Session 6: Valuation Framework and Stochastic Volatility Models

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QF620 Stochastic Modelling in Finance

Pricing Models *vs* Reporting Models

So far we have been formulating our models as **pricing models**.

⇒ As the name suggests, pricing models are used to <u>price and risk-manage derivatives</u>.

⇒ The **dynamic** of the pricing models ought to conform to the modeler's intuition of the underlying asset's **evolution over time**.

⇒ The **Greeks** of the pricing models should accurately capture the **sensitivities** of the derivatives.

All financial institutions with a trading desk tend to have their own choices of pricing models, with the model parameters (e.g. σ, β, etc.) <u>calibrated</u> to the liquid option markets.

Pricing Models *vs* Reporting Models

Apart from pricing models, many (option) exchanges have also adopted the notion of **reporting models**.

⇒ A reporting model is used merely to report market option prices—these prices are driven by **supply and demand**.

⇒ Since it is often more elegant to report implied volatilities instead of prices (why?), a reporting model is required to perform this **conversion from price to volatility**.

⇒ Reporting models tend to make simplifying assumption about the asset dynamics, given that the primary objective is to arrive at an analytical tractable pricing formula for **price-volatility conversion**.

⇒ This reporting model's parameters (e.g. implied volatilities) can then be displayed on brokers' screens to communicate live option prices.

Option Market Screenshot

Example SPX index option chain, expiration on 15-Oct-2021.

CALLS							PUTS					
BID SIZE	BID	ASK	ASK SIZE	IMPL. BID VOL.	IMPL. ASK VOL.	STRIKE	BID SIZE	BID	ASK	ASK SIZE	IMPL. BID VOL.	IMPL. ASK VOL.
20	79.80	81.50	20	22.568%	23.495%	4290	75	13.00	13.40	75	23.071%	23.225%
20	75.80	77.50	20	22.373%	23.271%	4295	75	13.90	14.40	50	22.861%	23.010%
30	71.90	73.50	30	22.200%	23.143%	4300	76	15.00	15.40	21	22.600%	22.817%
30	68.00	69.60	30	21.974%	22.890%	4305	70	16.10	16.50	20	22.432%	22.573%
30	64.20	65.70	30	21.763%	22.656%	4310	70	17.20	17.70	63	22.210%	22.347%
30	60.40	62.00	30	21.566%	22.303%	4315	71	18.40	18.90	43	21.934%	22.135%
38	56.80	58.20	53	21.311%	22.097%	4320	63	19.70	20.20	43	21.737%	21.933%
53	53.10	54.50	38	21.193%	21.835%	4325	20	21.10	21.60	43	21.482%	21.675%
38	49.60	50.90	38	21.139%	21.580%	4330	40	22.50	23.00	43	21.170%	21.422%
42	46.60	47.40	38	20.893%	21.266%	4335	40	23.70	24.60	40	20.924%	21.173%
44	43.20	44.00	39	20.586%	20.954%	4340	41	25.60	26.20	20	20.616%	20.862%
49	39.90	40.70	38	20.337%	20.702%	4345	21	27.30	28.00	20	20.367%	20.610%
45	36.80	37.50	60	20.081%	20.384%	4350	40	29.20	29.80	40	20.111%	20.353%
44	33.70	34.40	60	19.756%	20.118%	4355	41	31.00	31.70	20	19.666%	20.088%
45	30.70	31.40	60	19.479%	19.781%	4360	23	32.80	33.80	20	19.268%	19.751%
69	27.80	28.50	60	19.187%	19.491%	4365	83	34.90	35.90	20	19.036%	19.460%
66	25.10	25.80	60	18.877%	19.183%	4370	43	37.10	38.40	63	18.602%	19.274%
67	22.50	23.20	61	18.544%	18.854%	4375	43	39.50	40.80	30	18.265%	19.009%
62	20.10	20.70	40	18.249%	18.564%	4380	39	42.10	43.40	30	17.964%	18.722%
101	17.70	18.40	60	17.924%	18.248%	4385	30	44.60	46.10	30	17.632%	18.474%
79	15.60	16.20	25	17.634%	17.968%	4390	30	47.40	49.00	30	17.333%	18.201%
86	13.60	14.20	25	17.380%	17.656%	4395	30	50.40	52.10	30	16.997%	17.898%
81	11.80	12.40	50	17.091%	17.381%	4400	30	53.60	55.30	30	16.690%	17.704%
90	10.10	10.70	50	16.765%	17.146%	4405	30	56.90	58.60	30	16.341%	17.410%
145	8.60	9.20	141	16.556%	16.879%	4410	30	60.30	62.10	30	16.022%	17.238%
108	7.30	7.80	141	16.228%	16.576%	4415	30	63.90	65.70	30	15.742%	16.960%
122	6.10	6.60	286	16.042%	16.323%	4420	30	67.70	69.60	30	15.417%	16.827%
90	5.10	5.50	291	15.832%	16.137%	4425	30	71.70	73.60	30	15.038%	16.683%
108	4.20	4.60	42	15.599%	15.935%	4430	20	75.70	77.70	20	14.836%	16.530%
89	3.50	3.80	124	15.469%	15.718%	4435	20	80.00	81.90	20	14.479%	16.489%

Implied Volatility

Based on the observed option prices traded in the market, we can calculate the **implied volatilities**:

⇒ they are defined as the volatility parameter (σ) that we need to substitute into the Black-Scholes formula to match the option prices we observe.

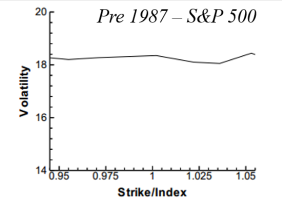
In general, for each strike K, we will need to have an implied volatility parameter σ:

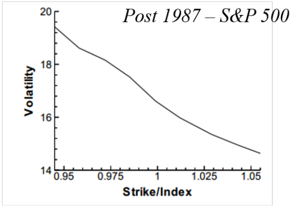
Strikes	Prices	Implied Volatilities
K_1	$C(K_1),\ P(K_1)$	BlackScholes($S,\ K_1,\ r,\ \sigma_{K_1},\ T$)
K_2	$C(K_2),\ P(K_2)$	BlackScholes($S,\ K_2,\ r,\ \sigma_{K_2},\ T$)
K_3	$C(K_3),\ P(K_3)$	BlackScholes($S,\ K_3,\ r,\ \sigma_{K_3},\ T$)
K_4	$C(K_4),\ P(K_4)$	BlackScholes($S,\ K_4,\ r,\ \sigma_{K_4},\ T$)
$\vdots$	$\vdots$	$\vdots$

Volatility Smile

Black-Scholes model assumes that the volatility of stock returns is **constant through time and strikes**. Is this true?

If the Black-Scholes assumptions are correct, then the implied volatilities of options should fall on a horizontal line when plotted against strikes.





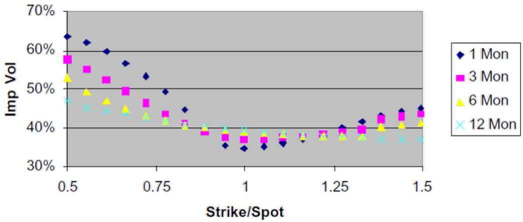
Prior to the 1987 Black Monday crash, this was roughly valid empirically. However, a distinct **volatility smile** manifested after the 1987 crash across a wide range of market—in anticipation of extreme market moves.

Volatility Smile

Background

- According to classical formulation, the Black-Scholes implied volatility of an option should be independent of its strike and expiration.
- Prior to the stock market crash of October 1987, the volatility smile of equity index options was indeed approximately flat.
- The Black-Scholes model assumes that a stock's return volatility is a constant, independent of strike and time to expiration.
- The volatility smile's appearance after the 1987 crash was due to the market's shock of discovering, for the first time since 1929, that a huge market could drop by 20% or more in a short period of time.
- In a *liquid* option market, option prices are determined by supply and demand, not by a valuation formula.

Volatility Smile



Volatility smile is generally steepest for short expiries, and is flatter for longer expiries.

Higher implied volatilities translate to higher option prices. The figure above shows that lower strike options are more in demand.

Market generally trades **out-of-the-money (OTM)** and **at-the-money (ATM) options**. **In-the-money (ITM)** options are relatively less liquid. This translate to more demand in the market for equity index put options.

Fitting Market Prices

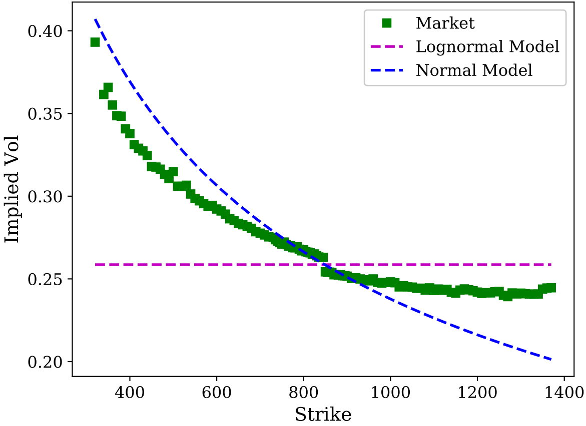
Suppose we are using the Black-Scholes or Bachelier model. The only model parameter we can vary is the volatility parameters (σ_{LN} or σ_{N}).

Since the at-the-money option (ATM) is the most important, we should choose the volatility parameter to fit the ATM option.

Strikes	Imp-Vol	Black-Scholes	Bachelier
K_1	σ_{K_1}	BlackScholes(S, K_1, σ_{LN}, T)	Bachelier(S, K_1, σ_{N}, T)
K_2	σ_{K_2}	BlackScholes(S, K_2, σ_{LN}, T)	Bachelier(S, K_2, σ_{N}, T)
K_3	σ_{K_3}	BlackScholes(S, K_3, σ_{LN}, T)	Bachelier(S, K_3, σ_{N}, T)
K_4	σ_{K_4}	BlackScholes(S, K_4, σ_{LN}, T)	Bachelier(S, K_4, σ_{N}, T)
$\vdots$	$\vdots$	$\vdots$	

Fitting Market Implied Volatilities

Example Consider Google's call and put options on 2013-08-30. We look at options expiring on 2015-01-17, the spot stock price is 846.9, and the at-the-money volatility is ≈ 0.26.



Displaced-Diffusion Model – Shifted Lognormal

In 1983, Mark Rubinstein introduced the **displaced-diffusion model**. Consider the following forward price process:

$$dF_t = \sigma F_t dW_t$$

We say that F_T follows a **lognormal** distribution. Based on this definition, we call the following a **shifted lognormal** (or displaced-diffusion) process:

$$d(F_t + \alpha) = \sigma(F_t + \alpha)dW_t, \qquad \alpha \in \mathbb{R}.$$

Since α is a constant, the process can be written as

$$d(F_t + \alpha) = dF_t = \sigma(F_t + \alpha)dW_t$$

Let $X_t = F_t + \alpha$, we can readily see that:

$$dX_t = \sigma X_t dW_t, \qquad X_T = F_T + \alpha.$$

Displaced-Diffusion Model – Option Pricing

The following stochastic differential equation is the most commonly used form for displaced-diffusion process

$$dF_t = \sigma[\beta F_t + (1-\beta)F_0]dW_t, \qquad \beta \in [0,1].$$

Note that the SDE now comprises a **geometric** and an **arithmetic** Brownian motion.

To solve this, we apply Itô formula to the function

$$X_t = f(F_t), \qquad \text{where } f(x) = \log[\beta x + (1-\beta)F_0]$$

to obtain

$$F_T = \frac{F_0}{\beta}\exp\left[-\frac{\beta^2\sigma^2 T}{2} + \beta\sigma W_T\right] - \frac{1-\beta}{\beta}F_0.$$

Displaced-Diffusion Model – Option Pricing

Question Given that

$$\text{Black} : F_T = F_0 e^{-\frac{\sigma^2 T}{2} + \sigma W_T}$$

$$\text{Displaced-Diffusion} : F_T = \frac{F_0}{\beta} \exp\left[-\frac{\beta^2\sigma^2 T}{2} + \beta\sigma W_T\right] - \frac{1-\beta}{\beta} F_0.$$

suppose we have implemented the option pricing function

$$\texttt{BlackCall}(F,\ K,\ \sigma,\ T)$$

can we price a European call option price under displaced-diffusion model using the same `BlackCall` function?

Displaced-Diffusion Model

From the graph, it appears that the implied volatility smile we observed in the market is between normal and lognormal.

We have seen earlier that the **displaced-diffusion** (shifted lognormal) model comprises features of the normal and lognormal models. Under a displaced-diffusion model, we have:

$$dF_t = \sigma[\beta F_t + (1-\beta)F_0]dW_t^*$$

Recall that the solution is given by

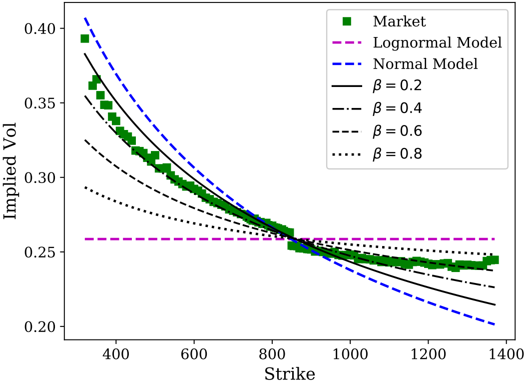
$$F_T = \frac{F_0}{\beta} e^{-\frac{\beta^2\sigma^2 T}{2} + \beta\sigma W_T^*} - \frac{1-\beta}{\beta} F_0$$

The option price under the displaced-diffusion model is

$$\text{Displaced-Diffusion} = \text{Black}\left(\frac{F_0}{\beta},\ K + \frac{1-\beta}{\beta}F_0,\ \sigma\beta,\ T\right)$$

Fitting Market Implied Volatilities

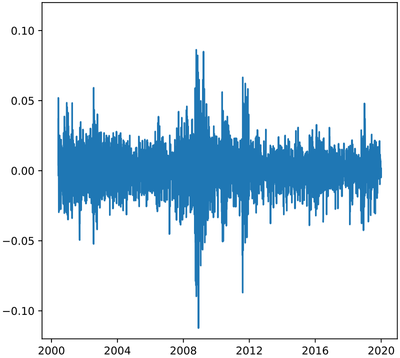
Observe that we are able to obtain a closer fit to the market using the displaced diffusion model by choosing the right β parameter.

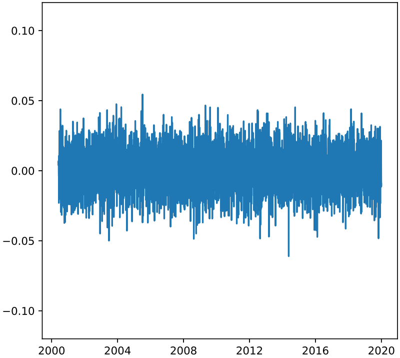


However, the fit is still not sufficiently accurate. How can we improve this?

Fitting Market Implied Volatilities

Which one is the “real” returns of a financial asset?





Stochastic Volatility

Direct observation of the daily log-return of any underlying should convince us that volatility is stochastic instead of deterministic.

In other words, instead of treating it as a constant, it should also be described by a stochastic differential equation. As a simple extension, we let it follow a driftless lognormal process

$$d\sigma_t = \nu \sigma_t dW_t^\sigma,$$

where ν is the **volatility of volatility**. We can solve this SDE to obtain the volatility process

$$\begin{aligned} \sigma_T &= \sigma_0 \exp\left[-\frac{1}{2}\nu^2 T + \nu W_T^\sigma\right] \\ &= \sigma_0 \exp\left[-\frac{1}{2}\nu^2 T + \nu\sqrt{T} N(0,1)\right]. \end{aligned}$$

In other words, instead of letting volatility be a constant, it is now evolving according to its own SDE, hence σ is also a stochastic process.

Heston Model

The **Heston Model** a stochastic volatility model formulated by Steven Heston in 1993, and is given by the stochastic differential equations:

$$\begin{cases} dS_t = rS_t dt + \sqrt{V_t} S_t dW_t^S \\ dV_t = \kappa(\theta - V_t)dt + \nu\sqrt{V_t} dW_t^V \end{cases}$$

where $dW_t^S dW_t^V = \rho\, dt$.

Heston models the variance as a stochastic process, following a mean-reverting square-root diffusion process.

The value of vanilla European options are determined by a 1-d integral which has to be evaluated numerically.

Heston model is popular among the equity desks.

SABR Model

The **SABR Model** (stochastic alpha-beta-rho) is pioneered by Patrick Hagan in 2002, and is characterised by the SDEs

$$\begin{cases} dF_t = \alpha_t F_t^{\beta} dW_t^F \\ d\alpha_t = \nu \alpha_t dW_t^{\alpha} \end{cases}$$

where $dW_t^F dW_t^{\alpha} = \rho\, dt$.

The volatility is stochastic and follows a zero-drift lognormal dynamics. Hagan derived the formula for implied volatility σ_{SABR} as an analytical function of the model parameters.

To value vanilla European options, we just need to calculate σ_{SABR} and substitute this implied volatility into the Black formula to convert to price.

This is much quicker than the Heston model. SABR model is widely used across a range of asset classes.

SABR Model

$$
\begin{aligned}
&\sigma_{\mathsf{SABR}}(F_0, K, \alpha, \beta, \rho, \nu) \\
&= \frac{\alpha}{(F_0K)^{(1-\beta)/2}\left\{1+\frac{(1-\beta)^2}{24}\log^2\left(\frac{F_0}{K}\right)+\frac{(1-\beta)^4}{1920}\log^4\left(\frac{F_0}{K}\right)+\cdots\right\}} \\
&\quad \times \frac{z}{x(z)} \times \left\{1+\left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(F_0K)^{1-\beta}}+\frac{1}{4}\frac{\rho\beta\nu\alpha}{(F_0K)^{(1-\beta)/2}}+\frac{2-3\rho^2}{24}\nu^2\right]T+\cdots\right.
\end{aligned}
$$

where

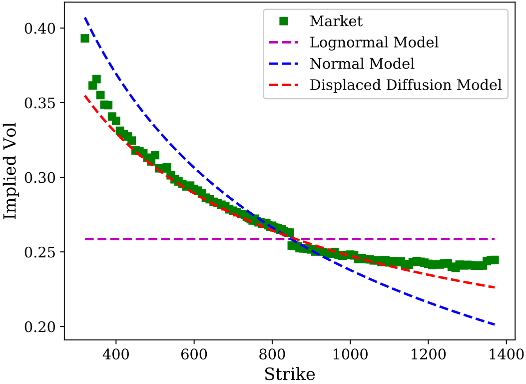
$$
z = \frac{\nu}{\alpha}(F_0K)^{(1-\beta)/2}\log\left(\frac{F_0}{K}\right),
$$

and

$$
x(z) = \log\left[\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right].
$$

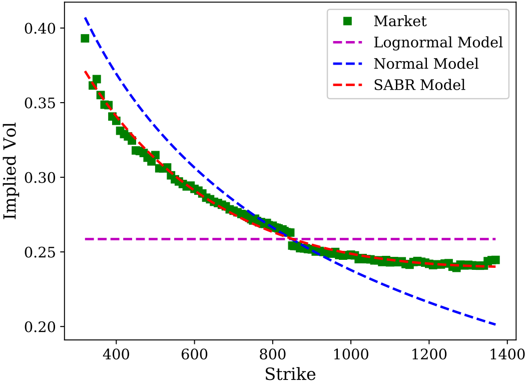
Code is provided in a separate Jupyter-Notebook

SABR Model



Displaced-diffusion model does not have sufficient **degree of freedom** to fit to market implied volatilities.

SABR Model



SABR model is able to fit the implied volatility surface well—it is a popular model due to the ease of calculation.