Sum To One

Question: On pressing a button, a random number is generated uniformly between 0 and 1. You keep on generating these numbers until the sum exceeds 1. What is the probability that you need to press the button more than n times? What is the expected number of times you need to press the button?

Solution: Let $X_1, X_2, \ldots, X_n, \ldots \overset{\text{i.i.d.}}{\sim} \text{Unif}(0,1)$ and $S_n = \sum_{i=1}^n X_i$.

For $n \geq 2$,

$$\begin{aligned}
P(S_n \leq 1) &= \int_0^1 P(S_n \leq 1 | X_n = x) dP_{X_n}(x) \\
&= \int_0^1 P(S_{n-1} \leq 1 - x | X_n = x) dx \\
&= \int_0^1 P(S_{n-1} \leq 1 - x) dx \\
&= \int_0^1 P(S_{n-1} \leq x) dx
\end{aligned}$$

This leads to define $P_n(x) := P(S_n \leq x)$. Applying the same calculation as above, $P_n(x) = \int_0^x P(S_{n-1} \leq y)dy = \int_0^x P_{n-1}(y)dy$. Then $P_n(x)' = P_{n-1}(x)$. Thus, $P_1(x) = P_n^{(n-1)}(x)$. Since $P_1(x) = P(X_1 \leq x) = x$, $P_n(x) = \frac{x^n}{n!}$. Thus, $P(S_n \leq 1) = \frac{1}{n!}$.

Let N be the least number of random numbers that sum over 1. There are two ways to compute $E(N)$:

- Note $P(N = n) = P(S_n > 1, S_{n-1} \leq 1) = P(S_{n-1} \leq 1) - P(S_n \leq 1) = \frac{1}{(n-1)!} - \frac{1}{n!}$ since $\{S_n \leq 1\} \subset \{S_{n-1} \leq 1\}$.

  Thus,

$$\begin{aligned}
E(N) &= \sum_{n=2}^{\infty} n \cdot P(N = n) \\
&= \sum_{n=2}^{\infty} n(\frac{1}{(n-1)!} - \frac{1}{n!}) \\
&= \sum_{n=2}^{\infty} \frac{1}{(n-2)!} \\
&= e.
\end{aligned}$$

- In fact, $P(N \geq n) = P(S_{n-1} \leq 1) = \frac{1}{(n-1)!}$.

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\begin{aligned}
E(N) &= \sum_{n=2}^{\infty} n \cdot P(N = n) \\
&= P(N \geq 2) + P(N \geq 2) + P(N \geq 3) + P(N \geq 4) + \cdots \\
&= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \\
&= e.
\end{aligned}
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