Factor Loading Matrix Rescaling in Rank Reduction

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Suppose the full rank factor loading matrix from eigen-decomposition $(E \cdot \sqrt{\Lambda})$ is:

$$H = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix}$$

Clearly, the diagonal entries of $H \cdot H^T$ should be all equal to 1. Next, suppose that we want to perform a rank reduction with a target rank equal to 3.

- The first step is to keeping only the first 3 columns of H which gives

$$H_1 = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ \vdots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} \end{pmatrix}$$

- Due to the truncation of the matrix, the diagonal entries of $H_1 \cdot H_1^T$ will not be equal to 1 which means H_1 is not yet a proper factor loading matrix. We need to rescale its entries to obtain a rescaled factor loading matrix H_2 so that the diagonal entries of $H_2 \cdot H_2^T$ will be equal to 1. How do we do this? First, we observe that the diagonal entries of $H_1 \cdot H_1^T$ are:

$$\begin{aligned} d_{1,1} &= a_{1,1}^2 + a_{1,2}^2 + a_{1,3}^2, \\ d_{2,2} &= a_{2,1}^2 + a_{2,2}^2 + a_{2,3}^2, \\ \vdots & \;\vdots\; \vdots \\ d_{n,n} &= a_{n,1}^2 + a_{n,2}^2 + a_{n,3}^2, \end{aligned}$$

so what we need to do is simply to set the rescaled matrix as follows:

$$H_2 = \begin{pmatrix} \frac{a_{1,1}}{\sqrt{d_{1,1}}} & \frac{a_{1,2}}{\sqrt{d_{1,1}}} & \frac{a_{1,3}}{\sqrt{d_{1,1}}} \\ \frac{a_{2,1}}{\sqrt{d_{2,2}}} & \frac{a_{2,2}}{\sqrt{d_{2,2}}} & \frac{a_{2,3}}{\sqrt{d_{2,2}}} \\ \vdots & \vdots & \vdots \\ \frac{a_{n,1}}{\sqrt{d_{n,n}}} & \frac{a_{n,2}}{\sqrt{d_{n,n}}} & \frac{a_{n,3}}{\sqrt{d_{n,n}}} \end{pmatrix}$$

One can verify that in this way the diagonal entries of $H_2 \cdot H_2^T$ will be be equal to 1. This is effectively what the rescaling matrix does as represented in class.