QF609 Risk Analysis
Lecture Notes 5

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Overview

Main topics:

- Recap
- Basic Statistics: Correlation Matrix Decomposition
- MC Simulation: Demo
- Value-at-Risk (VaR) Model

Recap

In the last lecture, we reviewed some basic statistics knowlege:

- random variables: discrete/continuous, PMF/PDF, CDF etc.
- covariance and correlation matrix
- LLN and CLT
- MC cimulation

Important Results:

- Let X be a continuous RV whereby its CDF $F(x)$ is strictly increasing. Then,
  - The RV $Y = F(X)$ has a Uniform(0,1) distribution
  - The RV $Z = F^{-1}(U)$ has the same distribution as X, where U is a Uniform(0,1) RV.

  This result is important as it implies that one can sample from an arbitrary distribution $F(x)$ by first sampling from the Uniform(0,1) distribution and then mapping the samples by applying $F^{-1}(u)$.

Recap

- MC simulation is often used to estimate the expected value of some function of RVs, $f(X_1, \cdots, X_n)$, where the distribution of f is either not explicitly known or not tractable analytically. Nevertheless, provided that one can simulate samples of $X_1, \cdots, X_n$, one can obtain a MC estimate of the expected value:

$$E\left[f(X_1, \cdots, X_n)\right] = \lim_{m \to \infty} \frac{\sum_{i=1}^{m} f(X_1^i, \cdots, X_n^i)}{m}$$

  where for each i, $(X_1^i, \cdots, X_n^i)$ denotes an independent sample of the RVs.

Recap

- In most applications, the problem reduces to simulating (potentially correlated) standard normal RVs. To simulate n standard normal RVs, $X_1, \cdots, X_n$, with a correlation matrix A:
  - Identify a decomposition Γ such that $A = \Gamma \cdot \Gamma^T$ where Γ is referred to as a factor loading matrix
  - Simulate a sample of n independent standard normal RVs $Z_1, \cdots, Z_n$ and denote it by $z^i = (z_1^i, \cdots, z_n^i)^T$. Then, $x^i = A \cdot z^i$ is a sample of the correlated standard normal RVs.

Given a valid correlation matrix A, there are various way to identify a decomposition $A = \Gamma \cdot \Gamma^T$. Such decomposition is particularly useful in simulation of correlated normal RVs. Here, we introduced two such decompositions which are widely used in practice. There should be modules in Python that allow you to calculate these decompositions.

Cholesky Decomposition

With this decomposition, the factor loading matrix Γ is a lower triangular matrix with non-negative diagonal entries. There are various agorithms to compute the Cholesky decomposition (you can refer to Wiki or linear algebra texts for further interests). A simple example of the Cholesky Decomposition:

$$A = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \quad \rightarrow \quad \Gamma = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix}$$

Exercise: can you try to derive the Cholesky Decomposition for the following correlation matrix:

$$A = \begin{pmatrix} 1 & \rho_{1,2} & \rho_{1,3} \\ \rho_{1,2} & 1 & \rho_{2,3} \\ \rho_{1,3} & \rho_{2,3} & 1 \end{pmatrix}$$

Eigen Decomposition

A valid correlation matrix A (n-by-n) admits the following decomposition:

$$A \;=\; \Gamma \cdot \Gamma^T, \quad \Gamma \;=\; E \cdot \sqrt{\Lambda}$$

where E is an $n \times n$ matrix with its columns being the eigen-vectors of the A and Λ is a diagonal matrix with its diagonal entries being the eigenvalues of A, i.e.

$$E \;=\; (e_1 \;\; e_2 \;\; \cdots \;\; e_n), \quad e_i \;=\; \begin{pmatrix} e_{1,i} \\ e_{2,i} \\ \vdots \\ e_{n,i} \end{pmatrix}, \quad \Lambda \;=\; Diag(\lambda_i)_{i=1,\cdots,n},$$

where λ_i is the eigenvalue associated with the eigenvector e_i.

MC Simulation: Demo

Here, we demonstrate a few simulation examples starting froma samples of Uniform[0,1]:

a. dice roll
b. exponential RV
c. normal RV
d. correlated standard normal random variable
e. call option price under the Black-Scholes (BS) model

The details are provided on the Excel worksheet posted on the course website.

Value-at-Risk (VaR) Model

Value-at-risk (VaR) is a statistical measure of the riskiness of portfolios of assets, which gives the maximum dollar amount expected to be lost over a given time horizon, at a pre-defined confidence level. VaR has 3 basic parametrers:

- *Significance level.* This is is often set by an external body, such as a regulator. Under the Basel II Accord, banks using internal VaR models to assess their market risk capital requirement need to measure VaR at the 1% significance level.
- *Risk horizon.* The risk horizon is the period over which one measures the potential loss, which is typically measured in trading days rather than calendar days. In practice, the risk horizon is usually 1-day, 10-day, or 21-day (i.e. a business-month). Under the Basel banking regulations, the risk horizon is 10-day.
- *Reporting currency.* This is the currency in which the VaR should be reported in.

A formal definition of VaR:

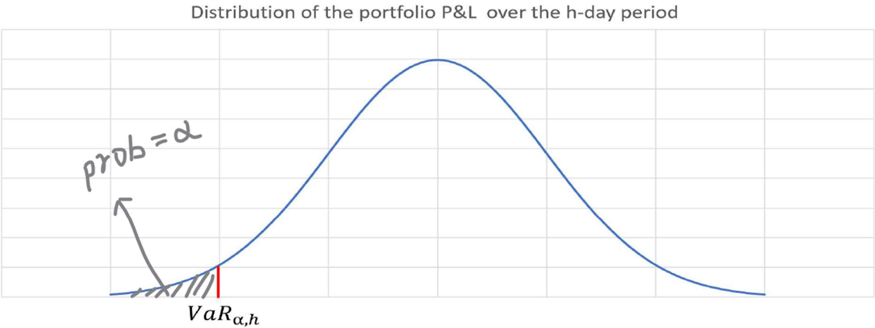
- Assume a portfolio of positions which can contain different types of assets (e.g. interest rates, FX, equity, commodity etc.).
- The current mark-to-market (MTM) value of the portfolio is P_0.
- Denote the future mark-to-market value of the portfolio in h business days by P_h (*assuming the portfolio positions remain unchanged over the period*).
- Denote the P&L (in a selected reporting currency) of the portfolio over the *h-day* horizon by a random variable $L = P_h - P_0$

Then, the *h-day* **VaR** at a signficance level of $\alpha\%$ for the portfolio is the number $VaR_{\alpha,h}$ such that

$$Probability\,(L \leq VaR_{\alpha,h}) \;=\; \alpha$$

Effectively, this means that $VaR_{\alpha,h}$ is the $\alpha\%$ percentile of the distribution of L. Following this definition, the key to calculate VaR is therefore to estimate the distribution of L.

Value-at-Risk (VaR) Model



Example: A portfolio consists of \$1 million in each of three stocks (stock A, B and C). The daily **relative** returns (i.e. $\frac{S(t+1day)}{S(t)} - 1$) of the stocks are normally distributed with the following means and covariance matrix:

$$\mu = \begin{pmatrix} 0.0356\% \\ 0.0267\% \\ 0.0133\% \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 0.00007 & 0.0001 & -0.000045 \\ 0.0001 & 0.0004 & -0.00008 \\ -0.000045 & -0.00008 & 0.000178 \end{pmatrix}$$

What will be the one-day VaR for the portfolio at a significant level of 99%?

Value-at-Risk (VaR) Model

Practical implementation of VaR calculation generally involves the following steps:

a. *define the VaR parameters: significance level, risk horizon, and reporting currency*
b. *identify the portfolio*
c. *identify the risk factors of the portfolio*
d. *model the joint distribution of the risk factors over the risk horizon*
e. *build the distribution of the portfolio P&L over the risk horizon*
f. *calculate the VaR*

In what follows, we shall discuss Step (b) - (f) in details.

Identifying the portfolio

This is to identify all transactions in the target portfolio which can include trades from various asset classes and denominated in different currencies. This may sound trivail at first. However, in a large bank, the number of trades in a portfolio can be in the order of a few 100k's, and there may exist different booking systems for different trade types. Thus, a good infrastructure is needed for trade query and aggregation.

Value-at-Risk (VaR) Model

Identifying the risk factors of the portfolio

For each trade in a VaR portfolio, there is generally a dedicated pricing and risk engine for calculating the official end-of-day (EOD) PV and risks of the trade on each business day. The market inputs to the pricing and risk engines change day-to-day and are effectively the underlying risk factors that drive the day-to-day PNL and risk changes of the trades. Such risk factors can include:

- a specific equity index price
- an FX spot rate
- a commodity index price
- an interest rate curve
- an implied volatility surface
- ...

These risk factors need to be identified for the portfolio for VaR calculation.

Examples of Risk Factors

- **USD 3M Libor swap.** In this case, the risk factors are inputs to construct the following curves:
  - The SOFR curve (for discounting)
  - The USD Libor 3M curve (used for forward rate projection)
- **SORA-SOFR cross currency swap** In this case, the risk factors are inputs to construct the following curves::
  - The SOFR curve (for discounting and SOFR forward rate projection)
  - The SORA curve (for SORA forward rate projection)
  - The SGD FX discount curve (for discounting SOFR-funded SGD cashflow)
- **USDJPY FX Option.** One way to price this is via the Black-Scholes formula which takes as inputs: 1) an FX forward, (2) an implied volatility (IV) from the corresponding USDJPY IV surface, and (3) a discount curve. In this case, the risk factors are:
  - Inputs to build the JPY FX disocunt curve, the USD SOFR curve, and the JPY TONA curve
  - Spot FX (USDJPY)
  - Inputs to construct the USDJPY IV surface

Value-at-Risk (VaR) Model

In the USD LIBOR 3M swap example above, the market instruments for building the 3M Libor curve can include some futures which are based on absolute dates (e.g. H20 means 3rd Weds of Mar-2020) rather than a relative tenor such as 1M, 2M etc. For absolute dated instruments, it may be difficult to analyze historical distribution of its changes directly (as they were not traded in the past), which we shall see is needed for VaR calculation. For this reason and also efficiency, a VaR engine typically standardizes interest rate curve risk factors to a selected set of relative tenor instruments.

For example, one may define the VaR risk factors for an interest rate curve to the zero rates for the standardized tenors 1D, 1M, 2M, 3M, 6M, 9M, 1Y, 2Y, 3Y, 4Y, 5Y, 10Y, 20Y, 30Y, 40Y, 50Y.

Value-at-Risk (VaR) Model

Modeling the joint distribution of the risk factor changes

Once the VaR risk factors are identified, the next step is to build a joint distribution of the changes of these factors over the risk horizon. This is effectively to determine how scenarios of risk factor changes are generated in order to derive a distribution for portfolio PNL from which the VaR can be computed.

This is aslo where different types of VaR models come in:

- **Parametric VaR**
- **Monte Carlo VaR**
- **Historical VaR**

Each of the 3 models applies a different approach to generate the joint distribution of the risk factor changes. We will look at the 3 different models in details later.

Value-at-Risk (VaR) Model

Choices of modelling the risk factor changes

For a given risk factor, one can specify whether to model its absolute or relative change. Let x_0 be the current value of a particular risk factor and x_h its value in h-day. There are different ways to model the change of the risk factor over the h-day horizon:

- **absolute return:** $\Delta x_{0,h} =_{def} x_h - x_0$
- **simple return:** $\Delta x_{0,h} =_{def} \frac{x_h - x_0}{x_0}$
- **log return:** $\Delta x_{0,h} =_{def} \ln \frac{x_h}{x_0}$

The simple return and log return are more appropriate for positive risk factors (e.g. equity prices, FX spot etc..). The advantage of log return is that it is additive:

$$\Delta x_{0,h} \quad = \quad \ln \frac{x_h}{x_0} = \ln \frac{x_1}{x_0} + \ln \frac{x_2}{x_1} + \cdots + \ln \frac{x_h}{x_{h-1}} = \sum_{i=1}^{h} \Delta x_{i-1,i}$$

Neverthess, if the risk horizon is short and small changes, log return and simple return should be reasonably close (Why?). In recent years, negative rates become normal for some currency. In such cases, absolute changes

Value-at-Risk (VaR) Model

building the distribution of the portfolio P&L over the risk horizon

Given the joint distribution of the risk factor changes over the risk horizon, one can determine the distribution of the portfolio PNL over the risk horizon. In practical implemention, this typically is done as follows:

a. calculate the base scenario portfolio value (P_0) which is simply the MTM value of the portfolio on the base date
b. sample from the joint distribution of the risk factor changes
c. calculate the portfolio value P_i under each scenario i obtained from the previous step, and the PNL under that scenario is given by $PNL_i = P_i - P_0$.
d. $\{PNL_i\}$ effectively provides a sample distribution of the portfolio PNL

In Step (c) above, the calculation of portfolio PNL under a scenario, there exist mainly two different approaches:

- Full-Revaluation
- Risk-Based

Caclulating the VaR

Once the PNL distribution is obtained, the VaR of the portfolio can be calculated accordingly following its definition, i.e. as the α-percentile of the PNL distribution where is α is the specified significance level of the VaR.

Note that the risk factor changes may be based on daily changes. In that case, the PNL distribution obtained in the previous step may refer to the distribution of **daily** PNL. If we were to calculate VaR over a 10-day horizon, one typically method is to first calculate the 1-dday VaR and then scale it to a 10-day VaR. We will see how this can be done later.