QF609 Risk Analysis
Lecture Notes 6

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Overview

Main topics:

- Recap
- Value-at-Risk (VaR) Model (Cont.)
- Group Assignment #1 Discussion

Recap

In the last lecture, we started to look into VaR modelling. VaR has 3 basic parameters:

- *Significance level.* This is is often set by an external body, such as a regulator. Under the Basel II Accord, banks using internal VaR models to assess their market risk capital requirement need to measure VaR at the 1% significance level.
- *Risk horizon.* The risk horizon is the period over which one measures the potential loss, which is typically measured in trading days rather than calendar days. In practice, the risk horizon is usually 1-day, 10-day, or 21-day (i.e. a business-month). Under the Basel banking regulations, the risk horizon is 10-day.
- *Reporting currency.* This is the currency in which the VaR should be reported in.

Denote the portfolio profilt and loss over the risk horizon by L. Technically, the *h-day* **VaR** at a signficance level of $\alpha\%$ for the portfolio is the number $VaR_{\alpha,h}$ given by:

$$VaR_{\alpha,h} = |J|, \quad where \ \ Probability\,(L \leq J) = \alpha.$$

Effectively, this means that $VaR_{\alpha,h}$ is the α-percentile of the distribution of L.

Recap

Exercise: A portfolio consists of \$1 million in each of three stocks (stock A, B and C). The daily **relative** returns (i.e. $\frac{S(t+1day)}{S(t)} - 1$) of the stocks are normally distributed with the following means and covariance matrix:

$$\mu = \begin{pmatrix} 0.0356\% \\ 0.0267\% \\ 0.0133\% \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 0.00007 & 0.0001 & -0.000045 \\ 0.0001 & 0.0004 & -0.00008 \\ -0.000045 & -0.00008 & 0.000178 \end{pmatrix}$$

What will be the one-day VaR for the portfolio at a significant level of 1%?

Ans: Since relative daily returns of the stocks are correlated normal, the one-day portfolio value change L is normally distributed with the followign mean and variance:

$$\begin{aligned} \mu_P &= w \cdot \mu = 756, \\ \sigma_P^2 &= w \cdot \Sigma \cdot \mu = 598,000,000. \end{aligned}$$

where $w = (1mil, \ 1mil, \ 1mil)$. The 1-day VaR at a 1% significance level is then the 1%-percentile of L which is given by:

$$VaR_{1d,0.01} = |756 + (-2.3263) \cdot \sqrt{(598,000,000)}| = 56,131.$$

Recap

Practical implementation of VaR calculation generally involves the following steps:

- *define the VaR parameters: significance level, risk horizon, and reporting currency*
- *identify the portfolio*
- *identify the risk factors of the portfolio*
- *model the joint distribution of the risk factors over the risk horizon*
- *build the distribution of the portfolio P&L over the risk horizon*
- *calculate the VaR*

There are 3 main VaR modelling methods:

- Parametric VaR
- Monte Carlo VaR
- Historical VaR

The main differences between the 3 models lie in the assumptions and the approach applied to generate the distribution of risk factor changes.

Parametric VaR Model

This model assumes:

- the daily VaR risk factor changes (which can be relative or absolute change) are multivariate-normally distributed with means and covariance matrix estimated from historical data
- the daily changes in each VaR risk factor are i.i.d. and are additive (i.e. the change in a risk factor over an h-day period is the sum of the corresponding h daily changes)
- the portfolio PV change over the risk horizon is a linear function of the VaR risk factor changes

As a result of the above, the portfolio P&L distribution over the risk horizon is normally distributed from which the VaR can be readily calculated.

Value-at-Risk (VaR) Model

Let's look at a technical formulation of the parametric VaR model:

- Assume an h-day risk horizon, a VaR significance level of α, and that all positions in the underlying portfolio are denominated in the same currency.
- There are m underlying VaR risk factors for the portfolio, denoted by $\{x^{(k)}\}_{k=1}^{m}$. The **daily** changes in the risk factors are assumed to be multivariate normal with the following means and variances:

$$\vec{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_m \end{pmatrix}, \quad \Sigma = \begin{pmatrix} v_{1,1} & v_{1,2} & \cdots & v_{1,m} \\ v_{1,2} & v_{2,2} & \cdots & v_{2,m} \\ \vdots & \vdots & \vdots & \vdots \\ v_{1,m} & v_{2,m} & \cdots & v_{m,m} \end{pmatrix}$$

- The daily changes are additive in that the change of each risk factor over the h-day period is the sum of its daily changes over the period.
- The portfolio value change over the h-day period is assumed to be:

$$L = P_h - P_0 = \sum_{k=1}^{m} a^{(k)} \cdot \Delta x_{0,h}^{(k)}$$

where $\Delta x_{0,h}^{(k)}$ denotes the change of the k-th risk factor over the h-day period, which can be either absolute or relative change, and $\{a^{(k)}\}_{k=1}^{m}$ are m constant weights.

Value-at-Risk (VaR) Model

Following these assumptions, the changes in the risk factors over the h-day risk horizon is multivariate-normally distributed with mean $\{\mu_k \cdot h\}_{k=1}^{m}$ and covariance matrix $\Sigma \cdot h$. To see this, denote by $\Delta x_{i,i+1}^{(j)}$ the daily change of the j-th risk factor from day i to day $i+1$. Let us consider the changes in two different factors $x^{(j_1)}$ and $x^{(j_2)}$ over the h-day horizon. Following our assumptions, we have:

$$
\begin{aligned}
\Delta x_{0,h}^{(j_1)} &= \sum_{i=0}^{h-1} \Delta x_i^{(j_1)}, \quad \Delta x_{0,h}^{(j_2)} = \sum_{i=0}^{h-1} \Delta x_i^{(j_2)} \\
E\left[\Delta x_{0,h}^{(j_1)}\right] &= \sum_{i=0}^{h-1} E\left[\Delta x_i^{(j_1)}\right] = \mu_{j_1} \cdot h \\
E\left[\Delta x_{0,h}^{(j_2)}\right] &= \sum_{i=0}^{h-1} E\left[\Delta x_i^{(j_2)}\right] = \mu_{j_2} \cdot h \\
V\left[\Delta x_{0,h}^{(j_1)}\right] &= \sum_{i=0}^{h-1} V\left[\Delta x_{i,i+1}^{(j_1)}\right] = v_{j_1,j_1} \cdot h \\
V\left[\Delta x_{0,h}^{(j_2)}\right] &= \sum_{i=0}^{h-1} V\left[\Delta x_{i,i+1}^{(j_2)}\right] = v_{j_2,j_2} \cdot h
\end{aligned}
$$

Value-at-Risk (VaR) Model

$$
\begin{aligned}
COV\left[\Delta x_{0,h}^{(j_1)}, \Delta x_{0,h}^{(j_2)}\right] &= COV\left(\sum_{i=0}^{h-1} \Delta x_{i,i+1}^{(j_1)}, \sum_{i=0}^{h-1} \Delta x_{i,i+1}^{(j_2)}\right) \\
&= \sum_{i=0}^{h-1} COV\left(\Delta x_{i,i+1}^{(j_1)}, \Delta x_{i,i+1}^{(j_2)})\right) \\
&= v_{j_1,j_2} \cdot h
\end{aligned}
$$

Consequently, the portfolio P&L, L, over the risk horizon is normally distributed with the following mean and variance:

$$
\begin{aligned}
\mu_P &= \sum_{k=1}^{m} a^{(k)} \cdot \mu_k \cdot h, \\
\sigma_P^2 &= V\left[\sum_{k=1}^{m} a^{(k)} \cdot \Delta x_{0,h}^{(k)}\right] = w \cdot (\Sigma \cdot h) \cdot w^T
\end{aligned}
$$

where $w = (a^{(1)}, a^{(2)}, \cdots, a^{(m)})$. The VaR is given by:

$$
VaR_{\alpha,h} = |\mu_P + z_\alpha \cdot \sigma_P|
$$

where z_α is the α-percentile of the standard normal distribution.

Value-at-Risk (VaR) Model

From the above, we see that the parametric VaR model requires the following inputs:

- estimates of the mean and variance of the daily risk factor changes
- the weights $\{a^{(k)}\}_{k=1}^{m}$

The mean and variance of the daily risk factor changes can be estimated from historical daily changes of the risk factors. The weights $\{a^{(k)}\}_{k=1}^{m}$ are effectively the sensitivities/risks of the portfolio value w.r.t. the VaR risk factors which can be calculated using the pricing/risk engines of the positions. These sensitivities need to be suitably defined so that the following formula gives a proper estimate of the portfolio P&L:

$$L = P_h - P_0 = \sum_{k=1}^{m} a^{(k)} \cdot \Delta x_{0,h}^{(k)}.$$

For example, if relative daily change is used for the k-th risk factor, the sensitivity $a^{(k)}$ should be defined w.r.t. a relative change in the risk factor. This is not uncommon for FX rates or equity price risk factors. In essence, the formula above is a risk-based estimate of the portfolio P&L.

Value-at-Risk (VaR) Model

Monte Carlo VaR Model

For the Monte Carlo VaR model, one needs to first specify the distributions of the daily changes of the underlying VaR risk factors. The parameters of the distributions are fitted to historical data of daily risk factor changes.

The normal distribution remains the most common distributional assumption used for Monte Carlo VaR model. One reason for this is that non-normal distributions are more technically challenging to work with, especially in modelling correlations between the risk factor changes.

Once the distributions of the risk factors are specified and their parameters are fitted, Monte Carlo simulation is used to generate scenarios of daily risk factors changes. For each simulated scenario, one needs to compute the corresponding daily portfolio P&L under the scenario. There are two approaches for this:

- **Full revaluation approach.** Each simulated scenario of risk factor changes is applied to the base risk factor values to generate a scenario of the risk factors, which are then fed into the position valuation engines to produce a corresponding portfolio P&L scenario.
- **Risk-based approach.** This applies the same the portfolio P&L formula as in the case of parametric VaR to obtain a daily portfolio P&L scenario for each simulated scenario of daily risk factor changes.

$$L \quad = \quad P_h - P_0 \; = \; \sum_{k=1}^{m} a^{(k)} \cdot \Delta x^{(k)}.$$

where $\{a^{(k)}\}_{k=1}^{m}$ are suitably defined sensitivities/risks of the portfolio value w.r.t. the VaR risk factors.

Following the above, one effectively obtains a simulated sample from the **daily** porfolio P&L distribution. The 1-day VaR can be calculated from the empirical distribution given by the porfolio P&L sample. Since the empirical distribution of the porfolio P&L is a discrete distribution, one may need to apply interpolation to obtain the VaR for an arbitrary VaR significance level.

Example:

Suppose that the Monte Carlo VaR Model has generated a sample of 155 portfolio P&L scenarios. The empirical distribution of the sample assigns a probability of 1/155 to each of P&L scenarios. Assume that we have arranged the sample P&L's in an increasing order: $L_1 < L_2 < \cdots < L_{155}$. We can verify that L_1 is the 0.65% percentile and L_2 is the 1.29% percentile. Thus, for a significance level of 1%, we can apply linear interpolation to obtain the 1% VaR:

$$VaR_{0.01,1d} \quad = \quad L_1 + \frac{L_2 - L_1}{1.29\% - 0.65\%} \cdot (1.00\% - 0.65\%).$$

Next, if the risk horizon is different from 1-day, say h-day, one would typically apply some scaling rule to convert the 1-day VaR to an h-day VaR:

$$VaR_{0.01,h} \quad \approx \quad \sqrt{h} \cdot VaR_{0.01,1d}.$$

Value-at-Risk (VaR) Model

Historical VaR Model

The historical VaR model differs from the Monte Carlo VaR only in how the distributions of the daily changes of the VaR risk factors are obtained. Under the historical VaR model, the distributions are given by the empirical (joint) distributions of the historical daily changes of the risk factors. This has some obvious advantages, e.g.

- no need for estimation of distributional parameters
- correlations between the risk factors by default are built in in the their historical empirical (joint) distributions

Apart from the above, the historical and Monte Carlo VaR models follow essentially the same ideas. In particular, the portfolio P&L scenarios can be calcualted either using a full revaluation approach or a risk-based approach.

Value-at-Risk (VaR) Model

Remarks

There are other more advance methods for generating scenarios (hence, distributions) of risk factor changes. For example, some time series approaches may be used to model risk factor changes which can take into the dynamic nature of the risk factors (e.g. autocorrelation and volatility clustering etc.). Some variations of the VaR model may also apply adjusted weights to the generated scenarios. For example, in the historical VaR, one may apply higher weights to more recent historical risk factor changes compared to the older ones.

Nevertheless, given that a typical VaR portfolio may have hundreds or risk factors and tens of thousands of financial positions, it is important that the implemented VaR methodologies are scalable in practice. For this reason, the standard historical VaR remains one of the most widely used models.

Discussion Group Assignment #1

- problem explanation
- Q&A