Graphs and Flows in Networks

Lecture 4

St. Petersburg State University, Russia

St. Petersburg, 2022

Independent and Dominating Sets — The Set Covering Problem

Given a graph $G = (X, \Gamma)$, one is often interested in finding subsets of the set X of vertices of G which possess some predefined property.

For example,

- what is the maximum cardinality of a subset $S \subseteq X$ so that subgraph $< S >$ is totally disconnected?
- what is the maximum cardinality of a subset $S \subseteq X$ so that the subgraph $< S >$ is complete?

The answer to the first question is known as the independence number of G and the answer to the second question as the clique number.

Another problem is to find the minimum cardinality of a subset S so that every vertex of $X - S$ can be reached from a vertex of S by a single arc. The answer to this problem is known as the dominance number of G.

Maximal complete subgraphs (cliques)

A concept which is the opposite of that of the maximal independent set is that of a maximal complete subgraph.

Clique

A clique is a subset of vertices of an undirected graph G such that every two distinct vertices in the clique are adjacent.

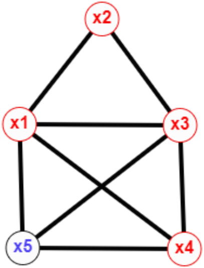
Maximal clique

A maximal clique is a clique that cannot be extended by including one more adjacent vertex, that is, a clique which does not exist exclusively within the vertex set of a larger clique.

Maximum clique

A maximum clique of a graph, G, is a clique, such that there is no clique with more vertices. Moreover, the clique number $\omega(G)$ of a graph G is the number of vertices in a maximum clique in G.

Hence, in contrast to a maximal independent set for which no two vertices are adjacent, the set of vertices of a clique are all adjacent to each other.



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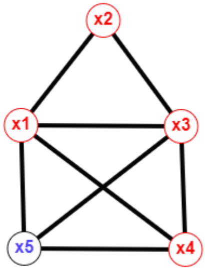

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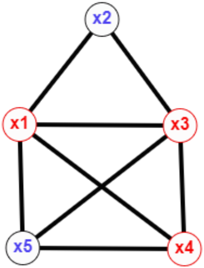

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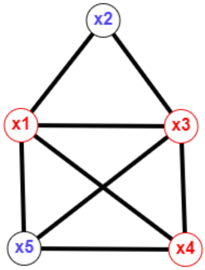

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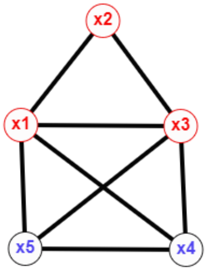

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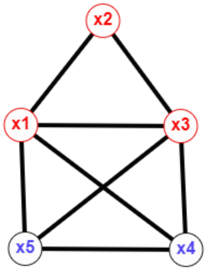

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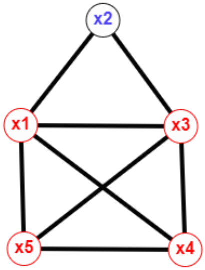

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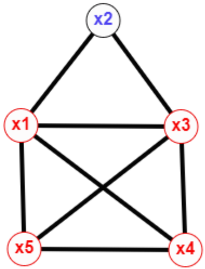
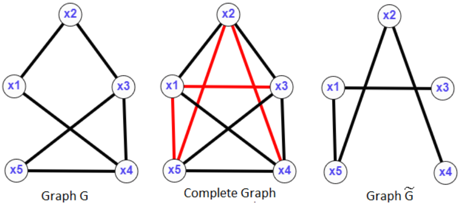
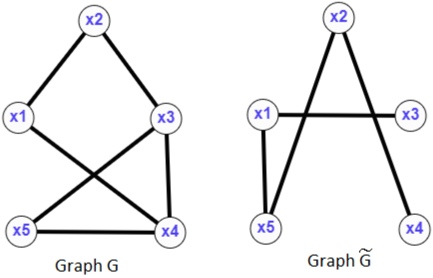


Maximum clique

The complement of a graph G

The complement or inverse of a graph G is a graph $\widetilde{G}$ on the same vertices such that two distinct vertices of $\widetilde{G}$ are adjacent if and only if they are not adjacent in G. That is, to generate the complement of a graph, one fills in all the missing edges required to form a complete graph, and removes all the edges that were previously there.





$\{x_3, x_4, x_5\}$, $\{x_1, x_2\}$, $\{x_2, x_3\}$, $\{x_1, x_4\}$ – maximal independent sets of $\widetilde{G}$.

It is quite obvious, therefore, that the maximal independent set of a graph $\widetilde{G}$ corresponds to a clique of the graph G and vice versa, where $\widetilde{G}$ is the graph complementary to G.

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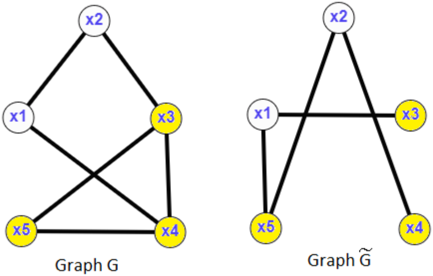


$\{x_3, x_4, x_5\}$ is a maximum independent sets of $\widetilde{G}$,
$\{x_3, x_4, x_5\}$ is a maximum clique of G, $\omega(G) = 3 = \alpha(\widetilde{G})$.

Dominating Sets

A dominating vertex set

For a graph $G = (X, \Gamma)$ a dominating vertex set (also known as an *externally stable set*), is a set of vertices $S \subseteq X$ chosen so that for every vertex x_j not in S, there is an arc from a vertex in S to x_j.

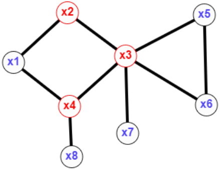
Thus S is a dominating vertex set (or simply a dominating set — when no confusion arises) if:

$$S \cup \Gamma(S) = X \tag{1}$$

A *minimal dominating vertex set*

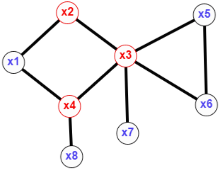
A dominating set is called minimal if there is no other dominating set which is contained in it.
Thus, a set S is a minimal dominating set if it satisfies eqn 1 and there is no other set $H \subset S$ which also satisfies eqn 1.

Dominating Sets



Is this set dominating? Yes!
Is this set minimal dominating?
No!

Dominating Sets



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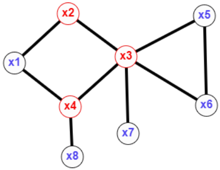

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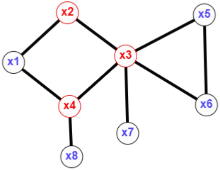

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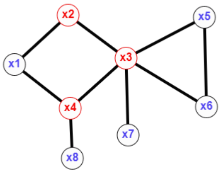

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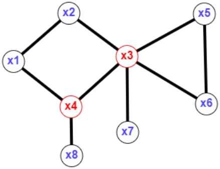

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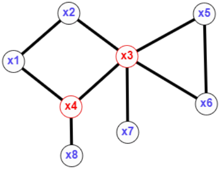
Dominating Sets



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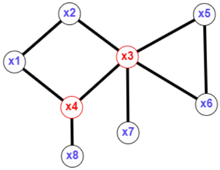
Dominating Sets

A dominance number

If P represents the family of minimal dominating sets then the number

$$\beta[G] = \min_{S \in P} |S| \qquad (2)$$

is called the dominance number of a graph G, and the set S^* from which it is derived is called the minimum dominating set.



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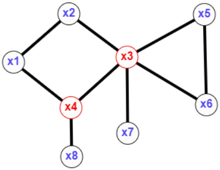

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Dominating Sets

Proposition 1

Any one vertex in a complete graph constitutes a minimum dominating set.

Proposition 2

Every maximal independent set is a dominating set.

Proposition 3

An independent set has the dominance property only if it is a maximal independent set.

Dominating Sets

Proposition 4

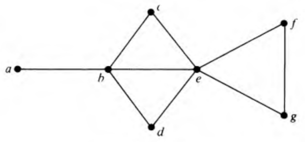
A minimal dominating set may or may not be independent.

Proposition 5

In a graph G

$$\alpha[G] \geq \beta[G].$$

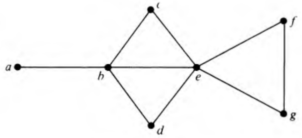
Construction of minimal dominating sets



The adjacency matrix

$$A = \begin{array}{c|ccccccc} & a & b & c & d & e & f & g \\ \hline a & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ b & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ c & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ d & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ e & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ f & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ g & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}$$

Construction of minimal dominating sets



The adjacency matrix where all elements of the main diagonal are equal to unity

$$A^* = \begin{array}{c|ccccccc} & a & b & c & d & e & f & g \\ \hline a & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ b & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ c & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ d & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ e & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ f & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ g & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}$$

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Construction of minimal dominating sets. Boolean arithmetics

We assign to any vertex of the graph x_i a Boolean variable.
Assume S – minimal dominating set. Let $x_i = 1$, if $x_i \in S$ and $x_i = 0$, otherwise.
Constructing a dominating set to dominate a vertex x_i we must either include x_i or any of the vertices adjacent to x_i.
Then

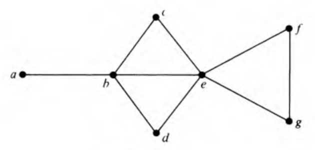
$$x_i + x_{i_1} + x_{i_2} + \ldots + x_{i_d} = 1,$$

where $x_{i_1}, x_{i_2}, \ldots, x_{i_d}$ are the vertices adjacent to x_i and d is a degree of x_i.
Taking the product over all vertices, we obtain the equation

$$\Theta = \Pi(x_i + x_{i_1} + x_{i_2} + \ldots + x_{i_d}) = 1.$$

We conclude that each term in Θ will represent a minimal dominating set.

Construction of minimal dominating sets. Boolean arithmetics



$$\Theta = (a+b)(b+c+d+e+a)(c+b+e)(d+b+e)\cdot$$

$$\cdot(e+b+c+d+f+g)(f+e+g)(g+e+f)$$

Since in Boolean arithmetic $(x+y)x = x$,

$$\Theta = (a+b)(c+b+e)(d+b+e)(g+e+f) =$$

$$= ae + be + bf + bg + acdf + acdg$$

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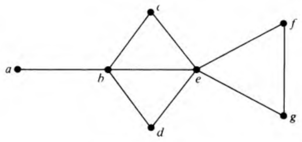

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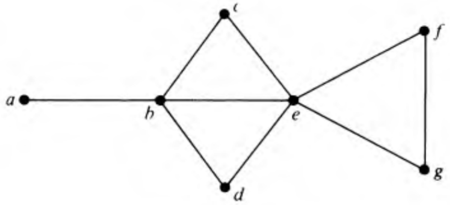

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$$\{a, c, d, f\}, \{a, c, d, g\}, \{b, g\}, \{b, f\}, \{a, e\}$$

– maximal independent sets.

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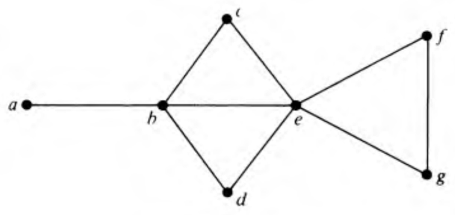

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