QF605 Fixed-Income Securities
Assignment 4, Due Date: 20-Mar-2024

1. Let S_t denote a forward swap rate at time t. Suppose a CMS product has the following payoff on maturity T:

$$g(S_T) = \begin{cases} 0, & S_T < K_1 \\ S_T - K_1 & K_1 \leq S_T \leq K_2 \\ K_2 - K_1 & S_T > K_2 \end{cases}$$

   Starting with

$$\int_0^\infty h(K) \frac{\partial^2 V^{pay}(K)}{\partial K^2} \, dK$$

   where $h(K) = \frac{g(K)}{\text{IRR}(K)}$, derive the static replication formula for this payoff.

2. The Ho-Lee interest rate model is given by

$$dr_t = \theta(t) dt + \sigma dW_t^*,$$

   where W_t^* is a standard Brownian motion under the measure $\mathbb{Q}^*$. Determine the mean and variance of the integral

$$\int_0^T r_u \, du.$$

3. Suppose we use a discrete ($\Delta t = 1y$) binomial-tree approximation of the Ho-Lee model, where at every step the rate can move up or down by 0.5%, and the risk-neutral probabilities of an up or down move are both 0.5. We observe the following discount factors:

Instrument	Value
$D(0, 1y)$	0.9656
$D(0, 2y)$	0.9224
$D(0, 3y)$	0.8903

   Draw the Ho-Lee binomial tree and determine the no-arbitrage values for θ_0 and θ_1.

1、 $$\int_0^\infty h(k)\, \frac{\partial^2 V^{pay}(k)}{\partial k^2}\, dk$$

$$= \int_{k_1}^{k_2} h(k)\, \frac{\partial^2 V^{pay}(k)}{\partial k^2}\, dk + \int_{k_2}^\infty h(k)\, \frac{\partial^2 V^{pay}(k)}{\partial k^2}\, dk$$

$$= \frac{\partial V^{pay}(k)}{\partial k} h(k)\Big|_{k_1}^{k_2} - \int_{k_1}^{k_2} \frac{\partial V^{pay}(k)}{\partial k} h'(k)\, dk + \frac{\partial V^{pay}(k)}{\partial k} h(k)\Big|_{k_2}^{\infty} - \int_{k_2}^\infty \frac{\partial V^{pay}(k)}{\partial k} h'(k)\, dk$$

$$= -\frac{\partial V^{pay}(k_1)}{\partial k_1} h(k_1) - \int_{k_1}^{k_2} h'(k)\, d V^{pay}(k) - \int_{k_2}^\infty h'(k)\, d V^{pay}(k)$$

$$= -\frac{\partial V^{pay}(k_1)}{\partial k_1} h(k_1) - V^{pay}(k) h'(k)\Big|_{k_1}^{k_2} + \int_{k_1}^{k_2} V^{pay}(k) h''(k)\, dk$$

$$- h'(k) V^{pay}(k)\Big|_{k_2}^{\infty} + \int_{k_2}^\infty V^{pay}(k) h''(k)\, dk$$

$$= -\frac{\partial V^{pay}(k_1)}{\partial k_1} h(k_1) + V^{pay}(k_1) h'(k_1) + \int_{k_1}^{k_2} V^{pay}(k) h''(k)\, dk$$

$$+ \int_{k_2}^\infty V^{pay}(k) h''(k)\, dk$$

$$= \frac{1}{IRR(k_1)} V^{pay}(k_1) + \int_{k_1}^{k_2} V^{pay}(k) h''(k)\, dk$$

$$+ \int_{k_2}^\infty V^{pay}(k) h''(k)\, dk$$

$$V^{pay}(k) = D(t,T) \int_k^\infty IRR(s)(s-k) f(s)\, ds$$

$$h(k) = \frac{g(k)}{IRR(k)}$$

$$h'(k) = \frac{g'(k)\, IRR(k) - IRR'(k) g(k)}{IRR^2(k)}$$

$$h''(k) = \frac{g'(k)\, IRR'(k) - IRR''(k) g(k)}{IRR^2(k)}$$

$$- \frac{2\, IRR'(k)[g'(k)\, IRR(k) - IRR'(k) g(k)]}{IRR^3(k)}$$

$$= \frac{2[IRR'(k)]^2 g(k)}{IRR^3(k)}$$

$$- \frac{g'(k)\, IRR'(k) - IRR''(k) g(k)}{IRR^2(k)}$$

2、 $dr_t = \theta(t)\,dt + \sigma\,dW_t^*$

$$r_t = r_0 + \int_0^t \theta(s)\,ds + \int_0^t \sigma\,dW_s^*$$

$$\int_0^T r_t\,dt = r_0 T + \int_0^T\int_0^t \theta(s)\,ds\,dt + \int_0^T\int_0^t \sigma\,dW_s^*\,dt$$

$$= r_0 T + \int_0^T\int_s^T \theta(s)\,dt\,ds + \int_0^T\int_s^T \sigma\,dt\,dW_s^*$$

$$= r_0 T + \int_0^T (T-s)\,\theta(s)\,ds + \int_0^T (T-s)\,\sigma\,dW_s^*$$

$$E\left[\int_0^T r_t\,dt\right] = r_0 T + \int_0^T (T-s)\,\theta(s)\,ds$$

$$V\left[\int_0^T r_t\,dt\right] = \int_0^T (T-s)^2\sigma^2\,ds = \frac{T^3\sigma^2}{3}$$

3、

$R(0,1) = -\ln(0.9656) = 3.5\%$

$D(1,2) = D(0,2)/D(0,1) = 0.95526$

$$3.5\% \quad \begin{cases} e^{-(4\% + \theta_0)} & \begin{cases} e^{-(4.5\% + \theta_0 + \theta_1)} \\ e^{-(3.5\% + \theta_0 + \theta_1)} \end{cases} \\ e^{-(3\% + \theta_0)} & \begin{cases} e^{-(3.5\% + \theta_0 + \theta_1)} \\ e^{-(2.5\% + \theta_0 + \theta_1)} \end{cases} \end{cases}$$

$$\frac{1}{2} e^{-(4\% + \theta_0)} + \frac{1}{2} e^{-(3\% + \theta_0)} = D(1,2)$$

$$\theta_0 = -\ln\left[2D(1,2) / (e^{-3\%} + e^{-4\%})\right] = 1.07832\%$$

$$D(2,3) = D(0,3) / D(0,2) = 0.9652$$

$$\theta_1 = -\ln\left[4D(2,3) / (e^{-(2.5\% + \theta_0)} + 2e^{-(3.5\% + \theta_0)} + e^{-(4.5\% + \theta_0)})\right]$$

$$= -1.033772\%$$