MDS 6106 — Optimization and Modeling

Exercise Sheet 3

Assignment A3.1 (Bisection and Golden Section Method): (approx. *16* points)
In this first exercise, we investigate the performance of the bisection and golden section method. Consider the minimization problem

$$\min_{x \in \mathbb{R}} \; g(x) := x \cdot \sin(x) \cdot \arctan(x) \qquad \text{s.t.} \qquad x \in [-3.5, 7.5].$$

where arctan is the arcus (inverse) tangent.

Implement the bisection and golden section method to solve this problem. Compare the number of iterations required to recover a solution in $[-3.5, 7.5]$ with accuracy less or equal than 10^{-5}. Do the bisection and golden section method converge to the same points?

Assignment A3.2 (Descent Directions): (approx. *10* points)
Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function and consider $\boldsymbol{x} \in \mathbb{R}^n$ with $\nabla f(\boldsymbol{x}) \neq 0$. Verify the following statements:

a) Set $\boldsymbol{d} = -(\nabla f(\boldsymbol{x})_j) \cdot \boldsymbol{e}_j = -\frac{\partial f}{\partial x_j}(\boldsymbol{x}) \cdot \boldsymbol{e}_j$, where $\boldsymbol{e}_j \in \mathbb{R}^n$ is the j-th unit vector and $j \in \{1, \dots, n\}$ is an index satisfying

$$\left| \frac{\partial f}{\partial x_j}(\boldsymbol{x}) \right| = \max_{1 \le i \le n} \left| \frac{\partial f}{\partial x_i}(\boldsymbol{x}) \right| = \|\nabla f(\boldsymbol{x})\|_\infty.$$

Then, $\boldsymbol{d}$ is a descent direction of f at $\boldsymbol{x}$.

b) Let $\boldsymbol{D} = \operatorname{diag}(\delta_1, \dots, \delta_n) \in \mathbb{R}^{n \times n}$ be a diagonal matrix with $\delta_i > 0$ for all i and define $\boldsymbol{d} = -\boldsymbol{D}^{-1} \nabla f(\boldsymbol{x})$. Show that $\boldsymbol{d}$ is a descent direction of f at $\boldsymbol{x}$.

Assignment A3.3 (Lipschitz Continuity): (approx. *24* points)
Discuss whether the gradient of the following mappings is Lipschitz continuous or not:

a) $f_1 : \mathbb{R}^3 \to \mathbb{R}$, $f_1(\boldsymbol{x}) := \frac{3}{2}x_1^2 + 2x_1x_2 - \frac{1}{3}x_3^3$.

b) $f_2 : \mathbb{R}^2 \to \mathbb{R}$, $f_2(\boldsymbol{x}) := \sqrt{1 + x_1^2 + x_2^2}$.

c) $f_3 : \mathbb{R}^2 \to \mathbb{R}$, $f_3(\boldsymbol{x}) := \ln(1 + x_1^2) + \ln(1 + x_2^2)$.

Report the corresponding Lipschitz constant L in case ∇f_i, $i = 1, 2, 3$, is Lipschitz continuous.

Sheet 3 is due on **Nov, 2nd**. Submit your solutions before **Nov, 2nd**, **11:00 pm**.