

1 Introduction

.....(introduction)

Define the kind of bundle we work with in this paper: Given a smooth homology sphere M , define a *framed* (M, ∞) -bundle $(\pi : E \rightarrow B, \sigma, \tau, F)$ (abbreviate all these to π) to be a smooth fiber bundle $\pi : E \rightarrow B$ with fiber M , with a smooth section σ , a trivialization τ of the bundle near σ , and a smooth vertical framing F of π “standard” near σ .

Define the bracket operation, $\pi_1, \pi_2 \rightarrow [\pi_1, \pi_2]$ on such bundles, in an intuitively clear but not necessarily rigorous way.

Define cobracket and coproduct on graph cohomology (everything is over \mathbb{Q}):

- First, define the graph complex \mathcal{G}' —the \mathbb{Q} -vector space spanned by (with correct orientation definition, omitted here) connected graphs containing either a univalent vertex or a simple loop (an edge starting and ending at the same vertex). The coboundary operation δ is given by contracting an edge. In δ and all the operations on graphs below, whenever a graph not in \mathcal{G}' appears (a graph that has a univalent vertex or simple loop), we set it to 0.
- Taking the homology of \mathcal{G}' with respect to δ , denote by $H^*\mathcal{G}'$.
- Define the cobracket operation to be the linear map

$$\begin{aligned} \Delta_{[\cdot]} : \mathcal{G}' &\longrightarrow \mathcal{G}' \otimes \mathcal{G}' \\ \Gamma &\longrightarrow \sum_{\Gamma' \leq \Gamma} (\Gamma' \otimes \Gamma/\Gamma' + (-1)^{\cdot} \Gamma/\Gamma' \otimes \Gamma'), \end{aligned} \tag{1}$$

where Γ' ranges through all full subgraphs of Γ that is connected, with no univalent vertex or simple loop.

- Check that $\Delta_{[\cdot]}$ commutes with δ and $\delta \otimes \text{id} \pm \text{id} \otimes \delta$, so it descends to

$$\Delta_{[\cdot]} : H^*\mathcal{G}' \longrightarrow H^*(\mathcal{G}' \otimes \mathcal{G}') \approx H^*\mathcal{G}' \otimes H^*\mathcal{G}'.$$

- Finally we also define the coproduct operation on \mathcal{G}' (this makes more sense for disconnected graphs but w=for connected graphs it is extra simple):

$$\begin{aligned} \Delta : \mathcal{G}' &\longrightarrow \mathcal{G}' \otimes \mathcal{G}' \\ \Gamma &\longrightarrow \Gamma \otimes (\text{the empty graph}) + (\text{the empty graph}) \otimes \Gamma. \end{aligned}$$

Brief introduction to Kontsevich’s characteristic classes. Given a framed (M, ∞) -bundle $\pi : E \rightarrow B$ as above, denote by

$$K_\pi : H^*(\mathcal{G}') \longrightarrow H^*(B)$$

Kontsevich’s characteristic classes of π .

Theorem 1.1. *Suppose $d \geq 3$. For $i = 1, 2$, suppose M_i is a d -dimensional smooth homology sphere and suppose $\pi_i : E_i \rightarrow B_i$ is a framed (M, ∞) -bundle. (Now, $[\pi_1, \pi_2] : E \rightarrow S^d \times B_1 \times B_2$ is the bracket bundle.) Then, for all $\eta \in H^*\mathcal{G}'$,*

$$K_{[\pi_1, \pi_2]}(\eta) = \text{PD}_{S^d}[S^d] \otimes (K_{\pi_1} \otimes K_{\pi_2})(\Delta_{[\cdot]}(\eta)) + \text{PD}_{S^d}[pt] \otimes (K_{\pi_1} \otimes K_{\pi_2})(\Delta_{\cdot}(\eta)).$$

(Both LHS and RHS lives in

$$H^*(S^d \times B_1 \times B_2) \approx H^*(S^d) \otimes H^*(B_1) \otimes H^*(B_2).$$

PD_{S^d} means Poincaré dual on S^d ; $[S^d]$ stands for the fundamental class of S^d and $[pt]$ stands for the point class of S^d .)

.....(Then talk about the $(d+1)$ -fold loop space structure on $\text{BDiff}_\partial^{\text{fr}}(D^d)$ and the theorem/corollary that it doesn't extend.)

(Below is an outline of the proof of Theorem 1.1. Throughout, π_1, π_2 are given and fixed.)

1.1 Notation

Given a graph G , we denote by $V(G)$ its vertex set and $E(G)$ its edge set.

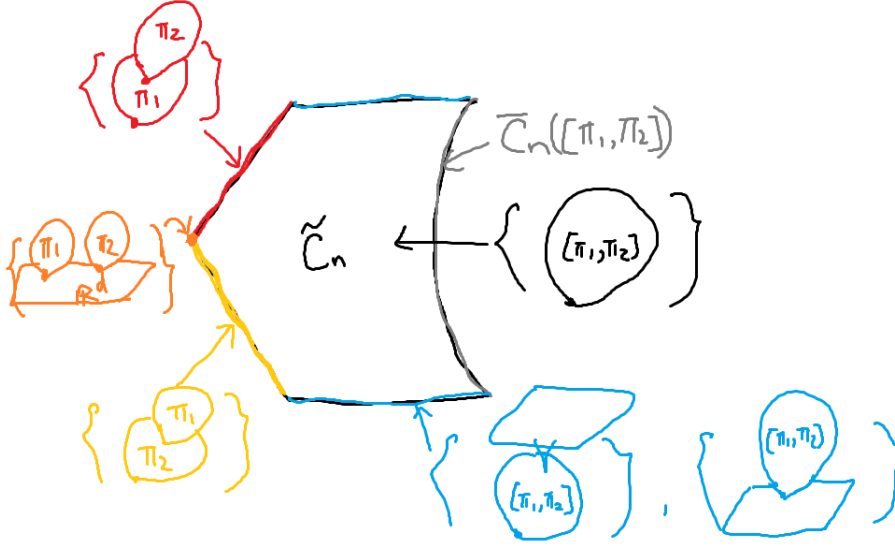
2 Conftilde

Construct the big configuration space \tilde{C}_A . Show that it is a smooth manifold with boundary and corners. (These are mostly already written in the file “conftilde” I sent a while ago.)

What we need are the following:

- \tilde{C}_A is a smooth manifold with boundary and corners;
- each S_T is a stratum of \tilde{C}_A ;
- $\bar{S}_T = \bigsqcup_{T'} S_{T'}$, where the disjoint union is taken over all A -labeled trees T' such that T can be obtained from T' by contracting some edges.

Here is a schematic picture of \tilde{C}_A (the marked points are not drawn; the actual stratification structure of \tilde{C}_A is more complicated than what is shown in the picture):



The boundary of \tilde{C}_A consists of the following parts:

- the gray part, denoted by \bar{S}_{gray} , is $\bar{C}_A([\pi_1, \pi_2])$; its interior, denoted by S_{gray} , is $C_A([\pi_1, \pi_2])$;
- $S_{\text{blue}} := \bigcup_{T \in \mathcal{T}_{\text{blue}}} S_T$, where $\mathcal{T}_{\text{blue}}$ is the set of all A -labeled trees whose shape and space labels are like $\begin{array}{c} \downarrow \mathbb{R}^d \\ \text{r} \quad [\pi_1, \pi_2] \end{array}$ or $\begin{array}{c} \downarrow [\pi_1, \pi_2] \\ \text{r} \quad \mathbb{R}^d \end{array}$; and let \bar{S}_{blue} be the closure of S_{blue} ;
- $S_{\text{red}} := \bigcup_{T \in \mathcal{T}_{\text{red}}} S_T$, where \mathcal{T}_{red} is the set of all A -labeled trees with the following shape and space labels: $\begin{array}{c} \downarrow \pi_2 \\ \text{r} \quad \pi_1 \end{array}$; and let \bar{S}_{red} be the closure of S_{red} ;
- $S_{\text{yellow}} := \bigcup_{T \in \mathcal{T}_{\text{yellow}}} S_T$, where $\mathcal{T}_{\text{yellow}}$ is the set of all A -labeled trees with the following shape and space labels: $\begin{array}{c} \downarrow \pi_1 \\ \text{r} \quad \pi_2 \end{array}$; and let \bar{S}_{yellow} be the closure of S_{yellow} ;

We also define $S_{\text{orange}} := \bigcup_{T \in \mathcal{T}_{\text{orange}}} S_T$, where $\mathcal{T}_{\text{orange}}$ is the set of all A -labeled trees with the following shape and space labels: $\begin{array}{c} \swarrow \pi_1 \quad \searrow \pi_2 \\ \text{r} \quad \mathbb{R}^d \end{array}$; and let \bar{S}_{orange} be the closure of S_{orange} . Then, $\bar{S}_{\text{orange}} = \bar{S}_{\text{red}} \cap \bar{S}_{\text{yellow}}$.

We define $\bar{C}_A^* = \bar{S}_{\text{red}} \cup \bar{S}_{\text{yellow}}$.

3 Propagators

Before starting the discussion on propagators, we first define another notion of “forgetful map”.

Given finite sets A (“set of point labels”) and B (“set of space labels”), recall the definition of an (A, B) -labeled tree in ...**(change the definition in confilde to allow arbitrary space labels!)**

In all the cases we care about, the elements of B will be (M, ∞) -bundles for some d -dimensional manifold M .

In this paper we only consider cases when $|B| \leq 2$. Define

$$\tilde{C}_A(B) = \begin{cases} \overline{C}_A(\mathbb{R}^d) & \text{if } B = \emptyset, \\ \overline{C}_A(\pi) & \text{if } B = \{\pi\} \text{ for some } (M, \infty) \text{ -- bundle } \pi, \\ \tilde{C}_A \setminus \overline{S}_{\text{gray}} & \text{if } B = \{\pi_1, \pi_2\} \text{ for some } (M_1, \infty)\text{-bundle } \pi_1 \text{ and some } (M_2, \infty)\text{-bundle } \pi_2. \end{cases}$$

Then, the strata of $\tilde{C}_A(B)$ are in 1-to-1 correspondence with (A, B) -labeled trees. Given such a stratum S , we denote by \mathcal{T}_S the tree corresponding to it and given such a tree T we denote by \mathcal{S}_T the stratum corresponding to it.¹ The condition $\mathcal{S}_{T'} \subset \overline{\mathcal{S}}_T$ is equivalent to that T can be obtained from T' by contracting some edges. In this case, the set of edges to be contracted to get from T' to T is unique and we denote by $\mathbf{c}_{T',T} : V(T') \rightarrow V(T)$ the map on the vertices induced by the contraction. Also define $\mathbf{i}_{T',T} : V(T) \rightarrow V(T')$ mapping v to the lowest vertex in $\mathbf{c}_{T',T}^{-1}(v)$. The following lemma is immediate:

Lemma 3.1. *Let T', T be (A, B) -labeled trees such that T can be obtained from T' by contracting some edges. Then, for every $v \in V(T)$, $lp(\geq v) = lp(\geq \mathbf{i}_{T',T}(v))$.*

Maybe move the above to an earlier section devoted to combinatorics.

For the rest of this section we only consider the case $A = \{1, 2\}$. Let B be a finite set such that every element of B is an (M, ∞) -bundles for some d -dimensional manifold M , and $|B| \leq 2$.

Definition 3.2. Let T be a $(\{1, 2\}, B)$ -labeled tree,² then $\mathcal{S}_T \approx \prod_{v \in V(T)} C_{lp(v) \cup cld(v)}(ls(v))$ is a stratum in $\tilde{C}_{\{1,2\}}(B)$. Define $\nu_T \in V(T)$ to be the vertex such that $\{1, 2\} \subset lp(\geq \nu_T)$ and for all $v > \nu_T$, $\{1, 2\} \not\subset lp(\geq v)$.³ Define $\mathfrak{s}_T := ls(\nu_T)$.⁴

- Define

$$\hat{f}_T : \mathcal{S}_T \longrightarrow C_2(\mathfrak{s}_T), \quad \hat{f}_T((c_v)_{v \in V(T)}) = c'_{\nu_T},$$

where $c'_{\nu_T} \in C_2(\mathfrak{s}_T)$ is obtained from c_{ν_T} by forgetting all the points except for two: $f_{\nu_T}(1)$ and $f_{\nu_T}(2)$. **(f_v is defined in confilde, at the beginning of Section 3.3.)**

- Suppose T' is a $(\{1, 2\}, B)$ -labeled tree such that T can be obtained from T' by contracting some edges. Abusing notation, we denote the subtree of T' spanned by vertices in $\mathbf{c}_{T',T}^{-1}(\nu_T)$ still by $\mathbf{c}_{T',T}^{-1}(\nu_T)$. Define $G_{T',T}$ to be the tree obtained from $\mathbf{c}_{T',T}^{-1}(\nu_T)$ by “stabilization with respect to $\{1, 2\}$ and \mathfrak{s}_T ”, namely: let $V' \subset V(\mathbf{c}_{T',T}^{-1}(\nu_T))$ (“set of unstable vertices”) consist of vertices v such that $\text{lsset}(ls(v)) \cap \text{lsset}(ls(\nu_T)) = \emptyset$ and $|lp(\geq v) \cap \{1, 2\}| < 2$; define $G_{T',T}$

¹(\mathcal{T} was used previously in confilde; maybe change it to \mathfrak{T} there.)

²The definition obviously extends to the case of an arbitrary number of marked points and forgetting to an arbitrary subset of marked points, but we only need this simple 2 point case here.

³(In this paper \subset means subset or equal. Specify this somewhere early.)

⁴(Actually, maybe use $\mathfrak{p}, \mathfrak{s}$ instead of $\mathfrak{fp}, \mathfrak{fs}$?)

to be obtained from $\mathfrak{c}_{T',T}^{-1}(\nu_T)$ by: for every vertex $v \in V'$, contracting the edge just below v . Then, $\mathcal{S}_{G_{T',T}}$ is a stratum of $\overline{C}_2(\mathfrak{s}_T)$. Define

$$\hat{f}_T : \mathcal{S}_{T'} \longrightarrow \mathcal{S}_{G_{T',T}} \subset \overline{C}_2(\mathfrak{s}_T), \quad \hat{f}_T((c_v)_{v \in V(T')}) = (c'_v)_{v \in V(G_{T',T})},$$

where each $c'_v \in C_2(ls(v))$ is as follows: let $v' \in V(\mathfrak{c}_{T',T}^{-1}(\nu_T)) \subset V(T')$ be the lowest vertex contracted to v , then c'_v is obtained from $c_{v'}$ by forgetting all points except for two: $f_{v'}(1)$ and $f_{v'}(2)$.

- We have therefore defined a map

$$\hat{f}_T : \overline{\mathcal{S}}_T \longrightarrow \overline{C}_2(\mathfrak{s}_T).$$

It is easy to verify that \hat{f}_T is smooth using the charts we constructed in Section (conftilde section).

Maybe introduce more notation when talking about the combinatorics of A -labeled trees, e.g., a pre-stable tree and how to get from a pre-stable tree to a stable tree by contraction.?

Note that if $G_{T',T}$ has only one vertex, then $\hat{f}_{T'} = \hat{f}_T|_{\overline{\mathcal{S}}_{T'}}$. Otherwise, this is not the case.

Example 3.3 ($|B| = 2$, in \tilde{C}_2). $T = \begin{array}{c} 1 \text{ } \pi_2 \\ \downarrow \\ 2 \text{ } \pi_1 \\ \downarrow \\ r \end{array}$, $T' = \begin{array}{c} 1 \text{ } \pi_2 \\ \downarrow \\ 2 \text{ } \pi_1 \\ \downarrow \\ r \end{array}$ (so $\mathcal{S}_{T'} \subset \overline{\mathcal{S}}_T$),

$$\hat{f}_T(\begin{array}{c} \bullet 2 \\ \circ \pi_2 \\ \bullet 1 \text{ } \pi_1 \end{array}) = \begin{array}{c} \bullet 2 \\ \circ \pi_1 \end{array}, \quad \hat{f}_T(\begin{array}{c} \bullet 1 \text{ } \pi_2 \\ \bullet 2 \\ \circ \pi_1 \end{array}) = \begin{array}{c} \bullet 1 \text{ } \pi_2 \\ \bullet 2 \\ \circ \pi_1 \end{array}, \quad \hat{f}_{T'}(\begin{array}{c} \bullet 1 \text{ } \pi_2 \\ \bullet 2 \\ \circ \pi_1 \end{array}) = \begin{array}{c} \bullet 1 \text{ } \pi_2 \\ \bullet 2 \end{array}.$$

Example 3.4 ($|B| = 2$, in \tilde{C}_2). $T = \begin{array}{c} 1,2 \text{ } \pi_1 \\ \downarrow \\ r \end{array}$, $T' = \begin{array}{c} 1,2 \text{ } \pi_1 \\ \downarrow \\ r \end{array}$ (so $\mathcal{S}_{T'} \subset \overline{\mathcal{S}}_T$),

$$\hat{f}_T(\begin{array}{c} \bullet 2 \\ \circ \pi_2 \\ \bullet 1 \text{ } \pi_1 \end{array}) = \begin{array}{c} \bullet 2 \\ \circ \pi_1 \end{array}, \quad \hat{f}_T(\begin{array}{c} \bullet 1 \text{ } \pi_2 \\ \bullet 2 \\ \circ \pi_1 \end{array}) = \begin{array}{c} \bullet 1 \text{ } \pi_2 \\ \bullet 2 \\ \circ \pi_1 \end{array}, \quad \hat{f}_{T'}(\begin{array}{c} \bullet 1 \text{ } \pi_2 \\ \bullet 2 \\ \circ \pi_1 \end{array}) = \begin{array}{c} \bullet 1 \text{ } \pi_2 \\ \bullet 2 \end{array}.$$

Example 3.5 ($|B| = 1$, in $\overline{C}_2(\pi)$ where π is an (M, ∞) -bundle). $T_1 = \begin{array}{c} 1,2 \\ \downarrow \\ \pi \\ \downarrow \\ r \end{array}$, $T_2 = \begin{array}{c} \pi \\ \downarrow \\ 1,2 \\ \downarrow \\ r \end{array}$, $T_3 = \begin{array}{c} 1 \text{ } \pi \\ \downarrow \\ 2 \\ \downarrow \\ r \end{array}$,

$$\hat{f}_{T_1}(\begin{array}{c} \bullet 1 \text{ } \pi \\ \bullet 2 \\ \circ \pi \end{array}) = \begin{array}{c} \bullet 1 \text{ } \pi \\ \bullet 2 \end{array}, \quad \hat{f}_{T_2}(\begin{array}{c} \bullet 1 \text{ } \pi \\ \bullet 2 \\ \circ \pi \end{array}) = \begin{array}{c} \bullet 1 \text{ } \pi \\ \bullet 2 \end{array}, \quad \hat{f}_{T_3}(\begin{array}{c} \bullet 1 \text{ } \pi \\ \bullet 2 \\ \circ \pi \end{array}) = \begin{array}{c} \bullet 1 \text{ } \pi \\ \bullet 2 \end{array}.$$

Note that $S_{T_1}, S_{T_2}, S_{T_3}$ are the only codimension-1 strata of $\overline{C}_2(\pi)$, and the codomain of $\hat{f}_{T_1}, \hat{f}_{T_2}, \hat{f}_{T_3}$ are all $\overline{C}_2(\mathbb{R}^d)$.

Corollary 3.6 (of Lemma 3.1). *Let T', T be $(\{1, 2\}, B)$ -labeled trees such that T can be obtained from T' by contracting some edges, then $\mathfrak{c}_{T', T}(\nu_{T'}) = \nu_T$. Moreover, if $\mathfrak{s}_T = \mathbb{R}^d$, then $\mathfrak{s}_{T'} = \mathbb{R}^d$.*

The following lemma is easy to check: ⁵

Lemma 3.7. *Let T', T be $(\{1, 2\}, B)$ -labeled trees such that T can be obtained from T' by contracting some edges, then*

$$\hat{f}_{T'} = \hat{f}_{G_{T', T}} \circ (\hat{f}_T|_{\overline{S}_{T'}}).$$

Definition 3.8. Suppose M is a d -dimensional \mathbb{Z} -homology sphere and π is an (M, ∞) -bundle. A *propagator* on $\overline{C}_2(\pi)$ (resp. $\overline{C}_2(\mathbb{R}^d)$) is a closed $(d-1)$ -form ω on $\overline{C}_2(\pi)$ satisfying: there exists a $(d-1)$ -form ω_0 on $S^{d-1} \approx \overline{C}_2(\mathbb{R}^d)$ such that $\int_{S^{d-1}} \omega_0 = 1$ and for every codimension-1 stratum $S \subset \partial\overline{C}_2(\pi)$ (resp. $S \subset \partial\overline{C}_2(\mathbb{R})$), $\omega|_{\overline{S}} = \hat{f}_S^* \omega_0$.

This definition is phrased differently from the usual definition of a propagator, see e.g. [5, Definition 3.9] or [6, Lemma 2.12], but can easily be seen to be equivalent.

Fix a volume form ω_0 on S^{d-1} . By [6, Lemma 2.12], there exist propagators ω_1 on $\overline{C}_2(\pi_1)$, ω_2 on $\overline{C}_2(\pi_2)$, and ω on $\overline{C}_2([\pi_1, \pi_2])$ such that the above condition in Definition 3.8 is satisfied with this ω_0 . We choose and fix such $\omega_1, \omega_2, \omega$.

In the rest of this section we construct a “propagator” on $\partial\tilde{C}_2$. This is done in two steps:

1. On each stratum S of $\partial\tilde{C}_2$, construct a closed $(d-1)$ -form ω_S on \overline{S} , such that, for two strata $S' \subset \overline{S}$, $\omega_S|_{\overline{S}'} = \omega_{S'}$.
2. Show that this collection $\{\omega_S\}_S$ extends to a closed form $\tilde{\omega}$ on \tilde{C}_2 ; namely, $\tilde{\omega}$ is such that $\tilde{\omega}|_S = \omega_S$ for every strata $S \subset \partial\tilde{C}_2$.

Step 1: There are two kinds of strata in $\partial\tilde{C}_2$: those that are not subsets of $\overline{S}_{\text{gray}}$ and those that are. The former kind of strata are of the form \mathcal{S}_T for some unique $(\{1, 2\}, \{\pi_1, \pi_2\})$ -labeled tree T with at least one edge. For these strata, define $\omega_{\mathcal{S}_T} = \hat{f}_T^* \omega_i$, where $i = 1$ if $\mathfrak{s}_T = \pi_1$, $i = 2$ if $\mathfrak{s}_T = \pi_2$, $i = 0$ if $\mathfrak{s}_T = \mathbb{R}^d$, and remove the subscript i if $\mathfrak{s}_T = [\pi_1, \pi_2]$. For the latter kind of strata: under the natural identification $\overline{S}_{\text{gray}} \approx \overline{C}_2([\pi_1, \pi_2])$, define ω_S to be the restriction of ω to S . This defines the collection $\{\omega_S\}_S$.

Now we need to show the compatibility condition: for $S' \subset \overline{S}$, $\omega_S|_{\overline{S}'} = \omega_{S'}$.

- If $S', S \subset \overline{S}_{\text{gray}}$, then this holds by definition.

⁵Hint: By continuity of the \hat{f} maps, it suffices to prove the equality on the open part $\mathcal{S}_{T'}$.

- If $S', S \not\subset \bar{S}_{\text{gray}}$: since $\omega_S = \hat{f}_{\mathcal{T}_S} \omega_i$ for some propagator ω_i ,

$$\omega_S|_{\bar{S}'} = (\hat{f}_{\mathcal{T}_S}|_{\bar{S}'})^* \omega_i = \begin{cases} \hat{f}_{\mathcal{T}_{S'}}^* \omega_i, & \text{if } G_{\mathcal{T}_{S'}, \mathcal{T}_S} \text{ has only 1 vertex} \\ (\hat{f}_{\mathcal{T}_S}|_{\bar{S}'})^* \hat{f}_{G_{\mathcal{T}_{S'}, \mathcal{T}_S}}^* \omega_0 = \hat{f}_{\mathcal{T}_{S'}}^* \omega_0, & \text{otherwise} \end{cases} = \omega_{S'},$$

where, for the second case, the second equality is because ω_i is a propagator as in Definition 3.8 and the third equality is because of Lemma 3.9.

- If $S' \subset \bar{S}_{\text{gray}}$, $S \not\subset \bar{S}_{\text{gray}}$: let T' be the $(\{1, 2\}, \{[\pi_1, \pi_2]\})$ -labeled tree $\mathcal{T}_{S'}$ under the identification $\bar{S}_{\text{gray}} \approx \bar{C}_2([\pi_1, \pi_2])$. We can also view T' as a $(\{1, 2\}, \{\pi_1, \pi_2\})$ -labeled tree; to avoid confusion let us call this $(\{1, 2\}, \{\pi_1, \pi_2\})$ -tree T . Then $\mathcal{S}_T \subset \bar{S}_{\text{blue}}$ is a stratum of \tilde{C}_2 , $\mathcal{S}_T \subset S$ and $\mathcal{S}_T \cap \bar{S}_{\text{gray}} = S'$. Since we have already shown $\omega_S|_{\bar{S}_T} = \omega_{\mathcal{S}_T}$, it suffices to show $\omega_{\mathcal{S}_T}|_{\bar{S}'} = \omega_{S'}$. We first state a lemma:

Lemma 3.9. *Let S be a stratum of $\partial \tilde{C}_2$. If $S \subset \bar{S}_{\text{blue}}$ and $S \not\subset \bar{S}_{\text{gray}}$, then $\mathfrak{s}_{\mathcal{T}_S} = \mathbb{R}^d$.*

Proof. If S is of top dimension in \bar{S}_{blue} , then \mathcal{T}_S has one vertex with space label $[\pi_1, \pi_2]$ and only one edge. The statement of the lemma follows from the last sentence of Example 3.5. If S is not of top dimension in \bar{S}_{blue} , then it is in the closure of a stratum of top dimension, and the lemma follows from the last sentence of Corollary 3.6. \square

Now, because the definition of \hat{f} is combinatorial and T, T' are the same tree, the two maps

$$\hat{f}_T|_{\bar{S}'} : \bar{S}' \longrightarrow \bar{C}_2(\mathbb{R}^d), \quad \hat{f}_{T'} : \bar{S}' \longrightarrow \bar{C}_2(\mathbb{R}^d)$$

are equal, where the first \bar{S}' is viewed as a subset of the stratum \bar{S}_T in \tilde{C}_2 while the second \bar{S}' is viewed as a boundary stratum of $\bar{C}_2([\pi_1, \pi_2])$ under the identification $\bar{S}_{\text{gray}} \approx \bar{C}_2([\pi_1, \pi_2])$. Since $\omega_{\mathcal{S}_T} = \hat{f}_T^* \omega_0$ and $\omega_{S'} = \hat{f}_{T'}^* \omega_0$, we conclude that they are equal on \bar{S}' .

Step 2: The following statement (Corollary A.4) is proved in Appendix A:

Suppose M is a compact manifold with embedded faces such that $H^k(M; \mathbb{R}) \rightarrow H^k(\partial M; \mathbb{R})$ is surjective. Suppose for every stratum $S \subset \partial M$, ω_S is a closed k -form on S , such that, if $S' \subset \bar{S}$, then $\omega_S|_{S'} = \omega_{S'}$. Then, there exists a closed k -form $\tilde{\omega}$ on M such that $\tilde{\omega}|_S = \omega_S$ for every S .

Therefore, to show the existence of a closed form $\tilde{\omega}$ on \tilde{C}_2 extending the $\{\omega_S\}_S$ we just defined, it suffices to show that the map

$$H^{d-1}(\tilde{C}_2) \xrightarrow{\text{restriction}} H^{d-1}(\partial \tilde{C}_2)$$

is surjective. It is therefore sufficient to show that $H^d(\tilde{C}_2, \partial \tilde{C}_2) = 0$. But

$$\begin{aligned} H^*(\tilde{C}_2, \partial \tilde{C}_2) &\approx H_{\dim(\tilde{C}_2)-*}(\tilde{C}_2 - \partial \tilde{C}_2) = H_{\dim(\tilde{C}_2)-*}(C_2([\pi_1, \pi_2]) \times (0, 1)) \\ &\approx H_{\dim(\tilde{C}_2)-*}(C_2([\pi_1, \pi_2])) \approx H^{*-1}(\bar{C}_2([\pi_1, \pi_2]), \partial \bar{C}_2([\pi_1, \pi_2])), \end{aligned}$$

and this is 0 when $* - 1 < d + 1$ by the proof of Lemma 2.12 in [6].

We choose and fix such an extension $\tilde{\omega}$ on \tilde{C}_2 .

4 Configuration space integrals

Note that B_I as defined in confilde Section 2.4 admits a submersion $\pi_{B_I} : B_I \rightarrow B_1 \times B_2$, making B_I a smooth fiber bundle (whose fibers are manifolds with corners) over $B_1 \times B_2$. Each fiber is homeomorphic to $S^d \times [0, \rho)$ and looks like the picture on page 10 of confilde. The map $\pi_{B_I} \circ \tilde{\pi}^A : \tilde{C}_A \rightarrow B_1 \times B_2$, and its restriction to every strata of \tilde{C}_A , are also submersions.

The base B of $[\pi_1, \pi_2]$ is B_{t_0} , the total space of the subbundle of B_I obtained by taking the subspace $S^d \times \{t_0\} \subset S^d \times [0, \rho)$ of each fiber. So, the fibers are diffeomorphic to S^d . Indeed, the fiber bundle $B \rightarrow B_1 \times B_2$ is trivial when regarded as a topological fiber bundle; namely, B is homeomorphic to $S^d \times B_1 \times B_2$. So, $H^*(B) \approx H^*(S^d) \otimes H^*(B_1) \otimes H^*(B_2)$.

We have the following diagram:

$$\begin{array}{ccc} \overline{C}_A([\pi_1, \pi_2]) & \hookrightarrow & \tilde{C}_A \\ \downarrow \pi_{\square}^A & & \downarrow \tilde{\pi}^A \\ B_{t_0} & \hookrightarrow & B_I \\ & \searrow \pi_{B_I}|_{B_{t_0}} & \downarrow \pi_{B_I} \\ & & B_1 \times B_2 \end{array},$$

where $\pi_{\square}^A, \pi_{B_I}$ are submersions but not $\tilde{\pi}^A$.

Lemma 4.1. *Let $A' \subset A$ be finite sets and $\mathfrak{f}_{A',A} : \tilde{C}_A \rightarrow \tilde{C}_{A'}$ the forgetful map forgetting all marked points not in A' . Then, for every (open) stratum S of \tilde{C}_A , $\mathfrak{f}_{A',A}(S)$ is again an open stratum of $\tilde{C}_{A'}$ and*

$$\mathfrak{f}_{A',A}|_S : S \longrightarrow \mathfrak{f}_{A',A}(S)$$

is a submersion.

In particular, when $A' = \emptyset$, this holds with $\tilde{C}_{A'} = B_I$ and $\mathfrak{f}_{A',A} = \tilde{\pi}$.

(Something about the above should be in the confilde section.)

Suppose $\sum_{i=1}^m \Gamma_i$ is a cocycle in graph cohomology. For convenience, in this section we use $\overline{S}_{\text{gray}}^i$ to denote the $\overline{S}_{\text{gray}}$ in $\tilde{C}_{V(\Gamma_i)}$, and same for the other colors (red, orange, yellow, blue) instead of gray. For every edge e of some Γ_i , we have the forgetful map

$$\mathfrak{f}_e : \tilde{C}_{V(\Gamma_i)} \longrightarrow \tilde{C}_2.$$

And, when restricted to $\overline{S}_{\text{gray}}^i \subset \tilde{C}_{V(\Gamma_i)}$ (resp. $\overline{C}_{V(\Gamma_i)}^* \subset \tilde{C}_{V(\Gamma_i)}$), it is the forgetful map

$$\mathfrak{f}_e : \overline{C}_{V(\Gamma_i)}([\pi_1, \pi_2]) \longrightarrow \overline{C}_2([\pi_1, \pi_2]) \quad (\text{resp. } \mathfrak{f}_e : \overline{C}_{V(\Gamma_i)}^* \longrightarrow \overline{C}_2^*).$$

Now we have the form $\bigwedge_{e \in E(\Gamma_i)} \mathfrak{f}_e^* \tilde{\omega}$ on $\tilde{C}_{V(\Gamma_i)}$. Recall that by the definition of Kontsevich's characteristic classes,

$$K_{[\pi_1, \pi_2]}([\sum_{i=1}^m \Gamma_i]) = [\sum_{i=1}^m (\pi_{\square}^{V(\Gamma_i)})_* \bigwedge_{e \in E(\Gamma_i)} \mathfrak{f}_e^* \tilde{\omega}] \in H^*(B) \approx H^*(S^d) \otimes H^*(B_1) \otimes H^*(B_2).$$

To determine $K_{[\pi_1, \pi_2]}([\sum_{i=1}^m \Gamma_i])$, it suffices to determine the intersection pairing

$$\langle K_{[\pi_1, \pi_2]}([\sum_{i=1}^m \Gamma_i]), \alpha_0 \otimes \alpha_1 \otimes \alpha_2 \rangle$$

for all homology classes $\alpha_0 \in H_*(S^d)$, $\alpha_1 \in H_*(B_1)$ and $\alpha_2 \in H_*(B_2)$. Below, we discuss the two cases $\alpha_0 = [pt]$ (the class of a point) and $\alpha_0 = [S^d]$ separately.

Case I: $\alpha_0 = [S^d]$

For $a = 1, 2$, suppose $\deg \alpha_a = k_a$ and α_a is represented by a piecewise-smooth singular chain $\sum_{j_a} \iota_a^{j_a}$, where $\iota_a^{j_a}$ are smooth maps from the standard k_a -dimensional simplex Δ^{k_a} to B_a .

Since for all finite set A , $\pi_{B_I} \circ \tilde{\pi}^A : \tilde{C}_A \rightarrow B_1 \times B_2$ and its restriction to every stratum is a fiber bundle, we can form the pullback bundles $(\iota_1^{j_1}, \iota_2^{j_2})^* \tilde{C}_A$, $(\iota_1^{j_1}, \iota_2^{j_2})^* \bar{C}_A([\pi_1, \pi_2])$, $(\iota_1^{j_1}, \iota_2^{j_2})^* \bar{C}_A^*$ over $\Delta^{k_1} \times \Delta^{k_2}$.

We then have

$$\langle K_{[\pi_1, \pi_2]}([\sum_{i=1}^m \Gamma_i]), [S^d] \otimes \alpha_1 \otimes \alpha_2 \rangle = \sum_{i, j_1, j_2} \int_{(\iota_1^{j_1}, \iota_2^{j_2})^* \bar{C}_{V(\Gamma_i)}([\pi_1, \pi_2])} \bigwedge_e \mathfrak{f}_e^* \omega = \sum_{i, j_1, j_2} \int_{(\iota_1^{j_1}, \iota_2^{j_2})^* \bar{S}_{\text{gray}}^i} \bigwedge_e \mathfrak{f}_e^* \tilde{\omega}. \quad (2)$$

Since

$$\partial((\iota_1^{j_1}, \iota_2^{j_2})^* \tilde{C}_{V(\Gamma_i)}) = (\iota_1^{j_1}, \iota_2^{j_2})^* \partial \tilde{C}_{V(\Gamma_i)} + ((\iota_1^{j_1}, \iota_2^{j_2})^* \tilde{C}_{V(\Gamma_i)})|_{\partial(\Delta^{k_1} \times \Delta^{k_2})},$$

by Stocks' Formula,

$$\begin{aligned} \int_{(\iota_1^{j_1}, \iota_2^{j_2})^* \partial \tilde{C}_{V(\Gamma_i)}} \bigwedge_e \mathfrak{f}_e^* \tilde{\omega} &= \int_{\partial((\iota_1^{j_1}, \iota_2^{j_2})^* \tilde{C}_{V(\Gamma_i)})} \bigwedge_e \mathfrak{f}_e^* \tilde{\omega} - \int_{(\iota_1^{j_1}, \iota_2^{j_2})^* \tilde{C}_{V(\Gamma_i)}|_{\partial(\Delta^{k_1} \times \Delta^{k_2})}} \bigwedge_e \mathfrak{f}_e^* \tilde{\omega} \\ &= \int_{(\iota_1^{j_1}, \iota_2^{j_2})^* \tilde{C}_{V(\Gamma_i)}} d\left(\bigwedge_e \mathfrak{f}_e^* \tilde{\omega}\right) - \int_{(\iota_1^{j_1}, \iota_2^{j_2})^* \tilde{C}_{V(\Gamma_i)}|_{\partial \Delta^{k_1} \times \Delta^{k_2}}} \bigwedge_e \mathfrak{f}_e^* \tilde{\omega} \pm \int_{(\iota_1^{j_1}, \iota_2^{j_2})^* \tilde{C}_{V(\Gamma_i)}|_{\Delta^{k_1} \times \partial \Delta^{k_2}}} \bigwedge_e \mathfrak{f}_e^* \tilde{\omega}. \end{aligned}$$

The first term is 0 because $\tilde{\omega}$ is closed. Because the singular chains $\sum_{j_1} \iota_1^{j_1}$, $\sum_{j_2} \iota_2^{j_2}$ are cycles, when summing over all j_1 , the second term is 0; when summing over all j_2 , the third term is 0. So,

$$(2) = \sum_{i, j_1, j_2} \left(\int_{(\iota_1^{j_1}, \iota_2^{j_2})^* \bar{S}_{\text{blue}}^i} \bigwedge_e \mathfrak{f}_e^* \tilde{\omega} + \int_{(\iota_1^{j_1}, \iota_2^{j_2})^* \bar{C}_{V(\Gamma_i)}^*} \bigwedge_e \mathfrak{f}_e^* \tilde{\omega} \right).$$

Since $\sum_i \Gamma_i$ is a cocycle in graph cohomology, the first term is 0 just like in the proof of the well-definedness of Kontsevich's classes, see e.g. [6, Appendix E]⁶, so

$$\langle K_{[\pi_1, \pi_2]}([\sum_i \Gamma_i]), [S^d] \otimes \alpha_1 \otimes \alpha_2 \rangle = \sum_{i, j_1, j_2} \int_{(\iota_1^{j_1}, \iota_2^{j_2})^* \bar{C}_{V(\Gamma_i)}^*} \bigwedge_e \mathfrak{f}_e^* \tilde{\omega}$$

⁶The argument goes as follows: a top-dimensional stratum S of \bar{S}_{blue}^i contains a screen with space label \mathbb{R}^d , consider 3 separate cases: (...)

$$= \sum_{i,j_1,j_2} \int_{(\iota_1^{j_1}, \iota_2^{j_2})^* S_{\text{red}}^i} \bigwedge_e \mathfrak{f}_e^* \tilde{\omega} + \sum_{i,j_1,j_2} \int_{(\iota_1^{j_1}, \iota_2^{j_2})^* S_{\text{yellow}}^i} \bigwedge_e \mathfrak{f}_e^* \tilde{\omega}. \quad (3)$$

It remains to show that

$$(3) = \langle (K_{\pi_1} \otimes K_{\pi_2})(\Delta_{[\cdot]}[\sum_i \Gamma_i]), \alpha_1 \otimes \alpha_2 \rangle. \quad (4)$$

The right hand side above equals to⁷

$$\begin{aligned} & \langle (K_{\pi_1} \otimes K_{\pi_2})([\sum_i \sum_{\Gamma'_i \leq \Gamma_i} (\Gamma'_i \otimes \Gamma_i / \Gamma'_i + (-1)^? \Gamma_i / \Gamma'_i \otimes \Gamma'_i)]), \alpha_1 \otimes \alpha_2 \rangle = \\ & \sum_{i,j_1,j_2} \sum_{\Gamma'_i \leq \Gamma_i} \left(\left(\int_{(\iota_1^{j_1})^* \overline{C}_{V(\Gamma'_i)}(\pi_1)} \bigwedge_{e \in E(\Gamma'_i)} \mathfrak{f}_e^* \omega_1 \right) \cdot \left(\int_{(\iota_2^{j_2})^* \overline{C}_{V(\Gamma_i/\Gamma'_i)}(\pi_2)} \bigwedge_{e \in E(\Gamma_i/\Gamma'_i)} \mathfrak{f}_e^* \omega_2 \right) \right. \\ & \quad \left. + (-1)^? \left(\int_{(\iota_1^{j_1})^* \overline{C}_{V(\Gamma_i/\Gamma'_i)}(\pi_1)} \bigwedge_{e \in E(\Gamma_i/\Gamma'_i)} \mathfrak{f}_e^* \omega_1 \right) \cdot \left(\int_{(\iota_2^{j_2})^* \overline{C}_{V(\Gamma'_i)}(\pi_2)} \bigwedge_{e \in E(\Gamma'_i)} \mathfrak{f}_e^* \omega_2 \right) \right). \end{aligned}$$

To prove (4), it suffices to show that, for all i, j_1, j_2 ,

$$\sum_{\Gamma'_i \leq \Gamma_i} \left(\int_{(\iota_1^{j_1})^* C_{V(\Gamma'_i)}(\pi_1)} \bigwedge_{e \in E(\Gamma'_i)} \mathfrak{f}_e^* \omega_1 \right) \cdot \left(\int_{(\iota_2^{j_2})^* C_{V(\Gamma_i/\Gamma'_i)}(\pi_2)} \bigwedge_{e \in E(\Gamma_i/\Gamma'_i)} \mathfrak{f}_e^* \omega_2 \right) = \int_{(\iota_1^{j_1}, \iota_2^{j_2})^* S_{\text{yellow}}^i} \bigwedge_{e \in E(\Gamma_i)} \mathfrak{f}_e^* \tilde{\omega} \quad (5)$$

and

$$\sum_{\Gamma'_i \leq \Gamma_i} \left(\int_{(\iota_1^{j_1})^* C_{V(\Gamma_i/\Gamma'_i)}(\pi_1)} \bigwedge_{e \in E(\Gamma_i/\Gamma'_i)} \mathfrak{f}_e^* \omega_1 \right) \cdot \left(\int_{(\iota_2^{j_2})^* C_{V(\Gamma'_i)}(\pi_2)} \bigwedge_{e \in E(\Gamma'_i)} \mathfrak{f}_e^* \omega_2 \right) = \int_{(\iota_1^{j_1}, \iota_2^{j_2})^* S_{\text{red}}^i} \bigwedge_{e \in E(\Gamma_i)} \mathfrak{f}_e^* \tilde{\omega}. \quad (6)$$

Below we only prove (5) since (6) is completely similar.

Since

$$S_{\text{yellow}}^i = \sum_{V_1, V_2: V_1 \sqcup V_2 = V(\Gamma_i)} C_{V_1}(\pi_1) \times C_{V_2 \sqcup \{\star\}}(\pi_2)$$

(where \star records the position of the node on π_2), and the bundle map $S_{\text{yellow}}^i \rightarrow B_1 \times B_2 \rightarrow B_1$ (resp. $\rightarrow B_2$) is by projecting first to the $C_{V_1}(\pi_1)$ (resp. $C_{V_2 \sqcup \{\star\}}(\pi_2)$) factor and then go along the bundle map $C_{V_1}(\pi_1) \rightarrow B_1$ (resp. $C_{V_2 \sqcup \{\star\}}(\pi_2) \rightarrow B_2$),

$$(\iota_1^{j_1}, \iota_2^{j_2})^* S_{\text{yellow}}^i = \sum_{V_1, V_2: V_1 \sqcup V_2 = V(\Gamma_i)} (\iota_1^{j_1})^* C_{V_1}(\pi_1) \times (\iota_2^{j_2})^* C_{V_2 \sqcup \{\star\}}(\pi_2).$$

So, by the construction of $\tilde{\omega}$ on S_{yellow} and by Fubini's Theorem, the right hand side of (5) is

$$\sum_{V_1, V_2: V_1 \sqcup V_2 = V(\Gamma_i)} \int_{(\iota_2^{j_2})^* C_{V_2 \sqcup \{\star\}}(\pi_2)} \int_{(\iota_1^{j_1})^* C_{V_1}(\pi_1)} \bigwedge_{\substack{e \in E(\Gamma_i) \\ \text{both endpoints of } e \text{ are in } V_1}} \mathfrak{f}_e^* \omega_1 \wedge \bigwedge_{\substack{e \in E(\Gamma) \\ \exists \text{ endpoint of } e \text{ in } V_2}} \mathfrak{f}_e^* \omega_2$$

⁷A little more argument needed for this claim, but it is true. (Say things like K_π is induced from a chain map from the graph complex to the differential forms on B . Maybe cite [6, Theorem 2.15(1)].)

$$= \sum_{V_1, V_2: V_1 \sqcup V_2 = V(\Gamma_i)} \left(\int_{(\iota_1^{j_1})^* C_{V_1}(\pi_1)} \bigwedge_{e \in E(\Gamma'(V_1))} \mathfrak{f}_e^* \omega_1 \right) \cdot \left(\int_{(\iota_2^{j_2})^* C_{V_2 \sqcup \{*\}}(\pi_2)} \bigwedge_{e \in E(\Gamma/\Gamma'(V_1))} \mathfrak{f}_e^* \omega_2 \right),$$

where, for $V_1 \subset V(\Gamma_i)$, $\Gamma'(V_1)$ denotes the subgraph of Γ_i spanned by vertices in V_1 . This proves (5).

Case II: $\alpha_0 = [pt]^8$

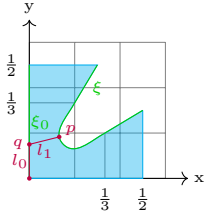
For simplicity of exposition, we assume for $a = 1, 2$, α_a can be represented by a smooth submanifold $B'_a \subset B_a$. This assumption is purely for the purpose of simplifying the notation: all the arguments below can be easily adapted to the case when B'_a is only a piecewise smooth singular chain, similar to Case I. (The readers familiar with the theory of pseudocycles should take B'_a to be a pseudocycle: the arguments below works verbatim and no assumption needs to be made.)

We consider the pullback of everything to $B'_1 \times B'_2$: let $B', B'_I, \tilde{C}'_A, \overline{C}'_A([\pi_1, \pi_2]), (\overline{S}^i_{\text{gray}})', (\overline{S}^i_{\text{blue}})', (\overline{S}^i_{\text{red}})', (\overline{S}^i_{\text{yellow}})', (\overline{S}^i_{\text{orange}})'$ be the pullback of $B, B_I, \tilde{C}_A, \overline{C}_A([\pi_1, \pi_2]), \overline{S}^i_{\text{gray}}, \overline{S}^i_{\text{blue}}, \overline{S}^i_{\text{red}}, \overline{S}^i_{\text{yellow}}, \overline{S}^i_{\text{orange}}$, respectively.

Recall that B_I is constructed by gluing together 3 pieces (cite Section 2.4 of confilde):

$$\mathring{E}_1(3) \times B_2 \times [0, \rho), \quad S^{d-1} \times L \times B_1 \times B_2, \quad B_1 \times \mathring{E}_2(3) \times [0, \rho),$$

where $L \subset \mathbb{R}^2$ is the (closure of the) blue region below:



We define ξ to be the curvy stratum of ∂L and ξ_0 to be $\{x = 0, 0 < y < 1/2\} \subset \partial L$. Then, $S^{d-1} \times \xi \times B_1 \times B_2$ is part of $B = B_{t_0} \subset B_I$, the base of $[\pi_1, \pi_2]$.

Let $v \in S^{d-1}$ be a generic point. Let p be a generic point on ξ and q be a generic point on ξ_0 . Let l_1 be a generic path (including endpoints) going from p to q , transverse to ∂L , and define l_0 to be the (closed) line segment from q to $(0, 0)$. Denote

$$\begin{aligned} M_p &:= v \times p \times B'_1 \times B'_2 \subset S^{d-1} \times \xi \times B'_1 \times B'_2 && \subset B' = B'_{t_0}, \\ M_q &:= v \times q \times B'_1 \times B'_2 \subset S^{d-1} \times \xi_0 \times B'_1 \times B'_2 && \subset B'_0, \\ M_{l_0} &:= v \times l_0 \times B'_1 \times B'_2 \subset S^{d-1} \times \xi_0 \times B'_1 \times B'_2 && \subset B'_0, \\ M_{l_1} &:= v \times l_1 \times B'_1 \times B'_2 \subset S^{d-1} \times L \times B'_1 \times B'_2 && \subset B'_I. \end{aligned}$$

⁸(The exposition is not so great; may need further polishing.)

Then, M_p is a section of the bundle $B' \rightarrow B'_1 \times B'_2$ and thus a representative of $[pt] \otimes \alpha_1 \otimes \alpha_2$. So,

$$\langle K_{[\pi_1, \pi_2]}([\sum_{i=1}^m \Gamma_i]), [pt] \otimes \alpha_1 \otimes \alpha_2 \rangle = \sum_i \int_{\tilde{C}'_{V(\Gamma_i)}([\pi_1, \pi_2])|_{M_p}} \bigwedge_e \mathfrak{f}_e^* \omega = \sum_i \int_{(\bar{S}_{\text{gray}}^i)'|_{M_p}} \bigwedge_e \mathfrak{f}_e^* \tilde{\omega}. \quad (7)$$

By Lemma 4.1, for every stratum S of $\tilde{C}'_{V(\Gamma_i)}$, the differential of $S \rightarrow B'_I$ is surjective at every point on $M_{l_1} \setminus M_q$, and on M_q the image of this differential covers $T_{S^{d-1}} \times \xi_0 \times B'_1 \times B'_2$. So, since the differential of l_1 at q covers the normal direction of $\xi_0 \subset L$, the map $S \rightarrow B'_I$ is transverse to M_{l_1} . Therefore, $\tilde{C}'_{V(\Gamma_i)}|_{M_{l_1}}$ is a manifold with corners, and

$$\begin{aligned} \partial(\tilde{C}'_{V(\Gamma_i)}|_{M_{l_1}}) &= (\partial\tilde{C}'_{V(\Gamma_i)})|_{M_{l_1}} + \tilde{C}'_{V(\Gamma_i)}|_{M_q} - \tilde{C}'_{V(\Gamma_i)}|_{M_p} \\ &= (\bar{S}_{\text{blue}}^i)'|_{M_{l_1}} + (\bar{S}_{\text{yellow}}^i)'|_{M_q} - (\bar{S}_{\text{gray}}^i)'|_{M_p}. \end{aligned}$$

By Stocks' Formula and that $\tilde{\omega}$ is closed we have

$$(7) = \sum_i \left(\int_{(\bar{S}_{\text{blue}}^i)'|_{M_{l_1}}} \bigwedge_e \mathfrak{f}_e^* \tilde{\omega} + \int_{(\bar{S}_{\text{yellow}}^i)'|_{M_q}} \bigwedge_e \mathfrak{f}_e^* \tilde{\omega} \right) = \sum_i \int_{(\bar{S}_{\text{yellow}}^i)'|_{M_q}} \bigwedge_e \mathfrak{f}_e^* \tilde{\omega}, \quad (8)$$

where, just as in Case I, the \bar{S}_{blue} term vanishes by the same argument as in showing the well-definedness of Kontsevich's classes.

Let $(\bar{S}_{\text{yellow}}^i)'|_{l_0}$ (resp. $(\bar{S}_{\text{yellow}}^i)'|_{l_0 \setminus q}$) be the preimage of $S^{d-1} \times l_0 \times B'_1 \times B'_2$ (resp. $S^{d-1} \times (l_0 \setminus q) \times B'_1 \times B'_2$) in $(\bar{S}_{\text{yellow}}^i)'$. Then, $(\bar{S}_{\text{yellow}}^i)'|_{l_0 \setminus q}$ is an open subset of $(\bar{S}_{\text{yellow}}^i)'$ and, since every stratum of $(\bar{S}_{\text{yellow}}^i)'$ is transverse to q , $(\bar{S}_{\text{yellow}}^i)'|_{l_0}$ is a manifold with corners. The boundary strata of $(\bar{S}_{\text{yellow}}^i)'|_{l_0}$ are of two kinds: the strata of $(\bar{S}_{\text{yellow}}^i)'|_{l_0 \setminus q}$ and the strata of $(\bar{S}_{\text{yellow}}^i)'$ restricted to $S^{d-1} \times q \times B'_1 \times B'_2$. Moreover, the map $(\bar{S}_{\text{yellow}}^i)'|_{l_0} \rightarrow S^{d-1}$ is a submersion when restricted to on every stratum, so the preimage of v under this map, $(\bar{S}_{\text{yellow}}^i)'|_{M_{l_0}}$, is again a manifold with corners, and we have

$$\partial(\bar{S}_{\text{yellow}}^i)'|_{M_{l_0}} = (\partial(\bar{S}_{\text{yellow}}^i)')|_{M_{l_0 \setminus (M_p, M_q)}} + (\bar{S}_{\text{yellow}}^i)'|_{M_q} - (\bar{S}_{\text{yellow}}^i)'|_{M_{(0,0)}}.$$

The first term consisting of strata with at least one screen with space label \mathbb{R}^d and the points on that screen is not a priori constrained; so, again by the same argument as showing well-definedness of Kontsevich's classes, these strata do not contribute to the integrals below. By Stocks' Formula,

$$(8) = \sum_i \int_{(\bar{S}_{\text{yellow}}^i)'|_{M_{(0,0)}}} \bigwedge_e \mathfrak{f}_e^* \tilde{\omega} = \sum_i \int_{(\bar{S}_{\text{yellow}}^i)'|_{M_q}} \bigwedge_e \mathfrak{f}_e^* \tilde{\omega} = \sum_i \int_{(S_{\text{orange}}^i)'|_v} \bigwedge_e \mathfrak{f}_e^* \tilde{\omega}. \quad (9)$$

Recall

$$S_{\text{orange}}^i = \sum_{V_1, V_2, V_3: V_1 \sqcup V_2 \sqcup V_3 = V(\Gamma_i)} C_{V_1}(\pi_1) \times C_{V_2}(\pi_2) \times C_{V_3 \sqcup \{\star_1, \star_2\}}(\mathbb{R}^d)$$

and restricting to v means restricting the $C_{V_3 \sqcup \{\star_1, \star_2\}}(\mathbb{R}^d)$ term to the case $\frac{\star_1 - \star_2}{[\star_1 - \star_2]} = v$; we can thus view $\star_1 = 0, \star_2 = v$ (here we implicitly identify S^{d-1} as the unit sphere in \mathbb{R}^d) to be two fixed points in \mathbb{R}^d and define

$$C_{V_3}^*(\mathbb{R}^d) = \{(x_v \in \mathbb{R}^d)_{v \in V_3} \mid \forall v \in V_3, x_v \neq 0, x_v \neq v; \forall v, w \in V_3, x_v \neq x_w\}.$$

For $V_1, V_2 \subset V(\Gamma_i)$, define $\Gamma_i^{\text{sub}}(V_1), \Gamma_i^{\text{sub}}(V_2)$ to be the subgraph of Γ_i spanned by the vertices in V_1 and V_2 , respectively. Define

$$\Gamma_i/(V_1, V_2) := (\Gamma_i/\Gamma_i^{\text{sub}}(V_1))/\Gamma_i^{\text{sub}}(V_2)$$

the quotient graph by contracting $\Gamma_i^{\text{sub}}(V_1)$ to a new vertex \star_1 and contracting $\Gamma_i^{\text{sub}}(V_2)$ to a new vertex \star_2 . For each edge $e \in E(\Gamma_i/(V_1, V_2))$, we still have the forgetful map $f_e : C_{V_3}^*(\mathbb{R}^d) \rightarrow S^{d-1}$. Then, by the way $\tilde{\omega}$ is constructed,

$$\begin{aligned} \int_{(S_{\text{orange}}^i)'|_v} \bigwedge_e f_e^* \tilde{\omega} &= \sum_{V_1, V_2, V_3: V_1 \sqcup V_2 \sqcup V_3 = V(\Gamma_i)} \left(\int_{C_{V_1}'(\pi_1)} \bigwedge_{e \in E(\Gamma_i^{\text{sub}}(V_1))} f_e^* \omega_1 \right) \\ &\quad \left(\int_{C_{V_2}'(\pi_2)} \bigwedge_{e \in E(\Gamma_i^{\text{sub}}(V_2))} f_e^* \omega_2 \right) \cdot \left(\int_{C_{V_3}^*(\mathbb{R}^d)} \bigwedge_{e \in E(\Gamma_i/(V_1, V_2))} f_e^* \omega_0 \right) \end{aligned} \quad (10)$$

We claim that if $\Gamma_i/(V_1, V_2)$ is not the empty graph, then the term $\int_{C_{V_3}^*(\mathbb{R}^d)} \bigwedge_{e \in E(E(\Gamma_i) \setminus (V_1, V_2))} f_e^* \omega_0$ is always 0: this is again the same argument as in showing the well-definedness of Kontsevich's classes: if $\Gamma_i/(V_1, V_2)$ has a univalent vertex, then there is not enough constraints to determine the position of this vertex; if the $\Gamma_i/(V_1, V_2)$ has a bivalent vertex, then there is a $\mathbb{Z}/2\mathbb{Z}$ symmetry making the relevant terms canceling out; if all the vertices of $\Gamma_i/(V_1, V_2)$ are at least trivalent, then the degree cannot match – there are too many constraints in the problem than freedoms.

Therefore, the only contributions to (10) comes from terms when $\Gamma_i/(V_1, V_2)$ is the empty graph. Since Γ_i is connected, this only happens if $V_1 = V(\Gamma_i)$ or $V_2 = V(\Gamma_i)$. Therefore,

$$(10) = \int_{C_{V(\Gamma_i)}'(\pi_1)} \bigwedge_{e \in E(\Gamma_i)} f_e^* \omega_1 + \int_{C_{V(\Gamma_i)}'(\pi_2)} \bigwedge_{e \in E(\Gamma_i)} f_e^* \omega_2.$$

We conclude that

$$\begin{aligned} (9) &= \sum_i (10) = \sum_i \int_{C_{V(\Gamma_i)}'(\pi_1)} \bigwedge_{e \in E(\Gamma_i)} f_e^* \omega_1 + \sum_i \int_{C_{V(\Gamma_i)}'(\pi_2)} \bigwedge_{e \in E(\Gamma_i)} f_e^* \omega_2 \\ &= \langle K_{\pi_1}([\sum_i \Gamma_i]), \alpha_1 \rangle + \langle K_{\pi_2}([\sum_i \Gamma_i]), \alpha_2 \rangle = \langle (K_{\pi_1} \otimes K_{\pi_2})(\Delta. [\sum_i \Gamma_i]), \alpha_1 \otimes \alpha_2 \rangle. \end{aligned}$$

This completes the proof of Theorem 1.1.

Appendix A Extending differential forms on a manifold with corners from boundary to interior

We first clarify the notation used in the present paper concerning manifold with corners, mostly following [3] and [4]. Here is a dictionary for our notation:

- *(smooth) manifold with corners*: as in [4, page 3] or, equivalently, [3, Definition 3].
- *smooth map between manifolds with corners*: [3, Definition 4] (“weakly smooth map” in [4, Definition 3.1])

- *tangent and cotangent spaces of manifolds with corners*: [3, Definition 10] (equivalently, [4, Definition 2.2])
- *cotangent bundle, tensor and exterior powers of cotangent bundle of manifolds with corners*: follows from the definition of cotangent spaces
- *differential form on manifolds with corners*: smooth section of exterior powers of the cotangent bundle

For example, in $(\mathbb{R}^{\geq 0})^n$ with coordinates denoted by (x_1, \dots, x_n) , a differential form of degree k can be written as

$$\sum_{1 \leq i_1 < \dots < i_k \leq n} f_{i_1 \dots i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k},$$

where $f_{i_1 \dots i_k}$ are functions on $(\mathbb{R}^{\geq 0})^n$, smooth in the sense that there exist smooth functions $\hat{f}_{i_1 \dots i_k}$ on an open neighborhood of $(\mathbb{R}^{\geq 0})^n$ in \mathbb{R}^n , such that $f_{i_1 \dots i_k} = \hat{f}_{i_1 \dots i_k}|_{(\mathbb{R}^{\geq 0})^n}$. For $0 \leq m < n$, the restriction (pullback by inclusion map) of the above differential form to $(\mathbb{R}^{\geq 0})^m \approx \{x_{m+1} = \dots = x_n = 0\} \subset (\mathbb{R}^{\geq 0})^n$ will be

$$\sum_{1 \leq i_1 < \dots < i_k \leq m} f_{i_1 \dots i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k};$$

namely, we remove the terms containing any dx_i with $i > m$.

- ∂M , *the boundary of a manifold with corners* M : the image of the map $i_M : \partial M \rightarrow M$ in [4, Definition 2.6]
 - $\tilde{\partial} M$: ∂M in [4, Definition 2.6]
 - *codimension- d (depth- d) stratum*: [3, Definition 7] (a connected component of a depth- d stratum in [4, Definition 2.3])
- For example, a solid cube has 6 codim-1 strata and 12 codim-2 strata. Its boundary is the surface of the cube, and its $\tilde{\partial}$ is the disjoint union of 6 closed squares.
- *manifold with embedded faces*: [3, Definition 18]: a manifold with corners such that the closure of every codim-1 strata is an embedded manifold with corners.

In this appendix we prove the following

Proposition A.1. *Let M be a compact manifold with embedded faces.⁹ Suppose for each stratum $S \subset \partial M$ of M , ω_S is a closed differential form of degree k on \overline{S} , such that, for each pair of strata $T \subset \overline{S} \subset \partial M$, $\omega_T = \omega_S|_{\overline{T}}$. Then, there exists an open neighborhood U of ∂M and a closed differential form ω on U such that $\omega|_{\overline{S}} = \omega_S$ for all strata $S \subset \partial M$.*

In light of Proposition A.1, we also define

- *degree- k differential form on ∂M* : a compatible collection of forms, $\{\omega_S\}_S$, as in the assumption of Proposition A.1.

⁹Neither the compactness or embedded faces condition should be necessary; this proposition likely holds for all manifolds with corners. We assume the stronger condition here to shorten the proof.

Remark A.2. It is easy to see that the \mathbb{R} -singular cohomology of ∂M (or more generally, any manifold with corners and “hinges”¹⁰) is equivalent to the de Rham cohomology defined using these forms, but we do not need it in the present paper.

Lemma A.3. *Let M be a manifold with corners and U an open neighborhood of ∂M . Suppose ω is a closed k -form on U such that $[\omega]$ lies in the image of the restriction map $H^k(M; \mathbb{R}) \rightarrow H^k(U; \mathbb{R})$, then there is a neighborhood $U'' \subset U$ of ∂M such that $\omega|_{U''}$ can be extended to a closed k -form on M .*

Proof (same as (9802062 below Lemma 1.2)). Let α be a closed d -form on $M \setminus \partial M$ such that the restriction of $[\alpha]$ to U is $[\omega]$. Then, there exists a $(k-1)$ -form β on U such that $d\beta = \omega - \alpha|_U$. We can find¹¹ open subsets U', U'' of M such that $\partial M \subsetneq U'' \subsetneq U' \subsetneq U$, and a smooth function $f : M \rightarrow [0, 1]$ such that $f|_{U''} \equiv 0$, $f|_{M \setminus U'} \equiv 1$. Define $\omega' = \omega - d(f\beta)$ on U . Then $\omega' \equiv \omega$ on U'' , $\omega' \equiv \alpha$ on $U \setminus U'$, and $d\omega' = 0$. So we can extend ω' to a form on M , defining it to be α out of U . \square

Corollary A.4. *Suppose M is a compact manifold with embedded faces such that $H^k(M; \mathbb{R}) \rightarrow H^k(\partial M; \mathbb{R})$ is surjective. Then, every closed k -form on ∂M can be extended to a closed k -form on M .*

The rest of this section is devoted to the proof of Proposition A.1. First we prove the statement locally, in $(\mathbb{R}^{\geq 0})^d \times \mathbb{R}^{n-d}$, where $0 \leq d \leq n$ and $0 \leq k \leq n$ are integers. For $I \subset \{1, \dots, d\}$, define $H_I := \{(x_1, \dots, x_n) \mid \forall i \in I, x_i = 0\} \subset (\mathbb{R}^{\geq 0})^d \times \mathbb{R}^{n-d}$. Write $H_i := H_{\{i\}}$ and $H_{i,j} = H_{\{i,j\}}$.

Lemma A.5. *Suppose for each $1 \leq i \leq d$, ω_i is a degree- k differential form on H_i , such that, for all $1 \leq i, j \leq d$, $\omega_i|_{H_{i,j}} = \omega_j|_{H_{i,j}}$. Then, there exists a degree- k differential form ω on $(\mathbb{R}^{\geq 0})^d \times \mathbb{R}^{n-d}$, such that, for all $1 \leq i \leq d$, $\omega|_{H_i} = \omega_i$. Moreover, if all ω_i are closed, ω can be taken to be closed as well.*

Proof. From the condition $\omega_i|_{H_{i,j}} = \omega_j|_{H_{i,j}}$ it is clear that for any $I \subset \{1, \dots, d\}$, the forms $\omega_i|_{H_I}$ are the same for all $i \in I$. We hence denote it by ω_I , which is on H_I . Let

$$p_I : (\mathbb{R}^{\geq 0})^d \times \mathbb{R}^{n-d} \longrightarrow H_I$$

be the projection map, sending all I -coordinates to 0 and not changing the other coordinates. We take ω to be the alternating sum

$$\omega = \sum_{1 \leq i \leq d} p_{\{i\}}^* \omega_i - \sum_{1 \leq i < j \leq d} p_{\{i,j\}}^* \omega_{\{i,j\}} + \sum_{1 \leq i < j < k \leq d} p_{\{i,j,k\}}^* \omega_{\{i,j,k\}} - \dots + (-1)^{d-1} p_{\{1,\dots,d\}}^* \omega_{\{1,\dots,d\}}.$$

To see that $\omega|_{H_i} = \omega_i$ for a given $i \in \{1, \dots, d\}$, note that, for each $I \subset \{1, \dots, d\}$ with $I \neq \emptyset$ and $i \notin I$,

$$(p_I^* \omega_I)|_{H_i} = p_I^*(\omega_I|_{H_i}) = p_{I \sqcup \{i\}}^*(\omega_{I \sqcup \{i\}}|_{H_i}) = (p_{I \sqcup \{i\}}^* \omega_{I \sqcup \{i\}})|_{H_i},$$

so these two terms cancel with each other. If all ω_i s are closed, ω is clearly also closed. \square

¹⁰By “hinges” we mean the manifold is locally modeled on $\mathbb{R}^a \times (\mathbb{R}^{\geq 0})^b \times ((\mathbb{R}^{\geq 0})^c \setminus (\mathbb{R}^{>0})^c)$ for $a, b, c \in \mathbb{Z}^{\geq 0}$. One can imagine a theory of differential forms on such manifolds, defined similar to here.

¹¹For example, using a metric on M and properly modify the distance function to ∂M .

Next, we patch the forms constructed locally to a global one. Without the closeness condition, this would be immediate by applying a partition of unity. With the closeness condition it is much subtler. The argument uses the same technique as translating between Čech and de Rham cohomology (see, e.g. [1]), which can also be viewed as a generalization of the technique in the proof of Lemma A.3. Note that from now on, the index set I has a different meaning than in Lemma A.5.

Given $p, q \in \mathbb{Z}^{\geq 0}$, a manifold M and a locally finite open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of M , recall a (skew-symmetric) p -Čech cochain of q -forms on M is: for each sequence $(i_0, i_1, \dots, i_p) \in I^{p+1}$, a differential q -form $\alpha_{i_0 i_1 \dots i_p}$ on $\bigcap_{j=0}^p U_{i_j}$; such that for all j , $\alpha_{i_0 \dots i_j i_{j+1} \dots i_p} = -\alpha_{i_0 \dots i_{j+1} i_j \dots i_p}$. We denote the \mathbb{R} -vector space of p -Čech cochain of q -forms on M by $\check{C}_{\mathcal{U}}^p(M; \mathcal{A}^q)$. The Čech differential is

$$\check{\delta} : \check{C}_{\mathcal{U}}^p(M; \mathcal{A}^q) \longrightarrow \check{C}_{\mathcal{U}}^{p+1}(M; \mathcal{A}^q)$$

$$\check{\delta}(\alpha_{i_0 \dots i_p})_{(i_0 \dots i_p) \in I^{p+1}} = (\beta_{i_0 \dots i_{p+1}})_{(i_0 \dots i_{p+1}) \in I^{p+2}}, \quad \beta_{i_0 \dots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \alpha_{i_0 \dots \hat{i}_j \dots i_{p+1}}|_{U_{i_0} \cap \dots \cap U_{i_{p+1}}}.$$

We also still denote by d the termwise differential of forms:

$$d : \check{C}_{\mathcal{U}}^p(M; \mathcal{A}^q) \longrightarrow \check{C}_{\mathcal{U}}^p(M; \mathcal{A}^{q+1}), \quad d(\alpha_{i_0 \dots i_p})_{(i_0 \dots i_p) \in I^{p+1}} = (d\alpha_{i_0 \dots i_p})_{(i_0 \dots i_p) \in I^{p+1}}.$$

It is clear that $\check{\delta}d = d\check{\delta}$, and both d and $\check{\delta}$ commute with pull-back maps between manifolds.

Lemma A.6. *Suppose M is a manifold with corners. Denote $N := \partial M$ and $\iota : N \rightarrow M$ the inclusion map.¹² Suppose $\mathcal{U} = \{U_i\}_{i \in I}$ is a locally finite open cover of M satisfying the condition that for all subset $I' \subset I$, if $U_{I'} := \bigcap_{i \in I'} U_i$ is non-empty, then*

1. *all de Rham cohomology groups of $U_{I'}$ are the same as those of a point,*
2. *$\iota^{-1}(U_{I'}) \neq \emptyset$, and*
3. *if σ is a closed form¹³ on $\iota^{-1}(U_{I'})$, then there exists a closed form $\tilde{\sigma}$ on $U_{I'}$ with $\iota^* \tilde{\sigma} = \sigma$.*

Then, the following proposition \mathcal{P}_p^q holds for all $p \geq 0, q \geq 1$:

\mathcal{P}_p^q : *Suppose $\alpha = (\alpha_{i_0 \dots i_p})_{(i_0 \dots i_p) \in I^{p+1}} \in \check{C}_{\mathcal{U}}^p(M; \mathcal{A}^q)$ satisfies $\check{\delta}\alpha = 0$, $\iota^* \alpha = 0$ and $d\alpha = 0$, then, there exists $\beta = (\beta_{i_0 \dots i_p})_{(i_0 \dots i_p) \in I^{p+1}} \in \check{C}_{\mathcal{U}}^p(M; \mathcal{A}^{q-1})$ such that $\check{\delta}\beta = 0$, $\iota^* \beta = 0$ and $d\beta = \alpha$.*

Proof. Two steps: we first show that $\mathcal{P}_{p+1}^{q-1} \implies \mathcal{P}_p^q$, then show that \mathcal{P}_p^1 holds for all $p \geq 0$.

Step 1: Suppose $q \geq 2$ and α is as in the condition of \mathcal{P}_p^q . By condition 1 above, there is $\beta'' \in \check{C}_{\mathcal{U}}^p(M; \mathcal{A}^{q-1})$ such that $d\beta'' = \alpha$. Then, $d\iota^* \beta'' = 0$, so condition 3 above implies that there is $\tilde{\beta}'' \in \check{C}_{\mathcal{U}}^p(M; \mathcal{A}^{q-1})$ such that $d\tilde{\beta}'' = 0$ and $\iota^* \tilde{\beta}'' = \iota^* \beta''$. Setting $\beta' = \beta'' - \tilde{\beta}''$ we have $d\beta' = \alpha$ and $\iota^* \beta' = 0$. Since $d\check{\delta}\beta' = \check{\delta}\alpha = 0$ and $\iota^* \check{\delta}\beta' = 0$, $\check{\delta}\beta'$ (replacing α) satisfies the condition of \mathcal{P}_{p+1}^{q-1} . If we assume \mathcal{P}_{p+1}^{q-1} holds, then there exists $\gamma \in \check{C}_{\mathcal{U}}^{p+1}(M; \mathcal{A}^{q-2})$ with $\check{\delta}\gamma = 0$, $\iota^* \gamma = 0$ and $d\gamma = \check{\delta}\beta'$.

¹²This lemma and Lemma A.7 hold if N is replaced with an arbitrary manifold with corners and $\iota : N \rightarrow M$ an arbitrary smooth map.

¹³Differential forms on open subsets of ∂M are defined as in the assumption of Proposition A.1.

Let $\{f_i : U_i \rightarrow [0, 1]\}_{i \in I}$ be a partition of unity subordinate to \mathcal{U} . we define β by taking

$$\beta_{i_0 \dots i_p} = \beta'_{i_0 \dots i_p} - \sum_{i \in I} d(f_i \cdot \gamma_{ii_0 \dots i_p}).$$

It is clear that $d\beta = d\beta' = \alpha$ and, since $\iota^* \gamma = 0$, $\iota^* \beta = \iota^* \beta' - \sum_{i \in I} d((f_i \circ \iota) \cdot \iota^* \gamma_{ii_0 \dots i_p}) = 0$. And

$$\begin{aligned} (\check{\delta}\beta' - \check{\delta}\beta)_{i_0 \dots i_{p+1}} &= \sum_{j=0}^{p+1} (-1)^j (\beta' - \beta)_{i_0 \dots \hat{i}_j \dots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \sum_{i \in I} d(f_i \cdot \gamma_{ii_0 \dots \hat{i}_j \dots i_{p+1}}) \\ &= d \sum_{i \in I} \left(f_i \cdot \sum_{j=0}^{p+1} (-1)^j \gamma_{ii_0 \dots \hat{i}_j \dots i_{p+1}} \right) = d \sum_{i \in I} f_i \cdot (\gamma_{i_0 \dots i_{p+1}} - (\check{\delta}\gamma)_{ii_0 \dots i_{p+1}}) = d\gamma_{i_0 \dots i_{p+1}}, \end{aligned} \tag{11}$$

where the last equality is due to $\check{\delta}\gamma = 0$, and the 4-th equality holds because

$$\sum_{j=0}^{p+1} (-1)^j \gamma_{ii_0 \dots \hat{i}_j \dots i_{p+1}} = -(\check{\delta}\gamma)_{ii_0 \dots i_{p+1}} + \gamma_{i_0 \dots i_{p+1}}.$$

Therefore, we have β such that $\check{\delta}\beta = 0$, $\iota^* \beta = 0$ and $d\beta = \alpha$, as desired.

Step 2: Suppose α is as in the condition of \mathcal{P}_p^1 . the argument at the beginning of Step 1 says there is β such that $\iota^* \beta = 0$ and $d\beta = \alpha$. Since α consists of 1-forms, β , as well as $\check{\delta}\beta$, consist of smooth functions. Since $d\check{\delta}\beta = \check{\delta}d\beta = 0$ and the domain of each $(\check{\delta}\beta)_{i_0 \dots i_p}$, if not empty, is connected by condition 1, $(\check{\delta}\beta)_{i_0 \dots i_p}$ are constant functions. Also $\iota^* \check{\delta}\beta = \check{\delta}\iota^* \beta = 0$, so, by condition 2 in the lemma, each $(\check{\delta}\beta)_{i_0 \dots i_p}$ with non-empty domain is 0 at at least one point, hence must be 0. Therefore $\check{\delta}\beta = 0$ and β satisfies the requirement in \mathcal{P}_p^1 . \square

Lemma A.7. Suppose ι, N, M, \mathcal{U} are as in the condition of Lemma A.6; $p \geq 0, q \geq 1$. Suppose $\alpha \in \check{C}_{\mathcal{U}}^p(M; \mathcal{A}^q)$ satisfies $d\alpha = 0$ and $\check{\delta}\iota^* \alpha = 0$, then there exists $\alpha' \in \check{C}_{\mathcal{U}}^p(M; \mathcal{A}^q)$ such that $d\alpha' = 0$, $\check{\delta}\alpha' = 0$ and $\iota^* \alpha' = \iota^* \alpha$.

Proof. We have $d(\check{\delta}\alpha) = \check{\delta}d\alpha = 0$, $\check{\delta}(\check{\delta}\alpha) = 0$, and $\iota^*(\check{\delta}\alpha) = \check{\delta}\iota^* \alpha = 0$. So, applying \mathcal{P}_{p+1}^q to $\check{\delta}\alpha$ we obtain $\beta \in \check{C}_{\mathcal{U}}^{p+1}(M; \mathcal{A}^{q-1})$ such that $\check{\delta}\beta = 0$, $\iota^* \beta = 0$ and $d\beta = \check{\delta}\alpha$. Again, let $\{f_i : U_i \rightarrow [0, 1]\}_{i \in I}$ be a partition of unity subordinate to \mathcal{U} and we define α' by taking

$$\alpha'_{i_0 \dots i_p} = \alpha_{i_0 \dots i_p} - \sum_{i \in I} d(f_i \cdot \beta_{ii_0 \dots i_p}),$$

then $d\alpha' = 0$ and $\iota^* \alpha' = \iota^* \alpha - \sum_{i \in I} d((f_i \circ \iota) \cdot \iota^* \beta_{ii_0 \dots i_p}) = \iota^* \alpha$. And, similar to (11),

$$\begin{aligned} (\check{\delta}\alpha - \check{\delta}\alpha')_{i_0 \dots i_{p+1}} &= \sum_{j=0}^{p+1} (-1)^j (\alpha - \alpha')_{i_0 \dots \hat{i}_j \dots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \sum_{i \in I} d(f_i \cdot \beta_{ii_0 \dots \hat{i}_j \dots i_{p+1}}) \\ &= d \sum_{i \in I} \left(f_i \cdot \sum_{j=0}^{p+1} (-1)^j \beta_{ii_0 \dots \hat{i}_j \dots i_{p+1}} \right) = d \sum_{i \in I} f_i \cdot (\beta_{i_0 \dots i_{p+1}} - (\check{\delta}\beta)_{ii_0 \dots i_{p+1}}) = d\beta_{i_0 \dots i_{p+1}}, \end{aligned}$$

Therefore, $\check{\delta}\alpha' = \check{\delta}\alpha - d\beta = 0$, as desired. \square

Lemma A.8. *Suppose M is a compact manifold with embedded faces. Then, there exist an open neighborhood U of ∂M , a finite open cover \mathcal{U} of U , such that \mathcal{U} satisfies the condition in Lemma A.7 when we plug in ∂M for N , U for M , and the inclusion map for ι .*

Proof. By [3, Theorem 17], M has a system of compatible collar neighborhoods ([3, Definition 35]).¹⁴ Fix a metric on M . For a point p in a depth- k strata S , we take a convex neighborhood U'_p of p in S and define $U_p = U'_p \times [0, \epsilon)^k \subset M$ (here we implicitly use the identification given by the collar neighborhood $S \times [0, \epsilon)^k \rightarrow M$). Then, for any finite set of points $\{p_i\}$ in ∂M , $\cap_i U_{p_i}$ is diffeomorphic to $(\mathbb{R}^{\geq 0})^d \times \mathbb{R}^{\dim M - d}$ for some d , and $(\cap_i U_{p_i}) \cap \partial M \neq \emptyset$. So, by Lemma A.5, $\cap_i U_i$ satisfied condition 3 for $U_{I'}$ imposed in Lemma A.6. Take \mathcal{U} to be a finite subcover of $\{U_p\}_{p \in \partial M}$ and $U = \bigcup_{U_p \in \mathcal{U}} U_p$. \square

Proof of Proposition A.1. Let U, \mathcal{U} be given as in Lemma A.8. Then, for every $V \in \mathcal{U}$, by Lemma A.5, we can find a closed form ω_V on V such that $\omega_V|_{S \cap V} = \omega_S|_{S \cap V}$ for all strata S of ∂M . The collection $\{\omega_V\}_{V \in \mathcal{U}}$ defines an $\alpha \in \check{C}^0_{\mathcal{U}}(M; \mathcal{A}^k)$, which satisfies the condition of Lemma A.7. So, there exists α' as in the conclusion of Lemma A.7. To get to the conclusion of Proposition A.1, take $\omega = \alpha'$. \square

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¹⁴(Delete! This is wrong!!!) In our application M is \tilde{C}_n , and its charts that we explicitly constructed in Section 2 is actually a system of compatible collar neighborhoods.