QF605 Fixed-Income Securities
Assignment 3, Due Date: 6-Mar-2024

1. (a) Write down the LIBOR Market Model (LMM), and identify under what numeraire is the LIBOR process a martingale.

   (b) A contract pays

   $$\Delta_i \times \sqrt{L_i(T)}$$

   at $T = T_{i+1}$. Derive a valuation formula for this contract using LIBOR market model.

   (c) Consider a contract with the following payoff at time $T = T_{i+1}$:

   $$\begin{cases} \$1 & \text{if } K_1 \leq L_i(T) \leq K_2 \\ 0 & \text{otherwise} \end{cases}$$

   Derive a valuation formula for this contract using LIBOR market model.

2. Under the Swap Market Model (SMM), the forward swap rate follows the stochastic differential equation

   $$dS_{n,N}(t) = \sigma_{n,N} S_{n,N}(t) dW^{n+1,N}.$$

   (a) What is the numeraire security associated with the risk-neutral measure $\mathbb{Q}^{n+1,N}$, under which $W^{n+1,N}$ is a standard Brownian motion?

   (b) A floating-leg-or-nothing digital option pays

   $$P_{n+1,N}(T) S_{n,N}(T) \mathbb{1}_{S_{n,N}(T)>K}$$

   on maturity T, where $P_{n+1,N}$ is the ***present value of a basis point***. Derive a valuation formula for this contract.

   (c) A contract pays

   $$S_{n,N}(T)$$

   on maturity T. Briefly explain why we cannot value this simple contract directly using the Swap Market Model without applying convexity correction.

1. (a) $$\Delta_i L_i(t) = \frac{D_i(t) - D_{i+1}(t)}{D_{i+1}(t)}$$

As Δ_i is a constant, $L_i(t)$ is a martingale under Q^{i+1} measure.

$$dL_i(t) = \sigma_i L_i(t)\, dW^{i+1}(t) \Rightarrow L_i(t) = L_i(0)\, e^{-\frac{1}{2}\sigma_i^2 t + \sigma_i W^{i+1}(t)}$$

(b) $$dL_i(t) = \sigma_i L_i(t)\, dW^{i+1}(t)$$

$$L_i(T) = L_i(0)\, e^{-\frac{\sigma_i^2}{2}T + \sigma_i W^{i+1}(T)}$$

$$\frac{V_0}{D_{i+1}(0)} = E^{i+1}\left[\frac{V_{T_{i+1}}}{D_{i+1}(T_{i+1})}\right]$$

$$V_0 = D_{i+1}(0)\, E^{i+1}\left[V_{T_{i+1}}\right]$$

$$= D_{i+1}(0)\, \Delta_i\, E^{i+1}\left[\sqrt{L_i(T)}\right]$$

$$= D_{i+1}(0)\, \Delta_i\, E^{i+1}\left[\sqrt{L_i(0)}\, e^{-\frac{\sigma_i^2}{4}T + \frac{1}{2}\sigma_i W^{i+1}(T)}\right]$$

$$= D_{i+1}(0)\, \Delta_i \sqrt{L_i(0)}\, e^{-\frac{\sigma_i^2 T}{8}}$$

$$= D(0, T_{i+1})\, \Delta_i \sqrt{L_0(T_i, T_{i+1})}\, e^{-\frac{\sigma_i^2 T}{8}}$$

(c) $$V_0 = D_{i+1}(0)\, E^{i+1}\left[\mathbb{1}_{K_1 \le L_i(T) \le K_2}\right]$$

$$= D_{i+1}(0) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbb{1}_{K_1 \le L_i(T) \le K_2}\, e^{-\frac{x^2}{2}}\, dx$$

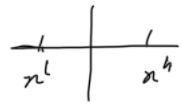
$$= D_{i+1}(0)\left[\Phi(-x^l) - \Phi(x^h)\right]$$

$$= D(0, T_{i+1})\left[\Phi(-x^l) - \Phi(x^h)\right]$$

$$K_1 \le L_i(0)\, e^{-\frac{\sigma_i^2}{2}T + \sigma_i\sqrt{T}x} \le K_2$$

$$\frac{\ln\frac{K_1}{L_i(0)} + \frac{\sigma_i^2}{2}T}{\sigma_i\sqrt{T}} \le x = x^l$$

$$\frac{\ln\frac{K_2}{L_i(0)} + \frac{\sigma_i^2}{2}T}{\sigma_i\sqrt{T}} \ge x = x^h$$



$$\Phi(-x^l) - \Phi(x^h)$$

2.

(a) use $P_{n+1,N}(t) = \sum_{i=n+1}^{N} \Delta_{i-1} D_i(t)$ as a numeraire under $Q^{n+1,N}$

$$-\frac{x^2 - 2\sigma_{n,N}\sqrt{T}x + \sigma_{n,N}^2 T}{2}$$

(b) $$\frac{V_0}{P_{n+1,N}(0)} = E^{n+1,N}\left[\frac{V_T}{P_{n+1,N}(T)}\right]$$

$$x^l = x > \frac{\ln\frac{K}{S_{n,N}(0)} + \frac{1}{2}\sigma_{n,N}^2 T}{\sigma_{n,N}\sqrt{T}}$$

$$V_0 = P_{n+1,N}(0)\, E^{n+1,N}\left[S_{n,N}(T)\, \mathbb{1}_{S_{n,N}(T) > K}\right]$$

$$= P_{n+1,N}(0) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} S_{n,N}(0)\, e^{-\frac{1}{2}\sigma_{n,N}^2 T + \sigma_{n,N}\sqrt{T}x}\, \mathbb{1}_{S_{n,N}(T) > K}\, e^{-\frac{x^2}{2}}\, dx$$

$$= P_{n+1,N}(0)\, S_{n,N}(0)\, \Phi(-x^l + \sigma_{n,N}\sqrt{T})$$

(c) $$V_0 = P_{n+1,N}(0)\, E^{n+1,N}\left[\frac{S_{n,N}(T)}{P_{n+1,N}(T)}\right] = P_{n+1,N}(0)\, E^{n+1,N}\left[D_n(t) - D_N(t)\right]$$

not in the right measure