Chess Tournament

Question: A chess tournament has K levels and 2^K players with skills $1 > 2 > ... > 2^K$. At each level, random pairs are formed and one person from each pair proceeds to next level. When two opponents play, the one with better skills always wins. What is the probability that players 1 and 2 will meet in the final level?

Solution: By the law of total probability,

$$
\begin{aligned}
&P(\text{meet in the final})\\
=&P(\text{not meet from 1 to } K-1)\\
=&P(\text{not meet } K-1|\text{ not meet from 1 to } K-2) \times P(\text{not meet } K-2|\text{ not meet from 1 to } K-3)\times\\
&\cdots \times P(\text{not meet 3 — not meet 1 and 2}) \times P(\text{not meet 2 — not meet 1}) \times P(\text{not meet 1})
\end{aligned}
$$

$P(\text{not meet } 1) = 1 - \frac{(2^K-2)!\times 2^{K-1}\times 2}{2^K!} = 1 - \frac{1}{2^K-1} = \frac{2^K-2}{2^K-1}$

$P(\text{not meet } K-k+1|\text{ not meet from 1 to } K-k) = \frac{2^k-2}{2^k-1}$

$P(\text{meet in the final}) = \frac{2^K-2}{2^K-1} \times \prod_{k=2}^{K-1} \frac{2(2^{k-1}-1)}{2^k-1} = \frac{2^K-2}{2^K-1} \times 2^{K-2} \times \frac{2^1-1}{2^{K-1}-1} = \frac{2^{K-1}}{2^K-1}$