Graphs and Flows in Networks

Lecture 3

St. Petersburg State University, Russia

St. Petersburg, 2022

Independent Sets

Consider a nondirected graph $G = (X, \Gamma)$.

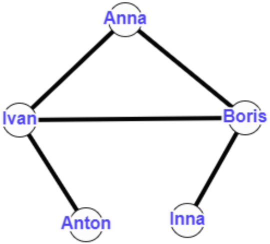
An independent vertex set

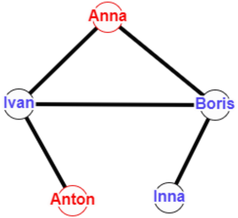
An independent vertex set (also known as *an internally stable set*) is a set of vertices of G so that no two vertices of the set are adjacent; i.e. no two vertices are joined by a link. Hence, any set $S \subset X$ which satisfies the relation:

$$S \cap \Gamma(S) = \emptyset \qquad (1)$$

is an independent vertex set.

For example, you want to invite as many of your friends to your party, but many pairs do not get along, represented by edges between them, and you do not want to invite two enemies

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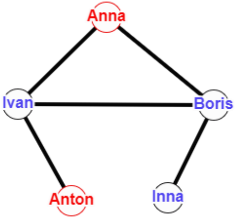
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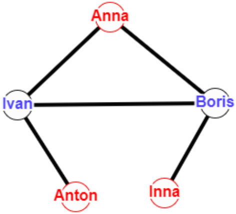
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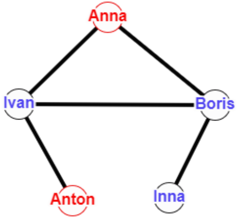
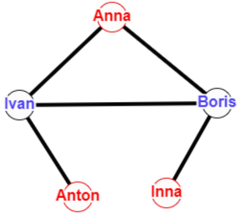
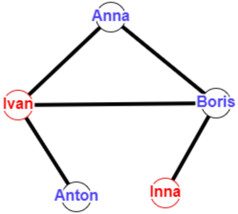
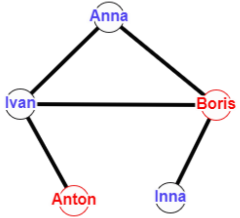
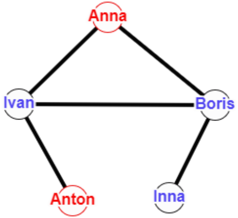


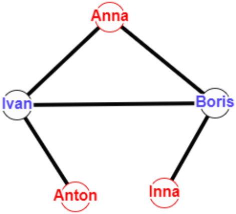


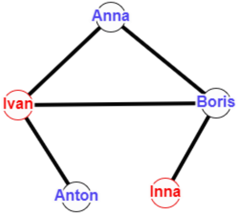


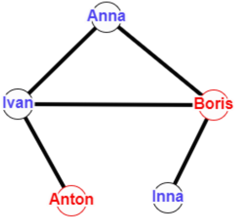


$\{Anna, Anton\}, \{Anna, Inna\}, \{Anna, Anton, Inna\}, \{Ivan, Inna\}, \{Anton, Boris\}$ – independent sets









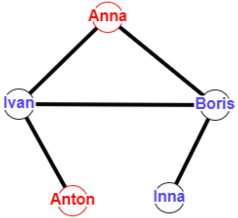
$\{Anna, Anton\}, \{Anna, Inna\}, \{Anna, Anton, Inna\},$
$\{Ivan, Inna\}, \{Anton, Boris\}$ – independent sets

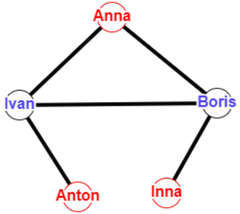
A maximal independent set

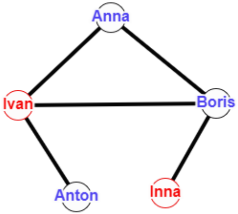
An independent set is called maximal when there is no other independent set that contains it. Thus, a set S is a maximal independent set if

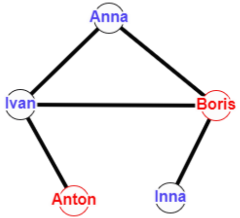
$$S \cap \Gamma(S) = \emptyset,$$

$$H \cap \Gamma(H) \neq \emptyset, \quad \forall H \supset S. \tag{2}$$

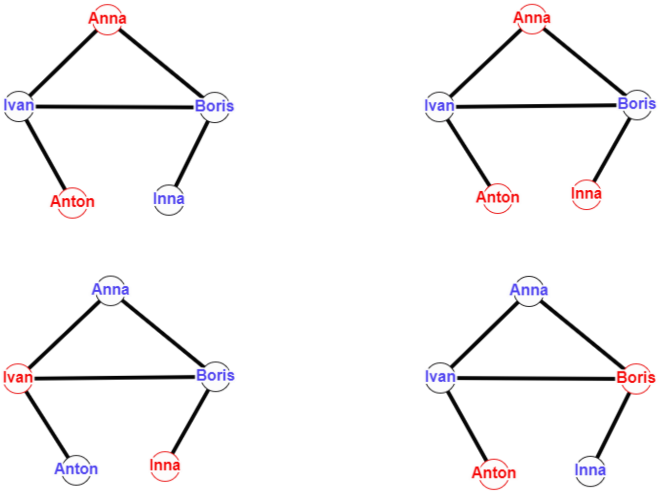








$\{Anna, Anton, Inna\}, \{Ivan, Inna\}, \{Anton, Boris\}$ – maximal independent sets



$\{Anna, Anton, Inna\}, \{Ivan, Inna\}, \{Anton, Boris\}$ – maximal independent sets

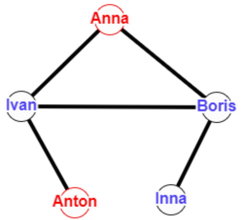
One should also note that the number of elements (vertices) in the various maximal independent sets is not the same for all the sets as can be seen from the above example.

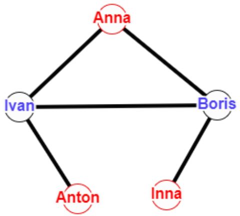
An independence number

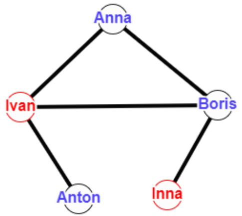
If Q is the family of independent sets then the number

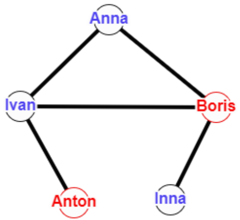
$$\alpha[G] = \max_{S \in Q} |S| \tag{3}$$

is called the independence number of a graph G, and the set S^* from which it is derived is called a maximum independent set.

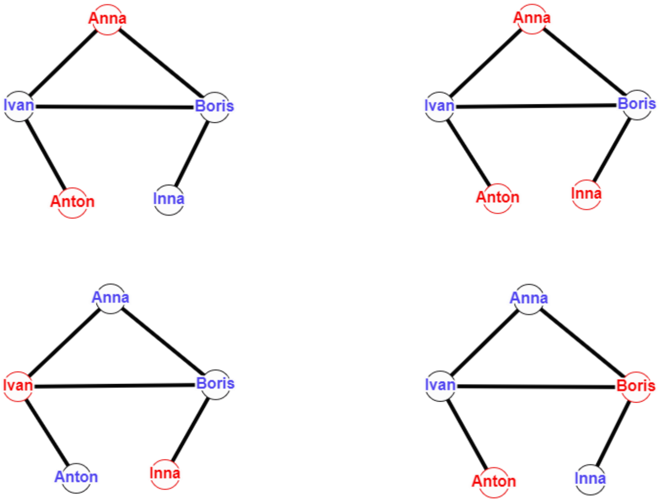








$\{Anna, Anton, Inna\}$, – maximum independent sets, $\alpha[G] = 3$.



$\{Anna, Anton, Inna\}$, – maximum independent sets, $\alpha[G] = 3$.

The computation of all maximal independent sets. Bron-Kerbosch Algorithm

The supposition is certainly true for small graphs of, say, up to 20 vertices, but as the number of vertices increases this method of generation becomes computationally unwieldy. This is so, not so much because the number of maximal independent sets becomes excessive, but owing to the fact that during the process a very large number of independent sets are formed and subsequently rejected because they are found to be contained in other previously generated sets and are, therefore, not maximal in themselves.

In the current section we will describe a systematic enumerative method due to Bron and Kerbosch, which substantially overcomes this difficulty so that independent sets – once generated — need not be checked for maximality against the previously generated sets.

The Basis of the Algorithm

S_k: at some stage k — an independent set of vertices S_k is augmented by the addition of another suitably chosen vertex to produce an independent set S_{k+1}, at stage $k+1$, until no further augmentation is possible and the set becomes a maximal independent set.

Q_k — the largest set of vertices (at stage k) for which

$$S_k \cap Q_k = \emptyset,$$

i.e. any vertex from Q_k added to S_k produces a set S_{k+1} which is independent.

$Q_k = Q_k^- \cup Q_k^+$:

- vertices in Q_k^- which have already been used earlier in the search to augment S_k
- vertices in Q_k^+ which have not yet been used.

A forward branching during the tree search then involves choosing a vertex $x_{i_k} \in Q_k^+$, appending it to S_k to produce

$$S_{k+1} = S_k \cup \{x_{ik}\} \tag{4}$$

and creating new sets Q_{k+1} as:

$$Q_{k+1}^+ = Q_k^+ - \Gamma(x_{ik}) - \{x_{i_k}\} \tag{5}$$

and

$$Q_{k+1}^- = Q_k^- - \Gamma(x_{i_k}) \tag{6}$$

A backtracking step of the algorithm involves the removal of x_{i_k} from S_{k+1} to revert back to S_k, and the removal of x_{i_k} from the old set Q_k^+ and its addition to the old set Q_k^- to form two new sets Q_k^+ and Q_k^-.

The following observations are quite obvious. A set S_k is a maximal independent set only if it cannot be augmented further, i.e. only if $Q_k^+ = \emptyset$.

- If $Q_k^- \neq \emptyset$, it immediately follows that the current S_k was at some previous stage augmented by some vertex in Q_k^- and is therefore not maximal.
- If $Q_k^- = \emptyset$, the current S_k has not been previously augmented, and since sets are generated without duplication, S_k is a maximal independent set.

Thus,

The necessary and sufficient conditions for S_k to be a maximal independent set are:

$$Q_k^+ = Q_k^- = \emptyset \tag{7}$$

It is now quite apparent that if a stage is reached when some vertex $x \in Q_k^-$ exists for which $\Gamma(x) \cap Q_k^+ = \emptyset$, then regardless of which vertex from Q_k^+ is used to augment S_k in any number of forward branchings, vertex x can never be removed from Q_p^- at any future step $p > k$. Thus,

the condition:

$$\exists x \in Q_k^- \text{ so that } \Gamma(x) \cap Q_k^+ = \emptyset \tag{8}$$

is sufficient for a backtracking step to be taken since no maximal independent set can result from any forward branching from S_k.

During a forward step one can choose any vertex $x_{i_k} \in Q_k^+$ with which to augment set S_k, and on backtracking x_{i_k} will be removed from Q_k^+ and inserted into Q_k^-. Now if x_{i_k} is chosen to be a vertex from $\Gamma(x)$ for some $x \in Q_k^-$, then at the corresponding backtracking step the number:

$$\Delta(x) = |\Gamma(x) \cap Q_k^+| \tag{9}$$

will be decreased by one (from its value prior to the completion of the forward and backward steps) so that condition 8 is now more nearly true.

Thus, one possible way of choosing the vertex x_{i_k} with which to augment S_k, is to first determine the vertex $x^* \in Q_k^-$ with the smallest possible value of $\Delta(x^*)$ and then choose x_{i_k} from the set $\Gamma(x^*) \cap Q_k^+$.

Bron-Kerbosch Algorithm

Initialisation

Step 1. Set $S_0 = Q_0^- = \emptyset$, $Q_0^+ = X$, $k = 0$.

Forward step

Step 2. Choose a vertex $x_{i_k} \in Q_k^+$ as mentioned earlier and form S_{k+1}, Q_{k+1}^- and Q_{k+1}^+ keeping Q_k^- and Q_k^+ intact. Set $k = k + 1$.

Test

Step 3. If condition 8 is satisfied to go to step 5, else go to step 4.

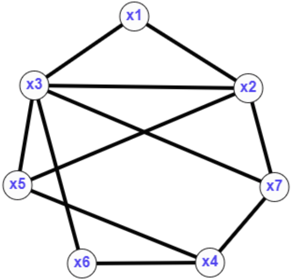
Step 4.
- If $Q_k^+ = Q_k^- = \emptyset$ print out the maximal independent set S_k and go to step 5.
- If $Q_k^+ = \emptyset$ but $Q_k^- \neq \emptyset$ go to step 5.
- Otherwise go to step 2.

Backtrack

Step 5. Set $k = k - 1$. Remove x_{i_k} from S_{k+1} to produce S_k. Retrieve Q_k^- and Q_k^+, remove x_{i_k} from Q_k^+ and add it to Q_k^-.

- If $k = 0$ and $Q_0^+ = \emptyset$, Stop. (All maximal independent sets have been printed out).
- Otherwise goto step 3.

Example



Obtaining maximal independent sets using Boolean arithmetic

Let each vertex in the graph be treated as a Boolean variable.

Let the logical (or Boolean) sum $a + b$ denote the operation of including vertex a or b or both.

Let the logical multiplication ab denote the operation of including both vertices a and b, and let the Boolean complement a' denote that vertex a is not included.

For a given graph G we must find a maximal subset of vertices that does not include the two end vertices of any edge in G. Let S is a maximal independent set, and $x = 1$, if $x \in S$, else $x = 0$. Let us express an edge (x, y) as a Boolean product, xy, of its end vertices x and y, and let us sum all such products in G to get a Boolean expression

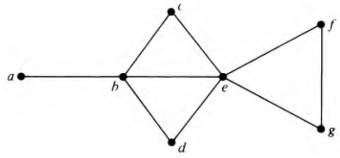
$$\varphi = \sum xy, \text{ for all } (x, y) \text{ in } G.$$

Let us further take the Boolean complement φ' of this expression, and express it as a sum of Boolean products:

$$\varphi' = f_1 + f_2 + \ldots + f_k.$$

A vertex set S is a maximal independent set if and only if $\varphi = 0$ (logically false), which is possible if and only if $\varphi' = 1$ (true), which is possible if and only if at least one $f_i = 1$, which is possible if and only if each vertex appearing in f_i (in complemented form) is excluded from the vertex set of S. Thus each f_i, will yield a maximal independent set, and every maximal independent set will be produced by this method.

Example



$$\varphi = ab + bc + bd + be + ce + de + ef + eg + fg,$$

$$\varphi' = (a'+b')(b'+c')(b'+d')(b'+e')(c'+e')(d'+e')(e'+f')(e'+g')(f'+g')$$

Example

$$\varphi' = (a'+b')(b'+c')(b'+d')(b'+e')(c'+e')(d'+e')(e'+f')(e'+g')(f'+g')$$

Multiplying these out and employing the usual identities of Boolean arithmetic, such as

$$aa = a, \quad a + a = a,$$
$$a + ab = a$$

we get

$$\varphi' = b'e'f' + b'e'g' + a'c'd'e'f' + a'c'd'e'g' + b'c'd'f'g'.$$

Now if we exclude from the vertex set of G vertices appearing in any one of these five terms, we get a maximal independent set. The five maximal independent sets are

$$acdf, acdg, bg, bf, ae.$$

These are all the maximal independent sets of the graph.