IRR-Settled Swaptions

An **Internal-Rate-of-Return (IRR)-settled swaption** has the following payoff:

$$\text{Payer Swaption} = \Big[\text{IRR}(S_{n,N}(T))(S_{n,N}(T) - K)\Big]^+$$

$$\text{Receiver Swaption} = \Big[\text{IRR}(S_{n,N}(T))(K - S_{n,N}(T))\Big]^+$$

where

$$\text{IRR}(S) = \sum_{i=1}^{(T_N - T_n)\times m} \frac{\frac{1}{m}}{\left(1+\frac{S}{m}\right)^i}$$

and $\frac{1}{m} = \Delta$ is the day count fraction corresponding to the payment frequency (m) of the swap.

IRR-settled swaptions are settled in cash based on the value of the payoff observed on the maturity date.

Swap-settled swaptions are common in the USD market, while IRR-settled swaptions are common in the European (EUR & GBP) markets.

swap-settled

$\mathbb{Q}^{n+1,N}$

$$dS_{n,N}(t) = \sigma_{n,N} S_{n,N}(t)\, dW^{n+1,N}(t)$$

Numeraire is $P_{n+1,N}(t)$

$$\frac{V^{pay}(0)}{P_{n+1,N}(0)} = \mathbb{E}^{n+1,N}\left[\frac{V^{pay}(T)}{P_{n+1,N}(T)}\right]$$

$$= \mathbb{E}^{n+1,N}\left[\frac{\cancel{P_{n+1,N}(T)} \cdot \left(S_{n,N}(T) - K\right)^{+}}{\cancel{P_{n+1,N}(T)}}\right]$$

$$V^{pay}(0) = P_{n+1,N}(0)\, \mathbb{E}^{n+1,N}\left[\left(S_{n,N}(T) - K\right)^{+}\right]$$

IRR-settled

$$\mathbb{Q}^{n+1,N}: \quad \frac{V^{pay}_{IRR}(0)}{P_{n+1,N}(0)} = \mathbb{E}^{n+1,N}\left[\frac{V^{pay}_{IRR}(T)}{P_{n+1,N}(T)}\right]$$

$$= \mathbb{E}^{n+1,N}\left[\frac{IRR(S_{n,N}(T))\left(S_{n,N}(T)-K\right)^+}{P_{n+1,N}(T)}\right]$$

$$\mathbb{Q}^{T}: \quad \frac{V^{pay}_{IRR}(0)}{D(0,T)} = \mathbb{E}^{T}\left[\frac{V^{pay}_{IRR}(T)}{\cancel{D(T,T)}^{\,1}}\right]$$

$$V^{pay}_{IRR}(0) = D(0,T)\,\mathbb{E}^{T}\left[IRR(S_{n,N}(T))\left(S_{n,N}(T)-K\right)^+\right]$$

$$V^{pay}_{IRR}(0) \simeq D(0,T)\, IRR\left(S_{1,N}(0)\right) \mathbb{E}^{T}\left[\left(S_{1,N}(T)-K\right)^{+}\right]$$

$$\simeq D(0,T)\, IRR\left(S_{1,N}(0)\right) \mathbb{E}^{1+1,N}\left[\left(S_{1,N}(T)-K\right)^{+}\right]$$

IRR-Settled Swaptions

The Market Model used to value IRR-settled swaptions is:

$$V_{n,N}(0) \approx D(0,T) \cdot \mathsf{IRR}(S_{n,N}(0)) \cdot \mathsf{Black}(S_{n,N}(0), K, \sigma_{n,N}, T)$$

Historical Note:

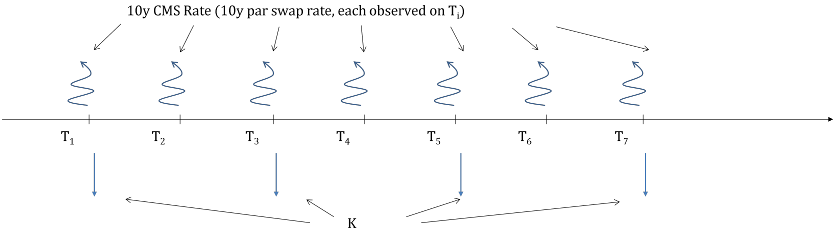
- In the USD market, participants agree on the value of the PV01 $P_{n+1,N}$, i.e. there is no dispute on the discount factors.
- In the earlier days, market participants disagree on the PV01 value in the Euro and Sterling market.
- To avoid ambiguity, market participants agree to use the IRR formula to discount cashflows in the EUR and GBP market.
- The rational was that since $D(0,T) = \frac{1}{(1+r)^T}$, a good approximation would be to use the observed swap rate $S_{n,N}(T)$ for discounting.

Constant Maturity Swap

A **constant maturity swap** (CMS) pays a swap rate rather than a LIBOR rate on its floating leg.

⇒ Can be either quoted **in arrears** or **in advance**.

⇒ The payment can be **capped** or **floored**.



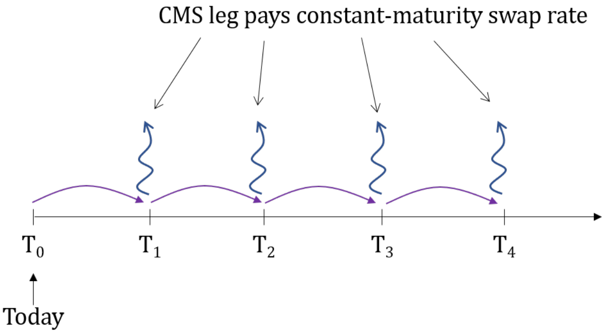
CMS is an instrument having cashflows "paid at the wrong timing":

⇒ A $10y$ CMS rate to be paid one year later is not exactly equal to the forward swap rate $S_{1y,10y}$.

⇒ **Convexity correction** is required to obtain the right price.

CMS Leg

A CMS leg pays the constant-maturity swap rate periodically over time:

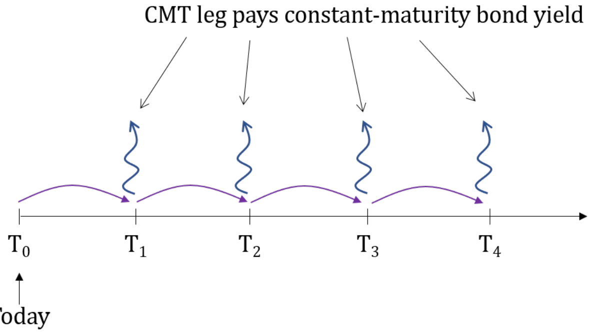


The CMS rate you receive at time T_{i+1} the par swap rate in the market at T_i.

(FYI)

CMT Leg

A closely related product is CMT, which pays the constant-maturity bond yield periodically over time:



The CMT bond yield you receive at time T_{i+1} the bond yield in the market at T_i.

CMS (or CMT) products give you an easy way to gain exposure to fixed-length longer-term interest rates.

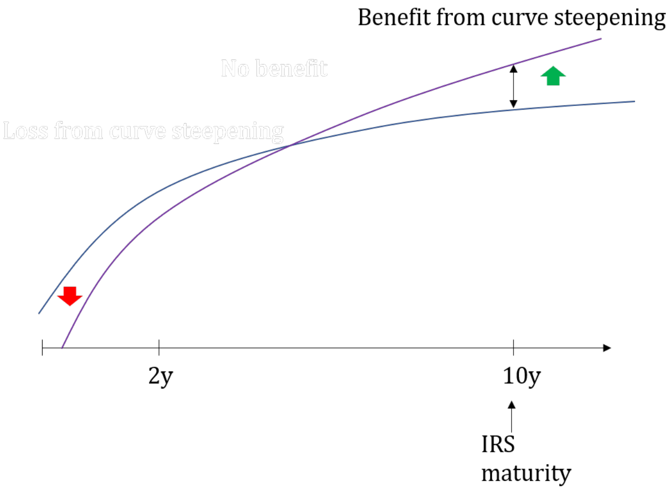
$\Rightarrow$ You can use it to express a view on a fixed point on the yield curve.

In contrast, if you use an IRS, then your exposure will progressively become shorter-term over time.

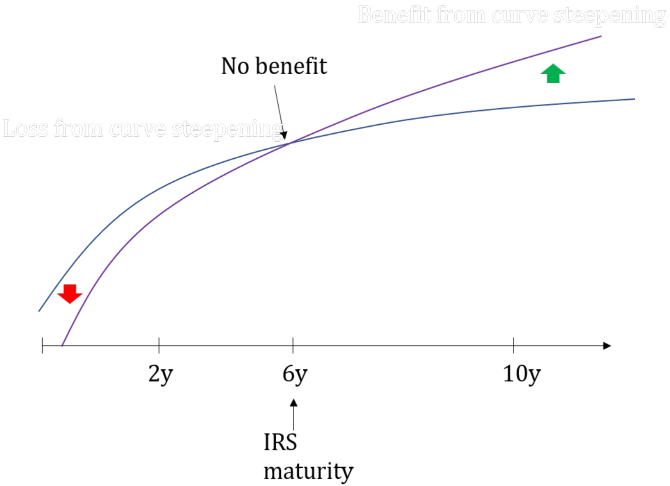
For example, suppose you think that the yield curve will steepen, so that $10y$ swap rate will increase, while $2y$ swap rate will decrease.

To this end, you long a $10y$ payer IRS. If the yield curve steepens, you benefit. If the yield curve flattens, you lose.

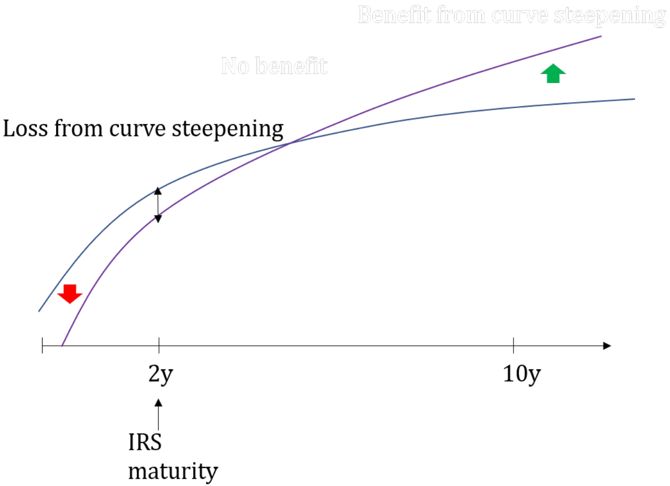
Trade day (initial)



4 years later



8 years later



Insurance companies or pension funds have long dated obligations — generally speaking, the exposure does not age time.

Year	Exposure
2y	-
5y	-
10y	-
15y	-
20y	-
30y	-
40y	-
50y	-

If they use IRS to hedge their exposure, the IRS sensitivity will progressively become shorter-term over time.

For hedge funds and other institutional clients, they use CMS products to speculate on the movement of the yield curve.

- Receive long-maturity CMS rate if they think yield will steepen
  $\Rightarrow$ Spread trade: Receive 10y pay 2y CMS
  + spd

- Pay long-maturity CMS rate if they think yield will flatten
  $\Rightarrow$ Spread trade: Pay 10y receive 2y CMS

- CMS spread options

Risk-Neutral Density of the Forward Swap Rate

The value of a **CMS payoff** is a function of the distribution of the swap rate.

The standard practice in the market is to use the **static-replication** method to obtain a **model-independent convexity correction**.

Let us begin with an IRR-settled payer swaption:

$$V^{pay}(K) = D(t,T)\int_K^\infty \mathsf{IRR}(s)\cdot(s-K)\ f(s)\ ds$$

Differentiating the formula twice yields

$$\frac{\partial V^{pay}(K)}{\partial K} = -D(t,T)\int_K^\infty \mathsf{IRR}(s)\ f(s)\ ds$$

$$\frac{\partial^2 V^{pay}(K)}{\partial K^2} = D(t,T)\ \mathsf{IRR}(K)\ f(K)$$

This can be rewritten as

$$f(K) = \frac{\partial^2 V^{pay}(K)}{\partial K^2}\times\frac{1}{D(t,T)\ \mathsf{IRR}(K)} = \frac{\partial^2 V^{rec}(K)}{\partial K^2}\times\frac{1}{D(t,T)\ \mathsf{IRR}(K)}$$

Static Replication Approach

We will also obtain the same result by differentiating the IRR-settled receiver swaption formula twice.

Suppose we want to value a contract paying $g(S_{n,N}(T))$ at time T, we let

$$h(K) = \frac{g(K)}{\mathsf{IRR}(K)},$$

and write (let $F = S_{n,N}(0)$ denote the forward swap rate)

$$\begin{aligned} V_0 &= D(0,T)\int_0^\infty g(K)\ f(K)\ dK \\ &= \int_0^F h(K)\ \frac{\partial^2 V^{rec}(K)}{\partial K^2}\ dK + \int_F^\infty h(K)\ \frac{\partial^2 V^{pay}(K)}{\partial K^2}\ dK \end{aligned}$$

Integration-by-parts twice, we will get

$$\begin{aligned} V_0 = D(0,T)g(F) + h'(F)[V^{pay}(F) - V^{rec}(F)] \\ + \int_0^F h''(K)V^{rec}(K)dK + \int_F^\infty h''(K)V^{pay}(K)dK \end{aligned}$$

Static Replication of CMS Payoffs

Using **quotient rule**, the first and second order derivatives of $h(K)$ are given by:

$$
\begin{aligned}
h(K) &= \frac{g(K)}{\mathsf{IRR}(K)} \\
h'(K) &= \frac{\mathsf{IRR}(K)g'(K) - g(K)\mathsf{IRR}'(K)}{\mathsf{IRR}(K)^2} \\
h''(K) &= \frac{\mathsf{IRR}(K)g''(K) - \mathsf{IRR}''(K)g(K) - 2\cdot\mathsf{IRR}'(K)g'(K)}{\mathsf{IRR}(K)^2} \\
&\quad + \frac{2\cdot\mathsf{IRR}'(K)^2 g(K)}{\mathsf{IRR}(K)^3}.
\end{aligned}
$$

$g(F) = F$

$g'(F) = 1$, $g''(F) = 0$

CMS Rate

Example Show that a CMS rate payment for the swap rate $S_{n,N}(T)$ at time T can be valued as (where $F = S_{n,N}(0)$)

$$D(0,T)F + \int_0^F h''(K)V^{rec}(K)dK + \int_F^\infty h''(K)V^{pay}(K)dK$$

using static replication, where

$$h''(K) = \frac{-\mathrm{IRR}''(K)\cdot K - 2\cdot \mathrm{IRR}'(K)}{\mathrm{IRR}(K)^2} + \frac{2\cdot \mathrm{IRR}'(K)^2\cdot K}{\mathrm{IRR}(K)^3}.$$

$g(F) = F - L$

CMS Caplet

$g'(F) = 1$, $g''(F) = 0$

Example Show that an at-the-money (ATM) CMS caplet struck at the forward swap rate $L = S_{n,N}(0) = F$ maturing at T can be valued as

$$\text{CMS Caplet} = V^{pay}(L)h'(L) + \int_L^\infty h''(K)V^{pay}(K)dK$$

using static replication, where

$$h'(K) = \frac{\text{IRR}(K) - \text{IRR}'(K)\cdot(K-L)}{\text{IRR}(K)^2}$$

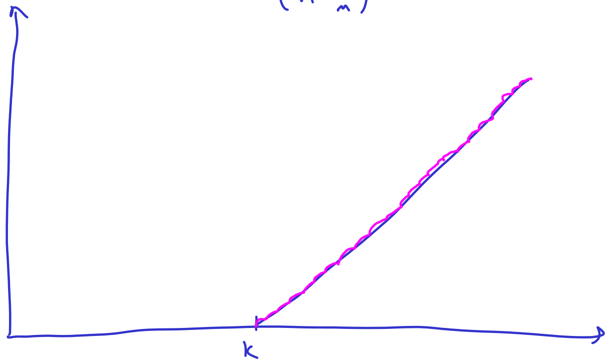
$$h''(K) = \frac{-\text{IRR}''(K)(K-L) - 2\cdot\text{IRR}'(K)}{\text{IRR}(K)^2} + \frac{2\cdot\text{IRR}'(K)^2\cdot(K-L)}{\text{IRR}(K)^3}$$

CMS caplet : $(S - L)^+$

IRR $\overset{\text{payer}}{\wedge}$ swaption : $\text{IRR}(S)\,(S - L)^+$

$$\uparrow$$

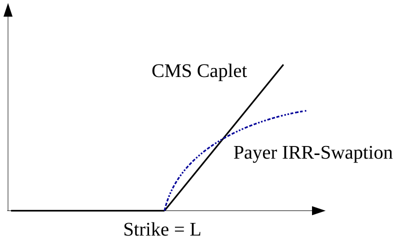
$$\sum_i \frac{\frac{1}{m}}{\left(1+\frac{S}{m}\right)^i}$$

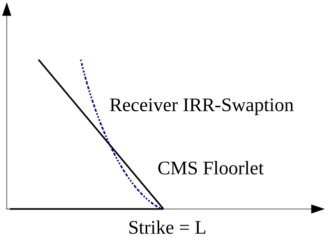


CMS Replication – Intuition

Note that the CMS caplet payoff and the IRR-settled payer swaption payoff are both functions of the same swap rate $S_{n,N}(T)$'s distribution.

Beyond the strike rate of L, CMS caplet payoff is **linear** and payer swaption payoff is **concave** of the swap rate $S_{n,N}(T)$:

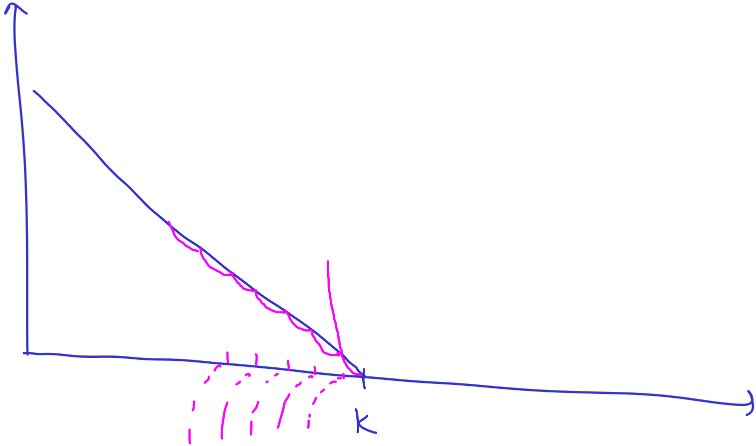




Since swaptions are vanilla derivatives and more liquid, we can **replicate the CMS caplet** payoff using a basket of IRR-settled payer swaptions with increasing strikes starting with the CMS caplet strike L.

CMS Floorlet : $(L - S)^+$

IRR rec swaption : $IRR(S)\,(L-S)^+$



CMS Replication – Intuition

Using a series of IRR-settled payer swaptions, we can **statically replicate** the CMS caplet as follows:

$$\text{CMS Caplet} = V^{pay}(L)h'(L) + \int_L^\infty h''(K)V^{pay}(K)dK$$
$$\approx V^{pay}(L)h'(L) + \sum_{i=1}^\infty h''(L+i\cdot\Delta K)\, V^{pay}(L+\ i\cdot\Delta K)\,\Delta K$$