Random Walk

Question: You are initially located at origin in the x-axis. You start a random walk with equal probability of moving left or right one step at a time. What is the probability that you will reach point a before reaching point $-b$? What is the expected number of steps to reach either a or $-b$? (a, b are natural numbers.)

Solution: Let $X_0 = 0$ and X_n be position after n steps. Then $E(X_{n+1}|X_n, \ldots, X_0) = (X_n + 1) \cdot 1/2 + (X_n - 1) \cdot 1/2 = X_n$. Thus, $\{X_n\}$ is a martingale.

Let $Y_n = X_n^2 - n$. Then $E(Y_{n+1}|Y_n, \ldots, Y_0) = (X_n + 1)^2 \cdot 1/2 + (X_n - 1)^2 \cdot 1/2 - n - 1 = Y_n$. Thus, $\{Y_n\}$ is also a martingale.

Let $N = \min\{n > 0 \colon X_n = a \text{ or } -b\}$. Let $P_a := P(X_N = a)$ and $P_b := P(X_N = -b) = 1 - P_a$. Then $E(X_N) = a \cdot P_a + (-b) \cdot P_b$.

By Doob's Optional Stopping Theorem, $E(X_N) = E(X_0) = 0$. Thus, $P_a = \frac{b}{(a+b)}$.

By the same argument, $E(Y_N) = E(X_N^2 - N) = a^2 \cdot \frac{b}{(a+b)} + (-b)^2 \cdot \frac{a}{(a+b)} - E(N) = 0$. Thus, $E(N) = ab$.