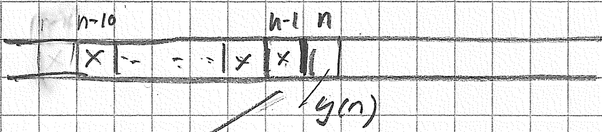
① $y(n) = 0.1\,[x(n) + x(n-1) + \dots + x(n-9)]$ nonrecursive system.

a)

$y(n) = y(n-1) + 0.1\,x(n) - 0.1\,x(n-10)$ recursive system.



$$\begin{cases} y(n-1) = 0.1\,[x(n-1) + x(n-2) + \dots + x(n-10)] \\ y(n) = 0.1\,[x(n) + x(n-1) + \dots\, x(n-9)] \end{cases}$$

$$\to \quad y(n) = y(n-1) + 0.1\,x(n) - 0.1\,x(n-10)$$

---

b)

$x(n) = \delta(n) \Rightarrow y(n) = h(n)$

Nonrecursive system: $h_1(n) = 0.1\,[\delta(n) + \delta(n-1) + \dots + \delta(n-9)]$

$$\Rightarrow h_1(n) = \{\underset{\uparrow}{\overset{0}{0.1}}, \overset{1}{0.1}, \dots\dots, \overset{9}{0.1}\}$$

Recursive system: $h_2(n) = h(n-1) + 0.1\,\delta(n) - 0.1\,\delta(n-10)$

$$h(0) = \underbrace{h(-1)}_{=0} + 0.1\,\delta(0) - 0.1\,\delta(-10) = 0.1$$

$$h(1) = h(0) + 0.1\,\delta(1) - 0.1\,\delta(-9) = 0.1$$

$$h(2) = h(1) + 0.1\,\delta(2) - 0.1\,\delta(-8) = 0.1$$

$$\vdots$$

$$h(9) = h(8) + 0.1\,\delta(9) - 0.1\,\delta(-1) = 0.1$$

$$h(10) = \underbrace{h(9)}_{=0.1} + 0.1\,\delta(10) - 0.1 \cdot \underbrace{\delta(0)}_{=1} = 0$$

$$h(11) = h(10) + 0.1\,\delta(11) - 0.1\,\delta(1) = 0$$

$$\Rightarrow h_2(n) = \{\underset{\uparrow}{\overset{0}{0.1}}, \overset{1}{0.1}, \dots\dots, \overset{9}{0.1}\}$$

$h_1(n) = h_2(n)$ as it should be!

---

① Cont. c)

$$y(n) = y(n-1) + 0.1x(n) - 0.1x(n-10)$$

$$\downarrow z$$

$$Y(z)\left[1 - z^{-1}\right] = X(z)\left(0.1 - 0.1z^{-10}\right)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{0.1\left(1 - z^{-10}\right)}{\left(1 - z^{-1}\right)} \quad \frac{\times z^{10}}{\times z^{10}} = \frac{0.1\left(z^{10} - 1\right)}{z^{10} - z^{9}}$$

$$= \frac{0.1\left(z^{10} - 1\right)}{z^{9}(z-1)}$$

zeros: o $\quad z^{10} - 1 = 0 \Rightarrow z^{10} = 1 \cdot e^{j2\pi k} \quad k = 0, 1, 2, 3, \ldots$

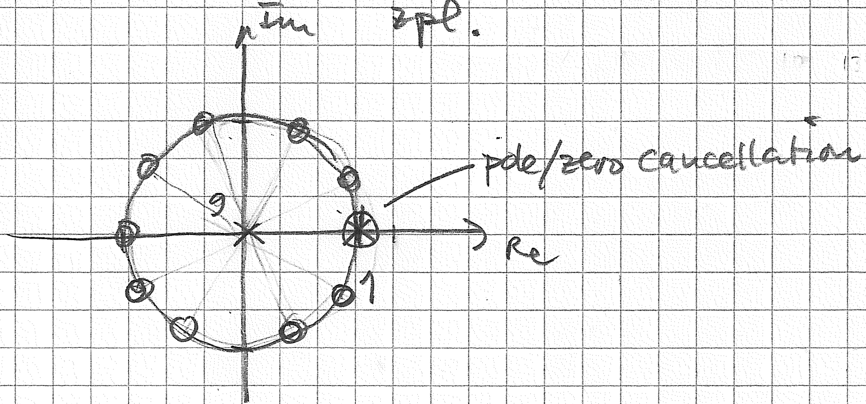
$z = r \cdot e^{j\theta}$

$\rightarrow r^{10} = 1$ and $r = 1$

$10 \cdot \theta = 2\pi k$ and $\theta = \frac{2\pi}{10} \cdot k$

$k = 0, 1, 2, \ldots 9$

Poles: × $\quad \begin{cases} p_1 = p_2 = \cdots p_9 = 0 \\ p_{10} = 1 \end{cases}$



(ALT) nonrecursive system.

$$y(n) = 0.1 \sum_{k=0}^{9} x(n-k) = 0.1\left[x(n) + x(n-1) + \cdots + x(n-9)\right]$$

$$z \downarrow$$

$$Y(z) = 0.1\left[X(z) + X(z) \cdot z^{-1} + \cdots X(z) \cdot z^{-9}\right]$$

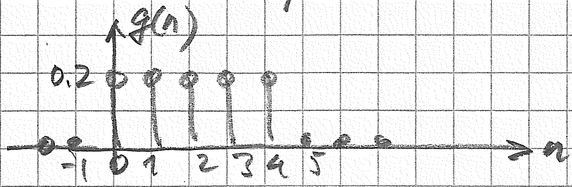
$$H(z) = \frac{Y(z)}{X(z)} = 0.1 \sum_{n=0}^{9} \left(z^{-1}\right)^n = 0.1\left(\frac{1 - z^{-10}}{1 - z^{-1}}\right) \quad \frac{\times z^{10}}{\times z^{10}} = 0.1\left[\frac{\left(z^{10} - 1\right)}{\left(z^{10} - z^{9}\right)}\right]$$

$$= 0.1 \frac{\left(z^{10} - 1\right)}{z^{9}(z-1)}$$

(2)

$$h(n) = \{\overset{-2}{0.2}, \overset{-1}{0.2}, \underset{\uparrow}{\overset{0}{0.2}}, \overset{1}{0.2}, \overset{2}{0.2}\}$$

$$g(n) = h(n-2) = \{\underset{\uparrow}{\overset{0}{0.2}}, \overset{1}{0.2}, \overset{2}{0.2}, \overset{3}{0.2}, \overset{4}{0.2}\}$$



a)

$$G(\omega) = \sum_{n=-\infty}^{\infty} g(n) e^{-j\omega n} = \sum_{n=0}^{4} 0.2\, e^{-j\omega n} = 0.2 \sum_{n=0}^{4} \left(e^{-j\omega}\right)^n$$

$$= \begin{cases} 0.2\, \dfrac{1-e^{-j\omega 5}}{1-e^{-j\omega}} & \omega \neq 0 \\ 0.2 \cdot 5 = 1 & \omega = 0 \end{cases}$$

$$\frac{1-e^{-j\omega 5}}{1-e^{-j\omega}} = \frac{e^{-j\omega \frac{5}{2}}\left(e^{j\omega\frac{5}{2}} - e^{-j\omega\frac{5}{2}}\right)}{e^{-j\frac{\omega}{2}}\left(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right)}$$

$$= e^{-j2\omega} \cdot \frac{\sin\left(\frac{5}{2}\cdot\omega\right)}{\sin\left(\frac{\omega}{2}\right)}$$

б)

$$G(\omega) = \begin{cases} 0.2\, \dfrac{\sin\left(\frac{5}{2}\omega\right)}{\sin\left(\frac{\omega}{2}\right)} e^{-j2\omega} & \omega \neq 0 \\ 1 & \omega = 0 \end{cases}$$

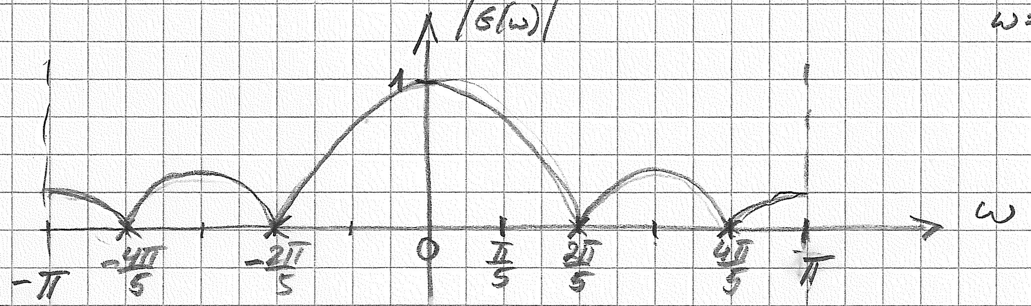
② b) Sketch the magnitude- and phase response
cont.

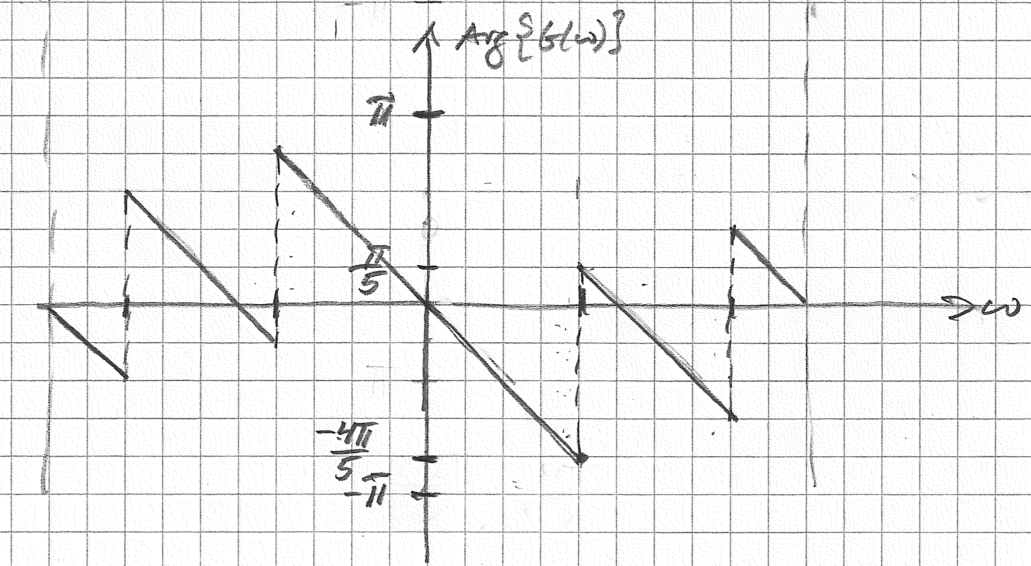
$$|G(\omega)| = \begin{cases} 0.2\,\dfrac{|\sin(\frac{5}{2}\omega)|}{|\sin(\frac{\omega}{2})|} & \omega \neq 0 \\ 1 & \omega = 0 \end{cases}$$

$$\text{Arg}\{G(\omega)\} = -2\omega + \begin{cases} 0 \\ \pi \end{cases}$$

$\frac{5}{2}\omega = k\cdot\pi \; ; k = \pm 1, \pm 2, \ldots$

$\omega = \frac{2\pi}{5}\cdot k.$





③

$$h(n) = \delta(n) + \left(\tfrac{1}{\sqrt{2}}\right)^{n-1} \cos\left(\tfrac{\pi}{4}(n-1)\right) u(n-1)$$

$$\Big\downarrow \mathcal{Z}$$

a)

$$H(z) = 1 + z^{-1} \frac{\left(1 - z^{-1} \frac{1}{\sqrt{2}} \cos\left(\frac{\pi}{4}\right)\right)}{\left(1 - z^{-1} 2 \cdot \frac{1}{\sqrt{2}} \cos\left(\frac{\pi}{4}\right) + \left(\frac{1}{\sqrt{2}}\right)^2 z^{-2}\right)}$$

$$= 1 + z^{-1} \cdot \frac{\left(1 - \frac{1}{2} z^{-1}\right)}{\left(1 - z^{-1} + \frac{1}{2} z^{-2}\right)}$$

$$= \frac{\left(1 - z^{-1} + \frac{1}{2} z^{-2}\right) + \left(z^{-1} - \frac{1}{2} z^{-2}\right)}{\left(1 - z^{-1} + \frac{1}{2} z^{-2}\right)} = \frac{1}{\left(1 - z^{-1} + \frac{1}{2} z^{-2}\right)}$$

---

b)

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{\left(1 - z^{-1} + \frac{1}{2} z^{-2}\right)}$$

$$Y(z) - z^{-1} Y(z) + \frac{1}{2} z^{-2} Y(z) = X(z)$$

$$\Big\downarrow \mathcal{Z}^{-1}$$

$$\boxed{y(n) - y(n-1) + \frac{1}{2} y(n-2) = x(n)} \quad \text{Diff. eq.}$$

$y(-1) = 1$
$y(-2) = 0$

$$\Big\downarrow \mathcal{Z}^{+}$$

$$Y^+(z) - \Big[z^{-1} Y^+(z) + \overset{=1}{y(-1)}\Big] + \frac{1}{2}\Big[z^{-2} Y^+(z) + \overset{=0}{y(-2)} + \overset{=1}{y(-1)} \cdot z^{-1}\Big] = X(z)$$

$$Y^+(z)\left[1 - z^{-1} + \tfrac{1}{2} z^{-2}\right] = X(z) + \left(1 - \tfrac{1}{2} z^{-1}\right)$$

$$Y^+(z) = \underbrace{\frac{1}{\left(1 - z^{-1} + \frac{1}{2} z^{-2}\right)} \cdot X(z)} + \underbrace{\frac{1 - \frac{1}{2} z^{-1}}{\left(1 - z^{-1} + \frac{1}{2} z^{-2}\right)}}$$

$x(n) = u(n)$

$$Y_1^+(z) = \frac{1}{\left(1 - z^{-1} + \frac{1}{2} z^{-2}\right)} \cdot \frac{1}{\left(1 - z^{-1}\right)}$$

$$\Big\downarrow \mathcal{Z}^{-1}$$

$$y_2(n) = \left(\tfrac{1}{\sqrt{2}}\right)^n \cos\left(\tfrac{\pi}{4} \cdot n\right) \cdot u(n)$$

(3) cont.

$$Y_1^+(z) = \frac{1}{(1-z^{-1}+\frac{1}{2}z^{-2})}\,\frac{1}{(1-z^{-1})}\quad \frac{\times z^3}{\times z^3}$$

$$= \frac{z^3}{(z^2-z+\frac{1}{2})(z-1)}$$

$$\frac{Y_1^+(z)}{z} = \frac{z^2}{(z-p_1)(z-p_1^*)(z-1)} = \frac{A_1}{(z-p_1)} + \frac{A_1^*}{(z-p_1^*)} + \boxed{\frac{A_3}{(z-1)}}$$

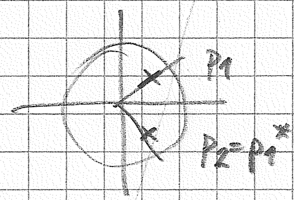
p_1 and p_2:

$$z^2 - z + \frac{1}{2} = 0$$

$$z_{1,2} = \frac{1}{2} \pm \sqrt{\underbrace{\left(\frac{1}{2}\right)^2 - \frac{1}{2}}_{-\frac{1}{4}}} = \frac{1}{2} \pm j\frac{1}{2}$$

so,

$$\begin{cases} p_1 = \frac{1}{2} + j\frac{1}{2} = \frac{1}{\sqrt{2}}\cdot e^{j\pi/4} = r_1\cdot e^{j\beta_1\cdot n} \\ p_2 = p_1^* = \frac{1}{2} - j\frac{1}{2} = \frac{1}{\sqrt{2}}\, e^{-j\pi/4} \end{cases}$$



$$A_1 = (z-p_1)\frac{Y_1^+(z)}{z}\Big|_{z=p_1} = \frac{p_1^2}{(p_1-p_1^*)(p_1-1)} = \frac{(\frac{1}{2}+j\frac{1}{2})^2}{j(-\frac{1}{2}+j\frac{1}{2})} = \frac{\frac{1}{4}-\frac{1}{4}+j\frac{1}{2}}{-\frac{1}{2}-j\frac{1}{2}}$$

$$= \frac{j\frac{1}{2}}{(-\frac{1}{2}-j\frac{1}{2})} = \frac{j\frac{1}{2}(-\frac{1}{2}+j\frac{1}{2})}{1/2} = -\frac{1}{2} - j\frac{1}{2}$$

$$= \underline{\frac{1}{\sqrt{2}}\, e^{j(-\frac{3\pi}{4})}}$$

$$A_3 = (z-1)\frac{Y_1^+(z)}{z}\Big|_{z=1} = \frac{1^2}{(1-p_1)(1-p_1^*)} = \frac{1}{(\frac{1}{2}-j\frac{1}{2})(\frac{1}{2}+j\frac{1}{2})} = \frac{1}{\frac{1}{2}} = \underline{2}$$

z^{-1}

$$2\cdot|A_1|\cdot(r_1)^n\cos(\beta_1 n + \mathrm{Arg}\{A_1\})\cdot u(n)$$

$$= 2\cdot\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)^n\cos\left(\frac{\pi}{4}n - \frac{3\pi}{4}\right)\cdot u(n)$$

$$\frac{2}{(1-z^{-1})} \xrightarrow{z^{-1}} 2\cdot u(n)$$

$$y(n) = \underbrace{\left(\frac{1}{\sqrt{2}}\right)^n\cos\left(\frac{\pi}{4}n\right)\cdot u(n) + 2\cdot\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)^n\cos\left(\frac{\pi}{4}n-\frac{3\pi}{4}\right)\cdot u(n)}_{\text{transient response } \to 0 \text{ when } n \text{ large}} + \underbrace{2\cdot u(n)}_{\text{steady state response}}$$

c)

(4)

a) "Peaks" at frequencies:

$$k = 40, 80, 160 \quad (352, 432, 472)$$

negative freq.

$$\omega_k = \frac{2\pi}{N} \cdot k$$

$$f_k = \frac{k}{N} \quad ; N = 512$$

$$F = f \cdot F_s \; ; \; F_s = 20\,kHz$$

$$\Rightarrow f_{40} = \frac{40}{512} \Longrightarrow F_{40} = \frac{40}{512} \cdot 20\,kHz \approx 1.6\,kHz$$

$$f_{80} = \frac{80}{512} \qquad F_{80} = \frac{80}{512} \cdot 20\,kHz \approx 3.1\,kHz$$

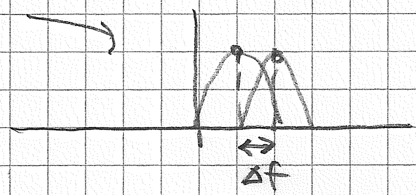
$$f_{160} = \frac{160}{512} \qquad F_{160} = \frac{160}{512} \cdot 20\,kHz \approx 6.3\,kHz.$$

b) Rectangular window. $F_s = 20\,kHz$

$$W(f) = \frac{\sin(\pi \cdot f \cdot M)}{\sin(\pi \cdot f)} \quad ; M = \text{length of the window}$$

half-width of the main lobe $\leftrightarrow$ first zero crossing

when $\pi \cdot f \cdot M = \pi \Rightarrow f = \frac{1}{M}$



So, $\underbrace{\Delta f}_{(f_2 - f_1)} = \frac{1}{M}$

$$\Updownarrow$$

$$\frac{\Delta F}{F_s} = \frac{1}{M} \Rightarrow \Delta F = \frac{F_s}{M} \quad \text{Analog freq. resolution}$$

$$\left.\begin{matrix} F_s = 20\,kHz \\ M = L = 256 \end{matrix}\right\} \Rightarrow \Delta F = \frac{20\,kHz}{256} \approx 80\,Hz$$

5

cont. c) $F_S = 30\,kHz$

$$\Delta F = \frac{F_S}{256} = \frac{30\,kHz}{256} \approx 120\,Hz$$

To get better freq. resolution you must increase the length L of the window.

$$\Delta F = \frac{F_S}{L} = \frac{20\,kHz}{256}$$

(ΔF in 5)

$$\Rightarrow L = \frac{F_S}{20\,kHz} \cdot 256 = \frac{30}{20} \cdot 256 = 384$$