Enclosing The Center

Question: n points are chosen at random on the circumference of a circle. A convex n-gon (n sided polygon) is drawn by joining these n points. What is the probability that the center of circle lies inside the region of n-gon?

Solution: First note that the center of circle lies inside the n-gon is when the n points are not in common semi-circle.

Let the n points be $x_1, x_2, \ldots, x_n$. Let A_i be the set of events that all other $n-1$ points are in the semi-circle started from x_i counter-clockwise. Then $\{n \text{ points in semi-circle}\} = \bigsqcup_{i=1}^{n} A_i$ is a disjoint union.

$P(A_i) = \int_{\text{cirlce}} P(A_i|x_i = x) dP_{x_i}(x) = \int_{\text{cirlce}} (\frac{1}{2})^{n-1} dP_{x_i}(x) = (\frac{1}{2})^{n-1}$. Thus, $P(n \text{ points in semi-circle}) = \sum_{i=1}^{n} P(A_i) = \frac{n}{2^{n-1}}$. Thus, $P(\text{center of circle lies inside } n\text{-gon}) = 1 - \frac{n}{2^{n-1}}$.