

Consecutive Heads

Question: What is the expected number of coin tosses required to get n consecutive heads?

Solution: Let p be the probability of Head. Let X_n be the number of tosses required to get n consecutive heads. Let Y be toss after X_{n-1} .

Then

$$\begin{aligned} E(X_n) &= E(X_n|Y = \text{Head}) \cdot p + E(X_n|Y = \text{Tail}) \cdot (1 - p) \\ &= (E(X_{n-1}) + 1)p + (E(X_{n-1}) + 1 + E(X_n))(1 - p) \\ &= E(X_{n-1}) + 1 + (1 - p)E(X_n) \end{aligned}$$

Thus, $pE(X_n) = E(X_{n-1}) + 1$. Note $E(X_1) = 1/p$. Thus, $E(X_n) = \sum_{i=1}^n 1/p^i$.