① $y(n) = x(n) + 2x(n-1) + x(n-2)$

a) $y(n) = h(n)$ when $x(n) = \delta(n)$

$h(n) = \delta(n) + 2\,\delta(n-1) + 1\cdot\delta(n-2)$

$\therefore$

$h(n) = [\underset{\uparrow}{1}, 2, 1]$

BIBO stable if: $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$

so the system is stable. (FIR-system!)

b) $H(\omega) = \sum_{n=0}^{2} h(n)\, e^{-j\omega n}$

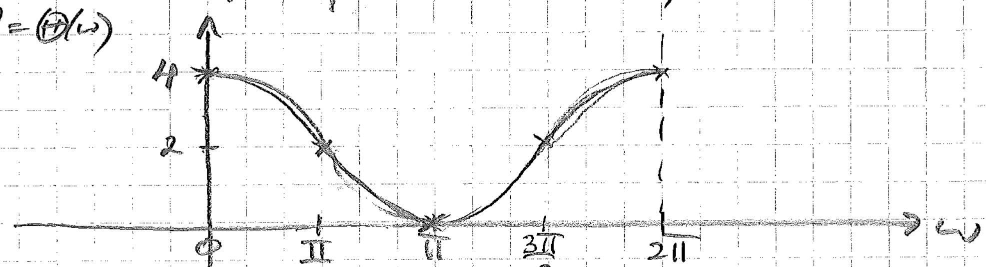
$= 1 + 2e^{-j\omega} + e^{-j2\omega}$

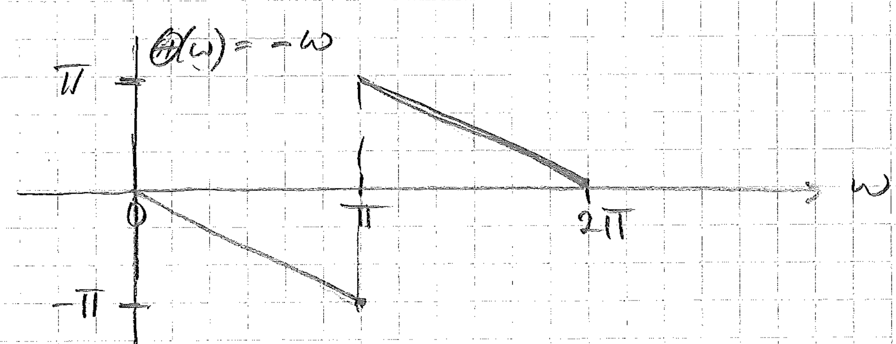
$= e^{-j\omega}\left(e^{j\omega} + e^{-j\omega}\right) + 2\cdot e^{-j\omega}$

$= e^{-j\omega}\left(2 + 2\cos(\omega)\right) = e^{-j\omega}\underbrace{2\cdot\left(1+\cos(\omega)\right)}_{H_{real}(\omega)}$

Sketch $|H(\omega)|$ and $Arg\{H(\omega)\} = \Theta(\omega)$

$|H(\omega)| = 2(1+\cos(\omega)) \geq 0 \;\; \forall \omega$





① cont.

c) $H_1(\omega) = H(\omega+\pi)$

$$H_1(\omega) = H(\omega+\pi) = \sum_{n=0}^{2} h(n)\, e^{-j(\omega+\pi)\cdot n}$$

$$= \sum_{n=0}^{2} h(n)\, e^{-j\omega n} \cdot e^{-j\pi \cdot n}$$

$$= \sum_{n=0}^{2} \underbrace{h(n) \cdot e^{-j\pi \cdot n}}_{= h_1(n)} \cdot e^{-j\omega n}$$

So $h_1(n) = h(n) \cdot e^{-j\pi \cdot n} = h(n)\left(e^{-j\pi}\right)^n = h(n)\,(-1)^n$

$h_1(0) = h(0)(-1)^0 = h(0)$
$h_1(1) = h(1)(-1)^1 = -h(1)$
$h_1(2) = h(2)(-1)^2 = h(2)$

$h_1(n) = [1, -2, 1]$

(2) a) $x_1(n) = 0{,}5\cos\left(\frac{\pi}{3}\cdot n\right) + 0.8\sin\left(\frac{\pi}{5}\cdot n\right) \quad -\infty < n < \infty$

Periodic signal is repr by $\{c_k\}$ in the freq. domain.

F-serie expansion:

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j\left(\frac{2\pi}{N}\cdot k\right)\cdot n} \quad ; N \text{ is the period of } x(n).$$

$$x_1(n) = 0.5\cos\left(\frac{2\pi}{6}\cdot n\right) + 0.8\sin\left(\frac{2\pi}{10}\cdot n\right)$$

$$= 0.5\cos\left(\frac{2\pi\cdot 5}{30}\cdot n\right) + 0.8\sin\left(\frac{2\pi\cdot 3}{30}\cdot n\right)$$

$N = 30$ (Minimum common period)

F-serie expansion when $N = 30$

$$x(n) = \sum_{k=0}^{29} c_k e^{\left(j\frac{2\pi}{30}\cdot k\right)\cdot n}$$

$$\underbrace{\mathcal{F}\{x_1(n)\}}_{c_k} \overset{\text{linear}}{=} \mathcal{F}\{\underbrace{0.5\cos\left(\frac{2\pi\cdot 5}{30}\cdot n\right)}_{c_k^{s1}}\} + \mathcal{F}\{\underbrace{0.8\sin\left(\frac{2\pi\cdot 3}{30}\cdot n\right)}_{c_k^{s2}}\}$$

Use Euler id:

$$0.5\cos\left(\frac{2\pi}{30}\cdot 5\cdot n\right) = 0.5\cdot\frac{1}{2}\left[e^{j\frac{2\pi}{30}\cdot 5\cdot n} + e^{-j\frac{2\pi}{30}\cdot 5\cdot n}\right]$$

$$= \frac{1}{4} e^{j\frac{2\pi}{30}\cdot 5\cdot n} + \frac{1}{4} e^{j\frac{2\pi}{30}(-5)\cdot n}$$

$$= c_5^{s1} \qquad c_{-5}^{s1} = c_{30-5}^{s1} = c_{25}^{s1}$$

periodic $N = 30$.

$$0.8\sin\left(\frac{2\pi\cdot 3}{30}\cdot n\right) = 0.8\frac{1}{2j}\left[e^{j\frac{2\pi}{30}\cdot 3\cdot n} - e^{-j\frac{2\pi}{30}\cdot 3\cdot n}\right]$$

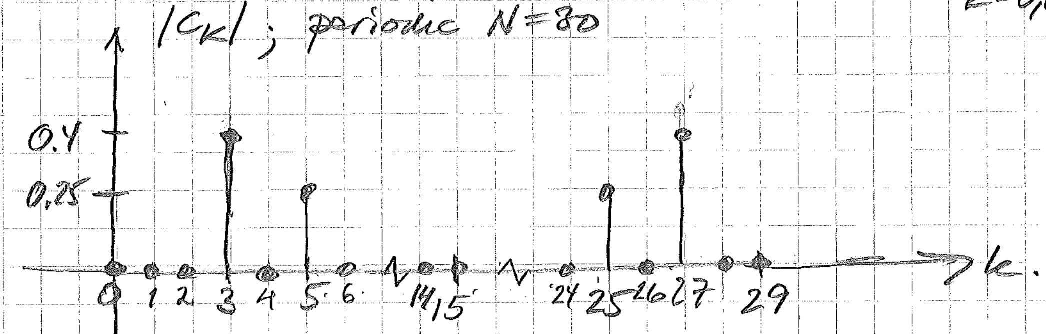
$$= \frac{0.4}{j} e^{j\frac{2\pi}{30}\cdot 3\cdot n} - \frac{0.4}{j} e^{j\frac{2\pi}{30}(-3)\cdot n}$$

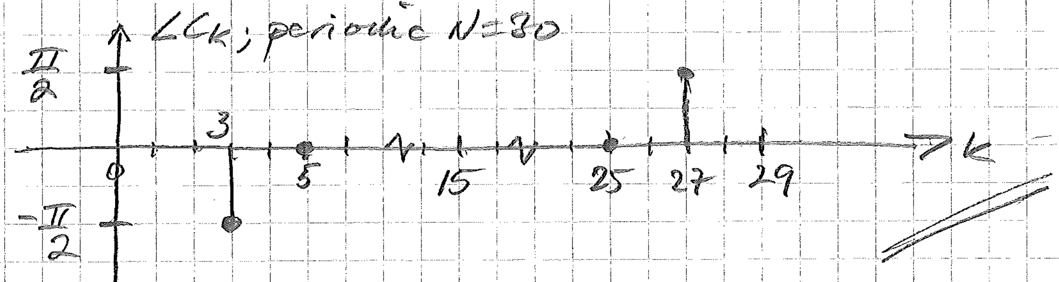
$$= c_3^{s2} \qquad = c_{-3}^{s2} = c_{30-3}^{s2} = c_{27}^{s2}$$

(2) cont.

$$C_k = C_k^{S1} + C_k^{S2} \Rightarrow \begin{cases} c_3 = \frac{0.4}{j} = 0.4\, e^{-j\pi/2} \\ c_5 = \frac{1}{4} = 0.25 \\ c_{25} = \frac{1}{4} = 0.25 \\ c_{27} = -\frac{0.4}{j} = 0.4\, e^{j\pi/2} \\ c_k = 0 \quad \text{for "other" } k\text{-values} \end{cases}$$

$k = 0, 1, 2, \ldots, 29$





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b)

$$x_2(n) = x_1(n) \cdot w(n) = w(n) \cdot 0.5 \cos\left(\tfrac{\pi}{3} n\right) + w(n) \cdot 0.8 \sin\left(\tfrac{\pi}{5} n\right)$$

$$X_2(\omega) = 0.5\, \frac{1}{2}\left[W\left(\omega - \tfrac{\pi}{3}\right) + W\left(\omega + \tfrac{\pi}{3}\right)\right] + 0.8 \cdot \frac{1}{2j}\left[W\left(\omega - \tfrac{\pi}{5}\right) - W\left(\omega + \tfrac{\pi}{5}\right)\right]$$

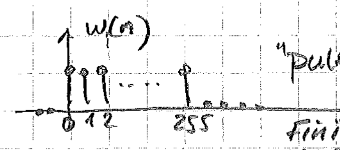
So determine

$$W(\omega) = \mathcal{F}\{w(n)\}$$

where
$$w(n) = \begin{cases} 1 & 0 \le n \le 255 \\ 0 & \text{otherwise} \end{cases}$$

② cont.

$$w(n) = u(n) - u(n-256)$$



Finite dur. causal signal
ROC: $z \neq 0$

$$\mathcal{Z} \downarrow$$

$$W(z) = \frac{1}{(1-z^{-1})} - \frac{z^{-256}}{(1-z^{-1})} = \frac{1-z^{-256}}{(1-z^{-1})}$$

$$W(\omega) = W(z)\Big|_{z=e^{j\omega}} = \frac{1-e^{-j\omega 256}}{1-e^{-j\omega}}$$

$$= \frac{e^{-j128\omega}\left(e^{j128\omega} - e^{-j128\omega}\right)}{e^{-j\omega/2}\left(e^{j\omega/2} - e^{-j\omega/2}\right)}$$

$$= e^{-j\frac{255}{2}\omega} \frac{\sin(128\omega)}{\sin(\omega/2)} \qquad \omega \neq 0$$

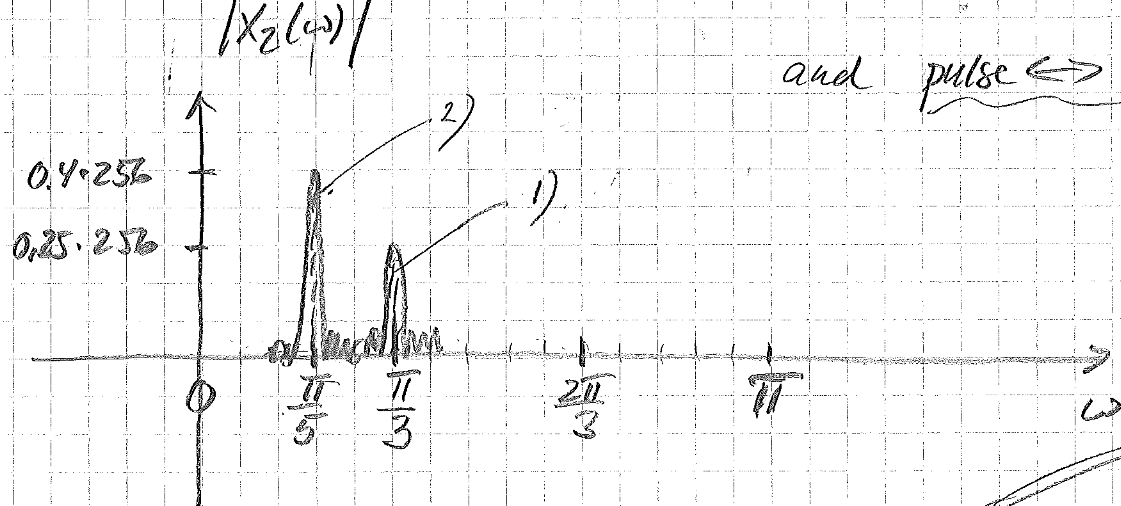
$$W(0) = \sum_{n=0}^{255} 1 = 256$$

zero-crossings when

$128\omega = k\cdot\pi \quad k = \cancel{0}, \pm1, \pm2, \ldots$

$\omega = \frac{\pi}{128}\cdot k$

and pulse $\Leftrightarrow$ sinc



1) $\left|\frac{0.5}{2} W\left(\omega - \frac{\pi}{3}\right)\right| = \frac{0.5}{2}\left|W\left(\omega - \frac{\pi}{3}\right)\right|$

2) $\left|\frac{0.8}{2j} W\left(\omega - \frac{\pi}{5}\right)\right| = \frac{0.8}{2}\left|W\left(\omega - \frac{\pi}{5}\right)\right|$

③

$$H(z) = \frac{0.1\cdot(1-z^{-2})}{1-0.9z^{-1}+0.81z^{-2}}$$

a) $$H(z)\frac{\times z^2}{\times z^2} = \frac{0.1\,(z^2-1)}{z^2-0.9z+0.81} = \frac{0.1\,(z+1)(z-1)}{(z-p_1)(z-p_2)}$$

p_1 and p_2 ? $\quad z^2-0.9z+0.81=0$

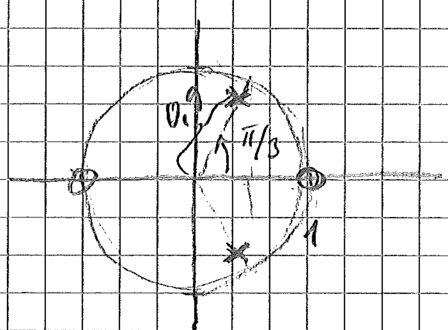
$$z_{1,2} = \frac{0.9}{2} \pm \sqrt{\left(\frac{0.9}{2}\right)^2-0.81}$$

$$= \frac{0.9}{2} \pm \sqrt{-\frac{3\cdot 0.81}{4}} = \frac{0.9}{2} \pm j\frac{\sqrt{3}}{2}\cdot 0.9$$

$$= 0.9\left(\frac{1}{2} \pm j\frac{\sqrt{3}}{2}\right)$$

$$\begin{cases} p_1 = 0.9\,e^{j\pi/3} \\ p_2 = p_1^* = 0.9\,e^{-j\pi/3} \end{cases}$$

Pole-zero pattern :



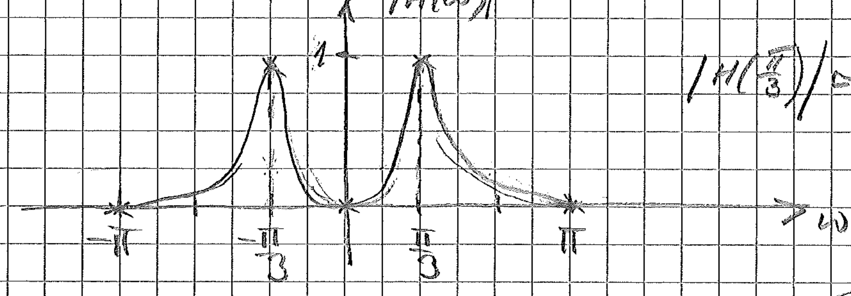
$$\begin{cases} p_1 = 0.9\,e^{j\pi/3} \\ p_2 = p_1^* \end{cases} \quad \text{poles}$$

$$\begin{cases} z_1 = 1 \\ z_2 = -1 \end{cases} \quad \text{zeros}$$

Magnitude function $H(\omega) = H(z)\big|_{z=e^{j\omega}}$

$H(z)$ peaks close to a pole, i.e at $\omega = \pm\frac{\pi}{3}$

$H(z)=0$ for $z=1$ and $z=-1$ (zeros) i.e at $\omega=0$ and $\pm\pi$



$$\left|H\left(\frac{\pi}{3}\right)\right| \simeq 1$$

(3) cont.

$$x(n) = 0.4 + 0.4\cos\left(\frac{\pi}{3}(n-2)\right) \qquad -\infty < n < \infty$$

Steady-state response of the output:

$$y(n) = 0.4\underbrace{|H(0)|} + 0.4\underbrace{\left|H\left(\frac{\pi}{3}\right)\right|}\cos\left(\frac{\pi}{3}(n-2) + \underbrace{\Theta\left(\frac{\pi}{3}\right)}\right)$$

zero at $\omega = 0$

$$H(0) = H(z)\Big|_{z=1} = 0 \qquad \text{(check the pole-zero pattern)}$$

$$H\left(\frac{\pi}{3}\right) = H(z)\Big|_{z=e^{j\frac{\pi}{3}}} = \frac{0.1\left(e^{j\frac{2\pi}{3}} - 1\right)}{\left(e^{j\frac{\pi}{3}} - 0.9e^{j\pi/3}\right)\left(e^{j\pi/3} - 0.9e^{-j\pi/3}\right)}$$

$$= \frac{0.1\left(-0.5 + j\frac{\sqrt{3}}{2} - 1\right)}{e^{j\pi/3}(1-0.9)\left(e^{j\pi/3} - 0.9e^{-j\pi/3}\right)} =$$

$$= \frac{0.1\left(-\frac{3}{2} + j\frac{\sqrt{3}}{2}\right)}{0.1\left(e^{j2\pi/3} - 0.9\right)} = \frac{\left(-\frac{3}{2} + j\frac{\sqrt{3}}{2}\right)}{\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2} - 0.9\right)}$$

$$= \frac{\left(-1.5 + j\frac{\sqrt{3}}{2}\right)}{\left(-1.4 + j\frac{\sqrt{3}}{2}\right)} \approx 1.05\, e^{j1.7^\circ}$$

So,

$$y(n) = \underbrace{0.4 \cdot 1.05}_{\approx 0.42} \cdot \cos\left(\frac{\pi}{3}(n-2) + 1.7^\circ\right)$$

(4)

a)

$$F_1 = 1500\,Hz,\quad F_2 = 4600\,Hz,\quad F_3 = 5800\,Hz \quad \Rightarrow \quad F_S = 8\,kHz,\quad f = \frac{F}{F_S}$$

$$f_1 = \frac{15}{80}$$

$$f_2 = \frac{46}{80} > \frac{1}{2}$$

$$f_3 = \frac{58}{80} > \frac{1}{2}$$

$$f_2 = \frac{46}{80} = -\frac{34}{80} + 1 \Rightarrow f_{2\,alias} = \frac{34}{80}$$

$$f_3 = \frac{58}{80} = -\frac{22}{80} + 1 \Rightarrow f_{3\,alias} = \frac{22}{80}$$

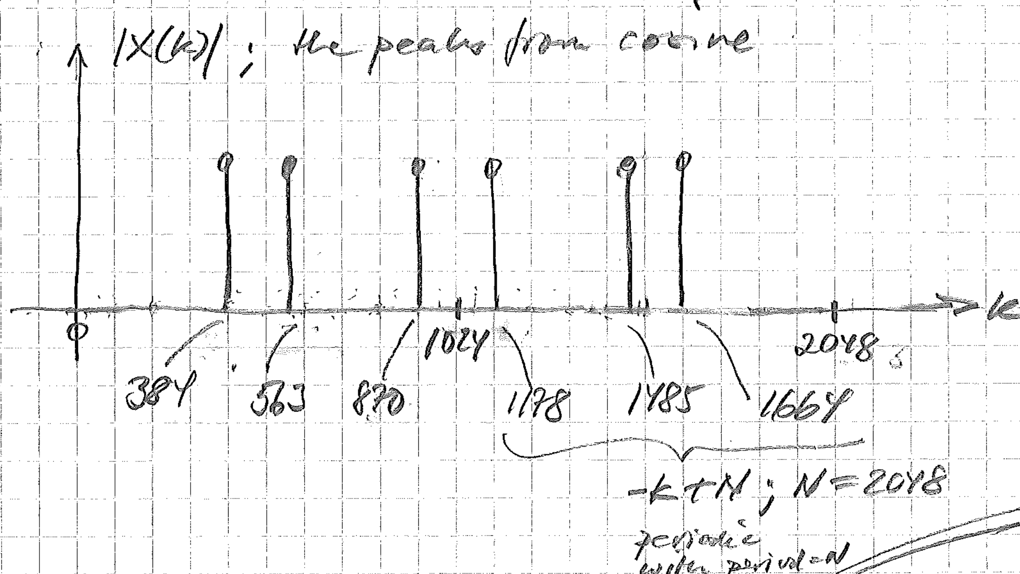
DFT in $N = 2048$ points is computed.

$$f = 1 \Longleftrightarrow k = 2048$$

$$f_1 = \frac{15}{80} \Longrightarrow k = \frac{15}{80} \cdot 2048 \simeq 384$$

$$f_{2\,alias} = \frac{34}{80} \Rightarrow k = \frac{34}{80} \cdot 2048 = 870$$

$$f_{3\,alias} = \frac{22}{80} \Rightarrow k = \frac{22}{80} \cdot 2048 = 563$$



(4) cont.

b) $F_s > 2F_{max} = 2 \cdot 5800 = 11.6$ kHz

Select $F_s = 14$ kHz.

$$f = \frac{F}{F_s} \Rightarrow \quad f_1 = \frac{15}{140} < \frac{1}{2}$$

$$f_2 = \frac{46}{140} < \frac{1}{2} \qquad \text{no aliasing!}$$

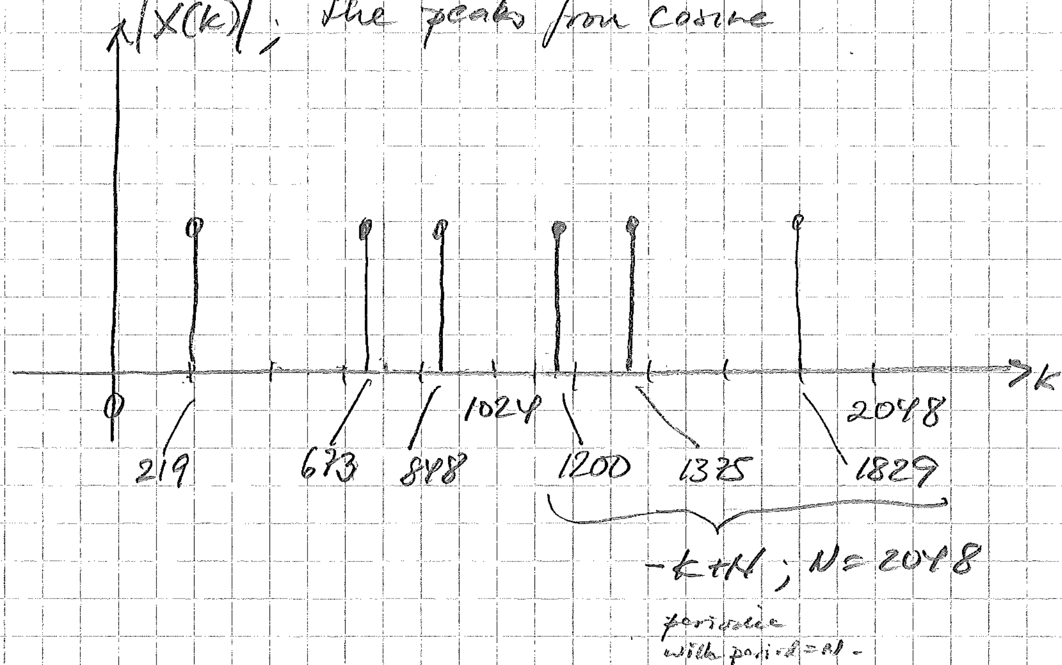
$$f_3 = \frac{58}{140} < \frac{1}{2} \qquad \text{— " —}$$

DFT in 2048 points

$$\Rightarrow \quad f_1 = \frac{15}{140} \rightarrow k = \frac{15}{140} \cdot 2048 \simeq 219$$

$$f_2 = \frac{46}{140} \rightarrow k = \frac{46}{140} \cdot 2048 \simeq 673$$

$$f_3 = \frac{58}{140} \rightarrow k = \frac{58}{140} \cdot 2048 \simeq 848$$



④ cont. c)

$$h(n) = [\underset{\uparrow}{\tfrac{1}{2}}, -\tfrac{1}{2}] \quad ; \text{ length } = 2$$

$$x(n) = [\underset{\uparrow}{-1}, -1, -1, 1, 1, 1] \quad ; \text{ length} = 6$$

$y = x(n) * h(n)$ is of length $(6+2-1) = 7$

So DFT must be computed in at least 7 points ; $\underline{\underline{N \geq 7}}$

Select $N = 8$

$$\left.\begin{aligned} Y(k) &= \underset{⑧}{\mathrm{DFT}}\{X(k)\} \cdot \underset{⑧}{\mathrm{DFT}}\{H(k)\} \\ y(n) &= \underset{⑧}{\mathrm{IDFT}}\{Y(k)\} \end{aligned}\right\}$$

can be computed in the time domain by circ. conv.

Graphical solution in the time domain: (zero-padding to length = 8)

-1	-1	-1	1	1	1	0	0

$\leftarrow$ ...

0	0	0	0	0	0	-1/2	1/2	0	0	0	0	0	0	-1/2	1/2

... periodic ; period = 8

-1/2	0	0	1	0	0	-1/2	0
0	1	2	3	4	5	6	7

periodic ; period = 8

$= y(n)$

$$y(n) = [\underset{\uparrow}{-\tfrac{1}{2}}, 0, 0, 1, 0, 0, -\tfrac{1}{2}]$$