KALMAN FILTER

MK 8005 – Intelligent vehicles

Lecture 8

Kalman Filter

- Model

$$x_k = F_k x_{k-1} + B_k u_k + w_k$$

- F_k is the state transition model which is applied to previous state x_{k-1}
- B_k is the control-input model which is applied to the control vector u_k
- W_k is the process noise which is assumed to be drawn from zero mean multivariable normal distribution with covariance Q_k

$$w_k \sim N(0, Q_{k-1})$$

Kalman Filter

- At time k an observation (or measurement) z_k of the true state x_k is made according to

$$z_k = H_k x_k + v_k$$

- Where H_k is the observation model which maps the true state space into the observed space and v_k is the observation noise which is assumed to be zero mean Gaussian white noise with covariance R_k

$$v_k \sim N(0, R_{k-1})$$

Kalman Filter - Phase 1: prediction

- Predicted (a priori) state

$$\hat{x}_{k|k-1} = F_k \hat{x}_{k-1|k-1} + B_k u_k$$

- Predicted (a priori) estimate covariance

$$P_{k|k-1} = F_k P_{k-1|k-1} F_k^T + Q_{k-1}$$

- There Q_{k-1} are the covariance of the process noise $w_k \sim N(0, Q_k)$, i.e. $\mathrm{cov}(B_k u_k)$

$$\mathrm{cov}(B_k u_k) = E\left[(B_k u_k)^2\right] - \underbrace{E[B_k u_k]^2}_{=0} = B_k E\left[u_k^2\right] B_k^T = B_k \Sigma_u B_k^T$$

Kalman Filter Phase 2: Update

- Innovation or measurement residual

  $$\tilde{y}_k = z_k - H_k \hat{x}_{k|k-1}$$

- Innovation (or residual) covariance

  $$S_k = H_k P_{k|k-1} H_k^T + R_k$$

- Optimal Kalman gain

  $$K_k = P_{k|k-1} H_k^T S_k^{-1}$$

- Updated (a posteriori) state estimate

  $$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k \tilde{y}_k$$

- Updated (a posteriori) estimate covariance

  $$P_{k|k} = \left(I - K_k H_k\right) P_{k|k-1}$$

Extended Kalman Filter

- In the extended Kalman filter, the state transition and observation models **need not to be linear** functions of the state but may instead be differentiable functions

$$x_k = f(x_{k-1}, u_{k-1}) + w_k$$

$$z_k = h(x_k) + v_k$$

- Where w_k and v_k are the process and observation noises which are both assumed to be zero mean multivariate Gaussian noises with covariance Q_k and R_k respectively
- However, *f* and *h* cannot be applied to the covariance directly. Instead a matrix of partial derivatives (the Jacobian) is computed.

Extended Kalman Filter - Phase 1: prediction

- Predicted (a priori) state

$$\hat{x}_{k|k-1} = f(\hat{x}_{k-1|k-1}, u_k)$$

- Predicted (a priori) estimate covariance

$$P_{k|k-1} = F_k P_{k-1|k-1} F_k^T + Q_{k-1}$$

- There Q_{k-1} are the covariance of the process noise $w_k \sim N(0, Q_k)$, i.e. $cov(B_k u_k)$

$$Q_{k-1} = B_k \Sigma_u B_k^T$$

Jacobians

$$F_k = \left.\frac{\partial f}{\partial x}\right|_{\hat{x}_{k-1|k-1}, u_k}$$

$$B_k = \left.\frac{\partial f}{\partial u}\right|_{\hat{x}_{k-1|k-1}, u_k}$$

Extended Kalman Filter - Phase 2: Update

- Innovation or measurement residual

$$\tilde{y}_k = z_k - h(\hat{x}_{k|k-1})$$

- Innovation (or residual) covariance

$$S_k = H_k P_{k|k-1} H_k^T + R_k$$

- (Optimal?) Kalman gain

$$K_k = P_{k|k-1} H_k^T S_k^{-1}$$

- Updated (a posteriori) state estimate

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k \tilde{y}_k$$

- Updated (a posteriori) estimate covariance

$$P_{k|k} = \left(I - K_k H_k\right) P_{k|k-1}$$

Jacobians

$$H_k = \left.\frac{\partial h}{\partial x}\right|_{\hat{x}_{k|k-1}}$$

Example EKF – 1. Prediction

- Predicted (a priori) state

$$\hat{x}_{k|k-1} = f(\hat{x}_{k-1|k-1}, u_k)$$

$$\hat{x}_{k|k-1} = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} + \begin{bmatrix} \Delta s \cos(\theta + \Delta\theta/2) \\ \Delta s \sin(\theta + \Delta\theta/2) \\ \Delta\theta \end{bmatrix} \qquad \Delta s = \frac{\Delta s_r + \Delta s_l}{2} \qquad \Delta\theta = \frac{\Delta s_r + \Delta s_l}{b}$$

- Predicted (a priori) estimate covariance

$$P_{k|k-1} = F_k P_{k-1|k-1} F_k^T + B_k \Sigma_u B_k^T$$

$$F_k = \begin{bmatrix} 1 & 0 & -\Delta s \sin(\theta + \Delta\theta/2) \\ 0 & 1 & \Delta s \cos(\theta + \Delta\theta/2) \\ 0 & 0 & 1 \end{bmatrix}$$

$$B_k = \begin{bmatrix} \frac{1}{2}\cos(\theta + \Delta\theta/2) - \frac{\Delta s}{2b}\sin(\theta + \Delta\theta/2) & \frac{1}{2}\cos(\theta + \Delta\theta/2) + \frac{\Delta s}{2b}\sin(\theta + \Delta\theta/2) \\ \frac{1}{2}\sin(\theta + \Delta\theta/2) + \frac{\Delta s}{2b}\cos(\theta + \Delta\theta/2) & \frac{1}{2}\cos(\theta + \Delta\theta/2) - \frac{\Delta s}{2b}\sin(\theta + \Delta\theta/2) \\ \frac{1}{b} & -\frac{1}{b} \end{bmatrix}$$

$$\Sigma_u = \begin{bmatrix} k_r |\Delta s_r| & 0 \\ 0 & k_l |\Delta s_l| \end{bmatrix}$$

Example EKF - 2: Update

- Innovation or measurement residual
  $$\tilde{y}_k = z_k - h(\hat{x}_{k|k-1})$$
- Innovation (or residual) covariance
  $$S_k = H_k P_{k|k-1} H_k^T + R_k$$
- (Optimal?) Kalman gain
  $$K_k = P_{k|k-1} H_k^T S_k^{-1}$$
- Updated (a posteriori) state estimate
  $$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k \tilde{y}_k$$
- Updated (a posteriori) estimate covariance
  $$P_{k|k} = \left(I - K_k H_k\right) P_{k|k-1}$$

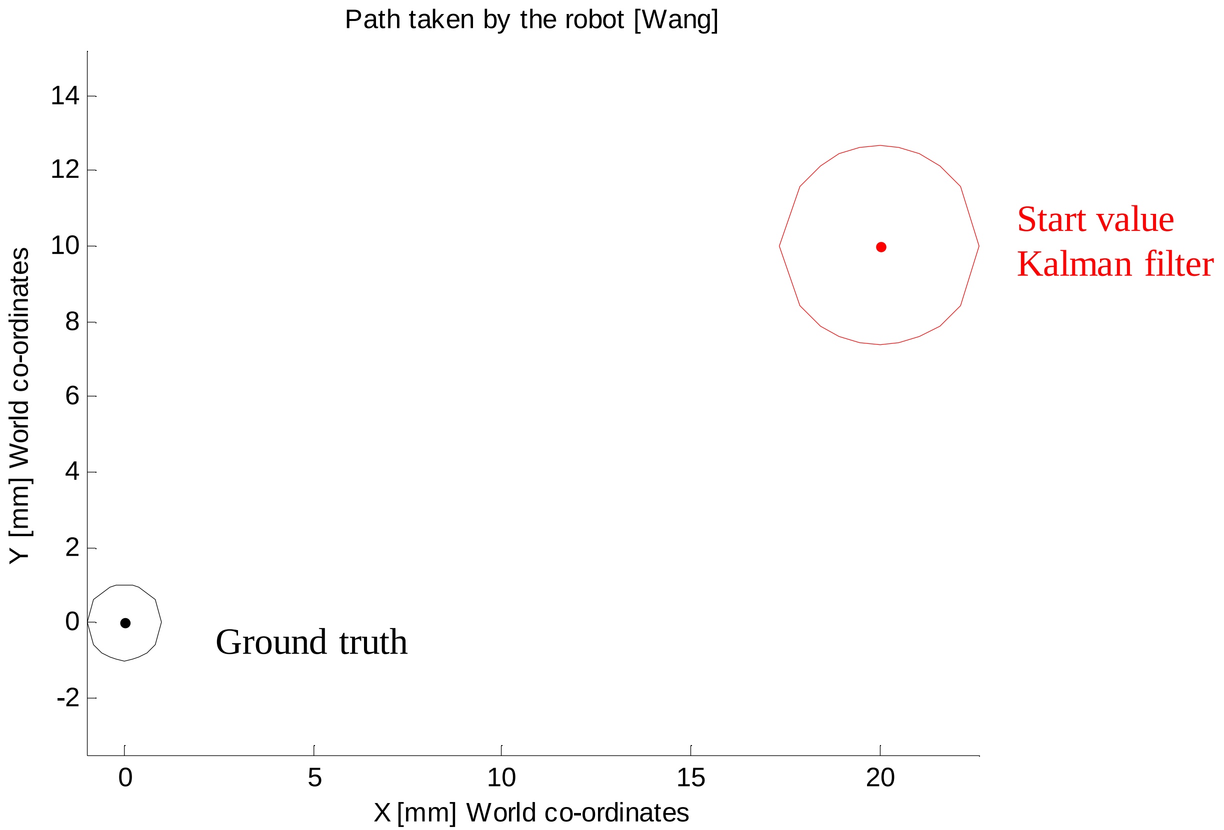
Beacon based navigation system that gives position (not orientation) with fix covariance.

$$z_k = \begin{bmatrix} x_{beacon} & y_{beacon} \end{bmatrix}^T$$

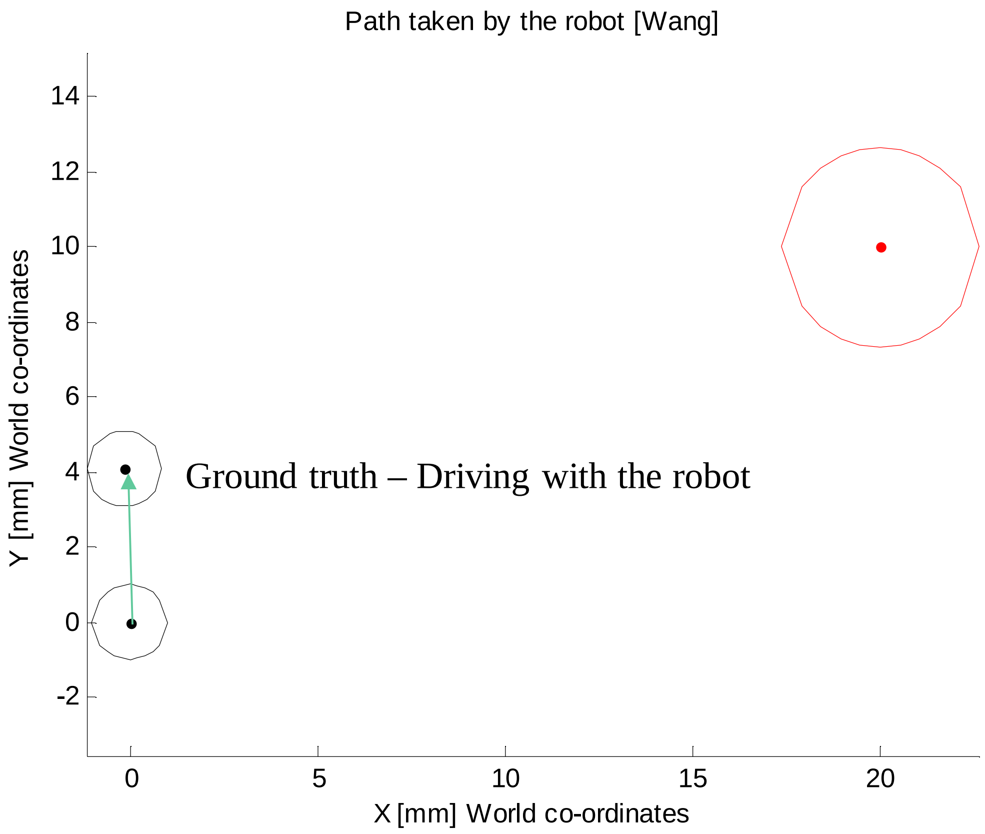
$$R_k = \begin{bmatrix} \sigma_{x_{beacon}}^2 & 0 \\ 0 & \sigma_{y_{beacon}}^2 \end{bmatrix}$$

$$H_k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

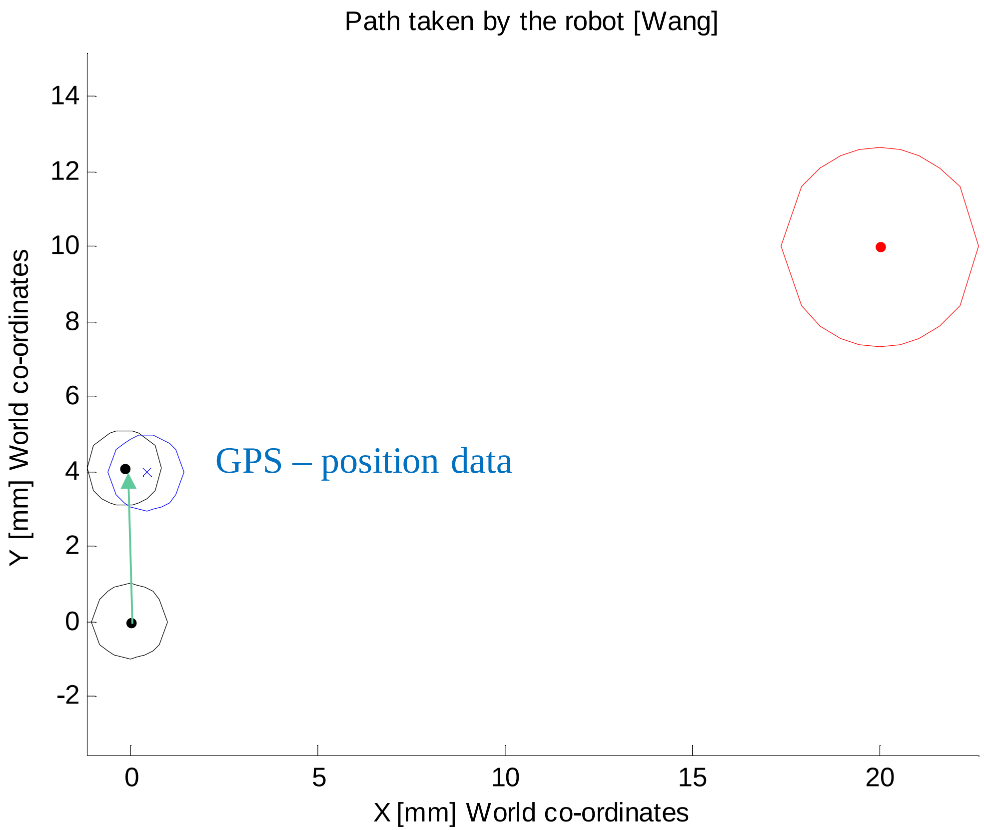
Example


Path taken by the robot [Wang]
Y [mm] World co-ordinates
X [mm] World co-ordinates
14
12
10
8
6
4
2
0
-2
0
5
10
15
20
Start value
Kalman filter
Ground truth

Example

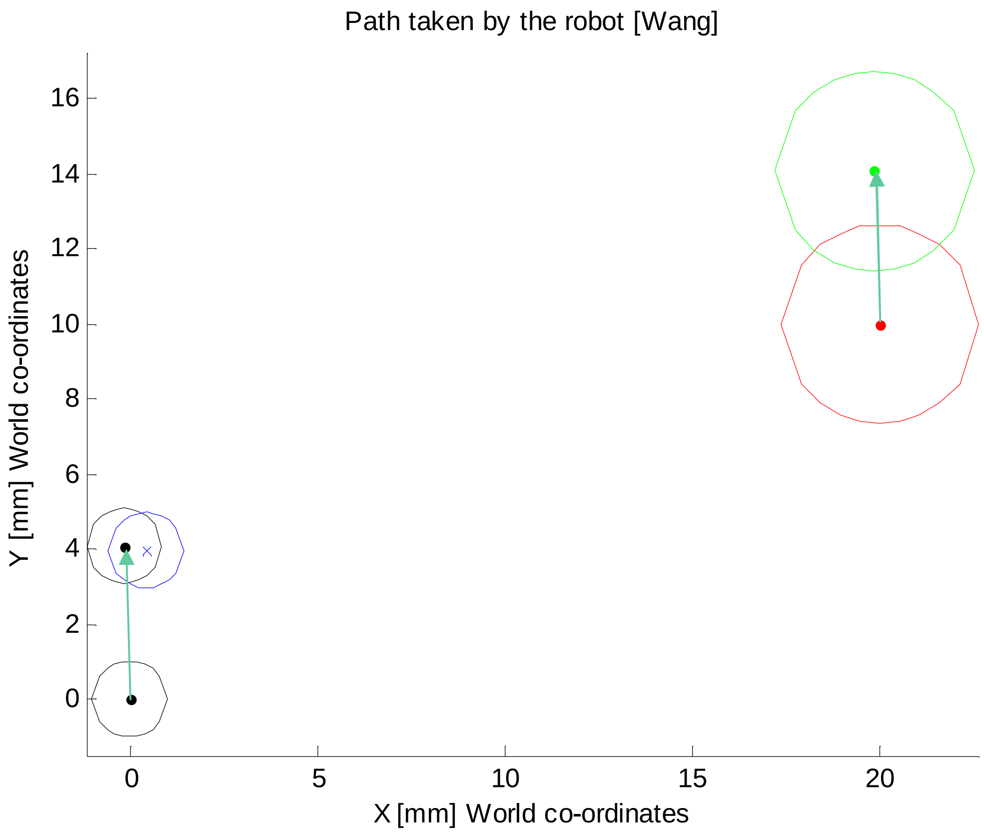

Path taken by the robot [Wang]
Y [mm] World co-ordinates
X [mm] World co-ordinates
14
12
10
8
6
4
2
0
-2
0
5
10
15
20
Ground truth – Driving with the robot

Example



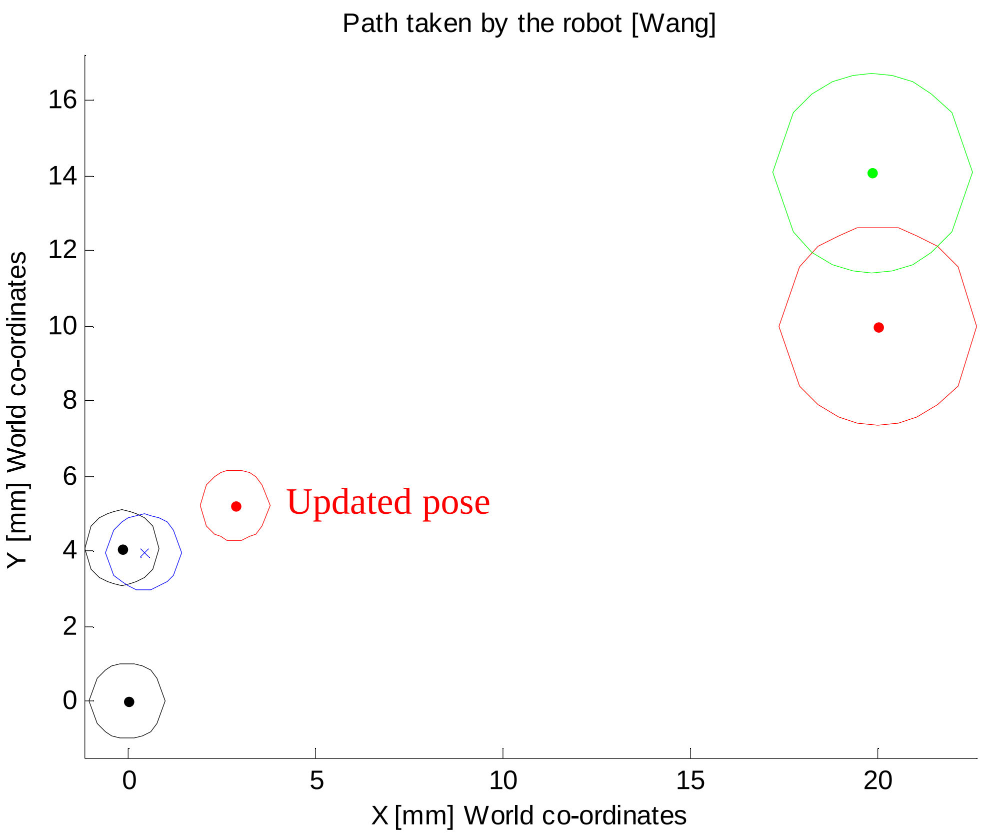
Path taken by the robot [Wang]
GPS – position data
Y [mm] World co-ordinates
X [mm] World co-ordinates
14
12
10
8
6
4
2
0
-2
0
5
10
15
20

Example



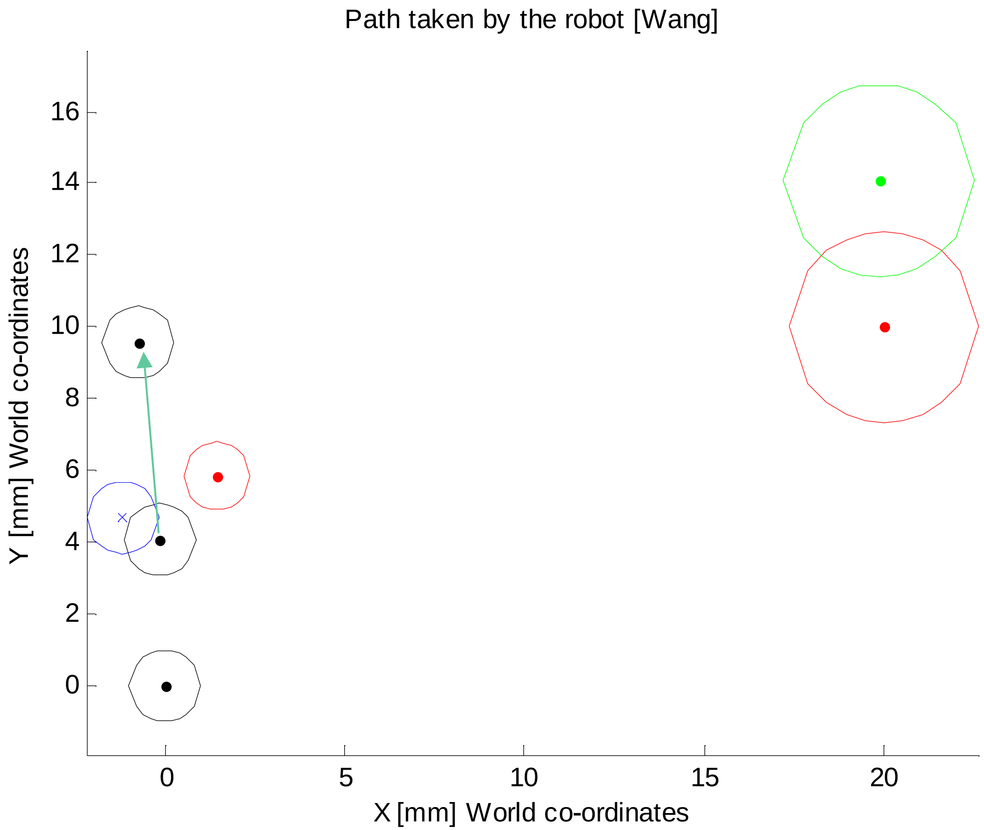
Estimated position based on odometry

Example



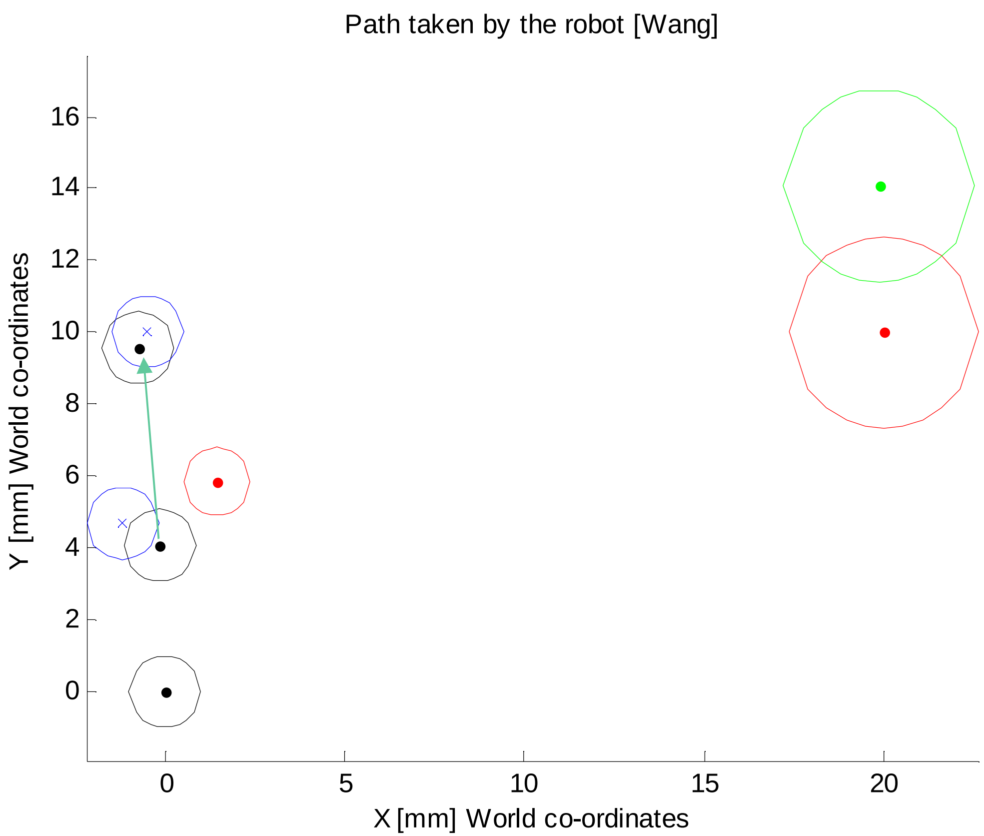
Path taken by the robot [Wang]
Updated pose
Y [mm] World co-ordinates
X [mm] World co-ordinates
16
14
12
10
8
6
4
2
0
0
5
10
15
20

Example



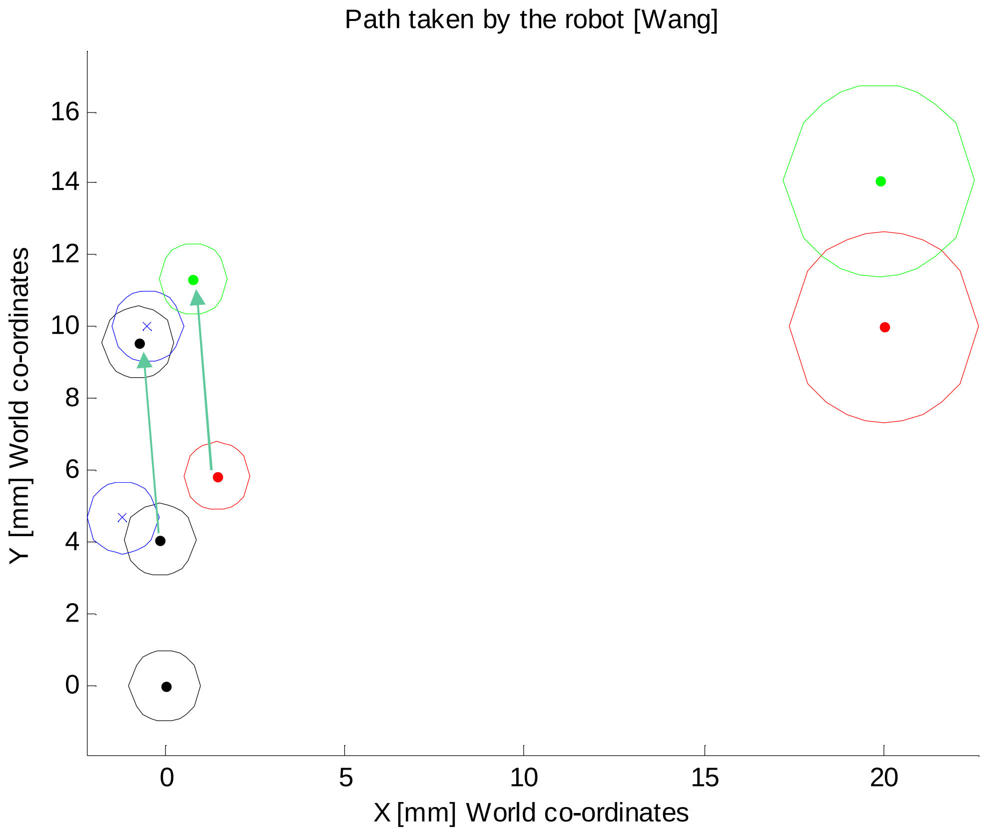
Path taken by the robot [Wang]
16
14
12
10
8
6
4
2
0
Y [mm] World co-ordinates
0
5
10
15
20
X [mm] World co-ordinates

Example



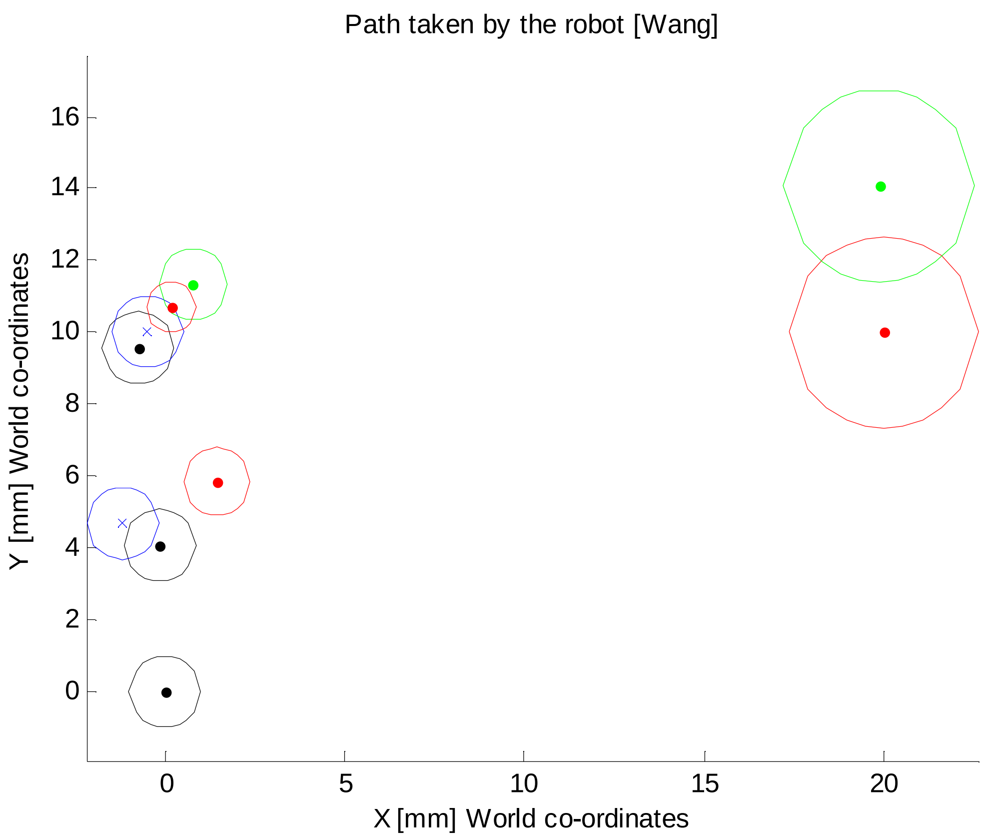
Path taken by the robot [Wang]
16
14
12
10
8
6
4
2
0
Y [mm] World co-ordinates
0
5
10
15
20
X [mm] World co-ordinates

Example



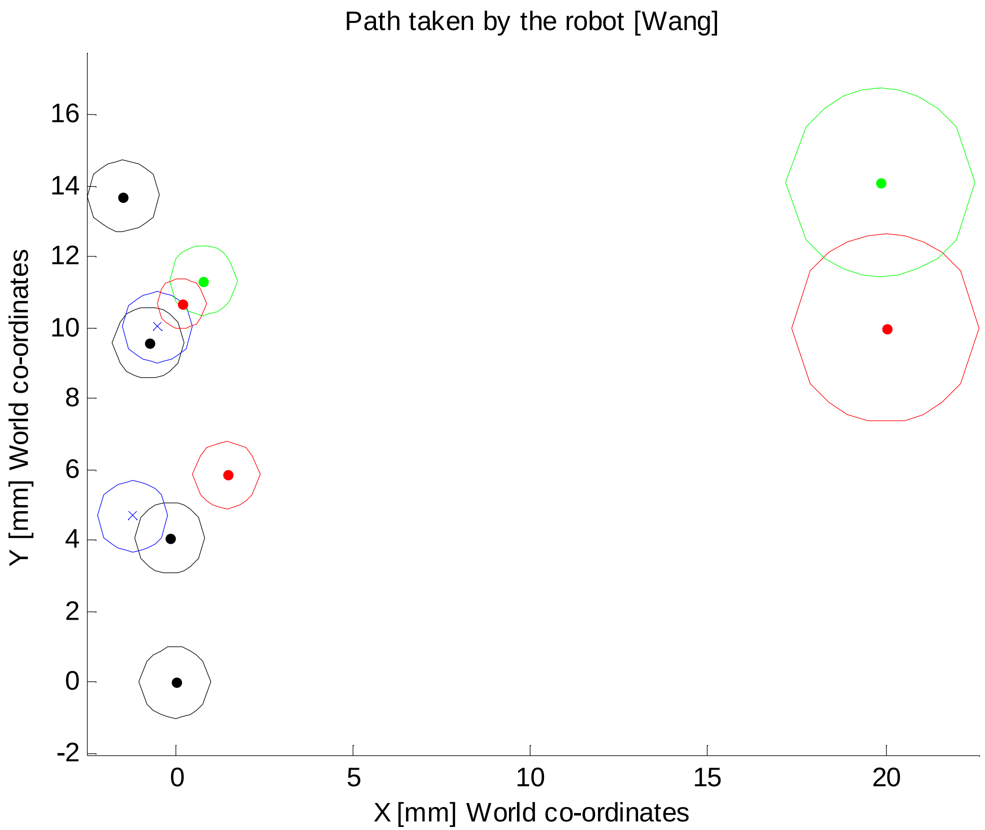
Path taken by the robot [Wang]
Y [mm] World co-ordinates
16
14
12
10
8
6
4
2
0
0
5
10
15
20
X [mm] World co-ordinates

Example



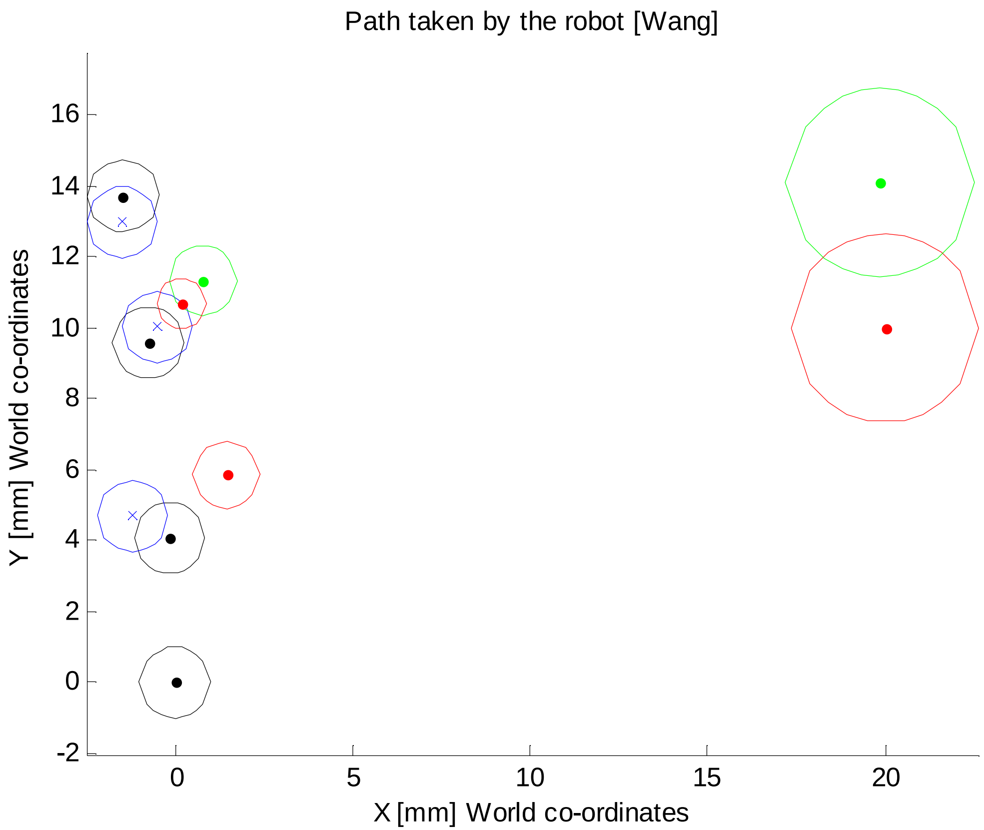
Path taken by the robot [Wang]
Y [mm] World co-ordinates
X [mm] World co-ordinates
16
14
12
10
8
6
4
2
0
0
5
10
15
20

Example



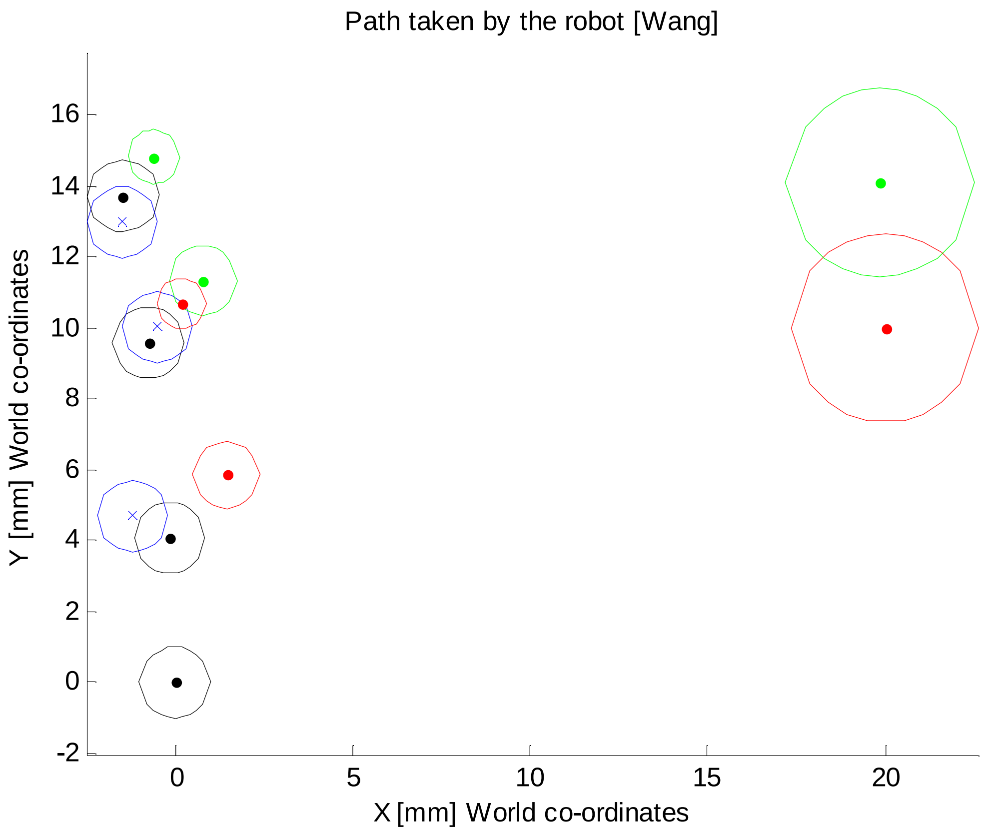
Path taken by the robot [Wang]
Y [mm] World co-ordinates
X [mm] World co-ordinates
16
14
12
10
8
6
4
2
0
-2
0
5
10
15
20

Example



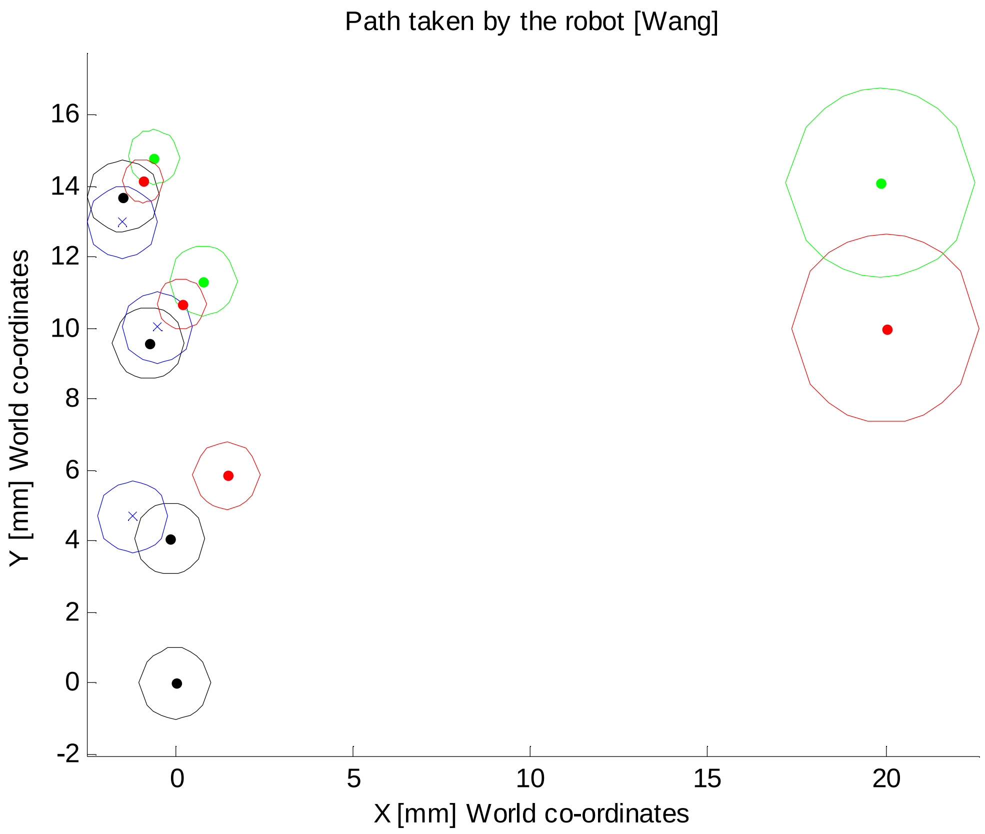
Path taken by the robot [Wang]
Y [mm] World co-ordinates
X [mm] World co-ordinates
16
14
12
10
8
6
4
2
0
-2
0
5
10
15
20

Example

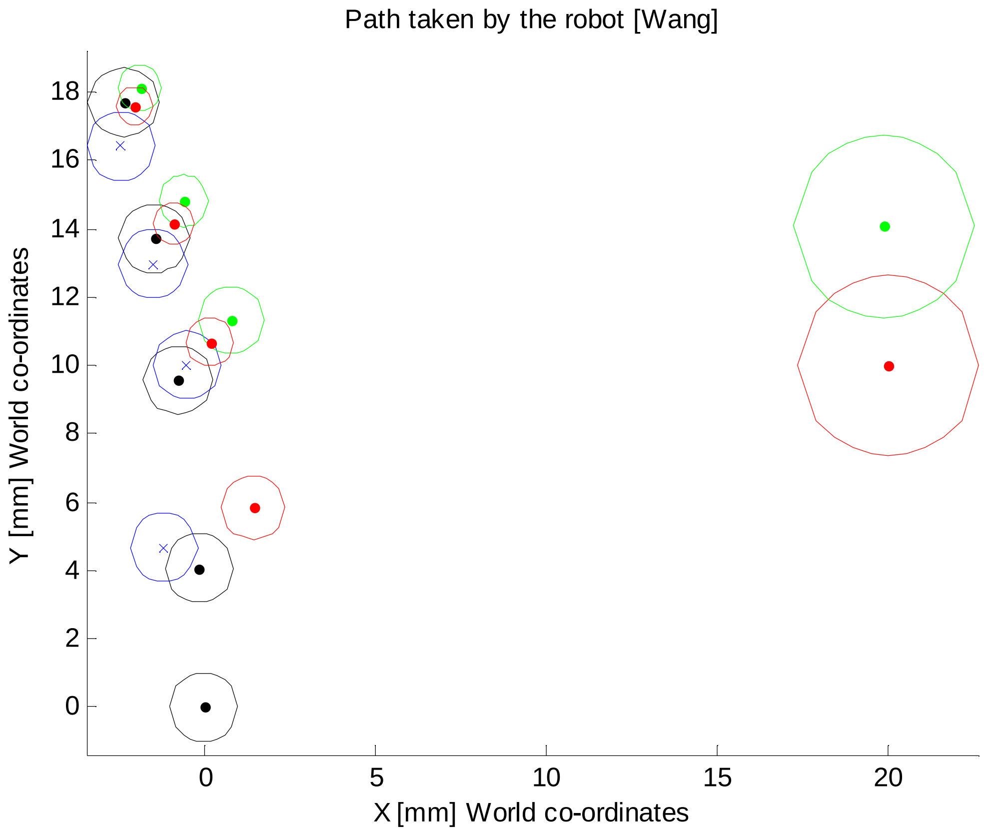


Path taken by the robot [Wang]
Y [mm] World co-ordinates
X [mm] World co-ordinates
16
14
12
10
8
6
4
2
0
-2
0
5
10
15
20

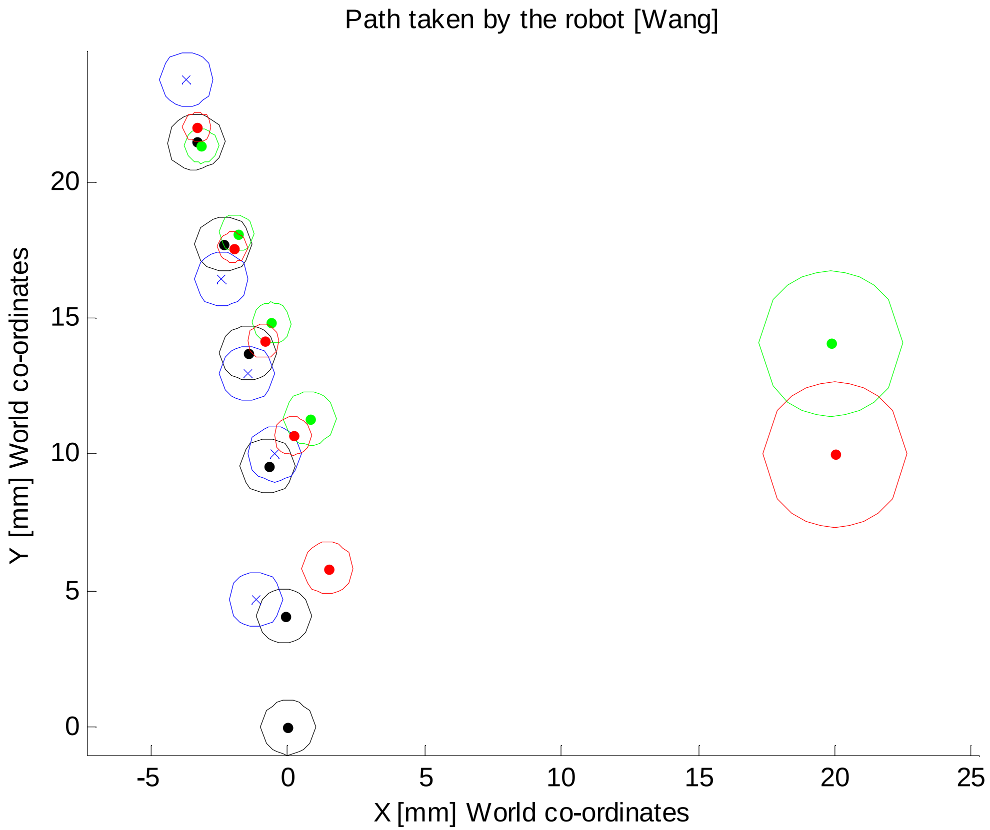
Example


Path taken by the robot [Wang]
Y [mm] World co-ordinates
X [mm] World co-ordinates
16
14
12
10
8
6
4
2
0
-2
0
5
10
15
20

Example

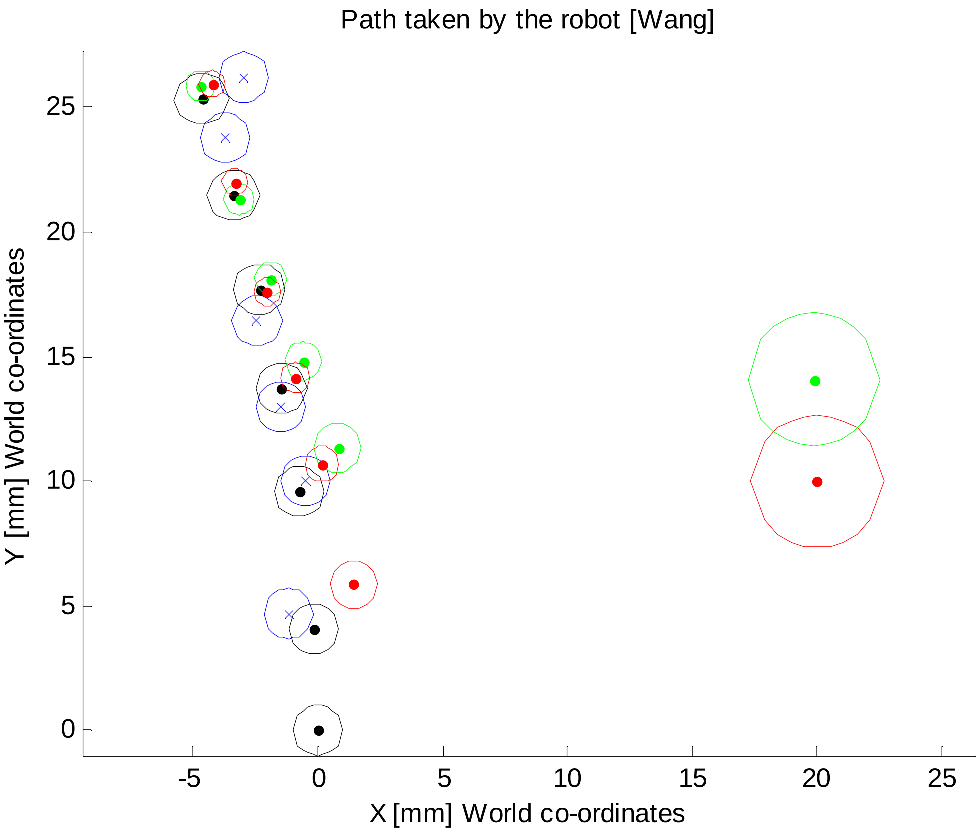

Path taken by the robot [Wang]
Y [mm] World co-ordinates
X [mm] World co-ordinates
18
16
14
12
10
8
6
4
2
0
0
5
10
15
20

Example



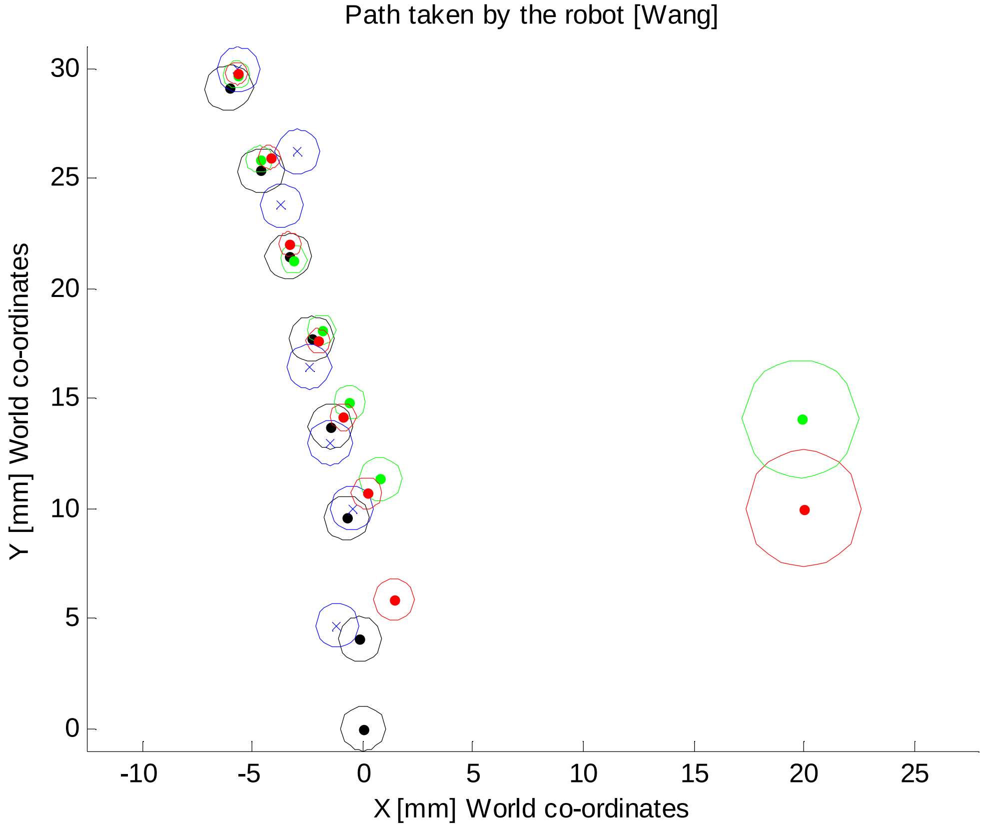
Path taken by the robot [Wang]
Y [mm] World co-ordinates
X [mm] World co-ordinates
20
15
10
5
0
-5
0
5
10
15
20
25

Example



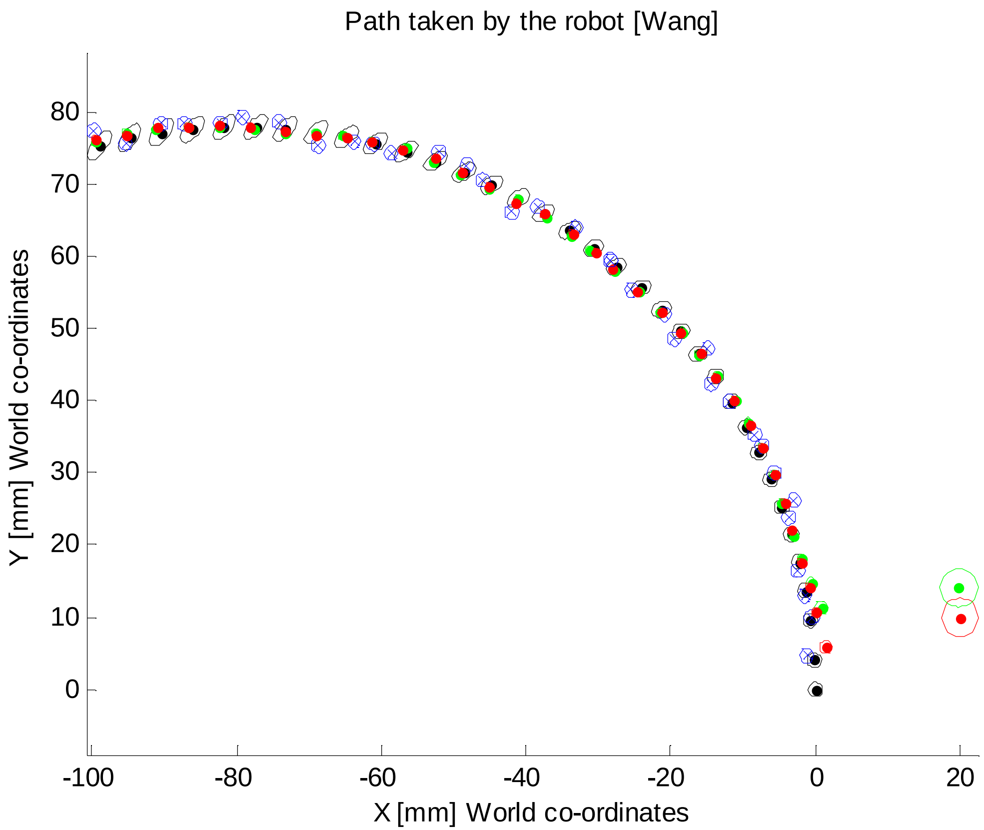
Path taken by the robot [Wang]
Y [mm] World co-ordinates
X [mm] World co-ordinates
25
20
15
10
5
0
-5
0
5
10
15
20
25

Example



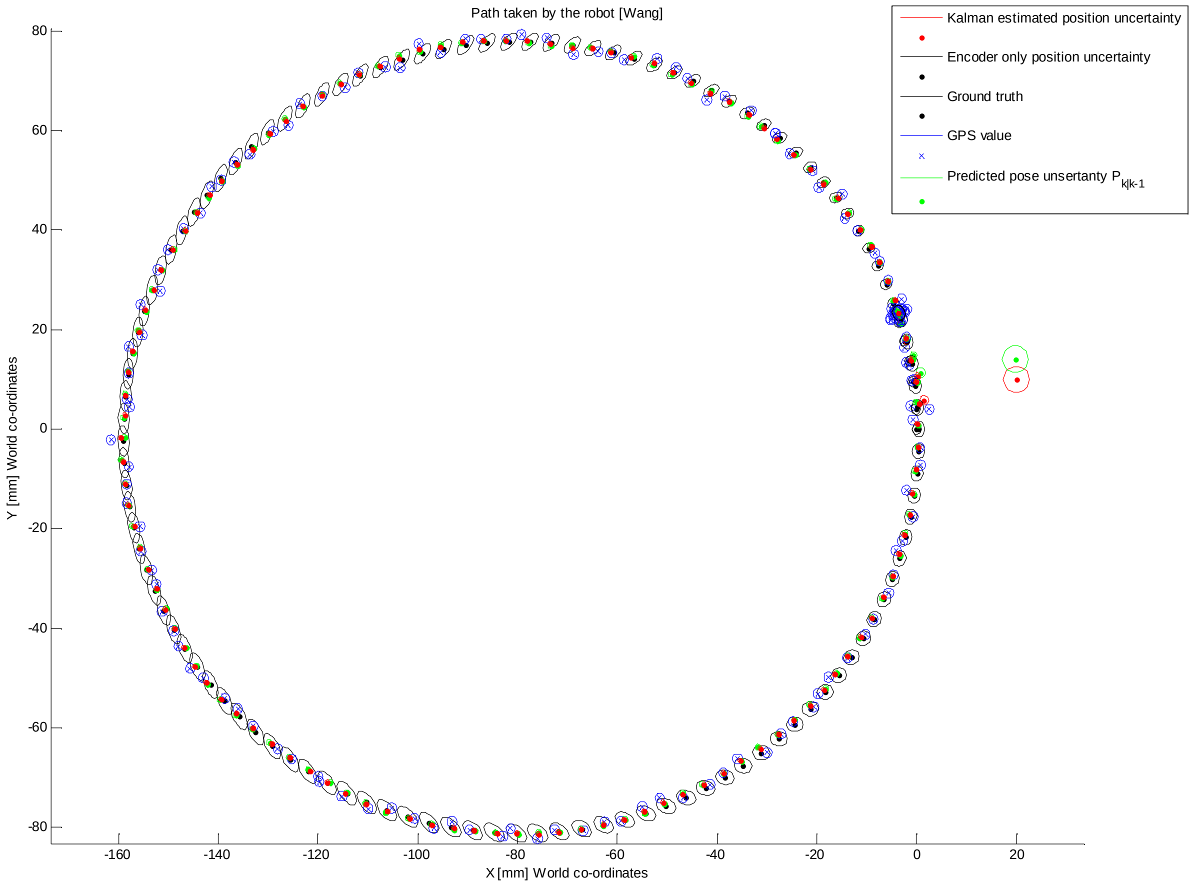
Path taken by the robot [Wang]
Y [mm] World co-ordinates
X [mm] World co-ordinates
30
25
20
15
10
5
0
-10
-5
0
5
10
15
20
25

Example



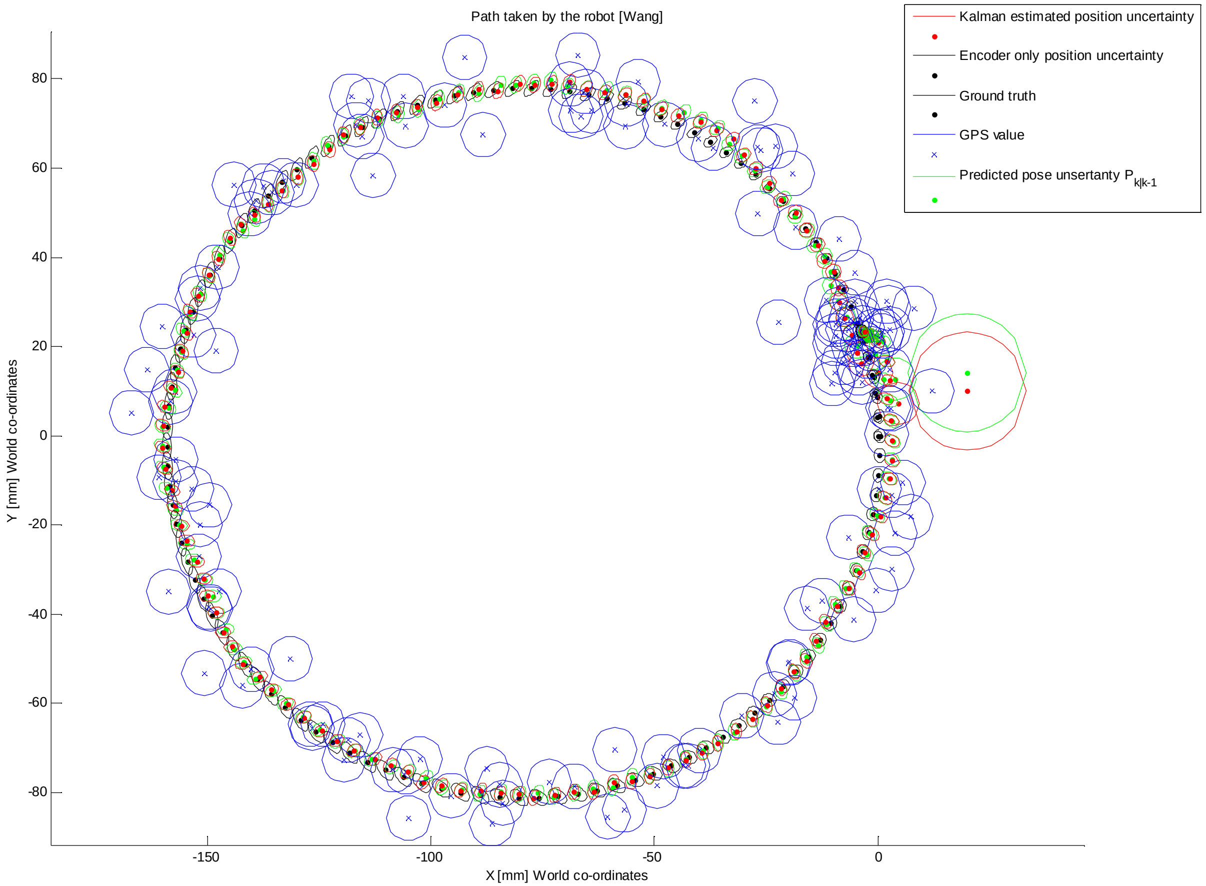
Path taken by the robot [Wang]
Y [mm] World co-ordinates
X [mm] World co-ordinates
80
70
60
50
40
30
20
10
0
-100
-80
-60
-40
-20
0
20

Example



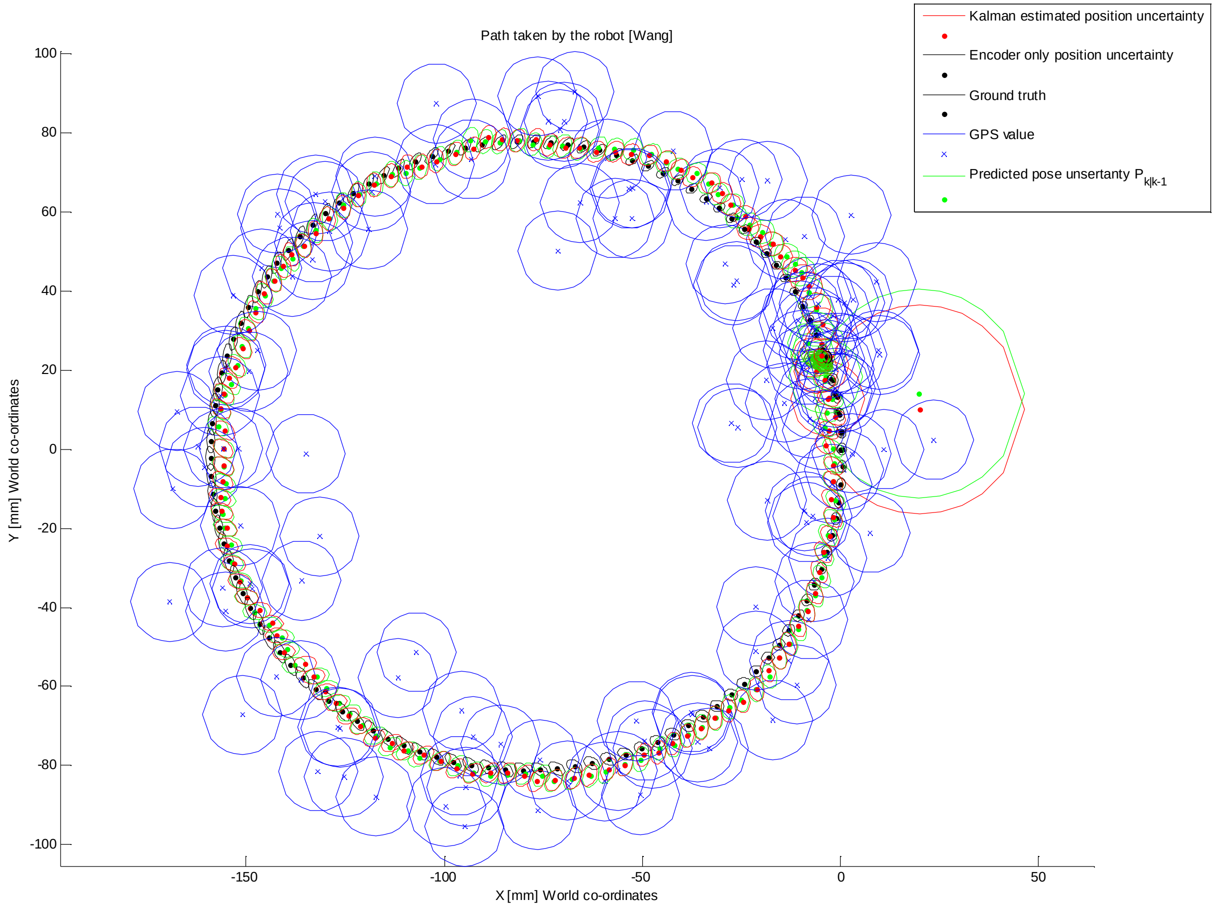
$\sigma_{GPS} = 1$

Example



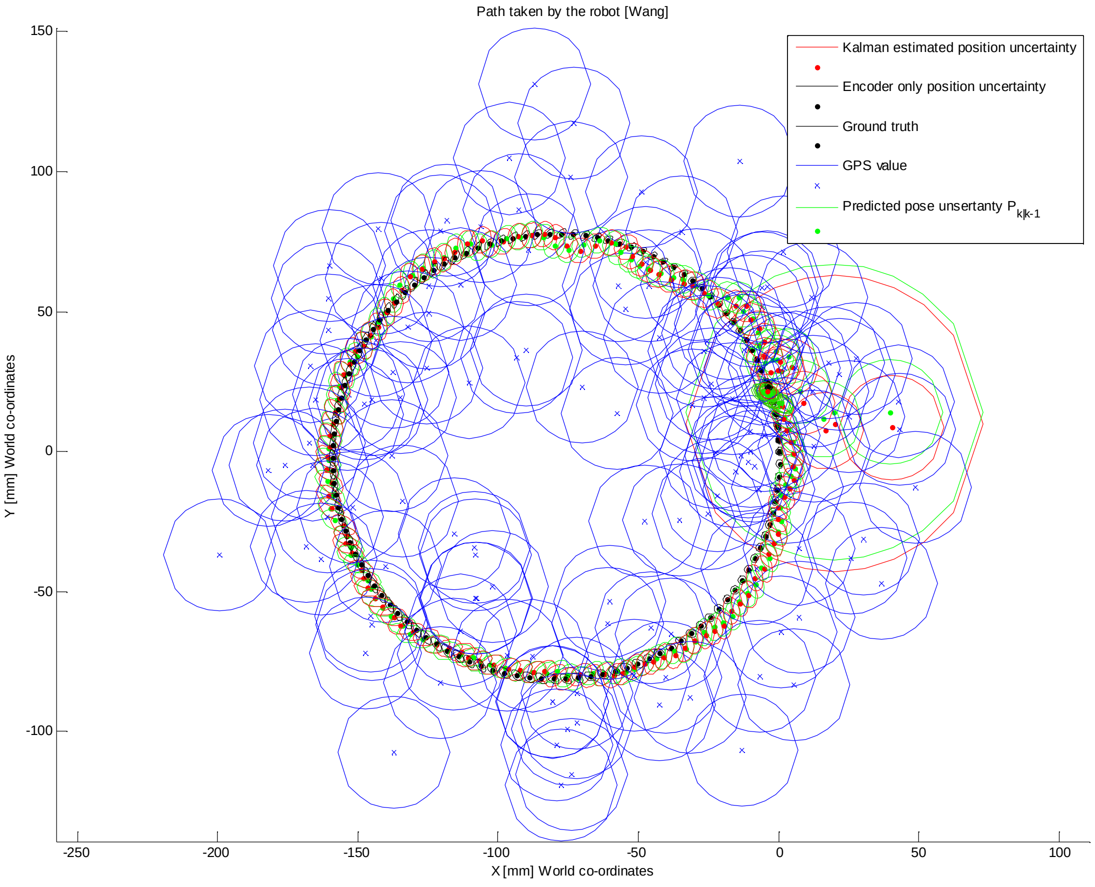
$\sigma_{GPS} = 5$

Example



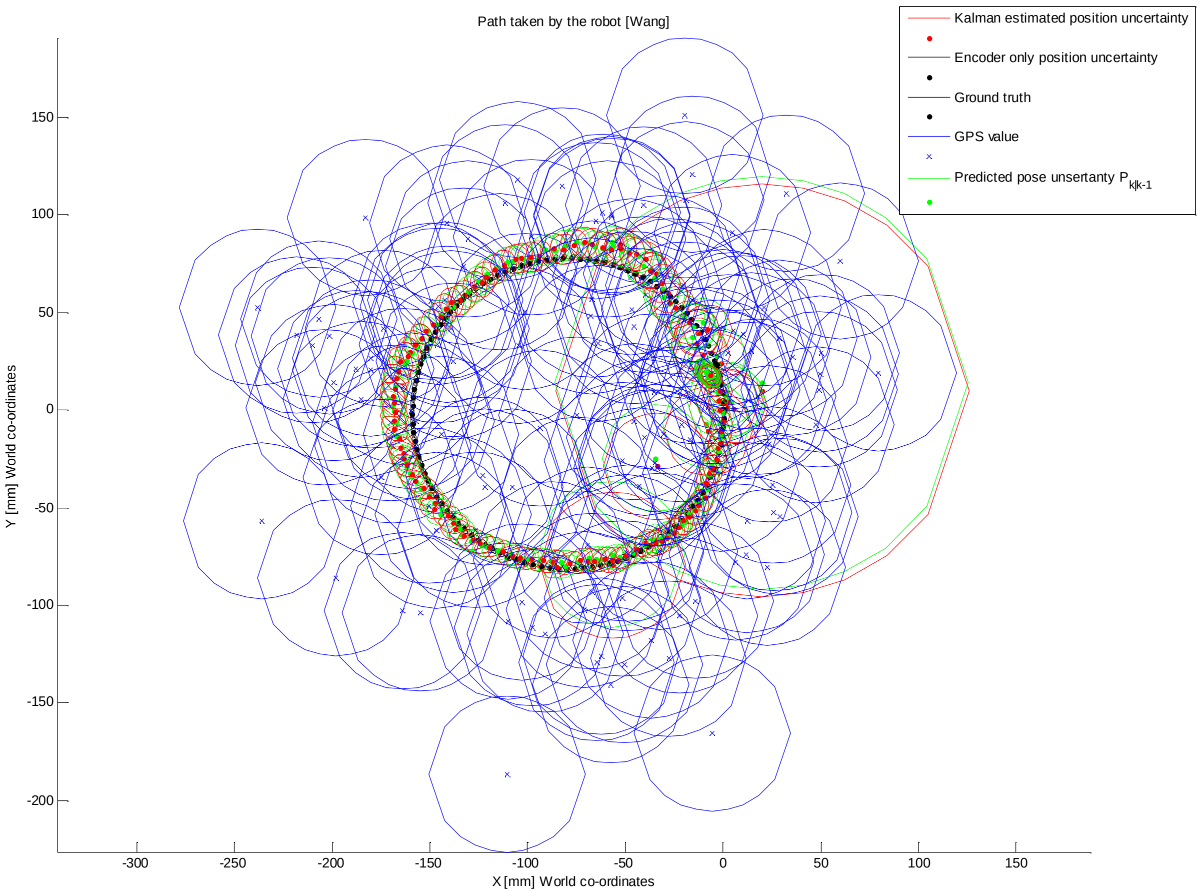
$\sigma_{GPS} = 10$

Example



$\sigma_{GPS} = 20$

Example



$\sigma_{GPS} = 40$