Variant 1.

1) Find a stationary point of the function

$$f(x) = x_1 x_2^2 x_3^3 (5 - x_1 - 2x_2 - 3x_3).$$

2) Solve the constrained optimization problem

$$f(x) = x_1 x_2 x_3 \to \text{extr}_X. \quad X = \{x \mid x_1^2 + x_2^2 + x_3^2 = 1,\ x_1 + x_2 + x_3 = 0\}.$$

$5 - \cancel{2}x_1 - 2x_2 - 3x_3^2$

$5 - 2x_2 - 3x_3^2 = 0.$

$x_3 =$ $\quad x_3 = \frac{2 \pm \sqrt{4x_2 + 60}}{-6}$

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1): stationary point:

$$\frac{\partial f(x)}{\partial x_1} = 5x_2^2 x_3^3 - 2x_1 x_2^2 x_3^3 - 2x_2^3 x_3^3 - 3x_2^2 x_3^4$$

$$\frac{\partial f(x)}{\partial x_2} = 10 x_1 x_2 x_3^3 - 2x_1^2 x_2 x_3^3 - 6x_1 x_2^2 x_3^3 - 6x_1 x_2 x_3^4$$

$$\frac{\partial f(x)}{\partial x_3} = 15 x_1 x_2^2 x_3^2 - 3x_1^2 x_2^2 x_3^2 - 6x_1 x_2^3 x_3^2 - 12 x_1 x_2^2 x_3^3$$

when $\left.\frac{\partial f(x)}{\partial x}\right|_{x^*} = 0$, x^* is a st. point.

a): $x_2 = 0$, $\forall\ x^* = (m, 0, n)$ is a st. point

b): $x_3 = 0$, $\forall\ x^* = (a, b, 0)$ is a st. point

d): $x_1 = 0$, $5 - 2x_2 - 3x_3 = 0$. $x_3 = \frac{5 - 2x_2}{3}$. $(0, x_2, \frac{5-2x_2}{3})$ st. point.

c): $x_1 \cdot x_2 \cdot x_3 \neq 0$,

$$\frac{\partial f(x)}{\partial x_1} = 5 - 2x_1 - 2x_2 - 3x_3 = 0,$$

$$\frac{\partial f(x)}{\partial x_2} = 10 - 2x_1 - 6x_2 - 6x_3 = 0$$

$$\frac{\partial f(x)}{\partial x_3} = 15 - 3x_1 - 6x_2 - 12x_3 = 0.$$

$$A = \begin{bmatrix} 2 & 2 & 3 \\ 2 & 6 & 6 \\ 3 & 6 & 12 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 10 \\ 15 \end{bmatrix}$$

$$X = A^{-1} \cdot b \Rightarrow x = \begin{bmatrix} 0.714 \\ 0.714 \\ 0.714 \end{bmatrix}$$

So, the st. point x^* $\begin{cases} (m, 0, n) \\ (a, b, 0) \\ (0.714, 0.714, 0.714) \end{cases}$

m, n, a, b - consts.

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(2). $f(x) = x_1 x_2 x_3$, $X = \{x \mid x_1^2 + x_2^2 + x_3^2 = 1,$
$x_1 + x_2 + x_3 = 0\}$.

$$L(x, \lambda) = x_1 x_2 x_3 + \lambda_1 (x_1^2 + x_2^2 + x_3^2 - 1) + \lambda_2 (x_1 + x_2 + x_3)$$

$$\begin{cases} \frac{\partial L}{\partial x_1} = x_2 x_3 + 2x_1 \lambda_1 + \lambda_2 \\ \frac{\partial L}{\partial x_2} = x_1 x_3 + 2x_2 \lambda_1 + \lambda_2 \end{cases}$$

$(\triangle)$

$$\left\{\begin{array}{l} \dfrac{\partial L}{\partial x_3} = x_1 x_2 + 2x_3\lambda_1 + \lambda_2 \\[2ex] \dfrac{\partial L}{\partial \lambda_1} = x_1^2 + x_2^2 + x_3^2 - 1 \\[2ex] \dfrac{\partial L}{\partial \lambda_2} = x_1 + x_2 + x_3 \end{array}\right.$$

When $(\Delta) = 0$.

$$\left\{\begin{array}{ll} x_2 x_3 + 2x_1\lambda_1 + \lambda_2 = 0 & ① \\ x_1 x_3 + 2x_2\lambda_1 + \lambda_2 = 0 & ② \\ x_1 x_2 + 2x_3\lambda_1 + \lambda_2 = 0 & ③ \\ x_1^2 + x_2^2 + x_3^2 - 1 = 0 & ④ \\ x_1 + x_2 + x_3 = 0 & ⑤ \end{array}\right.$$

$x_2x_3 + x_1x_3 + x_1x_2 + 3\lambda_2 = 0.$

$x_1x_2 + x_1x_3 + x_2x_3 = -\frac{1}{2}$

① - ② $\Rightarrow$ $(x_2 - x_1)x_3 + 2\lambda_1(x_1 - x_2) = 0.$

$(x_1 - x_2)(x_1 + x_2 + 2\lambda_1) = 0$

② - ③ $\Rightarrow$ $(x_3 - x_2)x_1 + 2\lambda_1(x_2 - x_3) = 0.$

$(x_2 - x_3)(x_2 + x_3 + 2\lambda_1) = 0.$

By solving this system of equations, we can find 6. stationing points.

a): $\lambda_1 = -\frac{1}{2\sqrt{6}}$, $\lambda_2 = \frac{1}{6}$, $\left(\sqrt{\frac{2}{3}}, -\frac{\sqrt{6}}{6}, -\frac{\sqrt{6}}{6}\right)$

$\left(-\frac{\sqrt{6}}{6}, \sqrt{\frac{2}{3}}, -\frac{\sqrt{6}}{6}\right)$

$\left(-\frac{\sqrt{6}}{6}, \frac{-\sqrt{6}}{6}, \sqrt{\frac{2}{3}}\right)$

b). $\lambda_1 = \frac{1}{2\sqrt{6}}$, $\lambda_2 = \frac{1}{6}$ $\left(\frac{-\sqrt{2}}{3}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

$\left(\frac{1}{\sqrt{6}}. \frac{1}{\sqrt{6}}, -\frac{\sqrt{2}}{\sqrt{3}}\right)$

$\left(\frac{1}{\sqrt{6}} \quad -\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{6}}\right)$

in case a). $d^2L(x,\lambda)\Big|_{x^*} = \frac{\partial^2 L}{\partial x_1^2} + 2\frac{\partial^2 L}{\partial x_1 \partial x_2} + 2\frac{\partial^2 L}{\partial x_1 x_3} + 2\frac{\partial^2 L}{\partial x_2 x_3} + \frac{\partial^2 L}{\partial x_2^2} + \frac{\partial^2 L}{\partial x_3^2} = 6\lambda_1 < 0.$

$f(x)$ has local max. $f(x) = \frac{\sqrt{6}}{18}$

in case b). $d^2L(x,\lambda) = 6\lambda_1 > 0.$

$f(x)$ has local min $f(x) = \frac{-\sqrt{6}}{18}$.

Variant 2.

1) Find a stationary point of the function

$$f(x) = x_1^2 + x_2^2 + x_3^2 + 2x_1 + 4x_2 - 6x_3.$$

2) Solve the constrained optimization problem

$$f(x) = x_1 + x_2 + x_3 \to \text{extr}_X. \quad X = \{x \mid x_1^2 + x_2^2 = 1,\ x_1 + x_2 - x_3 = 0\}.$$

1). $\frac{\partial f(x)}{\partial x_1} = 2x_1 + 2 = 0$

$\frac{\partial f(x)}{\partial x_2} = 2x_2 + 4 = 0 \Rightarrow$ $x_1 = -1$, $x_2 = -2$, $x_3 = 3$

$\frac{\partial f(x)}{\partial x_3} = 2x_3 - 6 = 0$

$x^* = (-1, -2, 3)$

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(2). $L(x, \lambda) = x_1 + x_2 + x_3 + \lambda_1(x_1^2 + x_2^2 - 1) + \lambda_2(x_1 + x_2 - x_3)$

$\frac{\partial L}{\partial x_1} = 1 + 2\lambda_1 x_1 + \lambda_2 = 0$ $\qquad x_1 = -\frac{1}{\lambda_1}$

$\frac{\partial L}{\partial x_2} = 1 + 2\lambda_1 x_2 + \lambda_2 = 0$ $\qquad x_2 = -\frac{1}{\lambda_1}$

$\frac{\partial L}{\partial x_3} = 1 - \lambda_2 = 0 \Rightarrow \lambda_2 = 1$

$\frac{\partial L}{\partial \lambda_1} = x_1^2 + x_2^2 - 1 = 0$

$\frac{\partial L}{\partial \lambda_2} = x_1 + x_2 - x_3 = 0$

if $\lambda_1 = 0$. $\Rightarrow \lambda_2 = -1$, $\lambda_2 = 1$,

no solutions

so. $\lambda_1 \neq 0$, $\Rightarrow$ $\lambda_1 = \pm\sqrt{2}$, $\lambda_2 = 1$

$x_1 = x_2 = \pm\frac{\sqrt{2}}{2}$. $x_3 = \pm\sqrt{2}$.

if: $x_1 = x_2 = \frac{\sqrt{2}}{2}$, $x_3 = \sqrt{2}$ $\Rightarrow$ $f(x) = \frac{3\sqrt{2}}{2}$ $\rightarrow$ max

if: $x_1 = x_2 = -\frac{\sqrt{2}}{2}$, $x_3 = -\sqrt{2}$ $\Rightarrow$ $f(x) = -\frac{3\sqrt{2}}{2}$ $\rightarrow$ min.

Variant 3.

1) Find a stationary point of the function

$$f(x) = x_1^2 + x_2^2 - x_3^2 + 4x_1x_2.$$

2) Solve the constrained optimization problem

$$f(x) = x_1^2 + x_2^2 + x_3^2 \to \text{extr}_X, \ X = \left\{x \in \mathbb{R}^3 \ \middle| \ \frac{x_1^2}{4} + \frac{x_2^2}{9} + \frac{x_3^2}{16} = 1\right\}.$$

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1). $\frac{\partial f}{\partial x_1} = 2x_1 + 4x_2 = 0$

$\frac{\partial f}{\partial x_2} = 2x_2 - 4x_1 = 0$ $\Rightarrow$ $\begin{cases} x_3 = 0 \\ x_1 = 0 \\ x_2 = 0. \end{cases}$

$\frac{\partial f}{\partial x_3} = -2x_3 = 0$.

$x^* = (0, 0, 0)$

2) $L(x.\lambda) = x_1^2 + x_2^2 + x_3^2 + \lambda\left(\frac{x_1^2}{4} + \frac{x_2^2}{9} + \frac{x_3^2}{16} - 1\right)$

$\frac{\partial L}{\partial x_1} = 2x_1 + \frac{1}{2}\lambda x_1 = 0$ $\quad x_1(2+\frac{1}{2}\lambda) = 0$

$\frac{\partial L}{\partial x_2} = 2x_2 + \frac{2}{9}\lambda x_2 = 0$ $\quad x_2(1+\frac{1}{9}\lambda) = 0$

$\frac{\partial L}{\partial x_3} = 2x_3 + \frac{1}{8}\lambda x_3 = 0$ $\quad x_3(2+\frac{1}{8}\lambda) = 0.$

$\frac{\partial L}{\partial \lambda} = \frac{x_1^2}{4} + \frac{x_2^2}{9} + \frac{x_3^2}{16} - 1 = 0$

a): $x_1 = x_2 = 0$, $\lambda = -16$, $x_3 = \pm 4$

b): $x_1 = x_3 = 0$, $\lambda = -9$, $x_2 = \pm 3$

c): $x_2 = x_3 = 0$. $\lambda = -4$, $x_1 = \pm 2$

$\frac{\partial L^2(x,\lambda)}{\partial x_1^2} = 2 + \frac{1}{2}\lambda$ $\quad \frac{\partial L^2(x,\lambda)}{\partial x_2^2} = 2 + \frac{2}{9}\lambda$

$\frac{\partial L^2(x,\lambda)}{\partial x_1 x_2} = 0$ $\quad \frac{\partial L^2(x,\lambda)}{\partial x_2 x_3} = 0.$

$\frac{\partial L^2(x,\lambda)}{\partial x_1 x_3} = 0.$ $\quad \frac{\partial L^2(x,\lambda)}{\partial x_3^2} = 2 + \frac{1}{8}\lambda$

$$d^2 L(x^*, \lambda^*) = 2\frac{\partial^2 L}{\partial x_1 \partial x_2} dx_1 dx_2 + 2\frac{\partial^2 L}{\partial x_1 \partial x_3} dx_1 dx_3$$

$$+2\frac{\partial^2 L}{\partial x_2 \partial x_3} dx_2 dx_3 + \frac{\partial^2 L}{\partial x_1^2} dx_1^2 + \frac{\partial^2 L}{\partial x_2^2} dx_2^2$$

$$+ \frac{\partial^2 L}{\partial x_3^2} dx_3^2$$

$$= 6 + \frac{61}{72}\lambda$$

if $\lambda = -4$, $d^2 L(x^*, \lambda^*) = 6 - 4 \cdot \frac{61}{72} = \frac{47}{18} > 0$.

$f(x) = 4 \rightarrow \min$

if $\lambda = -9$, $d^2 L(x^*, \lambda^*) = 6 - 9 \cdot \frac{61}{72} = -\frac{13}{8} < 0$.

$f(x) = 9 \rightarrow \max$

if $\lambda = -16$, $d^2 L(x^*, \lambda^*) = 6 - 16 \cdot \frac{61}{72} = -\frac{68}{9}$

$f(x) = 16 \rightarrow \max$.

so, $f(x)$ has two local max. $f(x) = 9$, $f(x) = 16$

has one local min. $f(x) = 4$.