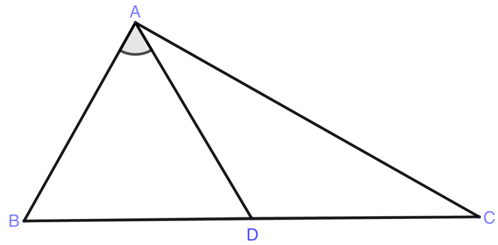
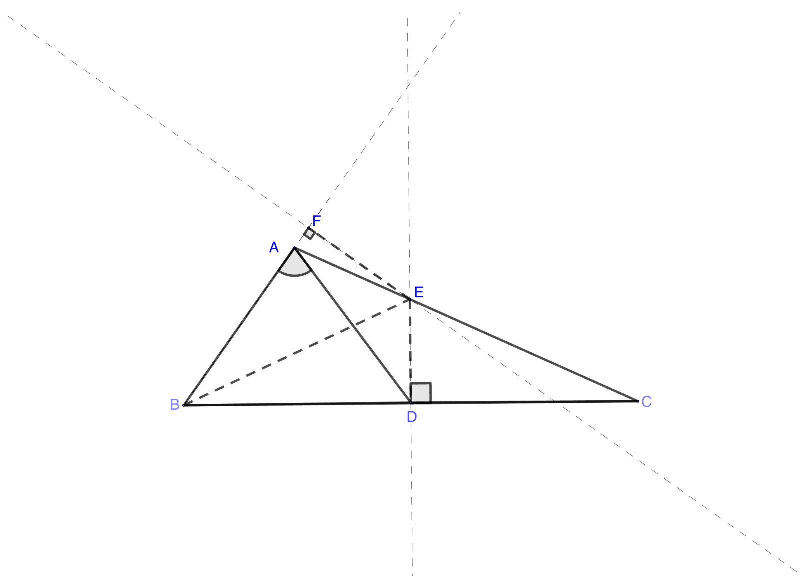
Question: In the following triangle $\triangle ABC$, $\angle B = 2\angle C$, $\angle BAD = 60°$ and $BD = DC$. Show $\angle BAC = 90°$ without using law of sines or cosines.



Proof:



Proof by contradiction. Suppose $\angle BAC > 90°$.

First since $BD = DC$, we can draw a perpendicular bisector of BC from point D, intersecting AC at E. Then we connect point B and point E. Lastly, we draw a perpendicular line from point E to side BA, intersecting at point F.

Since $\angle BAC > 90°$, point F is on the extension of side BA. Thus, $BF > BA$.

Since ED perpendicularly bisects BC, $\triangle EDB \cong \triangle EDC$. Thus, $\angle EBD = \angle C$. Since $\angle ABD = 2\angle C$, $\angle ABD = 2\angle EBD$ and $\angle EBA = \angle EBD$. Thus, E is on the angle bisector of $\angle ABD$. Thus, $\triangle EBF \cong \triangle EBD$ and $BD = BF > BA$. Therefore, $\angle BAD > \angle ADB$.

Let $\angle C = \alpha$. Then $\angle ABD = 2\alpha$. Since $\angle BAD = 60^\circ$, $\angle ADB = 180^\circ - 60^\circ - 2\alpha = 120^\circ - 2\alpha$. Since $\angle BAD > \angle ADB$, $60^\circ > 120^\circ - 2\alpha$ and thus $\alpha > 30^\circ$.

On the other hand, $\angle BAC = 180^\circ - \angle C - \angle ABC = 180^\circ - \alpha - 2\alpha = 180^\circ - 3\alpha > 90^\circ$ and thus $\alpha < 30^\circ$ which is a contradiction. Hence, $\angle BAC \leq 90^\circ$.

Similarly, we can show $\angle BAC \geq 90^\circ$. Hence, $\angle BAC = 90^\circ$.