An introduction to Kontsevich's invariants, and why it can detect exotic phenomena

Xujia Chen

March 6, 2024

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  Many developments since then.
- $M = S^4$: one of the most famous open conjectures in topology

Another question concerning exotic phenomena:

M: smooth manifold, possibly with boundary ∂M

$$\begin{aligned}\mathsf{Diff}_\partial(M) &:= \{f : M \to M \text{ diffeomorphism}, \\ &\qquad \exists \text{ neighborhood } U \text{ of } \partial M \text{ such that } f|_U = \mathsf{Id}\} \\ \mathsf{Homeo}_\partial(M) &:= \{f : M \to M \text{ homeomorphism}, \\ &\qquad \exists \text{ neighborhood } U \text{ of } \partial M \text{ such that } f|_U = \mathsf{Id}\}\end{aligned}$$

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Q: Is it a homotopy equivalence?

The simplest case

$M = D^n$: the standard n-dimensional disk (ball)

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$n = 2, 3$: Yes [Smale (1958), Cerf (1969), Hatcher (1983)]

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Is $\text{Diff}_\partial(D^n)$ contractible?

$n = 2, 3$: Yes [Smale (1958), Cerf (1969), Hatcher (1983)]
$n \geq 4$: No!
($n \geq 5$ case: known for decades; $n = 4$ case: [Watanabe (2018)])

In general, the homotopy groups of $\text{Diff}_\partial(D^n)$ are still largely unknown.

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Q: How to understand the role smooth structure plays in Kontsevich’s invariants?

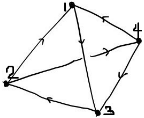
Today:

1, introduction to Kontsevich’s invariants

2, give a perspective on this question

Brief introduction to Kontsevich's invariants

M: d-dimensional smooth manifold

Γ: trivalent graph, e.g. 

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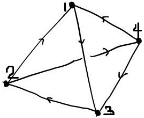
Γ: trivalent graph, e.g. 

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$\Delta = \{(x,y) \in M \times M \,|\, x = y\}$

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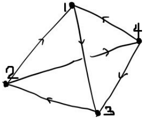

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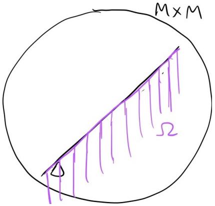

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For each edge e or Γ, choose a generic "submanifold" $\Omega_e \subset M \times M$ such that $\partial\Omega_e = \Delta$.

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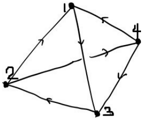

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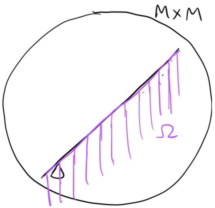

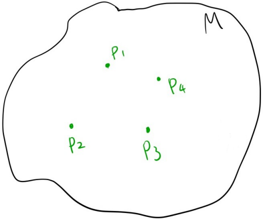

Kontsevich's invariant:
Count embeddings $V(\Gamma) \hookrightarrow M$ such that
for every edge e, the two points connected by e lie on Ω_e.

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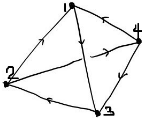

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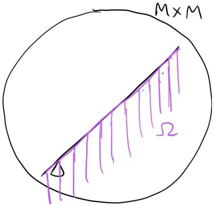

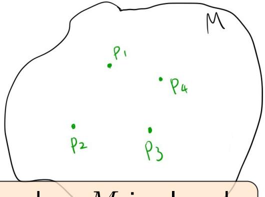

For each edge e or Γ, choose a generic "submanifold" $\Omega_e \subset M \times M$ such that $\partial\Omega_e = \Delta \cup p_\infty \times M \cup M \times p_\infty$.

Impossible when M is closed. Solution: assume M is a $\mathbb{Z}$-homology sphere; fix $p_\infty \in M$

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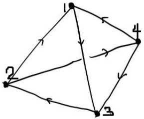
Assume d=3; or consider a family of manifolds ("fiber bundle") instead of just one



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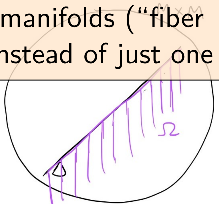

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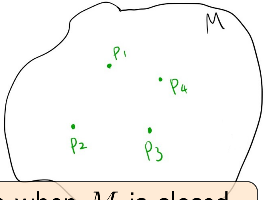

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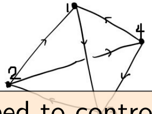

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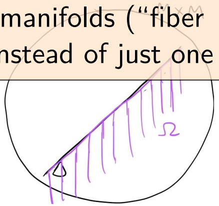
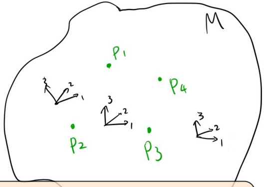
Need to control the directions Ω approaches Δ
Solution: assume a framing on $M \backslash p_\infty$, and use $\overline{\mathrm{Conf}}_S(M, p_\infty)$





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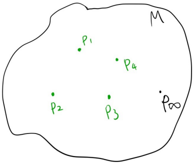
Given a finite set S,

$$\mathsf{Conf}_S(M, p_\infty) := \{(p_i)_{i\in S} \in M^S \mid p_i \neq p_j, \forall i \neq j; p_i \neq p_\infty, \forall i\}$$



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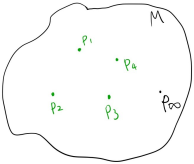

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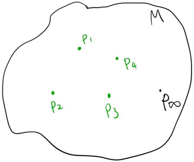

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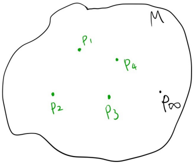

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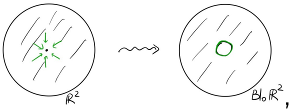

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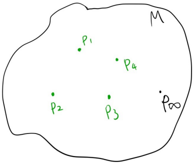

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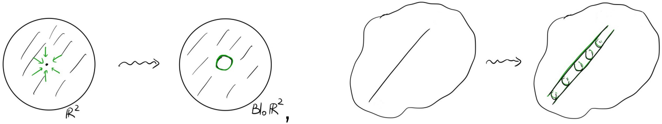

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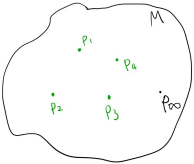

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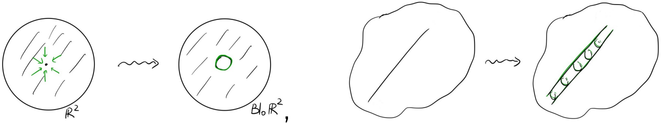
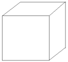


- is a manifold with boundary and corners.



$\overline{\mathsf{Conf}}_2(M, p_\infty) := \overline{\mathsf{Conf}}_{\{1,2\}}(M, p_\infty)$
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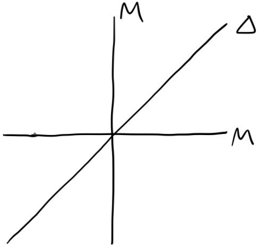

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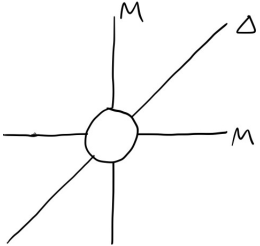

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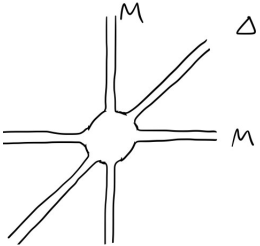

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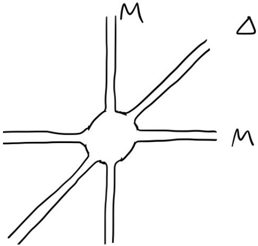

Assume there is a framing F on $M \backslash p_\infty$, "standard" near p_∞.
Then F induces a map

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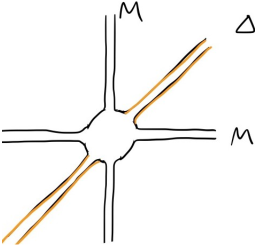

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$$S\mathcal{N}(\Delta \backslash (p_\infty \times p_\infty)) \approx ST(\Delta \backslash (p_\infty \times p_\infty)) \approx$$

$$ST(M \backslash p_\infty) \xrightarrow[\approx]{F} (M \backslash p_\infty) \times S^{d-1} \xrightarrow{\text{proj.}} S^{d-1}$$

Kontsevich's invariants for 3-manifolds

M: 3-dimensional $\mathbb{Z}$-homology sphere

$p_\infty \in M$ a point

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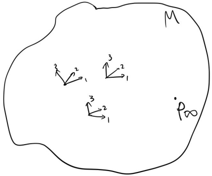
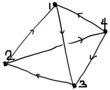
Γ: trivalent graph satisfying certain condition.





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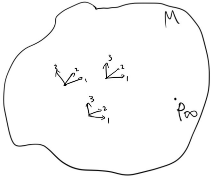
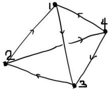
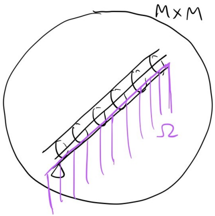
A propagator Ω is a "submanifold" in $\overline{\mathrm{Conf}}_2(M, p_\infty)$ such that $\exists v \in S^2$, $\partial\Omega = F^{-1}(v)$.







Kontsevich's invariants for 3-manifolds

M: 3-dimensional $\mathbb{Z}$-homology sphere
$p_\infty \in M$ a point
F: framing on $M \backslash p_\infty$, "standard" near p_∞
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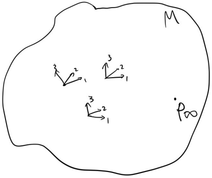

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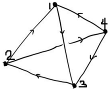

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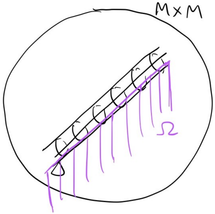

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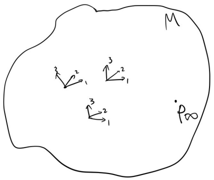

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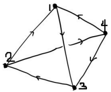

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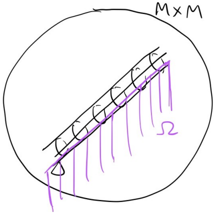
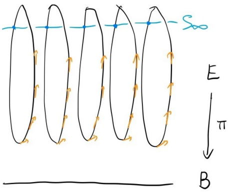


For every edge e, choose a generic propagator
Ω_e, then Kontsevich's invariant for (M, F, Γ) is

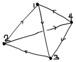
$$\#\Big(\bigcap_{e \in E(\Gamma)} f_e^{-1}(\Omega_e) \subset \overline{\mathrm{Conf}}_{V(\Gamma)}(M, \infty) \Big).$$

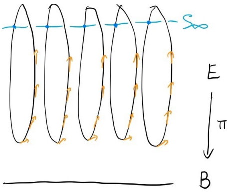
Kontsevich's invariants for fiber bundles



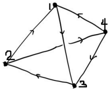
M: d-dimensional $\mathbb{Z}$-homology sphere

Suppose we are given

- a smooth fiber bundle $\begin{array}{c} E \\ \downarrow{\scriptstyle \pi} \\ B \end{array}$ with fiber M,
- a smooth section $s_\infty : B \to E$ and τ: a trivialization of π near s_∞,
- a vertical framing F on $E \backslash s_\infty$ that is "standard" near s_∞,
- a trivalent graph Γ satisfying certain condition, e.g. 



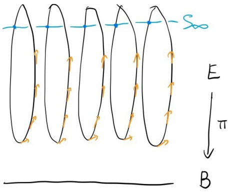
S_∞
E
π
B

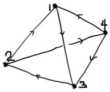


1
2
3
4

Fiberwise construction of $\overline{\mathsf{Conf}}_S(M, p_\infty)$

$\rightsquigarrow \quad \overline{\mathsf{Conf}}_S(\pi, s_\infty) \to B.$





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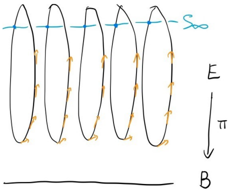
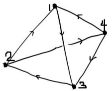


The framing F still induces a map

$$F : \partial\overline{\mathsf{Conf}}_2(\pi, s_\infty) \to S^{d-1}.$$



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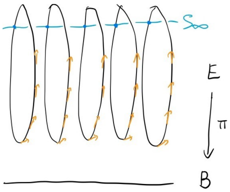

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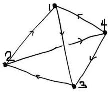

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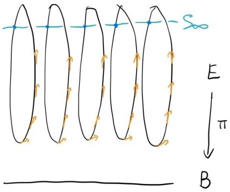

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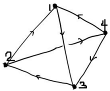

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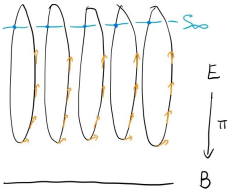

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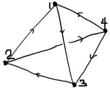

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Kontsevich’s invariant:

$$\text{count (with sign)} \qquad \bigcap_e f_e^{-1}(\Omega_e) \subset \overline{\text{Conf}}_{V(\Gamma)}(\pi, s_\infty).$$

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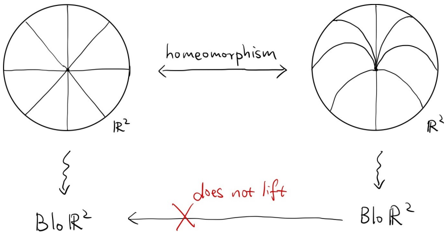

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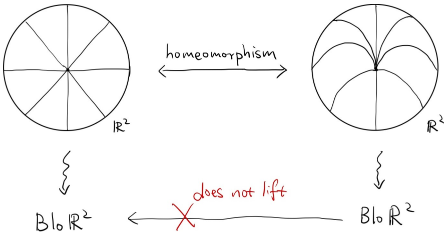

A: The smooth structure is "used up" at the stage of constructing $\overline{\mathrm{Conf}}_2(\pi, s_\infty)$ and the framing map F.

Theorem (C.'23)

Kontsevich's invariants only depend on the topology of

$$\overline{\mathsf{Conf}}_2(\pi, s_\infty) \overset{f_1}{\underset{f_2}{\rightrightarrows}} E \xrightarrow{\pi} B \ , \quad \partial\overline{\mathsf{Conf}}_2(\pi, s_\infty) \xrightarrow{F} S^{d-1}.$$

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Remark: related work Lin-Xie ('23) showing that Kontsevich's invariants only depend on the "formal smooth structure".
$\implies$ Kontsevich's invariants are not very helpful in detecting exotic smooth 4-manifolds.

Thank you!