Risk Analysis (QF609, AY2022-2023)

Exam Preparation Questions

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1. For (a) and (b):

$$\begin{aligned}
PV &= \sum_{i=1}^{20} \frac{25,000}{(1+6\%)^{\frac{i}{2}}} + \frac{1,000,000}{(1+6\%)^{10}} = 931,839 \\
Macaulay\ \ Duration &= \frac{\sum_{i=1}^{20} \frac{25,000\cdot\frac{i}{2}}{(1+6\%)^{\frac{i}{2}}} + \frac{1,000,000\cdot 10}{(1+6\%)^{10}}}{931,839} = 7.9034 \\
Modified\ \ Duration &= \frac{Macaulay\ \ Duration}{1+yield} = \frac{7.9034}{1+6\%} = 7.4560 \\
Convexity &= \frac{1}{PV}\cdot\frac{d^2PV}{dy^2} = \frac{1}{931,839}\cdot\left[\sum_{i=1}^{20}\frac{(\frac{i}{2}+1)\cdot\frac{i}{2}\cdot 25,000}{(1+6\%)^{\frac{i}{2}+2}} + \frac{1,000,000\cdot(10+1)\cdot 10}{(1+6\%)^{10+2}}\right] = 71.39
\end{aligned}$$

Similary, for (c):

Yield	6%	3%	8%
PV	931,839	1,173,779	805,279
Macaulay Duration	7.9034	8.1714	7.7131
Modified Duration	7.4560	7.9334	7.1417
Convexity	71.39	78.95	66.63

Finally, for (d):

$$\Delta PV \approx PV\cdot\left[-MD\cdot\Delta y + \frac{1}{2}\cdot convexity\cdot(\Delta y)^2\right]$$

Hence, the estimated impacts are:

$$\begin{aligned}
PV(yield=3\%) - PV(yield=6\%) &\approx 931,839\cdot[-7.4560\cdot(-0.03) + \frac{1}{2}\cdot 71.39\cdot(-0.03)^2 = 238,370 \\
PV(yield=8\%) - PV(yield=6\%) &\approx -125,651
\end{aligned}$$

where the exact impacts can be calcualted from results in (a)-(c) which are 241,940 and -126,560, respectively.

2. (a) $Combined\ \ Duration = \frac{100}{130}\cdot 5 + \frac{30}{130}\cdot 4.5 = 4.8846.$

   (b) $Leverage-adjusted\ \ D-Gap = 4.8846 - \frac{140}{130}\cdot 6 = -1.5769.$

   (c) The bond notional amount to be purchased should be such that the resulted leveraged adjusted D-Gap should be zero. Let this notional amount be X. We have

$$\frac{100\cdot 5 + 30\cdot 4.5 + \frac{X}{,1000,000}\cdot 931,839\cdot 7.9034}{100+30+\frac{X}{,1000,000}\cdot 931,839} = \frac{140}{100+30+\frac{X}{,1000,000}\cdot 931,839}\cdot 6 = 27.8355$$

3. (a) Applying linear interpolation to the CDF, the 1%-percentile of R, denoted by $r_{1\%}$, is given by

$$\frac{0.01-0.005}{r_{1\%}-(-0.2)} = \frac{0.125-0.005}{-0.1-(-0.2)}$$

   This gives $r_{1\%} = -0.1958$. As a result, the 1-day VaR is given by $|100\cdot -0.1958| = 19.58$.

   (b) In this case, the VaR is equal to $\left|100\cdot[1.8\% + \sqrt{0.8181\%}\cdot z_{1\%}]\right| = 19.24$ where $z_{1\%} = -2.32635$ is the 1%-percentile of the standard normal distribution.

4.

$$Expected\ Value \ = \ 1{,}000{,}000 \cdot (0.0356\% + 0.0267\% + 0.0133\%) \ = \ 756$$

$$\begin{aligned} Variance &= (1{,}000{,}000 \quad 1{,}000{,}000 \quad 1{,}000{,}000) \cdot \begin{pmatrix} 0.00007 & 0.0001 & -0.000045 \\ 0.0001 & 0.0004 & -0.00008 \\ -0.000045 & -0.00008 & 0.000178 \end{pmatrix} \cdot \begin{pmatrix} 1{,}000{,}000 \\ 1{,}000{,}000 \\ 1{,}000{,}000 \end{pmatrix} \\ &= 598{,}000{,}000 \end{aligned}$$

$$VaR_{1d,99\%} \ = \ \left| 756 + z_{0.01} \cdot \sqrt{598{,}000{,}000} \right| \ = \ 56133$$

where $z_{0.01} = -2.32635$ is the 1% percentile of the standard normal distribution.

5. Supposewe have the following Cholesky Decomposition:

$$\begin{pmatrix} 1 & 0.4 & 0.2 \\ 0.4 & 1 & 0.35 \\ 0.2 & 0.35 & 1 \end{pmatrix} \ = \ \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix} \cdot \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix}^T$$

It is clear that:

(a) $a = 1$ as the length of each row in the factor loading matrix has to be 1.

(b) $b = 0.4$, by observing the dot product between the two different rows gives the respective correlation coefficient. In this case, b effectively corresponds to the correlation between the first and second variate. In addition, $c = \sqrt{1 - 0.4^2} = 0.9165$ as the length of each row in the factor loading matrix has to be 1.

(c) Simlarly, $d = 0.2$. For e and f, they can be obtained from the following two conditions:

$$\begin{aligned} b \cdot d + c \cdot e &= 0.35 \\ d^2 + e^2 + f^2 &= 1 \end{aligned}$$

This gives $e = 0.2946$ and $f = 0.9345$.

6. (a) This is given by $A = E \cdot \sqrt{\Lambda}$, where E is the matrix with its columns being the eigenvectors and Λ is a diagonal matrix with its diagonal entries being the corresponding eigenvalues:

$$A \ = \ \begin{pmatrix} 0.9720 & 0.2061 & 0.1002 & 0.0504 & 0.0142 \\ 0.9863 & 0.1395 & -0.0323 & -0.0744 & -0.0329 \\ 0.9905 & 0.0000 & -0.1324 & 0.0000 & 0.0376 \\ 0.9863 & -0.1395 & -0.0323 & 0.0744 & -0.0329 \\ 0.9720 & -0.2061 & 0.1002 & -0.0504 & 0.0142 \end{pmatrix}$$

(b) Let's denote the rank-reduced factor loading matrix by $\hat{A}$, which can be obtained by

- removing the last two columns of A (or if you want to keep it as a 5-by-5 matrix, simply setting all entries of the last two columns to 0)

$$\begin{pmatrix} 0.9720 & 0.2061 & 0.1002 \\ 0.9863 & 0.1395 & -0.0323 \\ 0.9905 & 0.0000 & -0.1324 \\ 0.9863 & -0.1395 & -0.0323 \\ 0.9720 & -0.2061 & 0.1002 \end{pmatrix}$$

- rescale the entries in the resulted matrix in the last step - effectively dividing each row by a constant such that the diagonal entries of $\hat{A} \cdot \hat{A}^T$ are all 1 (which make it a proper correlation matrix)

$$\begin{pmatrix} 0.9733 & 0.2063 & 0.1004 \\ 0.9896 & 0.1400 & -0.0324 \\ 0.9912 & 0.0000 & -0.1325 \\ 0.9896 & -0.1400 & -0.0324 \\ 0.9733 & -0.2063 & 0.1004 \end{pmatrix}$$

(c) This is simply given by the product $\hat{A} \cdot \hat{A}^T$:

$$\begin{pmatrix} 1.0000 & 0.9888 & 0.9514 & 0.9311 & 0.9148 \\ 0.9888 & 1.0000 & 0.9852 & 0.9608 & 0.9311 \\ 0.9514 & 0.9852 & 1.0000 & 0.9852 & 0.9514 \\ 0.9311 & 0.9608 & 0.9852 & 1.0000 & 0.9888 \\ 0.9148 & 0.9311 & 0.9514 & 0.9888 & 1.0000 \end{pmatrix}$$

7. (a) The first 3 PCs are simply $e_1, \;\; e_2, \;\; e_3$.

   (b) Let $x = (0.3 \;\; 0.2 \;\; 0.15 \;\; 0.15 \;\; 0.1)$. The projected coordinates of the data point onto the first 3 PCs are respectively given by $x \cdot e_1 = 0.4021$, $x \cdot e_2 = 0.1369$, and $x \cdot e_= 0.0448$.