Some informal explanation on defining orientation on the moduli space of J-holomorphic disks

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Let $(E, F) \to (\mathbb{D}, \partial\mathbb{D})$ be a bundle pair, namely, $E \to \mathbb{D}$ is a complex vector bundle (say of complex dimension n, $n \geq 3$) and $F \to \partial\mathbb{D}$ is a totally real subbundle of $E|_{\partial\mathbb{D}}$. For simplicity we assume F is oriented. Let $D : \Gamma(E, F) \to \Gamma(T^{0,1}\mathbb{D} \otimes E)$ be a real Cauchy-Riemann operator on (E, F).

Question: how to define an orientation on $\det D$?

Fact 1: the space of real Cauchy-Riemann operators on (E, F) is contractible.

Fact 2: can define the determinant line bundle on the space of real Cauchy-Riemann operators on (E, F).

Therefore, to specify an orientation on the determinant of a given real Cauchy-Riemann operator D, we can first specify an orientation on the determinant of another (arbitrary) real Cauchy-Riemann operator D', then continuously extend this orientation to the determinant of D. This is what we will do.

The standard way of defining an orientation on $\det D$ is through a step called "pinching".

The pinching procedure is only for this purpose: reducing the above question to the special case when the Maslov index of (E, F) is 0.

So, what you do is: construct a new bundle pair $(E', F') \to (\Sigma, \partial\Sigma)$, where Σ is a nodal surface with one node—it is a sphere attached to the center of a disk. Let's call the sphere Σ_s and the disk Σ_d. Let S' be a circle in $\mathbb{D}$ obtained by slightly shifting $\partial\mathbb{D}$ inside. Let τ be a trivialization of F. Then, $(E', F') \to (\Sigma, \partial\Sigma)$ is obtained by pinching $E|_{S'}$ to $\mathbb{C}^n \to$ point using τ. (See Figure 8.1.1 of FOOO.)

I would like to remark that this process only defines a bundle pair; it does not define a real Cauchy-Riemann operator on (E', F'). It is clear from definition that the Maslov index of $(E', F')|_{\Sigma_d}$ is 0. So, this is a process of "putting all the non-trivial Maslov index to the sphere part, and only leave the Maslov index 0 part of the bundle to the disk".

Remark 0.1. This procedure does not depend on the choice of τ. (This is because we are not yet considering operators—only viewing E' as a complex vector bundle not a holomorphic one.) Up to homotopy, there are two choices of τ, differing by the non-trivial element in $\pi_1(SO(n))$, but since $\pi_1(SO(n)) \to \pi_1(U(n))$ is trivial, they induce trivializations of $E|_{\partial\mathbb{D}}$ that are homotopic to each other.

The rest of the argument goes as follows:

- Choose a real Cauchy-Riemann operator, D', on (E', F'). It simply consists of a complex linear Cauchy-Riemann operator D'_s on $E'|_{\Sigma_s} \to \Sigma_s$ and another real Cauchy-Riemann operator D'_d on $E'|_{\Sigma_d} \to \Sigma_d$.
- By writing down some simple short exact sequences we can conclude that an orientation on $\det D'_s$, an orientation on $\det D'_d$, and an orientation on $E|_{\text{node}}$ canonically induces in orientation on $\det D'$. Since two of the three above are complex, they have natural complex orientations. So, we have reduced the problem of orienting $\det D'$ to orienting $\det D'_d$. This is why we say that the purpose of the pinching process is to reduce the problem to the Maslov index 0 case.
- On the other hand, the gluing procedure gives
  - a family of Riemann surfaces over $[0,1]$ (denote by $\widetilde{\Sigma} \to [0,1]$), which are all disks over $(0,1]$ and degenerate to Σ over 0;
  - a bundle pair $(\widetilde{E}, \widetilde{F}) \to (\widetilde{\Sigma}, \partial\widetilde{\Sigma})$ that restricts to (E, F) over 1 and (E', F') over 0;
  - a family of real Cauchy-Riemann operators $\{\widetilde{D}_t\}_{t\in[0,1]}$, where $\widetilde{D}_t$ is on $(\widetilde{E}, \widetilde{F})|_t$.
- Suppose $\det D'$ is oriented (by the second bullet above, we only need to orient $\det D'_d$ for this to happen), then, by taking a path γ' from D' to $\widetilde{D}_0$ in the space of real Cauchy-Riemann operators over (E', F'), and another path γ from D to $\widetilde{D}_1$ in the space of real Cauchy-Riemann operators over (E, F), we can obtain an orientation on $\det D$ by continually extending the orientation on $\det D'$ first through γ', then through the family $\widetilde{D}_t$ above, then through γ.

To complete the above argument, it only remains to

- construct a real Cauchy-Riemann operator D' on (E', F'), consisting of a complex Cauchy-Riemann operator D'_s on $E'|_{\Sigma_s} \to \Sigma_s$ and a real Cauchy-Riemann operator D'_d on $E'|_{\Sigma_d} \to \Sigma_d$;
- orient $\det D'_d$.

We don't really care about D'_s—just choose an arbitrary one. We do care about D'_d. **This is where the Spin-structure comes in.** Again, suppose τ is a trivialization of F' (which is the same as F). Then τ induces a trivialization of $(E'|_{\Sigma_d}, F') \to \Sigma_d$:

$$
\begin{tikzcd}
(E'|_{\Sigma_d}, F') \arrow[rr, "\approx"] \arrow[dr] & & (\mathbb{C}^n \times \Sigma_d, \mathbb{R}^n \times S^1) \arrow[dl] \\
& (\Sigma_d, S^1) &
\end{tikzcd}
.
$$

The standard $\bar{\partial}$-operator on $\mathbb{C}^n \times \Sigma_d \to \Sigma_d$ thus gives a real Cauchy-Riemann operator on $(E', F')|_{\Sigma_d}$. This is our D'_d. It depends on τ. Moreover, we have a canonical identification $\ker D'_d \approx F'|_{\mathrm{pt}} \approx \mathbb{R}^n$. This orients $\ker D'_d$ and thus $\det D'_d$ (since $\bar{\partial}$ is surjective). **This orientation of $\det D'_d$ depends on τ. This is how the Spin-structure gets used.**

Remark 0.2. You might get confused because we mentioned above that the choice of the real Cauchy-Riemann operator doesn't matter to the orientation, yet here the only place difference choices of τ matter is that they determine different D'_d. The situation here is the following: say the two different choices of τ yield two different operators $D'_{d,a}$ and $D'_{d,b}$. What we meant by "the operator doesn't matter" before is that, if we choose a path (in the space of real Cauchy-Riemann operators) between $D'_{d,a}$ and $D'_{d,b}$, then it induces an identification between the orientations of $\det D'_{d,a}$ and $\det D'_{d,a}$ and this identification doesn't depend on the choice of the path. Let's call this identification id_{path}. However, the way we defined an orientation on $\det D'_{d,a}$ as well as $\det D'_{d,b}$ is completely different, not involving a continuation in the space of all real Cauchy-Riemann operators at all. And, actually, the orientations thus defined on $\det D'_{d,a}$ and $\det D'_{d,b}$ disagree with id_{path} (Proposition 8.1.7 in FOOO).

This completes the construction of an orientation on $\det D$, given a trivialization of F. It remains to check that it is well-defined—independent of various choices made along the way—and this can be done using Fact 1 and 2. (Leave as exercise.)

Similarly, one can check that in the setting of J-holomorphic curves in a symplectic manifold with Lagrangian boundary, if the trivialization of $F = u^*TL$ comes from a Spin-structure on L, then the above definition varies continuously as u varies in $\mathcal{M}$. (Leave as exercise.)