Convolution.

1. The characteristic function of a set.

Given a set $A \subseteq U$, we call "the characteristic function of A" a function $L : U \rightarrow R$ such that

$$L(a)= \begin{cases} 1 & \text{iff } a \in A \\ 0 & \text{iff } a \in U \setminus A \end{cases}$$

Example a): Characteristic function of a 2D interval :

Let $U = R^2$ and A be the interval [-X/2, X/2] x [-Y/2, Y/2] $\subseteq U$: then

$$L(x, y)= \begin{cases} 1, & \text{if } x \in [-X/2, X/2] \text{ and } y \in [-Y/2, Y/2] \\ 0, & \text{otherwise} \end{cases}$$

is the characteristic function of A. In this case, L is a separable function. That is:

$$L(x, y)=L_1(x) \bullet L_2(y)$$

Where L_1: R→R is the characteristic function of [−X/2, X/2]⊆R and L_2: R→R is the characteristic function of [−Y/2, Y/2] (*write down the definitions of these two functions explicitly*).

Since the 2−dimensional Fourier Transform (FT) also acts separately on the two coordinates, we have

$$FT\{L(x, y)\} = FT\{L_1(x)\} \bullet FT\{L_2(y)\}$$

Generate the image L and display it.

```
L=zeros(64,64);                 %zero matrix 64 x 64
L(29:35,29:35)=ones(7,7);       %put in a white square 7 x 7
subplot(2,2,1); imshow(L);      %show it
```

Calculate the FT of the image L and display the absolute value of the FT.

```
FL=fft2(L);    %2D DFT
subplot(2,2,2); imshow(fftshift(log10(abs(FL)))); %show the centralized DFT
```

Questions:

Make the interval (=white square) smaller (5 x 5) and subsequently larger (11 x 11) in the image L. Display the images and their absolute FT:s.
Express how scaling in one domain affects the scale of objects in the other domain.

Translate the white square, of size 7 x7, from the middle to one of the corners in the image. Display the image and its FT.
What is the effect of translation in the image domain on the Fourier transform?

Example b): Characteristic function of a 2D circle:
As another example, we consider a set A corresponding to a circle of radius r and center c:

$$A=\{(x,y) \in R^2 \text{ for } \|(x,y)-c\| \leq r\}$$

The characteristic function L (of which the graph is a white circle on a dark background) is not separable in this case.
Use the MATLAB function **d=circle(r)** to generate a circle of radius r=10. Display the circle and the magnitude of its FT.

d=circle(10); %generate a white circle on a black background
D=fft2(d); %2D DFT
subplot(2,2,1); imshow(d); %show the circle
subplot(2,2,2); imagesc(fftshift(abs(D))); colormap(gray); axis image %and its DFT

Note: The image d is circularly symmetric. This is preserved in the Fourier transform. The radial cross section of the absolute value of the transform is a Bessel function of the first order.

Scale the circle (use a radius of 20). Display it along with the absolute value of its FT.
Note the relation between the two domains when scaling.

2. Convolution.

Convolution in the image domain can be performed by multiplication in the Fourier domain.
That is:

$$(f * g)(x) = \text{IFT}\{ \text{FT}(f(x)) \bullet \text{FT}(g(x)) \}$$

Example a): synthetic images
Generate two images **f =ones(5,5);** and **g=ones(3,3);**.
Compute the convolution y = (f * g) in the spatial domain (**y=conv2(f,g);**).

Compute the convolution in the frequency domain by multiplication, by using DFT and inverse DFT (IDFT).
That is:

$$y = \text{IDFT}\{ \text{DFT}(f) \bullet \text{DFT}(g) \}$$

%do convolution by DFT, multiplication, and IDFT
F=fft2(f,5,5); % DFT in 5 x 5 points (N=5)
G=fft2(g,5,5); % both DFT must be of equal size (later on multiplication!)
Y=F.*G; % point multiplication
yy=real(ifft2(Y)); % convolved image

Print the matrices **y** and **yy** on the screen (and print them on a paper).

Questions:
Explain the differences between the matrices y and yy (hint: periodic images, period=N, when using the DFT).
Generate by hand the matrix yy from y (hint: indices are calculated modulo N).

Choose the correct value on N. Do now the convolution of f and g by using DFT.
Check that y equals yy!

Example b): real images
Load the image ’flowers.jpg’, crop it to 256 x 256 pixels and cast it to type double.

f=imread(’flowers.jpg’);
f=double(f(1:256,1:256));

Generate the (normalized) low pass filter **g=ones(21,21)/(21*21);**.
Compute the convolution y = (f * g) (**y=conv2(f,g);**).
Display the original image f and the convolved image y (**subplot(2,2,1); imshow(f/255);**).

Now compute the convolution in the frequency domain by multiplication. Use the DFT and the inverse DFT (IDFT) as follows:

$$yy = \text{IDFT}\{\ \text{DFT}(f) \bullet \text{DFT}(g)\ \}$$

First, compute yy1 by using 256 x 256 points in the DFT (which is the size of the original image). Display the convolved image yy1 (**subplot; imshow(yy1/255);**).

Second, compute yy2 by using 276 x 276 points in the DFT. Display the convolved image yy2 (**subplot; imshow(yy2/255);**).

Questions:
Compare the two images yy1 and yy2. Explain the differences between them and with respect to image y.
In the second case you used 276 x 276 points when calculating the DFT:s. Why just 276 x 276 points?

*“For fun”: try if you manage to reconstruct yy1 from the image yy2 (refer to the simpler case in Example a) above). This gives a new image (let’s call it “new”). Display the image “new” and compute the norm of the difference between the images “new” and “yy1” (**e=yy1−new; error = sum(sum(e.*e));**).*