1 Introduction

.....(introduction)

Define the kind of bundle we work with in this paper: Given a smooth homology sphere M, define a *framed* (M,∞)*-bundle* $(\pi : E \to B, \sigma, \tau, F)$ (abbreviate all these to π) to be a smooth fiber bundle $\pi : E \to B$ with fiber M, with a smooth section σ, a trivialization τ of the bundle near σ, and a smooth vertical framing F of π "standard" near σ.

Define the bracket operation, $\pi_1, \pi_2 \to [\pi_1, \pi_2]$ on such bundles, in an intuitively clear but not necessarily rigorous way.

Define cobracket and coproduct on graph cohomology (everything is over $\mathbb{Q}$):

- First, define the graph complex $\mathcal{G}'$—the $\mathbb{Q}$-vector space spanned by (with correct orientation definition, omitted here) connected graphs containing either a univalent vertex or a simple loop (an edge starting and ending at the same vertex). The coboundary operation δ is given by contracting an edge. In δ and all the operations on graphs below, whenever a graph not in $\mathcal{G}'$ appears (a graph that has a univalent vertex or simple loop), we set it to 0.
- Taking the homology of $\mathcal{G}'$ with respect to δ, denote by $H^*\mathcal{G}'$.
- Define the cobracket operation to be the linear map

$$\begin{aligned} \Delta_{[,]} : \mathcal{G}' &\longrightarrow \mathcal{G}' \otimes \mathcal{G}' \\ \Gamma &\longrightarrow \sum_{\Gamma' \leq \Gamma} \left(\Gamma' \otimes \Gamma/\Gamma' + (-1)^{\cdot\cdot} \Gamma/\Gamma' \otimes \Gamma' \right), \end{aligned}$$

  where Γ' ranges through all full subgraphs of Γ that is connected, with no univalent vertex or simple loop.
- Check that $\Delta_{[,]}$ commutes with δ and $\delta \otimes \mathrm{id} \pm \mathrm{id} \otimes \delta$, so it descends to

$$\Delta_{[,]} : H^*\mathcal{G}' \longrightarrow H^*(\mathcal{G}' \otimes \mathcal{G}') \approx H^*\mathcal{G}' \otimes H^*\mathcal{G}'.$$

- Finally we also define the coproduct operation on $\mathcal{G}'$ (this makes more sense for disconnected graphs but w=for connected graphs it is extra simple):

$$\begin{aligned} \Delta_{\cdot} : \mathcal{G}' &\longrightarrow \mathcal{G}' \otimes \mathcal{G}' \\ \Gamma &\longrightarrow \Gamma \otimes (\text{the empty graph}) + (\text{the empty graph}) \otimes \Gamma. \end{aligned}$$

Brief introduction to Kontsevich's characteristic classes. Given a framed (M,∞)-bundle $\pi : E \to B$ as above, denote by

$$K_\pi : H^*(\mathcal{G}') \longrightarrow H^*(B)$$

Kontsevich's characteristic classes of π.

Theorem 1.1. *Suppose $d \geq 3$. For $i = 1, 2$, suppose M_i is a d-dimensional smooth homology sphere and suppose $\pi_i : E_i \to B_i$ is a framed (M, ∞)-bundle. (Now, $[\pi_1, \pi_2] : E \to S^d \times B_1 \times B_2$ is the bracket bundle.) Then, for all $\eta \in H^*\mathcal{G}'$,*

$$K_{[\pi_1,\pi_2]}(\eta) = \mathrm{PD}_{S^d}[S^d] \otimes (K_{\pi_1} \otimes K_{\pi_2})(\Delta_{[,]}(\eta)) + \mathrm{PD}_{S^d}[pt] \otimes (K_{\pi_1} \otimes K_{\pi_2})(\Delta_{\cdot}(\eta)).$$

(Both LHS and RHS lives in

$$H^*(S^d \times B_1 \times B_2) \approx H^*(S^d) \otimes H^*(B_1) \otimes H^*(B_2).$$

PD_{S^d} *means Poincaré dual on S^d; $[S^d]$ stands for the fundamental class of S^d and $[pt]$ stands for the point class of S^d.)*

......(Then talk about the $(d+1)$-fold loop space structure on $\mathrm{BDiff}^{\mathrm{fr}}_{\partial}(D^d)$ and the theorem/corollary that it doesn't extend.)

(Below is an outline of the proof of Theorem 1.1. Throughout, π_1, π_2 are given and fixed.)

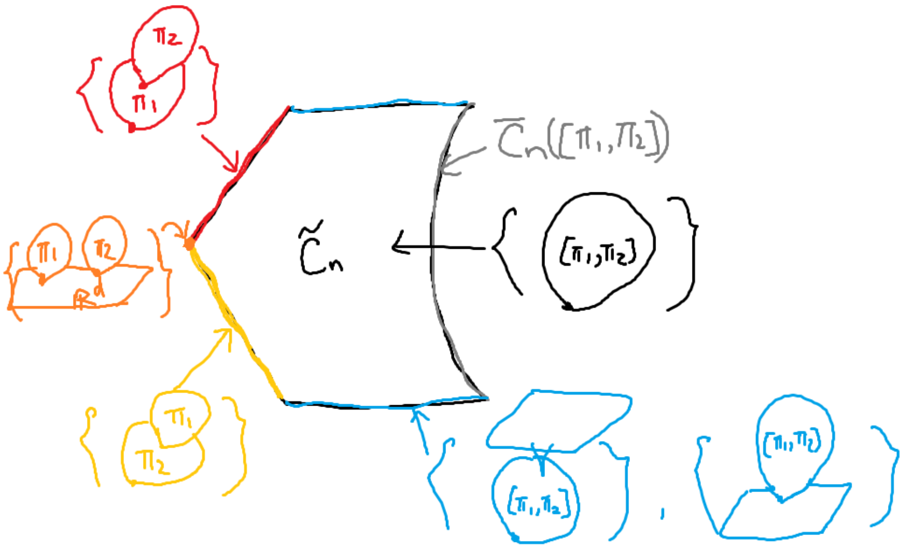
2 Conftilde

Construct the big configuration space $\widetilde{C}_A$. Show that it is a smooth manifold with boundary and corners. (These are mostly already written in the file "conftilde" I sent a while ago.)

What we need are the following:

- $\widetilde{C}_A$ is a smooth manifold with boundary and corners;
- each S_T is a stratum of $\widetilde{C}_A$;
- $\overline{S}_T = \bigsqcup_{T'} S_{T'}$, where the disjoint union is taken over all A-labeled trees T' such that T can be obtained from T' by contracting some edges.

Here is a schematic picture of $\widetilde{C}_A$ (the marked points are not drawn; the actual stratification structure of $\widetilde{C}_A$ is more complicated than what is shown in the picture):



The boundary of $\widetilde{C}_A$ consists of the following parts:

- the gray part, denoted by $\overline{S}_{\text{gray}}$, is $\overline{C}_A([\pi_1, \pi_2])$; its interior, denoted by S_{gray}, is $C_A([\pi_1, \pi_2])$;
- $S_{\text{blue}} := \bigcup_{T \in \mathcal{T}_{\text{blue}}} S_T$, where $\mathcal{T}_{\text{blue}}$ is the set of all A-labeled trees whose shape and space labels are like $\begin{array}{l} \bullet\ \mathbb{R}^d \\ | \\ \bullet\ [\pi_1, \pi_2] \\ \mathrm{r} \end{array}$ or $\begin{array}{l} \bullet\ [\pi_1, \pi_2] \\ | \\ \bullet\ \mathbb{R}^d \\ \mathrm{r} \end{array}$; and let $\overline{S}_{\text{blue}}$ be the closure of S_{blue};
- $S_{\text{red}} := \bigcup_{T \in \mathcal{T}_{\text{red}}} S_T$, where $\mathcal{T}_{\text{red}}$ is the set of all A-labeled trees with the following shape and space labels: $\begin{array}{l} \bullet\ \pi_2 \\ | \\ \bullet\ \pi_1 \\ \mathrm{r} \end{array}$; and let $\overline{S}_{\text{red}}$ be the closure of S_{red};
- $S_{\text{yellow}} := \bigcup_{T \in \mathcal{T}_{\text{yellow}}} S_T$, where $\mathcal{T}_{\text{yellow}}$ is the set of all A-labeled trees with the following shape and space labels: $\begin{array}{l} \bullet\ \pi_1 \\ | \\ \bullet\ \pi_2 \\ \mathrm{r} \end{array}$; and let $\overline{S}_{\text{yellow}}$ be the closure of S_{yellow};

We also define $S_{\text{orange}} := \bigcup_{T \in \mathcal{T}_{\text{orange}}} S_T$, where $\mathcal{T}_{\text{orange}}$ is the set of all A-labeled trees with the following shape and space labels: $\begin{array}{ccc} \bullet\ \pi_1 & & \bullet\ \pi_2 \\ & \diagdown\ \diagup & \\ & \bullet\ \mathbb{R}^d & \\ & \mathrm{r} & \end{array}$; and let $\overline{S}_{\text{orange}}$ be the closure of S_{orange}. Then, $\overline{S}_{\text{orange}} = \overline{S}_{\text{red}} \cap \overline{S}_{\text{yellow}}$.

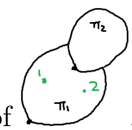
We define $\overline{C}^*_A = \overline{S}_{\text{red}} \cup \overline{S}_{\text{yellow}}$.

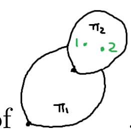
3 Propagators

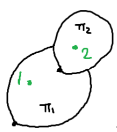
Fix a volume form ω_0 on S^{d-1}. For $i = 1, 2$, fix a propagator ω_i on $\overline{C}_2(\pi_i)$ such that on $\partial\overline{C}_2(\pi_i)$, ω_i is given by ω_0.

3.1 Propagator on confstar

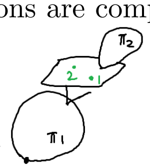
ω_1, ω_2 naturally induce a propagator ω_* on $\overline{C}_2^*$. On the main strata of $\overline{C}_2^*$, define ω_* as follows:

- On the stratum S_{11} consisting of 

, there is the forgetful map $f_{11} : S_{11} \to \overline{C}_2(\pi_1)$. We define $\omega_*|_{S_{11}} = f_{11}^*(\omega_1)$.
- On the stratum S_{22} consisting of 

, there is the forgetful map $f_{22} : S_{22} \to \overline{C}_2(\pi_2)$. We define $\omega_*|_{S_{22}} = f_{22}^*(\omega_2)$.
- On the stratum S_{12} consisting of 

, there is the forgetful map $f_{12} : S_{12} \to \overline{C}_2(\pi_1)$, where the second marked point is considered to be located at the node on π_1. We define $\omega_*|_{S_{22}} = f_{22}^*(\omega_2)$.
- ...(There are 5 more situations, all variations of the above, permuting $\pi_1, \pi_2, 1, 2$.)

We then check that the above definitions are compatible when the main strata glue together. For example, the stratum S' consisting of 

lies in the intersection of $\overline{S}_{11}$ and $\overline{S}_{22}$; but the ω_* constructed on S_{11} and on S_{22}, when extended to here, agree, since they both become $(f')^*\omega_0$, where $f' : S' \to S^{d-1}$ is the forgetful map recording the direction of the vector between the two points.

All other codimension-1 strata can be straight-forwardly checked like this as well. We may need to check higher-codimension strata as well but it shouldn't be hard.

3.2 Propagator on other parts of the boundary of conftilde

To define a propagator on the blue part of $\partial\widetilde{C}_2$, we have no choice but to define it as induced from ω_0 in the usual way.

To define a propagator in the gray part of $\partial\widetilde{C}_2$, we just choose any propagator ω on $\widetilde{C}_2([\pi_1, \pi_2])$, so that $\omega|_{\partial\widetilde{C}_2([\pi_1,\pi_2])}$ is induced from ω_0.

Now we have defined a propagator on all of $\partial\widetilde{C}_2$.

3.3 Extend the propagator to the interior

Lemma 3.1.

To show that there exists a closed form on $\widetilde{C}_2$ extending the propagator we just defined on $\partial\widetilde{C}_2$, it suffices to show that the map

$$H^{d-1}(\widetilde{C}_2) \xrightarrow{\text{restriction}} H^{d-1}(\partial\widetilde{C}_2)$$

is surjective. It is therefore sufficient to show that $H^d(\widetilde{C}_2, \partial\widetilde{C}_2) = 0$. But

$$\begin{aligned} H^*(\widetilde{C}_2, \partial\widetilde{C}_2) &\approx H_{\dim(\widetilde{C}_2)-*}(\widetilde{C}_2 - \partial\widetilde{C}_2) = H_{\dim(\widetilde{C}_2)-*}\big(C_2([\pi_1, \pi_2]) \times (0,1)\big) \\ &\approx H_{\dim(\widetilde{C}_2)-*}(C_2([\pi_1, \pi_2])) \approx H^{*-1}\big(\overline{C}_2([\pi_1, \pi_2]), \partial\overline{C}_2([\pi_1, \pi_2])\big), \end{aligned}$$

and this is 0 when $* - 1 < d + 1$; see the proof of Lemma 2.12 in Watanabe's addendum paper.

We choose and fix such an extension. This gives us a propagator $\widetilde{\omega}$ on $\widetilde{C}_2$.

4 Configuration space integrals

For simplicity here we will only sketch part of the proof of Theorem 1.1, namely we only justify the first term on the RHS. (The second term is slightly more complicated but not much: basically because we only work with the point class on S^d—it is only 0-dimensional so you can still easily achieve transversality as needed. The second term is not needed for the corollary about the loop space structure on $\mathrm{BDiff}^{\mathrm{fr}}_\partial(D^d)$ either.)

Given a graph G, we denote by $V(G)$ its vertex set and $E(G)$ its edge set.

Suppose Γ is a cocycle in graph cohomology. (For the simplicity of notation we assume Γ is one graph—when it is a formal sum everything works as well.)

For every edge e of Γ, we have the forgetful map

$$f_e : \widetilde{C}_{V(\Gamma)} \longrightarrow \widetilde{C}_2.$$

And, when restricted to the gray (resp. red, orange, and yellow) part of $\partial\widetilde{C}_{V(\Gamma)}$, it is the forgetful map

$$f_e : \overline{C}_{V(\Gamma)}([\pi_1, \pi_2]) \longrightarrow \overline{C}_2([\pi_1, \pi_2]) \qquad (\text{resp. } f_e : \overline{C}^*_{V(\Gamma)} \longrightarrow \overline{C}^*_2).$$

Now we have the form $\bigwedge_{e\in E(\Gamma)} f_e^*\widetilde{\omega}$ on $\widetilde{C}_{V(\Gamma)}$. When restricted to the gray part of $\partial\widetilde{C}_n$, it is used to define Kontsevich's classes for the bundle $[\pi_1,\pi_2]$.

To compute the $(\mathrm{PD}_{S^d}[S^d])$-part of $K_{[\pi_1,\pi_2]}([\Gamma])$, we only need to compute, given arbitrary homology classes $\alpha_1\in H_*(B_1)$ and $\alpha_2\in H_*(B_2)$, the evaluation

$$\left\langle K_{[\pi_1,\pi_2]}([\Gamma]), [S^d]\otimes\alpha_1\otimes\alpha_2\right\rangle.$$

For $i=1,2$, suppose α_i is represented by a sub-pseudo-manifold $\iota_i : B_i' \hookrightarrow B_i$. (For simplicity you can think of a smooth submanifold instead.)

For any n: notice that the projection map $\widetilde{p} : \widetilde{C}_n \to B_1\times B_2$ is a fiber bundle, and so is the restriction of $\widetilde{p}$ to each stratum of $\widetilde{C}_n$. Let us pull everything back along (ι_1,ι_2), so that the base gets changed to $B_1'\times B_2'$ instead of $B_1\times B_2$. Abusing notation, we still denote by $\widetilde{C}_n, \overline{C}_n([\pi_1,\pi_2]), \overline{C}_n^*$ their pull-backs.

Now, we have

$$\left\langle K_{[\pi_1,\pi_2]}([\Gamma]), [S^d]\otimes\alpha_1\otimes\alpha_2\right\rangle = \int_{\overline{C}_{V(\Gamma)}([\pi_1,\pi_2])} \bigwedge_e f_e^*\omega = \int_{\text{gray part of } \partial\widetilde{C}_{V(\Gamma)}} \bigwedge_e f_e^*\widetilde{\omega}. \tag{1}$$

By Stocks' Formula (and the fact that $\widetilde{\omega}$ is closed),

$$\int_{\partial\widetilde{C}_{V(\Gamma)}} \bigwedge_e f_e^*\widetilde{\omega} = \int_{\widetilde{C}_{V(\Gamma)}} d\Big(\bigwedge_e f_e^*\widetilde{\omega}\Big) = 0,$$

so,

$$(1) = \int_{\text{blue part of } \partial\widetilde{C}_{V(\Gamma)}} \bigwedge_e f_e^*\widetilde{\omega} + \int_{\overline{C}^*_{V(\Gamma)}} \bigwedge_e f_e^*\omega_*.$$

Since Γ is a cocycle in graph cohomology, the first term is 0 just like in the proof of the well-definedness of Kontsevich's classes, so

$$\left\langle K_{[\pi_1,\pi_2]}([\Gamma]), [S^d]\otimes\alpha_1\otimes\alpha_2\right\rangle = \int_{\overline{C}^*_{V(\Gamma)}} \bigwedge_e f_e^*\omega_*.$$

5 Configuration space integral on confstar

We continue with the notation from last section (in particular, everything is over B_1', B_2' instead of B_1, B_2).

It remains to show that

$$\left\langle (K_{\pi_1}\otimes K_{\pi_2})(\Delta_{[,]}[\Gamma]), \alpha_1\otimes\alpha_2\right\rangle = \int_{\overline{C}^*_{V(\Gamma)}} \bigwedge_e f_e^*\omega_*. \tag{2}$$

For a graph G, we denote by $V(G)$ its vertex set and $E(G)$ its edge set. The LHS above equals to[1]

$$\sum_{\Gamma'\le\Gamma} \Big(\int_{\overline{C}_{V(\Gamma')}(\pi_1)} \bigwedge_{e\in E(\Gamma')} f_e^*\omega_1\Big)\cdot\Big(\int_{\overline{C}_{V(\Gamma/\Gamma')}(\pi_2)} \bigwedge_{e\in E(\Gamma/\Gamma')} f_e^*\omega_2\Big)$$

[1]A little more argument needed for this claim, but it is true.

$$\pm\Big(\int_{\overline{C}_{V(\Gamma/\Gamma')}(\pi_1)}\bigwedge_{e\in E(\Gamma/\Gamma')} f_e^*\omega_1\Big)\cdot\Big(\int_{\overline{C}_{V(\Gamma')}(\pi_2)}\bigwedge_{e\in E(\Gamma')} f_e^*\omega_2\Big).$$

To prove (2), it suffices to show that

$$\sum_{\Gamma'\le\Gamma}\Big(\int_{\overline{C}_{V(\Gamma')}(\pi_1)}\bigwedge_{e\in E(\Gamma')} f_e^*\omega_1\Big)\cdot\Big(\int_{\overline{C}_{V(\Gamma/\Gamma')}(\pi_2)}\bigwedge_{e\in E(\Gamma/\Gamma')} f_e^*\omega_2\Big)=\int_{\text{yellow part of }\overline{C}^*_{V(\Gamma)}}\bigwedge_{e\in E(\Gamma)} f_e^*\omega_* \quad (3)$$

and

$$\sum_{\Gamma'\le\Gamma}\Big(\int_{\overline{C}_{V(\Gamma/\Gamma')}(\pi_1)}\bigwedge_{e\in E(\Gamma/\Gamma')} f_e^*\omega_1\Big)\cdot\Big(\int_{\overline{C}_{V(\Gamma')}(\pi_2)}\bigwedge_{e\in E(\Gamma')} f_e^*\omega_2\Big)=\int_{\text{red part of }\overline{C}^*_{V(\Gamma)}}\bigwedge_{e\in E(\Gamma)} f_e^*\omega_*; \quad (4)$$

here the "red" and "yellow" refers to the $\widetilde{C}_n$ picture in Section 2. Below we only prove (3) since (4) is completely similar.

The yellow part of $\overline{C}^*_{V(\Gamma)}$ is simply

$$\sum_{V_1,V_2:V_1\sqcup V_2=V(\Gamma)}\overline{C}_{V_1}(\pi_1)\times\overline{C}_{V_2\sqcup\{\star\}}(\pi_2),$$

where $\star$ records the position of the node on π_2. Therefore, the RHS of (3) is

$$\sum_{V_1,V_2:V_1\sqcup V_2=V(\Gamma)}\int_{\overline{C}_{V_2\sqcup\{\star\}}(\pi_2)}\int_{\overline{C}_{V_1}(\pi_1)}\bigwedge_{\substack{e\in E(\Gamma)\\ \text{both endpoints of } e \text{ are in } V_1}} f_e^*\omega_*\wedge\bigwedge_{\substack{e\in E(\Gamma)\\ \exists \text{ endpoint of } e \text{ in } V_2}} f_e^*\omega_*.$$

For $V_1\subset V(\Gamma)$, we denote by $\Gamma'(V_1)$ the subgraph of Γ spanned by vertices in V_1. Then, by the way ω_* is constructed, and by Fubini's Theorem, the above equals to

$$\sum_{V_1,V_2:V_1\sqcup V_2=V(\Gamma)}\Big(\int_{\overline{C}_{V_1}(\pi_1)}\bigwedge_{e\in E(\Gamma'(V_1))} f_e^*\omega_1\Big)\cdot\Big(\int_{\overline{C}_{V_2\sqcup\{\star\}}(\pi_2)}\bigwedge_{e\in E(\Gamma/\Gamma'(V_1))} f_e^*\omega_2\Big).$$

This proves (3).

Appendix A Extending differential forms on a manifold with corners from boundary to interior

We first clarify the notation used in the present paper concerning manifold with corners, mostly following [2] and [3]. Here is a dictionary for our notation:

- *(smooth) manifold with corners*: as in [3, page 3] or, equivalently, [2, Definition 3].
- *smooth map between manifolds with corners*: [2, Definition 4] ("weakly smooth map" in [3, Definition 3.1])
- *tangent and cotangent spaces of manifolds with corners*: [2, Definition 10] (equivalently, [3, Definition 2.2])

- *cotangent bundle, tensor and exterior powers of cotangent bundle of manifolds with corners*: follows from the definition of cotangent spaces
- *differential form on manifolds with corners*: smooth section of exterior powers of the cotangent bundle

  For example, in $(\mathbb{R}^{\geq 0})^n$ with coordinates denoted by $(x_1, \ldots, x_n)$, a differential form of degree k can be written as

  $$\sum_{1 \leq i_1 < \ldots < i_k \leq n} f_{i_1 \ldots i_k} dx_{i_1} \wedge \ldots \wedge dx_{i_k},$$

  where $f_{i_1 \ldots i_k}$ are functions on $(\mathbb{R}^{\geq 0})^n$, smooth in the sense that there exist smooth functions $\hat{f}_{i_1 \ldots i_k}$ on an open neighborhood of $(\mathbb{R}^{\geq 0})^n$ in $\mathbb{R}^n$, such that $f_{i_1 \ldots i_k} = \hat{f}_{i_1 \ldots i_k}|_{(\mathbb{R}^{\geq 0})^n}$. For $0 \leq m < n$, the restriction (pullback by inclusion map) of the above differential form to $(\mathbb{R}^{\geq 0})^m \approx \{x_{m+1} = \ldots = x_n = 0\} \subset (\mathbb{R}^{\geq 0})^n$ will be

  $$\sum_{1 \leq i_1 < \ldots < i_k \leq m} f_{i_1 \ldots i_k} dx_{i_1} \wedge \ldots \wedge dx_{i_k};$$

  namely, we remove the terms containing any dx_i with $i > m$.
- *∂M, the boundary of a manifold with corners M*: the image of the map $i_M : \partial M \to M$ in [3, Definition 2.6]
- *$\tilde{\partial} M$*: ∂M in [3, Definition 2.6]
- *codimension-d (depth-d) stratum*: [2, Definition 7] (a connected component of a depth-d stratum in [3, Definition 2.3])

  For example, a solid cube has 6 codim-1 strata and 12 codim-2 strata. Its boundary is the surface of the cube, and its $\tilde{\partial}$ is the disjoint union of 6 closed squares.
- *manifold with embedded faces*: [2, Definition 18]: a manifold with corners such that the closure of every codim-1 strata is an embedded manifold with corners.

In this appendix we prove the following

Proposition A.1. *Let M be a compact manifold with embedded faces.*[2] *Suppose for each stratum $S \subset \partial M$ of M, ω_S is a closed differential form of degree k on $\overline{S}$, such that, for each pair of strata $T \subset \overline{S} \subset \partial M$, $\omega_T = \omega_S|_{\overline{T}}$. Then, there exists an open neighborhood U of ∂M and a closed differential form ω on U such that $\omega|_{\overline{S}} = \omega_S$ for all strata $S \subset \partial M$.*

In light of Proposition A.1, we also define

- *degree-k differential form on ∂M*: a compatible collection of forms, $\{\omega_S\}_S$, as in the assumption of Proposition A.1.

Remark A.2. It is easy to see that the $\mathbb{R}$-singular cohomology of ∂M (or more generally, any manifold with corners and "hinges"[3]) is equivalent to the de Rham cohomology defined using these forms, but we do not need it in the present paper.

[2] Neither the compactness or embedded faces condition should be necessary; this proposition likely holds for all manifolds with corners. We assume the stronger condition here to shorten the proof.

[3] By "hinges" we mean the manifold is locally modeled on $\mathbb{R}^a \times (\mathbb{R}^{\geq 0})^b \times \left((\mathbb{R}^{\geq 0})^c \backslash (\mathbb{R}^{>0})^c\right)$ for $a, b, c \in \mathbb{Z}^{\geq 0}$. One can image a theory of differential forms on such manifolds, defined similar to here.

Lemma A.3. *Let M be a manifold with corners and U an open neighborhood of ∂M. Suppose ω is a closed k-form on U such that $[\omega]$ lies in the image of the restriction map $H^k(M;\mathbb{R}) \to H^k(U;\mathbb{R})$, then there is a neighborhood $U'' \subset U$ of ∂M such that $\omega|_{U''}$ can be extended to a closed k-form on M.*

Proof (same as (add ref)). Let α be a closed d-form on $M \backslash \partial M$ such that the restriction of $[\alpha]$ to U is $[\omega]$. Then, there exists a $(k-1)$-form β on U such that $d\beta = \omega - \alpha|_U$. We can find[4] open subsets U', U'' of M such that $\partial M \subsetneq U'' \subsetneq U' \subsetneq U$, and a smooth function $f : M \to [0,1]$ such that $f|_{U''} \equiv 0$, $f|_{M\backslash U'} \equiv 1$. Define $\omega' = \omega - d(f\beta)$ on U. Then $\omega' \equiv \omega$ on U'', $\omega' \equiv \alpha$ on $U\backslash U'$, and $d\omega' = 0$. So we can extend ω' to a form on M, defining it to be α out of U. □

Corollary A.4. *Suppose M is a manifold with corners such that $H^k(M;\mathbb{R}) \to H^k(\partial M; R)$ is surjective. Then, every closed k-form on ∂M can be extended to a closed k-form on M.*

The rest of this section is devoted to the proof of Proposition A.1. First we prove the statement locally, in $(\mathbb{R}^{\geq 0})^d \times \mathbb{R}^{n-d}$, where $0 \leq d \leq n$ and $0 \leq k \leq n$ are integers. For $I \subset \{1, \dots, d\}$, define $H_I := \big\{(x_1, \dots, x_n) \big| \forall i \in I, x_i = 0\big\} \subset (\mathbb{R}^{\geq 0})^d \times \mathbb{R}^{n-d}$. Write $H_i := H_{\{i\}}$ and $H_{i,j} = H_{\{i,j\}}$.

Lemma A.5. *Suppose for each $1 \leq i \leq d$, ω_i is a degree-k differential form on H_i, such that, for all $1 \leq i, j \leq d$, $\omega_i|_{H_{i,j}} = \omega_j|_{H_{i,j}}$. Then, there exists a degree-k differential form ω on $(\mathbb{R}^{\geq 0})^d \times \mathbb{R}^{n-d}$, such that, for all $1 \leq i \leq d$, $\omega|_{H_i} = \omega_i$. Moreover, if all ω_i are closed, ω can be taken to be closed as well.*

Proof. From the condition $\omega_i|_{H_{i,j}} = \omega_j|_{H_{i,j}}$ it is clear that for any $I \subset \{1, \dots, d\}$, the forms $\omega_i|_{H_I}$ are the same for all $i \in I$. We hence denote it by ω_I, which is on H_I. Let

$$p_I : (\mathbb{R}^{\geq 0})^d \times \mathbb{R}^{n-d} \longrightarrow H_I$$

be the projection map, sending all I-coordinates to 0 and not changing the other coordinates. We take ω to be the alternating sum

$$\omega = \sum_{1\leq i \leq d} p_{\{i\}}^* \omega_i - \sum_{1 \leq i < j \leq d} p_{\{i,j\}}^* \omega_{\{i,j\}} + \sum_{1 \leq i < j < k \leq d} p_{\{i,j,k\}}^* \omega_{\{i,j,k\}} - \ldots + (-1)^{d-1} p_{\{1,\dots,d\}}^* \omega_{\{1,\dots,d\}}.$$

To see that $\omega|_{H_i} = \omega_i$ for a given $i \in \{1, \dots, d\}$, note that, for each $I \subset \{1, \dots, d\}$ with $I \neq \emptyset$ and $i \notin I$,

$$(p_I^* \omega_I)|_{H_i} = p_I^*(\omega_I|_{H_i}) = p_{I \sqcup \{i\}}^*(\omega_{I \sqcup \{i\}}|_{H_i}) = (p_{I \sqcup \{i\}}^* \omega_{I \sqcup \{i\}})|_{H_i},$$

so these two terms cancel with each other. If all ω_is are closed, ω is clearly also closed. □

Next, we patch the forms constructed locally to a global one. Without the closeness condition, this would be immediate by applying a partition of unity. With the closeness condition it is much subtler. The argument uses the same technique as translating between Čech and de Rham cohomology (see, e.g. [1]), which can also be viewed as a generalization of the technique in the proof of Lemma A.3. Note that from now on, the index set I has a different meaning than in Lemma A.5.

[4] For example, using a metric on M and properly modify the distance function to ∂M.

Given $p, q \in \mathbb{Z}^{\geq 0}$, a manifold M and a locally finite open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of M, recall a (skew-symmetric) p-Čech cochain of q-forms on M is: for each sequence $(i_0, i_1, \ldots, i_p) \in I^{p+1}$, a differential q-form $\alpha_{i_0 i_1 \ldots i_p}$ on $\bigcap_{j=0}^p U_{i_j}$; such that for all j, $\alpha_{i_0\ldots i_j i_{j+1}\ldots i_p} = -\alpha_{i_0\ldots i_{j+1} i_j \ldots i_p}$. We denote the $\mathbb{R}$-vector space of p-Čech cochain of q-forms on M by $\check{C}^p_{\mathcal{U}}(M; \mathcal{A}^q)$. The Čech differential is

$$\check{\delta} : \check{C}^p_{\mathcal{U}}(M; \mathcal{A}^q) \longrightarrow \check{C}^{p+1}_{\mathcal{U}}(M; \mathcal{A}^q)$$

$$\check{\delta}\,(\alpha_{i_0\ldots i_p})_{(i_0\ldots i_p)\in I^{p+1}} = (\beta_{i_0\ldots i_{p+1}})_{(i_0\ldots i_{p+1})\in I^{p+2}}, \qquad \beta_{i_0\ldots i_{p+1}} = \sum_{j=0}^{p+1}(-1)^j \alpha_{i_0\ldots\hat{i}_j\ldots i_{p+1}}\big|_{U_{i_0}\cap\ldots\cap U_{i_{p+1}}}.$$

We also still denote by d the termwise differential of forms:

$$d : \check{C}^p_{\mathcal{U}}(M; \mathcal{A}^q) \longrightarrow \check{C}^p_{\mathcal{U}}(M; \mathcal{A}^{q+1}), \qquad d\,(\alpha_{i_0\ldots i_p})_{(i_0\ldots i_p)\in I^{p+1}} = (d\alpha_{i_0\ldots i_p})_{(i_0\ldots i_p)\in I^{p+1}}.$$

It is clear that $\check{\delta} d = d\check{\delta}$, and both d and $\check{\delta}$ commute with pull-back maps between manifolds.

Lemma A.6. *Suppose M is a manifold with corners. Denote $N := \partial M$ and $\iota : N \to M$ the inclusion map.*[5] *Suppose $\mathcal{U} = \{U_i\}_{i\in I}$ is a locally finite open cover of M satisfying the condition that for all subset $I' \subset I$, if $U_{I'} := \bigcap_{i\in I'} U_i$ is non-empty, then*

1. *all de Rham cohomology groups of $U_{I'}$ are the same as those of a point,*
2. *$\iota^{-1}(U_{I'}) \neq \emptyset$, and*
3. *if σ is a closed form*[6] *on $\iota^{-1}(U_{I'})$, then there exists a closed form $\tilde{\sigma}$ on $U_{I'}$ with $\iota^*\tilde{\sigma} = \sigma$.*

Then, the following proposition $\mathcal{P}^q_p$ holds for all $p \geq 0, q \geq 1$:

$\mathcal{P}^q_p$: Suppose $\alpha = (\alpha_{i_0\ldots i_p})_{(i_0\ldots i_p)\in I^{p+1}} \in \check{C}^p_{\mathcal{U}}(M; \mathcal{A}^q)$ satisfies $\check{\delta}\alpha = 0$, $\iota^*\alpha = 0$ and $d\alpha = 0$, then, there exists $\beta = (\beta_{i_0\ldots i_p})_{(i_0\ldots i_p)\in I^{p+1}} \in \check{C}^p_{\mathcal{U}}(M; \mathcal{A}^{q-1})$ such that $\check{\delta}\beta = 0$, $\iota^*\beta = 0$ and $d\beta = \alpha$.

Proof. Two steps: we first show that $\mathcal{P}^{q-1}_{p+1} \implies \mathcal{P}^q_p$, then show that $\mathcal{P}^1_p$ holds for all $p \geq 0$.

Step 1: Suppose $q \geq 2$ and α is as in the condition of $\mathcal{P}^q_p$. By condition 1 above, there is $\beta'' \in \check{C}^p_{\mathcal{U}}(M; \mathcal{A}^{q-1})$ such that $d\beta'' = \alpha$. Then, $d\iota^*\beta'' = 0$, so condition 3 above implies that there is $\tilde{\beta}'' \in \check{C}^p_{\mathcal{U}}(M; \mathcal{A}^{q-1})$ such that $d\tilde{\beta}'' = 0$ and $\iota^*\tilde{\beta}'' = \iota^*\beta''$. Setting $\beta' = \beta'' - \tilde{\beta}''$ we have $d\beta' = \alpha$ and $\iota^*\beta' = 0$. Since $d\check{\delta}\beta' = \check{\delta}\alpha = 0$ and $\iota^*\check{\delta}\beta' = 0$, $\check{\delta}\beta'$ (replacing α) satisfies the condition of $\mathcal{P}^{q-1}_{p+1}$. If we assume $\mathcal{P}^{q-1}_{p+1}$ holds, then there exists $\gamma \in \check{C}^{p+1}_{\mathcal{U}}(M; \mathcal{A}^{q-2})$ with $\check{\delta}\gamma = 0$, $\iota^*\gamma = 0$ and $d\gamma = \check{\delta}\beta'$.

Let $\{f_i : U_i \to [0,1]\}_{i\in I}$ be a partition of unity subordinate to $\mathcal{U}$. we define β by taking

$$\beta_{i_0\ldots i_p} = \beta'_{i_0\ldots i_p} - \sum_{i\in I} d(f_i \cdot \gamma_{i i_0\ldots i_p}).$$

It is clear that $d\beta = d\beta' = \alpha$ and, since $\iota^*\gamma = 0$, $\iota^*\beta = \iota^*\beta' - \sum_{i\in I} d\big((f_i\circ\iota)\cdot\iota^*\gamma_{ii_0\ldots i_p}\big) = 0$. And

$$(\check{\delta}\beta' - \check{\delta}\beta)_{i_0\ldots i_{p+1}} = \sum_{j=0}^{p+1}(-1)^j(\beta' - \beta)_{i_0\ldots\hat{i}_j\ldots i_{p+1}} = \sum_{j=0}^{p+1}(-1)^j \sum_{i\in I} d(f_i\cdot\gamma_{ii_0\ldots\hat{i}_j\ldots i_{p+1}})$$

[5] This lemma and Lemma A.7 hold if N is replaced with an arbitrary manifold with corners and $\iota : N \to M$ an arbitrary smooth map.

[6] Differential forms on open subsets of ∂M are defined as in the assumption of Proposition A.1.

$$= d\sum_{i\in I}\Big(f_i\cdot\sum_{j=0}^{p+1}(-1)^j\gamma_{ii_0\dots\hat{i}_j\dots i_{p+1}}\Big) = d\sum_{i\in I} f_i\cdot(\gamma_{i_0\dots i_{p+1}} - (\check{\delta}\gamma)_{ii_0\dots i_{p+1}}) = d\gamma_{i_0\dots i_{p+1}}, \quad (5)$$

where the last equality is due to $\check{\delta}\gamma = 0$, and the 4-th equality holds because

$$\sum_{j=0}^{p+1}(-1)^j\gamma_{ii_0\dots\hat{i}_j\dots i_{p+1}} = -(\check{\delta}\gamma)_{ii_0\dots i_{p+1}} + \gamma_{i_0\dots i_{p+1}}.$$

Therefore, we have β such that $\check{\delta}\beta = 0$, $\iota^*\beta = 0$ and $d\beta = \alpha$, as desired.

Step 2: Suppose α is as in the condition of $\mathcal{P}_p^1$. the argument at the beginning of Step 1 says there is β such that $\iota^*\beta = 0$ and $d\beta = \alpha$. Since α consists of 1-forms, β, as well as $\check{\delta}\beta$, consist of smooth functions. Since $d\check{\delta}\beta = \check{\delta}d\beta = 0$ and the domain of each $(\check{\delta}\beta)_{i_0\dots i_p}$, if not empty, is connected by condition 1, $(\check{\delta}\beta)_{i_0\dots i_p}$ are constant functions. Also $\iota^*\check{\delta}\beta = \check{\delta}\iota^*\beta = 0$, so, by condition 2 in the lemma, each $(\check{\delta}\beta)_{i_0\dots i_p}$ with non-empty domain is 0 at at least one point, hence must be 0. Therefore $\check{\delta}\beta = 0$ and β satisfies the requirement in $\mathcal{P}_p^1$. □

Lemma A.7. *Suppose $\iota, N, M, \mathcal{U}$ are as in the condition of Lemma A.6; $p \geq 0$,$q \geq 1$. Suppose $\alpha \in \check{C}^p_{\mathcal{U}}(M;\mathcal{A}^q)$ satisfies $d\alpha = 0$ and $\check{\delta}\iota^*\alpha = 0$, then there exists $\alpha' \in \check{C}^p_{\mathcal{U}}(M;\mathcal{A}^q)$ such that $d\alpha' = 0$, $\check{\delta}\alpha' = 0$ and $\iota^*\alpha' = \iota^*\alpha$.*

Proof. We have $d(\check{\delta}\alpha) = \check{\delta}d\alpha = 0$, $\check{\delta}(\check{\delta}\alpha) = 0$, and $\iota^*(\check{\delta}\alpha) = \check{\delta}\iota^*\alpha = 0$. So, applying $\mathcal{P}^q_{p+1}$ to $\check{\delta}\alpha$ we obtain $\beta \in \check{C}^{p+1}_{\mathcal{U}}(M;\mathcal{A}^{q-1})$ such that $\check{\delta}\beta = 0$, $\iota^*\beta = 0$ and $d\beta = \check{\delta}\alpha$. Again, let $\{f_i : U_i \to [0,1]\}_{i\in I}$ be a partition of unity subordinate to $\mathcal{U}$ and we define α' by taking

$$\alpha'_{i_0\dots i_p} = \alpha_{i_0\dots i_p} - \sum_{i\in I} d(f_i\cdot\beta_{ii_0\dots i_p}),$$

then $d\alpha' = 0$ and $\iota^*\alpha' = \iota^*\alpha - \sum_{i\in I} d((f_i\circ\iota)\cdot\iota^*\beta_{ii_0\dots i_p}) = \iota^*\alpha$. And, similar to (5),

$$\begin{aligned}(\check{\delta}\alpha - \check{\delta}\alpha')_{i_0\dots i_{p+1}} &= \sum_{j=0}^{p+1}(-1)^j(\alpha-\alpha')_{i_0\dots\hat{i}_j\dots i_{p+1}} = \sum_{j=0}^{p+1}(-1)^j\sum_{i\in I} d(f_i\cdot\beta_{ii_0\dots\hat{i}_j\dots i_{p+1}})\\ &= d\sum_{i\in I}\Big(f_i\cdot\sum_{j=0}^{p+1}(-1)^j\beta_{ii_0\dots\hat{i}_j\dots i_{p+1}}\Big) = d\sum_{i\in I} f_i\cdot(\beta_{i_0\dots i_{p+1}} - (\check{\delta}\beta)_{ii_0\dots i_{p+1}}) = d\beta_{i_0\dots i_{p+1}},\end{aligned}$$

Therefore, $\check{\delta}\alpha' = \check{\delta}\alpha - d\beta = 0$, as desired. □

Lemma A.8. *Suppose M is a compact manifold with embedded faces. Then, there exist an open neighborhood U of ∂M, a finite open cover $\mathcal{U}$ of U, such that $\mathcal{U}$ satisfies the condition in Lemma A.7 when we plug in ∂M for N, U for M, and the inclusion map for ι.*

Proof. By [2, Theorem 17], M has a system of compatible collar neighborhoods ([2, Definition 35]).[7] Fix a metric on M. For a point p in a depth-k strata S, we take a convex neighborhood

[7] In our application M is $\widetilde{C}_n$, and its charts that we explicitly constructed in Section 2 is actually a system of compatible collar neighborhoods.

U'_p of p in S and define $U_p = U'_p \times [0, \epsilon)^k \subset M$ (here we implicitly use the identification given by the collar neighborhood $S \times [0, \epsilon)^k \to M$). Then, for any finite set of points $\{p_i\}$ in ∂M, $\cap_i U_{p_i}$ is diffeomorphic to $(\mathbb{R}^{\geq 0}))^d \times \mathbb{R}^{\dim M - d}$ for some d, and $(\cap_i U_{p_i}) \cap \partial M \neq \emptyset$. So, by Lemma A.5, $\cap_i U_i$ satisfied condition 3 for $U_{I'}$ imposed in Lemma A.6. Take $\mathcal{U}$ to be a finite subcover of $\{U_p\}_{p \in \partial M}$ and $U = \bigcup_{U_p \in \mathcal{U}} U_p$. □

Proof of Proposition A.1. Let U,$\mathcal{U}$ be given as in Lemma A.8. Then, for every $V \in \mathcal{U}$, by Lemma A.5, we can find a closed form ω_V on V such that $\omega_V|_{S \cap V} = \omega_S|_{S \cap V}$ for all strata S of ∂M. The collection $\{\omega_V\}_{V \in \mathcal{U}}$ defines an $\alpha \in \check{C}^0_{\mathcal{U}}(M; \mathcal{A}^k)$, which satisfies the condition of Lemma A.7. So, there exists α' as in the conclusion of Lemma A.7. To get to the conclusion of Proposition A.1, take $\omega = \alpha'$. □

References

[1] R. Bott and L. Tu, *Differential Forms in Algebraic Topology*, Springer, 1982

[2] P. Hájek, *On manifolds with corners*, Master's Thesis, Ludwig Maximilian University of Munich, 2014

[3] D. Joyce, *On manifolds with corners*, arxiv/0910.3518v2