QF609 Risk Analysis
Lecture Notes 3

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Overview

Main topics:

- Recap
- Interest Rate Notions
- Coupon Bond Pricing
- Duration Model

Recap: Repring Model, ΔEVE, and ΔNII

Within the general framework of IRRBB, a repricing event typically refers to either the rollover of an instrument at its expiry or the nearest future interest rate fixing event for a floating rate instrument.

- In the first case (instrument rollover), the interest rate applying to the renewed instrument will be fixed only at the time of renewal and hence is subject to the risk of interest rate changes.
- In the second case (floating rate fixing event), interest rate changes impact all future interest payments whose underlying floating rates are not fixed yet.

Theoretically speaking, all realized cash flow coming in and out will also be subject to repricing risk at the time when the cash flow happens since all such cash flows will be reinvested/refinanced risk at a rate unknown to us as of now and will only be revealed at the cash flow time.

Recap: Repricing Model, ΔEVE, and ΔNII

The first-order repricing risk is on the pricing of notional amount. Repricing of interest rate payments is considered to be second-order and hence is not taken into account in the basic IRRBB framework.

Some rules of thumb:

- Repricing of a fixed rate instrument occurs typically at its maturity when the instrument get rolled over with a new and yet-to-determine fixed rate.
- Repricing of a floating rate instrument occurs typically at its nearest next interest rate fixing date.
- For forward starting instrument, one needs to consider if there is any financing/investment coming along with the forward-starting transaction.
- ΔEVE and ΔNII are two different IRRBB metrics focusing on a different perspective of IRRBB. ΔEVE looks at impacts on economic value of the balance sheet items while ΔNII looks at impacts on net interest rate earnings/expenses over a pre-defined window (typically short term, e.g. 1Y).
- The reconciliation example presented in class is a very special case where we possibly can reconcile the two metrics. However, do keep in mind that they are in gneral very different measures and may not make much sense to compare them.

Recap: Repricing Model, ΔEVE, and ΔNII

Example 1: A 100 dollar 5Y fixed rate loan paying 5% annually

Assuming the current discount curve is 5% flat. So the current EV_0 is 100. If interest rates suddenly change with a parallel shift of 200bps, there will be clearly a change in the EV of the loan as now we discount the loan cash flow with a discount rate of 7%. Can we try to understand why there is an EV change intuitively?

Example 2: A 100 dollar 2Y floating rate deposit paying 3M Libor quarterly

The first floating rate is already fixed. If interest rates suddenly change with a parallel shift of 200bps, how much do we expect the impact on the NII over the 1Y window to be? Intuitively speaking, we would expect an increasing interest expense of 2 dollar on each of the 3 remaining floating rate interest payment over the 1Y period, thus a total impact of -6 for the 1Y window.

Recap: Repricing Model, ΔEVE, and ΔNII

In the two examples above, we look at what should be the expected ΔEVE and ΔNII from first principle without referring to the definition of repricing gap and the calculation methods of ΔEVE, and ΔNII previously introduced. But we can see the expected ΔEVE and ΔNII we gave above are actually in line with those following from the repricing gap model and the calculation methods of ΔEVE, and ΔNII introduced before.

The two examples shred some light on why the repricing date and amount are defined in the way we presented. In essence, they serve to make the calculation methods and the results for ΔEVE and ΔNII make sense. With some more thoughts, it should become more natural and intuitive to you (Hopefully!).

Let us introduce some common notions of interest rates that are useful for our later discussions.

Nominal Interest Rate

A nominal interest rate is defined by a rate value as well as a specified annual compounding frequency m. Denote the rate by $y^{(m)}$. The discounting factor for a maturity t (in years) is given by:

$$DF(t) = \left(1 + \frac{y^{(m)}}{m}\right)^{-m \cdot t}.$$

Effective Interest Rate

This is simply the case with $m = 1$, which refers to as the effective annual interest rate.

Interest Rate Notions

If we are given a nominal rate $y^{(m)}$, how can we convert it to an equivalent nominal rate with a different compounding frequency, say n?

For this, we can obtain $y^{(n)}$ by enforcing an condition that both rates should result in the same discount factor for all t, i.e.:

$$DF(t) = \left(1 + \frac{y^{(m)}}{m}\right)^{-m \cdot t} = \left(1 + \frac{y^{(n)}}{n}\right)^{-n \cdot t}.$$

This gives:

$$y^{(n)} = \left[\left(1 + \frac{y^{(m)}}{m}\right)^{\frac{m}{n}} - 1\right] \cdot n$$

One can show that $y^{(m)} < y^{(n)}$ if $m > n$.

Zero Rate

This is just the limiting case with continuous compounding. Denoting the rate by r, we have:

$$DF(t) \;=\; \exp\left(-r \cdot t\right)$$

It is easy to show (exercise?) that:

$$r = \lim_{m \to \infty} y^{(m)}.$$

So far, we assume no term structure for interest rates - so the same rate is applied to all maturities. In practice, interest rates have a term structure, e.g. if you look at the zero rate curve for USD SOFR or SGD SORA discounting, it is far from being flat. Another way of saying this is that we need to index the rate by maturity, e.g. r(t), and $r(t)$ is not a constant function.

Coupon Bond Pricing

A coupon bond is a debt instrument that pays periodic coupons and a principal at maturity. Consider a coupon bond with:

- coupon payment frequency $= m$ times per year
- coupon rate $= C$
- notional $= \$100$
- maturity $=$ N years
- effective yield $= y$

The price of the bond is given by:

$$P = \sum_{i=1}^{N \cdot m} \frac{\frac{100 \cdot C}{m}}{(1+y)^{\frac{i}{m}}} + \frac{100}{(1+y)^N}$$

Coupon Bond Pricing

The effective bond yield can be converted to a nominal yield, e.g. with compounding frequency = m. In that case, we simply have:

$$
\begin{aligned}
y^{(m)} &= \left[(1+y)^{\frac{1}{m}} - 1\right] \cdot m, \\
P &= \sum_{i=1}^{N \cdot m} \frac{\frac{100 \cdot C}{m}}{(1+\frac{y^{(m)}}{m})^i} + \frac{100}{\left(1+\frac{y^{(m)}}{m}\right)^{N \cdot m}}
\end{aligned}
$$

Duration Model

Duration Model:

- Duration offers an alternative approach to measure interest rate risk. It is a market value based approach which takes into account the maturity and the timing of the cash flows of a financial position.
- The **Macaulay duration**, or simply duration, of an asset or liability is the weighted-average time-to-maturity of its cash flows using the present value (PV) of the cash flows as weights:

$$D = \frac{\sum_{i=1}^{n} CF_i \cdot DF(T_i) \cdot T_i}{\sum_{i=1}^{n} CF_i \cdot DF(T_i)} = \frac{\sum_{i=1}^{n} PV_i \cdot T_i}{PV}$$

  where $DF(T_i) = (1+R)^{-T_i}$ is the discount factor applied to the cash flow at time T_i and R is the annual compounding rate for discounting.

  Q: Roughly speaking, what would be a good guess of the duration for a 10Y Japanese Government bond?
- *Note that the above definition assumes that all cash flows $\{CF_i\}$ have the same sign.*

Duration Model

- Macaulay Duration is non-negative and is in the unit of years.
- It is easy to verify:

$$D = -\frac{dPV}{dR} \cdot \frac{1+R}{PV} = -\frac{\frac{dPV}{PV}}{\frac{dR}{1+R}}$$

  Hence, duration is also a measure of interest rate elasticity of the PV of a position.
- In general, duration gives a quick indication of the direction of the PV change in corresponding to a given change in interest rate. However, it is less direct in telling the actual PV change in corresponding the rate change. The **modified duration** complements the Macaulay duration in this respect.

Duration Model

- **Modified duration**

$$MD \quad \stackrel{def}{=} \quad -\frac{\frac{dPV}{dR}}{PV} = \frac{D}{1+R}$$

where D is the Macaulay duration. We have:

$$\frac{\Delta PV}{PV} \approx -MD \cdot \Delta R$$

Hence, $-MD \cdot 0.01$ gives the percentage PV change for $+100bp$ shift in the rate R.

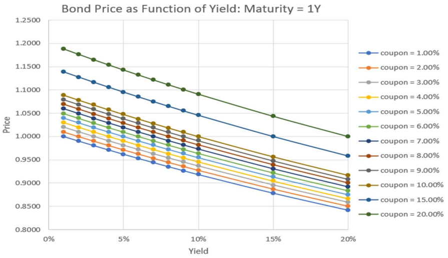
- **Dollar duration**

$$DD = -MD \cdot PV$$

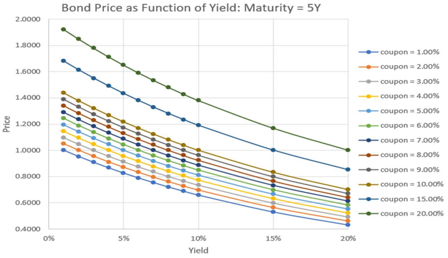
Hence, $DD \cdot 0.01$ gives the PV change for $+100bp$ shift in the rate R.

Coupon Bond Price and Duration Behaviors

* *assuming annual coupon payment and y is annual compounding yield*



Duration Model



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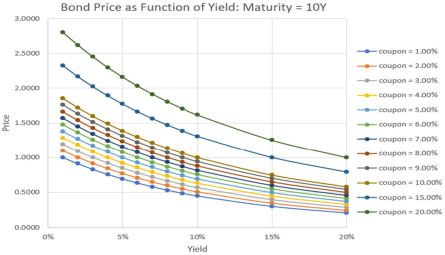

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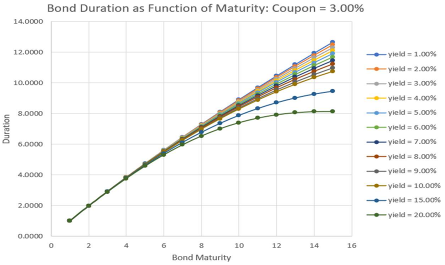

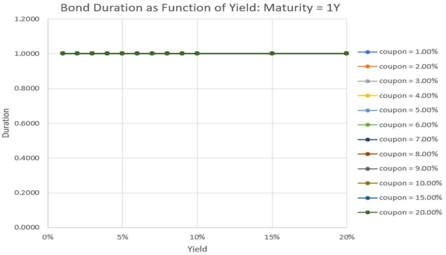

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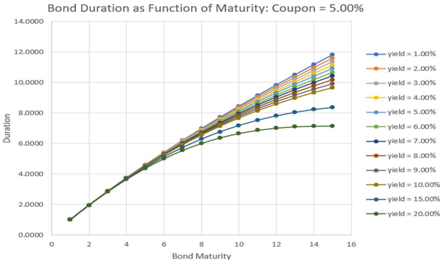

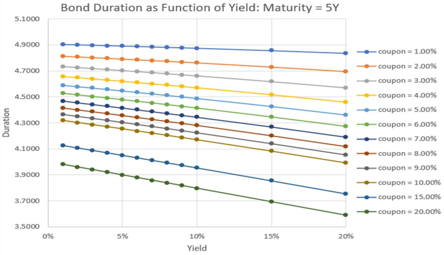

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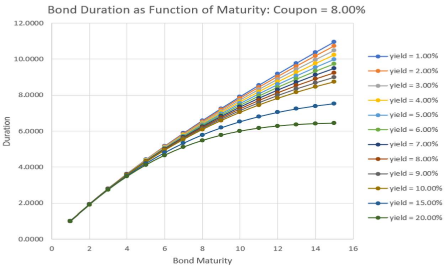

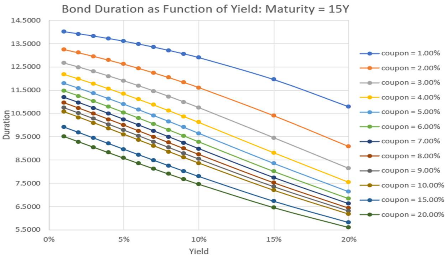

Duration Model

Properties of coupon bond price and durations:

- Price is increasing/invariant/decreasing in maturity when coupon rate is greater-than/equal-to/less-than yield, respectively.
- Price is decreasing in yield, at a decreasing rate.
- Duration is increasing in maturity, at a decreasing rate.
- Duration is decreasing in yield, but the rate of decreasing can be either increasing or decreasing.

Exercise: calculate the duration of a fixed-to-float IRS.

The cash flow of an IRS does not directly fit in the definition of duration as it contains floating rate cash flows. To consider the duration of an IRS, we note the followings:

- Adding the notional amount to both legs at maturity will not change the interest rate risk of the swap
- The float leg with a notional payment at maturity will be valued at Par (i.e. equal to the notional amount) at time 0.

Therefore, the float leg with a notional payment at maturity effectively is not exposed to rate risk. As a result, the duration of the IRS is equivalent to the duration of the fixed leg with a notional payment at maturity (or equivalently, a fixed rate coupon bond).

Duration Model

Duration can be used to translate bond yield volatility into bond price volatility. Recall:

$$\frac{dP}{P} = -\frac{D}{1+y}dy = -MD \cdot dy$$

where MD is the modified duration of the bond. Hence,

$$vol\left[\frac{\Delta P}{P}\right] \approx MD \cdot vol\left[\Delta y\right]$$

Duration Model

Asset and Liability Changes

Assume:

- the same intereste rate is applied to both assets and liabilities, say an annually compounding rate y
- The (positive) PV of assets and liabilities are denoted by A and L, respectively.

We have:

$$\begin{aligned}
\Delta A &= -D_A \cdot \frac{\Delta y}{1+y} \cdot A \\
\Delta L &= -D_L \cdot \frac{\Delta y}{1+y} \cdot L
\end{aligned}$$

where D_A and D_L are the **Macaulay duration** of the assets and liabilities, respectively.

Duration Model

The durations D_A and D_L can be calculated from individual asset/liability durations per below:

$$
\begin{aligned}
D_A &= \omega_A^1 \cdot D_A^1 + \cdots + \omega_A^m \cdot D_A^m, \quad \sum_{i=1}^{m} \omega_A^i = 1, \\
D_L &= \omega_L^1 \cdot D_L^1 + \cdots + \omega_L^n \cdot D_L^n, \quad \sum_{i=1}^{n} \omega_L^i = 1,
\end{aligned}
$$

where ω_A^i (resp. ω_L^i) is a weight given by the ratio between the individual asset (resp. liability) PV and the total asset (resp. liability) PV.

Immunizing Equity Change on Balance Sheet

- Recall the equity of a balance sheet is given by $E = A - L$. Hence $\Delta E = \Delta A - \Delta L$
- Given a shock Δy to the annually compounding rate y applied to assets and liabilities, we have:

$$\Delta E = -[D_A \cdot A - D_L \cdot L] \cdot \frac{\Delta y}{1+y} = -[D_A - D_L \cdot K] \cdot A \cdot \frac{\Delta y}{1+y}$$

  where $K = \frac{L}{A}$ is the leverage ratio of the bank or FI and $D_A - D_L \cdot K$ is called the **leverage-adjusted duration gap (D-Gap)** .
- So, to immunize equity value against interest rate change, the FI can adjust either D_A, D_L, A, or L (or combination of them) to achieve $D_A - D_L \cdot K = 0$.

Duration Model

Example:

Assume $A = 100$, $L = 50 + 40$, and $D_A = 5 \;\; years$. The liabilities consist of two items:

- Item 1: PV=50, duration =3.8 years.
- Item 2: PV=40, duration =2 years.

Then:

- What is D_L and the leverage-adjusted D-Gap?
- What will ΔE be if an interest rate rise of $+1\%$ is to be expected (current rate is 10%)?
- How to immunize the equity value change?

Duration Model

Regulatory Concerns

- Regulators set target ratios for an FI's capital (net worth) ratio, i.e. $\frac{E}{A}$
- Regulators require FIs to maintain capital ratio over the time, i.e. $\Delta\frac{E}{A} = 0$. This translates into requiring

$$D_A = D_L.$$

The proof of the above is left as an exercise.

Duration Model

Assume that the current values of the assets (A) and liabilities (L) on a bank's balance sheet are $A = 100 + 30$ and $L = 140$, respectively. The Macaulay duration of the liabilities is equal to 6 years. The assets consist of two items:

- Asset 1: PV = 100, Macaulay duration = 5 years
- Asset 2: PV = 30, Macaulay duration = 4.5 years

a. Calculate the combined Macaulay duration of the two assets.
b. Calculate the leverage-adjusted duration gap (D-Gap) of the balance sheet.
c. Consider the 10-year bond in question 1 with a 5% coupon rate and a 6% yield. What notional amount of the bond should the bank purchase in order to immunize its equity value change?

Answer:

a. $Combined \;\; Duration = \frac{100}{130} \cdot 5 + \frac{30}{130} \cdot 4.5 = 4.8846.$
b. $Leverage - adjusted \;\; D - Gap = 4.8846 - \frac{140}{130} \cdot 6 = -1.5769.$
c. The bond notional amount to be purchased should be such that the resulted leveraged adjusted D-Gap should be zero. Let this notional amount be X. We have

$$\frac{100 \cdot 5 + 30 \cdot 4.5 + \frac{X}{,1000,000} \cdot 931,839 \cdot 7.9034}{100 + 30 + \frac{X}{,1000,000} \cdot 931,839} = \frac{140}{100 + 30 + \frac{X}{,1000,000} \cdot 931,839} \cdot 6$$

which gives $X = 27.8355.$

Duration Model

Limitations of Duration-Based Immunization

- Immunizing the entire balance sheet can be time consuming and costly
  - derivative instruments can be used to hedge positions instead of rebalancing.
- Immunization is a dynamic process which involves high transaction fees
  - in practice, only approximately dynamically immunized at discrete intervals
- Large interest rate change effects are not accurately captured
  - higher order sensitivity can be included to improve accuracy
- It cannot be directly applied to transcations with cash flows and/or cash flows timing depending on interest rates.
- The basic version of duration analysis assumes a flat yield curve and parallel rate shift.

Duration Model

Convexity

Duration gives a reasonable estimate of price sensitivity of a position for small rate changes. For large changes, to achieve a better accuracy, one needs to include convexity:

- Duration is used for capturing first order effect in value change
- Convexity is used for capturing second order effect in value change

Let $P(y)$ denote the PV function of some financial position where y is the annually compounding rate used for discounting the cash flows of the position. Applying Taylor's expansion:

$$\frac{\Delta P}{P} = \frac{P(y+\Delta y)-P(y)}{P(y)} \approx \frac{1}{P}\cdot\frac{dP}{dy}\cdot\Delta y + \frac{1}{2}\cdot\frac{1}{P}\cdot\frac{d^2P}{dy^2}\cdot(\Delta y)^2$$

where the approximation keeps up to second order terms only and ignores the higher order terms. The term $\frac{1}{P}\cdot\frac{d^2P}{dy^2}$ is generally referred to as price convexity.

Duration Model

In terms of duration, we have:

$$\begin{aligned}\frac{\Delta P}{P} &= -MD \cdot \Delta y + \frac{1}{2} \cdot Convexity \cdot (\Delta y)^2 \\ Convexity &= \frac{1}{P} \cdot \frac{d^2 P}{dy^2} = -\frac{dMD}{dy} + MD^2\end{aligned}$$

where MD is the modified duration and the last equality follows from the modified duration formula developed earlier:

$$MD = -\frac{1}{P} \cdot \frac{dP}{dy}$$

The duration and convexity formulas for a coupon bond can be derived easily and will be left as an exercise.

Example:

Consider a 10-year coupon bond with notional = \$1,000,000 and semi-annual coupon payments at an annualized coupon rate of 5%. The current annually compounding yield of the coupon bond is 6%.

a. Calculate the PV of the bond.
b. Calculate the Macaulay duration, modified duration, and convexity of the bond.
c. Repeat (a)-(b) if the yield is 3% and 8%, respectively.
d. Using the modified duration and convexity obtained in (b), enstimate the PV impacts when the yield changes from 6% to 3% and 8%, respectively (compare them to the exact impacts calculated using (c)).

Answer:

For (a) and (b):

$$PV = \sum_{i=1}^{20} \frac{25,000}{(1+6\%)^{\frac{i}{2}}} + \frac{1,000,000}{(1+6\%)^{10}} = 931,839$$

$$Macaulay \quad Duration = \frac{\sum_{i=1}^{20} \frac{25,000 \cdot \frac{i}{2}}{(1+6\%)^{\frac{i}{2}}} + \frac{1,000,000 \cdot 10}{(1+6\%)^{10}}}{931,839} = 7.9034,$$

$$Modified \quad Duration = \frac{Macaulay \quad Duration}{1+y} = \frac{7.9034}{1+6\%} = 7.4560$$

$$Convexity = \frac{1}{PV} \cdot \frac{d^2 PV}{dy^2} = \frac{1}{931,839} \cdot \left[\sum_{i=1}^{20} \frac{(\frac{i}{2}+1) \cdot \frac{i}{2} \cdot 25,000}{(1+6\%)^{\frac{i}{2}+2}} + \frac{1,000,000 \cdot (10+1) \cdot 10}{(1+6\%)^{10+2}} \right] = 71.39$$

Similary, for (c):

Yield	6%	3%	8%
PV	931,839	1,173,779	805,279
Macaulay Duration	7.9034	8.1714	7.7131
Modified Duration	7.4560	7.9334	7.1417
Convexity	71.39	78.95	66.63

Duration Model

Finally, for (d):

$$\Delta PV \quad \approx \quad PV \cdot \left[-MD \cdot \Delta y + \frac{1}{2} \cdot convexity \cdot (\Delta y)^2\right]$$

Hence, the estimated impacts are:

$$\begin{aligned} PV(yield = 3\%) - PV(yield = 6\%) \quad &\approx \quad 931,839 \cdot [-7.4560 \cdot (-0.03) + \frac{1}{2} \cdot 71.39 \cdot (-0.03)^2] \;=\; 238,370 \\ PV(yield = 8\%) - PV(yield = 6\%) \quad &\approx \quad -125,651 \end{aligned}$$

where the exact impacts can be calcualted from results in (a)-(c) which are 241,940 and -126,560, respectively.