To Begin or Not to begin?

Question: A and B are alternately picking balls from a bag without replacement. The bag has k black balls and 1 red ball. Winner is the one who picks the red ball. Who is more likely to win, the one who starts first, or second?

Solution: Let X be the number of turns for the one who starts first and wins. Note X can only be odd number.

$$\begin{aligned} P(\text{first win}) &= \sum_{i=0}^{\lfloor k/2 \rfloor} P(X = 2i+1) \\ &= \sum_{i=0}^{\lfloor k/2 \rfloor} \frac{k}{k+1}\frac{k-1}{k}\cdots\frac{k+1-2i}{k+2-2i}\frac{1}{k+1-2i} \\ &= \sum_{i=0}^{\lfloor k/2 \rfloor} \frac{1}{k+1} \\ &= \frac{1+\lfloor k/2 \rfloor}{k+1} \end{aligned}$$

There are two cases:

- k is even, $\frac{1+\lfloor k/2 \rfloor}{k+1} > \frac{1}{2}$
- k is odd, $\frac{1+\lfloor k/2 \rfloor}{k+1} = \frac{1}{2}$

When k is odd, it does not matter who starts first. When k is even, whoever starts first is more likely to win.