(1)

$$H(z) = \frac{\left(3 - \frac{10}{3} z^{-1}\right)}{\left(1 - \frac{11}{6} z^{-1} + \frac{1}{2} z^{-2}\right)}$$

Find the poles for $H(z)$. and do partial fraction expansion.

$$H(z) = \frac{z\left(3z - \frac{10}{3}\right)}{\left(z^2 - \frac{11}{6} z + \frac{1}{2}\right)} = \frac{z\left(3z - \frac{10}{3}\right)}{(z-p_1)(z-p_2)}$$

Poles: $z^2 - \frac{11}{6} z + \frac{1}{2} = 0$

$$z_{1,2} = \frac{11}{12} \pm \sqrt{\underbrace{\frac{121}{144} - \frac{1}{2}}_{\frac{49}{144}}} = \frac{11}{12} \pm \frac{7}{12}$$

$$\begin{cases} p_1 = \frac{18}{12} = \underline{\underline{\frac{3}{2}}} \\ p_2 = \frac{4}{12} = \underline{\underline{\frac{1}{3}}} \end{cases}$$

$$\frac{H(z)}{z} = \frac{\left(3z - \frac{10}{3}\right)}{\left(z - \frac{3}{2}\right)\left(z - \frac{1}{3}\right)} = \frac{A_1}{\left(z - \frac{3}{2}\right)} + \frac{A_2}{\left(z - \frac{1}{3}\right)}$$

$$A_1 = \left(z - \frac{3}{2}\right) \frac{H(z)}{z} \Big|_{z=\frac{3}{2}} = \frac{3 \cdot \frac{3}{2} - \frac{10}{3}}{\left(\frac{3}{2} - \frac{1}{3}\right)} = \frac{7/6}{7/6} = 1$$

$$A_2 = \left(z - \frac{1}{3}\right) \frac{H(z)}{z} \Big|_{z=\frac{1}{3}} = \frac{3 \cdot \frac{1}{3} - \frac{10}{3}}{\left(\frac{1}{3} - \frac{3}{2}\right)} = \frac{-7/3}{-7/6} = 2$$

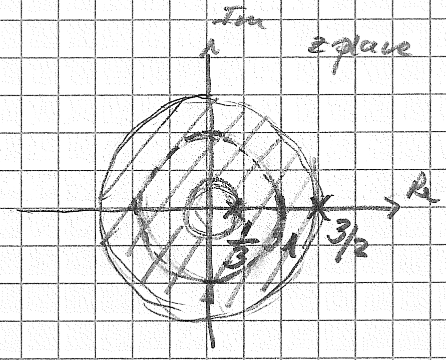
$$H(z) = \frac{1}{\left(1 - \frac{3}{2} z^{-1}\right)} + \frac{2}{\left(1 - \frac{1}{3} z^{-1}\right)}$$

(1) cont.

a) The system is <u>stable</u>

stable $\Leftrightarrow$ $|z|=1 \in ROC$

ROC: $\frac{1}{3} < |z| < \frac{3}{2}$



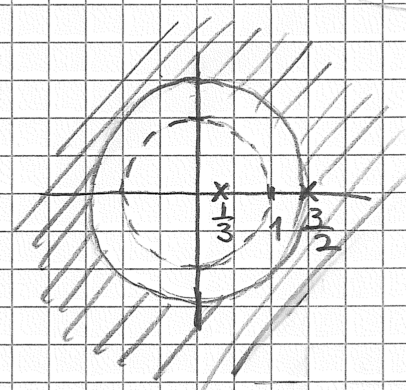
$$H(z) = \frac{2}{(1-\frac{1}{3}z^{-1})} + \frac{1}{(1-\frac{3}{2}z^{-1})}$$

$$\xrightarrow{z^{-1}}$$

$$h(n) = \underbrace{2\cdot\left(\frac{1}{3}\right)^n \cdot u(n)}_{\text{causal part}} - \underbrace{\left(\frac{3}{2}\right)^n u(-n-1)}_{\text{anti-causal part}}$$

b) The system is <u>causal</u>

ROC: $|z| > \frac{3}{2}$

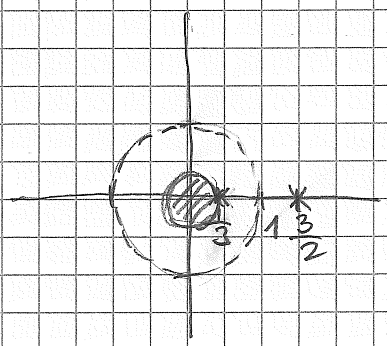


$$\xrightarrow{z^{-1}}$$

$$h(n) = 2\left(\frac{1}{3}\right)^n \cdot u(n) + \left(\frac{3}{2}\right)^n u(n)$$

c) The system is <u>anticausal</u>

ROC $|z| < \frac{1}{3}$



$$\xrightarrow{z^{-1}}$$

$$h(n) = -2\left(\frac{1}{3}\right)^n u(-n-1) - \left(\frac{3}{2}\right)^n u(-n-1)$$

② a) $x_1(n) = 0.8\cos\left(\frac{3\pi}{5}(n-1)\right) \quad -\infty < n < \infty$

F-serie
Infinite
duration
periodic
DT-signal

$$x(n) = \sum_{k=0}^{N-1} c_k\, e^{j\frac{2\pi}{N}\cdot k\cdot n}$$

Find the period = N !

$$x_1(n) = 0.8\cos\left(\frac{3\pi}{5}\cdot n - \frac{3\pi}{5}\right) = 0.8\cos\left(\frac{2\pi}{10}\cdot 3\cdot n - \frac{3\pi}{5}\right)$$

$$\Rightarrow N = 10$$

F-series $\Rightarrow$
$$x(n) = \sum_{k=0}^{9} c_k\, e^{j\frac{2\pi}{10}\cdot k\cdot n}$$
$$= c_0 + \dots + c_3 e^{j\frac{2\pi}{10}\cdot 3\cdot n} + \dots\dots$$

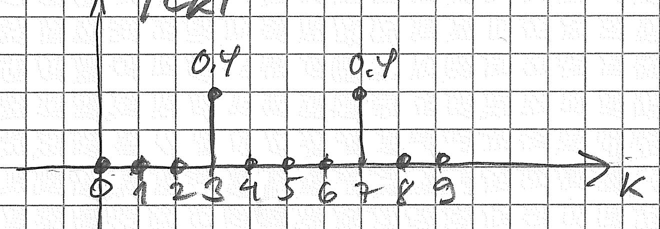
Euler id $\Rightarrow$

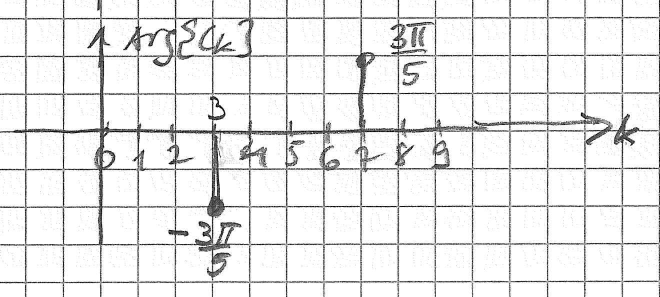
$$x_1(n) = 0.8\,\frac{1}{2}\left[e^{j\left(\frac{2\pi}{10}\cdot 3n - \frac{3\pi}{5}\right)} + e^{-j\left(\frac{2\pi}{10}\cdot 3\cdot n - \frac{3\pi}{5}\right)}\right]$$

$$= \underbrace{0.4\, e^{-j\frac{3\pi}{5}}}_{=c_3} e^{j\frac{2\pi}{10}\cdot 3\cdot n} + \underbrace{0.4\, e^{j\frac{3\pi}{5}}}_{=c_{-3} = c_7} e^{-j\frac{2\pi}{10}\cdot 3\cdot n}$$

$\boxed{c_{k+N} = c_k}$

Freq. descr $= \{c_k\}_{k=0}^{9} \Rightarrow \begin{cases} c_3 = 0.4\, e^{-j\frac{3\pi}{5}} \\ c_7 = 0.4\, e^{j\frac{3\pi}{5}} \\ \text{for other } k\text{-values} \\ c_k = 0 \end{cases}$





ALT. Solution

② a)

$$c_k = \frac{1}{10} \sum_{n=0}^{9} x_1(n)\, e^{-j\frac{2\pi}{10}\cdot k\cdot n} \qquad N=10$$

$$= \frac{1}{10} \sum_{n=0}^{9} \underbrace{0.8 \cos\left(\frac{3\pi}{5}(n-1)\right)}_{\text{Euler id}} e^{-j\frac{2\pi}{10}\cdot k\cdot n}$$

$$\frac{0.8}{2}\left[e^{j\frac{3\pi}{5}(n-1)} + e^{-j\frac{3\pi}{5}(n-1)}\right]$$

$$= 0.4\left[e^{j\frac{3\pi}{5}n} \cdot e^{-j\frac{3\pi}{5}} + e^{-j\frac{3\pi}{5}n}\, e^{j\frac{3\pi}{5}}\right]$$

$$= \frac{0.4}{10} e^{-j\frac{3\pi}{5}} \sum_{n=0}^{9} e^{-j\left(\frac{2\pi}{10}k - \frac{3\pi}{5}\right)n} + \frac{0.4}{10} e^{j\frac{3\pi}{5}} \sum_{n=0}^{9} e^{-j\left(\frac{2\pi}{10}k + \frac{3\pi}{5}\right)n}$$

$\neq 0$ only when $k=3$!
When $k=3$ the sum $=10$

$\neq 0$ only when $k=-3$
When $k=-3$ the sum $=10$

$$\Rightarrow \quad c_3 = 0.4\, e^{-j\frac{3\pi}{5}}$$

$$c_{-3} = 0.4\, e^{j\frac{3\pi}{5}} \quad ; \quad c_{-3} = c_7$$

periodic $=10$.

② cont. b)

$$w(n) = \begin{cases} 1 & 0 \le n \le 128 \\ 0 & \text{otherwize} \end{cases}$$

causal finite dur signal → ROC whole z-plane except $z=0$

$$w(n) = u(n) - u(n-129)$$

$z \downarrow$

$$W(z) = \frac{1}{1-z^{-1}} - \frac{z^{-129}}{1-z^{-1}} = \frac{1-z^{-129}}{1-z^{-1}}$$

$$W(\omega) = W(z)\Big|_{z=e^{j\omega}} \quad \text{if } |z|=1 \in \text{ROC of } W(z) \text{ / ok.}$$

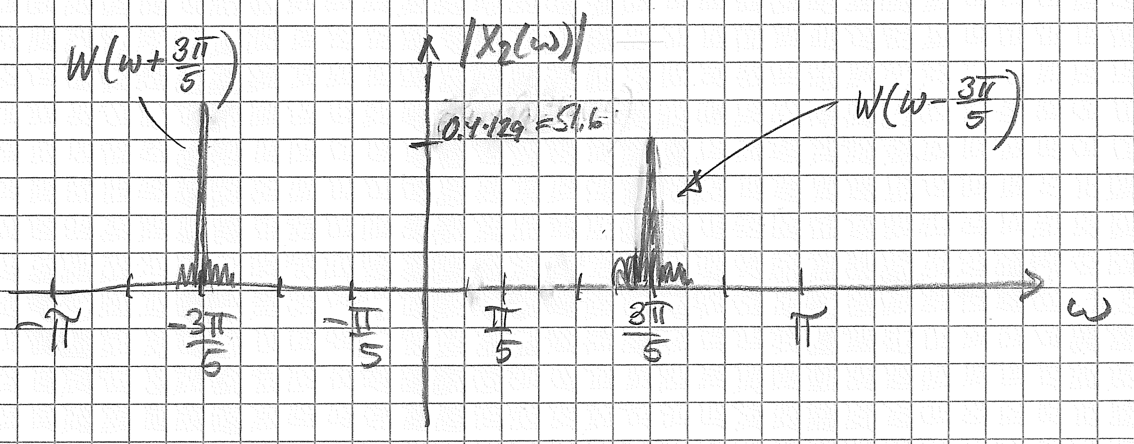
$$= \frac{1-e^{-j\omega 129}}{1-e^{-j\omega}} = \frac{e^{j\omega 129/2}\left(e^{j\omega 129/2} - e^{-j\omega 129/2}\right)}{e^{j\omega/2}\left(e^{j\omega/2} - e^{-j\omega/2}\right)}$$

$$= e^{-j64\omega} \frac{\sin\left(\frac{129}{2}\omega\right)}{\sin\left(\frac{\omega}{2}\right)} \quad \omega \neq 0$$

$$W(0) = \sum_{n=0}^{128} 1 = 129$$

zero-crossing? $\quad \frac{129}{2}\omega = k\pi \quad k = \pm 1, \pm 2, \ldots$

$$\omega = \frac{2\pi}{129} k$$



3

$$y(n) = \frac{1}{7} \cdot \sum_{k=0}^{6} x(n-k)$$

$$= \frac{1}{7}\left[x(n) + x(n-1) + \cdots x(n-6)\right]$$

a)

$$Y(z) = \frac{1}{7} X(z)\left[1 + z^{-1} + z^{-2} + \cdots + z^{-6}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{7}\left[1 + z^{-1} + z^{-2} + \cdots + z^{-6}\right]$$

$$= \frac{1}{7}\sum_{k=0}^{6} z^{-k} = \frac{1}{7}\sum_{k=0}^{6}\left(z^{-1}\right)^{k}$$

Finite geom series

$$= \begin{cases} \frac{1}{7} \cdot \frac{1-z^{-7}}{1-z^{-1}} & \text{if } z^{-1} \neq 1 \\ 1 & \text{if } z^{-1} = 1 \Leftrightarrow z = 1 \end{cases}$$

Pole-zero pattern:

$$H(z) = \frac{1}{7} \frac{(1-z^{-7})}{1-z^{-1}} \frac{\times z^7}{\times z^7} = \frac{1}{7} \frac{(z^7 - 1)}{z^6(z-1)}$$

Poles:

$p_1 = 1$

$p_2 = p_3 = \cdots = p_7 = 0$

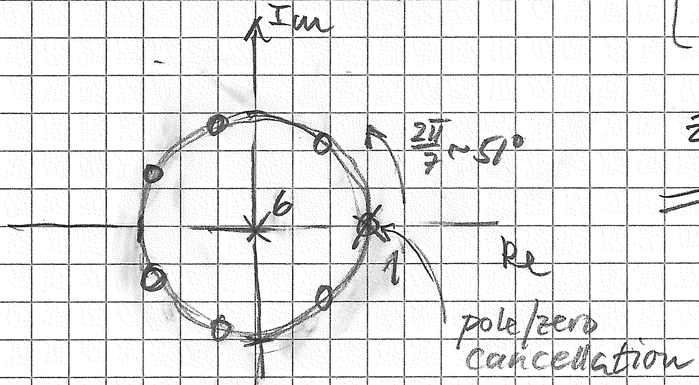
zeros:

$z^7 - 1 = 0$

$z^7 = 1 \Leftrightarrow \left(re^{j\theta}\right)^7 = 1 \cdot e^{jk \cdot 2\pi} \quad k = 0,1,2,\ldots,6$

$$\begin{cases} r^7 = 1 \rightarrow r = 1 \\ 7\theta = k \cdot 2\pi \rightarrow \theta = \frac{2\pi}{7} \cdot k \end{cases}$$

$$z_k = 1 \cdot e^{j\frac{2\pi}{7} \cdot k} \quad k = 0,1,2,\ldots,6$$



FIR-system

All-zero system

ROC: whole z-plane except $z = 0$.

3 cont.

b)

$H(\omega) = H(z)\big|_{z=e^{j\omega}}$ if $|z|=1 \in ROC$ of $H(z)$ / OK.

$$H(\omega) = \begin{cases} \frac{1}{7} \cdot \frac{1-e^{-j\omega 7}}{1-e^{-j\omega}} & \omega \neq 0 \\ 1 & \omega = 0 \end{cases}$$

$$\underline{H(\omega)} = \frac{1}{7} \cdot \frac{1-e^{-j\omega 7}}{1-e^{-j\omega}} = \frac{1}{7} \frac{e^{-j\frac{7}{2}\omega}\left(e^{j\frac{7}{2}\omega} - e^{-j\frac{7}{2}\omega}\right)}{e^{-j\frac{\omega}{2}}\left(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right)}$$

$$= \frac{1}{7} e^{-j3\omega} \frac{\sin\left(\frac{7}{2}\omega\right)}{\sin\left(\frac{\omega}{2}\right)}$$

$$|H(\omega)| = \frac{1}{7} \left| \frac{\sin\left(\frac{7}{2}\omega\right)}{\sin\left(\frac{\omega}{2}\right)} \right|$$

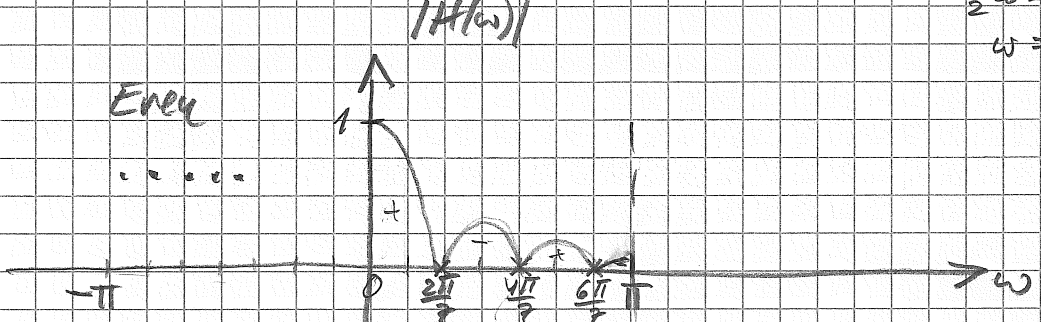
$$Arg\{H(\omega)\} = -3\omega + \begin{cases} 0 \\ \pi \end{cases}$$

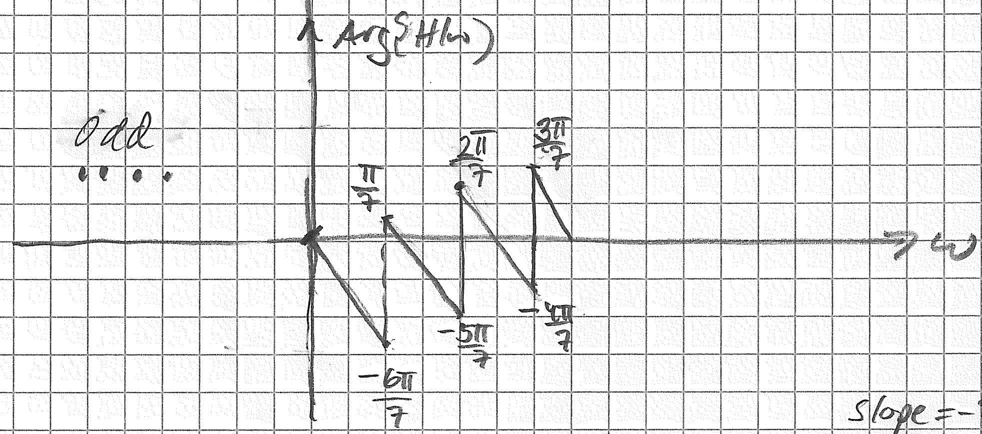
zero?

$\frac{7}{2}\omega = k \cdot \pi$

$\omega = \frac{2\pi}{7} \cdot k$

$k = \cancel{0}, 1, 2, 3$





slope $= -3\omega$

when $\omega = \frac{2\pi}{7}$

$\rightarrow -\frac{6\pi}{7}$

3 cont.

c) Steady state response

$$\Rightarrow y(n) = H(0)\cdot 1.5$$
$$+ \left|H\left(\tfrac{\pi}{7}\right)\right|\cdot 0.8 \cos\left(\tfrac{\pi}{7}n - \tfrac{\pi}{7} + \operatorname{Arg}\left\{H\left(\tfrac{\pi}{7}\right)\right\}\right)$$
$$+ \left|H\left(\tfrac{4\pi}{7}\right)\right|\cdot 0.3 \sin\left(\tfrac{4\pi}{7}n + \operatorname{Arg}\left\{H\left(\tfrac{4\pi}{7}\right)\right\}\right)$$

$H(0) = 1$

$\left|H\left(\tfrac{4\pi}{7}\right)\right| = 0$ zero at $\omega = \tfrac{4\pi}{7}$

$$\begin{cases} \left|H\left(\tfrac{\pi}{7}\right)\right| = \tfrac{1}{7}\left|\dfrac{\sin\left(\tfrac{7}{2}\cdot\tfrac{\pi}{7}\right)}{\sin\left(\tfrac{\pi}{7}\cdot\tfrac{1}{2}\right)}\right| = \tfrac{1}{7}\cdot\left|\dfrac{\sin\left(\tfrac{\pi}{2}\right)}{\sin\left(\tfrac{\pi}{14}\right)}\right| \approx 0,65 \\ \operatorname{Arg}\left\{H\left(\tfrac{\pi}{7}\right)\right\} = -3\cdot\tfrac{\pi}{7} + 0 = -\tfrac{3\pi}{7} \end{cases}$$

$$\underline{\underline{y(n)}} = 1\cdot 1.5 + 0.65\cdot 0.8 \cos\left(\tfrac{\pi}{7}n - \tfrac{\pi}{7} - \tfrac{3\pi}{7}\right)$$
$$= 1,5 + 0,52\cdot \underbrace{\cos\left(\tfrac{\pi}{7}n - \tfrac{4\pi}{7}\right)}_{\cos\left(\tfrac{\pi}{7}(n-4)\right)}$$

4

a)

$$h(n) = \{ \underset{\uparrow}{\tfrac{1}{3}}, \tfrac{1}{3}, \tfrac{1}{3} \}$$

$$x(n) = \{ \underset{\uparrow}{1}, 1, 1, -1, -1, -1 \}$$

Finite duration signals → graphical solution

1	1	1	-1	-1	-1

1/3	1/3	1/3

→

x	x	x	x	x	x	x	x

$$y(n) = \{ \underset{\uparrow}{\tfrac{1}{3}}, \tfrac{2}{3}, 1, \tfrac{1}{3}, -\tfrac{1}{3}, -1, -\tfrac{2}{3}, -\tfrac{1}{3} \}$$

b)

$$Y(k) = \underset{N}{DFT}\{x(n)\} \cdot \underset{N}{DFT}\{h(n)\} = X(k) \cdot H(k)$$

$$\longleftrightarrow$$

$$y(n) = x(n) \, \textcircled{N} \, h(n)$$ circ conv in the time domain

Graphical solution: $N=6$

$$x(n) = \{ \underset{\uparrow}{1}, 1, 1, -1, -1, -1 \}$$

$$h(n) = \{ \underset{\uparrow}{\tfrac{1}{3}}, \tfrac{1}{3}, \tfrac{1}{3}, 0, 0, 0 \}$$ zero-padded to length = 6.

						1	1	1	-1	-1	-1	$x(n)$
... 0	0	0	1/3	1/3	1/3	0	0	0	1/3	1/3	1/3	.... $h((-n))_6$
....						-1/3	1/3	1	1/3	-1/3	-1	.... $y_p(n)$

$$\hat{y}(n) = y_p(n) \quad , \; n = 0, 1, 2, \ldots, 5$$

$$\hat{y}(n) = \{ \underset{\uparrow}{-\tfrac{1}{3}}, \tfrac{1}{3}, 1, \tfrac{1}{3}, -\tfrac{1}{3}, -1 \}$$

ALT

$$y_p(n) = \sum_{l} y(n - l \cdot 6)$$

-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
							x	x	x	x	x	x	x	x	
x	x	x	x	x	x	x	x					x	x	x	x	x	x	x
							-1/3	1/3	1	1/3	-1/3	-1			

4 cont. c)

$F_1 = 1200$

$F_2 = 4200$

$F_3 = 6800$

Sample freq $F_S = 10kHz$

$$f = \frac{F}{F_S} \Rightarrow f_1 = \frac{1200}{10k} = \frac{12}{100} < \frac{1}{2} \text{ ok}$$

$$f_2 = \frac{4200}{10k} = \frac{42}{100} < \frac{1}{2} \text{ ok}$$

$$f_3 = \frac{6800}{10k} = \frac{68}{100} > \frac{1}{2} \text{ aliasing!}$$

and $\frac{68}{100} = -\frac{32}{100} + 1$

So f_3 is seen as the freq $\frac{32}{100}$

$$\begin{cases} f_k = \frac{k}{N} & k = 0,1,2,\ldots,N-1 \quad ; N = 1024 \\ \text{and} & F_k = f_k \cdot F_S \end{cases}$$

$$\Rightarrow F_{123} = \frac{123}{1024} \cdot 10k \approx 1200$$

$$F_{328} = \frac{328}{1024} \cdot 10k \approx 3200$$

$$F_{430} = \frac{430}{1024} \cdot 10k \approx 4200$$

Identify

$k = 123 \Rightarrow 1200Hz$

$k = 328 \rightarrow$ alias version of 6800 Hz

$k = 430 \rightarrow 4200 Hz$.