

Notes for Fixed-Income Securities Project

February 2, 2024

1 OIS Curve Fitting

Let f_0 denote the daily compounded overnight rate for $[0, 6m]$. Using the $6m$ OIS, we have:

$$PV_{fix}^{6m \text{ OIS}} = PV_{flt}^{6m \text{ OIS}}$$
$$D_o(0, 6m) \times 0.5 \times 0.25\% = D_o(0, 6m) \times \left[\left(1 + \frac{f_0}{360} \right)^{180} - 1 \right]$$

We can now solve for f_0 .

Having obtained this, we can then let f_1 denote the daily compounded overnight rate for $[6m, 1y]$. Using the $1y$ OIS, we have:

$$PV_{fix}^{1y \text{ OIS}} = PV_{flt}^{1y \text{ OIS}}$$
$$D_o(0, 1y) \times 0.3\% = D_o(0, 1y) \times \left[\left(1 + \frac{f_0}{360} \right)^{180} \left(1 + \frac{f_1}{360} \right)^{180} - 1 \right]$$

We can now solve for f_1 .

Next, we move on to the $2y$ OIS. Let f_2 denote the daily compounded overnight rate for $[1y, 2y]$, we have

$$PV_{fix}^{2y \text{ OIS}} = PV_{flt}^{2y \text{ OIS}}$$
$$\left[D_o(0, 1y) + D_o(0, 2y) \right] \times 0.325\% = D_o(0, 1y) \times \left[\left(1 + \frac{f_0}{360} \right)^{180} \left(1 + \frac{f_1}{360} \right)^{180} - 1 \right]$$
$$+ D_o(0, 2y) \times \left[\left(1 + \frac{f_2}{360} \right)^{360} - 1 \right]$$

2 Swaption Calibration

When calibrating to the (swap-settled) swaption data, tabulate your calibration results as follows:

Calibrated Displaced-Diffusion Model Parameters

<u>Sigma</u>					
Expiry\Tenor	1Y	2Y	3Y	5Y	10Y
1Y					
5Y					
10Y					
<u>Beta</u>					
Expiry\Tenor	1Y	2Y	3Y	5Y	10Y
1Y					
5Y					
10Y					

Calibrated SABR Model Parameters

<u>Alpha</u>					
Expiry\Tenor	1Y	2Y	3Y	5Y	10Y
1Y					
5Y					
10Y					
<u>Nu</u>					
Expiry\Tenor	1Y	2Y	3Y	5Y	10Y
1Y					
5Y					
10Y					
<u>Rho</u>					
Expiry\Tenor	1Y	2Y	3Y	5Y	10Y
1Y					
5Y					
10Y					

Suppose we now wish to price the $2y \times 10y$ 6% payer swaption using SABR model. We did not calibrate to any $2y$ expiry swaption. But having calibrated our SABR parameters, we can interpolate the $10y$ tenor (last column) α , ν , and ρ parameters between the $1y$ and $5y$ expiries to get the (α, ρ, ν) parameters we need for the $2y \times 10y$ swaption.

3 CMS Rates

The CMS rates in **Part III Q2** are calculated in the following way.

A CMS contract paying the swap rate $S_{n,N}(T)$ at time $T = T_n$ can be expressed as

$$\frac{V_0}{D(0,T)} = \mathbb{E}^T \left[\frac{V_T}{D(T,T)} \right] \quad \Rightarrow \quad V_0 = D(0,T) \mathbb{E}^T [S_{n,N}(T)]$$

By static-replication approach, and choosing the forward swap rate $F = S_{n,N}(0)$ as our expansion point, we can express this as

$$\begin{aligned} V_0 &= D(0,T)g(F) + h'(F)[V^{pay}(F) - V^{rec}(F)] \\ &\quad + \int_0^F h''(K)V^{rec}(K)dK + \int_F^\infty h''(K)V^{pay}(K)dK \\ &= D(0,T)g(F) + \int_0^F h''(K)V^{rec}(K)dK + \int_F^\infty h''(K)V^{pay}(K)dK \end{aligned}$$

In other words, we can write:

$$\underbrace{\mathbb{E}^T [S_{n,N}(T)]}_{\text{CMS Rate}} = g(F) + \frac{1}{D(0,T)} \left[\int_0^F h''(K)V^{rec}(K)dK + \int_F^\infty h''(K)V^{pay}(K)dK \right]$$

Here, the IRR-settled option pricer (V^{pay} or V^{rec}) is given by

$$V(K) = D(0,T) \cdot \text{IRR}(S_{n,N}(0)) \cdot \text{Black76}(S_{n,N}(0), K, \sigma_{\text{SABR}}, T)$$

so the discount factor $D(0,T)$ can be cancelled away.

Since the payoff function is simply $g(K) = K$, we have

$$h''(K) = \frac{-\text{IRR}''(K) \cdot K - 2 \cdot \text{IRR}'(K)}{\text{IRR}(K)^2} + \frac{2 \cdot \text{IRR}'(K)^2 \cdot K}{\text{IRR}(K)^3}.$$

4 Valuing CMS leg

The PV of CMS legs in **Part III Q1** are calculated in the following way.

A CMS leg is a collection of CMS rates paid over a period. For example, the PV of a leg receiving CMS10y semi-annually over the next 2 years is

$$\begin{aligned} PV &= D(0,6m) \cdot 0.5 \cdot \mathbb{E}^T [S_{6m,10y6m}(6m)] + D(0,1y) \cdot 0.5 \cdot \mathbb{E}^T [S_{1y,11y}(1y)] \\ &\quad + D(0,1y6m) \cdot 0.5 \cdot \mathbb{E}^T [S_{1y6m,11y6m}(1y6m)] + D(0,2y) \cdot 0.5 \cdot \mathbb{E}^T [S_{2y,12y}(2y)] \end{aligned}$$

In words, the PV is the sum of the discounted values of the CMS rates, multiplied by the day count fraction. Each CMS rate is calculated using the static-replication approach outlined in the previous section.