Binomial Tree Models

QF607 Numerical Methods

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Pricing Financial Derivatives

Flow or Vanilla options that can be priced analytically

- Swap / Forward / Futures
  - Linear products, priced by "Law of One Price"

- European option - an option that may only be exercised on expiry date, priced by Black-Scholes analytic formulas

Pricing Financial Derivatives

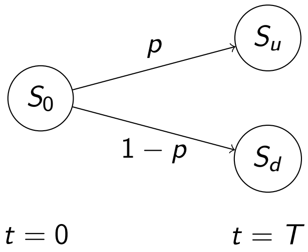
Exotic options require numerical pricer

- **American option** - option that allows exercise any time prior to the expiry date
- **Barrier option** - if the spot price touches the pre-defined barrier in a given time window, the option holder obtains (Knock-In Option) or loses (Knock-Out Option) an underlying payoff
  - the underlying payoff can be a European option,
  - or just a constant rebate (touch option)
- **Asian option** - an option where the payoff is determined by the average underlying price over some pre-set period of time
  - average of price has lower volatility, therefore a cheaper option for hedging purpose,
  - less sensitive to price manipulation on the expiry date
- Target redemption forward, Bermudan callable structure, etc.

Bionmial tree option pricing model

- A numerical method for the valuation of options.
- The model uses a discrete-time model of varying price over time of the underlying financial instrument.
- First proposed by Cox, Ross and Rubinstein in 1979 – the CRR binomial tree

One-Step Binomial Tree



- A single time step from 0 to T
- Two traded instruments in the market: stock S and a zero coupon bond with continuous yield r (risk free interest rate)
- Two possible states at time T: up and down
- What's the price of the call and put option at time 0?

Price by Replication

Consider an option with final payoff $V(S_T, T)$, we want to solve for it's present value $V(S_0, 0)$:

- Construct a portfolio of δ shares of the stock and $V(S_0, 0) - \delta S_0$ units of bond that grows at constant compounding rate r.

- At time T, we would like the portfolio to have value $V(S_T, T)$ no matter whether S_0 goes to the up state or the down state, so the below equations should hold

$$\begin{cases} \delta S_u + e^{rT}(V_0 - \delta S_0) = V_u \\ \delta S_d + e^{rT}(V_0 - \delta S_0) = V_d \end{cases} \tag{1}$$

Solution and Risk Neutral Probabilities

We know S_u, S_d, V_u and V_d in the equations, so two unknowns δ, V_0 and two equations we can solve

$$\begin{cases} \delta = \frac{V_u - V_d}{S_u - S_d} \\ V_0 = e^{-rT}\left(\underbrace{\frac{S_u - S_0 e^{rT}}{S_u - S_d}}_{q_d} V_d + \underbrace{\frac{S_0 e^{rT} - S_d}{S_u - S_d}}_{q_u} V_u\right) \end{cases} \tag{2}$$

We call q_u and q_d risk neutral probabilities, and the option's price is expectation of V_T under the risk neutral probability measure ($\mathbb{Q}$-measure):

$$V_0 = e^{-rT}\mathbb{E}_{\mathbb{Q}}[V_T] \tag{3}$$

Why Binomial Model?

Pros:

- Overly simplified, but surprisingly general after extensions
- More final states can be included with multiple steps
- Handle many payoffs and option types – the only assumption of the payoff V we make is that it depends on the terminal value of S_T
- Handle American options naturally
- Easy to implement

Cons:

- Difficult to handle path-dependent options

How Easy Is The Implementation

Inputs are:

- Current value of the underlying stock S_0
- Risk free interest rate r
- Up state S_u and down state S_d
- Option type: Call or Put for Now
- Option strike K and time to maturity T

Output: option price

One Step Binomial Tree Implementation

```python
from enum import Enum
import math
class PayoffType(str, Enum):
    Call = 'Call'
    Put = 'Put'

def oneStepBinomial(S:float, r:float, u:float, d:float, optType:PayoffType, K:
     float, T:float) -> float:
    p = (math.exp(r * T) - d) / (u-d)
    if optType == PayoffType.Call:
        return math.exp(-r*T) * (p*max(S*u-K, 0) + (1-p) * max(S*d-K, 0))
```

```python
oneStepBinomial(S=100, r=0.01, u=1.2, d=0.8, optType=PayoffType.Call, K=105, T=1.0)
7.798504987524955
```

Let's recap the input of our simple one step binomial model:

- Option type, strike K and time to maturity T,

- Current value of the underlying stock S_0, risk free interest rate r

- Up state S_u and down state S_d
  — HOW can we possibly know S_u and S_d in reality?

But does it mean our binomial tree model is impractical? No, it is in fact a discretized version of Black-Scholes model

Black-Scholes Model

The Black-Scholes model says, under the risk neutral measure, the non-dividend paying stock price follows a log-normal process:

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t \tag{4}$$

where

- σ is the volatility
- W_t is a standard Brownian motion
- r is the risk free interest rate

This is much closer to reality than the one step binomial model!

Black-Scholes Solution of the Stock Price

Applying Ito's lemma we can get the diffusion of $d \ln S_t$:

$$d \ln S_t = \frac{1}{S_t} dS_t - \frac{1}{2S_t^2} dS_t^2 = (r - \frac{1}{2}\sigma^2)dt + \sigma dW_t$$

So $\ln S_t$ is a drifted Brownian motion

$$\ln S_t = \ln S_0 + \int_0^t (r - \frac{1}{2}\sigma^2) ds + \int_0^t \sigma dW \tag{5}$$

$$= \ln S_0 + (r - \frac{1}{2}\sigma^2)t + \sigma W_t \tag{6}$$

And the solution of SDE (4) is

$$S_t = S_0 \exp\left((r - \frac{1}{2}\sigma^2)t + \sigma W_t\right) \tag{7}$$

Black-Scholes Formula

The present value ($t = 0$) of a call option with strike at K and expiry at T is

$$C(S_0, K, T) = e^{-rT}\mathbb{E}_{\mathbb{Q}}[(S_T - K)_+] = S_0 N(d_+) - Ke^{-rT}N(d_-) \quad (8)$$

where

- r is the risk-free rate
- $d_{\pm} = \dfrac{ln\frac{S_0}{K} + (r \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$
- $N(\cdot)$ is the standard cumulative normal function

And the price of a put option is

$$P(S_0, K, T) = e^{-rT}\mathbb{E}_{\mathbb{Q}}[(K - S_T)_+] = Ke^{-rT}N(-d_-) - S_0 N(-d_+) \quad (9)$$

Black-Scholes Formula Implementation

Input:

- option type, strike, current stock price, volatility, and interest rate

```
def cnorm(x):
    return (1.0 + math.erf(x / math.sqrt(2.0))) / 2.0
def bsPrice(S, r, vol, payoffType, K, T):
    fwd = S * math.exp(r * T)
    stdev = vol * math.sqrt(T)
    d1 = math.log(fwd / K) / stdev + stdev / 2
    d2 = d1 - stdev
    if payoffType == PayoffType.Call:
        return math.exp(-r * T) * (fwd * cnorm(d1) - cnorm(d2) * K)
    elif payoffType == PayoffType.Put:
        return math.exp(-r * T) * (K * cnorm(-d2) - cnorm(-d1) * fwd)
    else:
        raise Exception("not supported payoff type", payoffType)
# test ---
S, r, vol, K, T, u, d = 100, 0.01, 0.2, 105, 1.0, 1.2, 0.8
print("blackPrice: ", bsPrice(S, r, vol, T, K, PayoffType.Call))
print("oneStepTree: ", oneStepBinomial(S, r, u, d, PayoffType.Call, K, T))
```

Do they agree on the price? Why not?

Where is the Gap?

The inputs for the option are the same, how about the market inputs?

- One step binomial tree model:
  - current stock price S_0,
  - risk free interest rate r,
  - up state S_u and down state S_d
- Black-Schole model:
  - current stock price S_0,
  - risk free interest rate r,
  - volatility σ

The two models are making different assumption on the distribution of the stock price at T.

Possible to Build the Bridge?

Distributions of S_T

- Binomial tree: two state discrete distribution.
  - Risk neutral probability of S_u: $p = \frac{S_0 e^{rT} - S_d}{S_u - S_d}$
  - Risk neutral probability of S_d: $1 - p$
  - Mean of S_t: $S_0 e^{rT}$
  - Variance of S_t: $pS_u^2 + (1-p)S_d^2 - S_0^2 e^{2rT}$
- Black-Scholes: continuous log-normal distribution

$$S_t = S_0 e^{(r - \frac{1}{2}\sigma^2)t + \sigma W_t} \tag{10}$$

  - Mean of S_t: $S_0 e^{rT}$
  - Variance of S_t: $S_0^2 e^{2rt} e^{\sigma^2 t} - S_0^2 e^{2rt}$

They have the same mean, so can we match the variance?

Matching the Variance

Stock price is always positive, we can rewrite $S_u = uS_0, S_d = dS_0$ with $u > 0, d > 0$

The variance of the binomial tree becomes

$$S_0^2(pu^2 + (1-p)d^2) - S_0^2 e^{2rt} \tag{11}$$

To match the variance with Black-Sholes's, we need to satisfy:

$$e^{2rt+\sigma^2 t} = pu^2 + (1-p)d^2, \quad p = \frac{e^{rT} - d}{u - d} \tag{12}$$

One equation and two unknowns — Let us impose another constraint $d = \frac{1}{u}$: CRR binomial tree

Equation (12) now becomes

$$e^{2rT+\sigma^2 T} = \frac{e^{rT}u^2 - u + 1/u - e^{rT}(1/u)^2}{u - 1/u} = e^{rT}(u + 1/u) - 1 \tag{13}$$

$$e^{rT}u^2 - (e^{2rT+\sigma^2 T} + 1)u + e^{rT} = 0 \tag{14}$$

$$u^2 - (e^{rT+\sigma^2 T} + e^{-rT})u + 1 = 0 \tag{15}$$

Solving the quadratic equation (14) we have

$$u = \frac{b \pm \sqrt{b^2 - 4}}{2}, \quad \text{where} \quad b = e^{rT+\sigma^2 T} + e^{-rT} \tag{16}$$

The two solutions correspond to u and d, and we expect $u > 1$ since its the up state, so

$$u = \frac{b + \sqrt{b^2 - 4}}{2} \tag{17}$$

$$d = \frac{b - \sqrt{b^2 - 4}}{2} = \frac{1}{u} \tag{18}$$

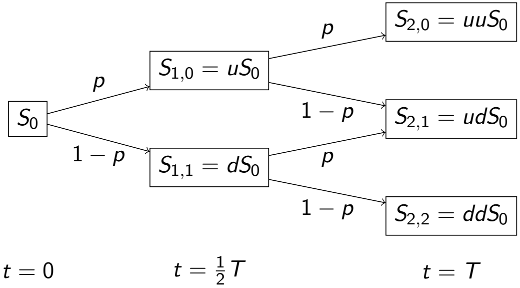
Now It Is Consistent

Now we can derive u and d from σ, making the input about the market the same for both pricers

```
def oneStepBinomial2(S, r, vol, optType, K, T):
    b = math.exp(vol * vol * T+r*T) + math.exp(-r * T)
    u = (b + math.sqrt(b*b - 4)) / 2
    d = 1/u
    p = (math.exp(r * T) - d) / (u-d)
    if optType == PayoffType.Call:
        return math.exp(-r * T) * (p * max(S * u - K, 0) + (1-p) * max(S * d -
     K, 0))
# test ---
S,r,vol,K,T,u,d = 100, 0.01, 0.2, 105, 1.0, 1.2, 0.8
print("blackPrice: \t", bsPrice(S, r, vol, PayoffType.Call, K, T))
print("oneStepTree1: \t", oneStepBinomial(S, r, u, d, PayoffType.Call, K, T))
print("oneStepTree2: \t", oneStepBinomial2(S, r, vol, PayoffType.Call, K, T))
```

Bringing It Closer

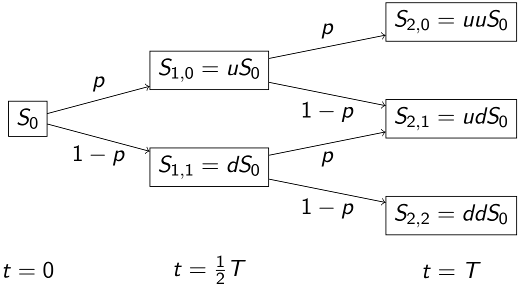
- Now we have first moment and second moment matched, have we closed the gap between the two pricers now? Sort of, but we are still subject to discretization errors – our one step tree is very coarse
- Natural way to extend our model is to make more steps and more states



Multi-Step Binomial Tree

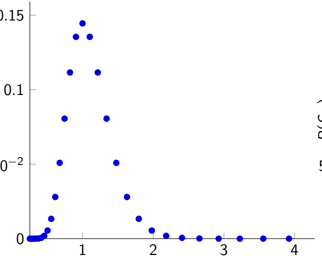
- Fortunately, our tree is **recombining**: the two intermediate states collapse to the same final state because $ud = du$. Otherwise the number of nodes will grow exponentially and will soon become unmanageable.
- The probability associated with each final state:

$$P(S_{2,0}) = p^2, P(S_{2,1}) = 2p(1-p), P(S_{2,2}) = (1-p)^2$$

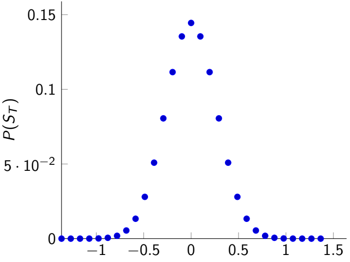


Multi-Step Binomial Tree

- Adding one time step will give us one more final state
- The probability of the state $S_{N,i}$ is $P(S_{N+1,i}) = \binom{N}{i} p^i (1-p)^{N-i}$
- And the probability density function of S_T converges to a log-normal distribution:



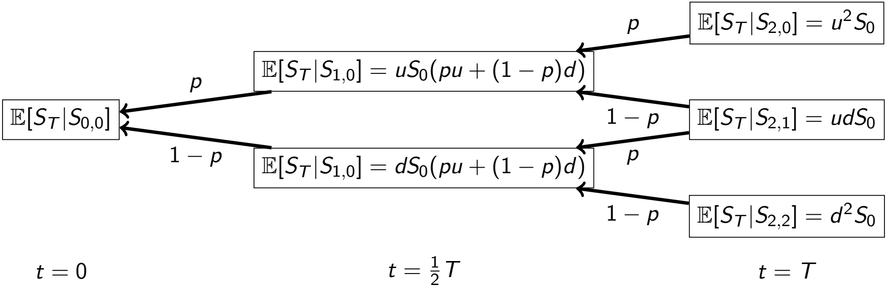
$S_T(S_0 = 1, N = 30, p = 0.5, u = 1.05, v = \frac{1}{u})$



$\ln S_T(S_0 = 1, N = 30, p = 0.5, u = 1.05, v = \frac{1}{u})$

- So, we can match as close as we want the Black-Scholes distribution by matching the first and second moments, and increasing the number of time steps

N-Step Binomial Tree — First Moment



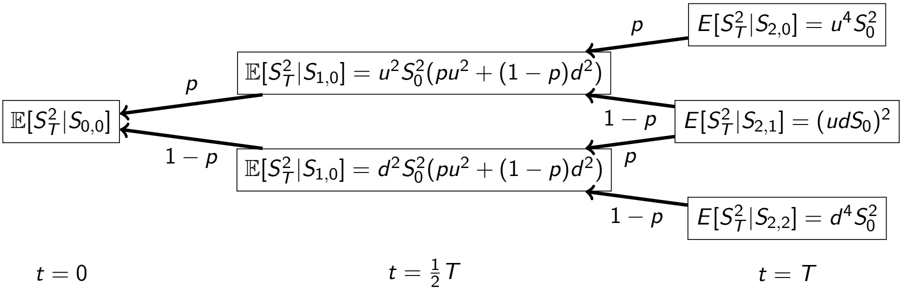
For two step model

$$\mathbb{E}[S_T|S_0] = S_0(pu + (1-p)d)^2 \tag{19}$$

By induction for N step model

$$\mathbb{E}[S_T|S_0] = S_0 \times (pu + (1-p)d)^N \tag{20}$$

N-Step Binomial Tree — Second Moment



For two step model

$$\mathbb{E}[S_T^2|S_0] = S_0^2(pu^2 + (1-p)d^2)^2 \tag{21}$$

By induction for N step model

$$\mathbb{E}[S_T^2|S_0] = S_0^2 \times (pu^2 + (1-p)d^2)^N \tag{22}$$

N-Step Binomial Tree — Matching With Black-Scholes

We split the N step in equal space, so each time step is $\Delta_t = \dfrac{T}{N}$

- To match the mean we just need the probability p to be risk neutral:

$$p = \frac{S_0 e^{r\Delta_t} - dS_0}{uS_0 - dS_0} = \frac{e^{r\Delta_t} - d}{u - d} \tag{23}$$

then

$$\mathbb{E}[S_t|S_0] = S_0(pu + (1-p)d)^N = S_0 e^{r\Delta_t N} = S_0 e^{rT} \tag{24}$$

- Similarly, we just need to replace T by Δ_t in our one step binomial model so as to match the second moment:

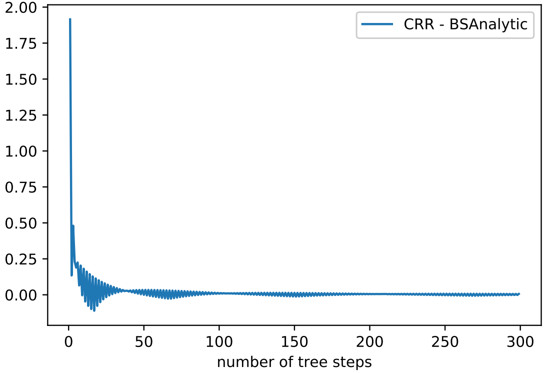
$$u = \frac{e^{(2r+\sigma^2)\Delta_t} + 1 + \sqrt{(e^{(2r+\sigma^2)\Delta_t} + 1)^2 - 4e^{2r\Delta_t}}}{2e^{r\Delta_t}}, \quad d = \frac{1}{u} \tag{25}$$

N-Step Binomial Tree Implementation

```python
def crrBinomial(S, r, vol, payoffType, K, T, n):
    t = T / n
    b = math.exp(vol * vol * t+r*t) + math.exp(-r * t)
    u = (b + math.sqrt(b*b - 4)) / 2
    p = (math.exp(r * t) - (1/u)) / (u - 1/u)
    # set up the last time slice, there are n+1 nodes at the last time slice
    payoffDict = {
        PayoffType.Call: lambda s: max(s-K, 0),
        PayoffType.Put: lambda s: max(K-s, 0),
    }
    vs = [payoffDict[payoffType]( S * u**(n-i-i)) for i in range(n+1)]
    # iterate backward
    for i in range(n-1, -1, -1):
        # calculate the value of each node at time slide i, there are i nodes
        for j in range(i+1):
            vs[j] = math.exp(-r * t) * (vs[j] * p + vs[j+1] * (1-p))
    return vs[0]
# test ---
S, r, vol, K, T = 100, 0.01, 0.2, 105, 1.0
print("blackPrice: \t", bsPrice(S, r, vol, T, K, PayoffType.Call))
print("crrNStepTree: \t", crrBinomial(S, r, vol, PayoffType.Call, K, T, 300))
```

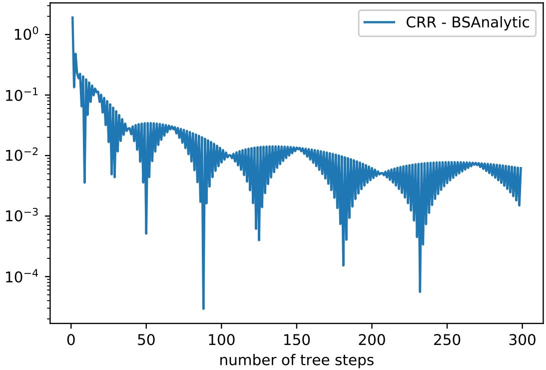
Difference between CRR tree and BS analytic

```
import matplotlib.pyplot as plt
n = 300
S, r, vol, K, T = 100, 0.01, 0.2, 105, 1.0
bsPrc = bsPrice(S, r, vol, PayoffType.Call, K, T)
crrErrs = [(crrBinomial(S,r,vol,PayoffType.Call,K,T,i) - bsPrc) for i in range(1, n)]
plt.plot(range(1, n), crrErrs, label = "CRR - BSAnalytic")
plt.xlabel('number of tree steps')
plt.legend()
plt.savefig('../figs/crrError.eps', format='eps')
```



Easier to look at error on log scale

```
import matplotlib.pyplot as plt
n = 300
S, r, vol, K, T = 100, 0.01, 0.2, 105, 1.0
bsPrc = bsPrice(S, r, vol, PayoffType.Call, K, T)
crrErrs = [abs(crrBinomial(S,r,vol,PayoffType.Call,K,T,i) - bsPrc) for i in range(1, n)] # abs error here
plt.plot(range(1, n), crrErrs, label = "CRR - BSAnalytic")
plt.xlabel('number of tree steps')
plt.yscale('log')  # plot on log scale
plt.legend()
plt.savefig('../figs/crrLogError.eps', format='eps')
```



What's the Next Step

- European call/put and digital options are not all — we have plenty of other products we would like to trade
  - European option with generic payoff
  - American options
  - Barrier options
  - And maybe more — you never know how the industry evolves but you should be ready for changes
- Not all of them have analytic formulae, but our binomial tree pricer can handle much more
- We would like the tree pricer to be implemented elegantly so that it is
  - easy to maintain
  - easy to extend

American Option

- At any point in time, or any node of the tree, we know the continuation value of the product — through calculating the conditional expectation

- American product allows the option holder to exercise the option any time

- This translates to the choice to make at any node of the tree: continue holding the option or take the intrinsic value
  - Continue holding the option — the option is worth its conditional expectation
  - Exercise now — the option is worth its intrinsic value
  - Optimal exercise strategy is to exercise when intrinsic value is worth more than the continuation value — taking the max

American Binomial Tree Pricer — A Trivial Extension

```python

def crrBinomialAmer(S, r, vol, payoffType, K, T, n):
    ... # set up tree
    ... # set up last time slice payoff
    # iterate backward
    for i in range(n-1, -1, -1):
        # calculate the value of each node at time slide i, there are i nodes
        for j in range(i+1):
            # here we deal with american early exercise:
            continuation = math.exp(-r * t) * (vs[j] * p + vs[j+1] * (1-p))
            vs[j] = max(continuation, payoffDict[payoffType](S * u**(i-j-j)))
    return vs[0]
```

Only change to the european pricer `crrBinomial()`:

- At each iteration, `max` is taken between
  - continuation value — not exercise, and
  - payoff value — exercise immediately

Problems?

- Trivial extension is OK for experimenting the algorithms

- Not serious code for production usage
  - Copy-and-paste should be avoided in general

  - Both "'crrBinomial"' and "'crrBinomialAmer"' do not leave too much room for payoff extension — what if I want to price a digital option or call spread?

First Step Extension

Encapsulate arguments that belong to an trade-able `instrument`

- For European style options:
  - expiry T
  - strike K
  - payoffType
- For American style options:
  - everything of an European option
  - eerly exercie feature
- So let us define classes for the trading instruments

European option class:

```python
class EuropeanOption():
    def __init__(self, expiry, strike, payoffType):
        self.expiry = expiry
        self.strike = strike
        self.payoffType = payoffType
    def payoff(self, S):
        if self.payoffType == PayoffType.Call:
            return max(S - self.strike, 0)
        elif self.payoffType == PayoffType.Put:
            return max(self.strike - S, 0)
        else:
            raise Exception("payoffType not supported: ", self.payoffType)
```

American option class:

```python
class AmericanOption():
    def __init__(self, expiry, strike, payoffType):
        self.expiry = expiry
        self.strike = strike
        self.payoffType = payoffType
    def payoff(self, S):
        if self.payoffType == PayoffType.Call:
            return max(S - self.strike, 0)
        elif self.payoffType == PayoffType.Put:
            return max(self.strike - S, 0)
        else:
            raise Exception("payoffType not supported: ", self.payoffType)
```

In this way we lift out the code that deals with payoff from our binomial tree pricer, so that it is more orthogonal to the trade being priced:

```python
def crrBinomial(S, r, vol, trade, n):
    t = trade.expiry / n
    b = math.exp(vol * vol * t+r*t) + math.exp(-r * t)
    u = (b + math.sqrt(b*b - 4)) / 2
    p = (math.exp(r * t) - (1/u)) / (u - 1/u)
    # d = 1 / u
    # set up the last time slice, there are n+1 nodes at the last time slice
    vs = [trade.payoff( S * u**(n-i-i)) for i in range(n+1)]
    # iterate backward
    for i in range(n-1, -1, -1):
        # calculate the value of each node at time slide i, there are i nodes
        for j in range(i+1):
            vs[j] = math.exp(-r * t) * (vs[j] * p + vs[j+1] * (1-p))
    return vs[0]
```

- The only requirement from `crrBinomial` to `trade` is the `expiry` and `payoff` function, everything else is generic

Dealing with American Option

In order not to "'copy paste"', our first attempt is to add a boolean flag "'isAmer"' to the signature of "'crrBinomial"'

```
def crrBinomial(S, r, vol, trade, isAmer, n):
    ...
    for i in range(n-1, -1, -1):
        # calculate the value of each node at time slide i, there are i nodes
        for j in range(i+1):
            vs[j] = math.exp(-r * t) * (vs[j] * p + vs[j+1] * (1-p))
            if isAmer:
               vs[j] = max(vs[j], trade.payoff(S * u**(i-j-j)))
    return vs[0]
```

- It does avoid "'copy paste"', but...
- It puts back attributes that belong to the tradeable to the pricing model, consider these two function calls

```
crrBinomial(S = 100, r = 0.01, vol = 0.2, trade = EuropeanOption(expiry=1, strike=105, payoffType =
        PayoffType.Call), isAmer = True, n = 1000)
crrBinomial(S = 100, r = 0.01, vol = 0.2, trade = AmericanOption(expiry=1, strike=105, payoffType =
        PayoffType.Call), isAmer = False, n = 1000)
```

- without looking at the code of "'crrBinomial"' it's hard to tell which one is pricing an American option and which one is European option.

Second Attempt

What if we require a function `isAmer() -> bool` from `trade`

```
def crrBinomial(S, r, vol, trade, n):
    ...
    for i in range(n-1, -1, -1):
        # calculate the value of each node at time slide i, there are i nodes
        for j in range(i+1):
            vs[j] = math.exp(-r * t) * (vs[j] * p + vs[j+1] * (1-p))
            if trade.isAmer():
               vs[j] = max(vs[j], trade.payoff(S * u**(i-j-j)))
    return vs[0]
```

Better that the attribute of the trade is provided by the trade, but

- the pricer is restricted to American trade and non-American trade
- Exercise strategy should be associated with the product, not the pricer

Look At Our Tree Algorithm Again

Algorithm 1 succeed = PlaceQ (prevQ, i)

```
1: Set up the tree and parameters
2: Initialize the last time slice with final payoff
3: for k = N − 1 to 0 do
4:     for i = 0 to k do
5:         Calculate the continuation value (discounted expectation)
6:         Given the information of the tree node, calculate the option value
           at Node(k, i)
7:     end for
8: end for
```

- Step 5 belongs to the tree pricer
- Step 6 is fully determined by the product:
  - **Input**: stock price S, continuation value V, current time t
  - **Output**: current option value

Pricing American Products With Tree

- It makes sense to encapsulate step 6 in the tree algorithm into the trade-able classes

```
class AmericanOption():
    def __init__(self, expiry, strike, payoffType):
        self.expiry = expiry
        self.strike = strike
        self.payoffType = payoffType
    def payoff(self, S):
        if self.payoffType == PayoffType.Call:
            return max(S - self.strike, 0)
        elif self.payoffType == PayoffType.Put:
            return max(self.strike - S, 0)
        else:
            raise Exception("payoffType not supported: ", self.payoffType)
    # step 6 in tree algo, exercise logic
    def valueAtNode(self, t, S, continuation):
        return max(self.payoff(S), continuation)
```

- For European option, `valueAtNode` just pass on the continuation value

```
class EuropeanOption():
    def __init__(self, expiry, strike, payoffType):
        self.expiry = expiry
        self.strike = strike
        self.payoffType = payoffType
    def payoff(self, S):
        if self.payoffType == PayoffType.Call:
            return max(S - self.strike, 0)
        elif self.payoffType == PayoffType.Put:
            return max(self.strike - S, 0)
        else:
            raise Exception("payoffType not supported: ", self.payoffType)

    def valueAtNode(self, t, S, continuation):
        return continuation
```

Pricing American Products With Tree

The tree pricer now becomes

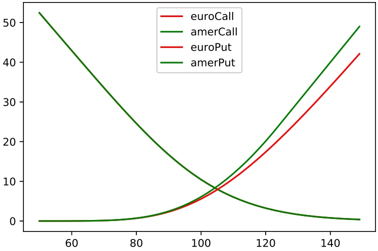
```python
def crrBinomialG(S, r, vol, trade, n):
    t = trade.expiry / n
    b = math.exp(vol * vol * t+r*t) + math.exp(-r * t)
    u = (b + math.sqrt(b*b - 4)) / 2
    p = (math.exp(r * t) - (1/u)) / (u - 1/u)
    # d = 1 / u
    # set up the last time slice, there are n+1 nodes at the last time slice
    vs = [trade.payoff( S * u**(n-i-i)) for i in range(n+1)]
    # iterate backward
    for i in range(n-1, -1, -1):
        # calculate the value of each node at time slide i, there are i nodes
        for j in range(i+1):
            nodeS = S * u**(i-j-j)
            continuation = math.exp(-r * t) * (vs[j] * p + vs[j+1] * (1-p))
            vs[j] = trade.valueAtNode(t*i, nodeS, continuation)
    return vs[0]
```

There is no more ambiguity pricing European and American options:

```python
euroPrc.append(crrBinomialG(S, r, vol, EuropeanOption(expiry, strike, PayoffType.Call), 300))
amerPrc.append(crrBinomialG(S, r, vol, AmericanOption(expiry, strike, PayoffType.Call), 300))
```

Testing Binomial Tree Pricer with American Option

```
euroPrc, amerPrc = [],[]
S, r, vol = 100, 0.05, 0.2
ks = range(50, 150)
for k in ks:
    euroPrc.append(crrBinomialG(S, r, vol, EuropeanOption(1, float(k), PayoffType.Call), 300))
    amerPrc.append(crrBinomialG(S, r, vol, AmericanOption(1, float(k), PayoffType.Call), 300))
plt.plot(ks, euroPrc, 'r', label='euroCall')
plt.plot(ks, amerPrc, 'g', label='amerCall')
euroPrc, amerPrc = [], []
for k in ks:
    euroPrc.append(crrBinomialG(S, r, vol, EuropeanOption(1, float(k), PayoffType.Put), 300))
    amerPrc.append(crrBinomialG(S, r, vol, AmericanOption(1, float(k), PayoffType.Put), 300))
plt.plot(ks, euroPrc, 'r', label='euroPut')
plt.plot(ks, amerPrc, 'g', label='amerPut')
plt.legend()
plt.show()
```



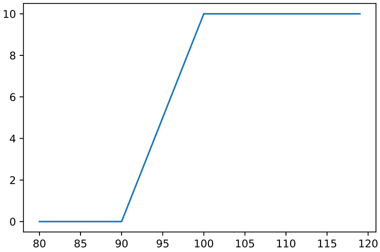
Generalize the Payoff Function

Now the CRR binomial tree pricer is generic, and if we want to price European / American exercise style option with any payoff function, we just need to create a tradeable

```python
class EuropeanPayoff():
    def __init__(self, expiry, payoffFun):
        self.expiry = expiry
        self.payoffFun = payoffFun
    def payoff(self, S):
        return self.payoffFun(S)
    def valueAtNode(self, t, S, continuation):
        return continuation

class AmericanPayoff():
    def __init__(self, expiry, payoffFun):
        self.expiry = expiry
        self.payoffFun = payoffFun
    def payoff(self, S):
        return self.payoffFun(S)
    def valueAtNode(self, t, S, continuation):
        return max(self.payoff(S), continuation)
```

Example: Pricing a Call Spread

```
S, r, vol = 100, 0.05, 0.2
callSpread = lambda S: min(max(S-90, 0), 10)
plt.plot(range(80, 120), [callSpread(i) for i in range(80, 120)] )
plt.show()
print("Euro callspread: ", crrBinomialG(S, r, vol, EuropeanPayoff(1, callSpread), 300))
print("Amer callspread: ", crrBinomialG(S, r, vol, AmericanPayoff(1, callSpread), 300))
```




```
Euro callspread:  6.259190489574921
Amer callspread:  10.0
```

Practice: plot a spot ladder of Euro/Amer call spread prices, explain what you see.

Pricing Barrier Options

Now let us extend the CRR binomial pricer to Barrier Options

- A barrier option defines
  - one or two barrier levels (up or/and down) and
  - a pre-defined time window (normally from now to option expiry) ¡br¿

  such that if the spot price touches the barrier in the given time window, the option holder obtain (**Knock-In**) or lose (**Knock-Out**) a **underlying payoff**

- The underlying payoff can be
  - a European option
  - or just a constant rebate (normally called touch option)

Pricing Barrier Options

- Knock-Out (KO) barrier option can be priced naturally with tree
  - At each point of the tree, if the spot price triggers the KO, then the option is worth 0, otherwise it is worth its continuation value
  - The continuation value (discounted future expectation) does not contribute to the nodes triggering the KO
- Knock-In (KI) barrier option is not so natural for pricing with tree:
  - Not-yet knocked in does not mean the value is 0
  - Requires an auxiliary variable to price with tree
  - KIKO parity: KI + KO = Vanilla static replication
- We consider knock-out barrier option for now

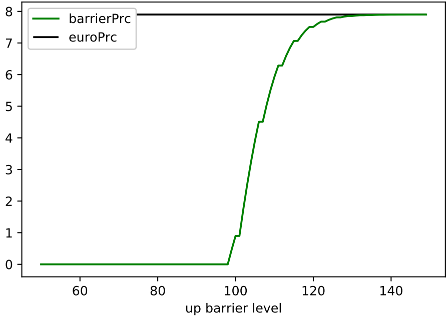
Barrier Option Class

```python
class BarrierOption():
    def __init__(self, downBarrier, upBarrier, barrierStart, barrierEnd,
     underlyingOption):
        self.underlyingOption = underlyingOption
        self.barrierStart = barrierStart
        self.barrierEnd = barrierEnd
        self.downBarrier = downBarrier
        self.upBarrier = upBarrier
        self.expiry = underlyingOption.expiry
    def payoff(self, S):
        return self.underlyingOption.payoff(S)
    def valueAtNode(self, t, S, continuation):
        if t > self.barrierStart and t < self.barrierEnd:
            if self.upBarrier != None and S > self.upBarrier:
                return 0
            elif self.downBarrier != None and S < self.downBarrier:
                return 0
        return continuation
```

Now we can enjoy the fruit of our generic CRR biniomial pricer — no change needed to the pricer

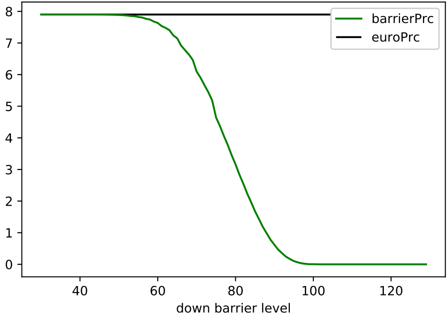
Testing Barrier Options — Up Barriers

```python
# varying up barrier
S, r, vol, K = 100, 0.05, 0.2, 105
eurOpt = EuropeanOption(1, k, PayoffType.Put)
euroPrc = crrBinomialG(S, r, vol, eurOpt, 300)
barrierPrc, ks = [], range(50, 150)
for barrierLevel in ks:
    prc = crrBinomialG(S, r, vol, BarrierOption(barrierStart = 0, barrierEnd = 1.0, downBarrier = None,
      upBarrier = barrierLevel, underlyingOption = eurOpt), n = 300)
    barrierPrc.append(prc)
plt.hlines(euroPrc, ks[0], ks[-1], label = 'euroPrc')
plt.plot(ks, barrierPrc, 'g', label='barrierPrc')
plt.xlabel('up barrier level'); plt.legend(); plt.savefig('../figs/upKO.eps', format='eps')
```



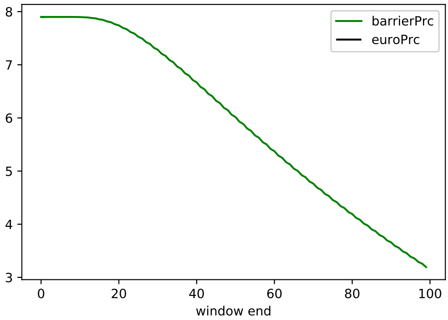
Testing Barrier Options — Down Barriers

```
# varying down barrier
S, r, vol, K = 100, 0.05, 0.2, 105
eurOpt = EuropeanOption(1, k, PayoffType.Put)
euroPrc = crrBinomialG(S, r, vol, eurOpt, 300)
barrierPrc, ks = [], range(30, 130)
for barrierLevel in ks:
    prc = crrBinomialG(S, r, vol, BarrierOption(barrierStart = 0, barrierEnd = 1.0, downBarrier =
      barrierLevel, upBarrier = None, underlyingOption = eurOpt), n = 300)
    barrierPrc.append(prc)
plt.hlines(euroPrc, ks[0], ks[-1], label = 'euroPrc')
plt.plot(ks, barrierPrc, 'g', label='barrierPrc')
plt.xlabel('down barrier level'); plt.legend(); plt.savefig('../figs/downKO.eps', format='eps')
```



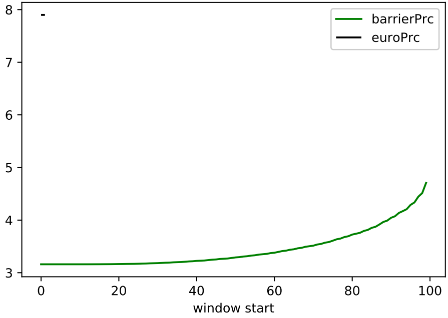
Testing Barrier Options — Window Barriers

```
# varying barrier window, barrier end
S, r, vol, k = 100, 0.05, 0.2, 105
eurOpt = EuropeanOption(1, k, PayoffType.Put)
euroPrc = crrBinomialG(S, r, vol, eurOpt, 300)
barrierPrc = []
ks = range(0, 100)
for t in ks:
    prc = crrBinomialG(S, r, vol, BarrierOption(barrierStart = 0, barrierEnd = t / 100.0, downBarrier =
      80, upBarrier = 150, underlyingOption = eurOpt), n = 300)
    barrierPrc.append(prc)
plt.hlines(euroPrc, ks[0], ks[-1] / 100, label = 'euroPrc')
plt.plot(ks, barrierPrc, 'g', label='barrierPrc')
plt.legend(); plt.xlabel('window end'); plt.savefig('../figs/winBarrier.eps', format='eps')
```



Testing Barrier Options — Window Barriers

```python
# varying barrier window, barrier start
S, r, vol, k = 100, 0.05, 0.2, 105
eurOpt = EuropeanOption(1, k, PayoffType.Put)
euroPrc = crrBinomialG(S, r, vol, eurOpt, 300)
barrierPrc = []
ks = range(0, 100)
for t in ks:
    prc = crrBinomialG(S, r, vol, BarrierOption(barrierStart = t, barrierEnd = 1.0, downBarrier = 80,
      upBarrier = 150, underlyingOption = eurOpt), n = 300)
    barrierPrc.append(prc)
plt.hlines(euroPrc, ks[0], ks[-1] / 100, label = 'euroPrc')
plt.plot(ks, barrierPrc, 'g', label='barrierPrc')
plt.legend(); plt.xlabel('window start'); plt.savefig('../figs/winBarrierStart.eps', format='eps')
```



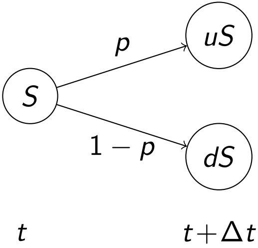
More On Binomial Tree Models

- Recall that we use CRR binomial tree model [1] where the additional constraint $u = \frac{1}{d}$ was imposed to restrict the solution space when trying to match the second moment

$$e^{2r\Delta t+\sigma^2\Delta t} = pu^2 + (1-p)d^2 \tag{26}$$

- Recall also that the first moment is matched by setting

$$p = \frac{e^{r\Delta t} - d}{u - d} \tag{27}$$



Alternative Binomial Tree Models

Alternative binomial tree models are built by imposing different constriants

- Jarrow-Rudd binomial tree
  - Equal probabily variation (not risk neutral): $p = \frac{1}{2}$
  - Risk neutral variation
- Tian binomial model: match the first three moments
- ...

M. Joshi presented a comparison of 11 different tree modesl: [2]

Jarrow-Rudd Models

Jarrow-Rudd Equal Probability Model (JREQ)

$$\begin{cases} p &= \frac{1}{2} \\ u &= e^{(r-\frac{\sigma^2}{2})\Delta t + \sigma\sqrt{\Delta t}} \\ d &= e^{(r-\frac{\sigma^2}{2})\Delta t - \sigma\sqrt{\Delta t}} \end{cases} \tag{28}$$

- Not risk neutral since $\frac{e^{r\Delta t}-d}{u-d} \neq \frac{1}{2}$

Jarrow-Rudd Risk Neutral Model (JRRN)

$$\begin{cases} p &= \frac{e^{r\Delta t}-d}{u-d} \\ u &= e^{(r-\frac{\sigma^2}{2})\Delta t + \sigma\sqrt{\Delta t}} \\ d &= e^{(r-\frac{\sigma^2}{2})\Delta t - \sigma\sqrt{\Delta t}} \end{cases} \tag{29}$$

- Use the same u and d as JREQ model, but replace the probability by risk neutral probability.

Tian's Model

Matching the first three moments:

$$\begin{cases} pu + (1-p)d = e^{r\Delta t} \\ pu^2 + (1-p)d^2 = e^{2r\Delta t + \sigma^2 \Delta t} \\ pu^3 + (1-p)d^3 = e^{3r\Delta t + 3\sigma^2 \Delta t} \end{cases} \tag{30}$$

This leads to the parameters:

$$\begin{cases} p &= \frac{e^{r\Delta t} - d}{u-d} \\ u &= 0.5 e^{r\Delta t} v (v + 1 + \sqrt{v^2 + 2v - 3}) \\ d &= 0.5 e^{r\Delta t} v (v + 1 - \sqrt{v^2 + 2v - 3}) \\ v &= e^{\sigma^2 \Delta t} \end{cases} \tag{31}$$

Note that all these models calibrate model parameters p, u, d to the drift r and volatility σ, the binomial tree algorithm does not change — another sign of generalization.

Our Generic Binomial Tree Pricer

```
def binomialPricer(S, r, vol, trade, n, calib):
    t = trade.expiry / n
    (u, d, p) = calib(r, vol, t)
    # set up the last time slice, there are n+1 nodes at the last time slice
    vs = [trade.payoff(S * u ** (n - i) * d ** i) for i in range(n + 1)]
    # iterate backward
    for i in range(n - 1, -1, -1):
        # calculate the value of each node at time slide i, there are i nodes
        for j in range(i + 1):
            nodeS = S * u ** (i - j) * d ** j
            continuation = math.exp(-r * t) * (vs[j] * p + vs[j + 1] * (1 - p))
            vs[j] = trade.valueAtNode(t * i, nodeS, continuation)
    return vs[0]
```

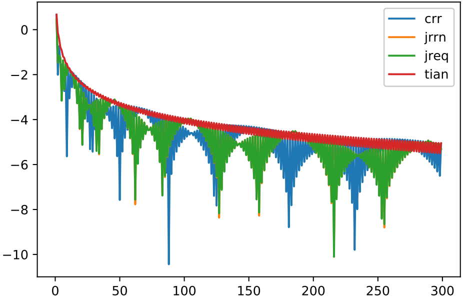
- The tree price expects `u, d, p` from the model
- Each model is responsible for the `calib` function:

Binommial Tree Models Calib

```python
def crrCalib(r, vol, t):
    b = math.exp(vol * vol * t + r * t) + math.exp(-r * t)
    u = (b + math.sqrt(b * b - 4)) / 2
    p = (math.exp(r * t) - (1 / u)) / (u - 1 / u)
    return (u, 1/u, p)

def jrrnCalib(r, vol, t):
    u = math.exp((r - vol * vol / 2) * t + vol * math.sqrt(t))
    d = math.exp((r - vol * vol / 2) * t - vol * math.sqrt(t))
    p = (math.exp(r * t) - d) / (u - d)
    return (u, d, p)

def jreqCalib(r, vol, t):
    u = math.exp((r - vol * vol / 2) * t + vol * math.sqrt(t))
    d = math.exp((r - vol * vol / 2) * t - vol * math.sqrt(t))
    return (u, d, 1/2)

def tianCalib(r, vol, t):
    v = math.exp(vol * vol * t)
    u = 0.5 * math.exp(r * t) * v * (v + 1 + math.sqrt(v*v + 2*v - 3))
    d = 0.5 * math.exp(r * t) * v * (v + 1 - math.sqrt(v*v + 2*v - 3))
    p = (math.exp(r * t) - d) / (u - d)
    return (u, d, p)
```

Test Binomial Models

```
opt = EuropeanOption(1, 105, PayoffType.Call)
S, r, vol, n = 100, 0.01, 0.2, 300
bsprc = bsPrice(S, r, vol, opt.payoffType, opt.strike, opt.expiry)
crrErrs = [math.log(abs(binomialPricer(S, r, vol, opt, i, crrCalib) - bsprc)) for i in range(1, n)]
jrrnErrs = [math.log(abs(binomialPricer(S, r, vol, opt, i, jrrnCalib) - bsprc)) for i in range(1, n)]
jreqErrs = [math.log(abs(binomialPricer(S, r, vol, opt, i, jreqCalib) - bsprc)) for i in range(1, n)]
tianErrs = [math.log(abs(binomialPricer(S, r, vol, opt, i, tianCalib) - bsprc)) for i in range(1, n)]
plt.plot(range(1, n), crrErrs, label = "crr")
plt.plot(range(1, n), jrrnErrs, label = "jrrn")
plt.plot(range(1, n), jreqErrs, label = "jreq")
plt.plot(range(1, n), tianErrs, label = "tian")
plt.legend(); plt.savefig('../figs/btrees.eps', format='eps')
```



References

John C. Cox, Stephen A. Ross, and Mark Rubinstein. *Option pricing: A simplified approach.*,Journal of Financial Economics, 7(3):229-263, 1979.

Mark Joshi *The convergence of binomial trees for pricing the American put*, Risk Net, 2009