Graphs and Flows in Networks

Lecture 1

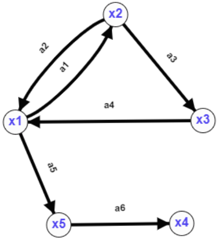
St. Petersburg State University, Russia

St. Petersburg, 2022

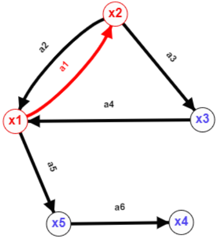
Definition

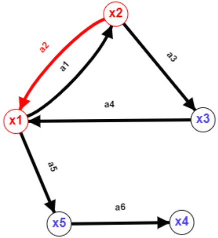
A graph $G = (X, A)$ is a set of vertices $x_1, x_2, \ldots, x_n$ (denoted by the set X), and a set of edges $a_1, a_2, \ldots, a_m$ (denoted by the set A) joining all or some of these vertices.

If the edges in A have a direction - which is usually shown by an arrow - they are called arcs and the resulting graph is called a directed graph.



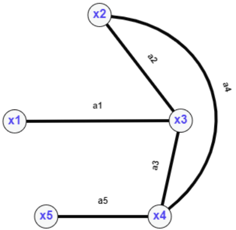
An arc is denoted by the pair of initial and final vertices, its direction will be assumed to be from the first vertex to the second.





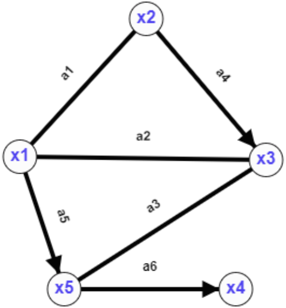
$a_1 \longrightarrow (x_1, x_2)$ $a_2 \longrightarrow (x_2, x_1)$

If the edges have no orientation the are called links and the graph is nondirected.



In the case where $G = (X, A)$ is a directed graph, but we want to disregard the direction of the arcs in A, the nondirected counterpart to G will be written as $\overline{G} = (X, \overline{A})$.

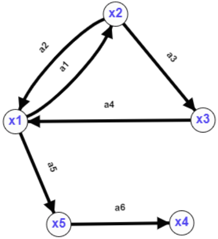
Mixed graphs



Correspondence Γ

An alternative way to describe a direct graph $G = (X, \Gamma)$, is by specifying:

- the set X of vertices
- a correspondence $\Gamma : X \to X$ which shows how the vertices are related to each other.



$$\Gamma(x_1) = \{x_2, x_5\},\ \Gamma(x_2) = \{x_1, x_3\}$$

$$\Gamma(x_3) = ?, \Gamma(x_4) = ?, \Gamma(x_5) = ?$$

Correspondence Γ

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- the set X of vertices
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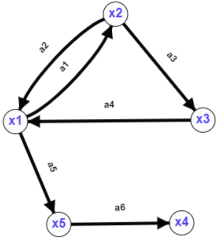
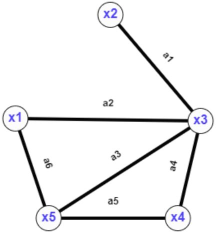


$$\Gamma(x_1) = \{x_2, x_5\},\ \Gamma(x_2) = \{x_1, x_3\}$$

$$\Gamma(x_3) = \{x_1\}, \Gamma(x_4) = \emptyset, \Gamma(x_5) = \{x_4\}$$

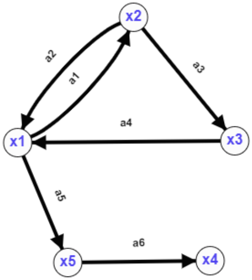
In the case of nondirected or mixed graph, the correspondence Γ will be assumed to be those as for an equivalent directed graph in which every link has been replaced by two arcs in opposite directions.



$$\Gamma(x_5) = \{x_1, x_3, x_4\},\ \Gamma(x_1) = \{x_3, x_5\}$$

Inverse correspondence

$\Gamma^{-1}(x_i)$ – the set of those vertices x_k for which an arc (x_k, x_i) exists in G.

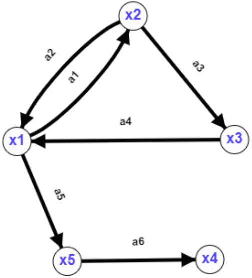


$$\Gamma^{-1}(x_1) = \{x_2, x_3\},\ \Gamma^{-1}(x_2) = \{x_1\}$$

$$\Gamma^{-1}(x_3) =?, \Gamma^{-1}(x_4) =?, \Gamma^{-1}(x_5) =?$$

Inverse correspondence

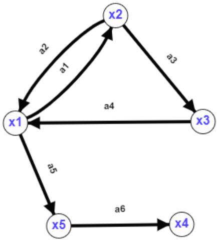
$\Gamma^{-1}(x_i)$ – the set of those vertices x_k for which an arc (x_k, x_i) exists in Γ.



$$\Gamma^{-1}(x_1) = \{x_2, x_3\},\ \Gamma^{-1}(x_2) = \{x_1\}$$

$$\Gamma^{-1}(x_3) = \{x_2\}, \Gamma^{-1}(x_4) = \{x_5\}, \Gamma^{-1}(x_5) = \{x_1\}$$

$$X_q = \{x_1, x_2, \ldots, x_q\}$$
$$\Gamma(X_q) = \Gamma(x_1) \cup \Gamma(x_2) \cup \ldots \cup \Gamma(x_q)$$

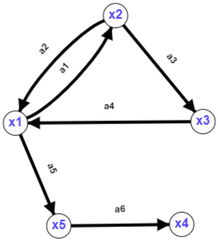


$$\Gamma(\{x_2, x_5\}) = \{x_1, x_3, x_4\},\ \Gamma(\{x_1, x_3\}) =?$$

The double correspondence

$$\Gamma^2(x_1) = \Gamma(\Gamma(x_1)) = \Gamma(\{x_2, x_5\}) = \{x_1, x_3, x_4\}$$
$$\Gamma^3(x_1) =?,\ \Gamma^{-2}(x_1) =?$$

$$X_q = \{x_1, x_2, \ldots, x_q\}$$
$$\Gamma(X_q) = \Gamma(x_1) \cup \Gamma(x_2) \cup \ldots \cup \Gamma(x_q)$$



$$\Gamma(\{x_2, x_5\}) = \{x_1, x_3, x_4\},\ \Gamma(\{x_1, x_3\}) = \{x_1, x_2, x_5\}$$

The double correspondence

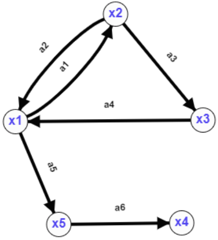
$$\Gamma^2(x_1) = \Gamma(\Gamma(x_1)) = \Gamma(\{x_2, x_5\}) = \{x_1, x_3, x_4\}$$
$$\Gamma^3(x_1) = \{x_2, x_5, x_1\},\ \Gamma^{-2}(x_1) = \{x_1, x_2\}$$

Definition

Arcs which have a common terminal vertex are called adjacent.

Definition

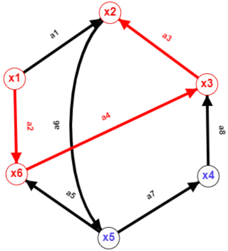
Two vertices x_i and x_j are called adjacent if either arc (x_i, x_j) or arc (x_j, x_i) or both exit in the graph.

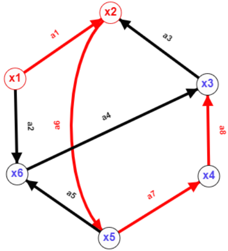


$$a_2, a_4; \quad a_2, a_5; \quad x_2, x_3; \quad x_4, x_5$$

Definition

A path in a directed graph is any sequence of arcs where the final vertex of one is the initial vertex of the next one.





a_2, a_4, a_3

a_1, a_6, a_7, a_8

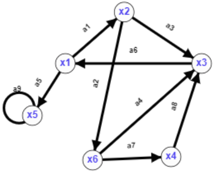
$a_3, a_6, a_5, a_2 -?$

Definition

A simple path is a path which does not use the same arc more than once.

Definition

An elementary path is a path which does not use the same vertex more than once.

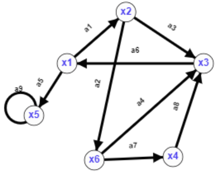


Definition

A loop is an arc whose initial and final vertices are the same

Definition

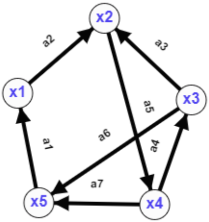
A circuit is a path $a_1, a_2, \ldots, a_q$ in which the initial vertex of a_1 coincides with the final vertex of a_q.



a_2, a_7, a_8, a_6, a_1 $\qquad$ a_1, a_3, a_6

Definition

An elementary circuit which passes through all the n vertices of a graph G is called a Hamiltonian circuit.



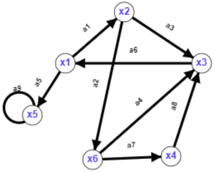
example - ?

Definition

A chain is a sequence of links $(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_q)$ in which every link $\bar{a}_i$, except perhaps the first and last links, is connected to the links $\bar{a}_{i-1}$ and $\bar{a}_{i+1}$ by its two terminal vertices.

Definition

A cycle is a chain $x_1, x_2, \ldots, x_q$ in which the beginning and end vertices are the same, i.e. in which $x_1 = x_q$.



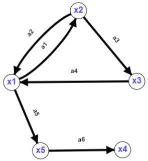
$\overline{a}_1, \overline{a}_3, \overline{a}_4;$ $\overline{a}_3, \overline{a}_4, \overline{a}_2$

Definition

The number of arcs which have a vertex x_i as their initial vertex is called the outdegree of vertex x_i.

Definition

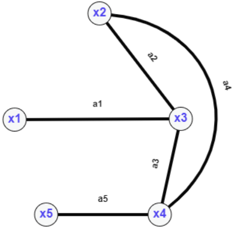
The number of arcs which have a vertex x_i as their final vertex is called the indegree of vertex x_i.



$$d_o(x_2) = |\Gamma(x_2)| = 2, \ d_t(x_2) = |\Gamma^{-1}(x_2)| = 1$$

$$\sum_{i=1}^{n} d_0(x_i) = \sum_{i=1}^{n} d_t(x_i) = m$$

For a nondirected graph: $d(x_i) = |\Gamma(x_i)|$ – the degree of a vertex x_i.



$$d(x_3) = 3$$

Let $G = (X, A)$.

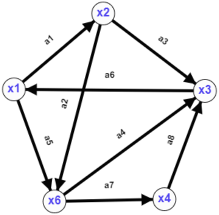
Definition

A partial graph G_p of G is the graph (X, A_p) with $A_p \subset A$. Thus a partial graph is a graph with the same number of vertices but with only a subset of the arcs of the original graph.

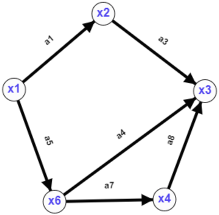
Definition

A subgraph G_s is the graph (X_s, Γ_s) with $X_s \subset X$ and for every $x_i \in X_s$, $\Gamma_s(x_i) = \Gamma(x_i) \cap X_s$. Thus, a subgraph has only a subset X_s of the set of vertices of the original graph but contains all the arcs whose initial and final vertices are both within this subset.

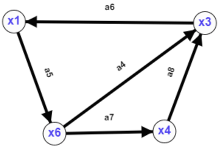
A partial subgraph is a partial graph of the subgraph.



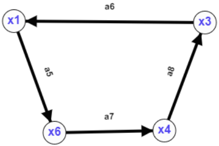
Graph



Partial graph



Subgraph

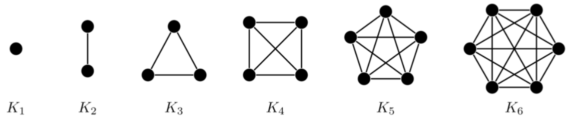


Partial subgraph

Definition

A graph $G = (X, A)$ is said to be complete if for every pair of vertices x_i and x_j in X, there exists a link $\overline{(x_i, x_j)}$ in $\overline{G} = (X, \overline{A})$ i.e. there must be at least one arc joining every pair of vertices.

The complete nondirected graph on n vertices is denoted by K_n.

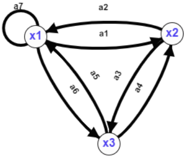


Definition

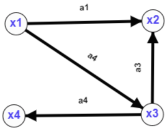
A graph (X, A) is said to be symmetric if, whenever an arc (x_i, x_j) is one of the arcs in the set A of arcs, the opposite arc (x_j, x_i) is also in the set A.

Definition

An antisymmetric graph is a graph in which whenever an arc $(x_i, x_j) \in A$, the opposite arc $(x_j, x_i) \notin A$.



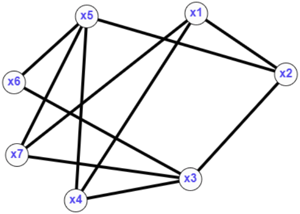
Symmetric graph

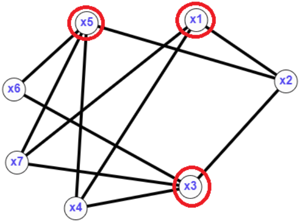


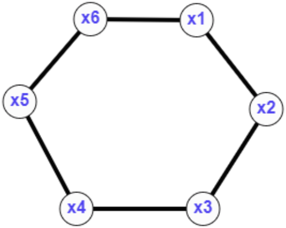
Antisymmetric graph

Definition

A nondirected graph $G = (X, A)$ is said to be bipartite, if the set X of its vertices can be partitioned into two subsets X^a and X^b so that all arcs have one terminal vertex in X^a and the other in X^b. A directed graph G is said to be bipartite if its nondirected counterpart G is bipartite.




x5
x1
x6
x2
x7
x3
x4


x6
x1
x5
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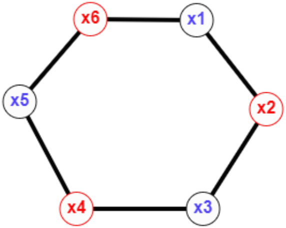
x3

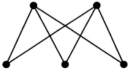

x6
x1
x5
x2
x4
x3

Theorem

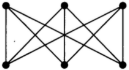
A nondirected graph G is bipartite if and only if it contains no circuits of odd cardinality.

Definition

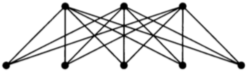
A bipartite graph $G = (X^a \cup X^b, A)$ is said to be complete if for every two vertices $x_i \in X^a$ and $x_j \in X^b$ there exists a link (x_i, x_j) in $G = (X, A)$.



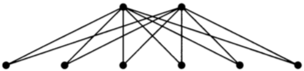
$K_{2,3}$



$K_{3,3}$



$K_{3,5}$

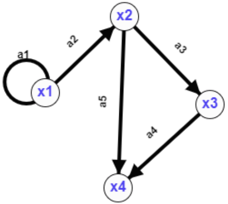


$K_{2,6}$

The adjacency matrix

Given a graph G, its adjacency matrix is denoted by $A = [a_{ij}]$ and is given by:

- $a_{ij} = 1$ if arc (x_i, x_j) exists in G
- $a_{ij} = o$ if arc (x_i, x_j) does not exist in G.



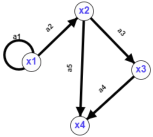
$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The incidence matrix

Given a graph G of n vertices and m arcs, the incidence matrix of G is denoted by $B = [b_{ij}]$ and is an $n \times m$ matrix defined as follows.

- $b_{ij} = 1$ if x_i is the initial vertex of arc a_j
- $b_{ij} = -1$ if x_i is the final vertex of arc a_j

and $b_{ij} = 0$ if x_i is not a terminal vertex of arc a_j or if a_j is a loop.



$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix}$$