Notes for Fixed-Income Securities Project

January 11, 2024

1 OIS Curve Fitting

Let f_0 denote the daily compounded overnight rate for $[0, 6m]$. Using the $6m$ OIS, we have:

$$
\begin{aligned}
PV_{fix}^{6m \text{ OIS}} &= PV_{flt}^{6m \text{ OIS}} \\
D_o(0, 6m) \times 0.5 \times 0.25\% &= D_o(0, 6m) \times \left[\left(1 + \frac{f_0}{360}\right)^{180} - 1 \right]
\end{aligned}
$$

We can now solve for f_0.

Having obtained this, we can then let f_1 denote the daily compounded overnight rate for $[6m, 1y]$. Using the $1y$ OIS, we have:

$$
\begin{aligned}
PV_{fix}^{1y \text{ OIS}} &= PV_{flt}^{1y \text{ OIS}} \\
D_o(0, 1y) \times 0.3\% &= D_o(0, 1y) \times \left[\left(1 + \frac{f_0}{360}\right)^{180} \left(1 + \frac{f_1}{360}\right)^{180} - 1 \right]
\end{aligned}
$$

We can now solve for f_1.

Next, we move on to the $2y$ OIS. Let f_2 denote the daily compounded overnight rate for $[1y, 2y]$, we have

$$
\begin{aligned}
PV_{fix}^{2y \text{ OIS}} &= PV_{flt}^{2y \text{ OIS}} \\
\Big[D_o(0, 1y) + D_o(0, 2y) \Big] \times 0.325\% &= D_o(0, 1y) \times \left[\left(1 + \frac{f_0}{360}\right)^{180} \left(1 + \frac{f_1}{360}\right)^{180} - 1 \right] \\
&\quad + D_o(0, 2y) \times \left[\left(1 + \frac{f_2}{360}\right)^{360} - 1 \right]
\end{aligned}
$$

2 Swaption Calibration

When calibrating to the (swap-settled) swaption data, tabulate your calibration results as follows:

Calibrated Displaced-Diffusion Model Parameters

Sigma					
Expiry\Tenor	1Y	2Y	3Y	5Y	10Y
1Y					
5Y					
10Y					
Beta					
Expiry\Tenor	1Y	2Y	3Y	5Y	10Y
1Y					
5Y					
10Y					

Calibrated SABR Model Parameters

Alpha					
Expiry\Tenor	1Y	2Y	3Y	5Y	10Y
1Y					
5Y					
10Y					
Nu					
Expiry\Tenor	1Y	2Y	3Y	5Y	10Y
1Y					
5Y					
10Y					
Rho					
Expiry\Tenor	1Y	2Y	3Y	5Y	10Y
1Y					
5Y					
10Y					

Suppose we now wish to price the $2y \times 10y$ 6% payer swaption using SABR model. We did not calibrate to any $2y$ expiry swaption. But having calibrated our SABR parameters, we can interpolate the $10y$ tenor (last column) α, ν, and ρ parameters between the $1y$ and $5y$ expiries to get the (α, ρ, ν) parameters we need for the $2y \times 10y$ swaption.

3 CMS Rates

The CMS rates in **Part III Q2** are calculated in the following way.

A CMS contract paying the swap rate $S_{n,N}(T)$ at time $T = T_n$ can be expressed as

$$\frac{V_0}{D(0,T)} = \mathbb{E}^T\left[\frac{V_T}{D(T,T)}\right] \quad \Rightarrow \quad V_0 = D(0,T)\mathbb{E}^T\Big[S_{n,N}(T)\Big]$$

By static-replication approach, and choosing the forward swap rate $F = S_{n,N}(0)$ as our expansion point, we can express this as

$$\begin{aligned} V_0 &= D(0,T)g(F) + h'(F)[V^{pay}(F) - V^{rec}(F)] \\ &\quad + \int_0^F h''(K)V^{rec}(K)dK + \int_F^\infty h''(K)V^{pay}(K)dK \\ &= D(0,T)g(F) + \int_0^F h''(K)V^{rec}(K)dK + \int_F^\infty h''(K)V^{pay}(K)dK \end{aligned}$$

In other words, we can write:

$$\underbrace{\mathbb{E}^T\Big[S_{n,N}(T)\Big]}_{\text{CMS Rate}} = g(F) + \frac{1}{D(0,T)}\left[\int_0^F h''(K)V^{rec}(K)dK + \int_F^\infty h''(K)V^{pay}(K)dK\right]$$

Here, the IRR-settled option pricer (V^{pay} or V^{rec}) is given by

$$V(K) = D(0,T) \cdot \text{IRR}(S_{n,N}(0)) \cdot \text{Black76}(S_{n,N}(0), K, \sigma_{\text{SABR}}, T)$$

so the discount factor $D(0,T)$ can be cancelled away.

Since the payoff function is simply $g(K) = K$, we have

$$h''(K) = \frac{-\text{IRR}''(K) \cdot K - 2 \cdot \text{IRR}'(K)}{\text{IRR}(K)^2} + \frac{2 \cdot \text{IRR}'(K)^2 \cdot K}{\text{IRR}(K)^3}.$$

4 Valuing CMS leg

The PV of CMS legs in **Part III Q1** are calculated in the following way.

A CMS leg is a collection of CMS rates paid over a period. For example, the PV of a leg receiving CMS10y semi-annually over the next 2 years is

$$\begin{aligned} PV &= D(0,6m) \cdot 0.5 \cdot \mathbb{E}^T\Big[S_{6m,10y6m}(6m)\Big] + D(0,1y) \cdot 0.5 \cdot \mathbb{E}^T\Big[S_{1y,11y}(1y)\Big] \\ &\quad + D(0,1y6m) \cdot 0.5 \cdot \mathbb{E}^T\Big[S_{1y6m,11y6m}(1y6m)\Big] + D(0,2y) \cdot 0.5 \cdot \mathbb{E}^T\Big[S_{2y,12y}(2y)\Big] \end{aligned}$$

In words, the PV is the sum of the discounted values of the CMS rates, multiplied by the day count fraction. Each CMS rate is calculated using the static-replication approach outlined in the previous section.