1 Introduction

.....(introduction)

Define the kind of bundle we work with in this paper: Given a smooth homology sphere M, define a *framed* (M, ∞)*-bundle* $(\pi : E \to B, \sigma, \tau, F)$ (abbreviate all these to π) to be a smooth fiber bundle $\pi : E \to B$ with fiber M, with a smooth section σ, a trivialization τ of the bundle near σ, and a smooth vertical framing F of π "standard" near σ.

Define the bracket operation, $\pi_1, \pi_2 \to [\pi_1, \pi_2]$ on such bundles, in an intuitively clear but not necessarily rigorous way.

Define cobracket and coproduct on graph cohomology (everything is over $\mathbb{Q}$):

- First, define the graph complex $\mathcal{G}$ (the coboundary operator δ is given by contracting an edge). Require all the graphs to be connected.
- We will be working with the quotient $\mathcal{G}' = \mathcal{G}/X$, where X is the subspace of $\mathcal{G}$ spanned by graphs containing either a univalent vertex or a simple loop (an edge starting and ending at the same vertex).
- Taking the homology of $\mathcal{G}'$ with respect to δ, denote by $H^*\mathcal{G}'$.
- Define the cobracket operation to be the linear map

$$\begin{aligned} \Delta_{[,]} : \mathcal{G}' &\longrightarrow \mathcal{G}' \otimes \mathcal{G}' \\ \Gamma &\longrightarrow \sum_{\Gamma' \leq \Gamma} \left(\Gamma' \otimes \Gamma/\Gamma' + (-1)^{\cdot\cdot} \Gamma/\Gamma' \otimes \Gamma' \right), \end{aligned}$$

  where Γ' ranges through all full subgraphs of Γ that is connected, with no univalent vertex or simple loop.
- Check that $\Delta_{[,]}$ commutes with δ and $\delta \otimes \mathrm{id} \pm \mathrm{id} \otimes \delta$, so it descends to

$$\Delta_{[,]} : H^*\mathcal{G}' \longrightarrow H^*(\mathcal{G}' \otimes \mathcal{G}') \approx H^*\mathcal{G}' \otimes H^*\mathcal{G}'.$$

- Finally we also define the coproduct operation on $\mathcal{G}'$ (this makes more sense for disconnected graphs but w=for connected graphs it is extra simple):

$$\begin{aligned} \Delta_{\cdot} : \mathcal{G}' &\longrightarrow \mathcal{G}' \otimes \mathcal{G}' \\ \Gamma &\longrightarrow \Gamma \otimes (\text{the empty graph}) + (\text{the empty graph}) \otimes \Gamma. \end{aligned}$$

Brief introduction to Kontsevich's characteristic classes. Given a framed (M, ∞)-bundle $\pi : E \to B$ as above, denote by

$$K_\pi : H^*(\mathcal{G}') \longrightarrow H^*(B)$$

Kontsevich's characteristic classes of π.

Theorem 1.1. *Suppose $d \geq 3$. For $i = 1, 2$, suppose M_i is a d-dimensional smooth homology sphere and suppose $\pi_i : E_i \to B_i$ is a framed (M, ∞)-bundle. (Now, $[\pi_1, \pi_2] : E \to S^d \times B_1 \times B_2$ is the bracket bundle.) Then, for all $\eta \in H^*\mathcal{G}'$,*

$$K_{[\pi_1,\pi_2]}(\eta) = \mathrm{PD}_{S^d}[S^d] \otimes (K_{\pi_1} \otimes K_{\pi_2})(\Delta_{[,]}(\eta)) + \mathrm{PD}_{S^d}[pt] \otimes (K_{\pi_1} \otimes K_{\pi_2})(\Delta_{\cdot}(\eta)).$$

(Both LHS and RHS lives in

$$H^*(S^d \times B_1 \times B_2) \approx H^*(S^d) \otimes H^*(B_1) \otimes H^*(B_2).$$

PD_{S^d} *means Poincaré dual on S^d; $[S^d]$ stands for the fundamental class of S^d and $[pt]$ stands for the point class of S^d.)*

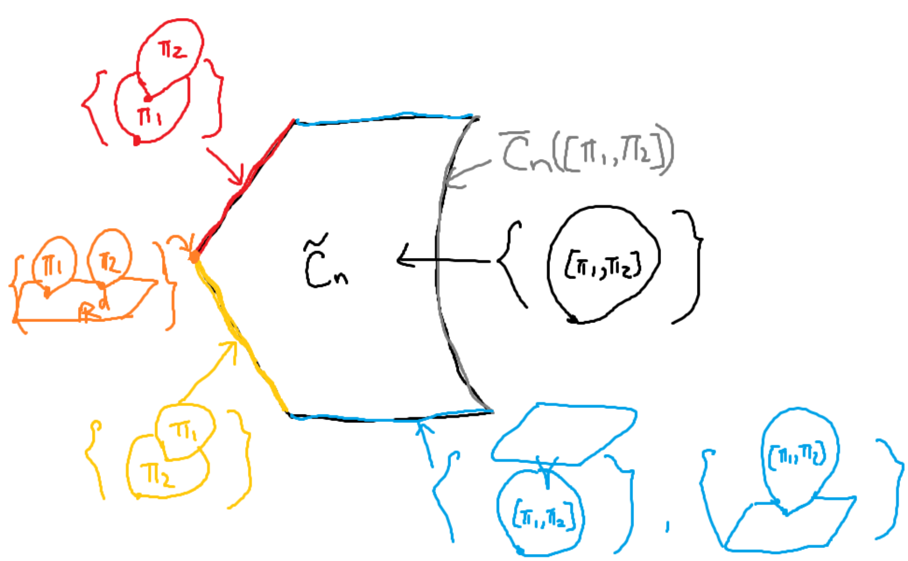
......(Then talk about the $(d+1)$-fold loop space structure on $\mathrm{BDiff}^{\mathrm{fr}}_{\partial}(D^d)$ and the theorem/corollary that it doesn't extend.)

(Below is an outline of the proof of Theorem 1.1. Throughout, π_1, π_2 are given and fixed.)

2 Conftilde

Construct the big configuration space $\widetilde{C}_n$. Show that it is a smooth manifold with boundary and corners. (These are mostly already written in the file "conftilde" I sent a while ago.)

Here is a schematic picture of $\widetilde{C}_n$ (the marked points are not drawn):



The boundary of $\widetilde{C}_n$ consists of 3 parts:

- the gray part is $\overline{C}_n([\pi_1, \pi_2])$;

- the blue part is a 1-parameter family of $\partial\overline{C}_n([\pi_1,\pi_2])$—its interior is diffeomorphic to

$$\partial\overline{C}_n([\pi_1,\pi_2])\times(0,1);$$

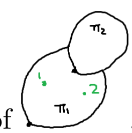
- $\overline{C}_n^*$, consisting of the red, orange, and yellow parts.

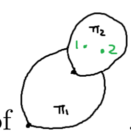
3 Propagators

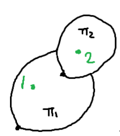
Fix a volume form ω_0 on S^{d-1}. For $i=1,2$, fix a propagator ω_i on $\overline{C}_2(\pi_i)$ such that on $\partial\overline{C}_2(\pi_i)$ ω_i is given by ω_0.

3.1 Propagator on confstar

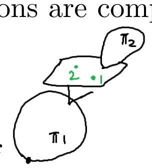
ω_1,ω_2 naturally induce a propagator ω_* on $\overline{C}_2^*$. On the main strata of $\overline{C}_2^*$, define ω_* as follows:

- On the stratum S_{11} consisting of 

, there is the forgetful map $f_{11}:S_{11}\to\overline{C}_2(\pi_1)$. We define $\omega_*|_{S_{11}}=f_{11}^*(\omega_1)$.
- On the stratum S_{22} consisting of 

, there is the forgetful map $f_{22}:S_{22}\to\overline{C}_2(\pi_2)$. We define $\omega_*|_{S_{22}}=f_{22}^*(\omega_2)$.
- On the stratum S_{12} consisting of 

, there is the forgetful map $f_{12}:S_{12}\to\overline{C}_2(\pi_1)$, where the second marked point is considered to be located at the node on π_1. We define $\omega_*|_{S_{22}}=f_{22}^*(\omega_2)$.
- ...(There are 5 more situations, all variations of the above, permuting $\pi_1,\pi_2,1,2$.)

We then check that the above definitions are compatible when the main strata glue together. For example, the stratum S' consisting of 

lies in the intersection of $\overline{S}_{11}$ and $\overline{S}_{22}$; but the ω_* constructed on S_{11} and on S_{22}, when extended to here, agree, since they both become $(f')^*\omega_0$, where $f':S'\to S^{d-1}$ is the forgetful map recording the direction of the vector between the two points.

All other codimension-1 strata can be straight-forwardly checked like this as well. We may need to check higher-codimension strata as well but it shouldn't be hard.

Remark 3.1. I lied a bit—"compatibility" of ω_* along the gluing is a stronger condition then it is presented above. Given two manifolds with boundary, X, Y, such that $\partial X \approx \partial Y =: W$, let Z be the space obtained by gluing X, Y together along their common boundary W, but we do not specify a smooth structure near W; then, say a form α on X is compatible with a form β on Y if

- $\alpha|_W = \beta|_W$;
- there are collar neighborhoods N_X of W in X and N_Y of W in Y (with projection maps $p_X : N_X \to W$, $p_Y : N_Y \to W$) such that $\alpha|_{N_X} = p_X^* \alpha|_W$, $\beta|_{N_Y} = p_Y^* \beta|_W$.

The second condition says that in a collar neighborhood of the boundary the forms are trivial in the normal direction. In our situation, this stronger compatibility condition should be achievable as well, if we impose a similar condition ("parallel to boundary" near the boundary) on ω_1, ω_2.

3.2 Propagator on other parts of the boundary of conftilde

To define a propagator on the blue part of $\partial\widetilde{C}_2$, we have no choice but to define it as induced from ω_0 in the usual way.

To define a propagator in the gray part of $\partial\widetilde{C}_2$, we just choose any propagator ω on $\widetilde{C}_2([\pi_1, \pi_2])$, so that $\omega|_{\partial\widetilde{C}_2([\pi_1,\pi_2])}$ is induced from ω_0.

Now we have defined a propagator on all of $\partial\widetilde{C}_2$.

3.3 Extend the propagator to the interior

To show that there exists a closed form on $\widetilde{C}_2$ extending the propagator we just defined on $\partial\widetilde{C}_2$, it suffices to show that the map

$$H^{d-1}(\widetilde{C}_2) \xrightarrow{\text{restriction}} H^{d-1}(\partial\widetilde{C}_2)$$

is surjective. It is therefore sufficient to show that $H^d(\widetilde{C}_2, \partial\widetilde{C}_2) = 0$. But

$$\begin{aligned} H^*(\widetilde{C}_2, \partial\widetilde{C}_2) &\approx H_{\dim(\widetilde{C}_2)-*}(\widetilde{C}_2 - \partial\widetilde{C}_2) = H_{\dim(\widetilde{C}_2)-*}\big(C_2([\pi_1,\pi_2]) \times (0,1)\big) \\ &\approx H_{\dim(\widetilde{C}_2)-*}(C_2([\pi_1,\pi_2])) \approx H^{*-1}\big(\overline{C}_2([\pi_1,\pi_2]), \partial\overline{C}_2([\pi_1,\pi_2])\big), \end{aligned}$$

and this is 0 when $* - 1 < d + 1$; see the proof of Lemma 2.12 in Watanabe's addendum paper.

We choose and fix such an extension. This gives us a propagator $\widetilde{\omega}$ on $\widetilde{C}_2$.

4 Configuration space integrals

For simplicity here we will only sketch part of the proof of Theorem 1.1, namely we only justify the first term on the RHS. (The second term is slightly more complicated but not much: basically because we only work with the point class on S^d—it is only 0-dimensional so you can still easily achieve transversality as needed. The second term is not needed for the corollary about the loop space structure on $\mathrm{BDiff}^{\mathrm{fr}}_\partial(D^d)$ either.)

Suppose Γ is a cocycle in graph cohomology. (For the simplicity of notation we assume Γ is one graph—when it is a formal sum everything works as well.) Suppose Γ has n vertices numbered $1, \ldots, n$.

For every edge e of Γ, we have the forgetful map

$$f_e : \widetilde{C}_n \longrightarrow \widetilde{C}_2.$$

And, when restricted to the gray (resp. red, orange, and yellow) part of $\partial\widetilde{C}_n$, it is the forgetful map

$$f_e : \overline{C}_n([\pi_1, \pi_2]) \longrightarrow \overline{C}_2([\pi_1, \pi_2]) \qquad (\text{resp. } f_e : \overline{C}^*_n \longrightarrow \overline{C}^*_2).$$

Now we have the form $\bigwedge_e f_e^*\widetilde{\omega}$ on $\widetilde{C}_n$, where e ranges over all edges of Γ. When restricted to the gray part of $\partial\widetilde{C}_n$, it is used to define Kontsevich's classes for the bundle $[\pi_1, \pi_2]$.

To compute the $(\mathrm{PD}_{S^d}[S^d])$-part of $K_{[\pi_1,\pi_2]}([\Gamma])$, we only need to compute, given arbitrary homology classes $\alpha_1 \in H_*(B_1)$ and $\alpha_2 \in H_*(B_2)$, the evaluation

$$\big\langle K_{[\pi_1,\pi_2]}([\Gamma]), [S^d] \otimes \alpha_1 \otimes \alpha_2 \big\rangle.$$

For $i = 1, 2$, suppose α_i is represented by a sub-pseudo-manifold $\iota_i : B'_i \hookrightarrow B_i$. (For simplicity you can think of a smooth submanifold instead.)

(Notice that the projection map $\widetilde{p} : \widetilde{C}_n \to B_1 \times B_2$ is a fiber bundle, and so is the restriction of $\widetilde{p}$ to each stratum of $\widetilde{C}_n$.) Let us pull everything back along (ι_1, ι_2), so that the base gets changed to $B'_1 \times B'_2$ instead of $B_1 \times B_2$. Abusing notation, we still denote by $\widetilde{C}_n, \overline{C}_n([\pi_1, \pi_2]), \overline{C}^*_n$ their pull-backs. Now, we have

$$\big\langle K_{[\pi_1,\pi_2]}([\Gamma]), [S^d] \otimes \alpha_1 \otimes \alpha_2 \big\rangle = \int_{\overline{C}_n([\pi_1,\pi_2])} \bigwedge_e f_e^*\omega = \int_{\text{gray part of } \partial\widetilde{C}_n} \bigwedge_e f_e^*\widetilde{\omega}. \tag{1}$$

By Stocks' Formula (and the fact that $\widetilde{\omega}$ is closed),

$$\int_{\partial\widetilde{C}_n} \bigwedge_e f_e^*\widetilde{\omega} = \int_{\widetilde{C}_n} d\Big(\bigwedge_e f_e^*\widetilde{\omega}\Big) = 0,$$

so,

$$(1) = \int_{\text{blue part of } \partial\widetilde{C}_n} \bigwedge_e f_e^*\widetilde{\omega} + \int_{\overline{C}^*_n} \bigwedge_e f_e^*\omega_*.$$

Since Γ is a cocycle in graph cohomology, the first term is 0 just like in the proof of the well-definedness of Kontsevich's classes, so

$$\big\langle K_{[\pi_1,\pi_2]}([\Gamma]), [S^d] \otimes \alpha_1 \otimes \alpha_2 \big\rangle = \int_{\overline{C}^*_n} \bigwedge_e f_e^*\omega_*.$$

5 Configuration space integral on confstar

We continue with the notation from last section (in particular, everything is over B_1', B_2' instead of B_1, B_2).

It remains to show that

$$\left\langle (K_{\pi_1} \otimes K_{\pi_2})(\Delta_{[,]}[\Gamma]), \alpha_1 \otimes \alpha_2 \right\rangle = \int_{\overline{C}_n^*} \bigwedge_e f_e^* \omega_*. \tag{2}$$

For a graph G, we denote by $V(G)$ its vertex set and $E(G)$ its edge set. The LHS above equals to[1]

$$\sum_{\Gamma' \le \Gamma} \Big(\int_{\overline{C}_{V(\Gamma')}(\pi_1)} \bigwedge_{e \in E(\Gamma')} f_e^* \omega_1 \Big) \cdot \Big(\int_{\overline{C}_{V(\Gamma/\Gamma')}(\pi_2)} \bigwedge_{e \in E(\Gamma/\Gamma')} f_e^* \omega_2 \Big)$$
$$\pm \Big(\int_{\overline{C}_{V(\Gamma/\Gamma')}(\pi_1)} \bigwedge_{e \in E(\Gamma/\Gamma')} f_e^* \omega_1 \Big) \cdot \Big(\int_{\overline{C}_{V(\Gamma')}(\pi_2)} \bigwedge_{e \in E(\Gamma')} f_e^* \omega_2 \Big).$$

To prove (2), it suffices to show that

$$\sum_{\Gamma' \le \Gamma} \Big(\int_{\overline{C}_{V(\Gamma')}(\pi_1)} \bigwedge_{e \in E(\Gamma')} f_e^* \omega_1 \Big) \cdot \Big(\int_{\overline{C}_{V(\Gamma/\Gamma')}(\pi_2)} \bigwedge_{e \in E(\Gamma/\Gamma')} f_e^* \omega_2 \Big) = \int_{\text{yellow part of } \overline{C}_n^*} \bigwedge_{e \in E(\Gamma)} f_e^* \omega_* \tag{3}$$

and

$$\sum_{\Gamma' \le \Gamma} \Big(\int_{\overline{C}_{V(\Gamma/\Gamma')}(\pi_1)} \bigwedge_{e \in E(\Gamma/\Gamma')} f_e^* \omega_1 \Big) \cdot \Big(\int_{\overline{C}_{V(\Gamma')}(\pi_2)} \bigwedge_{e \in E(\Gamma')} f_e^* \omega_2 \Big) = \int_{\text{red part of } \overline{C}_n^*} \bigwedge_{e \in E(\Gamma)} f_e^* \omega_*; \tag{4}$$

here the "red" and "yellow" refers to the $\widetilde{C}_n$ picture. Below we only prove (3) since (4) is completely similar.

The yellow part of $\overline{C}_n^*$ is simply

$$\sum_{V_1, V_2 : V_1 \sqcup V_2 = V(\Gamma)} \overline{C}_{V_1}(\pi_1) \times \overline{C}_{V_2 \sqcup \{\star\}}(\pi_2),$$

where $\star$ records the position of the node on π_2. Therefore, the RHS of (3) is

$$\sum_{V_1, V_2 : V_1 \sqcup V_2 = V(\Gamma)} \int_{\overline{C}_{V_2 \sqcup \{\star\}}(\pi_2)} \int_{\overline{C}_{V_1}(\pi_1)} \bigwedge_{\substack{e \in E(\Gamma) \\ \text{both endpoints of } e \text{ are in } V_1}} f_e^* \omega_* \wedge \bigwedge_{\substack{e \in E(\Gamma) \\ \exists \text{ endpoint of } e \text{ in } V_2}} f_e^* \omega_*.$$

For $V_1 \subset V(\Gamma)$, we denote by $\Gamma'(V_1)$ the subgraph of Γ spanned by vertices in V_1. Then, by the way ω_* is constructed, and by Fubini's Theorem, the above equals to

$$\sum_{V_1, V_2 : V_1 \sqcup V_2 = V(\Gamma)} \Big(\int_{\overline{C}_{V_1}(\pi_1)} \bigwedge_{e \in E(\Gamma'(V_1))} f_e^* \omega_1 \Big) \cdot \Big(\int_{\overline{C}_{V_2 \sqcup \{\star\}}(\pi_2)} \bigwedge_{e \in E(\Gamma/\Gamma'(V_1))} f_e^* \omega_2 \Big).$$

[1] A little more argument needed for this claim, but it is true.