1 Basic notion

For any $n \in \mathbb{Z}^{>0}, \epsilon \in \mathbb{R}^{>0}, p \in \mathbb{R}^d$, denote by $\mathbb{D}^n_p(\epsilon) \subset \mathbb{R}^n$ the (closed) n-dimensional ball centered at p and of radius ϵ; denote by $\mathring{\mathbb{D}}^n_p(\epsilon)$ its interior. If $p = 0$ or $\epsilon = 1$, we omit it from the notation. Denote by S^n the standard n-dimensional sphere.

For a topological space B and a continuous function $g : B \to \mathbb{R}^{>0}$, define

$$\mathcal{D}_B(g) \xrightarrow{\pi^{\mathcal{D}}_B} B \qquad (\text{resp. } \mathring{\mathcal{D}}_B(g) \xrightarrow{\pi^{\mathcal{D}}_B} B)$$

to be the smooth sub-fiber bundle of $\mathbb{R}^d \times B$ such that the fiber over $b \in B$ is $\mathbb{D}^d(g(b))$ (resp. $\mathring{\mathbb{D}}^d(g(b))$). Let $\sigma^{\mathcal{D}}_B : B \to \mathcal{D}_B(g)$ (resp. $\mathring{\mathcal{D}}_B(g)$) be the zero-section. Similarly, define

$$\mathcal{A}_B(g) \xrightarrow{\pi^{\mathcal{A}}_B} B \qquad (\text{resp. } \mathring{\mathcal{A}}_B(g) \xrightarrow{\pi^{\mathcal{A}}_B} B)$$

to be the smooth sub-fiber bundle of $\mathbb{R}^d \times B$ such that the fiber over $b \in B$ is $\mathbb{R}^d \backslash \mathring{\mathbb{D}}^d(g(b))$ (resp. $\mathbb{R}^d \backslash \mathbb{D}^d(g(b))$).

Let $d \geq 4$ be an integer. Suppose M is a d-dimensional $\mathbb{Z}$-homology sphere.

Definition 1.1. An *(M, ∞)-bundle* is a tuple $(\pi : E \to B, \sigma, \tau)$, where

- $\pi : E \to B$ is a smooth fiber bundle whose fibers are diffeomorphic to M;
- $\sigma : B \to E$ is a smooth section of π;
- τ is a germ of trivializations of π near $\sigma(B)$, namely:
  - say (ϕ, U, g) is a trivialization of π near $\sigma(B)$ if
    - $U \subset E$ is a neighborhood of $\sigma(B)$, $g : B \to \mathbb{R}^{>0}$ is smooth and
    - $\phi : U \to \mathring{\mathcal{D}}_B(g)$ is a diffeomorphism satisfying

$$\pi^{\mathcal{D}}_B \circ \phi = \pi \qquad \phi \circ \sigma = \sigma^{\mathcal{D}}_B;$$

  - say two such (ϕ_1, U_1, g_1), (ϕ_2, U_2, g_2) are equivalent if there exists an open subset $U' \subset U_1 \cap U_2$ such that $\phi_1|_{U'} = \phi_2|_{U'}$;
  - τ is an equivalence class of such trivializations of π near $\sigma(B)$.

For later convenience, given such a (ϕ, U, g) in the equivalence class τ, we write $U' = U \backslash (\sigma(B))$, and define

$$\phi' : U' \to \mathring{\mathcal{A}}_B(1/g) \qquad \phi' = \big((b, x) \to (b, x/|x|^2)\big) \circ \phi|_{U'}.$$

Definition 1.2. We call such a $(\phi', U', 1/g)$ a *representative* of τ.

Definition 1.3. Let $(\pi : E \to B, \sigma, \tau)$ be an (M, ∞)-bundle. Denote by $T^v E$ the vertical tangent bundle of E. A *framing* F on π is a trivialization of $T^v E$ over $E \backslash \sigma(B)$,

$$F : T^v E|_{E \backslash \sigma(B)} \approx \mathbb{R}^d \times E \backslash \sigma(B),$$

such that, there exists (ϕ', U', g) a representative of τ, such that $\phi'_*(F)$ is the standard framing on $\mathring{\mathcal{A}}_B(1/g)$.

Given such a framing F and $p \in E \backslash \sigma(B)$, denote by $F|_p : T^v_p E \to \mathbb{R}^d$ the restriction of F to the vertical tangent space at p.

Definition 1.4. Let $(\pi : E \to B, \sigma, \tau)$ be an (M, ∞)-bundle and let F be a framing on π. Define the *exponential map*

$$\exp : T^v E|_{E\backslash\sigma(B)} \to E\backslash\sigma(B)$$

as follows. Suppose $v \in T^v_p E$. Suppose

$$F(v) = (v', p) \in \mathbb{R}^d \times E\backslash\sigma(B).$$

Then, $F^{-1}\big(\{v'\} \times E\backslash\sigma(B)\big)$ is a (vertical) vector field on $E\backslash\sigma(B)$, containing v. Let $\gamma_v : \mathbb{R}^{\geq 0} \to E\backslash\sigma(B)$ be the integral curve of this vector field such that $\gamma_v(0) = p$. Then, define $\exp(v) = \gamma_v(1)$.

Given $p \in E$, denote by $\exp_p : T^v_p E \to E$ the restriction of exp to the vertical tangent space at p.

Fact: For every $p \in E\backslash\sigma(B)$, there exists $\epsilon > 0$, such that, the restriction of

$$\exp_p \circ (F|_p)^{-1} : \mathbb{R}^d \longrightarrow E$$

to $\mathbb{D}^d(\epsilon)$ is an embedding. (cite some source for this)

For an (M, ∞)-bundle $(\pi : E \to B, \sigma, \tau)$, and for a finite set A, denote

$$C_A(\pi) = \{(x_i \in E)_{i\in A} \big| \forall i, x_i \notin \sigma(B) \text{ and } \forall i \neq j, x_i \neq x_j, \pi(x_i) = \pi(x_j)\};$$

it is an open subset of $\underbrace{E \times_B E \times_B \ldots \times_B E}_{A-\text{times}}$. Denote by $\overline{C}_A(\pi)$ its Fulton-MacPherson compactification. Abusing notation, we also denote by

$$\pi : C_A(\pi)(\text{resp. } \overline{C}_A(\pi)) \longrightarrow B$$

the map of a configuration to the base point the fiber over which the points in the configuration lie on.

Similarly, denote

$$C_A(\mathbb{R}^d) = \Big(\underbrace{\mathbb{R}^d \times \ldots \times \mathbb{R}^d}_{A-\text{times}} - (\text{the big diagonal})\Big) \Big/ \text{scaling and translations.}$$

Denote by $\overline{C}_A(\mathbb{R}^d)$ its Fulton-MacPherson compactification.

1.1 a choice and more notation

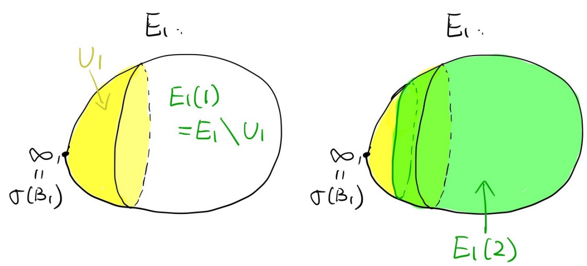
For the rest of this note, suppose we are given two d-dimensional $\mathbb{Z}$-homology spheres M_1, M_2, an (M_1, ∞)-bundle $(\pi_1 : E_1 \to B_1, \sigma_1, \tau_1)$ with framing F_1, and an (M_2, ∞)-bundle $(\pi_2 : E_2 \to B_2, \sigma_2, \tau_2)$ with framing F_2. Assume B_1, B_2 are compact.

For $i = 1, 2$, we choose a representative (τ'_i, U'_i, g_i) of τ_i and fix this choice for the rest of this document. Without loss of generality of all the arguments in this document, we assume $g_i \equiv 1$.

Some notation:

- For $N \geq 1$, denote

$$E_i(N) = E_i - \Big((\tau_i')^{-1}(B_i \times \mathbb{R}^d \backslash \mathbb{D}^d(N))\Big) \qquad E_i(N)^\circ = E_i - \Big((\tau_i')^{-1}(B_i \times \mathbb{R}^d \backslash \mathring{\mathbb{D}}^d(N))\Big).$$



- For $p \in E_i$, define "radius$^{\max}(p)$" to be the biggest $\epsilon > 0$ such that the map $\exp_p \circ (F|_p)^{-1}|_{\mathbb{D}^d(\epsilon)}$ is an embedding. For $0 < \epsilon < \text{radius}^{\max}(p)$, we define

$$D_p(\epsilon) := \exp_p \circ (F|_p)^{-1}(\mathbb{D}^d(\epsilon)) \subset E_i|_{\pi_i(p)}. \tag{1}$$

- Define $\rho_0 = \frac{1}{2}\min\big(\{\text{radius}^{\max}(p)\}_{p \in E_1 \cup E_2} \sqcup \{1\}\big)$ and $\rho = \frac{1}{4}\rho_0^2$.

2 Defining the bracket bundle

In this section, for $0 < t < \rho$, we define the bracket bundle, $\pi^t : E^t \to B^t$ (t is the "gluing parameter").

2.1 Defining the base

First, the base B^t is the same space for all t, so we call this space B', which we define now. It is obtained by gluing together three pieces:

$$E_1(3)^\circ \times B_2, \quad S^{d-1} \times (-\frac{1}{2}, \frac{1}{2}) \times B_1 \times B_2, \quad B_1 \times E_2(3)^\circ. \tag{2}$$

The gluing is done as follows: we glue part of the first piece

$$(E_1(3)^\circ - E_1(2)) \times B_2 \ \subset \ E_1(3)^\circ \times B_2$$

to part of the second piece

$$(-\frac{1}{2}, -\frac{1}{3}) \times B_1 \times B_2 \ \subset \ (-\frac{1}{2}, \frac{1}{2}) \times B_1 \times B_2$$

using the diffeomorphism

$$(E_1(3) - E_1(2)) \times B_2 \overset{\tau_1'}{\approx} (\mathring{\mathbb{D}}^d(3) - \mathbb{D}^d(2)) \times B_1 \times B_2 \approx S^{d-1} \times (2,3) \times B_1 \times B_2 \approx S^{d-1} \times (-\frac{1}{2}, -\frac{1}{3}) \times B_1 \times B_2,$$

where the last map is given by

$$(2,3) \to (-\frac{1}{2}, -\frac{1}{3}), \qquad x \to -1/x.$$

Similarly, we glue part of the third piece

$$B_1 \times (E_2(3)^\circ - E_2(2)) \ \subset \ B_1 \times E_2(3)^\circ$$

to part of the second piece

$$(\frac{1}{3}, \frac{1}{2}) \times B_1 \times B_2 \ \subset \ (-\frac{1}{2}, \frac{1}{2}) \times B_1 \times B_2$$

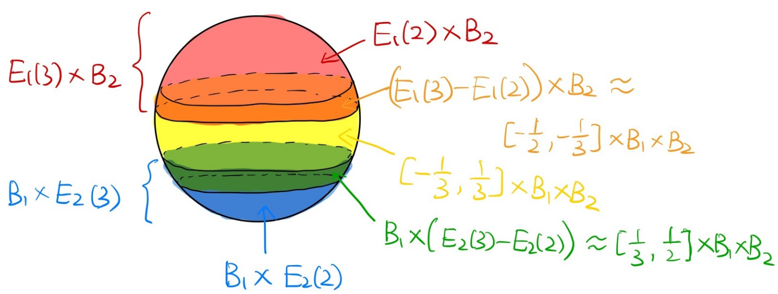
using the diffeomorphism

$$B_1 \times (E_2(3)^\circ - E_2(2)) \overset{\tau_2'}{\approx} B_1 \times (\mathring{\mathbb{D}}^d(3) - \mathbb{D}^d(2)) \times B_2 \approx S^{d-1} \times (2,3) \times B_1 \times B_2$$
$$\overset{\text{antipodal map on } S^{d-1}}{\approx} S^{d-1} \times (2,3) \times B_1 \times B_2 \approx (\frac{1}{3}, \frac{1}{2}) \times B_1 \times B_2,$$

where the last map is given by

$$(2,3) \to (\frac{1}{3}, \frac{1}{2}), \qquad x \to 1/x.$$

A picture of B':



Remark 2.1. The orientation on B' needs a bit of care.

This completes the definition of B'. We define $B_{\mathring{I}} = B' \times (0, \rho)$, and $\bar{\pi} : B_{\mathring{I}} \to (0, \rho)$ the projection to the second factor. Define $B^t = \bar{\pi}^{-1}(t)$.

2.2 Defining the total space

Now we define a bundle $\pi_{\mathring{I}} : E_{\mathring{I}} \to B_{\mathring{I}}$ that restricts to π^t for each $t \in (0, \rho)$.

2.2.1 Over the first part of the base

We first construct $E_{\mathring{I}}|_{E_1(3)^\circ \times B_2 \times (0,\rho)}$, as follows:

Define the tautological bundle $(\mathrm{pr}^1_{B_1})^*E_1 \to E_1(3)^\circ \times B_2 \times (0,\rho)$ as the pull back bundle

$$\begin{tikzcd}
(\mathrm{pr}^1_{B_1})^*E_1 \arrow[r, "f_1"] \arrow[d, "(\mathrm{pr}^1_{B_1})^*(\pi_1)"] & E_1 \arrow[d, "\pi_1"] \\
E_1(3)^\circ \times B_2 \times (0,\rho) \arrow[r, "\mathrm{pr}^1_{B_1}"] & B_1
\end{tikzcd}$$

where $\mathrm{pr}^1_{B_1}$ is by projecting to the first factor and then mapping by π_1. It has a canonical section

$$\begin{aligned}
s_1 : E_1(3)^\circ \times B_2 \times (0,\rho) &\longrightarrow (\mathrm{pr}^1_{B_1})^*E_1 \\
(p, b_2, t) &\longrightarrow f_1^{-1}(p);
\end{aligned}$$

here, although f_1 itself is not a diffeomorphism, its restriction to each fiber is, and the "f_1^{-1}" here means the inverse of the restriction of f_1 to the fiber over (p, b_2, t). For a function

$$\lambda : E_1(3)^\circ \times B_2 \times (0,\rho) \longrightarrow (0, 2\rho_0),$$

let

$$\mathcal{D}(\lambda) \longrightarrow E_1(3)^\circ \times B_2 \times (0,\rho)$$

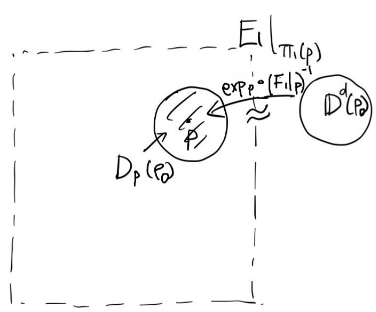
be the fiber bundle whose fiber over (p, b_2, t) is $\mathbb{D}^d(\lambda(p, b_2, t))$. We have a fiber bundle map

$$\begin{aligned}
\mathcal{D}(\lambda) &\longrightarrow (\mathrm{pr}^1_{B_1})^*E_1 \\
\big(v \in \mathbb{D}^d(\lambda(p, b_2, t)), (p, b_2, t) \in E_1(3)^\circ \times B_2 \times (0,\rho)\big) &\longrightarrow f_1^{-1}\big(\exp_p((F_1|_p)^{-1}(v))\big)
\end{aligned} \tag{3}$$

which is an embedding. Define $\mathcal{N}(s_1, \lambda)$ to be the image of this map. It is a closed neighborhood of the image of s_1. (include a better picture?) For a function $g : (0,\rho) \to (0, 2\rho_o)$, define

$$\mathcal{D}(g(t)) = \mathcal{D}((p, b_2, t) \to g(t)), \qquad \mathcal{N}(s_1, g(t)) = \mathcal{N}(s_1, ((p, b_2, t) \to g(t))).$$

For a constant $c \in (0, 2\rho_0)$, define $\mathcal{D}(c) = \mathcal{D}(\lambda)$ and $\mathcal{N}(s_1, c) = \mathcal{N}(s_1, \lambda)$ where $\lambda \equiv c$ is the constant function.



Define also the following pull-back bundle:

$$\begin{tikzcd}
(\mathrm{pr}^1_{B_2})^*E_2 \arrow[r, "f_2'"] \arrow[d, "(\mathrm{pr}^1_{B_2})^*\pi_2"] & E_2 \arrow[d, "\pi_2"] \\
E_1(3)^\circ \times B_2 \times (0,\rho) \arrow[r, "\mathrm{pr}^1_{B_2}"] & B_2
\end{tikzcd}$$

where $\mathrm{pr}^1_{B_2}$ is the projection onto the B_2 factor. For a function

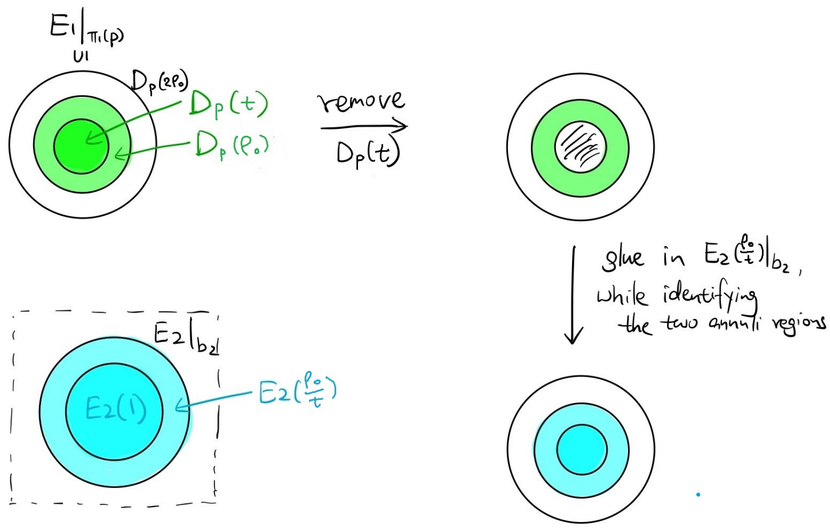
$$\lambda : E_1(3)^\circ \times B_2 \times (0,\rho) \longrightarrow [1,\infty),$$

define $(\mathrm{pr}^1_{B_2})^* E_2(\lambda)$ to be the disk sub-bundle of $(\mathrm{pr}^1_{B_2})^* E_2$ whose fiber over (p, b_2, t) is $E_2(\lambda(p, b_2, t))|_{b_2}$. (include a picture?) For a function $g : (0,\rho) \to (0, 2\rho_o)$, define

$$(\mathrm{pr}^1_{B_2})^* E_2(g(t)) = (\mathrm{pr}^1_{B_2})^* E_2((p, b_2, t) \to g(t)).$$

Now, $E_{\mathring{I}}|_{E_1(3)^\circ \times B_2 \times (0,\rho)}$ is constructed by removing $\mathcal{N}(s_1, t)$ from $(\mathrm{pr}^1_{B_1})^* E_1$, and glue back $(\mathrm{pr}^1_{B_2})^* E_2(\rho_0/t)$. The gluing is done via the bundle-isomorphism

$$\begin{aligned}(\mathrm{pr}^1_{B_2})^* E_2(\rho_0/t) - (\mathrm{pr}^1_{B_2})^* E_2(1) &\overset{\tau_2'}{\approx} (\mathbb{D}^d(\rho_0/t) - \mathbb{D}^d(1)) \times E_1(3)^\circ \times B_2 \times (0,\rho) \\ &\overset{\text{scaling by } t}{\approx} \mathcal{D}(\rho_0) - \mathcal{D}(t) \overset{(3)}{\approx} \mathcal{N}(s_1, \rho_0) - \mathcal{N}(s_1, t). \qquad (4)\end{aligned}$$



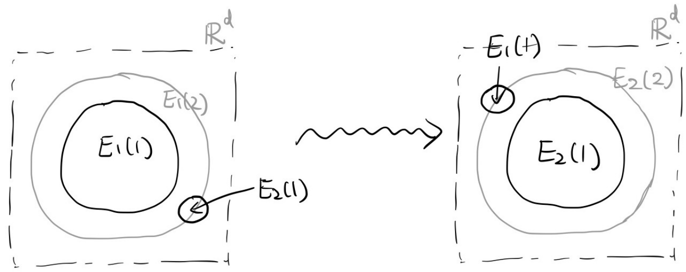
This completes the construction of $E_{\mathring{I}}|_{E_1(3)^\circ \times B_2 \times (0,\rho)}$. Clearly, the projection map

$$E_{\mathring{I}}|_{E_1(3)^\circ \times B_2 \times (0,\rho)} \to E_1(3)^\circ \times B_2 \times (0,\rho)$$

is a submersion. The construction of $E_{\mathring{I}}|_{B_1 \times E_2(3)^\circ \times (0,\rho)}$ is similar.

2.2.2 Over the second part of the base

It remains to construct $E_{\mathring{I}}|_{S^{d-1} \times (-\frac{1}{2}, \frac{1}{2}) \times B_1 \times B_2 \times (0,\rho)}$. The intuition here is that we gradually increase the size of $E_2(1)$ and decrease the size of $E_1(1)$ in $\mathbb{R}^d$, as the picture below.

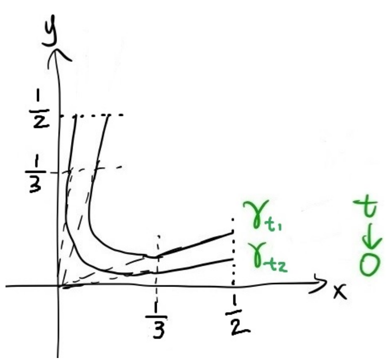


The bit of technicality here is to make this procedure smooth with respect to $E_{\mathring{I}}|_{E_1(3)^\circ \times B_2 \times (0,\rho)}$ and $E_{\mathring{I}}|_{B_1 \times E_2(3)^\circ \times (0,\rho)}$. To do this, we view $E_1(1), E_2(1)$ as (taking up the spaces of) two disks inside of $\mathbb{R}^d$, and considered only up to scaling and translation of $\mathbb{R}^d$. So, to normalize, we can choose the scale in such a way that the distance between the centers of $E_1(1)$ and $E_2(1)$ is 1. And the only variables are the radii of the two disks $E_1(1), E_2(2)$, which we (temporarily) denote by x and y in the remaining of this section. When

$$b = (p, b_1, b_2) \in (E_1(3) - E_1(1)) \times B_2 \approx (\mathring{\mathbb{D}}^d(3) - \mathbb{D}^d(1)) \times B_1 \times B_2,$$

the distance between the centers of $E_1(1)$, $E_2(2)$ is $|p|$, while the radius of $E_1(1)$ is 1 and the radius of $E_2(1)$ is t. So, after normalizing the distance between the centers of the disks to be 1, the radius of $E_1(1)$, i.e. x, becomes $1/|p|$, and the radius of $E_2(1)$, i.e. y, becomes $t/|p|$. As $|p|$ increases to 3, (x, y) decreases to $(\frac{1}{3}, \frac{t}{3})$, while keeping the ratio $y/x = t$. The case when $b \in B_1 \times E_2(3)$ is similar, except that the roles of x, y are swapped.

So, to make the smooth transition in between, we fix a choice of a 1-parameter family of curves, $\gamma_t : \mathbb{R} \to \mathbb{R}^2$ $(0 < t < \rho)$, in the first quarter of $\mathbb{R}^2$, like this:



satisfying the following conditions:

- $\gamma : (0, \rho) \times [-1/2, 1/2] \to \mathbb{R}^2$,defined by $\gamma(t, s) = \gamma_t(s)$ is a smooth embedding, and its image lies in the $(x > 0, y > 0)$ quarter of $\mathbb{R}^2$;
- for $s \le -1/3$, $\gamma_t(s) = (-ts, -s)$;
- for $s \ge 1/3$, $\gamma_t(s) = (s, ts)$;
- for all $s \in [-1/2, 0]$, $\lim_{t\to 0} \gamma_t(s) = (0, -s)$;
- for all $s \in [0, 1/2]$, $\lim_{t\to 0} \gamma_t(s) = (s, 0)$.

We can now define $E_{\mathring{I}}|_{S^{d-1}\times[-\frac{1}{2},\frac{1}{2}]\times B_1\times B_2\times(0,\rho)}$.

Take the trivial bundle $\mathbb{R}^d \times S^{d-1} \times [-\frac{1}{2}, \frac{1}{2}] \times B_1 \times B_2 \times (0, \rho)$. For a constant $1 \leq c \leq 2$, let $Z_1(c)$ (resp. $Z_2(c)$) be the disk sub-bundle of this trivial bundle whose fiber over

$$b = (\theta, \eta, b_1, b_2, t) \in S^{d-1} \times [-\frac{1}{2}, \frac{1}{2}] \times B_1 \times B_2 \times (0, \rho)$$

is $\mathbb{D}_{p_1}(cx)$ (resp. $\mathbb{D}_{p_2}(cy)$), where $p_1, p_2 \in \mathbb{R}^d$ are such that

$$|p_2 - p_1| = 1, \frac{p_2 - p_1}{|p_2 - p_1|} = \theta, p_1 + p_2 = 0; \text{ and } (x, y) = \gamma_t(\eta). \tag{5}$$

For $i = 1, 2$ and $1 \leq c \leq 2$, we also define the pull-back bundle

$$\begin{array}{ccc} (\mathrm{pr}^2_{B_i})^* E_i(c) & \longrightarrow & E_i(c) \\ \downarrow & & \downarrow{\scriptstyle \pi_i} \\ S^{d-1} \times [-\frac{1}{2}, \frac{1}{2}] \times B_1 \times B_2 \times (0, \rho) & \xrightarrow{\mathrm{pr}^2_{B_i}} & B_i \end{array}$$

where $\mathrm{pr}^2_{B_i}$ is the projection to the B_i factor.

Now, define $E_{\mathring{I}}|_{S^{d-1}\times[-\frac{1}{2},\frac{1}{2}]\times B_1\times B_2}$ by removing $Z_1(1)$ and $Z_2(1)$ from the trivial bundle $\mathbb{R}^d \times S^{d-1} \times [-\frac{1}{2}, \frac{1}{2}] \times B_1 \times B_2$ and glue back $(\mathrm{pr}^2_{B_1})^* E_1(2)$ and $(\mathrm{pr}^2_{B_2})^* E_2(2)$, respectively, via the following diffeomorphisms that preserves framing up to scaling by a constant:

$$Z_1(2) - Z_1(1) \overset{\tau_1'}{\approx} (\mathrm{pr}^2_{B_1})^* E_1(2) - (\mathrm{pr}^2_{B_1})^* E_1(1), \qquad Z_2(2) - Z_2(1) \overset{\tau_2'}{\approx} (\mathrm{pr}^2_{B_2})^* E_2(2) - (\mathrm{pr}^2_{B_2})^* E_2(1).$$

It is easy to see that, over the overlapping parts of the there pieces (2) of the base, the definitions in Section 2.2.1 and in Section 2.2.2 agree, after we properly scale and translate $\mathbb{R}^d$.

This completes the definition of $\pi_{\mathring{I}} : E_{\mathring{I}} \to E_{\mathring{I}}$. Note that we clearly have a "section at ∞",

$$\sigma_{\mathring{I}} : B_{\mathring{I}} \longrightarrow E_{\mathring{I}}$$

induced from σ_1 and σ_2. And $\pi_{\mathring{I}}$ is an $(M_1 \# M_2, \infty)$-bundle, where # denotes connect sum.

2.3 Specifying an induced framing on the bracket bundle

On fibers over $S^{d-1} \times [-\frac{1}{2}, \frac{1}{2}] \times B_1 \times B_2 \times (0, \rho)$ it is the obvious one.

Next, we specify a vertical framing $F_{\mathring{I}}$ on $E_{\mathring{I}}|_{E_1(3)^\circ\times B_2\times(0,\rho)}$. (The $E_{\mathring{I}}|_{B_1\times E_2(3)^\circ\times(0,\rho)}$ case is similar, so we omit it below.) Recall that when defining $E_{\mathring{I}}|_{E_1(3)^\circ\times B_2\times(0,\rho)}$, we remove $\mathcal{N}(s_1, t)$ from $(\mathrm{pr}^1_{B_1})^* E_1$ and glue back $(\mathrm{pr}^1_{B_2})^* E_2(\rho_0/t)$ using the diffeomorphism (4):

$$\begin{aligned} (\mathrm{pr}^1_{B_2})^* E_2(\rho_0/t) - (\mathrm{pr}^1_{B_2})^* E_2(1) &\overset{\tau_2'}{\approx} (\mathbb{D}^d(\rho_0/t) - \mathbb{D}^d(1)) \times E_1(3) \times B_2 \times (0, \rho) \\ &\overset{\text{scaling by } t}{\approx} \mathcal{D}(\rho_0) - \mathcal{D}(t) \overset{\exp}{\approx} \mathcal{N}(s_1, \rho_0) - \mathcal{N}(s_1, t). \end{aligned}$$

Now,

- On $(\mathrm{pr}^1_{B_1})^* E_1 - \mathcal{N}(s_1, 2\sqrt{t})$, let $F_{\mathring{I}}$ be $f_1^* F_1$. (Note that since $t < \rho$, $2\sqrt{t} < \rho_0$.)
- On $(\mathrm{pr}^1_{B_2})^* E_2(1/\sqrt{t})$, let $F_{\mathring{I}}$ be $(f_2')^* F_2$. (Note that since $\rho < 1$, $1/\sqrt{t} > 1$.)
- It remains to specify $F_{\mathring{I}}$ on the region

$$(\mathrm{pr}^1_{B_2})^* E_2(2/\sqrt{t}) - (\mathrm{pr}^1_{B_2})^* E_2(1/\sqrt{t}) \overset{(4)}{\approx} \mathcal{N}(s_1, 2\sqrt{t}) - \mathcal{N}(s_1, \sqrt{t}). \tag{6}$$

Since the diffeomorphism (4) is not framing-preserving, we need to choose a homotopy in-between. The existence of such a homotopy is clear. So we choose and fix a framing on the region (6), that smoothly extends $F_{\mathring{I}}$ on the parts it has already been defined; and define $F_{\mathring{I}}$ on the region (6) using this framing we chose.

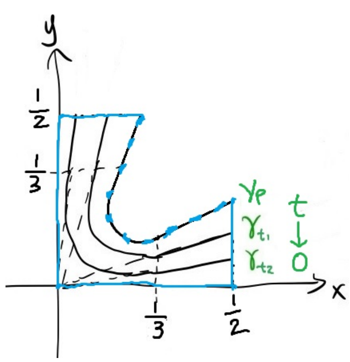
Now that $F_{\mathring{I}}$ is defined, for $p \in E_{\mathring{I}}$, we define $\exp_p$, $\mathrm{radius}^{\max}(p)$, $D_p(\epsilon)$ in the same way as in Section 1.1.

2.4 Defining the compactified base of the family of bracket bundles

In this section, we define B_I, the compactified base of the family of bracket bundles. The space B_I is a compactification of $B' \times (0, \rho)$ on the 0 side. More precisely, it is obtained by gluing together the following 3 pieces:

$$E_1(3)^\circ \times B_2 \times [0, \rho), \quad S^{d-1} \times L \times B_1 \times B_2, \quad B_1 \times E_2(3)^\circ \times [0, \rho), \tag{7}$$

where $L \subset \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0}$ is the L-shaped region enclosed by the blue lines below. (The solid blue lines are within L and the dashed blue curve is not in L but just part of the boundary of L.)



We have a homeomorphism (if we define γ_0 in the obvious way – it goes along the left and bottom blue lines, from $(0, 1/2)$ to $(1/2, 0)$)

$$\begin{aligned} [-\frac{1}{2}, \frac{1}{2}] \times [0, \rho) &\longrightarrow L \\ (s, t) &\longrightarrow \gamma_t(s), \end{aligned} \tag{8}$$

which is a diffeomorphism except at the point $(0, 0) \in L$.

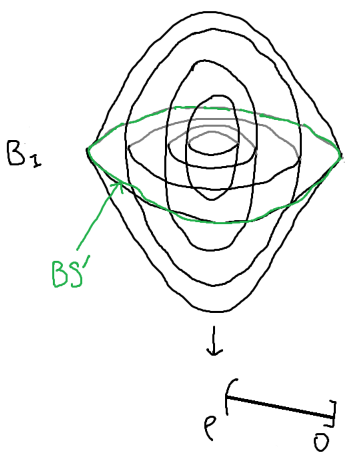
With the identification (8), we define B_I by gluing the 3 pieces in (7), in the exact same way as how we defined B' in Section 2.1, but now with everything timed with $[0, \rho)$. The result is a manifold with boundary and corners. The main stratum is $B_{\mathring{I}}$. It has 2 codim-1 boundary strata:

- one is by gluing together $E_1(3)^\circ \times B_2 \times \{0\}$ and $S^{d-1} \times (-\frac{1}{2}, 0) \times \{0\} \times B_1 \times B_2$ above; we call this stratum BS^+;
- the other is by gluing together $B_1 \times E_2(2)^\circ \times \{0\}$ and $S^{d-1} \times (0, \frac{1}{2}) \times \{0\} \times B_1 \times B_2$ above; we call this stratum BS^-.

And it has 1 codim-2 corner stratum which we call BS':

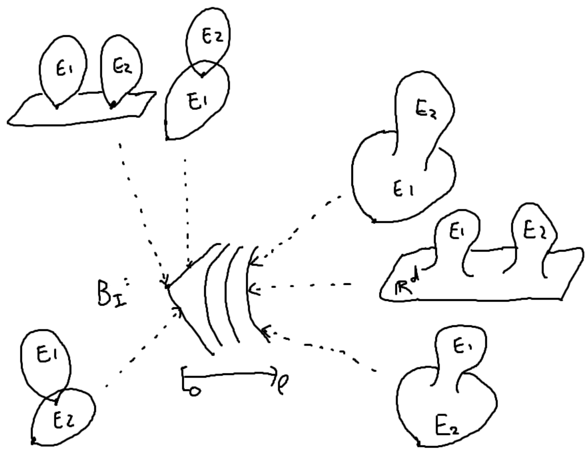
- $BS' = S^{d-1} \times \{0\} \times \{0\} \times B_1 \times B_2$, a subset of the second piece in (7).

Below is a picture of B_I when $d = 2$, $B_1 = B_2 =$point, with its map to $[0, \rho)$. Here B_I is visualized as a family of spheres parameterized by $[0, \rho)$ (in the picture, the sphere with bigger parameter is more inside). The outside most one (the one parameterized by 0), is actually not a smooth sphere—the upper and lower hemispheres are not glued together in a smooth way.



In Section 4 later, we will define, for a finite set A, $\widetilde{C}_A$ (what I called $\widetilde{Conf}_A$ in the emails)—the "family" configuration space. $\widetilde{C}_\emptyset$ is just B_I here.

B_I should be thought of a the parameter space of the fibers of bracket bundles (with different smoothing parameter $t \in [0, \rho)$). A picture of what the points of B_I are parameterizing:



We can't quite make this statement rigorous by defining a smooth "universal family" over B_I, because it will have singularities along the nodes.

2.5 Specifying a "trivialization near infinity" on the bracket bundle

In this section we specify a trivialization

$$\tau_{\mathring{I}} : U_{\mathring{I}} \longrightarrow B_{\mathring{I}} \times (\mathbb{R}^d \backslash \mathbb{D}^d(3))$$

of $E_{\mathring{I}} - \sigma_{\mathring{I}}(B_{\mathring{I}})$ in some neighborhood $U_{\mathring{I}}$ or $\sigma_{\mathring{I}}(B_{\mathring{I}})$. Once this is done, we define, for $N \geq 3$,

$$E_{\mathring{I}}(N) = E_{\mathring{I}} - \Big((\tau_{\mathring{I}})^{-1}\big(B_{\mathring{I}} \times (\mathbb{R}^d \backslash \mathbb{D}^d(N))\big)\Big), \qquad E_{\mathring{I}}(N)^\circ = E_{\mathring{I}} - \Big((\tau_{\mathring{I}})^{-1}\big(B_{\mathring{I}} \times (\mathbb{R}^d \backslash \mathring{\mathbb{D}}^d(N))\big)\Big).$$

2.5.1 when one bundle is inside another, trivialization near infinity of the bigger bundle

As before, for $i = 1, 2$, define the tautological bundle

$$\pi_i^* E_i \xrightarrow{\pi_i^* \pi_i} E_i - \sigma_1(B_i)$$

as the pull-back

$$\begin{array}{ccc} \pi_i^* E_i & \xrightarrow{\quad f_i \quad} & E_i \\ \Big\downarrow{\scriptstyle \pi_i^* \pi_i} & & \Big\downarrow{\scriptstyle \pi_i} \\ E_i - \sigma_i(B_i) & \xrightarrow{\quad \pi_i \quad} & B_i, \end{array}$$

and define

$$s_i : E_i - \sigma_i(B_i) \to \pi_i^* E_i, \qquad s_i(p) = f_i^{-1}(p)$$

its canonical section. Recall we have specified trivializations

$$\tau_i' : U_i' = E_i - \sigma_i(B_i) - E_i(1) \xrightarrow{\approx} B_i \times (\mathbb{R}^d \backslash \mathbb{D}^d).$$

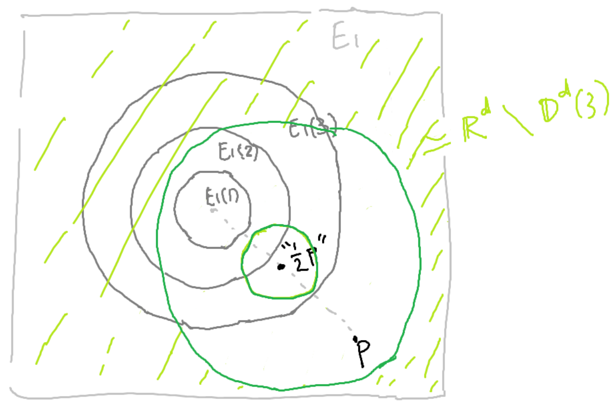
Write $\tau_i^{\mathbb{R}^d}: E_i - \sigma_i(B_i) - E_i(1) \to \mathbb{R}^d \backslash \mathbb{D}^d$ the composition of τ_i' with the projection to the $\mathbb{R}^d \backslash \mathbb{D}^d$ factor.

Now, we specify a trivialization

$$\hat{\tau}_i' : \hat{U}_i' \longrightarrow \big(E_i - \sigma_i(B_i)\big) \times \big(\mathbb{R}^d \backslash \mathbb{D}^d(3)\big)$$

of $\pi_i^* \pi_i$ in a (center-removed) neighborhood $\hat{U}_i'$ of the pulled-back section $\pi_i^* \sigma_i$. Before spelling out the details, what we want to do is the following:

- On fibers over $E_i(2)$, $\hat{\tau}_i'$ is just the pull-back $\pi_i^* \tau_i'$.
- On a fiber over $p \in E_i - \sigma_i(B_i) - E_i(3)$, $\hat{\tau}_i'$ is defined as follows.
  - First, shifting $\pi_i^* \tau_i'$, via a translation in $\mathbb{R}^d$, so that the "center" becomes the point "$1/2p$". Here is an illustration:



    In this picture, the spheres centered at 0 in $\mathbb{R}^d$, under $\pi_i^* \tau_i'$, are the gray circles; the light-green identification represents $\hat{\tau}_i'$.
  - Next, as p goes further away from $E_i(3)$, we also re-scale so that, although the radius of $(\hat{U}_i')|_p$ gets bigger and bigger (because $(\hat{U}_i')|_p$ cannot intersect $f_i^{-1}(E_i(1))$), with respect to $\pi_i^* \tau_i'$, we want the radius of $(\hat{U}_i')|_p$ to stay constant with respect to $\hat{\tau}_i'$.
- Over $E_i(3) - E_i(2)$, we gradually increases the shifting from 0 (the first case above) to shifting by $\frac{1}{2}p$ (the second case above).

Once $\hat{\tau}_i'$ is defined, we denote

$$\hat{E}_i(N) = \pi_i^* E_i - \Big((\hat{\tau}_i')^{-1} \big((E_i - \sigma_i(B_i)) \times \mathbb{R}^d \backslash \mathbb{D}^d(N) \big) \Big).$$

(The rest of Section 2.5 can be skipped when first reading.)

The details of defining $\hat{\tau}_i'$ is as follows:

First, define a smooth function $g_i : E_i - \sigma_i(B_i) \to \mathbb{R}^{\geq 1}$ by: when $p \in E_i(2)$, $g_i(p) = 3$; when $p \in E_i - \sigma_i(B_i) - E_i(3)$, $g_i(p) = 3|\tau_i^{\mathbb{R}^d}(p)|$; when $p \in E_i(3) - E_i(2)$, g_i is an arbitrarily chosen increasing function that makes $g_i(p)$ smooth.

- Over $E_i(2)$, define $\hat{U}_i' = f_i^{-1}(E_i - \sigma_i(B_i) - E_i(3))$ and $\hat{\tau}_i' = \pi_i^* \tau_i'|_{E_i - \sigma_i(B_i) - E_i(3)}$.
- Over $E_i - \sigma_i(B_i) - E_i(3)$, define $\hat{U}_i' \subset \pi_i^* E_i$ such that, for $p \in E_i - \sigma_i(B_i) - E_i(3)$,

$$\hat{U}_i' \cap (\pi_i^* \pi_i)^{-1}(p) = f_i^{-1}\Big(E_i|_{\pi_i(p)} - \sigma_i(B_i) - E_i(1) - (\tau_i^{\mathbb{R}^d})^{-1}\big(\mathbb{D}^d_{\frac{1}{2}\tau_i^{\mathbb{R}^d}(p)}(g_i(p))\backslash \mathbb{D}^d\big)\Big)$$

and, for $\tilde{p} \in \hat{U}_i' \cap (\pi_i^* \pi_i)^{-1}(p)$,

$$\hat{\tau}_i'(\tilde{p}) = \Big(p,\ \frac{3}{g_i(p)} \cdot \big(\tau_i^{\mathbb{R}^d}\big(f_i(\tilde{p})\big) - \frac{1}{2}\tau_i^{\mathbb{R}^d}(p)\big)\Big) \in \big(E_i - \sigma_i(B_i)\big) \times \mathbb{R}^d \backslash \mathbb{D}^d(3).$$

- Over $E_i(3) - E_i(2)$: let $\lambda : [2,3] \to [0,1/2]$ be a smooth monotonically increasing function such that $\lambda'(0) - \lambda'(1) = 0$; given $p \in E_i(3) - E_i(2)$, define $p' = \lambda(|\tau_i^{\mathbb{R}^d}(p)|) \cdot \tau_i^{\mathbb{R}^d}(p) \in \mathbb{R}^d$; then, for $p \in E_i(3) - E_i(2)$, define

$$\hat{U}_i' \cap (\pi_i^* \pi_i)^{-1}(p) = f_i^{-1}\Big(E_i|_{\pi_i(p)} - \sigma_i(B_i) - E_i(1) - (\tau_i^{\mathbb{R}^d})^{-1}\big(\mathbb{D}^d_{p'}(g_i(p))\backslash \mathbb{D}^d\big)\Big)$$

and, for $\tilde{p} \in \hat{U}_i' \cap (\pi_i^* \pi_i)^{-1}(p)$,

$$\hat{\tau}_i'(\tilde{p}) = \Big(p,\ \frac{3}{g_i(p)} \cdot \big(\tau_i^{\mathbb{R}^d}\big(f_i(\tilde{p})\big) - p'\big)\Big).$$

Now that we have defined $\hat{\tau}_1'$ (resp. $\hat{\tau}_2'$), by simply taking a product with $B_2 \times (0,\rho)$ (resp. $B_1 \times (0,\rho)$), it induces a trivialization of

$$\begin{array}{lll} & (\mathrm{pr}^1_{B_1})^* E_1 \xrightarrow{(\mathrm{pr}^1_{B_1})^*(\pi_1)} E_1(3)^\circ \times B_2 \times (0,\rho) & \\ \big(\text{resp.} & (\mathrm{pr}^2_{B_2})^* E_2 \xrightarrow{(\mathrm{pr}^2_{B_2})^*(\pi_2)} B_1 \times E_2(3)^\circ \times (0,\rho) & \big) \end{array}$$

near $(\mathrm{pr}^1_{B_1})^* \sigma_1(B_1)$ (resp. $(\mathrm{pr}^2_{B_2})^* \sigma_2(B_2)$). Since the construction of $E_{\mathring{I}}$ over $E_1(3)^\circ \times B_2 \times (0,\rho)$ and $B_1 \times E_2(3) \times (0,\rho)$ is done by doing some surgery on $(\mathrm{pr}^1_{B_1})^* E_1$ and $(\mathrm{pr}^2_{B_2})^* E_2$, $\hat{\tau}_1'$ induces a trivialization of $E_{\mathring{I}}|_{E_1(3)^\circ \times B_2 \times (0,\rho) \cup B_1 \times E_2(3)^\circ \times (0,\rho)}$ near $\sigma_{\mathring{I}}(B_{\mathring{I}})$.

2.5.2 Trivialization of the bracket bundle near infinity

Recall that $E_{\mathring{I}}$ is defined by gluing the 3 pieces

$$E_{\mathring{I}}|_{E_1(3)^\circ \times B_2 \times (0,\rho)}, \qquad E_{\mathring{I}}|_{S^{d-1} \times (-\frac{1}{2}, \frac{1}{2}) \times B_1 \times B_2 \times (0,\rho)}, \qquad E_{\mathring{I}}|_{B_1 \times E_2(3)^\circ \times (0,\rho)}$$

together. We want to define a trivialization $\tau_{\mathring{I}}$ of $E_{\mathring{I}}$ near $\sigma_{\mathring{I}}$. On the first and third part of $E_{\mathring{I}}$ above, let $\tau_{\mathring{I}}$ be given as in Section 2.5.1. It remains to define $\tau_{\mathring{I}}$ on the second part of $E_{\mathring{I}}$. Over $S^{d-1} \times \big((-\frac{1}{2}, -\frac{1}{3}) \cup (\frac{1}{3}, \frac{1}{2})\big) \times B_1 \times B_2 \times (0,\rho)$, it coincides with the definition of $\tau_{\mathring{I}}$ on the other two pieces, via gluing. Over $S^{d-1} \times [-\frac{1}{3}, \frac{1}{3}] \times B_1 \times B_2 \times (0,\rho)$, we define $\tau_{\mathring{I}}$ to be given by the standard trivialization on the trivial bundle $\mathbb{R}^d \times S^{d-1} \times [-\frac{1}{3}, \frac{1}{3}] \times B_1 \times B_2 \times (0,\rho)$, from which $E_{\mathring{I}}|_{S^{d-1} \times [-\frac{1}{2}, \frac{1}{2}] \times B_1 \times B_2 \times (0,\rho)}$ is obtained by surgery. Note that, by (5), this means the mid-point of the centers of the two disks to be removed when performing this surgery is 0 under this trivialization. The definition in Section 2.5.1 ensures that the $\tau_{\mathring{I}}$ defined this way is smooth.

3 The Strata of $\widetilde{C}_A$

For a finite set S, denote by $|S|$ the number of elements in S.

Let A be a finite set. In this section we describe the strata of $\widetilde{C}_A$.

3.1 The combinatorics

Definition 3.1. An *A-labeled tree* consists of

- a tree T—we denote by $V(T), E(T)$ the vertex and edge set of T, respectively;
- $r \in V(T)$, "the root";

  for two vertices $v \neq w \in V(T)$, if the unique path between v and r passes through w, we say w is an *ancestor* v and v is a *descendant* of w, and denote $v > w$; if, additionally, v and w are also adjacent, then we say w is the *parent* of v and v is a *child* of w;

  denote the set of children of v by $cld(v)$;

  for an edge $e \in E(T)$, say the vertex adjacent to e and closer to r *just below* e, denoted by $v_-(e)$ and the vertex adjacent to e and further away from r *just above* e, denoted by $v_+(e)$;
- every vertex $v \in V(T)$ is associated with one of the four labels: $\mathbb{R}^d, \pi_1, \pi_2, \pi_{\mathring{I}}$; call it the "space label", denoted by $ls(v)$;
- every vertex $v \in V(T)$ is associated with a subset of A; call it the "points label", denoted by $lp(v)$;

satisfying the following conditions

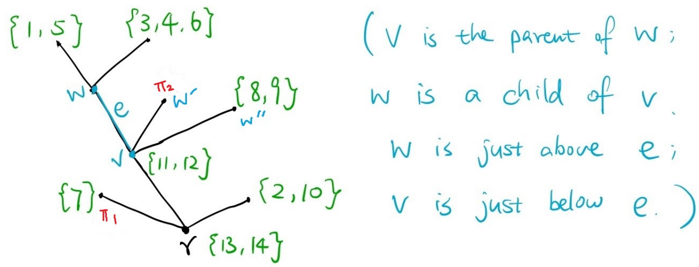
- either

  both π_1, π_2 are associated to exactly one vertex, and no vertex is associated to $\pi_{\mathring{I}}$;

  or

  $\pi_{\mathring{I}}$ is associated to exactly one vertex, and no vertex is associated to π_1 or π_2;
- if $v \neq w \in V(T)$, then $lp(v) \cap lp(w) = \emptyset$;
- $\bigsqcup_{v \in V(T)} lp(v) = A$;
- if $v \in V(T)$ is such that $ls(v) = \mathbb{R}^d$, then $|cld(v)| + |lp(v)| \geq 2$.

Definition 3.2. For notational convenience, define $\overline{cld}(v) = cld(v) \sqcup lp(v)$.



(In this example, $A = \{1, 2, \ldots, 14\}$; the point labels are green and space labels are red. When a space label is $\mathbb{R}^d$, it is omitted.)

Denote by $\mathcal{T}(A)$ the set of A-labeled trees. The strata of $\widetilde{C}_A$ will be labeled by elements of $\mathcal{T}(A)$. Note that the conditions above implies that, if $T \in \mathcal{T}(A)$ has only one vertex, then its space label must be $\pi_{\mathring{I}}$ and its points label must be A.

Definition 3.3. We define an "addition" operation on the set of space labels $\{\mathbb{R}^d, \pi_1, \pi_2, \pi_{\mathring{I}}\}$: first define a map ($\mathcal{P}$ means power set)

$$\begin{gathered} \text{lsset} : \{\mathbb{R}^d, \pi_1, \pi_2, \pi_{\mathring{I}}\} \longrightarrow \mathcal{P}\{1,2\} \\ \mathbb{R}^d \to \emptyset, \qquad \pi_i \to \{i\}, \qquad \pi_{\mathring{I}} \to \{1,2\}. \end{gathered} \tag{9}$$

Then, for $X, Y \in \{\mathbb{R}^d, \pi_1, \pi_2, \pi_{\mathring{I}}\}$, define $X + Y = \text{lsset}^{-1}(\text{lsset}(X) \cup \text{lsset}(Y))$.

Definition 3.4. For $T \in \mathcal{T}(A)$, let $T/e \in \mathcal{T}(A)$ be the tree obtained by contracting e—merging the two vertices connected to e into a single vertex, taking a union of their point labels and sum (in the sense above) of their space labels. Similarly, if $I \subset E(T)$, denote by $T/I \in \mathcal{T}(A)$ the tree obtained by contracting all edges in I.

We denote the new vertex in $V(T/e)$ by $[e]$. From the definition of T/e, there are obvious maps

$$\text{contr}^V_{T;e} : V(T) \longrightarrow V(T/e)$$

which maps $v_-(e), v_+(e)$ to $[e]$ and every other vertex to itself, and

$$\text{contr}^E_{T;e} : E(T) \backslash \{e\} \longrightarrow E(T/e)$$

which maps every edge to itself.

3.2 Describing each stratum

Let T be an A-labeled tree. For $v \in V(T)$, define

$$\text{Space}(v) = C_{\overline{cld(v)}}(ls(v));$$

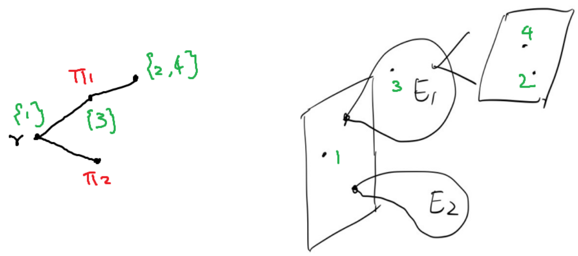
i.e., the (uncompactified) configuration space of $\overline{cld}(v)$-labeled points in $ls(v)$. For example, for the vertex v in the picture above,

$$ls(v) = \mathbb{R}^d, lp(v) = \{11, 12\}, cld(v) = \{w, w', w''\},$$

so $\text{Space}(v) = C_{\{11,12,w,w',w''\}}(\mathbb{R}^d)$.

Define $S_T = \prod_{v \in V(T)} \mathrm{Space}(v)$. In particular, when T has only one vertex, $S_T = C_A(\pi_{\mathring{I}})$. It follows from definition that S_T is a smooth manifold; it is also the total space of a smooth fiber bundle over $B_1 \times B_2$.

Example: if $A = \{1, 2, 3, 4\}$ and T is the tree below on the left, then S_T consists of configurations like the picture below on the right.



Define $\widetilde{C}_A = \bigsqcup_{T \in \mathcal{T}(A)} S_T$ as a set.

3.3 More notation; normalization

Given $T \in \mathcal{T}(A)$ and $v \in V(T)$, define

$$lp(\geq v) = \{a \in A \big| \, \exists v' \in V(T), v' \geq v, a \in lp(v')\}$$

and

$$ls(> v) = \{x \in \{\pi_1, \pi_2\} \big| \, \exists v' \in V(T), v' > v, \mathrm{lsset}(x) \subset \mathrm{lsset}(ls(v'))\}.$$

We then define a "forgetful map"

$$f_v : lp(\geq v) \sqcup ls(> v) \longrightarrow \overline{cld}(v)$$

$$f_v(x) = \begin{cases} x, \ \text{if } x \in lp(v); \\ w \in cld(v) \text{ such that } \exists v' \geq w, x \in lp(v') \text{ or } \mathrm{lsset}(x) \subset \mathrm{lsset}(ls(v')), \ \text{otherwise.} \end{cases}$$

For example, if T is as in the example in Section 3.2, then

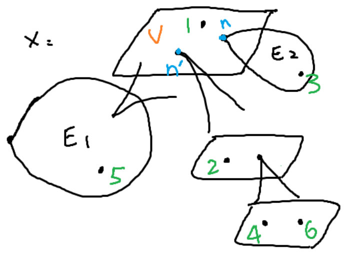
$$f_r(2) = f_r(4) = f_r(3) = f_r(\pi_1) = \text{the vertex carrying } \pi_1 \text{ and } 3 \in cld(r).$$

Suppose $x = (x_v)_{v \in V(T)} \in S_T = \prod_{v \in V(T)} \mathrm{Space}(v)$ and $v \in V(T)$ is such that $ls(v) = \mathbb{R}^d$.

Definition 3.5. The *weight* of an element $n \in \overline{cld}(v)$ is

$$\begin{cases} 1, \ \text{if } n \in lp(v) \\ |f_v^{-1}(n)|, \ \text{if } n \in cld(v) \text{ and } f_v^{-1}(n) \cap ls(> v) = \emptyset \\ \infty, \ \text{if } f_v^{-1}(n) \cap ls(> v) \neq \emptyset. \end{cases}$$

For example, if x and v are like this



then, on the v-screen, the marked point 1 has weight 1, the node n has weight ∞, and the node n' has weight 3.

(Note the newly defined notation $\overline{cld}(v) := lp(v) \sqcup cld(v)$.)

Definition 3.6. Suppose $i, j \in \overline{cld}(v), i \neq j$. We define a *standard representative (depending on i, j)*, $x_v' \in (\mathbb{R}^d)^{\overline{cld}(v)}$, in the equivalence class $x_v \in C_{\overline{cld}(v)}(\mathbb{R}^d)$ (recall the equivalence relation here is modulo translation and scaling of $\mathbb{R}^d$): x_v' is such that

- $0 \in \mathbb{R}^d$ is located at the "center of gravity" of points in $\overline{cld}(v)$ (for example, in the picture above, the center of gravity of the v-screen coincides with the node n); note there is a bit of ambiguity: in the case there are 2 elements in $\overline{cld}(v)$ with weight ∞, we define the center of gravity to be the mid-point of these two elements;
- the scaling is such that the distance between the points i and j is 1.

For each $T \in \mathcal{T}(A)$ and $v \in V(T)$, we choose, arbitrarily, $i \neq j \in \overline{cld}(v)$, and fix these choices for the rest of the document. We can therefore talk about the standard representative x_v' of x_v.

In Section 4, we will define a smooth manifold structure on $\widetilde{C}_A$ by "smoothing out the nodes". To prepare for this, we next define the largest and smallest radii on each screen where the smoothing can take place.

Definition 3.7. Suppose $x = (x_v)_{v \in V(T)} \in S_T$, $v \in V(T)$. Define $R^{\min}(x; v)$ as follows:

- if $ls(v) = \mathbb{R}^d$, then $R^{\min}(x; v) =$
$$2 \cdot \inf\{R \in \mathbb{R}^{\geq 1} \big| \text{in the configuration } x_v', \text{ all marked points and nodes are inside } \mathbb{D}^d(R)\};$$
- if $ls(v) = \pi_i$, $i = 1$ or 2 or $\mathring{I}$, and $ls(> v) = \emptyset$, then $R^{\min}(x; v) =$
$$2 \cdot \inf\{R \in \mathbb{R}^{\geq 1} \big| \text{in the configuration } x_v, \text{ all marked points and nodes are inside } E_i(R)\};$$
- if $ls(v) = \pi_i$, $i = 1$ or 2, and $ls(> v) = \{\pi_{3-i}\}$: denote by $\star_v \in E_i - \sigma_i(B_i)$ the position of the marked point $f_v(\pi_{3-i})$ in the configuration x_v, then we can view x_v as an element in
$$C_{\overline{cld}(v) \setminus \{f_v(E_{3-i})\}}(\pi_i^* E_i)|_{\star_v},$$
where $\pi_i^* E_i$ is the total space of the tautological bundle as in Section 2.5.1, and $(\pi_i^* E_i)|_{\star_v}$ denotes the fiber of $\pi_i^*(E_i)$ over the point $\star_v$; now define $R^{\min}(x; v) =$
$$2 \cdot \inf\{R \in \mathbb{R}^{\geq 3} \big| \text{in the configuration } x_v, \text{ all marked points and nodes are inside } \hat{E}_i(R)\}.$$

Definition 3.8. Suppose $x = (x_v)_{v\in V(T)} \in S_T$, and $e \in E(T)$. Note that $v_+(e) \in cld(v_-(e))$. If $ls(v_-(e)) \neq \mathbb{R}^d$, we denote by $\star_e \in ls(v_-(e))$ the location of the $v_+(e)$-marked point in the configuration $x_{v_-(e)}$; if $ls(v_-(e)) = \mathbb{R}^d$, we denote by $\star_e \in \mathbb{R}^d$ the location of the $v_+(e)$-marked point in the standard representative $x'_{v_-(e)}$. Define $r^{\max}(x;e)$ as follows:

- if $ls(v_-(e)) = \mathbb{R}^d$, then

$$r^{\max}(x;e) = \frac{1}{2}\cdot \sup\{r \in \mathbb{R}^{<1} \,|\, \text{in the standard representative } x'_{v_-(e)}, \text{ all marked points and nodes, except for } \star_e, \text{ are outside } \mathbb{D}_{\star_e}(r)\};$$

- if $ls(v_-(e)) = \pi_{\mathring{I}}$ or π_i, $i = 1$ or 2, then (recall the definition of $D_{\star_e}(r)$ from (1) using the exponential map)

$$r^{\max}(x;e) = \frac{1}{2}\cdot \sup\{r \in \mathbb{R}^{<\mathrm{radius}^{\max}(\star_e)} \,|\, \text{in the configuration } x_{v_-(e)}, \text{ all marked points and nodes, except for } \star_e, \text{ are outside } D_{\star_e}(r)\}.$$

Definition 3.9. For all $T \in \mathcal{T}(A)$ and $e \in E(T)$, define the *maximal smoothing parameter function*

$$\epsilon^{\max}_{T;e} : S_T \longrightarrow \mathbb{R}^{>0}, \qquad \epsilon^{\max}_{T;e}(x) = r^{\max}(x;e)/R^{\min}(x;v_+(e)).$$

It is easy to see that $\epsilon^{\max}_{T;e}$ is smooth. Note that the infimum of $\epsilon^{\max}_{T;e}$ is 0.

4 The manifold structure on $\widetilde{C}_A$

Recall we defined $\widetilde{C}_A = \bigsqcup_{T\in\mathcal{T}(A)} S_T$ as a set. In this section we define the smooth-manifold-with-corners structure on $\widetilde{C}_A$.

For each $T \in \mathcal{T}(A)$, define ("dom" stands for "domain")

$$\mathrm{dom}(T) = \big\{\big(x, (\epsilon_e)_{e\in E(T)}\big) \in S_T \times (\mathbb{R}^{\geq 0})^{E(T)} \,\big|\, \forall\, e, \epsilon_e < \epsilon^{\max}_{T;e}(x)\big\}.$$

(The definition of $\mathrm{dom}(T)$ may need to be changed to make the range of those ϵ even smaller.)

The strategy now is as follows: since we have already defined each stratum of $\widetilde{C}_A$ as a smooth manifold, we define the smooth structure on $\widetilde{C}_A$ by considering how to "glue the neighborhoods of each S_T together". By "the neighborhood of S_T" we mean $\mathrm{dom}(T)$. More precisely, for each T, we will construct a map

$$\psi_T : \mathrm{dom}(T) \longrightarrow \widetilde{C}_A$$

such that,

1. ψ_T is injective;

2. $\psi_T|_{S_T \times (0,\ldots,0)}$ is the identity map on S_T;

3. for each $I \subset E(T)$, define the "I-face of $\mathrm{dom}(T)$"

$$\mathrm{dom}^I(T) := \{(x, (\epsilon_e)_e) \in \mathrm{dom}(T) |\, \epsilon_e = 0 \text{ iff } e \in I\} \subset \mathrm{dom}(T),$$

   then, $\psi_T(\mathrm{dom}^I(T)) \subset S_{T/(E(T)-I)}$ and

$$\psi_T|_{\mathrm{dom}^I(T)} : \mathrm{dom}^I(T) \longrightarrow S_{T/(E(T)-I)}$$

   is smooth;

4. $\psi_T|_{\mathrm{dom}^I(T)}$ above has smooth inverse from its image;

5. for all $T_1, T_2 \in \mathcal{T}(A)$, $\psi_{T_2}^{-1}(\mathrm{image}(\psi_{T_1}) \cap \mathrm{image}(\psi_{T_2}))$ is an open subset of $\mathrm{dom}(T_2)$, and the "transition map"

$$\psi_{T_1}^{-1} \circ \psi_{T_2}|_{\psi_{T_2}^{-1}(\mathrm{image}(\psi_{T_1}) \cap \mathrm{image}(\psi_{T_2}))} : \psi_{T_2}^{-1}(\mathrm{image}(\psi_{T_1}) \cap \mathrm{image}(\psi_{T_2})) \longrightarrow \mathrm{dom}(T_1)$$

   is smooth.

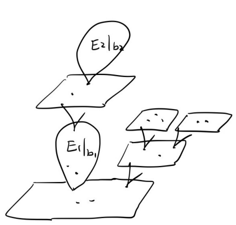
Notice that the above definition includes when T has only one vertex, in which case $\mathrm{dom}_T = S_T$ which is the main stratum, and $\psi_T = \mathrm{Id}_{S_T}$.

Lemma 4.1. *These ψ_T induces a topology and a smooth atlas on $\widetilde{C}_A$.*

Proof. (For each point $p \in \widetilde{C}_A$, using either the main stratum of $\widetilde{C}_A$ or some ψ_T whose image contains p, we can define a smooth chart on a neighborhood of p; the last condition above ensures that for different choices of ψ_T, these charts are compatible.) □

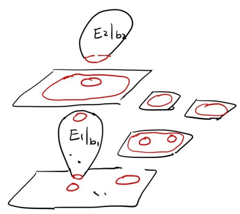
The process of defining ψ_T is by smoothing out the nodes, which we describe now.

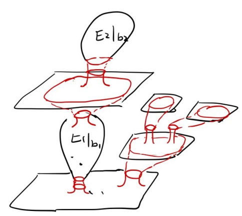
Suppose $x = (x_v)_{v \in V(T)} \in S_T$ is pictorially represented by a tree of configurations, e.g. the one below,



by using the "standard representations" specified in Definition 3.6, we can view x as literally a "nodal" space – a space $X = \bigvee_{v \in V(T)} X_v$ obtained by wedging finitely many manifolds together (for the example here, a copy of $E_1|_{b_1}$, $E_2|_{b_2}$ and some copies of $S^d = \mathbb{R}^d \sqcup \{\infty\}$ are wedged together), together with A-many marked points scattered in it. Each component X_v, minus the lower node ("∞"), now has a metric on it – on the S^d-component it is the standard metric on R^d and on the $E_1, E_2, E_{\mathring{I}}$ components they are the metrics specified in Sections 1,2. Given a component X_v of this nodal space and a point $p \in X_v$, if p is not the lower node of X_v, we can talk about the "ball of radius r in X_v centered at p" to be given by the metric and the induced exponential map on X_v. If p is the lower node, since each X_v also has a specified "identification with $\mathbb{R}^d$ near infinity" near its lower node, we define the "ball of radius $1/R$ centered at p" to be $\{\infty\} \sqcup \mathbb{R}^d \backslash \mathbb{D}^d(R)$ under this identification.

To construct $\psi_T(x, (\epsilon_e)_{e \in E(T)})$, for each node corresponding to $e \in E(T)$, we remove the ball of radius r centered at the node from $X_{v_-(e)}$ and the ball of radius $1/R$ centered at the node from X_{v_+e}, where $r, R \geq 0$ are chosen arbitrarily satisfying $r/R = \epsilon_e$, $r < r^{\min}(x; e)$, $R > R^{\max}(x; v_+(e))$; we then glue the two boundaries together, forming a new (maybe nodal) space with A-many marked points on it. (For those e such that $\epsilon_e = 0$, we do not do anything to the corresponding node.) This space, together with the marked points, represent an element in $\widetilde{C}_A$ [1], which we define $\psi_T(x, (\epsilon_e)_{e \in E(T)})$ to be.





(These pictures are not optimal. I'll draw them a bit differently later.)

This gives us the intuitive definition of ψ_T. Below we give the precise definition. (If you think the above description is enough, we can also just say "the precise description is easy but tedious so we don't write.)

Definition 4.2. (Define $\psi_T(x; (\epsilon_e)_e)$. It seems to me that the most convenient way of doing so (at least if we smooth all nodes at once) is to consider the following 3 cases separately:
(1) one of the vertices of T has space label $\pi_{\mathring{I}}$, and no vertices with space label π_1 or π_2;
(2) T has two vertices with space label π_1 and π_2, respectively, and the path connecting these two vertices in T goes through the root r;
(3) same as (2), but the path does not go through r.
For cases (2) and (3), we first specify what the space would be (if all nodes along the path have positive ϵ, then this means specifying a fiber of $\pi_{\mathring{I}}$) after the smoothing, then specify, one by one maybe, where the marked points are located after smoothing, probably using a formula involving exponential maps.)

Remark 4.3. If we have used a different i, j in Definition 3.6, and denote by ψ'_T instead of ψ_T the map thus obtained, then $(\psi'_T)^{-1} \circ \psi_T$ can be easily seen to also be a smooth map defined on an open subset of $\mathrm{dom}(T)$. So, the choices of i, j in defining the standard representative does not matter for our purpose here.

The following corollary is clear:

Corollary 4.4. *The conditions 2, 3 above follows immediately from the definition.*

[1](One needs to be careful here to rigorously define how to translate this "nodal space with points" description back to an element in $\widetilde{C}_A$. That this can be done is because that we constructed $\pi_{\mathring{I}}$ very explicitly in Section 4.)

Lemma 4.5. *ψ_T is injective.*

Proof. (This follows from the "Riemannian center of mass" notion in [1].) $\square$

Lemma 4.6. *Condition 4 above holds.*

Proof. (This uses that, by construction, $E_{\mathring{I}}$ remembers where the "centers" of the E_1 or/and E_2 making up to it are; it also uses that the Riemannian center of mass depends smoothly on the marked points.) $\square$

Lemma 4.7. *Suppose $T_1 \in \mathcal{T}(A)$, $I \subset E(T_1)$, and $T_2 = T_1/I \in \mathcal{T}(A)$. Then:*

- $\psi_{T_1}^{-1}(\mathrm{image}(\psi_{T_1}) \cap \mathrm{image}(\psi_{T_2})) = \mathrm{dom}(T_1) \backslash \cup_{I' \subset I} \mathrm{dom}^I(T_1)$*, hence is an open subset of* $\mathrm{dom}(T_1)$.
- $\psi_{T_2}^{-1}(\mathrm{image}(\psi_{T_1}) \cap \mathrm{image}(\psi_{T_2})) \subset \mathrm{dom}(T_2)$ *is of the form*
$$\left\{ \left(x, (\epsilon_e)_{e \in E(T_2)}\right) \in S_{T_2} \times (\mathbb{R}^{\geq 0})^{E(T_2)} \middle| \, x \in S', \text{ and } \forall\, e, \epsilon_e < \epsilon'_e(x) \right\},$$
for some open subset $S' \subset S_{T_2}$ and smooth functions $\epsilon'_e \leq \epsilon^{\max}_{T_2;e}$ on S'. So, it is also an open subset of $\mathrm{dom}(T_1)$.
- $$\psi_{T_1}^{-1} \circ \psi_{T_2}\big|_{\psi_{T_2}^{-1}(\mathrm{image}(\psi_{T_1}) \cap \mathrm{image}(\psi_{T_2}))} : \left\{ \left(x, (\epsilon_e)_{e \in E(T_2)}\right) \in S' \times (\mathbb{R}^{\geq 0})^{E(T_2)} \middle| \, \forall\, e, \epsilon_e < \epsilon'_e(x) \right\}$$
$$\longrightarrow \left\{ \left(x, (\epsilon_e)_{e \in E(T_1)}\right) \in S_{T_1} \times (\mathbb{R}^{\geq 0})^{E(T_1)} \middle| \, \forall\, e, \epsilon_e < \epsilon^{\max}_{T_1;e}(x) \right\}$$
is given by:
$$\left(x, (\epsilon_e)_{e \in E(T_2)}\right) \longrightarrow \left(\psi_{T_1}\left(x, (\delta_{e,I} \cdot \epsilon_e)_{e \in E(T_2)}\right), (\text{smooth functions in } x \text{ and } (\epsilon_e)_{e \in E(T_2)}\right),$$
where $\delta_{e,I} = \begin{cases} 1, & \text{if } e \in I \\ 0, & \text{if } e \notin I. \end{cases}$
- $\psi_{T_2}^{-1} \circ \psi_{T_1}\big|_{\psi_{T_1}^{-1}(\mathrm{image}(\psi_{T_1}) \cap \mathrm{image}(\psi_{T_2}))}$ *is given by: (?)*

Proof. $\square$

Corollary 4.8. *The ψ_T we constructed satisfies the required conditions, so $\widetilde{C}_A$ is a well-defined smooth manifold with corners. For $T \in \mathcal{T}(A)$, $S_T \subset \widetilde{C}_A$ is a codimension-$|E(T)|$ stratum.*

Suppose $A' \subset A$ is a finite subset, then we have an obviously-defined forgetful map $\mathfrak{f}_{A',A} : \widetilde{C}_A \to \widetilde{C}_{A'}$.

Proposition 4.9. *$f_{A,A'} : \widetilde{C}_A \to \widetilde{C}_{A'}$ is smooth. Moreover, for every (open) stratum S of $\widetilde{C}_A$, $\mathfrak{f}_{A',A}(S)$ is again an open stratum of $\widetilde{C}_{A'}$ and*
$$\mathfrak{f}_{A',A}|_S : S \longrightarrow \mathfrak{f}_{A',A}(S)$$
is a submersion.

Proof. $\square$

(Below is the old version; not needed here.)

4.1 Smoothing out one node

Suppose $T \in \mathcal{T}(A)$ and $e \in E(T)$. We construct a map

$$\psi_{T;e} : \mathrm{dom}(T;e) \longrightarrow S_T \sqcup S_{T/e}.$$

Define $\psi_{T;e}|_{S_T \times 0}$ to map by the identity to S_T.

For $x = \big(x_v \in C_{cld(v) \sqcup lp(v)}(ls(v))\big)_{v \in V(T)}$ and $\epsilon > 0$, define

$$\psi_{T;e}(x,\epsilon) = \big(y_v \in C_{cld(v) \sqcup lp(v)}(ls(v))\big)_{v \in V(T/e)}$$

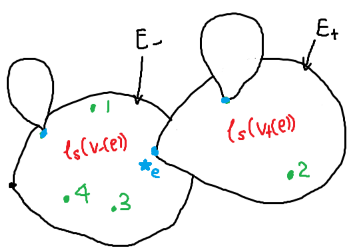
as follows: take $y_{\mathrm{contr}^V_{T;e}(v)} = x_v$; it remains to define

$$y_{[e]} \in C_{lp(v_+(e)) \sqcup lp(v_-(e)) \sqcup cld(v_+(e)) \sqcup (cld(v_-(e)) \backslash v_+(e))}(ls(v_-(e)) + ls(v_+(e))).$$

Before going into the detail of this, we explain how intuitively $y_{[e]}$ is defined. It is obtained from $x_{v_-(e)}$ and $x_{v_+(e)}$ as follows:

Take, arbitrarily, $R > R^{\min}(x; v_+(e))$ and $r < r^{\max}(x; e)$ such that $r/R = \epsilon$.

Draw $x_{v_-(e)}$ and $x_{v_+(e)}$ as below:

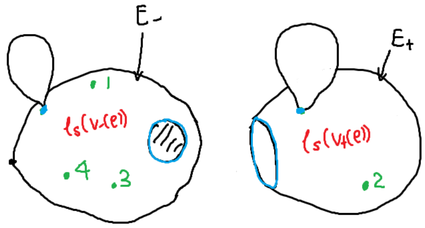


where $x_{v_-(e)}$ (resp. $x_{v_+(e)}$) is the configuration of green and blue points on $ls(v_-(e))$ (resp. $ls(v_+(e))$). Here the space where the marked points lie on, denoted by $E_\pm$, is the fiber of the bundle $ls(v_\pm(e))$ on which $x_{v_\pm(e)}$ lies, if $ls(v_\pm(e)) \neq \mathbb{R}^d$; if $ls(v_\pm(e)) = \mathbb{R}^d$, then $E_\pm = \mathbb{R}^d$, and the positions of points on it are given by $x'_{v_\pm(e)}$.

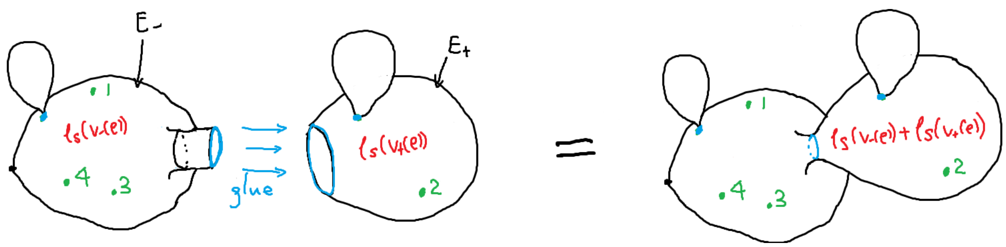
Next, we cut off a disk of radius r in E_-, centered at $\star_e$. Here, if $E_- \neq \mathbb{R}^d$, "radius r" is made sense of by the exponential map.

We also cut off a neighborhood of $\star_e$ in E_+. This neighborhood is the one identified with $\mathbb{R}^d \backslash \mathbb{D}^d(R)$ via a "trivialization at ∞" of $ls(v_+(e))$. This "trivialization at ∞" depends on $ls(v_+(e))$ and

$ls(> v_\pm(e))$: if $ls(v_+(e)) = \mathbb{R}^d$, then it is the standard trivialization on $\mathbb{R}^d$; if $ls(v_+(e)) = \pi_i$ and $ls(> v_+(e)) = \emptyset$, then it is τ_i'; if $ls(v_+(e)) \neq \mathbb{R}^d$ and $ls(> v_-(e)) = \{\pi_1, \pi_2\}$, then it is induced from the trivializations $\hat{\tau}_i'$ or $\tau_{\mathring{I}}$ defined in Section 2.5.



Last, we glue this two pieces together to form $y_{[e]}$. Notice that the result doesn't depend on the choices of r, R, as long as they satisfy the required conditions: $R > R^{\min}(x; v_+(e))$, $r < r^{\max}(x; e)$, and $r/R = \epsilon$.



The above is the intuition of constructing $y_{[e]}$. Now, the detailed description of defining $y_{[e]}$:

- If $ls(v_+(e)) = \mathbb{R}^d, ls(v_-(e)) \neq \mathbb{R}^d$: let E_e be the fiber of $ls(v_-(e))$ on which $x_{v_-(e)}$ lies; let $\star_e \in E_e$ be the location of the marked point $v_+(e)$ in the configuration $x_{v_-(e)}$; then, $y_{[e]}$ is obtained from $x_{v_-(e)}$ by first removing the marked point $v_+(e) \in cld(v_-(e))$ and then adding the marked points in $cld(v_+(e)) \sqcup lp(v_+(e))$ to E_e, as the image of $x'_{v_+(e)}$ under the map

$$(\mathbb{R}^d)^{cld(v_+(e)) \sqcup lp(v_+(e))} \xrightarrow{(\exp_{\star_e}, \ldots, \exp_{\star_e}) \circ \text{ scaling by } \epsilon} (E_e)^{cld(v_+(e)) \sqcup lp(v_+(e))}.$$

- If $ls(v_+(e)) = \mathbb{R}^d, ls(v_-(e)) = \mathbb{R}^d$: let $\star_e \in \mathbb{R}^d$ be the location of the marked point $v_+(e)$ in the configuration $x'_{v_-(e)}$; then, a representative of $y_{[e]}$ is obtained from $x'_{v_-(e)}$ by first removing the marked point $v_+(e) \in cld(v_-(e))$ and then adding the marked points in $cld(v_+(e)) \sqcup lp(v_+(e))$ to $\mathbb{R}^d$, as the image of $x'_{v_+(e)}$ under the map

$$(\mathbb{R}^d)^{cld(v_+(e)) \sqcup lp(v_+(e))} \longrightarrow (\mathbb{R}^d)^{cld(v_+(e)) \sqcup lp(v_+(e))}$$
$$(w_v \in \mathbb{R}^d)_{v \in cld(v_+(e)) \sqcup lp(v_+(e))} \longrightarrow \big(\epsilon \cdot w_v + \star_e\big)_{v \in cld(v_+(e)) \sqcup lp(v_+(e))}.$$

- If $ls(v_+(e)) = \pi_1, \pi_2$ or $\pi_{\mathring{I}}$, $ls(v_-(e)) = \mathbb{R}^d$: let $\star_e \in \mathbb{R}^d$ be be the location of the marked point $v_+(e)$ in the configuration $x'_{v_-(e)}$; let E_e be the fiber of $ls(v_+(e))$ in which the configuration $x_{v_+(e)}$ lies; then, $y_{[e]}$ is obtained from $x_{v_+(e)}$ by adding the marked points in $cld(v_-(e)) \sqcup lp(v_-(e))$ to E_e, as the image of $x'_{v_-(e)}$ under the map

$$(\mathbb{R}^d \backslash \mathbb{D}^d_{\star_e}(r^{\max}(x; e)))^{cld(v_-(e)) \sqcup lp(v_-(e))} \longrightarrow (E_e)^{cld(v_-(e)) \sqcup lp(v_-(e))},$$
$$(w_v \in \mathbb{R}^d)_{v \in cld(v_-(e)) \sqcup lp(v_-(e))} \to (\tau|_{E_e})^{-1}\big(1/\epsilon \cdot (w_v + \star_e)\big)_{v \in cld(v_-(e)) \sqcup lp(v_-(e))}$$

where

$$\tau = \begin{cases} \tau_i', \text{ if } ls(v_+(e)) = \pi_i, i = 1 \text{ or } 2, \text{ and } ls(> v_+(e)) = \emptyset \\ \hat{\tau}_i', \text{ if } ls(v_+(e)) = \pi_i, i = 1 \text{ or } 2, \text{ and } ls(> v_+(e)) = \{\pi_{3-i}\} \\ \tau_{\mathring{I}}, \text{ if } ls(v_+(e)) = \pi_{\mathring{I}}. \end{cases}$$

- If $ls(v_-(e)) = \pi_i$ and $ls(v_+(e)) = \pi_{3-i}$ for $i = 1$ or 2: let E_e^- (resp. E_e^+) be the fiber of π_i (resp. π_{3-i}) in which the configuration $x_{v_-(e)}$ (resp. $x_{v_+(e)}$) lies; let $\star_e \in E_e^-$ be the location of the marked point $v_+(e)$ in the configuration $x_{v_-(e)}$. Then,
  - if $\star_e \in E_i(3)$, then $y_{[e]}$, as a configuration of points on $\pi_{\mathring{I}}$, lies on the fiber over $(\star_e, \pi_{3-i}(x_{v_+(e)}), \epsilon) \in E_i(3) \times B_{3-i} \times (0, \rho)$;
  - if $\star_e \in E_i - \sigma_i(B_i) - E_i(2)$, write

  $$\tau_i'(\star_e) = \big(\pi_i(x_{v_-(e)}), l, \theta\big) \in B_i \times \mathbb{R}^d \backslash \mathbb{D}^d(3),$$

  where $(l \in [3, \infty), \theta \in S^{d-1})$ is the polar coordinate of the $\mathbb{R}^d \backslash \mathbb{D}^d(3)$-factor, then $y_{[e]}$, as a configuration of points on $\pi_{\mathring{I}}$, lies on the fiber over

  $$\begin{aligned} \big((-1)^{i-1}\theta, \pi_i(x_{v_-(e)}), \pi_{3-i}(x_{v_+(e)}), \gamma^{-1}(x, y)\big) &\in S^{d-1} \times B_i \times B_{3-i} \times (0, \rho) \times [-\frac{1}{2}, \frac{1}{2}] \\ &\approx S^{d-1} \times [-\frac{1}{2}, \frac{1}{2}] \times B_1 \times B_2 \times (0, \rho) \end{aligned}$$

  where $x = 1/l, y = \epsilon$, and $\gamma : (0, \rho) \times [-1/2, 1/2] \to \mathbb{R}^2$ is the map in Section 2.2.2, defined right under the second picture in that section.

  Recall in Section 2.2 we constructed $E_{\mathring{I}}$ by doing surgery on E_1 and E_2, so parts of $E_{\mathring{I}}$ are naturally identified with parts of E_1 and E_2. We define $y_{[e]}$ by putting together the marked points in $x_{v_-(e)}$ and $x_{v_+(e)}$, which are viewed as points in $E_{\mathring{I}}$ under this identification.

This completes the definition of $\psi_{T;e}$. It is easy to be convinced that

$$\psi_{T;e}|_{\mathrm{dom}(T;e) - S_T \times 0} : \mathrm{dom}(T; e) - S_T \times 0 \to S_{T/e}$$

is smooth. (Maybe re-write the above in a better way to make this obvious.)

To show $\psi_{T;e}$ has a smooth inverse on a sufficiently small neighborhood of S_T: in the case $ls(> v_-(e))$ is non-empty or $ls(v_-(e)) = \mathbb{R}^d$, this is clear (the case $ls(v_+(e)) = \pi_i, ls(v_-(e)) = \pi_{3-i}$ follows by how the bracket bundle is constructed); in the case $ls(> v_-(e))$ is empty and $ls(v_-(e)) \neq \mathbb{R}^d$, all the difficulty here is, given a collection of points $x_1, \ldots, x_n$ in a small region on a manifold M, whether there is a unique point $p \in M$ nearby such that $x_1, \ldots, x_n$ can be obtained from a configuration of points in T_pM by exponential map at p. This is precisely the definition of the Riemannian center of mass in [1]. The uniqueness of p is given by [1, Proposition 3.1].

4.2 Smoothing out multiple nodes

Given $T \in \mathcal{T}(A)$, and an (arbitrary) order on $E(T)$, we define $\psi_T : \mathrm{dom}(T) \to \widetilde{C}_A$ by smoothing out the nodes one by one. (From here to just above Lemma 4.10 can be skipped on first reading.)

More precisely, given

$$\big((x_v)_{v\in V(T)}, (\epsilon_e)_{e\in E(T)}\big) \in \mathrm{dom}(T) = S_T \times (\mathbb{R}^{\geq 0})^{E(T)},$$

$\psi_T\big((x_v),(\epsilon_e)\big)$ is determined by the following inductive process: first, write

$$E'(T) = \{e \in E(T) \big| \, \epsilon_e \neq 0\} \subset E(T),$$

then:

- Start with $T_0 = T$, $E'(T)$, the induced order on $E'(T)$, and $X_0 = \big((x_v)_{v\in T}, (\epsilon_e)_{e\in E'(T)}\big)$.
- Given $T_i \in \mathcal{T}(A)$, a subset $E'(T_i) \subset E(T_i)$, an order on $E'(T_i)$,and

  $$X_i = \big((x_{i,v})_{v\in V(T_i)} \in S_{T_i}, (\epsilon_e \in \mathbb{R}^{>0})_{e\in E'(T_i)}\big),$$

  we construct T_{i+1}, $E'(T_{i+1})$, an order on $E'(T_{i+1})$, and X_{i+1}:
  - Let $e' \in E'(T_i)$ be the first element of $E'(T_i)$, then

    $$T_{i+1} = T_i/e', \qquad E'(T_{i+1}) = \mathrm{contr}^E_{T_i;e'}\big(E'(T_i)\backslash\{e'\}\big)$$

    and the order on $E'(T_{i+1})$ is induced from the order on $E'(T_i)$ and $\mathrm{contr}^E_{T_i;e'}$.
  - For all $v \in V(T_i)$, $x_{i+1,\mathrm{contr}^V_{T_i;e'}} = x_{i,v}$.
  - $x_{i+1,[e']} = \psi_{T_i;e'}\big((x_{i,v})_{v\in V(T_i)}, \epsilon_{e'}\big)$.
  - For all $e \neq e' \in E'(T_i)$, if $\{v_-(e), v_+(e)\} \cap \{v_-(e'), v_+(e')\} = \emptyset$, then $\epsilon_{\mathrm{contr}^E_{T_i;e'}(e)} = \epsilon_e$.
  - It remains to determine $\epsilon_{\mathrm{contr}^E_{T_i;e'}(e)}$ for other $e \neq e' \in E'(T_i)$:
    * If $ls(v_-(e')) = \mathbb{R}^d$, $ls(v_+(e')) \neq \mathbb{R}^d$, then, for $e \neq e' \in E(T_i)$:

      $$\begin{cases} v_-(e) = v_+(e') \implies \epsilon_{\mathrm{contr}^E_{T_i;e'}(e)} = \epsilon_e, \\ v_-(e) = v_-(e') \implies \epsilon_{\mathrm{contr}^E_{T_i;e'}(e)} = \epsilon_e/\epsilon_{e'}, \\ v_+(e) = v_-(e') \implies \epsilon_{\mathrm{contr}^E_{T_i;e'}(e)} = \epsilon_e \cdot \epsilon_{e'}. \end{cases}$$

    * If $ls(v_+(e')) = \mathbb{R}^d$, and either $ls(v_-(e')) \neq \mathbb{R}^d$ or

      $$ls(v_-(e')) = \mathbb{R}^d, v_+(e) \text{ is not the out-most marked point in } x_{i,v_-(e)},$$

      then, for $e \neq e' \in E(T_i)$:

      $$\begin{cases} v_-(e) = v_+(e') \implies \epsilon_{\mathrm{contr}^E_{T_i;e'}(e)} = \epsilon_e \cdot \epsilon_{e'}, \\ v_-(e) = v_-(e') \implies \epsilon_{\mathrm{contr}^E_{T_i;e'}(e)} = \epsilon_e, \\ v_+(e) = v_-(e') \implies \epsilon_{\mathrm{contr}^E_{T_i;e'}(e)} = \epsilon_e. \end{cases}$$

    * If $ls(v_-(e')) = ls(v_+(e')) = \mathbb{R}^d$ and $v_+(e)$ is the out-most marked point in $x_{i,v_-(e)}$, then... (not yet written; would be very annoying to write; you can probably see what I want to say; should find another way to write these, or change the second bullet in Definition 3.6)

* If $ls(v_-(e')) = \pi_i$, $ls(v_+(e')) = \pi_{3-i}$: let $\star_e \in E_i$ be the position of the marked point $v_+(e')$ in the configuration $x_{i,v_-(e)}$.
    - If $\star_e \in E_i(2)$, then, for all $e \neq e' \in E(T_i)$, $\epsilon_{\mathrm{contr}^E_{T_i;e'}(e)} = \epsilon_e$.
    - If $\star_e \in E_i - \sigma_i(E_i) - E_i(2)$: let $g_i : E_i - \sigma_i(B_i) \to \mathbb{R}^{\geq 1}$ be the function defined in Section 2.5.1. Then,

$$\begin{cases} v_-(e) = v_+(e') \implies \epsilon_{\mathrm{contr}^E_{T_i;e'}(e)} = \epsilon_e, \\ v_-(e) = v_-(e') \implies \epsilon_{\mathrm{contr}^E_{T_i;e'}(e)} = \epsilon_e \cdot g_i(\star_e)/3, \\ v_+(e) = v_-(e') \implies \epsilon_{\mathrm{contr}^E_{T_i;e'}(e)} = \epsilon_e \cdot 3/g_i(\star_e). \end{cases}$$

- Define $\psi_T\big((x_v), (\epsilon_e)\big) = X_{|E'(T)|}$.

This completes the definition of ψ_T.

Lemma 4.10. *ψ_T doesn't depend on the choice of order on $E(T)$.*

Proof. (If I have not made a mistake (which I probably have...), this should be true and easy to verify. The definitions in Section 3.3 are designed for this lemma to hold.) □

Corollary 4.11. *For $T \neq T' \in \mathcal{T}(A)$, the transition map $\psi_{T'}^{-1} \circ \psi_T|_{\psi_T^{-1}(\mathrm{image}(\psi_T) \cap \mathrm{image}(\psi_{T'}))}$ is smooth. Therefore, $\widetilde{C}_A$ is a well-defined smooth manifold with corners. For $T \in \mathcal{T}(A)$, $S_T \subset \widetilde{C}_A$ is a codimension-$|E(T)|$ stratum.*

If $A' \subset A$ is a finite subset, then we have an obviously-defined forgetful map $f_{A,A'} : \widetilde{C}_A \to \widetilde{C}_{A'}$.

Lemma 4.12. *$f_{A,A'} : \widetilde{C}_A \to \widetilde{C}_{A'}$ is smooth. (proof not yet written. Note that the forgetful maps are not smooth in the sense of Joyce's paper "Manifold with corners"; smoothness here means "weakly smooth" there.)*

References

[1] K. Grove and H. Karcher, *How to Conjugate C^1-Close Group Actions*, Math. Z., 1973