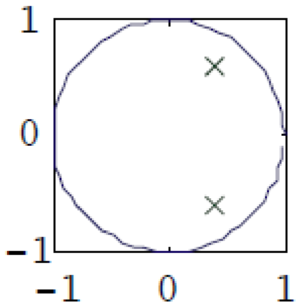
Digital Control – Exercise 1

1. $y(k) = K\frac{(1-z_1q^{-1})(1-z_2q^{-1})...(1-z_mq^{-1})}{(1-\lambda_1q^{-1})(1-\lambda_2q^{-1})...(1-\lambda_mq^{-1})}u(k)$, step response

A



From the zero-pole graph, we can see there are 2 poles inside the unit circle. Assume the 2 poles are $\frac{1}{2}+\frac{1}{2}j$ and $\frac{1}{2}-\frac{1}{2}j$ respectively, and let $K=1$.

So we can get:

$$y(k) = \frac{1}{[1-(\frac{1}{2}+\frac{1}{2}j)q^{-1}][1-(\frac{1}{2}-\frac{1}{2}j)q^{-1}]}u(k) = \frac{1}{1-q^{-1}+\frac{1}{2}q^{-2}}u(k).$$

Then we write it into the format of difference equation:

$y(k)(1-q^{-1}+\frac{1}{2}q^{-2}) = u(k)$, so $y(k)-y(k-1)+\frac{1}{2}y(k-2)=u(k)$.

$$y(0)=u(0)=1$$

$$y(1)=u(1)+y(0)=2$$

$$y(2)=u(2)+y(1)-\frac{1}{2}y(0)=2.5$$

$$y(3)=u(3)+y(2)-\frac{1}{2}y(1)=2.5$$

$$y(4)=u(4)+y(3)-\frac{1}{2}y(2)=2.25$$

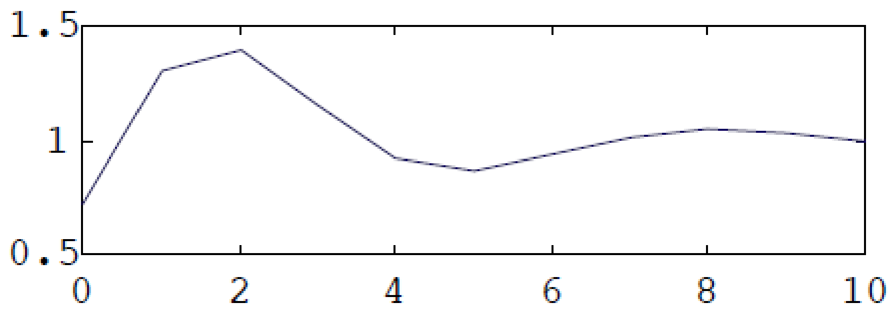
$$y(5)=u(5)+y(4)-\frac{1}{2}y(3)=2$$

...

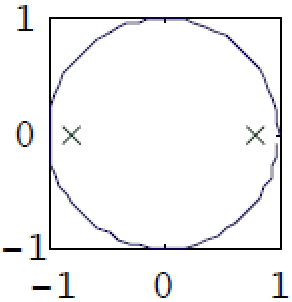
As all the poles of the system are inside the unit circle, this system is stable. So $y_\infty = G(1) = \frac{1}{1-1+\frac{1}{2}} = 2$.

The unit step response of this system looks like:

2.



B



From the zero-pole graph, we can see there are 2 poles inside the unit circle. Assume the 2 poles are $\frac{1}{2}$ and -$\frac{1}{2}$ respectively, and let $K=1$.

So we can get:

$$y(k)=\frac{1}{(1-\frac{1}{2}q^{-1})(1+\frac{1}{2}q^{-1})}u(k)=\frac{1}{1-\frac{1}{4}q^{-2}}u(k).$$

Then we write it into the format of difference equation:

$y(k)(1-\frac{1}{4}q^{-2})=u(k)$, so $y(k)-\frac{1}{4}y(k-2)=u(k)$.

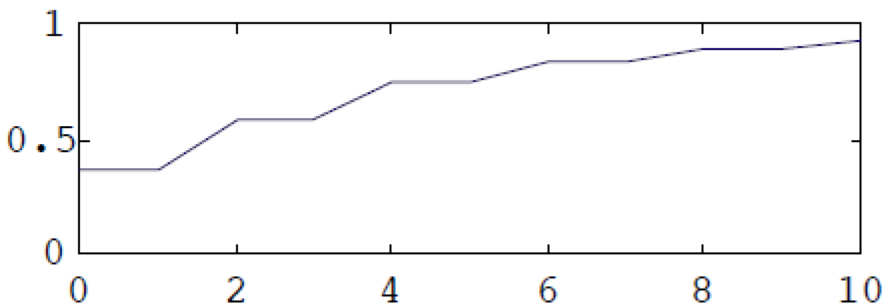
$$y(0)=u(0)=1$$
$$y(1)=u(1)=1$$
$$y(2)=u(2)+\frac{1}{4}y(0)=1.25$$
$$y(3)=u(3)+\frac{1}{4}y(1)=1.25$$
$$y(4)=u(4)+\frac{1}{4}y(2)=1.3125$$
$$y(5)=u(5)+\frac{1}{4}y(3)=1.3125$$
$$y(6)=u(6)+\frac{1}{4}y(4)=1.328125$$

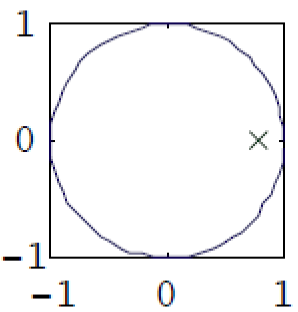
...

As all the poles of the system are inside the unit circle, this system is stable. So $y_\infty=G(1)=\frac{1}{1-\frac{1}{4}}=\frac{4}{3}$.

The unit step response of this system looks like:

4.



C 

From the zero-pole graph, we can see there are 1 pole inside the unit circle. Assume the pole is $\frac{1}{2}$, and let $K=1$.

So we can get: $y(k)=\frac{1}{1-\frac{1}{2}q^{-1}}u(k)$.

Then we write it into the format of difference equation:

$y(k)(1-\frac{1}{2}q^{-1})=u(k)$, so $y(k)-\frac{1}{2}y(k-1)=u(k)$.

$y(0)=u(0)=1$

$y(1)=u(1)+\frac{1}{2}y(0)=1.5$

$y(2)=u(2)+\frac{1}{2}y(1)=1.75$

$y(3)=u(3)+\frac{1}{2}y(2)=1.875$

$y(4)=u(4)+\frac{1}{2}y(3)=1.9375$

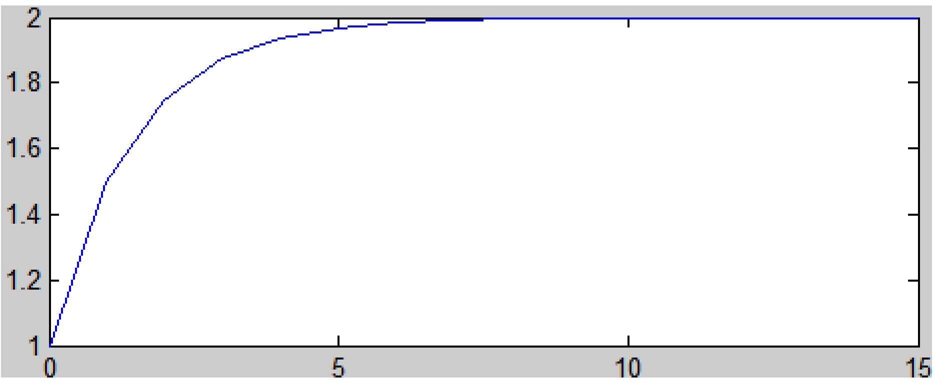
...

As the pole of the system is inside the unit circle, this system is stable. So $y_{\infty}=G(1)=\frac{1}{1-\frac{1}{2}}=2$.

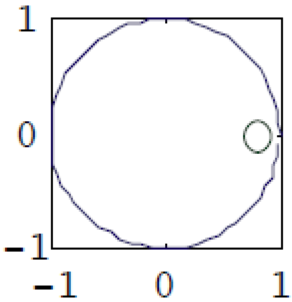
The unit step response graph of this system is:

5. No match

The exact graph should be this:



D



From the zero-pole graph, we can see there are 1 zero inside the unit circle. Assume the zero is $\frac{1}{2}$, and let $K=1$.

So we can get: $y(k)=(1-\frac{1}{2}q^{-1})u(k)$.

Then we write it into the format of difference equation:

$y(k)=u(k)-\frac{1}{2}u(k-1)$.

$$y(0)=u(0)=1$$

$$y(1)=u(1)-\frac{1}{2}u(0)=0.5$$

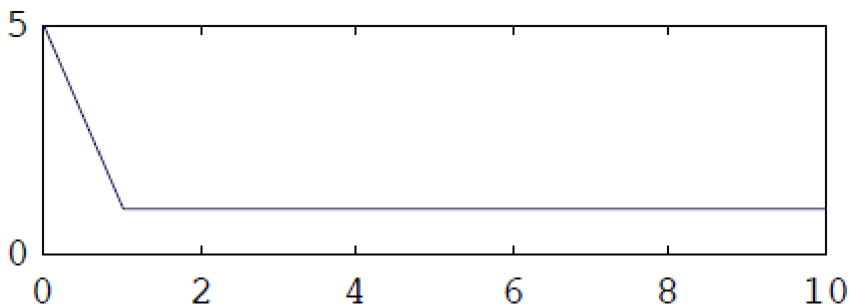
$$y(2)=u(2)-\frac{1}{2}u(1)=0.5$$

$$y(3)=u(3)-\frac{1}{2}u(2)=0.5$$

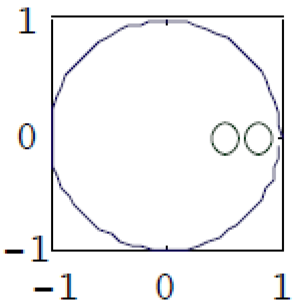
...

So $y(k)=\begin{cases}1 & k=0\\ 0.5 & k>0\end{cases}$. The unit step response of this system looks like:

3.



E



From the zero-pole graph, we can see there are 2 zeros inside the unit circle. Assume 2 zeros are $\frac{1}{2}$ and $\frac{3}{4}$ respectively, and let $K=1$.

So we can get: $y(k)=(1-\frac{1}{2}q^{-1})(1-\frac{3}{4}q^{-1})u(k)$.

Then we write it into the format of difference equation:

$$y(k)=(1-\frac{5}{4}q^{-1}+\frac{3}{8}q^{-2})u(k)=u(k)-\frac{5}{4}u(k-1)+\frac{3}{8}u(k-2).$$

$$y(0)=u(0)=1$$

$$y(1)=u(1)-\frac{5}{4}u(0)=-0.25$$

$$y(2)=u(2)-\frac{5}{4}u(1)+\frac{3}{8}u(0)=0.125$$

$$y(3)=u(3)-\frac{5}{4}u(2)+\frac{3}{8}u(1)=0.125$$

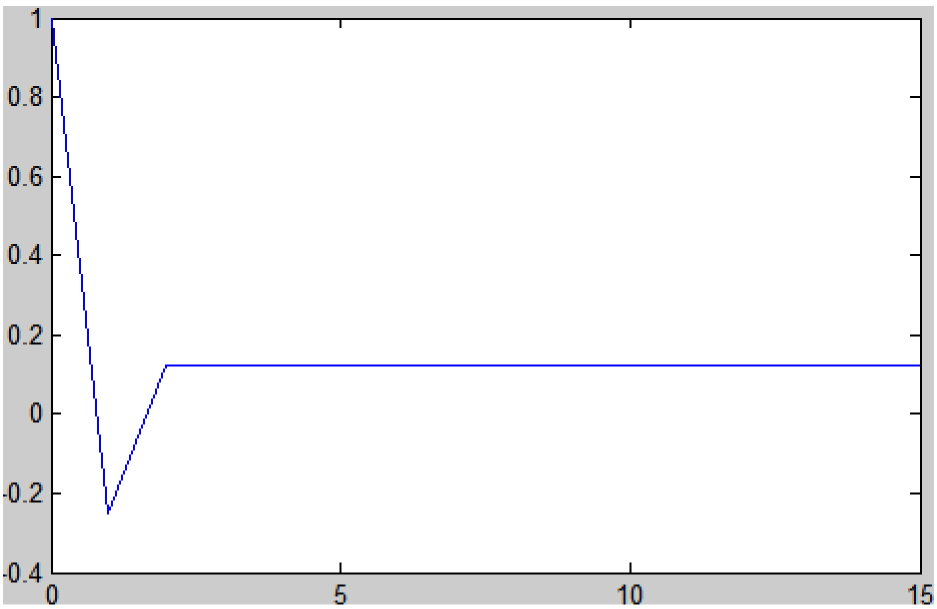
$$y(4)=u(4)-\frac{5}{4}u(3)+\frac{3}{8}u(2)=0.125$$

...

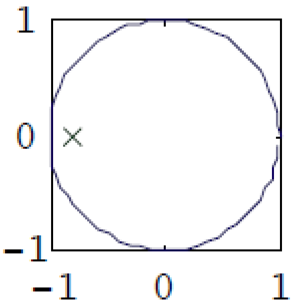
So $y(k)=\begin{cases}1 & k=0\\ -0.25 & k=1\\ 0.125 & k>1\end{cases}$. The unit step response graph of this system is:

5. No match

The exact graph should be this:



F



From the zero-pole graph, we can see there are 1 pole inside the unit circle. Assume the pole is $-\frac{1}{2}$, and let $K=1$.

So we can get: $y(k)=\dfrac{1}{1+\frac{1}{2}q^{-1}}u(k)$.

Then we write it into the format of difference equation:

$y(k)(1+\frac{1}{2}q^{-1})=u(k)$, so $y(k)+\frac{1}{2}y(k-1)=u(k)$.

$$y(0)=u(0)=1$$

$$y(1)=u(1)-\frac{1}{2}y(0)=0.5$$

$$y(2)=u(2)-\frac{1}{2}y(1)=0.75$$

$$y(3)=u(3)-\frac{1}{2}y(2)=0.625$$

$$y(4)=u(4)-\frac{1}{2}y(3)=0.6875$$

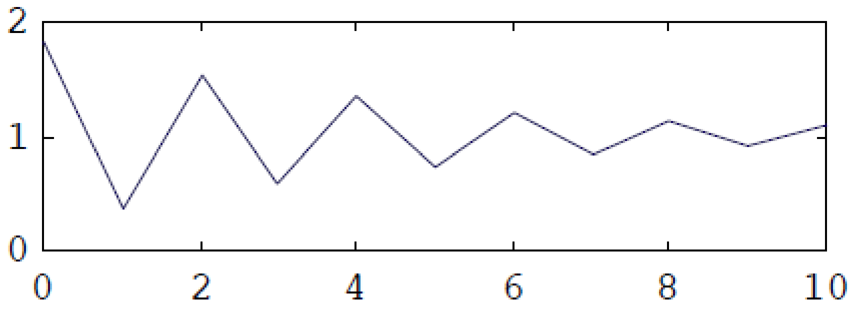
$$y(5)=u(5)-\frac{1}{2}y(4)=0.65625$$

...

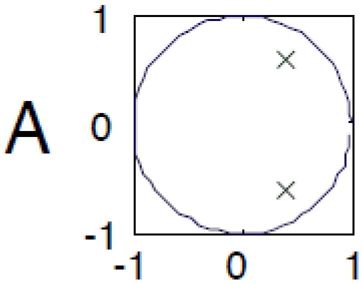
As the pole of the system is inside the unit circle, this system is stable.

So $y_\infty=G(1)=\dfrac{1}{1+\frac{1}{2}}=\dfrac{2}{3}$. The unit step response graph of this system looks like:

1.



2. $y(k) = K\dfrac{(1-z_1q^{-1})(1-z_2q^{-1})...(1-z_mq^{-1})}{(1-\lambda_1q^{-1})(1-\lambda_2q^{-1})...(1-\lambda_mq^{-1})}u(k)$, pulse response:



From the zero-pole graph, we can see there are 2 poles inside the unit circle. Assume the 2 poles are $\frac{1}{2}+\frac{1}{2}j$ and $\frac{1}{2}$-$\frac{1}{2}j$ respectively, and let $K=1$.

So we can get:

$$y(k) = \frac{1}{[1-(\frac{1}{2}+\frac{1}{2}j)q^{-1}][1-(\frac{1}{2}-\frac{1}{2}j)q^{-1}]}u(k) = \frac{1}{1-q^{-1}+\frac{1}{2}q^{-2}}u(k).$$

Then we write it into the format of difference equation:

$y(k)(1-q^{-1}+\frac{1}{2}q^{-2}) = u(k)$, so $y(k)-y(k-1)+\frac{1}{2}y(k-2) = u(k)$.

$$y(0) = u(0) = 1$$

$$y(1) = u(1) + y(0) = 1$$

$$y(2) = u(2) + y(1) - \frac{1}{2}y(0) = 0.5$$

$$y(3) = u(3) + y(2) - \frac{1}{2}y(1) = 0$$

$$y(4) = u(4) + y(3) - \frac{1}{2}y(2) = -0.25$$

$$y(5) = u(5) + y(4) - \frac{1}{2}y(3) = -0.25$$

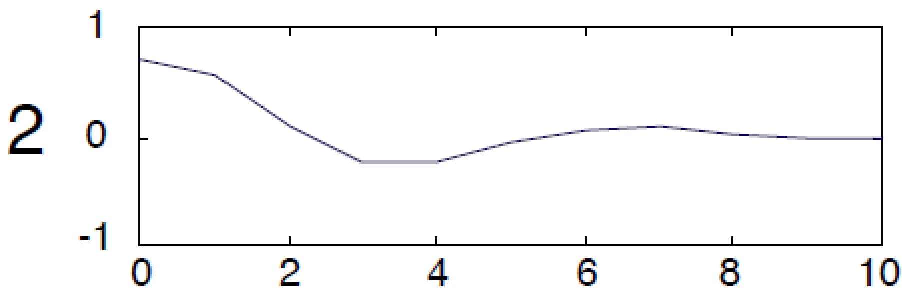
$$y(6) = u(6) + y(5) - \frac{1}{2}y(4) = -0.125$$

$$y(7) = u(7) + y(6) - \frac{1}{2}y(5) = 0$$

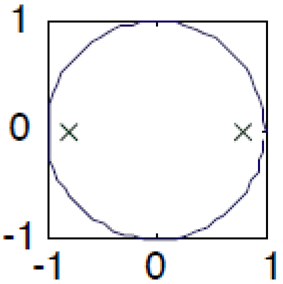
$$y(8) = u(8) + y(7) - \frac{1}{2}y(6) = 0.0625$$

...

The unit pulse response of this system looks like:



B



From the zero-pole graph, we can see there are 2 poles inside the unit circle. Assume the 2 poles are $\frac{1}{2}$ and -$\frac{1}{2}$ respectively, and let $K=1$.

So we can get:

$$y(k)=\frac{1}{(1-\frac{1}{2}q^{-1})(1+\frac{1}{2}q^{-1})}u(k)=\frac{1}{1-\frac{1}{4}q^{-2}}u(k).$$

Then we write it into the format of difference equation:

$y(k)(1-\frac{1}{4}q^{-2})=u(k)$, so $y(k)-\frac{1}{4}y(k-2)=u(k)$.

$$y(0)=u(0)=1$$

$$y(1)=u(1)=0$$

$$y(2)=u(2)+\frac{1}{4}y(0)=0.25$$

$$y(3)=u(3)+\frac{1}{4}y(1)=0$$

$$y(4)=u(4)+\frac{1}{4}y(2)=0.0625$$

$$y(5)=u(5)+\frac{1}{4}y(3)=0$$

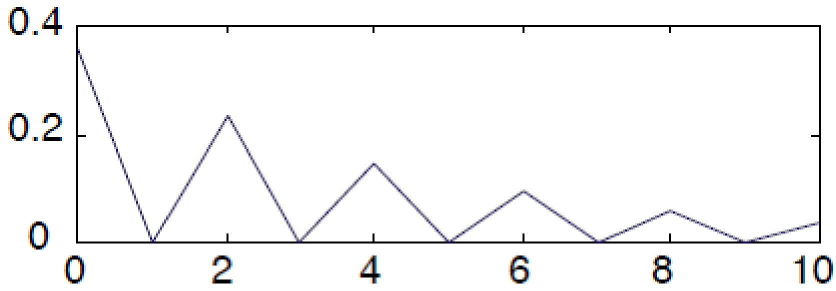
$$y(6)=u(6)+\frac{1}{4}y(4)=0.015625$$

$$y(7)=u(7)+\frac{1}{4}y(5)=0$$

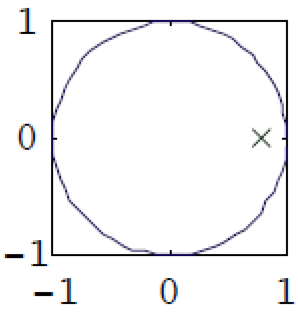
…

The unit pulse response of this system looks like:

1



C



From the zero-pole graph, we can see there are 1 pole inside the unit circle. Assume the pole is $\frac{1}{2}$, and let $K = 1$.

So we can get: $y(k) = \dfrac{1}{1-\frac{1}{2}q^{-1}}u(k)$.

Then we write it into the format of difference equation:

$y(k)(1-\frac{1}{2}q^{-1}) = u(k)$, so $y(k) - \frac{1}{2}y(k-1) = u(k)$.

$y(0) = u(0) = 1$

$y(1) = u(1) + \frac{1}{2}y(0) = 0.5$

$y(2) = u(2) + \frac{1}{2}y(1) = 0.25$

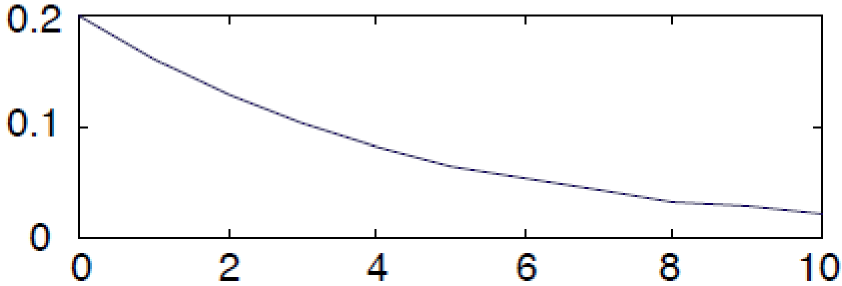
$y(3) = u(3) + \frac{1}{2}y(2) = 0.125$

$y(4) = u(4) + \frac{1}{2}y(3) = 0.0625$

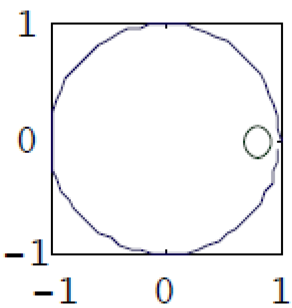
...

The unit pulse response graph of this system looks like:

3



D



From the zero-pole graph, we can see there are 1 zero inside the unit circle. Assume the zero is $\frac{1}{2}$, and let $K = 1$.

So we can get: $y(k) = (1 - \frac{1}{2}q^{-1})u(k)$.

Then we write it into the format of difference equation:

$y(k) = u(k) - \frac{1}{2}u(k-1)$.

$$y(0) = u(0) = 1$$

$$y(1) = u(1) - \frac{1}{2}u(0) = -0.5$$

$$y(2) = u(2) - \frac{1}{2}u(1) = 0$$

$$y(3) = u(3) - \frac{1}{2}u(2) = 0$$

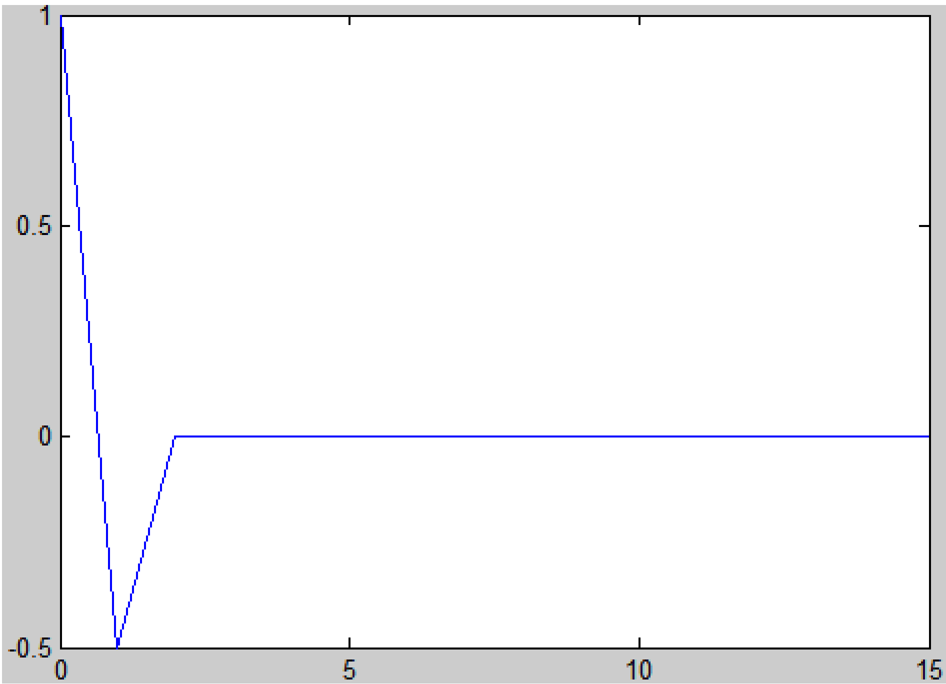
$$y(4) = u(4) - \frac{1}{2}u(3) = 0$$

...

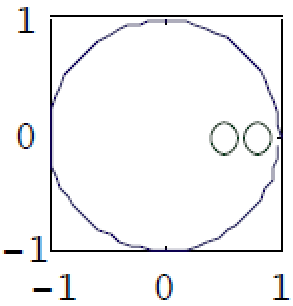
So $y(k) = \begin{cases} 1 & k = 0 \\ -0.5 & k = 1 \\ 0 & k > 1 \end{cases}$. The unit pulse response of this system looks like:

5. No match

The exact graph should be this:



E



From the zero-pole graph, we can see there are 2 zeros inside the unit circle. Assume 2 zeros are $\frac{1}{2}$ and $\frac{3}{4}$ respectively, and let $K=1$.

So we can get: $y(k)=(1-\frac{1}{2}q^{-1})(1-\frac{3}{4}q^{-1})u(k)$.

Then we write it into the format of difference equation:

$$y(k)=(1-\frac{5}{4}q^{-1}+\frac{3}{8}q^{-2})u(k)=u(k)-\frac{5}{4}u(k-1)+\frac{3}{8}u(k-2).$$

$$y(0)=u(0)=1$$

$$y(1)=u(1)-\frac{5}{4}u(0)=-1.25$$

$$y(2)=u(2)-\frac{5}{4}u(1)+\frac{3}{8}u(0)=0.375$$

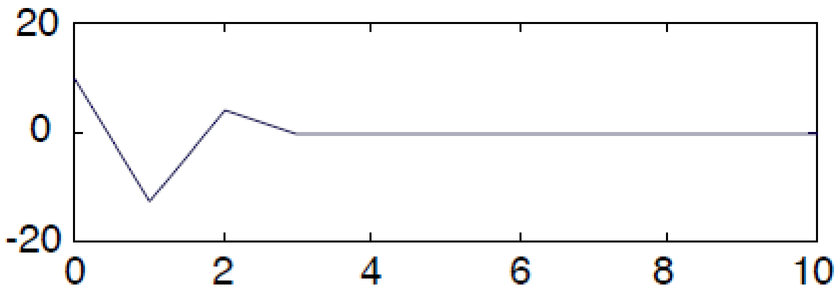
$$y(3)=u(3)-\frac{5}{4}u(2)+\frac{3}{8}u(1)=0$$

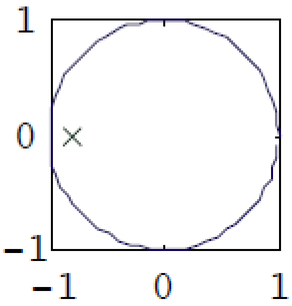
$$y(4)=u(4)-\frac{5}{4}u(3)+\frac{3}{8}u(2)=0$$

...

So $y(k)=\begin{cases}1 & k=0\\ -1.25 & k=1\\ 0.375 & k=2\\ 0 & k>2\end{cases}$. The unit pulse response graph of this system is:

4



F 

From the zero-pole graph, we can see there are 1 pole inside the unit circle. Assume the pole is $-\frac{1}{2}$, and let $K=1$.

So we can get: $y(k)=\dfrac{1}{1+\frac{1}{2}q^{-1}}u(k)$.

Then we write it into the format of difference equation:

$y(k)(1+\frac{1}{2}q^{-1})=u(k)$, so $y(k)+\frac{1}{2}y(k-1)=u(k)$.

$y(0)=u(0)=1$

$y(1)=u(1)-\frac{1}{2}y(0)=-0.5$

$y(2)=u(2)-\frac{1}{2}y(1)=0.25$

$y(3)=u(3)-\frac{1}{2}y(2)=-0.125$

$y(4)=u(4)-\frac{1}{2}y(3)=0.0625$

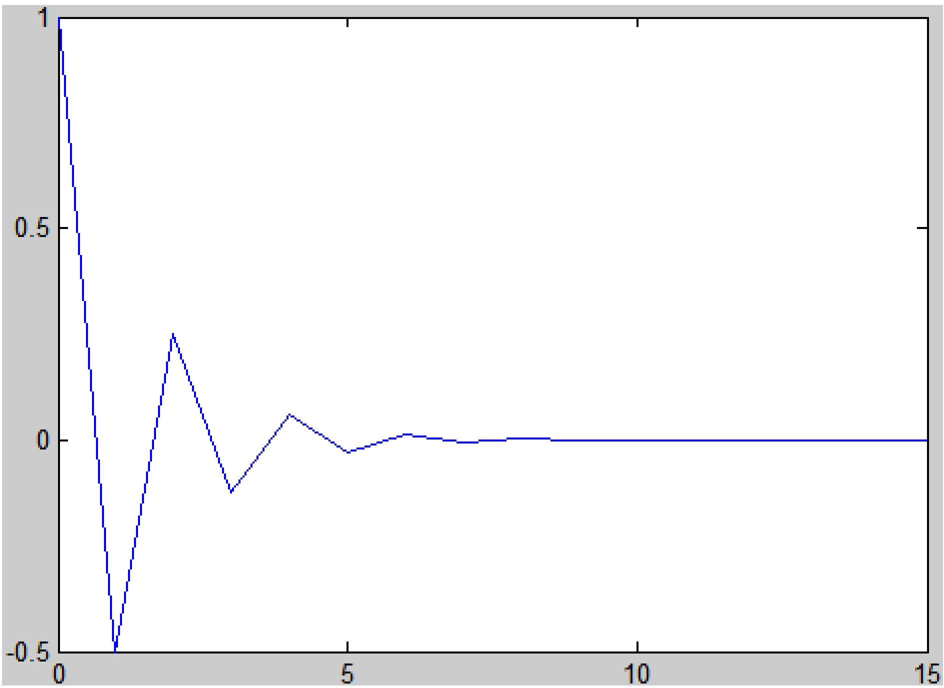
$y(5)=u(5)-\frac{1}{2}y(4)=-0.03125$

...

The unit step response graph of this system is:

5. No match

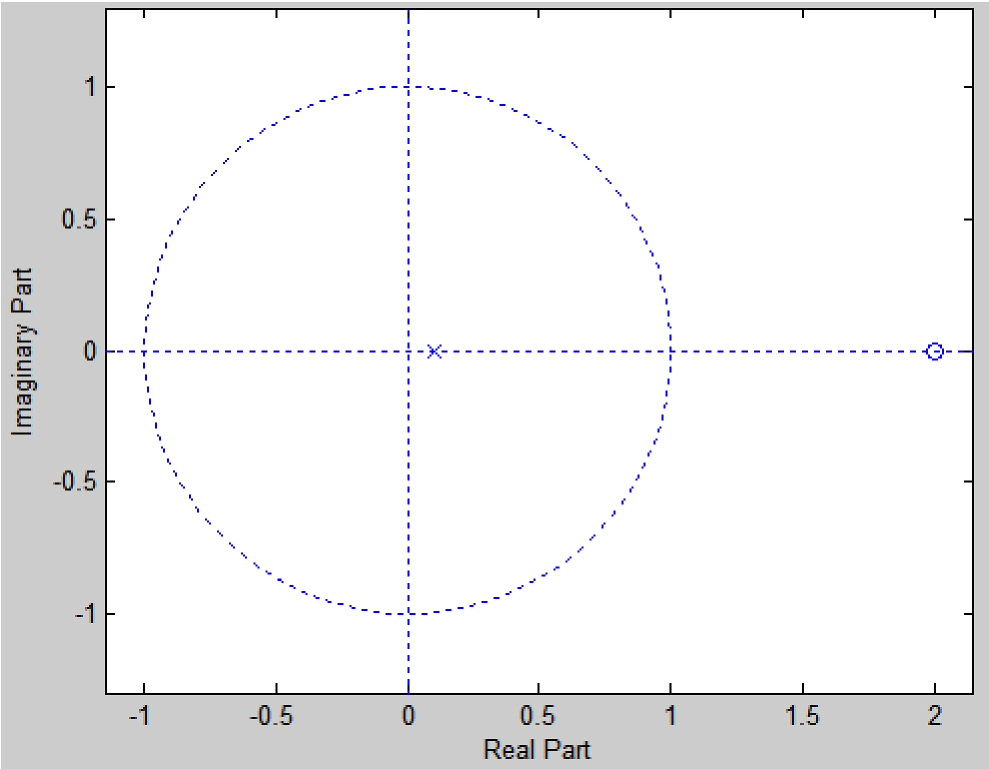
The exact graph should be this:



3. What are the steady-state (stationary) gain?

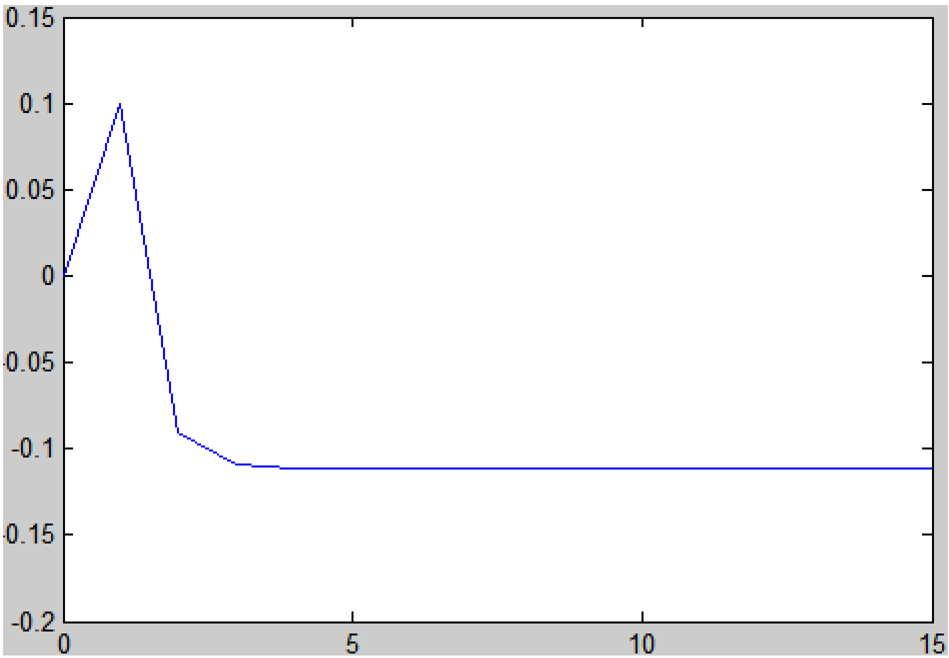
a) $y(k) = \dfrac{0.1q^{-1} - 0.2q^{-2}}{1 - 0.1q^{-1}} u(k)$

We can draw the zero-pole graph:



So all the poles are inside the unit circle. The system is stable.

The steady-state gain is: $y_\infty = G(1) = \dfrac{0.1 - 0.2}{1 - 0.1} = -\dfrac{1}{9}$



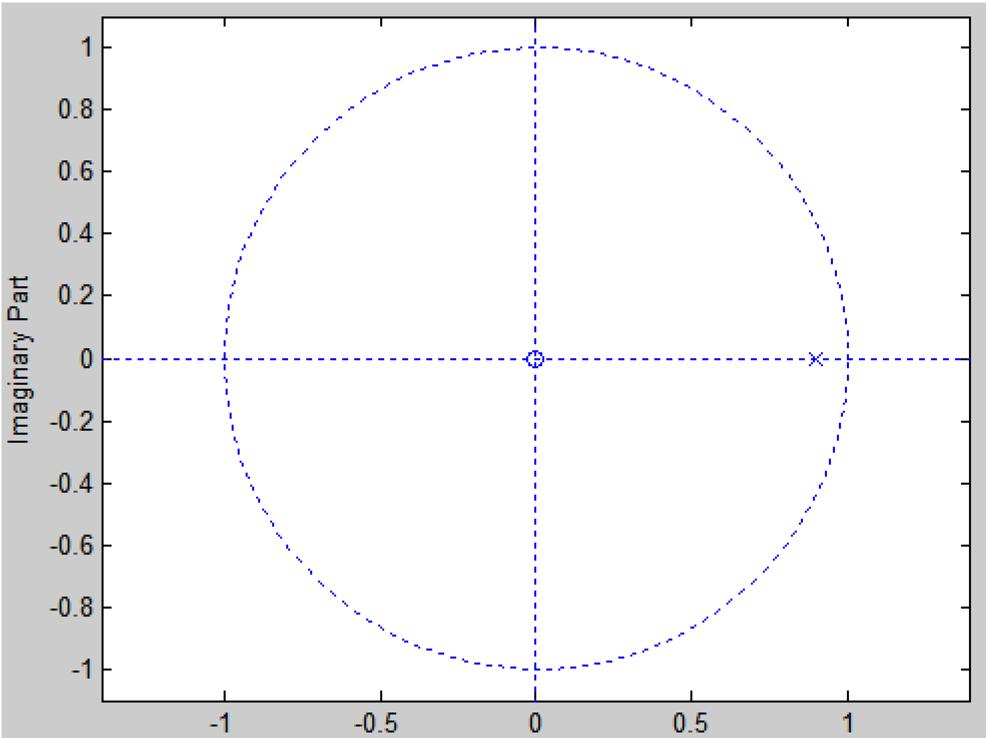
b) $y(k) = 0.9y(k-1) + 0.1u(k-1)$

Transfer the differential equation:

$$y(k) = 0.9q^{-1}y(k) + 0.1q^{-1}u(k) \Rightarrow y(k) - 0.9q^{-1}y(k) = 0.1q^{-1}u(k)$$

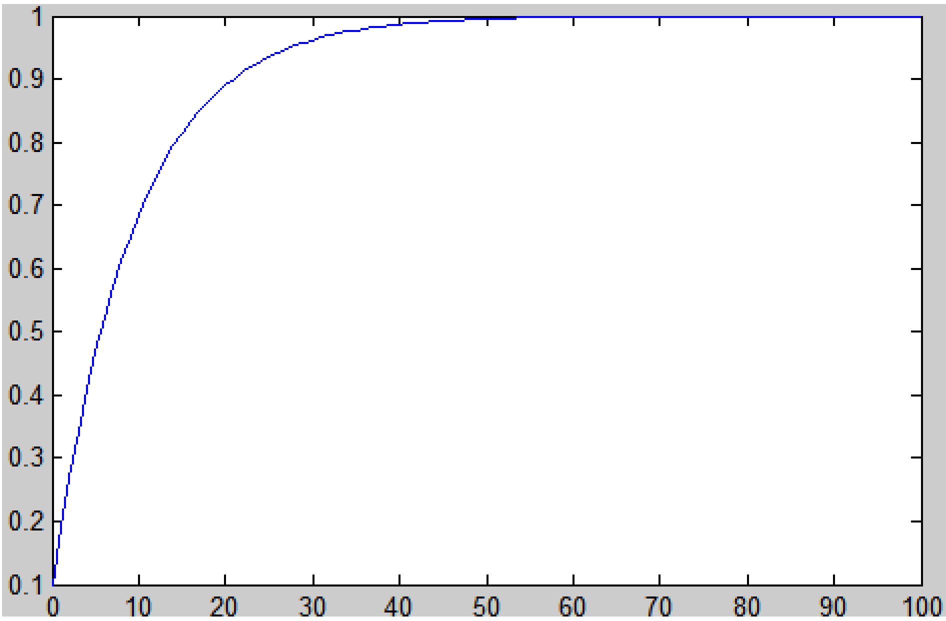
$$y(k) = \frac{0.1q^{-1}}{1-0.9q^{-1}}u(k)$$

We can draw the zero-pole graph:



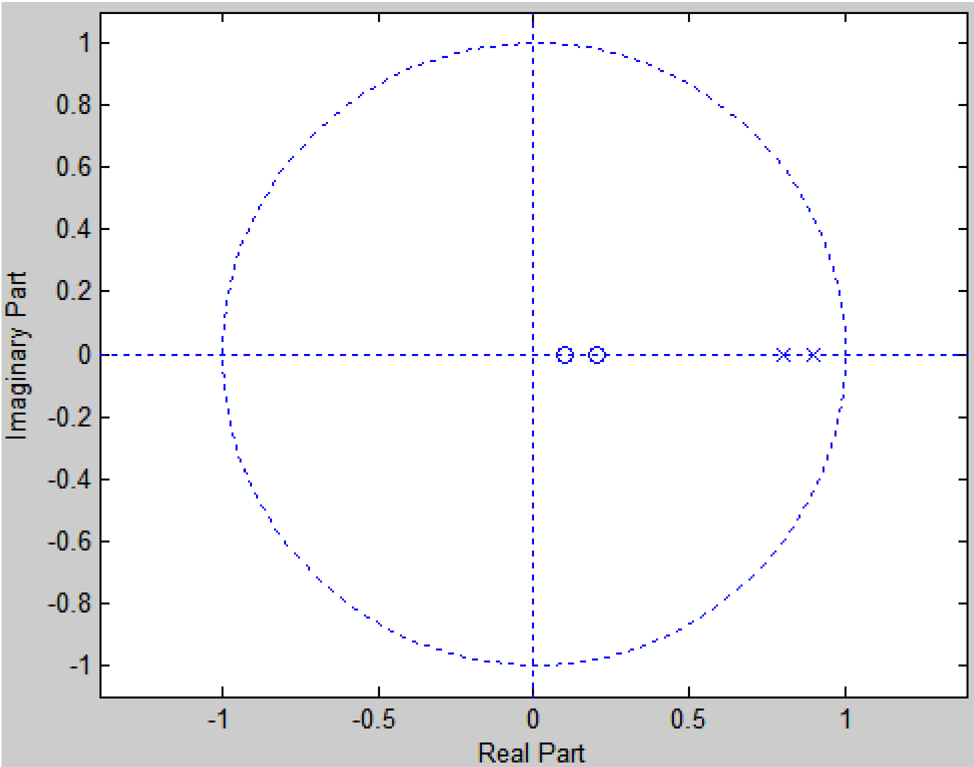
So all the poles are inside the unit circle. The system is stable.

The steady-state gain is: $y_{\infty} = G(1) = \frac{0.1}{1-0.9} = 1$



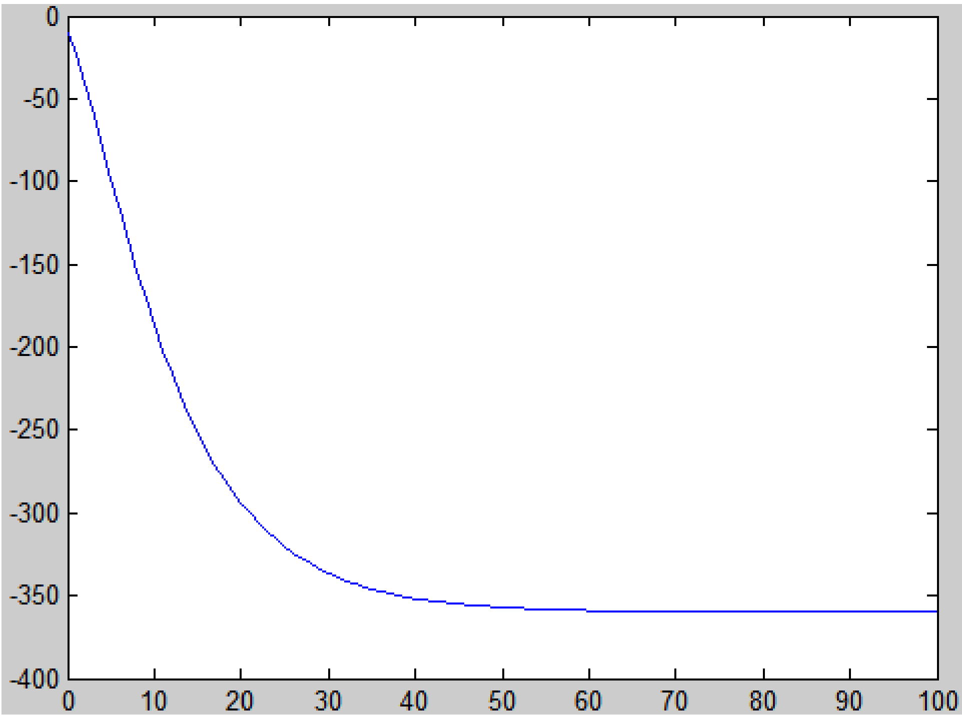
c) $y(k)=\dfrac{-10(1-0.1q^{-1})(1-0.2q^{-1})}{(1-0.9q^{-1})(1-0.8q^{-1})}u(k)$

We can draw the zero-pole graph:



So all the poles are inside the unit circle. The system is stable.

The steady-state gain is: $y_{\infty}=G(1)=\dfrac{-10\times(1-0.1)\times(1-0.2)}{(1-0.9)\times(1-0.8)}=-360$



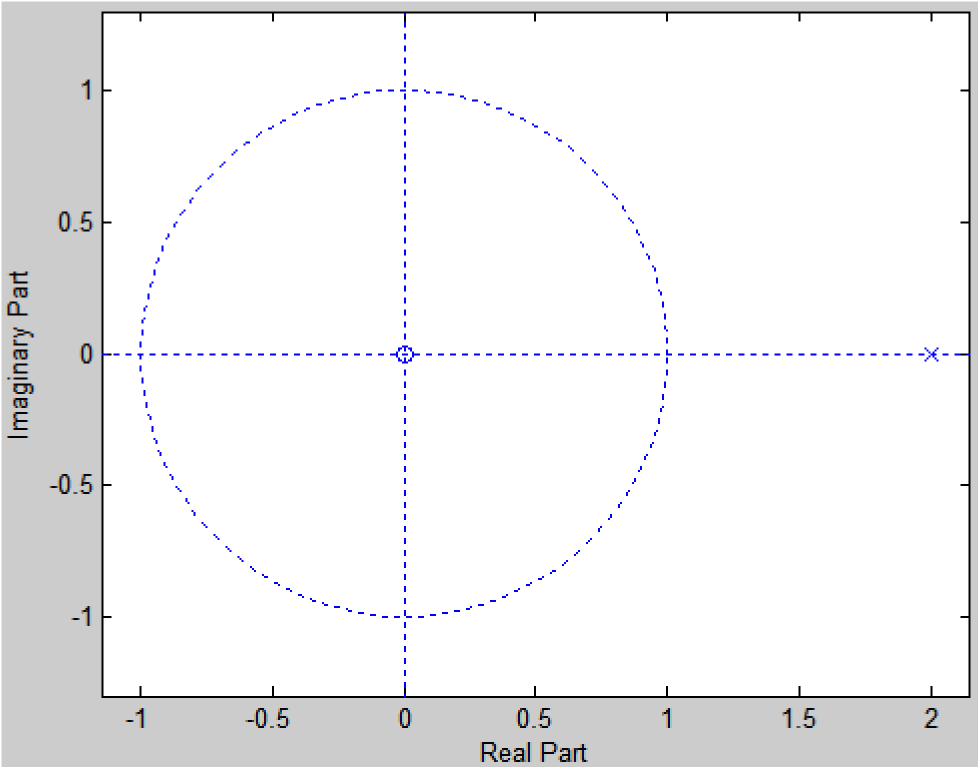
d) $y(k) - 2y(k-1) = 2u(k-10)$

Transfer the differential equation:

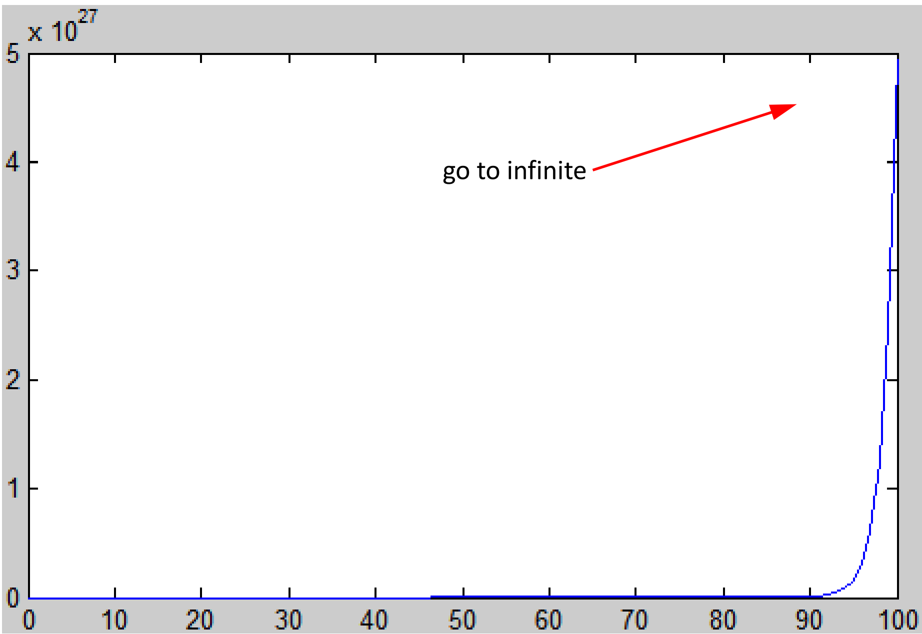
$$y(k) - 2q^{-1}y(k) = 2q^{-10}u(k)$$

$$y(k) = \frac{2q^{-10}}{1-2q^{-1}}u(k)$$

We can draw the zero-pole graph:

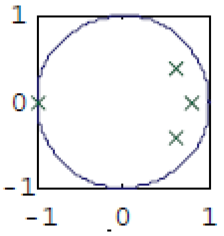


So the poles are outside the unit circle. This system is not stable. So there is no steady-state gain for this system.



4. $G(q^{-1}) = K \dfrac{(1-z_1q^{-1})(1-z_2q^{-1})...(1-z_mq^{-1})}{(1-\lambda_1q^{-1})(1-\lambda_2q^{-1})...(1-\lambda_mq^{-1})} q^{-2}$, step response

a



From the zero-pole graph, we can see there are 4 poles inside the unit circle. Assume the 4 poles are $\frac{1}{2}+\frac{1}{2}j, \frac{1}{2}-\frac{1}{2}j, \frac{3}{4}, -1$ respectively, and let $K=1$.

So we can get:

$$G(q^{-1}) = \frac{q^{-2}}{[1-(\frac{1}{2}+\frac{1}{2}j)q^{-1}][1-(\frac{1}{2}-\frac{1}{2}j)q^{-1}](1-\frac{3}{4}q^{-1})(1+q^{-1})} = \frac{q^{-2}}{1-\frac{3}{4}q^{-1}-\frac{1}{2}q^{-2}+\frac{7}{8}q^{-3}-\frac{3}{8}q^{-4}}$$

.

Then we write it into the format of difference equation:

$$y(k)(1-\frac{3}{4}q^{-1}-\frac{1}{2}q^{-2}+\frac{7}{8}q^{-3}-\frac{3}{8}q^{-4}) = u(k)q^{-2},$$

so $y(k)-\frac{3}{4}y(k-1)-\frac{1}{2}y(k-2)+\frac{7}{8}y(k-3)-\frac{3}{8}y(k-4) = u(k-2)$.

$$y(0) = u(-2) = 0$$

$$y(1) = u(-1)+\frac{3}{4}y(0) = 0$$

$$y(2) = u(0)+\frac{3}{4}y(1)+\frac{1}{2}y(0) = 1$$

$$y(3) = u(1)+\frac{3}{4}y(2)+\frac{1}{2}y(1)-\frac{7}{8}y(0) = 1.75$$

$$y(4) = u(2)+\frac{3}{4}y(3)+\frac{1}{2}y(2)-\frac{7}{8}y(1)+\frac{3}{8}y(0) = 2.8125$$

$$y(5) = u(3)+\frac{3}{4}y(4)+\frac{1}{2}y(3)-\frac{7}{8}y(2)+\frac{3}{8}y(1) = 3.1094$$

$$y(6) = u(4)+\frac{3}{4}y(5)+\frac{1}{2}y(4)-\frac{7}{8}y(3)+\frac{3}{8}y(2) = 3.5820$$

$$y(7) = u(5)+\frac{3}{4}y(6)+\frac{1}{2}y(5)-\frac{7}{8}y(4)+\frac{3}{8}y(3) = 3.4365$$

$$y(8) = u(6)+\frac{3}{4}y(7)+\frac{1}{2}y(6)-\frac{7}{8}y(5)+\frac{3}{8}y(4) = 3.7024$$

$$y(9) = u(7)+\frac{3}{4}y(8)+\frac{1}{2}y(7)-\frac{7}{8}y(6)+\frac{3}{8}y(5) = 3.5268$$

$$y(10) = u(8)+\frac{3}{4}y(9)+\frac{1}{2}y(8)-\frac{7}{8}y(7)+\frac{3}{8}y(6) = 3.8326$$

$$y(11) = u(9)+\frac{3}{4}y(10)+\frac{1}{2}y(9)-\frac{7}{8}y(8)+\frac{3}{8}y(7) = 3.6869$$

$$y(12)=u(10)+\frac{3}{4}y(11)+\frac{1}{2}y(10)-\frac{7}{8}y(9)+\frac{3}{8}y(8)=3.9840$$

$$y(13)=u(11)+\frac{3}{4}y(12)+\frac{1}{2}y(11)-\frac{7}{8}y(10)+\frac{3}{8}y(9)=3.8005$$

$$y(14)=u(12)+\frac{3}{4}y(13)+\frac{1}{2}y(12)-\frac{7}{8}y(11)+\frac{3}{8}y(10)=4.0535$$

$$y(15)=u(13)+\frac{3}{4}y(14)+\frac{1}{2}y(13)-\frac{7}{8}y(12)+\frac{3}{8}y(11)=3.8370$$

$$y(16)=u(14)+\frac{3}{4}y(15)+\frac{1}{2}y(14)-\frac{7}{8}y(13)+\frac{3}{8}y(12)=4.0730$$

$$y(17)=u(15)+\frac{3}{4}y(16)+\frac{1}{2}y(15)-\frac{7}{8}y(14)+\frac{3}{8}y(13)=3.8517$$

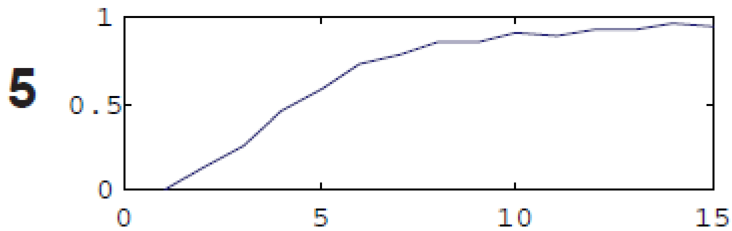
$$y(18)=u(16)+\frac{3}{4}y(17)+\frac{1}{2}y(16)-\frac{7}{8}y(15)+\frac{3}{8}y(14)=4.0880$$

$$y(19)=u(17)+\frac{3}{4}y(18)+\frac{1}{2}y(17)-\frac{7}{8}y(16)+\frac{3}{8}y(15)=3.8668$$

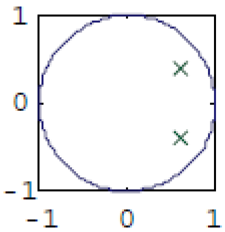
$$y(20)=u(18)+\frac{3}{4}y(19)+\frac{1}{2}y(18)-\frac{7}{8}y(17)+\frac{3}{8}y(16)=4.1012$$

...

As one of the poles of the system is just on the boundary of unit circle, this system is stable, but it will keep oscillating between two states(in this example are approximately 3.8857 and 4.1143) when k goes to infinite. The unit step response of this system looks like:



b



From the zero-pole graph, we can see there are 2 poles inside the unit circle. Assume the 2 poles are $\frac{1}{2}+\frac{1}{2}j$ and $\frac{1}{2}$-$\frac{1}{2}j$ respectively, and let $K=1$.
So we can get:

$$G(q^{-1})=\frac{q^{-2}}{[1-(\frac{1}{2}+\frac{1}{2}j)q^{-1}][1-(\frac{1}{2}-\frac{1}{2}j)q^{-1}]}=\frac{q^{-2}}{1-q^{-1}+\frac{1}{2}q^{-2}}.$$

Then we write it into the format of difference equation:

$y(k)(1-q^{-1}+\frac{1}{2}q^{-2})=u(k)q^{-2}$, so $y(k)-y(k-1)+\frac{1}{2}y(k-2)=u(k-2)$.

$$y(0)=u(-2)=0$$

$$y(1)=u(-1)+y(0)=0$$

$$y(2)=u(0)+y(1)-\frac{1}{2}y(0)=1$$

$$y(3)=u(1)+y(2)-\frac{1}{2}y(1)=2$$

$$y(4)=u(2)+y(3)-\frac{1}{2}y(2)=2.5$$

$$y(5)=u(3)+y(4)-\frac{1}{2}y(3)=2.5$$

$$y(6)=u(4)+y(5)-\frac{1}{2}y(4)=2.25$$

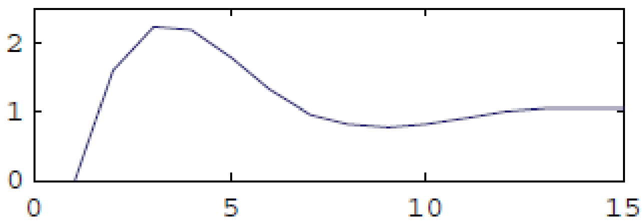
$$y(7)=u(5)+y(6)-\frac{1}{2}y(5)=2$$

...

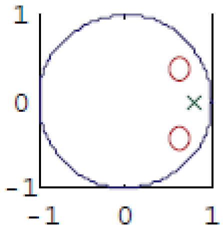
As all the poles of the system are inside the unit circle, this system is stable.

So $y_\infty=G(1)=\frac{1}{1-1+\frac{1}{2}}=2$. It's just the graph in 1.A shifting for 2 time units. The unit step response of this system looks like:

1



C



From the zero-pole graph, we can see there are 2 zeros and 1 pole inside the unit circle. Assume the 2 zeros are $\frac{1}{2}+\frac{1}{2}j$ and $\frac{1}{2}$-$\frac{1}{2}j$, and the pole is $\frac{3}{4}$ respectively, and let $K=1$.

So we can get:

$$G(q^{-1})=\frac{[1-(\frac{1}{2}+\frac{1}{2}j)q^{-1}][1-(\frac{1}{2}-\frac{1}{2}j)q^{-1}]q^{-2}}{1-\frac{3}{4}q^{-1}}=\frac{q^{-2}-q^{-3}+\frac{1}{2}q^{-4}}{1-\frac{3}{4}q^{-1}}.$$

Then we write it into the format of difference equation:

$y(k)(1-\frac{3}{4}q^{-1})=u(k)(q^{-2}-q^{-3}+\frac{1}{2}q^{-4})$,

so $y(k)-\frac{3}{4}y(k-1)=u(k-2)-u(k-3)+\frac{1}{2}u(k-4)$.

$$y(0)=u(-2)-u(-3)+\frac{1}{2}u(-4)=0$$

$$y(1)=u(-1)-u(-2)+\frac{1}{2}u(-3)+\frac{3}{4}y(0)=0$$

$$y(2)=u(0)-u(-1)+\frac{1}{2}u(-2)+\frac{3}{4}y(1)=1$$

$$y(3)=u(1)-u(0)+\frac{1}{2}u(-1)+\frac{3}{4}y(2)=0.75$$

$$y(4)=u(2)-u(1)+\frac{1}{2}u(0)+\frac{3}{4}y(3)=1.0625$$

$$y(5)=u(3)-u(2)+\frac{1}{2}u(1)+\frac{3}{4}y(4)=1.2969$$

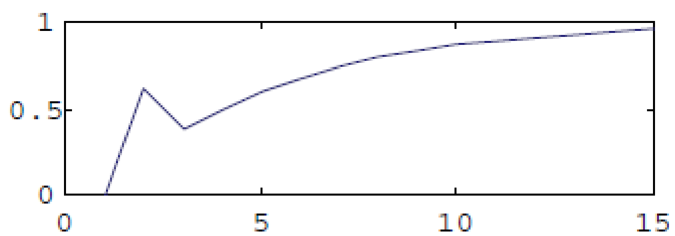
$$y(6)=u(4)-u(3)+\frac{1}{2}u(2)+\frac{3}{4}y(5)=1.4727$$

$$y(7)=u(5)-u(4)+\frac{1}{2}u(3)+\frac{3}{4}y(6)=1.6045$$

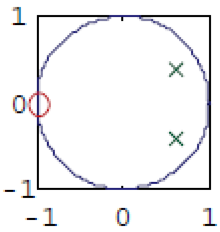
…

As all the poles of the system are inside the unit circle, this system is stable. So $y_\infty=G(1)=\frac{1-1+\frac{1}{2}}{1-\frac{3}{4}}=2$. The unit step response of this system looks like:

2



d



From the zero-pole graph, we can see there are 2 poles and 1 zero.

Assume the poles are $\frac{1}{2}+\frac{1}{2}j$ and $\frac{1}{2}-\frac{1}{2}j$ and the zero is -1 respectively, and let $K=1$.

So we can get:

$$G(q^{-1})=\frac{(1+q^{-1})q^{-2}}{[1-(\frac{1}{2}+\frac{1}{2}j)q^{-1}][1-(\frac{1}{2}-\frac{1}{2}j)q^{-1}]}=\frac{q^{-2}+q^{-3}}{1-q^{-1}+\frac{1}{2}q^{-2}}.$$

Then we write it into the format of difference equation:

$$y(k)(1-q^{-1}+\frac{1}{2}q^{-2})=u(k)(q^{-2}+q^{-3}),$$

so $y(k)-y(k-1)+\frac{1}{2}y(k-2)=u(k-2)+u(k-3)$.

$$y(0)=u(-2)+u(-3)=0$$

$$y(1)=u(-1)+u(-2)+y(0)=0$$

$$y(2)=u(0)+u(-1)+y(1)-\frac{1}{2}y(0)=1$$

$$y(3)=u(1)+u(0)+y(2)-\frac{1}{2}y(1)=3$$

$$y(4)=u(2)+u(1)+y(3)-\frac{1}{2}y(2)=4.5$$

$$y(5)=u(3)+u(2)+y(4)-\frac{1}{2}y(3)=5$$

$$y(6)=u(4)+u(3)+y(5)-\frac{1}{2}y(4)=4.75$$

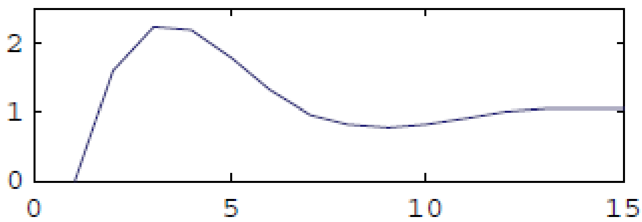
$$y(7)=u(5)+u(4)+y(6)-\frac{1}{2}y(5)=4.25$$

…

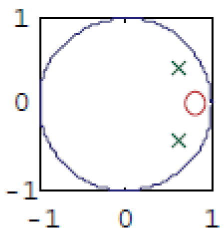
As all the poles of the system are inside the unit circle, this system is stable.

So $y_\infty=G(1)=\frac{1+1}{1-1+\frac{1}{2}}=4$. The unit step response of this system looks like:

1



e



From the zero-pole graph, we can see there are 2 poles and 1 zero. Assume the poles are $\frac{1}{2}+\frac{1}{2}j$ and $\frac{1}{2}-\frac{1}{2}j$ and the zero is $\frac{3}{4}$ respectively, and let $K=1$.

So we can get:

$$G(q^{-1})=\frac{(1-\frac{3}{4}q^{-1})q^{-2}}{[1-(\frac{1}{2}+\frac{1}{2}j)q^{-1}][1-(\frac{1}{2}-\frac{1}{2}j)q^{-1}]}=\frac{q^{-2}-\frac{3}{4}q^{-3}}{1-q^{-1}+\frac{1}{2}q^{-2}}.$$

Then we write it into the format of difference equation:

$$y(k)(1-q^{-1}+\frac{1}{2}q^{-2})=u(k)(q^{-2}-\frac{3}{4}q^{-3}),$$

so $y(k)-y(k-1)+\frac{1}{2}y(k-2)=u(k-2)-\frac{3}{4}u(k-3)$.

$$y(0)=u(-2)-\frac{3}{4}u(-3)=0$$

$$y(1)=u(-1)-\frac{3}{4}u(-2)+y(0)=0$$

$$y(2)=u(0)-\frac{3}{4}u(-1)+y(1)-\frac{1}{2}y(0)=1$$

$$y(3)=u(1)-\frac{3}{4}u(0)+y(2)-\frac{1}{2}y(1)=1.25$$

$$y(4)=u(2)-\frac{3}{4}u(1)+y(3)-\frac{1}{2}y(2)=1$$

$$y(5)=u(3)-\frac{3}{4}u(2)+y(4)-\frac{1}{2}y(3)=0.6250$$

$$y(6)=u(4)-\frac{3}{4}u(3)+y(5)-\frac{1}{2}y(4)=0.3750$$

$$y(7)=u(5)-\frac{3}{4}u(4)+y(6)-\frac{1}{2}y(5)=0.3125$$

$$y(8)=u(6)-\frac{3}{4}u(5)+y(7)-\frac{1}{2}y(6)=0.3750$$

$$y(9)=u(7)-\frac{3}{4}u(6)+y(8)-\frac{1}{2}y(7)=0.4688$$

$$y(10)=u(8)-\frac{3}{4}u(7)+y(9)-\frac{1}{2}y(8)=0.5313$$

$$y(11)=u(9)-\frac{3}{4}u(8)+y(10)-\frac{1}{2}y(9)=0.5469$$

$$y(12)=u(10)-\frac{3}{4}u(9)+y(11)-\frac{1}{2}y(10)=0.5313$$

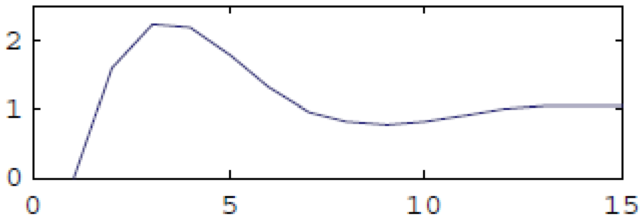
…

As all the poles of the system are inside the unit circle, this system is stable. So

$$y_\infty = G(1) = \frac{1-\frac{3}{4}}{1-1+\frac{1}{2}} = 0.5\,.$$

The unit step response of this system looks like:

1



Group problem:

From the given requirements, we can write the system function as:

$$G(q^{-1}) = K\frac{(1+q^{-1})}{(1-\lambda_1 q^{-1})(1-\lambda_2 q^{-1})}q^{-1}$$

We choose the $\lambda_1 = 0.9+0.2i$ to solve this problem. In this way,

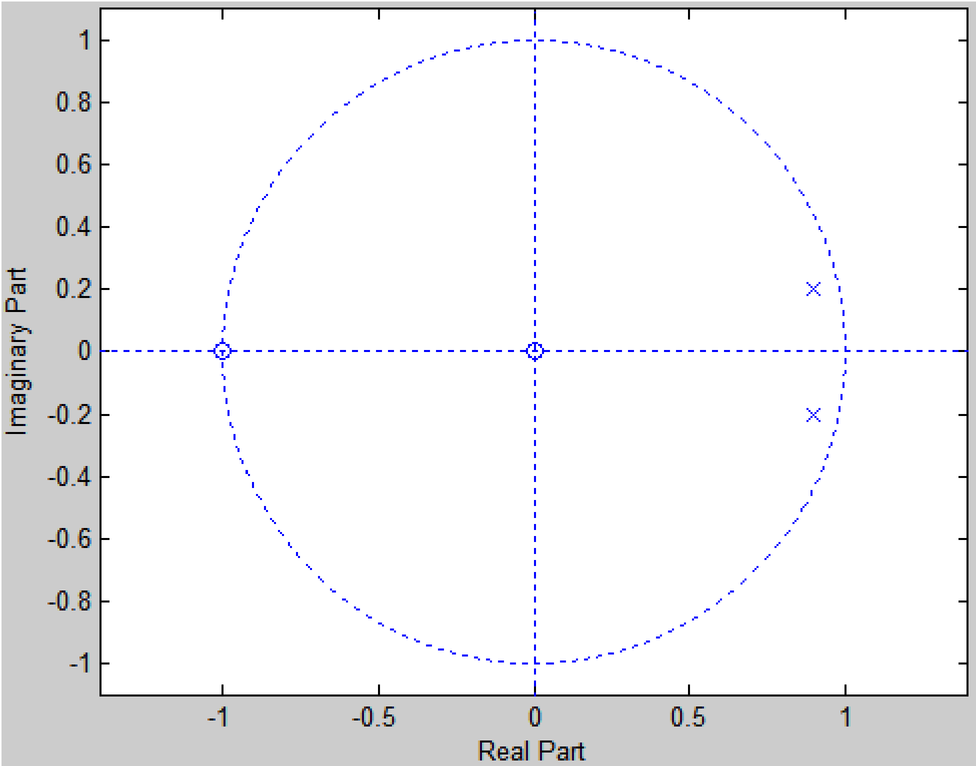
$$G(q^{-1}) = K\frac{(1+q^{-1})}{[1-(0.9+0.2i)q^{-1}][1-(0.9-0.2i)q^{-1}]}q^{-1} = \frac{K(q^{-1}+q^{-2})}{1-1.8q^{-1}+0.85q^{-2}}$$

Obviously, $|\lambda_1| = |\lambda_2| < 1$. So this system is stable.

The steady-state gain is $G(1) = \dfrac{K(1+1)}{1-1.8+0.85} = 2$, so $K = 0.05$.

So $G(q^{-1}) = \dfrac{0.05(1+q^{-1})}{[1-(0.9+0.2i)q^{-1}][1-(0.9-0.2i)q^{-1}]}q^{-1} = \dfrac{0.05q^{-1}+0.05q^{-2}}{1-1.8q^{-1}+0.85q^{-2}}$

1. Illustrate the poles an zero in a pole-zero diagram



2. Describe the system as a recursive difference equation and show how the step response can be calculated recursively ten steps

$$G(q^{-1}) = \frac{y(k)}{u(k)} = \frac{0.05q^{-1} + 0.05q^{-2}}{1 - 1.8q^{-1} + 0.85q^{-2}}$$

$$y(k)(1 - 1.8q^{-1} + 0.85q^{-2}) = u(k)(0.05q^{-1} + 0.05q^{-2})$$

$$y(k) - 1.8y(k-1) + 0.85y(k-2) = 0.05u(k-1) + 0.05u(k-2)$$

As we want to get the step response, $u(k) = \begin{cases} 0 & k < 0 \\ 1 & k \geq 0 \end{cases}.$

$$y(1) = 0.05u(0) + 0.05u(-1) = 0.05$$

$$y(2) = 1.8y(1) + 0.05u(1) + 0.05u(0) = 0.19$$

$$y(3) = 1.8y(2) - 0.85y(1) + 0.05u(2) + 0.05u(1) = 0.3995$$

$$y(4) = 1.8y(3) - 0.85y(2) + 0.05u(3) + 0.05u(2) = 0.6576$$

$$y(5) = 1.8y(4) - 0.85y(3) + 0.05u(4) + 0.05u(3) = 0.944105$$

$$y(6) = 1.8y(5) - 0.85y(4) + 0.05u(5) + 0.05u(4) = 1.240429$$

$$y(7)=1.8y(6)-0.85y(5)+0.05u(6)+0.05u(5)=1.53028295$$
$$y(8)=1.8y(7)-0.85y(6)+0.05u(7)+0.05u(6)=1.80014466$$
$$y(9)=1.8y(8)-0.85y(7)+0.05u(8)+0.05u(7)=2.0395198805$$
$$y(10)=1.8y(9)-0.85y(8)+0.05u(9)+0.05u(8)=2.2410128239$$

3. Use the backward-shift operator to describe the system in polynomial form. Then use this representation to calculate the step response.

$$G(q^{-1})=\frac{0.05(1+q^{-1})}{[1-(0.9+0.2i)q^{-1}][1-(0.9-0.2i)q^{-1}]}q^{-1}=\frac{0.05q^{-1}+0.05q^{-2}}{1-1.8q^{-1}+0.85q^{-2}}$$

Here is the step response: