathbb{Q}^{i+1}} \cdot \Delta_i L_i(T)\right] \\
&= D(0,T_i)\Delta_i\mathbb{E}^{i+1}\left[\frac{L_i(T_i) + \Delta_i L_i(T_i)^2}{1+\Delta_i L_i(0)}\right] \\
&= D(0,T_i)\frac{\Delta_i}{1+\Delta_i L_i(0)}\mathbb{E}^{i+1}\left[L_i(0)e^{-\frac{\sigma_i^2 T}{2}+\sigma_i W_{T_i}^{i+1}} + \Delta_i L_i(0)^2 e^{-\sigma_i^2 T + 2\sigma_i W_{T_i}^{i+1}}\right] \\
&= D(0,T_i)\frac{\Delta_i}{1+\Delta_i L_i(0)}\left[L_i(0) + \Delta_i L_i(0)^2 e^{\sigma_i^2 T_i}\right] \quad \triangleleft
\end{aligned}
$$

4. We have

$$dF_t = \sigma_X F_t dW_t^D \quad \Rightarrow \quad F_T = F_0 e^{-\frac{\sigma_X^2 T}{2} + \sigma_X W_T^D}$$

and

$$dL_i^D(t) = \sigma_i L_i^D(t) dW_t^{i+1} \quad \Rightarrow \quad L_i^D(T) = L_i^D(0) e^{-\frac{\sigma_i^2 T}{2} + \sigma_i W_T^{i+1}}$$

with $W_T^{i+1} \cdot W_T^D = \rho dt$. We apply multi-currency change of numeraire theorem to evaluate the expectation:

$$\begin{aligned}
\mathbb{E}^{i+1,F}\left[L_i^D(T)\right] &= \mathbb{E}^{i+1,D}\left[L_i^D(T) \cdot \frac{d\mathbb{Q}^{i+1,F}}{d\mathbb{Q}^{i+1,D}}\right] \\
&= \mathbb{E}^{i+1,D}\left[L_i^D(T) \cdot \frac{\frac{X_T D_{i+1}^F(T)}{X_0 D_{i+1}^F(0)}}{\frac{D_{i+1}^D(T)}{D_{i+1}^D(0)}}\right] \\
&= \mathbb{E}^{i+1,D}\left[L_i^D(T) \cdot \frac{\frac{X_T D_{i+1}^F(T)}{D_{i+1}^D(T)}}{\frac{X_0 D_{i+1}^F(0)}{D_{i+1}^D(0)}}\right] \\
&= \mathbb{E}^{i+1,D}\left[L_i^D(T) \cdot \frac{F_T}{F_0}\right] \\
&= \mathbb{E}^{i+1,D}\left[L_i^D(0) e^{-\frac{\sigma_i^2 T}{2} + \sigma_i W_T^{i+1}} e^{-\frac{\sigma_X^2 T}{2} + \sigma_X W_T^D}\right] \\
&= L_i^D(0) e^{-\frac{\sigma_i^2 T}{2}} e^{-\frac{\sigma_X^2 T}{2}} \mathbb{E}^{i+1,D}\left[e^{\sigma_i W_T^{i+1} + \sigma_X W_T^D}\right] \\
&= L_i^D(0) e^{-\frac{\sigma_i^2 T}{2}} e^{-\frac{\sigma_X^2 T}{2}} \mathbb{E}^{i+1,D}\left[e^{\sigma_i Z_T^1 + \sigma_X\left(\rho Z_T^1 + \sqrt{1-\rho^2} Z_T^2\right)}\right] \\
&= L_i^D(0) e^{-\frac{\sigma_i^2 T}{2}} e^{-\frac{\sigma_X^2 T}{2}} \mathbb{E}^{i+1,D}\left[e^{(\sigma_i + \rho\sigma_X) Z_T^1 + \sigma_X \sqrt{1-\rho^2} Z_T^2}\right] \\
&= L_i^D(0) e^{\sigma_i \sigma_X \rho T} \quad \triangleleft
\end{aligned}$$

5. Under Cholesky decomposition

$$\begin{aligned}
dW_t^f &= \alpha_{11} dZ_t^1 \\
dW_t^g &= \alpha_{12} dZ_t^1 + \alpha_{22} dZ_t^2 \\
dW_t^h &= \alpha_{13} dZ_t^1 + \alpha_{23} dZ_t^2 + \alpha_{33} dZ_t^3,
\end{aligned}$$

where Z_t^1, Z_t^2 and Z_t^3 are three mutually independent Brownian motions, we have

$$\begin{aligned}
\alpha_{11} &= 1 \\
\alpha_{12} &= \rho_{fg} \\
\alpha_{13} &= \rho_{fh} \\
\alpha_{22} &= \sqrt{1 - \alpha_{12}^2} \\
\alpha_{23} &= \frac{\rho_{gh} - \alpha_{12}\alpha_{13}}{\alpha_{22}} \\
\alpha_{33} &= \sqrt{1 - \alpha_{13}^2 - \alpha_{23}^3}
\end{aligned}$$