QF605 Additional Examples
Session 8: Short-Rate Models, Ho-Lee & Hull-White

1 Questions

1. Consider a stylized interest rate model

$$dr_t = \mu\, dt + \sigma\, dW_t^*,$$

where W_t^* is a standard Brownian motion under the risk-neutral measure $\mathbb{Q}^*$.

(a) Determine the distribution, mean, and variance of the integral

$$\int_t^T r_u\, du.$$

(b) Identify the expressions $A(t,T)$ and $B(t,T)$ in the following expectation:

$$D(t,T) = \mathbb{E}^*\left[e^{-\int_t^T r_u\, du}\right] = e^{A(t,T) - r_t B(t,T)}.$$

(c) Explain what is an affine interest rate model. Is the short rate model considered above an affine interest rate model?

2. Consider the Vasicek short rate model

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t^*.$$

Determine the mean and variance of the integral

$$\int_0^T r_u du,$$

and use this to evaluate the expectation

$$D(0,T) = \mathbb{E}^*\left[e^{-\int_0^T r_u du}\right].$$

3. Consider the Ho-Lee interest rate model

$$dr_t = \theta(t)\, dt + \sigma\, dW_t^*,$$

where W_t^* is a standard Brownian motion under the risk-neutral measure $\mathbb{Q}^*$. Show that

$$\theta(t) = -\frac{\partial^2}{\partial t^2} \log D(0,t) + \sigma^2 t.$$

4. Suppose we use a discrete ($\Delta t = 1y$) binomial-tree approximation of the Ho-Lee model, where at every step the rate can move up or down by 1%, and the risk-neutral probabilities of an up or down move are both 0.5. We observe the following discount factors:

Instrument	Value
$D(0, 1y)$	0.95
$D(0, 2y)$	0.88

Determine the no-arbitrage values for θ_0.

5. Suppose we use a discrete ($\Delta t = 1y$) binomial-tree approximation of the Ho-Lee model, where at every step the rate can move up or down by 0.5%, and the risk-neutral probabilities of an up or down move are both 0.5. We observe the following discount factors:

Instrument	Value
$D(0, 1y)$	0.95123
$D(0, 2y)$	0.86936
$D(0, 3y)$	0.78663

Draw the Ho-Lee binomial tree and determine the no-arbitrage values for θ_0 and θ_1.

2 Suggested Solutions

1. (a) First we integrate the stochastic differential equation from t to s to obtain:

$$r_s = r_t + \mu(s-t) + \int_t^s \sigma \, dW_u^*.$$

Next we integrate r_s from t to T to obtain

$$\begin{aligned}\int_t^T r_s \, ds &= r_t(T-t) + \mu \int_t^T (s-t) \, ds + \int_t^T \int_t^s \sigma \, dW_u^* \, ds \\ &= r_t(T-t) + \mu \left[\frac{s^2}{2} - ts \right]_t^T + \int_t^T \int_u^T \sigma \, ds \, dW_u^* \\ &= r_t(T-t) + \frac{\mu}{2}(T-t)^2 + \int_t^T \sigma(T-u) \, dW_u^*\end{aligned}$$

Hence the mean is given by

$$\mathbb{E}^* \left[\int_t^T r_s \, ds \right] = r_t(T-t) + \frac{\mu}{2}(T-t)^2, \quad \triangleleft$$

and the variance is given by

$$\begin{aligned}V \left[\int_t^T r_s \, ds \right] &= \int_t^T \sigma^2 (T-u)^2 du \\ &= \frac{\sigma^2 (T-t)^3}{3} \quad \triangleleft\end{aligned}$$

(b) Having identified the mean and variance of the short rate integral, we have

$$\begin{aligned}D(t,T) &= \mathbb{E}^* \left[e^{-\int_t^T r_u \, du} \right] \\ &= e^{-r_t(T-t) - \frac{\mu}{2}(T-t)^2 + \frac{1}{2}\frac{\sigma^2(T-t)^3}{3}}\end{aligned}$$

Comparing this against

$$D(t,T) = e^{A(t,T) - r_t B(t,T)},$$

we note that

$$\begin{aligned}A(t,T) &= -\frac{\mu}{2}(T-t)^2 + \frac{\sigma^2(T-t)^3}{6} \quad \triangleleft \\ B(t,T) &= (T-t) \quad \triangleleft\end{aligned}$$

(c) For affine interest rate model, the zero coupon bond prices can be written as

$$D(t,T) = e^{A(t,T) - r_t B(t,T)},$$

for some deterministic functions of $A(t,T)$ and $B(t,T)$ of t and T only. This implies that

$$R(t,T) = \frac{1}{T-t}\Big(-A(t,T) + r_t B(t,T) \Big),$$

i.e. the zero (spot) rates are affine functions of the short rate. $\triangleleft$

2. Consider the Vasicek short rate model

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t^*.$$

We can solve this stochastic differential equation by applying Itô's formula to the function $f(r_t, t) = e^{\kappa t} r_t$, and the solution is given by

$$r_t = r_0 e^{-\kappa t} + \theta\left(1 - e^{-\kappa t}\right) + \sigma \int_0^t e^{\kappa(u-t)}\, dW_u^*$$

Integrating both sides from 0 to T, we have

$$\int_0^T r_t\, dt = \int_0^T r_0 e^{-\kappa t}\, dt + \int_0^T \theta\left(1 - e^{-\kappa t}\right)\, dt + \underbrace{\int_0^T \int_0^t \sigma e^{\kappa(u-t)}\, dW_u^*\, dt}_{\text{double integral}}.$$

On the right hand side, the first and second integrals can be carried out directly. The double integral can be simplified by exchanging the order of integration (Fubini's Theorem):

$$\begin{array}{lll} \text{Inner Integral } u: & 0 \le u \le T & \\ \text{Outer Integral } t: & 0 \le t \le T & \end{array} \Rightarrow \begin{array}{ll} \text{Inner Integral } t: & u \le t \le T \\ \text{Outer Integral } u: & 0 \le u \le T \end{array}$$

So we have

$$\begin{aligned} \int_0^T \int_0^t \sigma e^{\kappa(u-t)}\, dW_u^*\, dt &= \int_0^T \int_u^T \sigma e^{\kappa(u-t)}\, dt\, dW_u^* \\ &= \int_0^T \left[-\frac{\sigma}{\kappa} e^{\kappa(u-t)}\right]_u^T dW_u^* \\ &= \frac{\sigma}{\kappa} \int_0^T \left(1 - e^{\kappa(u-T)}\right)\, dW_u^* \end{aligned}$$

So we can write the overall integral as:

$$\int_0^T r_t\, dt = \int_0^T r_0 e^{-\kappa t}\, dt + \int_0^T \theta\left(1 - e^{-\kappa t}\right)\, dt + \frac{\sigma}{\kappa} \int_0^T \left(1 - e^{\kappa(u-T)}\right)\, dW_u^*$$

Taking expectation on both sides gives us the mean of this integral

$$\begin{aligned} \mathbb{E}^*\left[\int_0^T r_t\, dt\right] &= \int_0^T r_0 e^{-\kappa t}\, dt + \int_0^T \theta\left(1 - e^{-\kappa t}\right)\, dt \\ &= \frac{r_0}{\kappa}\left(1 - e^{-\kappa T}\right) + \theta T - \frac{\theta}{\kappa}\left(1 - e^{-\kappa T}\right). \end{aligned}$$

Taking the variance, we obtain

$$\begin{aligned} V\left[\int_0^T r_t\, dt\right] &= V\left[\int_0^T r_0 e^{-\kappa t}\, dt + \int_0^T \theta\left(1 - e^{-\kappa t}\right)\, dt + \frac{\sigma}{\kappa} \int_0^T \left(1 - e^{\kappa(u-T)}\right)\, dW_u^*\right] \\ &= V\left[\frac{\sigma}{\kappa} \int_0^T \left(1 - e^{\kappa(u-T)}\right)\, dW_u^*\right] \\ &= \frac{\sigma^2}{\kappa^2} \int_0^T \left(1 - e^{\kappa(u-T)}\right)^2\, du \qquad \because \text{ Itô's Isometry} \\ &= \frac{\sigma^2}{\kappa^2} \int_0^T \left(1 - 2e^{\kappa(u-T)} + e^{2\kappa(u-T)}\right)\, du \\ &= \frac{\sigma^2}{\kappa^2}\left[T - \frac{2}{\kappa}\left(1 - e^{-\kappa T}\right) + \frac{1}{2\kappa}\left(1 - e^{-2\kappa T}\right)\right] \end{aligned}$$

Finally, we can express the discount factor as

$$
\begin{aligned}
D(0,T) &= \mathbb{E}^*\left[e^{-\int_0^T r_t\,dt}\right] \\
&= \exp\left(\underbrace{-\frac{r_0}{\kappa}\left(1-e^{-\kappa T}\right) - \theta T + \frac{\theta}{\kappa}\left(1-e^{-\kappa T}\right)}_{\text{mean}} + \frac{1}{2}\cdot\underbrace{\frac{\sigma^2}{\kappa^2}\left[T - \frac{2}{\kappa}\left(1-e^{-\kappa T}\right) + \frac{1}{2\kappa}\left(1-e^{-2\kappa T}\right)\right]}_{\text{variance}}\right)
\end{aligned}
$$

3. Ho-Lee interest rate model is given by

$$dr_t = \theta(t)dt + \sigma dW_t^*,$$

where W_t^* is a Brownian motion under the measure $\mathbb{Q}^*$. To fit the initial term structure, we require that

$$\theta(t) = -\frac{\partial^2}{\partial t^2}\log D(0,t) + \sigma^2 t.$$

To prove this, first write out the interest rate process in integral format

$$r_t = r_0 + \int_0^t \theta(s)ds + \int_0^t \sigma dW_s^*.$$

Next

$$
\begin{aligned}
\int_0^t r_u du &= \int_0^t r_0 du + \int_0^t\int_0^u \theta(s)dsdu + \int_0^t\int_0^u \sigma dW_s^* du \\
&= r_0 t + \int_0^t \theta(s)(t-s)ds + \int_0^t \sigma(t-s)dW_s^*.
\end{aligned}
$$

The mean of this stochastic integral is given by

$$\mathbb{E}\left[\int_0^t r_u du\right] = r_0 t + \int_0^t \theta(s)(t-s)ds,$$

and the variance is given by

$$V\left[\int_0^t r_u du\right] = \int_0^t \sigma^2(t-s)^2 dt = \frac{1}{3}\sigma^2 t^3,$$

where we have used Itô Isometry. Therefore, the zero-coupon discount bond is given by

$$D(0,t) = \mathbb{E}[e^{-\int_0^t r_u du}] = \exp\left[-r_0 t - \int_0^t \theta(s)(t-s)ds + \frac{1}{6}\sigma^2 t^3\right].$$

From here we can work out that

$$
\begin{aligned}
\log D(0,t) &= -r_0 t - \int_0^t \theta(s)(t-s)ds + \frac{1}{6}\sigma^2 t^3 \\
\frac{\partial}{\partial t}\log D(0,t) &= -r_0 - \int_0^t \theta(s)ds + \frac{1}{2}\sigma^2 t^2 \\
\frac{\partial^2}{\partial t^2}\log D(0,t) &= -\theta(t) + \sigma^2 t \\
\Rightarrow \quad \theta(t) &= -\frac{\partial^2}{\partial t^2}\log D(0,t) + \sigma^2 t \quad \triangleleft
\end{aligned}
$$

4. Given the first discount factor $D(0, 1y) = 0.95$, we can work out the discretized short rate starts at $r = -\log(0.95) = 5.129\%$. The binomial tree process is given by

$$
\begin{tikzcd}[column sep=huge]
 & \begin{array}{c} r{=}6.129\%{+}\theta_0 \\ D(1{,}2){=}e^{-(6.129\%+\theta_0)\cdot 1} \end{array} \\
\begin{array}{c} r{=}5.129\% \\ D(0{,}2){=}0.88 \end{array} \arrow[ur] \arrow[dr] & \\
 & \begin{array}{c} r{=}4.129\%{+}\theta_0 \\ D(1{,}2){=}e^{-(4.129\%+\theta_0)\cdot 1} \end{array}
\end{tikzcd}
$$

$$
\begin{aligned}
D(0, 2y) &= D(0, 1y) \times D(1y, 2y) \\
0.88 &= 0.95 \times 0.5 \times \left(e^{-6.129\%+\theta_0} + e^{-4.129\%+\theta_0}\right) \\
\Rightarrow \quad \theta_0 &= 0.0253 \quad \triangleleft
\end{aligned}
$$

5. First work out the initial short rate:

$$
D(0, 1y) = 0.95123 = e^{-R(0,1)\times 1} \qquad \Rightarrow \qquad R(0, 1) = r_0 = 5\%
$$

The 3-period binomial tree for the short-rate under Ho-Lee model is as follows:

$$
\begin{tikzcd}
 & & r = 6\% + \theta_0 + \theta_1 \\
 & r = 5.5\% + \theta_0 \arrow[ur] \arrow[dr] & \\
r = 5\% \arrow[ur] \arrow[dr] & & r = 5\% + \theta_0 + \theta_1 \\
 & r = 4.5\% + \theta_0 \arrow[ur] \arrow[dr] & \\
 & & r = 4\% + \theta_0 + \theta_1
\end{tikzcd}
$$

Using $D(0, 2y)$, we write

$$
\begin{aligned}
D(0, 2y) &= D(0, 1y) \times \mathbb{E}^*[D(1y, 2y)] \\
&= D(0, 1y) \times \left[\frac{1}{2} \times e^{-(\theta_0+0.055)} + \frac{1}{2} \times e^{-(\theta_0+0.045)}\right] \\
\Rightarrow \quad \theta_0 &= 0.04 \quad \triangleleft
\end{aligned}
$$

Next, we note that

$$
\begin{cases}
D_u(1, 3) = e^{-(\theta_0+0.055)} \times \left[\frac{1}{2} \times e^{-(\theta_0+\theta_1+0.06)} + \frac{1}{2} \times e^{-(\theta_0+\theta_1+0.05)}\right] \\
D_d(1, 3) = e^{-(\theta_0+0.045)} \times \left[\frac{1}{2} \times e^{-(\theta_0+\theta_1+0.05)} + \frac{1}{2} \times e^{-(\theta_0+\theta_1+0.04)}\right]
\end{cases}
$$

And hence

$$D(0,3y) = D(0,1y)\mathbb{E}^*[D(1,3)] = D(0,1y) \times \left[\frac{1}{2} \times D_u(1,3) + \frac{1}{2} \times D_d(1,3)\right]$$

$\Rightarrow \quad \theta_1 = 0.01 \quad \triangleleft$