1 Introduction

.....(introduction)

Define the kind of bundle we work with in this paper: Given a smooth homology sphere M, define a *framed* (M,∞)*-bundle* $(\pi : E \to B, \sigma, \tau, F)$ (abbreviate all these to π) to be a smooth fiber bundle $\pi : E \to B$ with fiber M, with a smooth section σ, a trivialization τ of the bundle near σ, and a smooth vertical framing F of π "standard" near σ.

Define the bracket operation, $\pi_1, \pi_2 \to [\pi_1, \pi_2]$ on such bundles, in an intuitively clear but not necessarily rigorous way.

Define cobracket and coproduct on graph cohomology (everything is over $\mathbb{Q}$):

- First, define the graph complex $\mathcal{G}'$—the $\mathbb{Q}$-vector space spanned by (with correct orientation definition, omitted here) connected graphs containing either a univalent vertex or a simple loop (an edge starting and ending at the same vertex). The coboundary operation δ is given by contracting an edge. In δ and all the operations on graphs below, whenever a graph not in $\mathcal{G}'$ appears (a graph that has a univalent vertex or simple loop), we set it to 0.
- Taking the homology of $\mathcal{G}'$ with respect to δ, denote by $H^*\mathcal{G}'$.
- Define the cobracket operation to be the linear map

$$\begin{aligned}\Delta_{[,]} : \mathcal{G}' &\longrightarrow \mathcal{G}' \otimes \mathcal{G}' \\ \Gamma &\longrightarrow \sum_{\Gamma' \leq \Gamma} \left(\Gamma' \otimes \Gamma/\Gamma' + (-1)^{\cdot\cdot} \Gamma/\Gamma' \otimes \Gamma' \right),\end{aligned}$$

  where Γ' ranges through all full subgraphs of Γ that is connected, with no univalent vertex or simple loop.
- Check that $\Delta_{[,]}$ commutes with δ and $\delta \otimes \mathrm{id} \pm \mathrm{id} \otimes \delta$, so it descends to

$$\Delta_{[,]} : H^*\mathcal{G}' \longrightarrow H^*(\mathcal{G}' \otimes \mathcal{G}') \approx H^*\mathcal{G}' \otimes H^*\mathcal{G}'.$$

- Finally we also define the coproduct operation on $\mathcal{G}'$ (this makes more sense for disconnected graphs but w=for connected graphs it is extra simple):

$$\begin{aligned}\Delta_{\cdot} : \mathcal{G}' &\longrightarrow \mathcal{G}' \otimes \mathcal{G}' \\ \Gamma &\longrightarrow \Gamma \otimes (\text{the empty graph}) + (\text{the empty graph}) \otimes \Gamma.\end{aligned}$$

Brief introduction to Kontsevich's characteristic classes. Given a framed (M,∞)-bundle $\pi : E \to B$ as above, denote by

$$K_\pi : H^*(\mathcal{G}') \longrightarrow H^*(B)$$

Kontsevich's characteristic classes of π.

Theorem 1.1. *Suppose $d \geq 3$. For $i = 1, 2$, suppose M_i is a d-dimensional smooth homology sphere and suppose $\pi_i : E_i \to B_i$ is a framed (M, ∞)-bundle. (Now, $[\pi_1, \pi_2] : E \to S^d \times B_1 \times B_2$ is the bracket bundle.) Then, for all $\eta \in H^*\mathcal{G}'$,*

$$K_{[\pi_1,\pi_2]}(\eta) = \mathrm{PD}_{S^d}[S^d] \otimes (K_{\pi_1} \otimes K_{\pi_2})(\Delta_{[,]}(\eta)) + \mathrm{PD}_{S^d}[pt] \otimes (K_{\pi_1} \otimes K_{\pi_2})(\Delta_{\cdot}(\eta)).$$

(Both LHS and RHS lives in

$$H^*(S^d \times B_1 \times B_2) \approx H^*(S^d) \otimes H^*(B_1) \otimes H^*(B_2).$$

PD_{S^d} *means Poincaré dual on S^d; $[S^d]$ stands for the fundamental class of S^d and $[pt]$ stands for the point class of S^d.)*

......(Then talk about the $(d+1)$-fold loop space structure on $\mathrm{BDiff}^{\mathrm{fr}}_\partial(D^d)$ and the theorem/corollary that it doesn't extend.)

(Below is an outline of the proof of Theorem 1.1. Throughout, π_1, π_2 are given and fixed.)

1.1 Notation

Given a graph G, we denote by $V(G)$ its vertex set and $E(G)$ its edge set.

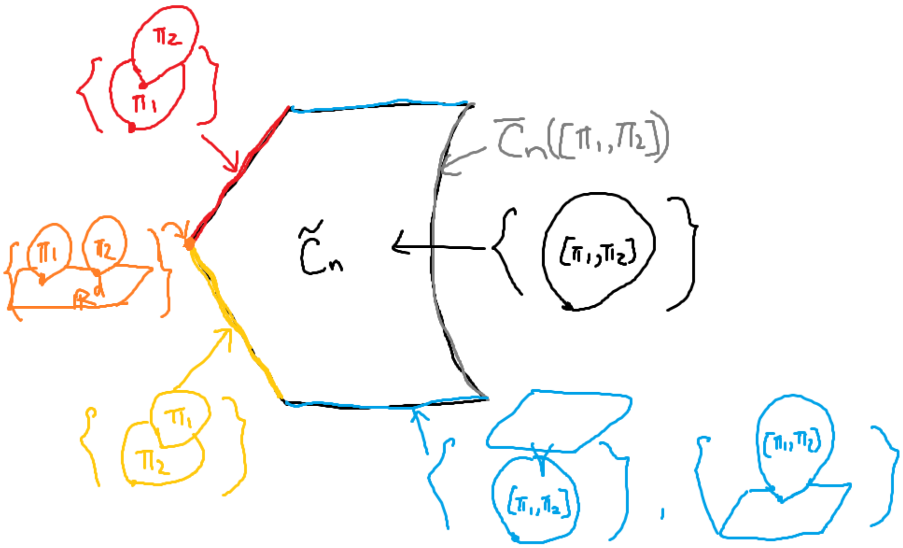
2 Conftilde

Construct the big configuration space $\widetilde{C}_A$. Show that it is a smooth manifold with boundary and corners. (These are mostly already written in the file "conftilde" I sent a while ago.)

What we need are the following:

- $\widetilde{C}_A$ is a smooth manifold with boundary and corners;
- each S_T is a stratum of $\widetilde{C}_A$;
- $\overline{S}_T = \bigsqcup_{T'} S_{T'}$, where the disjoint union is taken over all A-labeled trees T' such that T can be obtained from T' by contracting some edges.

Here is a schematic picture of $\widetilde{C}_A$ (the marked points are not drawn; the actual stratification structure of $\widetilde{C}_A$ is more complicated than what is shown in the picture):



The boundary of $\widetilde{C}_A$ consists of the following parts:

- the gray part, denoted by $\overline{S}_{\text{gray}}$, is $\overline{C}_A([\pi_1, \pi_2])$; its interior, denoted by S_{gray}, is $C_A([\pi_1, \pi_2])$;
- $S_{\text{blue}} := \bigcup_{T \in \mathcal{T}_{\text{blue}}} S_T$, where $\mathcal{T}_{\text{blue}}$ is the set of all A-labeled trees whose shape and space labels are like $\begin{array}{l} \bullet\ \mathbb{R}^d \\ \vert \\ \bullet\ [\pi_1, \pi_2] \\ \mathrm{r} \end{array}$ or $\begin{array}{l} \bullet\ [\pi_1, \pi_2] \\ \vert \\ \bullet\ \mathbb{R}^d \\ \mathrm{r} \end{array}$; and let $\overline{S}_{\text{blue}}$ be the closure of S_{blue};
- $S_{\text{red}} := \bigcup_{T \in \mathcal{T}_{\text{red}}} S_T$, where $\mathcal{T}_{\text{red}}$ is the set of all A-labeled trees with the following shape and space labels: $\begin{array}{l} \bullet\ \pi_2 \\ \vert \\ \bullet\ \pi_1 \\ \mathrm{r} \end{array}$; and let $\overline{S}_{\text{red}}$ be the closure of S_{red};
- $S_{\text{yellow}} := \bigcup_{T \in \mathcal{T}_{\text{yellow}}} S_T$, where $\mathcal{T}_{\text{yellow}}$ is the set of all A-labeled trees with the following shape and space labels: $\begin{array}{l} \bullet\ \pi_1 \\ \vert \\ \bullet\ \pi_2 \\ \mathrm{r} \end{array}$; and let $\overline{S}_{\text{yellow}}$ be the closure of S_{yellow};

We also define $S_{\text{orange}} := \bigcup_{T \in \mathcal{T}_{\text{orange}}} S_T$, where $\mathcal{T}_{\text{orange}}$ is the set of all A-labeled trees with the following shape and space labels: $\begin{array}{c} \pi_1 \bullet \quad \bullet\ \pi_2 \\ \diagdown\ \diagup \\ \bullet\ \mathbb{R}^d \\ \mathrm{r} \end{array}$; and let $\overline{S}_{\text{orange}}$ be the closure of S_{orange}. Then, $\overline{S}_{\text{orange}} = \overline{S}_{\text{red}} \cap \overline{S}_{\text{yellow}}$.

We define $\overline{C}^*_A = \overline{S}_{\text{red}} \cup \overline{S}_{\text{yellow}}$.

3 Propagators

Before starting the discussion on propagators, we first define another notion of "forgetful map".

Given finite sets A ("set of point labels") and B ("set of space labels"), recall the definition of an (A,B)-labeled tree in ...(change the definition in conftilde to allow arbitrary space labels!)

In all the cases we care about, the elements of B will be (M,∞)-bundles for some d-dimensional manifold M.

In this paper we only consider cases when $|B| \le 2$. Define

$$\widetilde{C}_A(B) = \begin{cases} \overline{C}_A(\mathbb{R}^d) \text{ if } B = \emptyset, \\ \overline{C}_A(\pi) \text{ if } B = \{\pi\} \text{ for some } (M,\infty)-\text{bundle } \pi, \\ \widetilde{C}_A \backslash \overline{S}_{\text{gray}} \text{ if } B = \{\pi_1, \pi_2\} \text{ for some } (M_1,\infty)\text{-bundle } \pi_1 \text{ and some } (M_2,\infty)\text{-bundle } \pi_2. \end{cases}$$

Then, the strata of $\widetilde{C}_A(B)$ are in 1-to-1 correspondence with (A,B)-labeled trees. Given such a stratum S, we denote by $\mathcal{T}_S$ the tree corresponding to it and given such a tree T we denote by $\mathcal{S}_T$ the stratum corresponding to it. [1] The condition $\mathcal{S}_{T'} \subset \overline{\mathcal{S}}_T$ is equivalent to that T can be obtained from T' by contracting some edges. In this case, the set of edges to be contracted to get from T' to T is unique and we denote by $\mathfrak{c}_{T',T} : V(T') \to V(T)$ the map on the vertices induced by the contraction. Also define $\mathfrak{i}_{T',T} : V(T) \to V(T')$ mapping v to the lowest vertex in $\mathfrak{c}^{-1}_{T',T}(v)$. The following lemma is immediate:

Lemma 3.1. *Let T', T be (A,B)-labeled trees such that T can be obtained from T' by contracting some edges. Then, for every $v \in V(T)$, $lp(\ge v) = lp(\ge \mathfrak{i}_{T',T}(v))$.*

Maybe move the above to an earlier section devoted to combinatorics.

For the rest of this section we only consider the case $A = \{1,2\}$. Let B be a finite set such that every element of B is an (M,∞)-bundles for some d-dimensional manifold M, and $|B| \le 2$.

Definition 3.2. Let T be a $(\{1,2\},B)$-labeled tree, [2] then $\mathcal{S}_T \approx \prod_{v\in V(T)} C_{lp(v)\cup cld(v)}(ls(v))$ is a stratum in $\widetilde{C}_{\{1,2\}}(B)$. Define $\nu_T \in V(T)$ to be the vertex such that $\{1,2\} \subset lp(\ge \nu_T)$ and for all $v > \nu_T$, $\{1,2\} \not\subset lp(\ge v)$.[3] Define $\mathfrak{s}_T := ls(\nu_T)$. [4]

- Define
$$\hat{f}_T : \mathcal{S}_T \longrightarrow C_2(\mathfrak{s}_T), \qquad \hat{f}_T\big((c_v)_{v\in V(T)}\big) = c'_{\nu_T},$$
where $c'_{\nu_T} \in C_2(\mathfrak{s}_T))$ is obtained from c_{ν_T} by forgetting all the points except for two: $f_{\nu_T}(1)$ and $f_{\nu_T}(2)$. (f_v is defined in conftilde, at the beginning of Section 3.3.)

- Suppose T' is a $(\{1,2\},B)$-labeled tree such that T can be obtained from T' by contracting some edges. Abusing notation, we denote the subtree of T' spanned by vertices in $\mathfrak{c}^{-1}_{T',T}(\nu_T)$ still by $\mathfrak{c}^{-1}_{T',T}(\nu_T)$. Define $G_{T',T}$ to be the tree obtained from $\mathfrak{c}^{-1}_{T',T}(\nu_T)$ by "stabilization with respect to $\{1,2\}$ and $\mathfrak{s}_T$", namely: let $V' \subset V(\mathfrak{c}^{-1}_{T',T}(\nu_T))$ ("set of unstable vertices") consist of vertices v such that $\text{lsset}(ls(v)) \cap \text{lsset}(ls(\nu)_T) = \emptyset$ and $|lp(\ge v) \cap \{1,2\}| < 2$; define $G_{T',T}$

[1] ($\mathcal{T}$ was used previously in conftilde; maybe change it to $\mathfrak{T}$ there.)

[2] The definition obviously extends to the case of an arbitrary number of marked points and forgetting to an arbitrary subset of marked points, but we only need this simple 2 point case here.

[3] (In this paper $\subset$ means subset or equal. Specify this somewhere early.)

[4] (Actually, maybe use $\mathfrak{p}, \mathfrak{s}$ instead of fp,fs?)

to be obtained from $\mathfrak{c}^{-1}_{T',T}(\nu_T)$ by: for every vertex $v \in V'$, contracting the edge just below v. Then, $\mathcal{S}_{G_{T',T}}$ is a stratum of $\overline{C}_2(\mathfrak{s}_T)$. Define

$$\hat{f}_T : \mathcal{S}_{T'} \longrightarrow \mathcal{S}_{G_{T',T}} \subset \overline{C}_2(\mathfrak{s}_T), \qquad \hat{f}_T\big((c_v)_{v\in V(T')}\big) = (c'_v)_{v\in V(G_{T',T})},$$

where each $c'_v \in C_2(ls(v))$ is as follows: let $v' \in V(\mathfrak{c}^{-1}_{T',T}(\nu_T)) \subset V(T')$ be the lowest vertex contracted to v, then c'_v is obtained from $c_{v'}$ by forgetting all points except for two: $f_{v'}(1)$ and $f_{v'}(2)$.

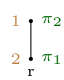
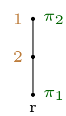
- We have therefore defined a map

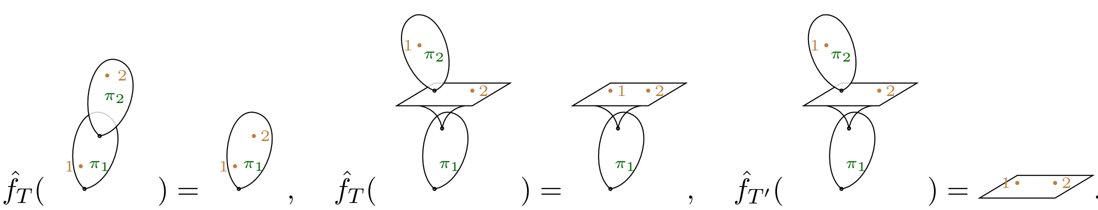
$$\hat{f}_T : \overline{\mathcal{S}}_T \longrightarrow \overline{C}_2(\mathfrak{s}_T).$$

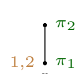
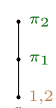
It is easy to verify that $\hat{f}_T$ is smooth using the charts we constructed in Section (conftilde section).

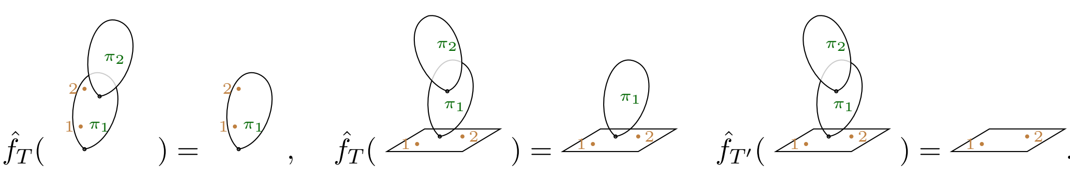
Maybe introduce more notation when talking about the combinatorics of A-labeled trees, e.g., a pre-stable tree and how to get from a pre-stable tree to a stabel tree by contraction.?

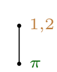
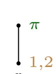
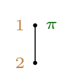
Note that if $G_{T',T}$ has only one vertex, then $\hat{f}_{T'} = \hat{f}_T|_{\overline{\mathcal{S}}_{T'}}$. Otherwise, this is not the case.

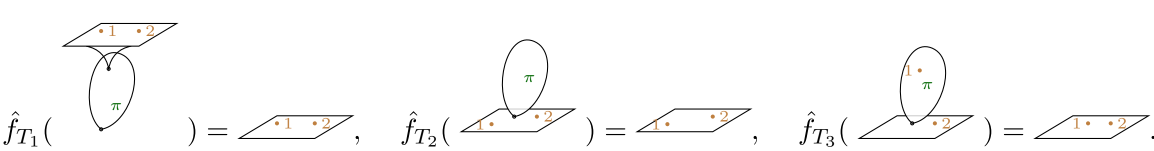
Example 3.3 ($|B| = 2$, in $\widetilde{C}_2$)**.** $T =$  , $T' =$  (so $\mathcal{S}_{T'} \subset \overline{\mathcal{S}}_T$),



Example 3.4 ($|B| = 2$, in $\widetilde{C}_2$)**.** $T =$  , $T' =$  (so $\mathcal{S}_{T'} \subset \overline{\mathcal{S}}_T$),



Example 3.5 ($|B| = 1$, in $\overline{C}_2(\pi)$ where π is an (M,∞)-bundle)**.** $T_1 =$  , $T_2 =$  , $T_3 =$  ,



Note that $S_{T_1}, S_{T_2}, S_{T_3}$ are the only codimension-1 strata of $\overline{C}_2(\pi)$, and the codomain of $\hat{f}_{T_1}$, $\hat{f}_{T_2}$, $\hat{f}_{T_3}$ are all $\overline{C}_2(\mathbb{R}^d)$.

Corollary 3.6 (of Lemma 3.1)**.** *Let T', T be $(\{1,2\}, B)$-labeled trees such that T can be obtained from T' by contracting some edges, then $\mathfrak{c}_{T',T}(\nu_{T'}) = \nu_T$. Moreover, if $\mathfrak{s}_T = \mathbb{R}^d$, then $\mathfrak{s}_{T'} = \mathbb{R}^d$.*

The following lemma is easy to check: [5]

Lemma 3.7. *Let T', T be $(\{1,2\}, B)$-labeled trees such that T can be obtained from T' by contracting some edges, then*

$$\hat{f}_{T'} = \hat{f}_{G_{T',T}} \circ (\hat{f}_T|_{\overline{\mathcal{S}}_{T'}}).$$

Definition 3.8. Suppose M is a d-dimensional $\mathbb{Z}$-homology sphere and π is an (M,∞)-bundle. A *propagator* on $\overline{C}_2(\pi)$ (resp. $\overline{C}_2(\mathbb{R}^d)$) is a closed $(d-1)$-form ω on $\overline{C}_2(\pi)$ satisfying: there exists a $(d-1)$-form ω_0 on $S^{d-1} \approx \overline{C}_2(\mathbb{R}^d)$ such that $\int_{S^{d-1}} \omega_0 = 1$ and for every codimension-1 stratum $S \subset \partial\overline{C}_2(\pi)$ (resp. $S \subset \partial\overline{C}_2(\mathbb{R})$), $\omega|_{\overline{S}} = \hat{f}^*_{T_S}\omega_0$.

This definition is phrased differently from the usual definition of a propagator, see e.g. [5, Definition 3.9] or [6, Lemma 2.12], but can easily be seen to be equivalent.

Fix a volume form ω_0 on S^{d-1}. By [6, Lemma 2.12], there exist propagators ω_1 on $\overline{C}_2(\pi_1)$, ω_2 on $\overline{C}_2(\pi_2)$, and ω on $\overline{C}_2([\pi_1,\pi_2])$ such that the above condition in Definition 3.8 is satisfied with this ω_0. We choose and fix such $\omega_1, \omega_2, \omega$.

In the rest of this section we construct a "propagator" on $\partial\widetilde{C}_2$. This is done in two steps:

1. On each stratum S of $\partial\widetilde{C}_2$, construct a closed $(d-1)$-form ω_S on $\overline{S}$, such that, for two strata $S' \subset \overline{S}$, $\omega_S|_{\overline{S'}} = \omega_{S'}$.
2. Show that this collection $\{\omega_S\}_S$ extends to a closed form $\widetilde{\omega}$ on $\widetilde{C}_2$; namely, $\widetilde{\omega}$ is such that $\widetilde{\omega}|_S = \omega_S$ for every strata $S \subset \partial\widetilde{C}_2$.

Step 1: There are two kinds of strata in $\partial\widetilde{C}_2$: those that are not subsets of $\overline{S}_{\text{gray}}$ and those that are. The former kind of strata are of the form $\mathcal{S}_T$ for some unique $(\{1,2\}, \{\pi_1,\pi_2\})$-labeled tree T with at least one edge. For these strata, define $\omega_{\mathcal{S}_T} = \hat{f}^*_T\omega_i$, where $i = 1$ if $\mathfrak{s}_T = \pi_1$, $i = 2$ if $\mathfrak{s}_T = \pi_2$, $i = 0$ if $\mathfrak{s}_T = \mathbb{R}^d$, and remove the subscript i if $\mathfrak{s}_T = [\pi_1,\pi_2]$. For the latter kind of strata: under the natural identification $\overline{S}_{\text{gray}} \approx \overline{C}_2([\pi_1,\pi_2])$, define ω_S to be the restriction of ω to S. This defines the collection $\{\omega_S\}_S$.

Now we need to show the compatibility condition: for $S' \subset \overline{S}$, $\omega_S|_{\overline{S'}} = \omega_{S'}$.

- If $S', S \subset \overline{S}_{\text{gray}}$, then this holds by definition.

[5]Hint: By continuity of the $\hat{f}$ maps, it suffices to prove the equality on the open part $\mathcal{S}_{T'}$.

- If $S', S \not\subset \overline{S}_{\text{gray}}$: since $\omega_S = \hat{f}_{\mathcal{T}_S}\omega_i$ for some propagator ω_i,

$$\omega_S|_{\overline{S}'} = (\hat{f}_{\mathcal{T}_S}|_{\overline{S}'})^*\omega_i = \begin{cases} \hat{f}^*_{\mathcal{T}_{S'}}\omega_i, \text{ if } G_{\mathcal{T}_{S'},\mathcal{T}_S} \text{ has only 1 vertex} \\ (\hat{f}_{\mathcal{T}_S}|_{\overline{S}'})^*\hat{f}^*_{G_{\mathcal{T}_{S'},\mathcal{T}_S}}\omega_0 = \hat{f}^*_{\mathcal{T}_{S'}}\omega_0, \text{ otherwise} \end{cases} = \omega_{S'},$$

  where, for the second case, the second equality is because ω_i is a propagator as in Definition 3.8 and the third equality is because of Lemma 3.9.

- If $S' \subset \overline{S}_{\text{gray}}$, $S \not\subset \overline{S}_{\text{gray}}$: let T' be the $(\{1,2\},\{[\pi_1,\pi_2]\})$-labeled tree $\mathcal{T}_{S'}$ under the identification $\overline{S}_{\text{gray}} \approx \overline{C}_2([\pi_1,\pi_2])$. We can also view T' as a $(\{1,2\},\{\pi_1,\pi_2\})$-labeled tree; to avoid confusion let us call this $(\{1,2\},\{\pi_1,\pi_2\})$-tree T. Then $\mathcal{S}_T \subset \overline{S}_{\text{blue}}$ is a stratum of $\widetilde{C}_2$, $\mathcal{S}_T \subset S$ and $\mathcal{S}_T \cap \overline{S}_{\text{gray}} = S'$. Since we have already shown $\omega_S|_{\overline{\mathcal{S}}_T} = \omega_{\mathcal{S}_T}$, it suffices to show $\omega_{\mathcal{S}_T}|_{\overline{S}'} = \omega_{S'}$. We first state a lemma:

  Lemma 3.9. *Let S be a stratum of $\partial\widetilde{C}_2$. If $S \subset \overline{S}_{\text{blue}}$ and $S \not\subset \overline{S}_{\text{gray}}$, then $\mathfrak{s}_{\mathcal{T}_S} = \mathbb{R}^d$.*

  Proof. If S is of top dimension in $\overline{S}_{\text{blue}}$, then $\mathcal{T}_S$ has one vertex with space label $[\pi_1,\pi_2]$ and only one edge. The statement of the lemma follows from the last sentence of Example 3.5. If S is not of top dimension in $\overline{S}_{\text{blue}}$, then it is in the closure of a stratum of top dimension, and the lemma follows from the last sentence of Corollary 3.6. □

  Now, because the definition of $\hat{f}$ is combinatorial and T, T' are the same tree, the two maps

$$\hat{f}_T|_{\overline{S}'} : \overline{S}' \longrightarrow \overline{C}_2(\mathbb{R}^d), \qquad \hat{f}_{T'} : \overline{S}' \longrightarrow \overline{C}_2(\mathbb{R}^d)$$

  are equal, where the first $\overline{S}'$ is viewed as a subset of the stratum $\overline{\mathcal{S}}_T$ in $\widetilde{C}_2$ while the second $\overline{S}'$ is viewed as a boundary stratum of $\overline{C}_2([\pi_1,\pi_2])$ under the identification $\overline{S}_{\text{gray}} \approx \overline{C}_2([\pi_1,\pi_2])$. Since $\omega_{\mathcal{S}_T} = \hat{f}^*_T\omega_0$ and $\omega_{S'} = \hat{f}_{T'}\omega_0$, we conclude that they are equal on $\overline{S}'$.

Step 2: The following statement (Corollary A.4) is proved in Appendix A:

Suppose M is a compact manifold with embedded faces such that $H^k(M;\mathbb{R}) \to H^k(\partial M; R)$ is surjective. Suppose for every stratum $S \subset \partial M$, ω_S is a closed k-form on S, such that, if $S' \subset \overline{S}$, then $\omega_S|_{S'} = \omega_{S'}$. Then, there exists a closed k-form $\widetilde{\omega}$ on M such that $\widetilde{\omega}|_S = \omega_S$ for every S.

Therefore, to show the existence of a closed form $\widetilde{\omega}$ on $\widetilde{C}_2$ extending the $\{\omega_S\}_S$ we just defined, it suffices to show that the map

$$H^{d-1}(\widetilde{C}_2) \xrightarrow{\text{restriction}} H^{d-1}(\partial\widetilde{C}_2)$$

is surjective. It is therefore sufficient to show that $H^d(\widetilde{C}_2, \partial\widetilde{C}_2) = 0$. But

$$\begin{aligned} H^*(\widetilde{C}_2, \partial\widetilde{C}_2) &\approx H_{\dim(\widetilde{C}_2)-*}(\widetilde{C}_2 - \partial\widetilde{C}_2) = H_{\dim(\widetilde{C}_2)-*}\big(C_2([\pi_1,\pi_2]) \times (0,1)\big) \\ &\approx H_{\dim(\widetilde{C}_2)-*}(C_2([\pi_1,\pi_2])) \approx H^{*-1}\big(\overline{C}_2([\pi_1,\pi_2]), \partial\overline{C}_2([\pi_1,\pi_2])\big), \end{aligned}$$

and this is 0 when $* - 1 < d + 1$ by the proof of Lemma 2.12 in [6].

We choose and fix such an extension $\widetilde{\omega}$ on $\widetilde{C}_2$.

4 Configuration space integrals

For simplicity here we will only sketch part of the proof of Theorem 1.1, namely we only justify the first term on the RHS. (The second term is slightly more complicated but not much: basically because we only work with the point class on S^d—it is only 0-dimensional so you can still easily achieve transversality as needed. The second term is not needed for the corollary about the loop space structure on $\mathrm{BDiff}^{\mathrm{fr}}_\partial(D^d)$ either.)

Suppose $\sum_{i=1}^m \Gamma_i$ is a cocycle in graph cohomology. For every edge e of some Γ_i, we have the forgetful map

$$\mathfrak{f}_e : \widetilde{C}_{V(\Gamma_i)} \longrightarrow \widetilde{C}_2.$$

And, when restricted to $\overline{S}_{\mathrm{gray}} \subset \widetilde{C}_{V(\Gamma_i)}$ (resp. $\overline{C}^*_{V(\Gamma_i)} \subset \widetilde{C}_{V(\Gamma_i)}$), it is the forgetful map

$$f_e : \overline{C}_{V(\Gamma_i)}([\pi_1, \pi_2]) \longrightarrow \overline{C}_2([\pi_1, \pi_2]) \qquad (\text{resp. } f_e : \overline{C}^*_{V(\Gamma_i)} \longrightarrow \overline{C}^*_2).$$

Now we have the form $\bigwedge_{e\in E(\Gamma_i)} f_e^*\widetilde{\omega}$ on $\widetilde{C}_{V(\Gamma_i)}$. Recall that by the definition of Kontsevich's characteristic classes,

$$K_{[\pi_1,\pi_2]}\big([\sum_{i=1}^m \Gamma_i]\big) = \sum_{i=1}^m \int_{\overline{S}_{\mathrm{gray}}} \bigwedge_{e\in E(\Gamma_i)} f_e^*\widetilde{\omega}|_{\overline{S}_{\mathrm{gray}}}.$$

To compute the $(\mathrm{PD}_{S^d}[S^d])$-part of $K_{[\pi_1,\pi_2]}([\sum_{i=1}^m \Gamma_i])$, we only need to compute, given arbitrary homology classes $\alpha_1 \in H_*(B_1)$ and $\alpha_2 \in H_*(B_2)$, the evaluation

$$\big\langle K_{[\pi_1,\pi_2]}([\sum_{i=1}^m \Gamma_i]), [S^d] \otimes \alpha_1 \otimes \alpha_2 \big\rangle.$$

For $j = 1, 2$, suppose α_j is represented by a piecewise-smooth singular chain $\sum_t \iota_j^t$, where ι_j^t is some smooth map from the standard $\deg(\alpha_j)$-dimensional simplex $\Delta^{\deg(\alpha_j)}$ to B_i.

Notice that, for any finite set A, the projection map $\widetilde{p} : \widetilde{C}_A \to B_1 \times B_2$ is a fiber bundle, and so is the restriction of $\widetilde{p}$ to each stratum of $\widetilde{C}_A$. So we can form the pull-backs $(\iota_t^j)^*\widetilde{C}_A$, $(\iota_t^j)^*\overline{C}_A([\pi_1, \pi_2])$, $(\iota_t^j)^*\overline{C}^*_A$.

Now, we have

$$\big\langle K_{[\pi_1,\pi_2]}([\sum_{i=1}^m \Gamma_i]), [S^d] \otimes \alpha_1 \otimes \alpha_2 \big\rangle = \sum_{i,j,t} \int_{(\iota_t^j)^*\overline{C}_{V(\Gamma_i)}([\pi_1,\pi_2])} \bigwedge_e f_e^*\omega = \int_{(\iota_t^j)^*\overline{S}_{\mathrm{gray}}} \bigwedge_e f_e^*\widetilde{\omega}. \qquad (1)$$

By Stocks' Formula (and the fact that $\widetilde{\omega}$ is closed),

$$\int_{\partial\widetilde{C}_{V(\Gamma)}} \bigwedge_e f_e^*\widetilde{\omega} = \int_{\widetilde{C}_{V(\Gamma)}} d\Big(\bigwedge_e f_e^*\widetilde{\omega}\Big) = 0,$$

so,

$$(1) = \int_{\text{blue part of } \partial\widetilde{C}_{V(\Gamma)}} \bigwedge_e f_e^*\widetilde{\omega} + \int_{\overline{C}^*_{V(\Gamma)}} \bigwedge_e f_e^*\omega_*.$$

Since Γ is a cocycle in graph cohomology, the first term is 0 just like in the proof of the well-definedness of Kontsevich's classes, so

$$\left\langle K_{[\pi_1,\pi_2]}([\Gamma]), [S^d] \otimes \alpha_1 \otimes \alpha_2 \right\rangle = \int_{\overline{C}^*_{V(\Gamma)}} \bigwedge_e f_e^* \omega_*.$$

5 Configuration space integral on confstar

We continue with the notation from last section (in particular, everything is over B_1', B_2' instead of B_1, B_2).

It remains to show that

$$\left\langle (K_{\pi_1} \otimes K_{\pi_2})(\Delta_{[,]}[\Gamma]), \alpha_1 \otimes \alpha_2 \right\rangle = \int_{\overline{C}^*_{V(\Gamma)}} \bigwedge_e f_e^* \omega_*. \tag{2}$$

For a graph G, we denote by $V(G)$ its vertex set and $E(G)$ its edge set. The LHS above equals to[6]

$$\sum_{\Gamma' \leq \Gamma} \Big(\int_{\overline{C}_{V(\Gamma')}(\pi_1)} \bigwedge_{e \in E(\Gamma')} f_e^* \omega_1 \Big) \cdot \Big(\int_{\overline{C}_{V(\Gamma/\Gamma')}(\pi_2)} \bigwedge_{e \in E(\Gamma/\Gamma')} f_e^* \omega_2 \Big)$$
$$\pm \Big(\int_{\overline{C}_{V(\Gamma/\Gamma')}(\pi_1)} \bigwedge_{e \in E(\Gamma/\Gamma')} f_e^* \omega_1 \Big) \cdot \Big(\int_{\overline{C}_{V(\Gamma')}(\pi_2)} \bigwedge_{e \in E(\Gamma')} f_e^* \omega_2 \Big).$$

To prove (2), it suffices to show that

$$\sum_{\Gamma' \leq \Gamma} \Big(\int_{\overline{C}_{V(\Gamma')}(\pi_1)} \bigwedge_{e \in E(\Gamma')} f_e^* \omega_1 \Big) \cdot \Big(\int_{\overline{C}_{V(\Gamma/\Gamma')}(\pi_2)} \bigwedge_{e \in E(\Gamma/\Gamma')} f_e^* \omega_2 \Big) = \int_{\text{yellow part of } \overline{C}^*_{V(\Gamma)}} \bigwedge_{e \in E(\Gamma)} f_e^* \omega_* \tag{3}$$

and

$$\sum_{\Gamma' \leq \Gamma} \Big(\int_{\overline{C}_{V(\Gamma/\Gamma')}(\pi_1)} \bigwedge_{e \in E(\Gamma/\Gamma')} f_e^* \omega_1 \Big) \cdot \Big(\int_{\overline{C}_{V(\Gamma')}(\pi_2)} \bigwedge_{e \in E(\Gamma')} f_e^* \omega_2 \Big) = \int_{\text{red part of } \overline{C}^*_{V(\Gamma)}} \bigwedge_{e \in E(\Gamma)} f_e^* \omega_*; \tag{4}$$

here the "red" and "yellow" refers to the $\widetilde{C}_n$ picture in Section 2. Below we only prove (3) since (4) is completely similar.

The yellow part of $\overline{C}^*_{V(\Gamma)}$ is simply

$$\sum_{V_1, V_2 : V_1 \sqcup V_2 = V(\Gamma)} \overline{C}_{V_1}(\pi_1) \times \overline{C}_{V_2 \sqcup \{\star\}}(\pi_2),$$

where $\star$ records the position of the node on π_2. Therefore, the RHS of (3) is

$$\sum_{V_1, V_2 : V_1 \sqcup V_2 = V(\Gamma)} \int_{\overline{C}_{V_2 \sqcup \{\star\}}(\pi_2)} \int_{\overline{C}_{V_1}(\pi_1)} \bigwedge_{\substack{e \in E(\Gamma) \\ \text{both endpoints of } e \text{ are in } V_1}} f_e^* \omega_* \wedge \bigwedge_{\substack{e \in E(\Gamma) \\ \exists \text{ endpoint of } e \text{ in } V_2}} f_e^* \omega_*.$$

[6]A little more argument needed for this claim, but it is true.

For $V_1 \subset V(\Gamma)$, we denote by $\Gamma'(V_1)$ the subgraph of Γ spanned by vertices in V_1. Then, by the way ω_* is constructed, and by Fubini's Theorem, the above equals to

$$\sum_{V_1,V_2:V_1\sqcup V_2=V(\Gamma)} \Big(\int_{\overline{C}_{V_1}(\pi_1)} \bigwedge_{e\in E(\Gamma'(V_1))} f_e^*\omega_1\Big)\cdot\Big(\int_{\overline{C}_{V_2\sqcup\{\star\}}(\pi_2)} \bigwedge_{e\in E(\Gamma/\Gamma'(V_1))} f_e^*\omega_2\Big).$$

This proves (3).

Appendix A Extending differential forms on a manifold with corners from boundary to interior

We first clarify the notation used in the present paper concerning manifold with corners, mostly following [3] and [4]. Here is a dictionary for our notation:

- *(smooth) manifold with corners*: as in [4, page 3] or, equivalently, [3, Definition 3].
- *smooth map between manifolds with corners*: [3, Definition 4] ("weakly smooth map" in [4, Definition 3.1])
- *tangent and cotangent spaces of manifolds with corners*: [3, Definition 10] (equivalently, [4, Definition 2.2])
- *cotangent bundle, tensor and exterior powers of cotangent bundle of manifolds with corners*: follows from the definition of cotangent spaces
- *differential form on manifolds with corners*: smooth section of exterior powers of the cotangent bundle

  For example, in $(\mathbb{R}^{\geq 0})^n$ with coordinates denoted by $(x_1,\dots,x_n)$, a differential form of degree k can be written as

  $$\sum_{1\leq i_1<\ldots<i_k\leq n} f_{i_1\ldots i_k} dx_{i_1}\wedge\ldots\wedge dx_{i_k},$$

  where $f_{i_1\ldots i_k}$ are functions on $(\mathbb{R}^{\geq 0})^n$, smooth in the sense that there exist smooth functions $\hat{f}_{i_1\ldots i_k}$ on an open neighborhood of $(\mathbb{R}^{\geq 0})^n$ in $\mathbb{R}^n$, such that $f_{i_1\ldots i_k} = \hat{f}_{i_1\ldots i_k}|_{(\mathbb{R}^{\geq 0})^n}$. For $0\leq m<n$, the restriction (pullback by inclusion map) of the above differential form to $(\mathbb{R}^{\geq 0})^m \approx \{x_{m+1}=\ldots=x_n=0\}\subset(\mathbb{R}^{\geq 0})^n$ will be

  $$\sum_{1\leq i_1<\ldots<i_k\leq m} f_{i_1\ldots i_k} dx_{i_1}\wedge\ldots\wedge dx_{i_k};$$

  namely, we remove the terms containing any dx_i with $i>m$.
- *∂M, the boundary of a manifold with corners M*: the image of the map $i_M:\partial M\to M$ in [4, Definition 2.6]
- $\tilde{\partial}M$: ∂M in [4, Definition 2.6]

- *codimension-d (depth-d) stratum*: [3, Definition 7] (a connected component of a depth-d stratum in [4, Definition 2.3])

  For example, a solid cube has 6 codim-1 strata and 12 codim-2 strata. Its boundary is the surface of the cube, and its $\tilde{\partial}$ is the disjoint union of 6 closed squares.

- *manifold with embedded faces*: [3, Definition 18]: a manifold with corners such that the closure of every codim-1 strata is an embedded manifold with corners.

In this appendix we prove the following

Proposition A.1. *Let M be a compact manifold with embedded faces.*[7] *Suppose for each stratum $S \subset \partial M$ of M, ω_S is a closed differential form of degree k on $\overline{S}$, such that, for each pair of strata $T \subset \overline{S} \subset \partial M$, $\omega_T = \omega_S|_{\overline{T}}$. Then, there exists an open neighborhood U of ∂M and a closed differential form ω on U such that $\omega|_{\overline{S}} = \omega_S$ for all strata $S \subset \partial M$.*

In light of Proposition A.1, we also define

- *degree-k differential form on ∂M*: a compatible collection of forms, $\{\omega_S\}_S$, as in the assumption of Proposition A.1.

Remark A.2. It is easy to see that the $\mathbb{R}$-singular cohomology of ∂M (or more generally, any manifold with corners and "hinges"[8]) is equivalent to the de Rham cohomology defined using these forms, but we do not need it in the present paper.

Lemma A.3. *Let M be a manifold with corners and U an open neighborhood of ∂M. Suppose ω is a closed k-form on U such that $[\omega]$ lies in the image of the restriction map $H^k(M;\mathbb{R}) \to H^k(U;\mathbb{R})$, then there is a neighborhood $U'' \subset U$ of ∂M such that $\omega|_{U''}$ can be extended to a closed k-form on M.*

Proof (same as (add ref)). Let α be a closed d-form on $M\backslash\partial M$ such that the restriction of $[\alpha]$ to U is $[\omega]$. Then, there exists a $(k-1)$-form β on U such that $d\beta = \omega - \alpha|_U$. We can find[9] open subsets U', U'' of M such that $\partial M \subsetneq U'' \subsetneq U' \subsetneq U$, and a smooth function $f : M \to [0,1]$ such that $f|_{U''} \equiv 0$, $f|_{M\backslash U'} \equiv 1$. Define $\omega' = \omega - d(f\beta)$ on U. Then $\omega' \equiv \omega$ on U'', $\omega' \equiv \alpha$ on $U\backslash U'$, and $d\omega' = 0$. So we can extend ω' to a form on M, defining it to be α out of U. □

Corollary A.4. *Suppose M is a compact manifold with embedded faces such that $H^k(M;\mathbb{R}) \to H^k(\partial M; R)$ is surjective. Then, every closed k-form on ∂M can be extended to a closed k-form on M.*

The rest of this section is devoted to the proof of Proposition A.1. First we prove the statement locally, in $(\mathbb{R}^{\geq 0})^d \times \mathbb{R}^{n-d}$, where $0 \leq d \leq n$ and $0 \leq k \leq n$ are integers. For $I \subset \{1, \ldots, d\}$, define $H_I := \big\{(x_1, \ldots, x_n) \big| \forall i \in I, x_i = 0\big\} \subset (\mathbb{R}^{\geq 0})^d \times \mathbb{R}^{n-d}$. Write $H_i := H_{\{i\}}$ and $H_{i,j} = H_{\{i,j\}}$.

[7] Neither the compactness or embedded faces condition should be necessary; this proposition likely holds for all manifolds with corners. We assume the stronger condition here to shorten the proof.

[8] By "hinges" we mean the manifold is locally modeled on $\mathbb{R}^a \times (\mathbb{R}^{\geq 0})^b \times \big((\mathbb{R}^{\geq 0})^c \backslash (\mathbb{R}^{>0})^c\big)$ for $a, b, c \in \mathbb{Z}^{\geq 0}$. One can image a theory of differential forms on such manifolds, defined similar to here.

[9] For example, using a metric on M and properly modify the distance function to ∂M.

Lemma A.5. *Suppose for each $1 \le i \le d$, ω_i is a degree-k differential form on H_i, such that, for all $1 \le i, j \le d$, $\omega_i|_{H_{i,j}} = \omega_j|_{H_{i,j}}$. Then, there exists a degree-k differential form ω on $(\mathbb{R}^{\ge 0})^d \times \mathbb{R}^{n-d}$, such that, for all $1 \le i \le d$, $\omega|_{H_i} = \omega_i$. Moreover, if all ω_i are closed, ω can be taken to be closed as well.*

Proof. From the condition $\omega_i|_{H_{i,j}} = \omega_j|_{H_{i,j}}$ it is clear that for any $I \subset \{1, \dots, d\}$, the forms $\omega_i|_{H_I}$ are the same for all $i \in I$. We hence denote it by ω_I, which is on H_I. Let

$$p_I : (\mathbb{R}^{\ge 0})^d \times \mathbb{R}^{n-d} \longrightarrow H_I$$

be the projection map, sending all I-coordinates to 0 and not changing the other coordinates. We take ω to be the alternating sum

$$\omega = \sum_{1 \le i \le d} p^*_{\{i\}} \omega_i - \sum_{1 \le i < j \le d} p^*_{\{i,j\}} \omega_{\{i,j\}} + \sum_{1 \le i < j < k \le d} p^*_{\{i,j,k\}} \omega_{\{i,j,k\}} - \dots + (-1)^{d-1} p^*_{\{1,\dots,d\}} \omega_{\{1,\dots,d\}}.$$

To see that $\omega|_{H_i} = \omega_i$ for a given $i \in \{1, \dots, d\}$, note that, for each $I \subset \{1, \dots, d\}$ with $I \ne \emptyset$ and $i \notin I$,

$$(p^*_I \omega_I)|_{H_i} = p^*_I(\omega_I|_{H_i}) = p^*_{I \sqcup \{i\}}(\omega_{I \sqcup \{i\}}|_{H_i}) = (p^*_{I \sqcup \{i\}} \omega_{I \sqcup \{i\}})|_{H_i},$$

so these two terms cancel with each other. If all ω_is are closed, ω is clearly also closed. $\square$

Next, we patch the forms constructed locally to a global one. Without the closeness condition, this would be immediate by applying a partition of unity. With the closeness condition it is much subtler. The argument uses the same technique as translating between Čech and de Rham cohomology (see, e.g. [1]), which can also be viewed as a generalization of the technique in the proof of Lemma A.3. Note that from now on, the index set I has a different meaning than in Lemma A.5.

Given $p, q \in \mathbb{Z}^{\ge 0}$, a manifold M and a locally finite open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of M, recall a (skew-symmetric) p-Čech cochain of q-forms on M is: for each sequence $(i_0, i_1, \dots, i_p) \in I^{p+1}$, a differential q-form $\alpha_{i_0 i_1 \dots i_p}$ on $\bigcap_{j=0}^p U_{i_j}$; such that for all j, $\alpha_{i_0 \dots i_j i_{j+1} \dots i_p} = -\alpha_{i_0 \dots i_{j+1} i_j \dots i_p}$. We denote the $\mathbb{R}$-vector space of p-Čech cochain of q-forms on M by $\check{C}^p_{\mathcal{U}}(M; \mathcal{A}^q)$. The Čech differential is

$$\check{\delta} : \check{C}^p_{\mathcal{U}}(M; \mathcal{A}^q) \longrightarrow \check{C}^{p+1}_{\mathcal{U}}(M; \mathcal{A}^q)$$

$$\check{\delta}\,(\alpha_{i_0 \dots i_p})_{(i_0 \dots i_p) \in I^{p+1}} = (\beta_{i_0 \dots i_{p+1}})_{(i_0 \dots i_{p+1}) \in I^{p+2}}, \qquad \beta_{i_0 \dots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \alpha_{i_0 \dots \hat{i}_j \dots i_{p+1}}|_{U_{i_0} \cap \dots \cap U_{i_{p+1}}}.$$

We also still denote by d the termwise differential of forms:

$$d : \check{C}^p_{\mathcal{U}}(M; \mathcal{A}^q) \longrightarrow \check{C}^p_{\mathcal{U}}(M; \mathcal{A}^{q+1}), \qquad d\,(\alpha_{i_0 \dots i_p})_{(i_0 \dots i_p) \in I^{p+1}} = (d\alpha_{i_0 \dots i_p})_{(i_0 \dots i_p) \in I^{p+1}}.$$

It is clear that $\check{\delta} d = d \check{\delta}$, and both d and $\check{\delta}$ commute with pull-back maps between manifolds.

Lemma A.6. *Suppose M is a manifold with corners. Denote $N := \partial M$ and $\iota : N \to M$ the inclusion map.*[10] *Suppose $\mathcal{U} = \{U_i\}_{i \in I}$ is a locally finite open cover of M satisfying the condition that for all subset $I' \subset I$, if $U_{I'} := \bigcap_{i \in I'} U_i$ is non-empty, then*

[10]This lemma and Lemma A.7 hold if N is replaced with an arbitrary manifold with corners and $\iota : N \to M$ an arbitrary smooth map.

1. *all de Rham cohomology groups of $U_{I'}$ are the same as those of a point,*
2. *$\iota^{-1}(U_{I'}) \neq \emptyset$, and*
3. *if σ is a closed form*[11] *on $\iota^{-1}(U_{I'})$, then there exists a closed form $\tilde{\sigma}$ on $U_{I'}$ with $\iota^*\tilde{\sigma} = \sigma$.*

Then, the following proposition $\mathcal{P}_p^q$ holds for all $p \geq 0$,$q \geq 1$:

$\mathcal{P}_p^q$: *Suppose $\alpha = (\alpha_{i_0\dots i_p})_{(i_0\dots i_p)\in I^{p+1}} \in \check{C}^p_{\mathcal{U}}(M;\mathcal{A}^q)$ satisfies $\check{\delta}\alpha = 0$, $\iota^*\alpha = 0$ and $d\alpha = 0$, then, there exists $\beta = (\beta_{i_0\dots i_p})_{(i_0\dots i_p)\in I^{p+1}} \in \check{C}^p_{\mathcal{U}}(M;\mathcal{A}^{q-1})$ such that $\check{\delta}\beta = 0$, $\iota^*\beta = 0$ and $d\beta = \alpha$.*

Proof. Two steps: we first show that $\mathcal{P}_{p+1}^{q-1} \implies \mathcal{P}_p^q$, then show that $\mathcal{P}_p^1$ holds for all $p \geq 0$.

Step 1: Suppose $q \geq 2$ and α is as in the condition of $\mathcal{P}_p^q$. By condition 1 above, there is $\beta'' \in \check{C}^p_{\mathcal{U}}(M;\mathcal{A}^{q-1})$ such that $d\beta'' = \alpha$. Then, $d\iota^*\beta'' = 0$, so condition 3 above implies that there is $\tilde{\beta}'' \in \check{C}^p_{\mathcal{U}}(M;\mathcal{A}^{q-1})$ such that $d\tilde{\beta}'' = 0$ and $\iota^*\tilde{\beta}'' = \iota^*\beta''$. Setting $\beta' = \beta'' - \tilde{\beta}''$ we have $d\beta' = \alpha$ and $\iota^*\beta' = 0$. Since $d\check{\delta}\beta' = \check{\delta}\alpha = 0$ and $\iota^*\check{\delta}\beta' = 0$, $\check{\delta}\beta'$ (replacing α) satisfies the condition of $\mathcal{P}_{p+1}^{q-1}$. If we assume $\mathcal{P}_{p+1}^{q-1}$ holds, then there exists $\gamma \in \check{C}^{p+1}_{\mathcal{U}}(M;\mathcal{A}^{q-2})$ with $\check{\delta}\gamma = 0$, $\iota^*\gamma = 0$ and $d\gamma = \check{\delta}\beta'$.

Let $\{f_i : U_i \to [0,1]\}_{i\in I}$ be a partition of unity subordinate to $\mathcal{U}$. we define β by taking

$$\beta_{i_0\dots i_p} = \beta'_{i_0\dots i_p} - \sum_{i\in I} d(f_i \cdot \gamma_{ii_0\dots i_p}).$$

It is clear that $d\beta = d\beta' = \alpha$ and, since $\iota^*\gamma = 0$, $\iota^*\beta = \iota^*\beta' - \sum_{i\in I} d\big((f_i\circ\iota)\cdot\iota^*\gamma_{ii_0\dots i_p}\big) = 0$. And

$$\begin{aligned}(\check{\delta}\beta' - \check{\delta}\beta)_{i_0\dots i_{p+1}} &= \sum_{j=0}^{p+1}(-1)^j(\beta'-\beta)_{i_0\dots\hat{i}_j\dots i_{p+1}} = \sum_{j=0}^{p+1}(-1)^j\sum_{i\in I} d(f_i\cdot\gamma_{ii_0\dots\hat{i}_j\dots i_{p+1}}) \\ &= d\sum_{i\in I}\Big(f_i\cdot\sum_{j=0}^{p+1}(-1)^j\gamma_{ii_0\dots\hat{i}_j\dots i_{p+1}}\Big) = d\sum_{i\in I} f_i\cdot(\gamma_{i_0\dots i_{p+1}} - (\check{\delta}\gamma)_{ii_0\dots i_{p+1}}) = d\gamma_{i_0\dots i_{p+1}}, \end{aligned} \tag{5}$$

where the last equality is due to $\check{\delta}\gamma = 0$, and the 4-th equality holds because

$$\sum_{j=0}^{p+1}(-1)^j\gamma_{ii_0\dots\hat{i}_j\dots i_{p+1}} = -(\check{\delta}\gamma)_{ii_0\dots i_{p+1}} + \gamma_{i_0\dots i_{p+1}}.$$

Therefore, we have β such that $\check{\delta}\beta = 0$, $\iota^*\beta = 0$ and $d\beta = \alpha$, as desired.

Step 2: Suppose α is as in the condition of $\mathcal{P}_p^1$. the argument at the beginning of Step 1 says there is β such that $\iota^*\beta = 0$ and $d\beta = \alpha$. Since α consists of 1-forms, β, as well as $\check{\delta}\beta$, consist of smooth functions. Since $d\check{\delta}\beta = \check{\delta}d\beta = 0$ and the domain of each $(\check{\delta}\beta)_{i_0\dots i_p}$, if not empty, is connected by condition 1, $(\check{\delta}\beta)_{i_0\dots i_p}$ are constant functions. Also $\iota^*\check{\delta}\beta = \check{\delta}\iota^*\beta = 0$, so, by condition 2 in the lemma, each $(\check{\delta}\beta)_{i_0\dots i_p}$ with non-empty domain is 0 at at least one point, hence must be 0. Therefore $\check{\delta}\beta = 0$ and β satisfies the requirement in $\mathcal{P}_p^1$. $\square$

Lemma A.7. *Suppose $\iota, N, M, \mathcal{U}$ are as in the condition of Lemma A.6; $p \geq 0$,$q \geq 1$. Suppose $\alpha \in \check{C}^p_{\mathcal{U}}(M;\mathcal{A}^q)$ satisfies $d\alpha = 0$ and $\check{\delta}\iota^*\alpha = 0$, then there exists $\alpha' \in \check{C}^p_{\mathcal{U}}(M;\mathcal{A}^q)$ such that $d\alpha' = 0$, $\check{\delta}\alpha' = 0$ and $\iota^*\alpha' = \iota^*\alpha$.*

[11] Differential forms on open subsets of ∂M are defined as in the assumption of Proposition A.1.

Proof. We have $d(\check{\delta}\alpha) = \check{\delta}d\alpha = 0$, $\check{\delta}(\check{\delta}\alpha) = 0$, and $\iota^*(\check{\delta}\alpha) = \check{\delta}\iota^*\alpha = 0$. So, applying $\mathcal{P}^q_{p+1}$ to $\check{\delta}\alpha$ we obtain $\beta \in \check{C}^{p+1}_{\mathcal{U}}(M; \mathcal{A}^{q-1})$ such that $\check{\delta}\beta = 0$, $\iota^*\beta = 0$ and $d\beta = \check{\delta}\alpha$. Again, let $\{f_i : U_i \to [0,1]\}_{i\in I}$ be a partition of unity subordinate to $\mathcal{U}$ and we define α' by taking

$$\alpha'_{i_0\dots i_p} = \alpha_{i_0\dots i_p} - \sum_{i\in I} d(f_i \cdot \beta_{ii_0\dots i_p}),$$

then $d\alpha' = 0$ and $\iota^*\alpha' = \iota^*\alpha - \sum_{i\in I} d((f_i \circ \iota)\cdot \iota^*\beta_{ii_0\dots i_p}) = \iota^*\alpha$. And, similar to (5),

$$\begin{aligned}(\check{\delta}\alpha - \check{\delta}\alpha')_{i_0\dots i_{p+1}} &= \sum_{j=0}^{p+1}(-1)^j(\alpha-\alpha')_{i_0\dots\hat{i}_j\dots i_{p+1}} = \sum_{j=0}^{p+1}(-1)^j\sum_{i\in I} d(f_i\cdot\beta_{ii_0\dots\hat{i}_j\dots i_{p+1}})\\ &= d\sum_{i\in I}\Big(f_i\cdot\sum_{j=0}^{p+1}(-1)^j\beta_{ii_0\dots\hat{i}_j\dots i_{p+1}}\Big) = d\sum_{i\in I} f_i\cdot(\beta_{i_0\dots i_{p+1}} - (\check{\delta}\beta)_{ii_0\dots i_{p+1}}) = d\beta_{i_0\dots i_{p+1}},\end{aligned}$$

Therefore, $\check{\delta}\alpha' = \check{\delta}\alpha - d\beta = 0$, as desired. □

Lemma A.8. *Suppose M is a compact manifold with embedded faces. Then, there exist an open neighborhood U of ∂M, a finite open cover $\mathcal{U}$ of U, such that $\mathcal{U}$ satisfies the condition in Lemma A.7 when we plug in ∂M for N, U for M, and the inclusion map for ι.*

Proof. By [3, Theorem 17], M has a system of compatible collar neighborhoods ([3, Definition 35]).[12] Fix a metric on M. For a point p in a depth-k strata S, we take a convex neighborhood U'_p of p in S and define $U_p = U'_p \times [0,\epsilon)^k \subset M$ (here we implicitly use the identification given by the collar neighborhood $S \times [0,\epsilon)^k \to M$). Then, for any finite set of points $\{p_i\}$ in ∂M, $\cap_i U_{p_i}$ is diffeomorphic to $(\mathbb{R}^{\geq 0}))^d \times \mathbb{R}^{\dim M - d}$ for some d, and $(\cap_i U_{p_i}) \cap \partial M \neq \emptyset$. So, by Lemma A.5, $\cap_i U_i$ satisfied condition 3 for $U_{I'}$ imposed in Lemma A.6. Take $\mathcal{U}$ to be a finite subcover of $\{U_p\}_{p\in\partial M}$ and $U = \bigcup_{U_p\in\mathcal{U}} U_p$. □

Proof of Proposition A.1. Let $U,\mathcal{U}$ be given as in Lemma A.8. Then, for every $V \in \mathcal{U}$, by Lemma A.5, we can find a closed form ω_V on V such that $\omega_V|_{S\cap V} = \omega_S|_{S\cap V}$ for all strata S of ∂M. The collection $\{\omega_V\}_{V\in\mathcal{U}}$ defines an $\alpha \in \check{C}^0_{\mathcal{U}}(M; \mathcal{A}^k)$, which satisfies the condition of Lemma A.7. So, there exists α' as in the conclusion of Lemma A.7. To get to the conclusion of Proposition A.1, take $\omega = \alpha'$. □

References

[1] R. Bott and L. Tu, *Differential Forms in Algebraic Topology*, Springer, 1982

[2] K. Grove and H. Karcher, *How to Conjugate C^1-Close Group Actions*, Math. Z., 1973

[3] P. Hájek, *On manifolds with corners*, Master's Thesis, Ludwig Maximilian University of Munich, 2014

[12] (Delete! This is wrong!!!)In our application M is $\widetilde{C}_n$, and its charts that we explicitly constructed in Section 2 is actually a system of compatible collar neighborhoods.

[4] D. Joyce, *On manifolds with corners*, arxiv/0910.3518v2

[5] C. Lescop *Invariants of links and 3–manifolds from graph configurations*, arxiv/2001.09929v2

[6] T. Watanabe *Addendum to: some exotic nontrivial elements of the rational homology groups of* $\mathrm{Diff}(S^4)$ *(homological interpretation)*, arxiv/2109.01609v3