HW_difgame_28022022_XuFeiran

Var3:

Pollution control problem

$$K_i(u,T) = \int_0^T (c_i u_i - k_i x)dt - D_i x(T);; \qquad \sum_{i=1}^{3} K_i(u,T) \to \max_u; \qquad (5)$$

$$x' = u_1^2 + u_2^2 + u_3^2, \qquad x(0) = 0. \qquad (6)$$

Solution :

Write the Hamiltonian function :

$H(x,\varphi,u) = c_1u_1 + c_2u_2 + c_3u_3 - kx + \varphi(u_1^2 + u_2^2 + u_3^2)$, where $k = k_1 + k_2 + k_3$

Taking the first derivative with respect to u_i

$\frac{\partial H}{\partial u_i} = c_i + 2\varphi u_i$ (1)

Taking the second derivative with respect to u_i

$\frac{\partial^2 H}{\partial u_i^2} = 2\varphi < 0$ (2)

Find the optimal control with condition (1) = 0

$u_i^* = -\frac{c_i}{2\varphi}, i = 1,2,3$ (3)

Rewrite the control system :

$$\begin{cases} \dot{x} = u_1^2 + u_2^2 + u_3^2 \\ \dot{\varphi} = k \end{cases}$$

$\begin{cases} \dot{x} = \frac{\overline{C}}{4\varphi^2} \\ \dot{\varphi} = k \end{cases}$ Where $\overline{C} = c_1^2 + c_2^2 + c_3^2$

According to

$\varphi(T) = -D, D = D_1 + D_2 + D_3$

$\varphi(t) = \varphi(0) + kt,$

We can get :

$\varphi(0) = -D - kT$

$\varphi(t) = kt - \overline{D}$, and $\overline{D} = D + kT$

So go back to our control system :

$\dot{x} = \frac{\overline{C}}{4(kt-\overline{D})^2}$

$x = \frac{\overline{C}}{4} \int \frac{1}{(kt-\overline{D})^2} dt$

$\int \frac{1}{(kt-\overline{D})^2} dt = \int \frac{1}{k} \frac{1}{(kt-\overline{D})^2} d(kt - \overline{D}) = \frac{-1}{k(kt-\overline{D})} + C^*$

We have condition $x(0) = 0$

$C^* = \frac{-1}{k\overline{\overline{D}}}$

So we get the optimal trajectory

$x^*(t) = \frac{c_1^2+c_2^2+c_3^2}{4}(\frac{1}{k(D+k(T-t))} - \frac{1}{k(D+kT)})$

The **optimal control :**

because our $u_i \in [0, A]$

$\frac{-c_i}{2(kt-\overline{D})} = A$

We can get $t* = \frac{-c_i}{2Ak} + T + \frac{D}{k}$

In this situation:

$$u_i^* = \begin{cases} A, & t > t^* \\ \frac{-c_i}{2(-D-k(T-t))}, & 0 \leq t \leq t^* \end{cases}$$