Number of Double Heads

Question: A coin is tossed 10 times and the output written as a string. What is the expected number of HH? Note that in HHH, number of $HH = 2$. (eg: expected number of HH in 2 tosses is 0.25, 3 tosses is 0.5)

Solution: Let X_n be the number of HH in n tosses.Note $E(X_1) = 0$, $E(X_2) = P(HH) = 0.25$.

Condition on the first toss:

$$\begin{aligned} E(X_n) &= E(X_n|T) \times P(T) + E(X_n|H) \times P(H) \\ &= E(X_{n-1}) \times 1/2 + E(X_n|H) \times 1/2 \end{aligned}$$

Compute $E(X_n|H)$:

Consider the following $n-1$ tosses. Then it has two groups depending on the second toss. Let $T?$ be the set of tosses that start with T and $H?$ be the set of tosses that start with H. For any toss t, let $HH(t)$ be number of HH in t.

Then by definition, $E(X_{n-1}) = \sum_{t \in T?} HH(t) \cdot P(t) + \sum_{t \in H?} HH(t) \cdot P(t)$.

Thus,

$$\begin{aligned} E(X_n|H) &= \sum_{t \in T?} HH(t) \cdot P(t) + \sum_{t \in H?} (HH(t) + 1) \cdot P(t) \\ &= \sum_{t \in T?} HH(t) \cdot P(t) + \sum_{t \in H?} HH(t) \cdot P(t) + \sum_{t \in H?} P(t) \\ &= E(X_{n-1}) + 1/2. \end{aligned}$$

Thus, $E(X_n) = E(X_{n-1}) \times 1/2 + (E(X_{n-1}) + 1/2) \times 1/2 = E(X_{n-1}) + 1/4$.

Thus, $E(X_n) = (n-1)/4$.