1

a) $x_1(n) = 3\cos\left(\frac{\pi}{3}(n-2)\right) = 3\cos\left(\frac{2\pi}{6}(n-2)\right)$

$f = \frac{1}{6} \Rightarrow N = 6$

F-serie expansion when $\underline{\underline{N=6}}$

$$x_1(n) = \sum_{k=0}^{5} c_k e^{j\frac{2\pi}{6}\cdot k\cdot n} = c_0 + c_1 e^{j\frac{2\pi}{6}\cdot 1\cdot n} + \dots + c_5 e^{j\frac{2\pi}{6}\cdot 5\cdot n}$$

But:

$$x_1(n) = 3\cos\left(\frac{2\pi}{6}(n-2)\right) \overset{\text{Euler}}{=} 3\cdot\frac{1}{2}\left[e^{j\frac{2\pi}{6}(n-2)} + e^{-j\frac{2\pi}{6}(n-2)}\right]$$

$$= \frac{3}{2}\left[e^{j\frac{2\pi}{6}\cdot n}\, e^{-j\frac{2\pi}{3}} + e^{-j\frac{2\pi}{6}\cdot n}\cdot e^{j\frac{2\pi}{3}}\right]$$

$$= \underbrace{\frac{3}{2}\cdot e^{-j\frac{2\pi}{3}}}_{=c_1} e^{j\frac{2\pi}{6}\cdot n} + \underbrace{\frac{3}{2}\cdot e^{j\frac{2\pi}{3}}}_{=c_{-1}} e^{-j\frac{2\pi}{6}\cdot n}$$

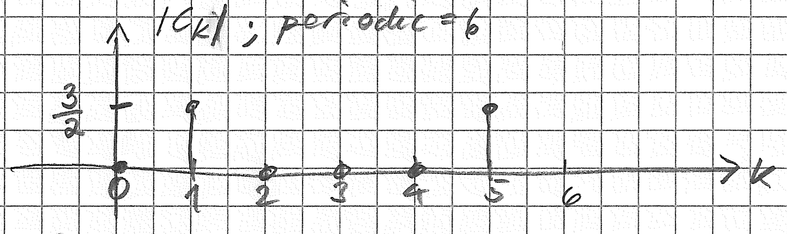
and c_k is periodic N, i.e. $c_{k+6} = c_k$

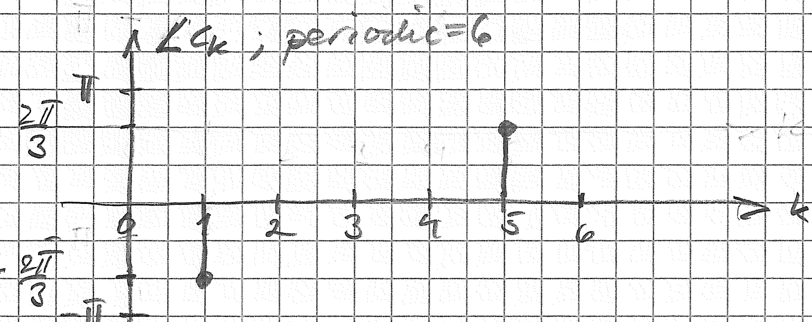
$\Rightarrow c_{-1} = c_5$

So $c_0 = c_2 = c_3 = c_4 = 0$

$c_1 = \frac{3}{2} e^{-j\frac{2\pi}{3}} \Rightarrow |c_1| = \frac{3}{2} \quad \angle c_1 = -\frac{2\pi}{3}$

$c_5 = c_{-1} \Rightarrow |c_5| = \frac{3}{2} \quad \angle c_5 = \frac{2\pi}{3}$





① cont.

b)

$$x_2(n) = 0.5\cos\left(\frac{2\pi}{3}\cdot n\right) + 0.8\sin\left(\frac{2\pi}{5}\cdot n\right)$$

$$= 0.5\cos\left(\frac{10\pi}{15}\cdot n\right) + 0.8\sin\left(\frac{6\pi}{15}\cdot n\right) \qquad \Rightarrow \underline{\underline{N=15}}$$

F-serie expansion when $N=15 \Rightarrow x_2(n) = \sum_{k=0}^{14} c_k e^{j\frac{2\pi}{15}\cdot k\cdot n}$

and linear $c_k = c_k^1 + c_k^2$

$$0.5\cos\left(\frac{10\pi}{15}\cdot n\right) \overset{\text{Euler}}{=} 0.5\,\frac{1}{2}\left[e^{j\frac{10\pi}{15}\cdot n} + e^{-j\frac{10\pi}{15}\cdot n}\right]$$

$$= \frac{1}{4}\cdot e^{j\frac{10\pi}{15}\cdot n} + \frac{1}{4}e^{-j\frac{10\pi}{15}\cdot n}$$

$$= \underbrace{\frac{1}{4}}_{=c_5^1} e^{j\frac{2\pi}{15}\cdot \underline{\underline{5}}\cdot n} + \underbrace{\frac{1}{4}}_{=c_{-5}^1} e^{-j\frac{2\pi}{15}\cdot \underline{\underline{5}}\cdot n}$$

$$= c_{-5}^1 \overset{\text{periodic } N=15}{=} c_{15-5}^1 = c_{10}^1$$

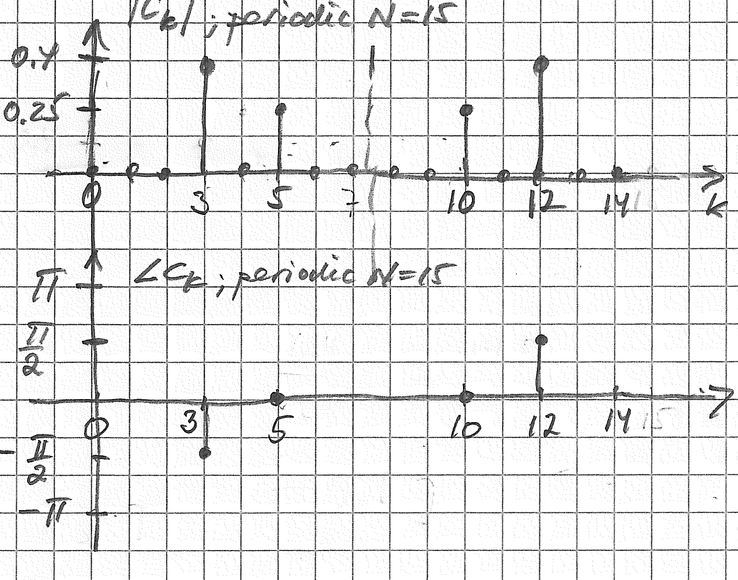
$$0{,}8\sin\left(\frac{6\pi}{15}\cdot n\right) \overset{\text{Euler}}{=} 0.8\,\frac{1}{2j}\left[e^{j\frac{6\pi}{15}\cdot n} - e^{-j\frac{6\pi}{15}\cdot n}\right]$$

$$= \underbrace{\frac{0.4}{j}}_{=c_3^2} e^{j\frac{2\pi}{15}\cdot\underline{\underline{3}}\cdot n} \underbrace{-\frac{0.4}{j}}_{c_{-3}^2} e^{-j\frac{2\pi}{15}\cdot\underline{\underline{3}}\cdot n}$$

$$c_{-3}^2 \overset{\text{periodic } N=15}{=} c_{15-3}^2 = c_{12}^2$$

$$\text{Linear} \Rightarrow c_k = c_k^1 + c_k^2 = \begin{cases} \frac{0.4}{j} = 0.4\cdot e^{-j\frac{\pi}{2}} & k=3 \\ \frac{1}{4} & k=5 \\ \frac{1}{4} & k=10 \\ -\frac{0.4}{j} = 0.4\,e^{j\frac{\pi}{2}} & k=12 \\ 0 & \text{otherwise} \end{cases}$$

$k = 0,1,2,\ldots 14$

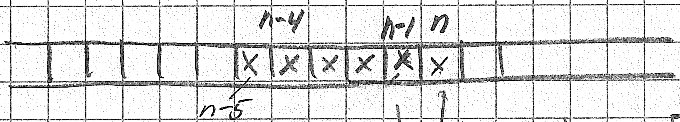


(2)

a) $y(n) = 0.2\left[x(n) + x(n-1) + \cdots + x(n-4)\right]$ (*) nonrecursive system

Recursive system?

$y(n) = y(n-1) + ?$



According to diff eq (*)

$\to y(n) = 0.2\left[x(n) + x(n-1) + \cdots + x(n-4)\right]$

$\to y(n-1) = 0.2\left[x(n-1) + x(n-2) + \cdots + x(n-4) + x(n-5)\right]$

$$\Rightarrow y(n) = y(n-1) + 0.2\,x(n) - 0.2\,x(n-5)$$

b) when $x(n) = \delta(n) \Rightarrow y(n) = h(n)$

from the diff. eq (*)

$h(n) = 0.2\left[\delta(n) + \delta(n-1) + \delta(n-2) + \delta(n-3) + \delta(n-4)\right]$

$$\Rightarrow h(n) = \{\underset{\uparrow}{\overset{0}{0.2}}, \overset{1}{0.2}, \overset{2}{0.2}, \overset{3}{0.2}, \overset{4}{0.2}\}$$

c) $$H(\omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-j\omega n} = \sum_{n=0}^{4} 0.2\, e^{-j\omega n}$$

$$= 0.2 \sum_{n=0}^{4} \left(e^{-j\omega}\right)^n = \begin{cases} 0.2\,\dfrac{1-e^{-j\omega 5}}{1-e^{-j\omega}} & \omega \neq 0 \\ 0.2 \cdot 5 = 1 & \omega = 0 \end{cases}$$

Write $H(\omega)$ as $H_r(\omega)e^{-j\omega \cdot 2}$; length of $h(n) = 5$

$$\frac{1-e^{-j\omega 5}}{1-e^{-j\omega}} = \frac{e^{-j\omega\frac{5}{2}}\left(e^{j\omega\frac{5}{2}} - e^{-j\omega\frac{5}{2}}\right)}{e^{-j\frac{\omega}{2}}\left(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right)}$$

$$= e^{-j\omega\cdot 2}\,\frac{\sin\left(\frac{5}{2}\cdot\omega\right)}{\sin\left(\frac{\omega}{2}\right)}$$

② cont.

So

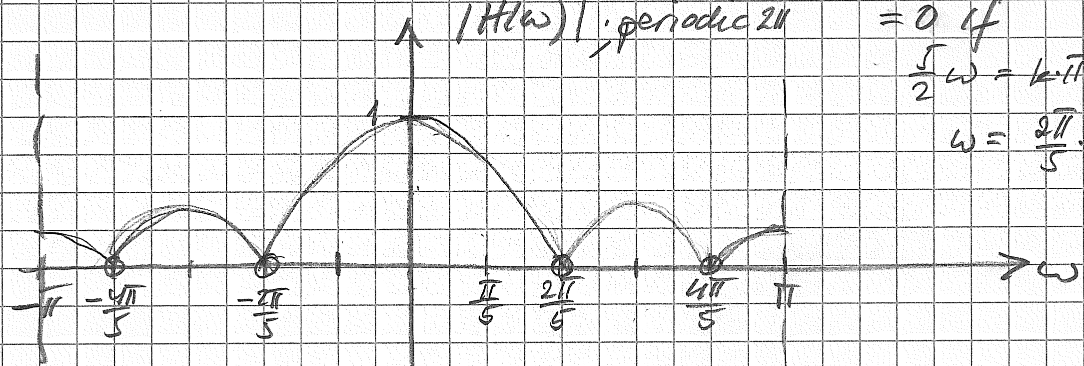
$$H(\omega)=\begin{cases} 0{,}2\cdot\dfrac{\sin\left(\frac{5}{2}\omega\right)}{\sin\left(\frac{\omega}{2}\right)}\cdot e^{-j2\omega} & \omega\neq 0 \\ 1 & \omega = 0 \end{cases}$$

Magnitude function

$$|H(\omega)| = \begin{cases} 0.2\left|\dfrac{\sin\left(\frac{5}{2}\omega\right)}{\sin\left(\frac{\omega}{2}\right)}\right| & \omega\neq 0 \\ 1 & \omega=0 \end{cases}$$

Phase function

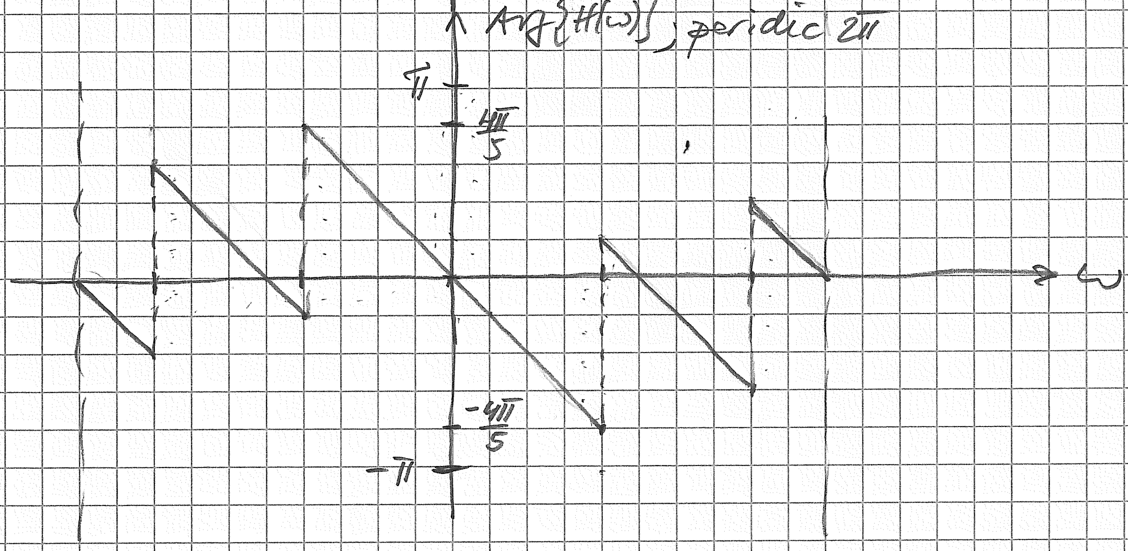
$$\mathrm{Arg}\{H(\omega)\} = -2\omega + \begin{cases} 0 \\ \pi \end{cases}$$



$= 0$ if

$\frac{5}{2}\omega = k\cdot\pi$; $k=\pm1,\pm2,\ldots$

$\omega = \frac{2\pi}{5}\cdot k$



③

a)

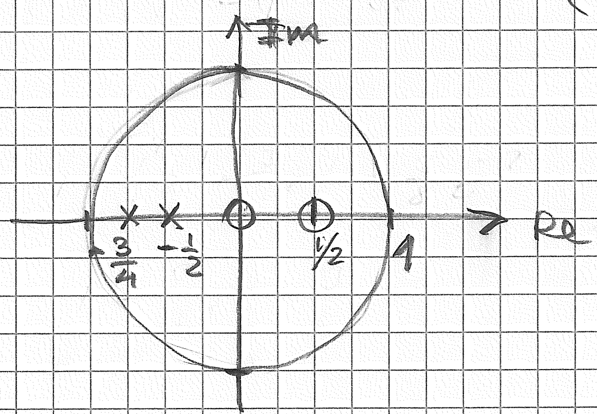
$$h(n) = \left[5\left(-\frac{3}{4}\right)^n - 4\left(-\frac{1}{2}\right)^n\right] u(n)$$

$$\downarrow \mathcal{Z}$$

$$H(z) = 5\,\frac{1}{1+\frac{3}{4}z^{-1}} - 4\,\frac{1}{1+\frac{1}{2}z^{-1}}$$

$$= \frac{5\left(1+\frac{1}{2}z^{-1}\right) - 4\left(1+\frac{3}{4}z^{-1}\right)}{\left(1+\frac{3}{4}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)} = \frac{\left(1-\frac{1}{2}z^{-1}\right)}{\left(1+\frac{5}{4}z^{-1}+\frac{3}{8}z^{-2}\right)} \begin{matrix}\times z^2 \\ \times z^2\end{matrix}$$

$$= \frac{z\left(z-\frac{1}{2}\right)}{\left(z+\frac{3}{4}\right)\left(z+\frac{1}{2}\right)} \Rightarrow \begin{cases} \text{zeros: } z_1 = 0\ ;\ z_2 = \frac{1}{2} \\ \text{poles: } p_1 = -\frac{3}{4}\ ;\ p_2 = -\frac{1}{2} \end{cases}$$



The system is stable!

causal system and $|p_k| < 1 \quad k = 1, 2$

b)

$$x(n) = 5.8 + 0.1\cos\left(\frac{5\pi}{6}\cdot n\right)$$

steady state response LTI-system

$$\Rightarrow y(n) = \underline{H(0)}\cdot 5.8 + \underline{H\left(\frac{5\pi}{6}\right)}\cdot 0.1\cos\left(\frac{5\pi}{6}\cdot n + \underline{\Theta\left(\frac{5\pi}{6}\right)}\right)$$

Compute $H(\omega) = H(z)\big|_{z=e^{j\omega}}$

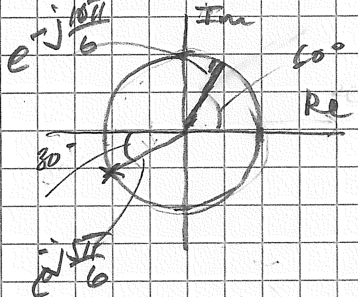
$$H(\omega) = \frac{\left(1-\frac{1}{2}z^{-1}\right)}{\left(1+\frac{5}{4}z^{-1}+\frac{3}{8}z^{-2}\right)}\Bigg|_{z=e^{j\omega}}$$

$$= \frac{\left(1-\frac{1}{2}e^{-j\omega}\right)}{\left(1+\frac{5}{4}e^{-j\omega}+\frac{3}{8}e^{-j2\omega}\right)}$$

③ cont.

$$H(0) = \frac{\left(1-\frac{1}{2}\right)}{\left(1+\frac{5}{4}+\frac{3}{8}\right)} = \frac{\frac{1}{2}}{\frac{21}{8}} = \frac{1\cdot 8}{2\cdot 21} = \frac{4}{21} \approx 0{,}2$$

$$H\left(\frac{5\pi}{6}\right) = \frac{\left(1-\frac{1}{2}e^{-j\frac{5\pi}{6}}\right)}{\left(1+\frac{5}{4}e^{-j\frac{5\pi}{6}}+\frac{3}{8}e^{-j\frac{10\pi}{6}}\right)}$$



$$\Rightarrow \begin{cases} e^{-j\frac{5\pi}{6}} = -\frac{\sqrt{3}}{2} - j\frac{1}{2} \\ e^{-j\frac{10\pi}{6}} = \frac{1}{2} + j\frac{\sqrt{3}}{2} \end{cases}$$

$$= \frac{\left(1-\frac{1}{2}\left(-\frac{\sqrt{3}}{2}-j\frac{1}{2}\right)\right)}{\left(1+\frac{5}{4}\left(-\frac{\sqrt{3}}{2}-j\frac{1}{2}\right)+\frac{3}{8}\left(\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)\right)}$$

$$= \frac{\left(1+\frac{\sqrt{3}}{4}+j\frac{1}{4}\right)}{\left(1-\frac{5\cdot\sqrt{3}}{8}-j\frac{5}{8}+\frac{3}{16}+j\frac{3\sqrt{3}}{16}\right)} = \frac{\left(\frac{4+\sqrt{3}}{4}+j\frac{1}{4}\right)}{\frac{19-10\cdot\sqrt{3}}{16}+j\frac{3\sqrt{3}-10}{16}}$$

$$= \frac{4\left(4+\sqrt{3}+j\right)}{\left(19-10\cdot\sqrt{3}\right)+j\left(3\sqrt{3}-10\right)} \approx \frac{22{,}9+j4}{1.7-j4.8}$$

$$\approx \frac{23{,}2\cdot e^{j10^\circ}}{5{,}1\cdot e^{-j70^\circ}} \approx 4.6\, e^{j80^\circ}$$

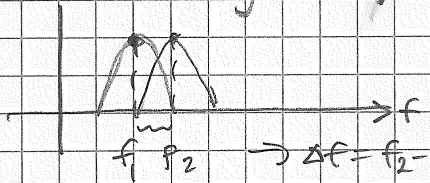
So $y(n) = 0{,}2\cdot 5{,}8 + 4{,}6\cdot 0.1\cos\left(\frac{5\pi}{6}n + 80^\circ\right)$

$$\approx 1{,}2 + 0.46\cos\left(\frac{5\pi}{6}n + 80^\circ\right)$$

(4)

a) $F_s = 15\,kHz$
$\Delta F = 60\,Hz$

Rectangular window with a length of M : $W(f) = \dfrac{\sin(\pi \cdot f \cdot M)}{\sin(\pi \cdot f)}$



$\rightarrow \Delta f = f_2 - f_1$

Δf = half width of the main lobe of $W(f)$

Half width $\Leftrightarrow$ 1:st zero crossing

$\Rightarrow \quad \pi \cdot f \cdot M = k \cdot \pi \; ; \; k = 1$

$f = \dfrac{1}{M}$

See figure $\Rightarrow \quad \Delta f = \dfrac{1}{M}$

$f_2 - f_1 = \dfrac{1}{M}$

$\Rightarrow \quad \dfrac{F_2}{F_s} - \dfrac{F_1}{F_s} = \dfrac{1}{M}$

$f = \dfrac{F}{F_s}$

$\dfrac{\Delta F}{F_s} = \dfrac{1}{M}$

$M = \dfrac{F_s}{\Delta F} = \dfrac{15k}{60} = \underline{\underline{250}}$

Select $M = 256$

---

b) $F_s = 15\,kHz \longrightarrow 20\,kHz$

$\Delta F = \dfrac{F_s}{M}$ from a)

$F_s = 15\,kHz \Rightarrow \dfrac{15k}{M}$

$F_s = 20\,kHz \Rightarrow \dfrac{20k}{M}$

$\left.\begin{matrix} \\ \end{matrix}\right\} \xrightarrow[\text{change}]{} \dfrac{15k}{M} \Big/ \dfrac{20k}{M} = \dfrac{3}{4}.$

So the freq resolution is lowered by 0.25.

---

(4) cont.

c) $F_s = 15\,kHz$

DFT in $N = 1024$ points

Peaks at $k = 102, 314, 396$ in the figure

(628, 710, 922) are neg. freq.

Scale of the freq axis:

$$k = N \longleftrightarrow F = F_s$$

$$\Rightarrow k = 1024 \longleftrightarrow F = 15\,kHz$$

$$k = 102 \Rightarrow F_1 = \frac{102}{1024} \cdot 15\,kHz \approx \underline{1500\,Hz}$$

$$k = 314 \Rightarrow F_2 = \frac{314}{1024} \cdot 15\,kHz \approx \underline{4600\,Hz}$$

$$k = 922 \Rightarrow F_3 = \frac{396}{1024} \cdot 15\,kHz \approx \underline{5800\,Hz}.$$