1 Introduction

.....(introduction)

Define the kind of bundle we work with in this paper: Given a smooth homology sphere M, define a *framed* (M,∞)*-bundle* $(\pi : E \to B, \sigma, \tau, F)$ (abbreviate all these to π) to be a smooth fiber bundle $\pi : E \to B$ with fiber M, with a smooth section σ, a trivialization τ of the bundle near σ, and a smooth vertical framing F of π "standard" near σ.

Define the bracket operation, $\pi_1, \pi_2 \to [\pi_1, \pi_2]$ on such bundles, in an intuitively clear but not necessarily rigorous way.

Define cobracket and coproduct on graph cohomology (everything is over $\mathbb{Q}$):

- First, define the graph complex $\mathcal{G}'$—the $\mathbb{Q}$-vector space spanned by (with correct orientation definition, omitted here) connected graphs containing either a univalent vertex or a simple loop (an edge starting and ending at the same vertex). The coboundary operation δ is given by contracting an edge. In δ and all the operations on graphs below, whenever a graph not in $\mathcal{G}'$ appears (a graph that has a univalent vertex or simple loop), we set it to 0.
- Taking the homology of $\mathcal{G}'$ with respect to δ, denote by $H^*\mathcal{G}'$.
- Define the cobracket operation to be the linear map

$$\begin{aligned} \Delta_{[,]} : \mathcal{G}' &\longrightarrow \mathcal{G}' \otimes \mathcal{G}' \\ \Gamma &\longrightarrow \sum_{\Gamma' \leq \Gamma} \big(\Gamma' \otimes \Gamma/\Gamma' + (-1)^{\cdot\cdot} \Gamma/\Gamma' \otimes \Gamma'\big), \end{aligned} \tag{1}$$

  where Γ' ranges through all full subgraphs of Γ that is connected, with no univalent vertex or simple loop.
- Check that $\Delta_{[,]}$ commutes with δ and $\delta \otimes \mathrm{id} \pm \mathrm{id} \otimes \delta$, so it descends to

$$\Delta_{[,]} : H^*\mathcal{G}' \longrightarrow H^*(\mathcal{G}' \otimes \mathcal{G}') \approx H^*\mathcal{G}' \otimes H^*\mathcal{G}'.$$

- Finally we also define the coproduct operation on $\mathcal{G}'$ (this makes more sense for disconnected graphs but w=for connected graphs it is extra simple):

$$\begin{aligned} \Delta_{\cdot} : \mathcal{G}' &\longrightarrow \mathcal{G}' \otimes \mathcal{G}' \\ \Gamma &\longrightarrow \Gamma \otimes (\text{the empty graph}) + (\text{the empty graph}) \otimes \Gamma. \end{aligned}$$

Brief introduction to Kontsevich's characteristic classes. Given a framed (M,∞)-bundle $\pi : E \to B$ as above, denote by

$$K_\pi : H^*(\mathcal{G}') \longrightarrow H^*(B)$$

Kontsevich's characteristic classes of π.

Theorem 1.1. *Suppose $d \geq 3$. For $i = 1, 2$, suppose M_i is a d-dimensional smooth homology sphere and suppose $\pi_i : E_i \to B_i$ is a framed (M, ∞)-bundle. (Now, $[\pi_1, \pi_2] : E \to S^d \times B_1 \times B_2$ is the bracket bundle.) Then, for all $\eta \in H^*\mathcal{G}'$,*

$$K_{[\pi_1,\pi_2]}(\eta) = \mathrm{PD}_{S^d}[S^d] \otimes (K_{\pi_1} \otimes K_{\pi_2})(\Delta_{[,]}(\eta)) + \mathrm{PD}_{S^d}[pt] \otimes (K_{\pi_1} \otimes K_{\pi_2})(\Delta_{\cdot}(\eta)).$$

(Both LHS and RHS lives in

$$H^*(S^d \times B_1 \times B_2) \approx H^*(S^d) \otimes H^*(B_1) \otimes H^*(B_2).$$

PD_{S^d} *means Poincaré dual on S^d; $[S^d]$ stands for the fundamental class of S^d and $[pt]$ stands for the point class of S^d.)*

......(Then talk about the $(d+1)$-fold loop space structure on $\mathrm{BDiff}^{\mathrm{fr}}_{\partial}(D^d)$ and the theorem/corollary that it doesn't extend.)

(Below is an outline of the proof of Theorem 1.1. Throughout, π_1, π_2 are given and fixed.)

1.1 Notation

Given a graph G, we denote by $V(G)$ its vertex set and $E(G)$ its edge set.

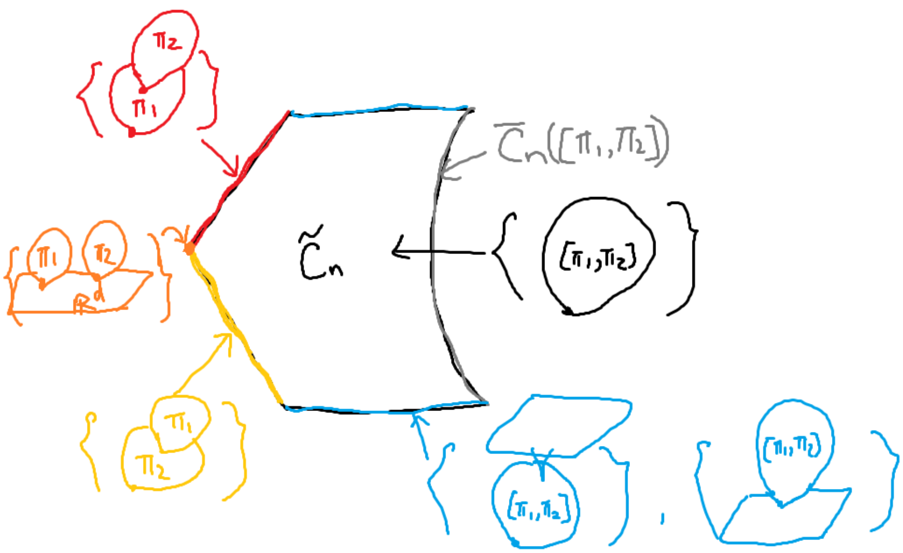
2 Conftilde

Construct the big configuration space $\widetilde{C}_A$. Show that it is a smooth manifold with boundary and corners. (These are mostly already written in the file "conftilde" I sent a while ago.)

What we need are the following:

- $\widetilde{C}_A$ is a smooth manifold with boundary and corners;
- each S_T is a stratum of $\widetilde{C}_A$;
- $\overline{S}_T = \bigsqcup_{T'} S_{T'}$, where the disjoint union is taken over all A-labeled trees T' such that T can be obtained from T' by contracting some edges.

Here is a schematic picture of $\widetilde{C}_A$ (the marked points are not drawn; the actual stratification structure of $\widetilde{C}_A$ is more complicated than what is shown in the picture):



The boundary of $\widetilde{C}_A$ consists of the following parts:

- the gray part, denoted by $\overline{S}_{\text{gray}}$, is $\overline{C}_A([\pi_1, \pi_2])$; its interior, denoted by S_{gray}, is $C_A([\pi_1, \pi_2])$;
- $S_{\text{blue}} := \bigcup_{T \in \mathcal{T}_{\text{blue}}} S_T$, where $\mathcal{T}_{\text{blue}}$ is the set of all A-labeled trees whose shape and space labels are like $\begin{array}{l} \bullet\ \mathbb{R}^d \\ | \\ \bullet\ [\pi_1, \pi_2] \\ \mathrm{r} \end{array}$ or $\begin{array}{l} \bullet\ [\pi_1, \pi_2] \\ | \\ \bullet\ \mathbb{R}^d \\ \mathrm{r} \end{array}$; and let $\overline{S}_{\text{blue}}$ be the closure of S_{blue};
- $S_{\text{red}} := \bigcup_{T \in \mathcal{T}_{\text{red}}} S_T$, where $\mathcal{T}_{\text{red}}$ is the set of all A-labeled trees with the following shape and space labels: $\begin{array}{l} \bullet\ \pi_2 \\ | \\ \bullet\ \pi_1 \\ \mathrm{r} \end{array}$; and let $\overline{S}_{\text{red}}$ be the closure of S_{red};
- $S_{\text{yellow}} := \bigcup_{T \in \mathcal{T}_{\text{yellow}}} S_T$, where $\mathcal{T}_{\text{yellow}}$ is the set of all A-labeled trees with the following shape and space labels: $\begin{array}{l} \bullet\ \pi_1 \\ | \\ \bullet\ \pi_2 \\ \mathrm{r} \end{array}$; and let $\overline{S}_{\text{yellow}}$ be the closure of S_{yellow};

We also define $S_{\text{orange}} := \bigcup_{T \in \mathcal{T}_{\text{orange}}} S_T$, where $\mathcal{T}_{\text{orange}}$ is the set of all A-labeled trees with the following shape and space labels: $\begin{array}{ccc} \bullet\ \pi_1 & & \bullet\ \pi_2 \\ & \diagdown\ \diagup & \\ & \bullet\ \mathbb{R}^d & \\ & \mathrm{r} & \end{array}$; and let $\overline{S}_{\text{orange}}$ be the closure of S_{orange}. Then, $\overline{S}_{\text{orange}} = \overline{S}_{\text{red}} \cap \overline{S}_{\text{yellow}}$.

We define $\overline{C}^*_A = \overline{S}_{\text{red}} \cup \overline{S}_{\text{yellow}}$.

3 Propagators

Before starting the discussion on propagators, we first define another notion of "forgetful map".

Given finite sets A ("set of point labels") and B ("set of space labels"), recall the definition of an (A, B)-labeled tree in ...(change the definition in conftilde to allow arbitrary space labels!)

In all the cases we care about, the elements of B will be (M, ∞)-bundles for some d-dimensional manifold M.

In this paper we only consider cases when $|B| \leq 2$. Define

$$\widetilde{C}_A(B) = \begin{cases} \overline{C}_A(\mathbb{R}^d) \text{ if } B = \emptyset, \\ \overline{C}_A(\pi) \text{ if } B = \{\pi\} \text{ for some } (M,\infty) - \text{bundle } \pi, \\ \widetilde{C}_A \backslash \overline{S}_{\text{gray}} \text{ if } B = \{\pi_1, \pi_2\} \text{ for some } (M_1,\infty)\text{-bundle } \pi_1 \text{ and some } (M_2,\infty)\text{-bundle } \pi_2. \end{cases}$$

Then, the strata of $\widetilde{C}_A(B)$ are in 1-to-1 correspondence with (A, B)-labeled trees. Given such a stratum S, we denote by $\mathcal{T}_S$ the tree corresponding to it and given such a tree T we denote by $\mathcal{S}_T$ the stratum corresponding to it. [1] The condition $\mathcal{S}_{T'} \subset \overline{\mathcal{S}}_T$ is equivalent to that T can be obtained from T' by contracting some edges. In this case, the set of edges to be contracted to get from T' to T is unique and we denote by $\mathfrak{c}_{T',T} : V(T') \to V(T)$ the map on the vertices induced by the contraction. Also define $\mathfrak{i}_{T',T} : V(T) \to V(T')$ mapping v to the lowest vertex in $\mathfrak{c}^{-1}_{T',T}(v)$. The following lemma is immediate:

Lemma 3.1. *Let T', T be (A, B)-labeled trees such that T can be obtained from T' by contracting some edges. Then, for every $v \in V(T)$, $lp(\geq v) = lp(\geq \mathfrak{i}_{T',T}(v))$.*

Maybe move the above to an earlier section devoted to combinatorics.

For the rest of this section we only consider the case $A = \{1, 2\}$. Let B be a finite set such that every element of B is an (M, ∞)-bundles for some d-dimensional manifold M, and $|B| \leq 2$.

Definition 3.2. Let T be a $(\{1, 2\}, B)$-labeled tree, [2] then $\mathcal{S}_T \approx \prod_{v \in V(T)} C_{lp(v) \cup cld(v)}(ls(v))$ is a stratum in $\widetilde{C}_{\{1,2\}}(B)$. Define $\nu_T \in V(T)$ to be the vertex such that $\{1, 2\} \subset lp(\geq \nu_T)$ and for all $v > \nu_T$, $\{1, 2\} \not\subset lp(\geq v)$.[3] Define $\mathfrak{s}_T := ls(\nu_T)$. [4]

- Define
$$\hat{f}_T : \mathcal{S}_T \longrightarrow C_2(\mathfrak{s}_T), \qquad \hat{f}_T\big((c_v)_{v \in V(T)}\big) = c'_{\nu_T},$$
where $c'_{\nu_T} \in C_2(\mathfrak{s}_T))$ is obtained from c_{ν_T} by forgetting all the points except for two: $f_{\nu_T}(1)$ and $f_{\nu_T}(2)$. (f_v is defined in conftilde, at the beginning of Section 3.3.)

- Suppose T' is a $(\{1, 2\}, B)$-labeled tree such that T can be obtained from T' by contracting some edges. Abusing notation, we denote the subtree of T' spanned by vertices in $\mathfrak{c}^{-1}_{T',T}(\nu_T)$ still by $\mathfrak{c}^{-1}_{T',T}(\nu_T)$. Define $G_{T',T}$ to be the tree obtained from $\mathfrak{c}^{-1}_{T',T}(\nu_T)$ by "stabilization with respect to $\{1, 2\}$ and $\mathfrak{s}_T$", namely: let $V' \subset V(\mathfrak{c}^{-1}_{T',T}(\nu_T))$ ("set of unstable vertices") consist of vertices v such that $\text{lsset}(ls(v)) \cap \text{lsset}(ls(\nu)_T) = \emptyset$ and $|lp(\geq v) \cap \{1, 2\}| < 2$; define $G_{T',T}$

[1] ($\mathcal{T}$ was used previously in conftilde; maybe change it to $\mathfrak{T}$ there.)

[2] The definition obviously extends to the case of an arbitrary number of marked points and forgetting to an arbitrary subset of marked points, but we only need this simple 2 point case here.

[3] (In this paper $\subset$ means subset or equal. Specify this somewhere early.)

[4] (Actually, maybe use $\mathfrak{p}, \mathfrak{s}$ instead of fp,fs?)

to be obtained from $\mathfrak{c}^{-1}_{T',T}(\nu_T)$ by: for every vertex $v \in V'$, contracting the edge just below v. Then, $\mathcal{S}_{G_{T',T}}$ is a stratum of $\overline{C}_2(\mathfrak{s}_T)$. Define

$$\hat{f}_T : \mathcal{S}_{T'} \longrightarrow \mathcal{S}_{G_{T',T}} \subset \overline{C}_2(\mathfrak{s}_T), \qquad \hat{f}_T\big((c_v)_{v\in V(T')}\big) = (c'_v)_{v \in V(G_{T',T})},$$

where each $c'_v \in C_2(ls(v))$ is as follows: let $v' \in V(\mathfrak{c}^{-1}_{T',T}(\nu_T)) \subset V(T')$ be the lowest vertex contracted to v, then c'_v is obtained from $c_{v'}$ by forgetting all points except for two: $f_{v'}(1)$ and $f_{v'}(2)$.

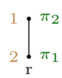
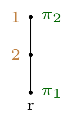
- We have therefore defined a map

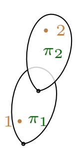
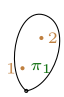
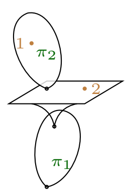
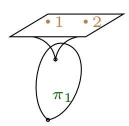
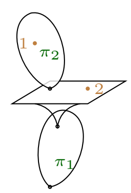
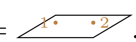
$$\hat{f}_T : \overline{\mathcal{S}}_T \longrightarrow \overline{C}_2(\mathfrak{s}_T).$$

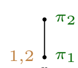
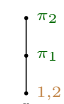
It is easy to verify that $\hat{f}_T$ is smooth using the charts we constructed in Section (conftilde section).

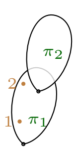
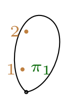
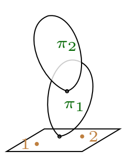
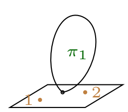
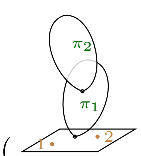
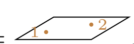
Maybe introduce more notation when talking about the combinatorics of A-labeled trees, e.g., a pre-stable tree and how to get from a pre-stable tree to a stabel tree by contraction.?

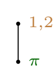
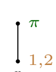
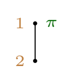
Note that if $G_{T',T}$ has only one vertex, then $\hat{f}_{T'} = \hat{f}_T|_{\overline{\mathcal{S}}_{T'}}$. Otherwise, this is not the case.

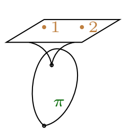
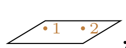
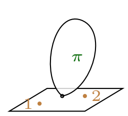
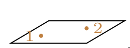
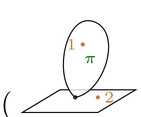
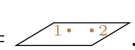
Example 3.3 ($|B| = 2$, in $\widetilde{C}_2$)**.** $T =$  , $T' =$  (so $\mathcal{S}_{T'} \subset \overline{\mathcal{S}}_T$),

$\hat{f}_T($  $) =$  , $\hat{f}_T($  $) =$  , $\hat{f}_{T'}($  $) =$  .

Example 3.4 ($|B| = 2$, in $\widetilde{C}_2$)**.** $T =$  , $T' =$  (so $\mathcal{S}_{T'} \subset \overline{\mathcal{S}}_T$),

$\hat{f}_T($  $) =$  , $\hat{f}_T($  $) =$  $\hat{f}_{T'}($  $) =$  .

Example 3.5 ($|B| = 1$, in $\overline{C}_2(\pi)$ where π is an (M, ∞)-bundle)**.** $T_1 =$  , $T_2 =$  , $T_3 =$  ,

$\hat{f}_{T_1}($  $) =$  , $\hat{f}_{T_2}($  $) =$  , $\hat{f}_{T_3}($  $) =$  .

Note that $S_{T_1}, S_{T_2}, S_{T_3}$ are the only codimension-1 strata of $\overline{C}_2(\pi)$, and the codomain of $\hat{f}_{T_1}$, $\hat{f}_{T_2}$, $\hat{f}_{T_3}$ are all $\overline{C}_2(\mathbb{R}^d)$.

Corollary 3.6 (of Lemma 3.1)**.** *Let T', T be $(\{1,2\}, B)$-labeled trees such that T can be obtained from T' by contracting some edges, then $\mathfrak{c}_{T',T}(\nu_{T'}) = \nu_T$. Moreover, if $\mathfrak{s}_T = \mathbb{R}^d$, then $\mathfrak{s}_{T'} = \mathbb{R}^d$.*

The following lemma is easy to check: [5]

Lemma 3.7. *Let T', T be $(\{1,2\}, B)$-labeled trees such that T can be obtained from T' by contracting some edges, then*

$$\hat{f}_{T'} = \hat{f}_{G_{T',T}} \circ (\hat{f}_T|_{\overline{\mathcal{S}}_{T'}}).$$

Definition 3.8. Suppose M is a d-dimensional $\mathbb{Z}$-homology sphere and π is an (M,∞)-bundle. A *propagator* on $\overline{C}_2(\pi)$ (resp. $\overline{C}_2(\mathbb{R}^d)$) is a closed $(d-1)$-form ω on $\overline{C}_2(\pi)$ satisfying: there exists a $(d-1)$-form ω_0 on $S^{d-1} \approx \overline{C}_2(\mathbb{R}^d)$ such that $\int_{S^{d-1}} \omega_0 = 1$ and for every codimension-1 stratum $S \subset \partial\overline{C}_2(\pi)$ (resp. $S \subset \partial\overline{C}_2(\mathbb{R})$), $\omega|_{\overline{S}} = \hat{f}^*_{T_S}\omega_0$.

This definition is phrased differently from the usual definition of a propagator, see e.g. [5, Definition 3.9] or [6, Lemma 2.12], but can easily be seen to be equivalent.

Fix a volume form ω_0 on S^{d-1}. By [6, Lemma 2.12], there exist propagators ω_1 on $\overline{C}_2(\pi_1)$, ω_2 on $\overline{C}_2(\pi_2)$, and ω on $\overline{C}_2([\pi_1, \pi_2])$ such that the above condition in Definition 3.8 is satisfied with this ω_0. We choose and fix such $\omega_1, \omega_2, \omega$.

In the rest of this section we construct a "propagator" on $\partial\widetilde{C}_2$. This is done in two steps:

1. On each stratum S of $\partial\widetilde{C}_2$, construct a closed $(d-1)$-form ω_S on $\overline{S}$, such that, for two strata $S' \subset \overline{S}$, $\omega_S|_{\overline{S'}} = \omega_{S'}$.

2. Show that this collection $\{\omega_S\}_S$ extends to a closed form $\widetilde{\omega}$ on $\widetilde{C}_2$; namely, $\widetilde{\omega}$ is such that $\widetilde{\omega}|_S = \omega_S$ for every strata $S \subset \partial\widetilde{C}_2$.

Step 1: There are two kinds of strata in $\partial\widetilde{C}_2$: those that are not subsets of $\overline{S}_{\text{gray}}$ and those that are. The former kind of strata are of the form $\mathcal{S}_T$ for some unique $(\{1,2\}, \{\pi_1, \pi_2\})$-labeled tree T with at least one edge. For these strata, define $\omega_{\mathcal{S}_T} = \hat{f}^*_T\omega_i$, where $i = 1$ if $\mathfrak{s}_T = \pi_1$, $i = 2$ if $\mathfrak{s}_T = \pi_2$, $i = 0$ if $\mathfrak{s}_T = \mathbb{R}^d$, and remove the subscript i if $\mathfrak{s}_T = [\pi_1, \pi_2]$. For the latter kind of strata: under the natural identification $\overline{S}_{\text{gray}} \approx \overline{C}_2([\pi_1, \pi_2])$, define ω_S to be the restriction of ω to S. This defines the collection $\{\omega_S\}_S$.

Now we need to show the compatibility condition: for $S' \subset \overline{S}$, $\omega_S|_{\overline{S'}} = \omega_{S'}$.

- If $S', S \subset \overline{S}_{\text{gray}}$, then this holds by definition.

[5]Hint: By continuity of the $\hat{f}$ maps, it suffices to prove the equality on the open part $\mathcal{S}_{T'}$.

- If $S', S \not\subset \overline{S}_{\text{gray}}$: since $\omega_S = \hat{f}_{\mathcal{T}_S}\omega_i$ for some propagator ω_i,

$$\omega_S|_{\overline{S}'} = (\hat{f}_{\mathcal{T}_S}|_{\overline{S}'})^*\omega_i = \begin{cases} \hat{f}^*_{\mathcal{T}_{S'}}\omega_i, \text{ if } G_{\mathcal{T}_{S'},\mathcal{T}_S} \text{ has only 1 vertex} \\ (\hat{f}_{\mathcal{T}_S}|_{\overline{S}'})^*\hat{f}^*_{G_{\mathcal{T}_{S'},\mathcal{T}_S}}\omega_0 = \hat{f}^*_{\mathcal{T}_{S'}}\omega_0, \text{ otherwise} \end{cases} = \omega_{S'},$$

  where, for the second case, the second equality is because ω_i is a propagator as in Definition 3.8 and the third equality is because of Lemma 3.9.

- If $S' \subset \overline{S}_{\text{gray}}$, $S \not\subset \overline{S}_{\text{gray}}$: let T' be the $(\{1,2\},\{[\pi_1,\pi_2]\})$-labeled tree $\mathcal{T}_{S'}$ under the identification $\overline{S}_{\text{gray}} \approx \overline{C}_2([\pi_1,\pi_2])$. We can also view T' as a $(\{1,2\},\{\pi_1,\pi_2\})$-labeled tree; to avoid confusion let us call this $(\{1,2\},\{\pi_1,\pi_2\})$-tree T. Then $\mathcal{S}_T \subset \overline{S}_{\text{blue}}$ is a stratum of $\widetilde{C}_2$, $\mathcal{S}_T \subset S$ and $\mathcal{S}_T \cap \overline{S}_{\text{gray}} = S'$. Since we have already shown $\omega_S|_{\overline{\mathcal{S}}_T} = \omega_{\mathcal{S}_T}$, it suffices to show $\omega_{\mathcal{S}_T}|_{\overline{S}'} = \omega_{S'}$. We first state a lemma:

  Lemma 3.9. *Let S be a stratum of $\partial\widetilde{C}_2$. If $S \subset \overline{S}_{\text{blue}}$ and $S \not\subset \overline{S}_{\text{gray}}$, then $\mathfrak{s}_{\mathcal{T}_S} = \mathbb{R}^d$.*

  Proof. If S is of top dimension in $\overline{S}_{\text{blue}}$, then $\mathcal{T}_S$ has one vertex with space label $[\pi_1,\pi_2]$ and only one edge. The statement of the lemma follows from the last sentence of Example 3.5. If S is not of top dimension in $\overline{S}_{\text{blue}}$, then it is in the closure of a stratum of top dimension, and the lemma follows from the last sentence of Corollary 3.6. □

  Now, because the definition of $\hat{f}$ is combinatorial and T, T' are the same tree, the two maps

$$\hat{f}_T|_{\overline{S}'} : \overline{S}' \longrightarrow \overline{C}_2(\mathbb{R}^d), \qquad \hat{f}_{T'} : \overline{S}' \longrightarrow \overline{C}_2(\mathbb{R}^d)$$

  are equal, where the first $\overline{S}'$ is viewed as a subset of the stratum $\overline{\mathcal{S}}_T$ in $\widetilde{C}_2$ while the second $\overline{S}'$ is viewed as a boundary stratum of $\overline{C}_2([\pi_1,\pi_2])$ under the identification $\overline{S}_{\text{gray}} \approx \overline{C}_2([\pi_1,\pi_2])$. Since $\omega_{\mathcal{S}_T} = \hat{f}^*_T\omega_0$ and $\omega_{S'} = \hat{f}_{T'}\omega_0$, we conclude that they are equal on $\overline{S}'$.

Step 2: The following statement (Corollary A.4) is proved in Appendix A:

Suppose M is a compact manifold with embedded faces such that $H^k(M;\mathbb{R}) \to H^k(\partial M; R)$ is surjective. Suppose for every stratum $S \subset \partial M$, ω_S is a closed k-form on S, such that, if $S' \subset \overline{S}$, then $\omega_S|_{S'} = \omega_{S'}$. Then, there exists a closed k-form $\widetilde{\omega}$ on M such that $\widetilde{\omega}|_S = \omega_S$ for every S.

Therefore, to show the existence of a closed form $\widetilde{\omega}$ on $\widetilde{C}_2$ extending the $\{\omega_S\}_S$ we just defined, it suffices to show that the map

$$H^{d-1}(\widetilde{C}_2) \xrightarrow{\text{restriction}} H^{d-1}(\partial\widetilde{C}_2)$$

is surjective. It is therefore sufficient to show that $H^d(\widetilde{C}_2, \partial\widetilde{C}_2) = 0$. But

$$\begin{aligned} H^*(\widetilde{C}_2, \partial\widetilde{C}_2) &\approx H_{\dim(\widetilde{C}_2)-*}(\widetilde{C}_2 - \partial\widetilde{C}_2) = H_{\dim(\widetilde{C}_2)-*}\big(C_2([\pi_1,\pi_2]) \times (0,1)\big) \\ &\approx H_{\dim(\widetilde{C}_2)-*}(C_2([\pi_1,\pi_2])) \approx H^{*-1}\big(\overline{C}_2([\pi_1,\pi_2]), \partial\overline{C}_2([\pi_1,\pi_2])\big), \end{aligned}$$

and this is 0 when $*-1 < d+1$ by the proof of Lemma 2.12 in [6].

We choose and fix such an extension $\widetilde{\omega}$ on $\widetilde{C}_2$.

4 Configuration space integrals

Note that B_I as defined in conftilde Section 2.4 admits a submersion $\pi_{B_I} : B_I \to B_1 \times B_2$, making B_I a smooth fiber bundle (whose fibers are manifolds with corners) over $B_1 \times B_2$. Each fiber is homeomorphic to $S^d \times [0, \rho)$ and looks like the picture on page 10 of conftilde. The map $\pi_{B_I} \circ \tilde{\pi}^A : \widetilde{C}_A \to B_1 \times B_2$, and its restriction to every strata of $\widetilde{C}_A$, are also submersions.

The base B of $[\pi_1, \pi_2]$ is B_{t_0}, the total space of the subbundle of B_I obtained by taking the subspace $S^d \times \{t_0\} \subset S^d \times [0, \rho)$ of each fiber. So, the fibers are diffeomorphic to S^d. Indeed, the fiber bundle $B \to B_1 \times B_2$ is trivial when regarded as a topological fiber bundle; namely, B is homeomorphic to $S^d \times B_1 \times B_2$. So, $H^*(B) \approx H^*(S^d) \otimes H^*(B_1) \otimes H^*(B_2)$.

We have the following diagram:

$$\begin{tikzcd}
\overline{C}_A([\pi_1, \pi_2]) \arrow[r, hook] \arrow[d, "\pi_{[]}^A"] & \widetilde{C}_A \arrow[d, "\tilde{\pi}^A"] \\
B_{t_0} \arrow[r, hook] \arrow[dr, "\pi_{B_I}|_{B_{t_0}}"'] & B_I \arrow[d, "\pi_{B_I}"] \\
& B_1 \times B_2
\end{tikzcd},$$

where $\pi_{[]}^A$, π_{B_I} are submersions but not $\tilde{\pi}^A$.

Lemma 4.1. *Let $A' \subset A$ be finite sets and $\mathfrak{f}_{A',A} : \widetilde{C}_A \to \widetilde{C}_{A'}$ the forgetful map forgetting all marked points not in A'. Then, for every (open) stratum S of $\widetilde{C}_A$, $\mathfrak{f}_{A',A}(S)$ is again an open stratum of $\widetilde{C}_{A'}$ and*

$$\mathfrak{f}_{A',A}|_S : S \longrightarrow \mathfrak{f}_{A',A}(S)$$

is a submersion.

In particular, when $A' = \emptyset$, this holds with $\widetilde{C}_{A'} = B_I$ and $\mathfrak{f}_{A',A} = \tilde{\pi}$.

(Something about the above should be in the conftilde section.)

Suppose $\sum_{i=1}^m \Gamma_i$ is a cocycle in graph cohomology. For convenience, in this section we use $\overline{S}^i_{\text{gray}}$ to denote the $\overline{S}_{\text{gray}}$ in $\widetilde{C}_{V(\Gamma_i)}$, and same for the other colors (red, orange, yellow, blue) instead of gray. For every edge e of some Γ_i, we have the forgetful map

$$\mathfrak{f}_e : \widetilde{C}_{V(\Gamma_i)} \longrightarrow \widetilde{C}_2.$$

And, when restricted to $\overline{S}^i_{\text{gray}} \subset \widetilde{C}_{V(\Gamma_i)}$ (resp. $\overline{C}^*_{V(\Gamma_i)} \subset \widetilde{C}_{V(\Gamma_i)}$), it is the forgetful map

$$\mathfrak{f}_e : \overline{C}_{V(\Gamma_i)}([\pi_1, \pi_2]) \longrightarrow \overline{C}_2([\pi_1, \pi_2]) \qquad (\text{resp. } \mathfrak{f}_e : \overline{C}^*_{V(\Gamma_i)} \longrightarrow \overline{C}^*_2).$$

Now we have the form $\bigwedge_{e \in E(\Gamma_i)} \mathfrak{f}_e^* \widetilde{\omega}$ on $\widetilde{C}_{V(\Gamma_i)}$. Recall that by the definition of Kontsevich's characteristic classes,

$$K_{[\pi_1,\pi_2]}\big([\sum_{i=1}^m \Gamma_i]\big) = \big[\sum_{i=1}^m (\pi_{[]}^{V(\Gamma_i)})_* \bigwedge_{e \in E(\Gamma_i)} \mathfrak{f}_e^* \omega\big] \in H^*(B) \approx H^*(S^d) \otimes H^*(B_1) \otimes H^*(B_2).$$

To determine $K_{[\pi_1,\pi_2]}\big(\big[\sum_{i=1}^m \Gamma_i\big]\big)$, it suffices to determine the intersection pairing

$$\big\langle K_{[\pi_1,\pi_2]}\big(\big[\sum_{i=1}^m \Gamma_i\big]\big), \alpha_0 \otimes \alpha_1 \otimes \alpha_2 \big\rangle$$

for all homology classes $\alpha_0 \in H_*(S^d)$, $\alpha_1 \in H_*(B_1)$ and $\alpha_2 \in H_*(B_2)$. Below, we discuss the two cases $\alpha_0 = [pt]$ (the class of a point) and $\alpha_0 = [S^d]$ separately.

Case I: $\alpha_0 = [S^d]$

For $a = 1, 2$, suppose $\deg \alpha_a = k_a$ and α_a is represented by a piecewise-smooth singular chain $\sum_{j_a} \iota_a^{j_a}$, where $\iota_a^{j_a}$ are smooth maps from the standard k_a-dimensional simplex Δ^{k_a} to B_a.

Since for all finite set A, $\pi_{B_I} \circ \tilde{\pi}^A : \widetilde{C}_A \to B_1 \times B_2$ and its restriction to every stratum is a fiber bundle, we can form the pullback bundles $(\iota_1^{j_1}, \iota_2^{j_2})^* \widetilde{C}_A$, $(\iota_1^{j_1}, \iota_2^{j_2})^* \overline{C}_A([\pi_1, \pi_2])$,$(\iota_1^{j_1}, \iota_2^{j_2})^* \overline{C}_A^*$ over $\Delta^{k_1} \times \Delta^{k_2}$.

We then have

$$\big\langle K_{[\pi_1,\pi_2]}([\sum_{i=1}^m \Gamma_i]), [S^d] \otimes \alpha_1 \otimes \alpha_2 \big\rangle = \sum_{i,j_1,j_2} \int_{(\iota_1^{j_1},\iota_2^{j_2})^* \overline{C}_{V(\Gamma_i)}([\pi_1,\pi_2])} \bigwedge_e \mathfrak{f}_e^* \omega = \sum_{i,j_1,j_2} \int_{(\iota_1^{j_1},\iota_2^{j_2})^* \overline{S}^i_{\text{gray}}} \bigwedge_e \mathfrak{f}_e^* \widetilde{\omega}. \tag{2}$$

Since

$$\partial\big((\iota_1^{j_1}, \iota_2^{j_2})^* \widetilde{C}_{V(\Gamma_i)}\big) = (\iota_1^{j_1}, \iota_2^{j_2})^* \partial \widetilde{C}_{V(\Gamma_i)} + \big((\iota_1^{j_1}, \iota_2^{j_2})^* \widetilde{C}_{V(\Gamma_i)}\big)\big|_{\partial(\Delta^{k_1} \times \Delta^{k_2})},$$

by Stocks' Formula,

$$\begin{aligned}
\int_{(\iota_1^{j_1},\iota_2^{j_2})^* \partial \widetilde{C}_{V(\Gamma_i)}} \bigwedge_e \mathfrak{f}_e^* \widetilde{\omega} &= \int_{\partial(\iota_1^{j_1},\iota_2^{j_2})^* \widetilde{C}_{V(\Gamma_i)}} \bigwedge_e \mathfrak{f}_e^* \widetilde{\omega} - \int_{(\iota_1^{j_1},\iota_2^{j_2})^* \widetilde{C}_{V(\Gamma_i)}|_{\partial(\Delta^{k_1} \times \Delta^{k_2})}} \bigwedge_e \mathfrak{f}_e^* \widetilde{\omega} \\
&= \int_{(\iota_1^{j_1},\iota_2^{j_2})^* \widetilde{C}_{V(\Gamma_i)}} d\Big(\bigwedge_e \mathfrak{f}_e^* \widetilde{\omega}\Big) - \int_{(\iota_1^{j_1},\iota_2^{j_2})^* \widetilde{C}_{V(\Gamma_i)}|_{\partial \Delta^{k_1} \times \Delta^{k_2}}} \bigwedge_e \mathfrak{f}_e^* \widetilde{\omega} \pm \int_{(\iota_1^{j_1},\iota_2^{j_2})^* \widetilde{C}_{V(\Gamma_i)}|_{\Delta^{k_1} \times \partial \Delta^{k_2}}} \bigwedge_e \mathfrak{f}_e^* \widetilde{\omega}.
\end{aligned}$$

The first term is 0 because $\widetilde{\omega}$ is closed. Because the singular chains $\sum_{j_1} \iota_1^{j_1}$, $\sum_{j_2} \iota_2^{j_2}$ are cycles, when summing over all j_1, the second term is 0; when summing over all j_2, the third term is 0. So,

$$(2) = \sum_{i,j_1,j_2} \Big(\int_{(\iota_1^{j_1},\iota_2^{j_2})^* \overline{S}^i_{\text{blue}}} \bigwedge_e \mathfrak{f}_e^* \widetilde{\omega} + \int_{(\iota_1^{j_1},\iota_2^{j_2})^* \overline{C}^*_{V(\Gamma_i)}} \bigwedge_e \mathfrak{f}_e^* \widetilde{\omega} \Big).$$

Since $\sum_i \Gamma_i$ is a cocycle in graph cohomology, the first term is 0 just like in the proof of the well-definedness of Kontsevich's classes, see e.g. [6, Appendix E][6], so

$$\big\langle K_{[\pi_1,\pi_2]}([\sum_i \Gamma_i]), [S^d] \otimes \alpha_1 \otimes \alpha_2 \big\rangle = \sum_{i,j_1,j_2} \int_{(\iota_1^{j_1},\iota_2^{j_2})^* \overline{C}^*_{V(\Gamma_i)}} \bigwedge_e \mathfrak{f}_e^* \widetilde{\omega}$$

[6] The argument goes as follows: a top-dimensional stratum S of $\overline{S}^i_{\text{blue}}$ contains a screen with space label $\mathbb{R}^d$, consider 3 separate cases: (...)

$$= \sum_{i,j_1,j_2} \int_{(\iota_1^{j_1},\iota_2^{j_2})^* S^i_{\text{red}}} \bigwedge_e \mathfrak{f}_e^* \widetilde{\omega} + \sum_{i,j_1,j_2} \int_{(\iota_1^{j_1},\iota_2^{j_2})^* S^i_{\text{yellow}}} \bigwedge_e \mathfrak{f}_e^* \widetilde{\omega}. \tag{3}$$

It remains to show that

$$(3) = \big\langle (K_{\pi_1} \otimes K_{\pi_2})(\Delta_{[,]}\big[\sum_i \Gamma_i\big]), \alpha_1 \otimes \alpha_2 \big\rangle. \tag{4}$$

The right hand side above equals to[7]

$$\big\langle (K_{\pi_1} \otimes K_{\pi_2})(\big[\sum_i \sum_{\Gamma_i' \le \Gamma_i} (\Gamma_i' \otimes \Gamma_i/\Gamma_i' + (-1)^? \Gamma_i/\Gamma_i' \otimes \Gamma_i')\big]), \alpha_1 \otimes \alpha_2 \big\rangle =$$

$$\sum_{i,j_1,j_2} \sum_{\Gamma_i' \le \Gamma_i} \Big(\Big(\int_{(\iota_1^{j_1})^* \overline{C}_{V(\Gamma_i')}(\pi_1)} \bigwedge_{e \in E(\Gamma_i')} \mathfrak{f}_e^* \omega_1 \Big) \cdot \Big(\int_{(\iota_2^{j_2})^* \overline{C}_{V(\Gamma_i/\Gamma_i')}(\pi_2)} \bigwedge_{e \in E(\Gamma_i/\Gamma_i')} \mathfrak{f}_e^* \omega_2 \Big)$$
$$+ (-1)^? \Big(\int_{(\iota_1^{j_1})^* \overline{C}_{V(\Gamma_i/\Gamma_i')}(\pi_1)} \bigwedge_{e \in E(\Gamma_i/\Gamma_i')} \mathfrak{f}_e^* \omega_1 \Big) \cdot \Big(\int_{(\iota_2^{j_2})^* \overline{C}_{V(\Gamma_i')}(\pi_2)} \bigwedge_{e \in E(\Gamma_i')} \mathfrak{f}_e^* \omega_2 \Big) \Big).$$

To prove (4), it suffices to show that, for all i, j_1, j_2,

$$\sum_{\Gamma_i' \le \Gamma_i} \Big(\int_{(\iota_1^{j_1})^* C_{V(\Gamma_i')}(\pi_1)} \bigwedge_{e \in E(\Gamma_i')} \mathfrak{f}_e^* \omega_1 \Big) \cdot \Big(\int_{(\iota_2^{j_2})^* C_{V(\Gamma_i/\Gamma_i')}(\pi_2)} \bigwedge_{e \in E(\Gamma_i/\Gamma_i')} \mathfrak{f}_e^* \omega_2 \Big) = \int_{(\iota_1^{j_1},\iota_2^{j_2})^* S^i_{\text{yellow}}} \bigwedge_{e \in E(\Gamma_i)} \mathfrak{f}_e^* \widetilde{\omega} \tag{5}$$

and

$$\sum_{\Gamma_i' \le \Gamma_i} \Big(\int_{(\iota_1^{j_1})^* C_{V(\Gamma_i/\Gamma_i')}(\pi_1)} \bigwedge_{e \in E(\Gamma_i/\Gamma_i')} \mathfrak{f}_e^* \omega_1 \Big) \cdot \Big(\int_{(\iota_2^{j_2})^* C_{V(\Gamma_i')}(\pi_2)} \bigwedge_{e \in E(\Gamma_i')} \mathfrak{f}_e^* \omega_2 \Big) = \int_{(\iota_1^{j_1},\iota_2^{j_2})^* S^i_{\text{red}}} \bigwedge_{e \in E(\Gamma_i)} \mathfrak{f}_e^* \widetilde{\omega}. \tag{6}$$

Below we only prove (5) since (6) is completely similar.

Since

$$S^i_{\text{yellow}} = \sum_{V_1, V_2 : V_1 \sqcup V_2 = V(\Gamma_i)} C_{V_1}(\pi_1) \times C_{V_2 \sqcup \{\star\}}(\pi_2)$$

(where $\star$ records the position of the node on π_2), and the bundle map $S^i_{\text{yellow}} \to B_1 \times B_2 \to B_1$ (resp. $\to B_2$) is by projecting first to the $C_{V_1(\pi_1)}$ (resp. $C_{V_2 \sqcup \{\star\}}(\pi_2)$) factor and then go along the bundle map $C_{V_1(\pi_1)} \to B_1$ (resp. $C_{V_2 \sqcup \{\star\}}(\pi_2) \to B_2$),

$$(\iota_1^{j_1}, \iota_2^{j_2})^* S^i_{\text{yellow}} = \sum_{V_1, V_2 : V_1 \sqcup V_2 = V(\Gamma_i)} (\iota_1^{j_1})^* C_{V_1}(\pi_1) \times (\iota_2^{j_2})^* C_{V_2 \sqcup \{\star\}}(\pi_2).$$

So, by the construction of $\widetilde{\omega}$ on S_{yellow} and by Fubini's Theorem, the right hand side of (5) is

$$\sum_{V_1, V_2 : V_1 \sqcup V_2 = V(\Gamma_i)} \int_{(\iota_2^{j_2})^* C_{V_2 \sqcup \{\star\}}(\pi_2)} \int_{(\iota_1^{j_1})^* C_{V_1}(\pi_1)} \bigwedge_{\substack{e \in E(\Gamma_i) \\ \text{both endpoints of } e \text{ are in } V_1}} \mathfrak{f}_e^* \omega_1 \wedge \bigwedge_{\substack{e \in E(\Gamma) \\ \exists \text{ endpoint of } e \text{ in } V_2}} \mathfrak{f}_e^* \omega_2$$

[7] A little more argument needed for this claim, but it is true.(Say things like K_π is induced from a chain map from the graph complex to the differential forms on B. Maybe cite [6, Theorem 2.15(1)].)

$$= \sum_{V_1,V_2:V_1\sqcup V_2=V(\Gamma_i)} \Big(\int_{(\iota_1^{j_1})^* C_{V_1}(\pi_1)} \bigwedge_{e\in E(\Gamma'(V_1))} \mathfrak{f}_e^*\omega_1\Big)\cdot\Big(\int_{(\iota_2^{j_2})^* C_{V_2\sqcup\{\star\}}(\pi_2)} \bigwedge_{e\in E(\Gamma/\Gamma'(V_1))} \mathfrak{f}_e^*\omega_2\Big),$$

where, for $V_1 \subset V(\Gamma_i)$, $\Gamma'(V_1)$ denotes the subgraph of Γ_i spanned by vertices in V_1. This proves (5).

Case II: $\alpha_0 = [pt]$[8]

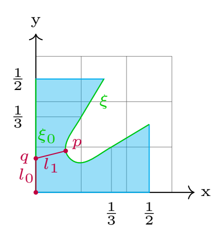
For simplicity of exposition, we assume for $a = 1, 2$, α_a can be represented by a smooth submanifold $B'_a \subset B_a$. This assumption is purely for the purpose of simplifying the notation: all the arguments below can be easily adapted to the case when B'_a is only a piecewise smooth singular chain, similar to Case I. (The readers familiar with the theory of pseudocycles should take B'_a to be a pseudocycle: the arguments below works verbatim and no assumption needs to be made.)

We consider the pullback of everything to $B'_1 \times B'_2$: let B', B'_I, $\widetilde{C}'_A$, $\overline{C}'_A([\pi_1,\pi_2])$, $(\overline{S}^i_{\text{gray}})'$, $(\overline{S}^i_{\text{blue}})'$, $(\overline{S}^i_{\text{red}})'$, $(\overline{S}^i_{\text{yellow}})'$, $(\overline{S}^i_{\text{orange}})'$ be the pullback of B, B_I, $\widetilde{C}_A$, $\overline{C}_A([\pi_1,\pi_2])$, $\overline{S}^i_{\text{gray}}$, $\overline{S}^i_{\text{blue}}$, $\overline{S}^i_{\text{red}}$, $\overline{S}^i_{\text{yellow}}$, $\overline{S}^i_{\text{orange}}$, respectively.

Recall that B_I is constructed by gluing together 3 pieces (cite Section 2.4 of conftilde):

$$\mathring{E}_1(3) \times B_2 \times [0,\rho), \quad S^{d-1} \times L \times B_1 \times B_2, \quad B_1 \times \mathring{E}_2(3) \times [0,\rho),$$

where $L \subset \mathbb{R}^2$ is the (closure of the) blue region below:



We define ξ to be the curvy stratum of ∂L and ξ_0 to be $\{x = 0, 0 < y < 1/2\} \subset \partial L$. Then, $S^{d-1} \times \xi \times B_1 \times B_2$ is part of $B = B_{t_0} \subset B_I$, the base of $[\pi_1,\pi_2]$.

Let $v \in S^{d-1}$ be a generic point. Let p be a generic point on ξ and q be a generic point on ξ_0. Let l_1 be a generic path (including endpoints) going from p to q, transverse to ∂L, and define l_0 to be the (closed) line segment from q to $(0,0)$. Denote

$$\begin{aligned}
M_p &:= v \times p \times B'_1 \times B'_2 \subset S^{d-1} \times \xi \times B'_1 \times B'_2 && \subset B' = B'_{t_0},\\
M_q &:= v \times q \times B'_1 \times B'_2 \subset S^{d-1} \times \xi_0 \times B'_1 \times B'_2 && \subset B'_0,\\
M_{l_0} &:= v \times l_0 \times B'_1 \times B'_2 \subset S^{d-1} \times \xi_0 \times B'_1 \times B'_2 && \subset B'_0,\\
M_{l_1} &:= v \times l_1 \times B'_1 \times B'_2 \subset S^{d-1} \times L \times B'_1 \times B'_2 && \subset B'_I.
\end{aligned}$$

[8] (The exposition is not so great; may need further polishing.)

Then, M_p is a section of the bundle $B' \to B'_1 \times B'_2$ and thus a representative of $[pt] \otimes \alpha_1 \otimes \alpha_2$. So,

$$\langle K_{[\pi_1,\pi_2]}([\sum_{i=1}^m \Gamma_i]), [pt] \otimes \alpha_1 \otimes \alpha_2 \rangle = \sum_i \int_{\overline{C}'_{V(\Gamma_i)}([\pi_1,\pi_2])|_{M_p}} \bigwedge_e \mathfrak{f}_e^* \omega = \sum_i \int_{(\overline{S}^i_{\text{gray}})'|_{M_p}} \bigwedge_e \mathfrak{f}_e^* \widetilde{\omega}. \tag{7}$$

By Lemma 4.1, for every stratum S of $\widetilde{C}'_{V(\Gamma_i)}$, the differential of $S \to B'_I$ is surjective at every point on $M_{l_1} \backslash M_q$, and on M_q the image of this differential covers $T_{S^{d-1}} \times \xi_0 \times B'_1 \times B'_2$. So, since the differential of l_1 at q covers the normal direction of $\xi_0 \subset L$, the map $S \to B'_I$ is transverse to M_{l_1}. Therefore, $\widetilde{C}'_{V(\Gamma_i)}|_{M_{l_1}}$ is a manifold with corners, and

$$\begin{aligned}\partial(\widetilde{C}'_{V(\Gamma_i)}|_{M_{l_1}}) &= (\partial \widetilde{C}'_{V(\Gamma_i)})|_{M_{l_1}} + \widetilde{C}'_{V(\Gamma_i)}|_{M_q} - \widetilde{C}'_{V(\Gamma_i)}|_{M_p} \\ &= (\overline{S}^i_{\text{blue}})'|_{M_{l_1}} + (\overline{S}^i_{\text{yellow}})'_{M_q} - (\overline{S}^i_{\text{gray}})'|_{M_p}.\end{aligned}$$

By Stocks' Formula and that $\widetilde{\omega}$ is closed we have

$$(7) = \sum_i \Big(\int_{(\overline{S}^i_{\text{blue}})'|_{\mathring{M}_{l_1}}} \bigwedge_e \mathfrak{f}_e^* \widetilde{\omega} + \int_{(\overline{S}^i_{\text{yellow}})'_{M_q}} \bigwedge_e \mathfrak{f}_e^* \widetilde{\omega} \Big) = \sum_i \int_{(\overline{S}^i_{\text{yellow}})'_{M_q}} \bigwedge_e \mathfrak{f}_e^* \widetilde{\omega}, \tag{8}$$

where, just as in Case I, the $\overline{S}_{\text{blue}}$ term vanishes by the same argument as in showing the well-definedness of Kontsevich's classes.

Let $(\overline{S}^i_{\text{yellow}})'|_{l_0}$ (resp. $(\overline{S}^i_{\text{yellow}})'|_{l_0 \backslash q}$) be the preimage of $S^{d-1} \times l_0 \times B'_1 \times B'_2$ (resp. $S^{d-1} \times (l_0 \backslash q) \times B'_1 \times B'_2$) in $(\overline{S}^i_{\text{yellow}})'$. Then, $(\overline{S}^i_{\text{yellow}})'|_{l_0 \backslash q}$ is an open subset of $(\overline{S}^i_{\text{yellow}})'$ and, since every stratum of $(\overline{S}^i_{\text{yellow}})'$ is transverse to q, $(\overline{S}^i_{\text{yellow}})'|_{l_0}$ is a manifold with corners. The boundary strata of $(\overline{S}^i_{\text{yellow}})'|_{l_0}$ are of two kinds: the strata of $(\overline{S}^i_{\text{yellow}})'|_{l_0 \backslash q}$ and the strata of $(\overline{S}^i_{\text{yellow}})'$ restricted to $S^{d-1} \times q \times B'_1 \times B'_2$. Moreover, the map $(\overline{S}^i_{\text{yellow}})'|_{l_0} \to S^{d-1}$ is a submersion when restricted to on every stratum, so the preimage of v under this map, $(\overline{S}^i_{\text{yellow}})'|_{M_{l_0}}$, is again a manifold with corners, and we have

$$\partial(\overline{S}^i_{\text{yellow}})'|_{M_{l_0}} = \big(\partial(\overline{S}^i_{\text{yellow}})'\big)|_{M_{l_0 \backslash (M_p, M_q)}} + (\overline{S}^i_{\text{yellow}})'|_{M_q} - (\overline{S}^i_{\text{yellow}})'|_{M_{(0,0)}}.$$

The first term consisting of strata with at least one screen with space label $\mathbb{R}^d$ and the points on that screen is not a priori constrainted; so, again by the same argument as showing well-definedness of Kontsevish's classes, these strata do not contribute to the integrals below. By Stocks' Formula,

$$(8) = \sum_i \int_{(\overline{S}^i_{\text{yellow}})'|_{M_{(0,0)}}} \bigwedge_e \mathfrak{f}_e^* \widetilde{\omega} = \sum_i \int_{(\overline{S}^i_{\text{yellow}})'|_{M_q}} \bigwedge_e \mathfrak{f}_e^* \widetilde{\omega} = \sum_i \int_{(S^i_{\text{orange}})'|_v} \bigwedge_e \mathfrak{f}_e^* \widetilde{\omega}. \tag{9}$$

Recall

$$S^i_{\text{orange}} = \sum_{V_1, V_2, V_3 : V_1 \sqcup V_2 \sqcup V_3 = V(\Gamma_i)} C_{V_1}(\pi_1) \times C_{V_2}(\pi_2) \times C_{V_3 \sqcup \{\star_1, \star_2\}}(\mathbb{R}^d)$$

and restricting to v means restricting the $C_{V_3 \sqcup \{\star_1, \star_2\}}(\mathbb{R}^d)$ term to the case $\frac{\star_1 - \star_2}{|\star_1 - \star_2|} = v$; we can thus view $\star_1 = 0, \star_2 = v$ (here we implicitly identify S^{d-1} as the unit sphere in $\mathbb{R}^d$) to be two fixed points in $\mathbb{R}^d$ and define

$$C^{\star}_{V_3}(\mathbb{R}^d) = \{(x_v \in \mathbb{R}^d)_{v \in V_3} | \, \forall v \in V_3, x_v \neq 0, x_v \neq v; \forall v, w \in V_3, x_v \neq x_w\}.$$

For $V_1, V_2 \subset V(\Gamma_i)$, define $\Gamma_i^{\mathrm{sub}}(V_1), \Gamma_i^{\mathrm{sub}}(V_2)$ to be the subgraph pf Γ_i spanned by the vertices in V_1 and V_2, respectively. Define

$$\Gamma_i/(V_1, V_2) := (\Gamma_i/\Gamma_i^{\mathrm{sub}}(V_1))/\Gamma_i^{\mathrm{sub}}(V_2)$$

the quotient graph by contracting $\Gamma_i^{\mathrm{sub}}(V_1)$ to a new vertex $\star_1$ and contracting $\Gamma_i^{\mathrm{sub}}(V_2)$ to a new vertex $\star_2$. For each edge $e \in E(\Gamma_i/(V_1, V_2))$, we still have the forgetful map $\mathfrak{f}_e : C^\star_{V_3}(\mathbb{R}^d) \to S^{d-1}$. Then, by the way $\widetilde{\omega}$ is constructed,

$$\begin{aligned}\int_{(S^i_{\mathrm{orange}})'|_v} \bigwedge_e \mathfrak{f}_e^* \widetilde{\omega} = \sum_{V_1, V_2, V_3 : V_1 \sqcup V_2 \sqcup V_3 = V(\Gamma_i)} &\left(\int_{C'_{V_1}(\pi_1)} \bigwedge_{e \in E(\Gamma_i^{\mathrm{sub}}(V_1))} \mathfrak{f}_e^* \omega_1 \right) \cdot \\ &\left(\int_{C'_{V_2}(\pi_2)} \bigwedge_{e \in E(\Gamma_i^{\mathrm{sub}}(V_2))} \mathfrak{f}_e^* \omega_2 \right) \cdot \left(\int_{C^\star_{V_3}(\mathbb{R}^d)} \bigwedge_{e \in E(\Gamma_i/(V_1, V_2))} \mathfrak{f}_e^* \omega_0 \right) \end{aligned} \tag{10}$$

We claim that if $\Gamma_i/(V_1, V_2)$ is not the empty graph, then the term $\int_{C^\star_{V_3}(\mathbb{R}^d)} \bigwedge_{e \in E(E(\Gamma_i \setminus (V_1, V_2)))} \mathfrak{f}_e^* \omega_0$ is always 0: this is again the same argument as in showing the well-definedness of Kontsevich's classes: if $\Gamma_i/(V_1, V_2)$ has a univalent vertex, then there is not enough constraints to determine the position of this vertex; if the $\Gamma_i/(V_1, V_2)$ has a bivalent vertex, then there is a $\mathbb{Z}/2\mathbb{Z}$ symmetry making the relevant terms canceling out; if all the vertices of $\Gamma_i/(V_1, V_2)$ are at least trivalent, then the degree cannot match – there are too many constraint in the problem than freedoms.

Therefore, the only contributions to (10) comes from terms when $\Gamma_i/(V_1, V_2)$ is the empty graph. Since Γ_i is connected, this only happens if $V_1 = V(\Gamma_i)$ or $V_2 = V(\Gamma_i)$. Therefore,

$$(10) = \int_{C'_{V(\Gamma_i)}(\pi_1)} \bigwedge_{e \in E(\Gamma_i)} \mathfrak{f}_e^* \omega_1 + \int_{C'_{V(\Gamma_i)}(\pi_2)} \bigwedge_{e \in E(\Gamma_i)} \mathfrak{f}_e^* \omega_2.$$

We conclude that

$$\begin{aligned}(9) = \sum_i (10) &= \sum_i \int_{C'_{V(\Gamma_i)}(\pi_1)} \bigwedge_{e \in E(\Gamma_i)} \mathfrak{f}_e^* \omega_1 + \sum_i \int_{C'_{V(\Gamma_i)}(\pi_2)} \bigwedge_{e \in E(\Gamma_i)} \mathfrak{f}_e^* \omega_2 \\ &= \big\langle K_{\pi_1}([\sum_i \Gamma_i]), \alpha_1 \big\rangle + \big\langle K_{\pi_2}([\sum_i \Gamma_i]), \alpha_2 \big\rangle = \big\langle (K_{\pi_1} \otimes K_{\pi_2})(\Delta.\big[\sum_i \Gamma_i\big]), \alpha_1 \otimes \alpha_2 \big\rangle.\end{aligned}$$

This completes the proof of Theorem 1.1.

Appendix A Extending differential forms on a manifold with corners from boundary to interior

We first clarify the notation used in the present paper concerning manifold with corners, mostly following [3] and [4]. Here is a dictionary for our notation:

- *(smooth) manifold with corners*: as in [4, page 3] or, equivalently, [3, Definition 3].
- *smooth map between manifolds with corners*: [3, Definition 4] ("weakly smooth map" in [4, Definition 3.1])

- *tangent and cotangent spaces of manifolds with corners*: [3, Definition 10] (equivalently, [4, Definition 2.2])
- *cotangent bundle, tensor and exterior powers of cotangent bundle of manifolds with corners*: follows from the definition of cotangent spaces
- *differential form on manifolds with corners*: smooth section of exterior powers of the cotangent bundle

  For example, in $(\mathbb{R}^{\geq 0})^n$ with coordinates denoted by $(x_1, \ldots, x_n)$, a differential form of degree k can be written as

  $$\sum_{1 \leq i_1 < \ldots < i_k \leq n} f_{i_1 \ldots i_k} dx_{i_1} \wedge \ldots \wedge dx_{i_k},$$

  where $f_{i_1 \ldots i_k}$ are functions on $(\mathbb{R}^{\geq 0})^n$, smooth in the sense that there exist smooth functions $\hat{f}_{i_1 \ldots i_k}$ on an open neighborhood of $(\mathbb{R}^{\geq 0})^n$ in $\mathbb{R}^n$, such that $f_{i_1 \ldots i_k} = \hat{f}_{i_1 \ldots i_k}|_{(\mathbb{R}^{\geq 0})^n}$. For $0 \leq m < n$, the restriction (pullback by inclusion map) of the above differential form to $(\mathbb{R}^{\geq 0})^m \approx \{x_{m+1} = \ldots = x_n = 0\} \subset (\mathbb{R}^{\geq 0})^n$ will be

  $$\sum_{1 \leq i_1 < \ldots < i_k \leq m} f_{i_1 \ldots i_k} dx_{i_1} \wedge \ldots \wedge dx_{i_k};$$

  namely, we remove the terms containing any dx_i with $i > m$.
- *∂M, the boundary of a manifold with corners M*: the image of the map $i_M : \partial M \to M$ in [4, Definition 2.6]
- $\tilde{\partial} M$: ∂M in [4, Definition 2.6]
- *codimension-d (depth-d) stratum*: [3, Definition 7] (a connected component of a depth-d stratum in [4, Definition 2.3])

  For example, a solid cube has 6 codim-1 strata and 12 codim-2 strata. Its boundary is the surface of the cube, and its $\tilde{\partial}$ is the disjoint union of 6 closed squares.
- *manifold with embedded faces*: [3, Definition 18]: a manifold with corners such that the closure of every codim-1 strata is an embedded manifold with corners.

In this appendix we prove the following

Proposition A.1. *Let M be a compact manifold with embedded faces.*[9] *Suppose for each stratum $S \subset \partial M$ of M, ω_S is a closed differential form of degree k on $\overline{S}$, such that, for each pair of strata $T \subset \overline{S} \subset \partial M$, $\omega_T = \omega_S|_{\overline{T}}$. Then, there exists an open neighborhood U of ∂M and a closed differential form ω on U such that $\omega|_{\overline{S}} = \omega_S$ for all strata $S \subset \partial M$.*

In light of Proposition A.1, we also define

- *degree-k differential form on ∂M*: a compatible collection of forms, $\{\omega_S\}_S$, as in the assumption of Proposition A.1.

[9] Neither the compactness or embedded faces condition should be necessary; this proposition likely holds for all manifolds with corners. We assume the stronger condition here to shorten the proof.

Remark A.2. It is easy to see that the $\mathbb{R}$-singular cohomology of ∂M (or more generally, any manifold with corners and "hinges"[10]) is equivalent to the de Rham cohomology defined using these forms, but we do not need it in the present paper.

Lemma A.3. *Let M be a manifold with corners and U an open neighborhood of ∂M. Suppose ω is a closed k-form on U such that $[\omega]$ lies in the image of the restriction map $H^k(M;\mathbb{R}) \to H^k(U;\mathbb{R})$, then there is a neighborhood $U'' \subset U$ of ∂M such that $\omega|_{U''}$ can be extended to a closed k-form on M.*

Proof (same as (9802062 below Lemma 1.2)). Let α be a closed d-form on $M\backslash\partial M$ such that the restriction of $[\alpha]$ to U is $[\omega]$. Then, there exists a $(k-1)$-form β on U such that $d\beta = \omega - \alpha|_U$. We can find[11] open subsets U', U'' of M such that $\partial M \subsetneq U'' \subsetneq U' \subsetneq U$, and a smooth function $f : M \to [0,1]$ such that $f|_{U''} \equiv 0$, $f|_{M\backslash U'} \equiv 1$. Define $\omega' = \omega - d(f\beta)$ on U. Then $\omega' \equiv \omega$ on U'', $\omega' \equiv \alpha$ on $U\backslash U'$, and $d\omega' = 0$. So we can extend ω' to a form on M, defining it to be α out of U. ☐

Corollary A.4. *Suppose M is a compact manifold with embedded faces such that $H^k(M;\mathbb{R}) \to H^k(\partial M; R)$ is surjective. Then, every closed k-form on ∂M can be extended to a closed k-form on M.*

The rest of this section is devoted to the proof of Proposition A.1. First we prove the statement locally, in $(\mathbb{R}^{\geq 0})^d \times \mathbb{R}^{n-d}$, where $0 \leq d \leq n$ and $0 \leq k \leq n$ are integers. For $I \subset \{1,\dots,d\}$, define $H_I := \{(x_1,\dots,x_n) \big| \forall i \in I, x_i = 0\} \subset (\mathbb{R}^{\geq 0})^d \times \mathbb{R}^{n-d}$. Write $H_i := H_{\{i\}}$ and $H_{i,j} = H_{\{i,j\}}$.

Lemma A.5. *Suppose for each $1 \leq i \leq d$, ω_i is a degree-k differential form on H_i, such that, for all $1 \leq i,j \leq d$, $\omega_i|_{H_{i,j}} = \omega_j|_{H_{i,j}}$. Then, there exists a degree-k differential form ω on $(\mathbb{R}^{\geq 0})^d \times \mathbb{R}^{n-d}$, such that, for all $1 \leq i \leq d$, $\omega|_{H_i} = \omega_i$. Moreover, if all ω_i are closed, ω can be taken to be closed as well.*

Proof. From the condition $\omega_i|_{H_{i,j}} = \omega_j|_{H_{i,j}}$ it is clear that for any $I \subset \{1,\dots,d\}$, the forms $\omega_i|_{H_I}$ are the same for all $i \in I$. We hence denote it by ω_I, which is on H_I. Let

$$p_I : (\mathbb{R}^{\geq 0})^d \times \mathbb{R}^{n-d} \longrightarrow H_I$$

be the projection map, sending all I-coordinates to 0 and not changing the other coordinates. We take ω to be the alternating sum

$$\omega = \sum_{1\leq i\leq d} p^*_{\{i\}}\omega_i - \sum_{1\leq i<j\leq d} p^*_{\{i,j\}}\omega_{\{i,j\}} + \sum_{1\leq i<j<k\leq d} p^*_{\{i,j,k\}}\omega_{\{i,j,k\}} - \ldots + (-1)^{d-1} p^*_{\{1,\dots,d\}}\omega_{\{1,\dots,d\}}.$$

To see that $\omega|_{H_i} = \omega_i$ for a given $i \in \{1,\dots,d\}$, note that, for each $I \subset \{1,\dots,d\}$ with $I \neq \emptyset$ and $i \notin I$,

$$(p_I^*\omega_I)|_{H_i} = p_I^*(\omega_I|_{H_i}) = p^*_{I\sqcup\{i\}}(\omega_{I\sqcup\{i\}}|_{H_i}) = (p^*_{I\sqcup\{i\}}\omega_{I\sqcup\{i\}})|_{H_i},$$

so these two terms cancel with each other. If all ω_is are closed, ω is clearly also closed. ☐

[10] By "hinges" we mean the manifold is locally modeled on $\mathbb{R}^a \times (\mathbb{R}^{\geq 0})^b \times \big((\mathbb{R}^{\geq 0})^c \backslash (\mathbb{R}^{>0})^c\big)$ for $a,b,c \in \mathbb{Z}^{\geq 0}$. One can image a theory of differential forms on such manifolds, defined similar to here.

[11] For example, using a metric on M and properly modify the distance function to ∂M.

Next, we patch the forms constructed locally to a global one. Without the closeness condition, this would be immediate by applying a partition of unity. With the closeness condition it is much subtler. The argument uses the same technique as translating between Čech and de Rham cohomology (see, e.g. [1]), which can also be viewed as a generalization of the technique in the proof of Lemma A.3. Note that from now on, the index set I has a different meaning than in Lemma A.5.

Given $p, q \in \mathbb{Z}^{\geq 0}$, a manifold M and a locally finite open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of M, recall a (skew-symmetric) p-Čech cochain of q-forms on M is: for each sequence $(i_0, i_1, \ldots, i_p) \in I^{p+1}$, a differential q-form $\alpha_{i_0 i_1 \ldots i_p}$ on $\bigcap_{j=0}^{p} U_{i_j}$; such that for all j, $\alpha_{i_0 \ldots i_j i_{j+1} \ldots i_p} = -\alpha_{i_0 \ldots i_{j+1} i_j \ldots i_p}$. We denote the $\mathbb{R}$-vector space of p-Čech cochain of q-forms on M by $\check{C}^p_{\mathcal{U}}(M; \mathcal{A}^q)$. The Čech differential is

$$\check{\delta} : \check{C}^p_{\mathcal{U}}(M; \mathcal{A}^q) \longrightarrow \check{C}^{p+1}_{\mathcal{U}}(M; \mathcal{A}^q)$$

$$\check{\delta}\,(\alpha_{i_0 \ldots i_p})_{(i_0 \ldots i_p) \in I^{p+1}} = (\beta_{i_0 \ldots i_{p+1}})_{(i_0 \ldots i_{p+1}) \in I^{p+2}}, \qquad \beta_{i_0 \ldots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \alpha_{i_0 \ldots \hat{i}_j \ldots i_{p+1}}\big|_{U_{i_0} \cap \ldots \cap U_{i_{p+1}}}.$$

We also still denote by d the termwise differential of forms:

$$d : \check{C}^p_{\mathcal{U}}(M; \mathcal{A}^q) \longrightarrow \check{C}^p_{\mathcal{U}}(M; \mathcal{A}^{q+1}), \qquad d\,(\alpha_{i_0 \ldots i_p})_{(i_0 \ldots i_p) \in I^{p+1}} = (d\alpha_{i_0 \ldots i_p})_{(i_0 \ldots i_p) \in I^{p+1}}.$$

It is clear that $\check{\delta} d = d \check{\delta}$, and both d and $\check{\delta}$ commute with pull-back maps between manifolds.

Lemma A.6. *Suppose M is a manifold with corners. Denote $N := \partial M$ and $\iota : N \to M$ the inclusion map.*[12] *Suppose $\mathcal{U} = \{U_i\}_{i \in I}$ is a locally finite open cover of M satisfying the condition that for all subset $I' \subset I$, if $U_{I'} := \bigcap_{i \in I'} U_i$ is non-empty, then*

1. *all de Rham cohomology groups of $U_{I'}$ are the same as those of a point,*
2. *$\iota^{-1}(U_{I'}) \neq \emptyset$, and*
3. *if σ is a closed form*[13] *on $\iota^{-1}(U_{I'})$, then there exists a closed form $\tilde{\sigma}$ on $U_{I'}$ with $\iota^* \tilde{\sigma} = \sigma$.*

Then, the following proposition $\mathcal{P}^q_p$ holds for all $p \geq 0$,$q \geq 1$:

$\mathcal{P}^q_p$: Suppose $\alpha = (\alpha_{i_0 \ldots i_p})_{(i_0 \ldots i_p) \in I^{p+1}} \in \check{C}^p_{\mathcal{U}}(M; \mathcal{A}^q)$ satisfies $\check{\delta}\alpha = 0$, $\iota^*\alpha = 0$ and $d\alpha = 0$, then, there exists $\beta = (\beta_{i_0 \ldots i_p})_{(i_0 \ldots i_p) \in I^{p+1}} \in \check{C}^p_{\mathcal{U}}(M; \mathcal{A}^{q-1})$ such that $\check{\delta}\beta = 0$, $\iota^*\beta = 0$ and $d\beta = \alpha$.

Proof. Two steps: we first show that $\mathcal{P}^{q-1}_{p+1} \implies \mathcal{P}^q_p$, then show that $\mathcal{P}^1_p$ holds for all $p \geq 0$.

Step 1: Suppose $q \geq 2$ and α is as in the condition of $\mathcal{P}^q_p$. By condition 1 above, there is $\beta'' \in \check{C}^p_{\mathcal{U}}(M; \mathcal{A}^{q-1})$ such that $d\beta'' = \alpha$. Then, $d\iota^*\beta'' = 0$, so condition 3 above implies that there is $\tilde{\beta}'' \in \check{C}^p_{\mathcal{U}}(M; \mathcal{A}^{q-1})$ such that $d\tilde{\beta}'' = 0$ and $\iota^*\tilde{\beta}'' = \iota^*\beta''$. Setting $\beta' = \beta'' - \tilde{\beta}''$ we have $d\beta' = \alpha$ and $\iota^*\beta' = 0$. Since $d\check{\delta}\beta' = \check{\delta}\alpha = 0$ and $\iota^*\check{\delta}\beta' = 0$, $\check{\delta}\beta'$ (replacing α) satisfies the condition of $\mathcal{P}^{q-1}_{p+1}$. If we assume $\mathcal{P}^{q-1}_{p+1}$ holds, then there exists $\gamma \in \check{C}^{p+1}_{\mathcal{U}}(M; \mathcal{A}^{q-2})$ with $\check{\delta}\gamma = 0$, $\iota^*\gamma = 0$ and $d\gamma = \check{\delta}\beta'$.

[12] This lemma and Lemma A.7 hold if N is replaced with an arbitrary manifold with corners and $\iota : N \to M$ an arbitrary smooth map.

[13] Differential forms on open subsets of ∂M are defined as in the assumption of Proposition A.1.

Let $\{f_i : U_i \to [0,1]\}_{i\in I}$ be a partition of unity subordinate to $\mathcal{U}$. we define β by taking

$$\beta_{i_0\ldots i_p} = \beta'_{i_0\ldots i_p} - \sum_{i\in I} d(f_i \cdot \gamma_{ii_0\ldots i_p}).$$

It is clear that $d\beta = d\beta' = \alpha$ and, since $\iota^*\gamma = 0$, $\iota^*\beta = \iota^*\beta' - \sum_{i\in I} d\big((f_i\circ\iota)\cdot\iota^*\gamma_{ii_0\ldots i_p}\big) = 0$. And

$$\begin{aligned}(\check\delta\beta' - \check\delta\beta)_{i_0\ldots i_{p+1}} &= \sum_{j=0}^{p+1}(-1)^j(\beta'-\beta)_{i_0\ldots\hat{i}_j\ldots i_{p+1}} = \sum_{j=0}^{p+1}(-1)^j\sum_{i\in I} d(f_i\cdot\gamma_{ii_0\ldots\hat{i}_j\ldots i_{p+1}})\\ &= d\sum_{i\in I}\Big(f_i\cdot\sum_{j=0}^{p+1}(-1)^j\gamma_{ii_0\ldots\hat{i}_j\ldots i_{p+1}}\Big) = d\sum_{i\in I} f_i\cdot(\gamma_{i_0\ldots i_{p+1}} - (\check\delta\gamma)_{ii_0\ldots i_{p+1}}) = d\gamma_{i_0\ldots i_{p+1}},\end{aligned} \tag{11}$$

where the last equality is due to $\check\delta\gamma = 0$, and the 4-th equality holds because

$$\sum_{j=0}^{p+1}(-1)^j\gamma_{ii_0\ldots\hat{i}_j\ldots i_{p+1}} = -(\check\delta\gamma)_{ii_0\ldots i_{p+1}} + \gamma_{i_0\ldots i_{p+1}}.$$

Therefore, we have β such that $\check\delta\beta = 0$, $\iota^*\beta = 0$ and $d\beta = \alpha$, as desired.

Step 2: Suppose α is as in the condition of $\mathcal{P}^1_p$. the argument at the beginning of Step 1 says there is β such that $\iota^*\beta = 0$ and $d\beta = \alpha$. Since α consists of 1-forms, β, as well as $\check\delta\beta$, consist of smooth functions. Since $d\check\delta\beta = \check\delta d\beta = 0$ and the domain of each $(\check\delta\beta)_{i_0\ldots i_p}$, if not empty, is connected by condition 1, $(\check\delta\beta)_{i_0\ldots i_p}$ are constant functions. Also $\iota^*\check\delta\beta = \check\delta\iota^*\beta = 0$, so, by condition 2 in the lemma, each $(\check\delta\beta)_{i_0\ldots i_p}$ with non-empty domain is 0 at at least one point, hence must be 0. Therefore $\check\delta\beta = 0$ and β satisfies the requirement in $\mathcal{P}^1_p$. □

Lemma A.7. *Suppose $\iota, N, M, \mathcal{U}$ are as in the condition of Lemma A.6; $p\ge 0$,$q \ge 1$. Suppose $\alpha \in \check{C}^p_{\mathcal{U}}(M;\mathcal{A}^q)$ satisfies $d\alpha = 0$ and $\check\delta\iota^*\alpha = 0$, then there exists $\alpha' \in \check{C}^p_{\mathcal{U}}(M;\mathcal{A}^q)$ such that $d\alpha' = 0$, $\check\delta\alpha' = 0$ and $\iota^*\alpha' = \iota^*\alpha$.*

Proof. We have $d(\check\delta\alpha) = \check\delta d\alpha = 0$, $\check\delta(\check\delta\alpha) = 0$, and $\iota^*(\check\delta\alpha) = \check\delta\iota^*\alpha = 0$. So, applying $\mathcal{P}^q_{p+1}$ to $\check\delta\alpha$ we obtain $\beta \in \check{C}^{p+1}_{\mathcal{U}}(M;\mathcal{A}^{q-1})$ such that $\check\delta\beta = 0$, $\iota^*\beta = 0$ and $d\beta = \check\delta\alpha$. Again, let $\{f_i : U_i \to [0,1]\}_{i\in I}$ be a partition of unity subordinate to $\mathcal{U}$ and we define α' by taking

$$\alpha'_{i_0\ldots i_p} = \alpha_{i_0\ldots i_p} - \sum_{i\in I} d(f_i\cdot\beta_{ii_0\ldots i_p}),$$

then $d\alpha' = 0$ and $\iota^*\alpha' = \iota^*\alpha - \sum_{i\in I} d((f_i\circ\iota)\cdot\iota^*\beta_{ii_0\ldots i_p}) = \iota^*\alpha$. And, similar to (11),

$$\begin{aligned}(\check\delta\alpha - \check\delta\alpha')_{i_0\ldots i_{p+1}} &= \sum_{j=0}^{p+1}(-1)^j(\alpha-\alpha')_{i_0\ldots\hat{i}_j\ldots i_{p+1}} = \sum_{j=0}^{p+1}(-1)^j\sum_{i\in I} d(f_i\cdot\beta_{ii_0\ldots\hat{i}_j\ldots i_{p+1}})\\ &= d\sum_{i\in I}\Big(f_i\cdot\sum_{j=0}^{p+1}(-1)^j\beta_{ii_0\ldots\hat{i}_j\ldots i_{p+1}}\Big) = d\sum_{i\in I} f_i\cdot(\beta_{i_0\ldots i_{p+1}} - (\check\delta\beta)_{ii_0\ldots i_{p+1}}) = d\beta_{i_0\ldots i_{p+1}},\end{aligned}$$

Therefore, $\check\delta\alpha' = \check\delta\alpha - d\beta = 0$, as desired. □

Lemma A.8. *Suppose M is a compact manifold with embedded faces. Then, there exist an open neighborhood U of ∂M, a finite open cover $\mathcal{U}$ of U, such that $\mathcal{U}$ satisfies the condition in Lemma A.7 when we plug in ∂M for N, U for M, and the inclusion map for ι.*

Proof. By [3, Theorem 17], M has a system of compatible collar neighborhoods ([3, Definition 35]).[14] Fix a metric on M. For a point p in a depth-k strata S, we take a convex neighborhood U'_p of p in S and define $U_p = U'_p \times [0, \epsilon)^k \subset M$ (here we implicitly use the identification given by the collar neighborhood $S \times [0, \epsilon)^k \to M$). Then, for any finite set of points $\{p_i\}$ in ∂M, $\cap_i U_{p_i}$ is diffeomorphic to $(\mathbb{R}^{\geq 0}))^d \times \mathbb{R}^{\dim M - d}$ for some d, and $(\cap_i U_{p_i}) \cap \partial M \neq \emptyset$. So, by Lemma A.5, $\cap_i U_i$ satisfied condition 3 for $U_{I'}$ imposed in Lemma A.6. Take $\mathcal{U}$ to be a finite subcover of $\{U_p\}_{p \in \partial M}$ and $U = \bigcup_{U_p \in \mathcal{U}} U_p$. □

Proof of Proposition A.1. Let U,$\mathcal{U}$ be given as in Lemma A.8. Then, for every $V \in \mathcal{U}$, by Lemma A.5, we can find a closed form ω_V on V such that $\omega_V|_{S \cap V} = \omega_S|_{S \cap V}$ for all strata S of ∂M. The collection $\{\omega_V\}_{V \in \mathcal{U}}$ defines an $\alpha \in \check{C}^0_{\mathcal{U}}(M; \mathcal{A}^k)$, which satisfies the condition of Lemma A.7. So, there exists α' as in the conclusion of Lemma A.7. To get to the conclusion of Proposition A.1, take $\omega = \alpha'$. □

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[14] (Delete! This is wrong!!!)In our application M is $\widetilde{C}_n$, and its charts that we explicitly constructed in Section 2 is actually a system of compatible collar neighborhoods.