3/28/2023

Kontsevich's invariants as topological invariants of configuration space bundles (arxiv: 2302.03021)

M : d-dim'l smooth mfld. R-homology sphere
$R \ni \mathbb{Z}$ or $\mathbb{Q}$ or $\mathbb{R}$

$\infty \in M$ chosen point

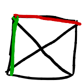
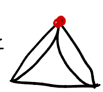
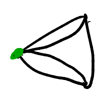
$\mathrm{Diff}(M, N\infty) := \{ \varphi \in \mathrm{Diff}(M) \mid \exists\, U\text{-nbhd of } \infty,\ \varphi|_U = id_U \}$

Kontsevich's invariants (characteristic classes):

Given ① $E \xrightarrow{\pi} B$ a smooth fiber bundle with fiber M and structure group $\mathrm{Diff}(M, N\infty)$ ("(M,∞)-bundle"), and

② a vertical framing F on π "standard near ∞"
(i.e. looks like the standard framing of $\mathbb{R}^d$ near ∞, under some canonical trivialization).

then, for ③ a trivalent graph Γ s.t. $d\Gamma = 0$ in graph homology.

(e.g. $\Gamma =$  , $d\Gamma =$  $+$  $+ \cdots$)

Kontsevich's invariant

$K_{\pi, F, \Gamma} \in H^*(B; R)$.

- S : set, $C_S(M,\infty) := M^S - \{ p_i = \infty \text{ for some } i \in S \}$
$- \{ p_i = p_j \text{ for some } i \neq j \in S \}$

$\overline{C}_S(M,\infty) :=$ Fulton–MacPherson compactification of $C_S(M,\infty)$.

$i \in S \rightsquigarrow f_i : \overline{C}_S(M,\infty) \longrightarrow M$ forgetful map.

- $E \xrightarrow{\pi} B$ an (M,∞)-bundle $\rightsquigarrow$

$C_S(\pi)$, $\overline{C}_S(\pi)$, f_i := fiberwise $C_S(M,\infty)$, $\overline{C}_S(M,\infty)$, f_i.

- F : vertical framing on π "standard near ∞". F induces map

$F : \partial \overline{C}_2(M,\infty) \longrightarrow S^{d-1}$

$\left(\text{on } (f_1,f_2)^{-1}(\Delta) \approx SN(\Delta \subset M \times M) \approx ST\Delta \approx STM \xrightarrow{F} S^{d-1}; \dots \right)$

- <u>Def</u> $\mathcal{G} := \Big\{ (\tilde{h}, h) \in \mathrm{Homeo}(\overline{C}_2(M,\infty)) \times \mathrm{Homeo}(M, N_\infty) \;\Big|$

$$\begin{tikzcd} \overline{C}_2(M,\infty) \arrow[r, "\tilde{h}"] \arrow[d, shift right, "f_1"'] \arrow[d, shift left, "f_2"] & \overline{C}_2(M,\infty) \arrow[d, shift right, "f_1"'] \arrow[d, shift left, "f_2"] \\ M \arrow[r, "h"] & M \end{tikzcd}$$ commutes $\Big\}$.

Thm Suppose $(E' \xrightarrow{\pi'} B', F')$, $(E'' \xrightarrow{\pi''} B'', F'')$ are two smooth (M,∞)-bundles with vertical framings F', F'' standard near ∞.

Suppose

① $\tilde{h} : \bar{C}_2(\pi') \to \bar{C}_2(\pi'')$, $h : E' \to E''$, $h_B : B' \to B''$

are continuous maps s.t. $\tilde{h}, h$ are fiberwise homeo.s,

and

$$\begin{tikzcd}
\bar{C}_2(\pi') \arrow[r, "\tilde{h}"] \arrow[d, shift right, "f_1"'] \arrow[d, shift left, "f_2"] & \bar{C}_2(\pi'') \arrow[d, shift right, "f_1"'] \arrow[d, shift left, "f_2"] \\
E' \arrow[r, "h"] \arrow[d, "\pi'"] & E'' \arrow[d, "\pi''"'] \\
B' \arrow[r, "h_B"] \arrow[u, bend left=60, "s'_\infty"] & B'' \arrow[u, bend right=60, "s''_\infty"']
\end{tikzcd}$$

commutes,

(i.e., $(\tilde{h}, h, h_B)$ is a $\mathcal{G}$-bundle map.)

② $$\begin{tikzcd}
\partial\bar{C}_2(\pi') \arrow[r, "\tilde{h}|_{\partial\bar{C}_2(\pi')}"] \arrow[d, "F'"] & \partial\bar{C}_2(\pi'') \arrow[d, "F''"] \\
S^{d-1} \arrow[r, "h_S"] & S^{d-1}
\end{tikzcd}$$

commutes for some homeomorphism h_S,

then $K_{\pi', F', \Gamma} = h_B^* K_{\pi'', F'', \Gamma}$ for all Γ.

Rmk: This theorem is (sort of) trivial for $d \geq 5$.

$d = 4$ I don't know.

How to prove: re-formulate the construction of Kontsevich's invariants in a way that all definitions are made only using the topological bundle structure of $\bar{C}_2(\pi) \underset{f_2}{\overset{f_1}{\rightrightarrows}} E \xrightarrow{\pi} B$ and the framing map $F: \partial\bar{C}_2(\pi) \to S^{d-1}$.

How to do this: modify the space $\bar{C}_{V(\Gamma)}(M,\infty)$ so that we mostly only work with cohomology classes instead of cochains.

set of vertices of Γ

Propagator class

Recall $F: \partial\bar{C}_2(\pi) \to S^{d-1}$.

Define $\sim_F$ on $\partial\bar{C}_2(\pi)$: $x \sim_F y$ if $F(x) = F(y)$.

Define $q: \bar{C}_2(\pi) \to \bar{C}_2(\pi)/\sim_F$ quotient map.

Def $\Omega \in H^{d-1}(\bar{C}_2(\pi)/\sim_F; R)$ is the <u>propagator class</u> if $\Omega|_{\partial\bar{C}_2(\pi)/\sim_F \approx S^{d-1}}$ is $PD_{S^{d-1}}([pt])$.

Well-definedness:

$$\cdots \to H^{d-1}(\bar{C}_2(\pi)/\sim_F, S^{d-1}) \to H^{d-1}(\bar{C}_2(\pi)/\sim_F) \to H^{d-1}(S^{d-1}) \to$$

$$\begin{matrix} \wr\wr \\ H^{d-1}(\bar{C}_2(\pi), \partial\bar{C}_2(\pi)) = 0 \end{matrix} = H^{d}(\bar{C}_2(\pi), \partial\bar{C}_2(\pi)) \cong H^{d}(\bar{C}_2(\pi)/\sim_F, S^{d-1}) \to \cdots$$

"Modify" $\overline{C}_{V(\Gamma)}(M)$

Ideal case: replace $\overline{C}_{V(\Gamma)}(M)$ by a space X s.t.

- $H^{d|V(\Gamma)|}(X) \approx R$
- has forgetful maps $f_e : X \to \overline{C}_2(M,\infty)/\sim_F$, $\forall e \in \underbrace{E(\Gamma)}_{\text{edge set of } \Gamma}$
- defined only using the topological structure of $\overline{C}_2(M,\infty) \underset{f_2}{\overset{f_1}{\rightrightarrows}} M$,

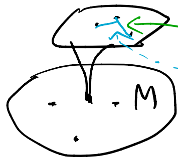
(i.e., X is naturally equipped with a $\mathcal{G}$-action induced from M).

If this is the case, then we take the bundle version $X(\pi)$ of X. (so $f_e : X(\pi) \to \overline{C}_2(\pi)/\sim_F$), and take

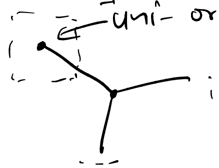
$$\Omega_\Gamma := \bigcup_{e\in E(\Gamma)} f_e^*(\Omega) \in H^*(X(\pi)),$$

and push it forward to $H^*(B; H^*(X)) \approx H^*(B; R)$.

Review: codim-1 boundary strata of $\overline{C}_{V(\Gamma)}(M)$: $A \subseteq V(\Gamma) \cup \{\infty\}$

$\overline{S}_A = \Big\{$  $\Big\}$

<u>Def</u> A is of

- type 1, if Γ_A:  $\exists$ uni- or zero-valent vertex
- type 2, if Γ_A:  $\exists$ bivalent vertex
- type 3, if all vertices of Γ_A are trivalent
- type 4, if $\Gamma_A =$ 

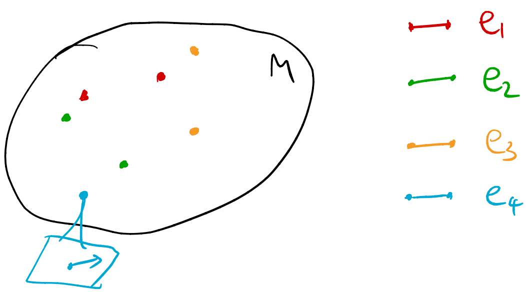
<u>To deal with type 1 strata:</u>

<u>Def</u> $\overline{C}_\Gamma(M) \subseteq \overline{C}_2(M,\infty)^{E(\Gamma)}$ to be the image of

$$f = (f_e)_{e \in E(\Gamma)} : \overline{C}_{V(\Gamma)}(M) \to \overline{C}_2(M,\infty)^{E(\Gamma)}$$

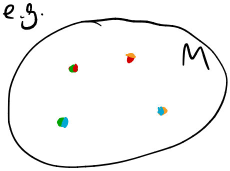
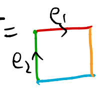
"weaker compactification of $C_{V(\Gamma)}(M)$" — only record directions of points approaching each other when they are connected by edges of Γ.

$\overline{C}_2(M,\infty)^{E(\Gamma)}$: an element looks like



- $f|_{C_{V(\Gamma)}(M)}$ is injective.

e.g.

$f(C_{V(\Gamma)}(M)) = \{$  $\}$ $(\Gamma =$  $)$

$X_1 =$ • •, $X_2 =$ • •

$f_1(X_1) = f_2(X_2)$

...

- $\overline{C}_\Gamma(M) = \overline{f(C_{V(\Gamma)}(M))}$, therefore

$\overline{C}_\Gamma(M)$ can be defined only using the topology of $\overline{C}_2(M,\infty) \rightrightarrows M$.

- $\overline{S}_A$ of type I are "contracted"

<u>Def</u> $\overline{S}_A^\Gamma = f(\overline{S}_A) \subseteq \overline{C}_\Gamma(M)$.

The definition can be made only using the topology of $\overline{C}_2(M,\infty) \rightrightarrows M$ too.

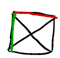
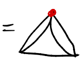
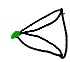
To deal with type 2 and type 4 strata

Morally: • each type 2 stratum has an ori. rev. involution

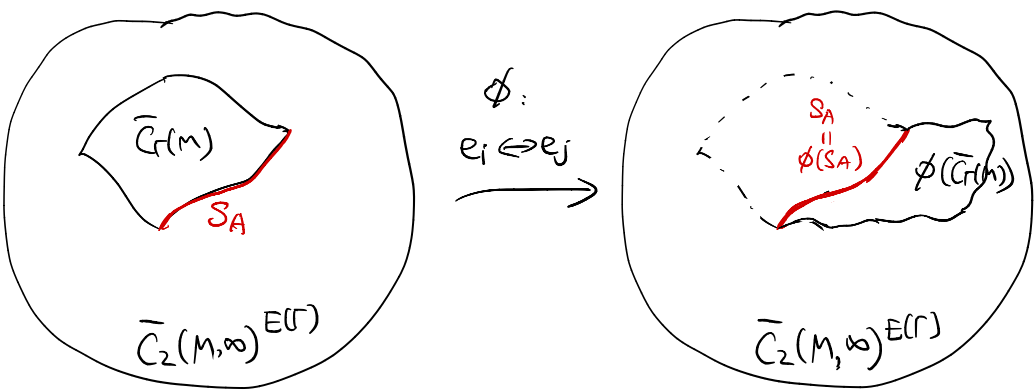
$S_A^\Gamma \ni \Big\{$  $\Big\} \xrightarrow{\phi} \Big\{$  $\Big\} \in S_A^\Gamma$

• type 4 strata can be divided into pairs s.t.

$\exists$ ori. rev. diffeomorphism between each pair.

(because $d\Gamma = 0$) e.g. $\Gamma =$  , $d\Gamma =$  $+$  $+ \cdots$

• These ori. rev. diffeomorphisms can be defined using the topology of $\overline{C}_2(M,\infty) \supseteq M$ only: type 2 case:



(the type 4 case is similar)

- So, we would like to glue $\overline{C}_\Gamma(M)$ to itself along these diffeomorphisms, to cancel out type 2 and type 4 boundary.

(But this imposes technical problems:
  - if we disregard some codim ≥ 2 stuff
    $\rightsquigarrow$ get nice space, but not compact
  - if we include everything
    $\rightsquigarrow$ compact, but the space obtained in not nice.)

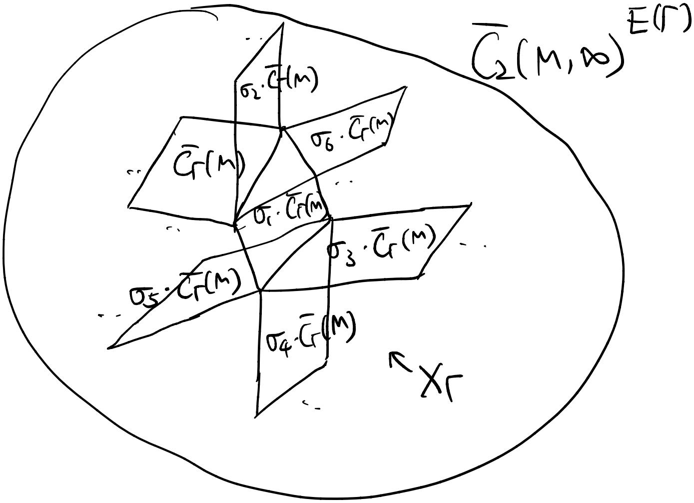
A nice space that makes everything work:

<u>Def</u> $\widetilde{S}_{E(\Gamma)} :=$ { permutations of $E(\Gamma)$ "with sign" }

e.g. $\begin{pmatrix} 1 \to +2 \\ 2 \to -3 \\ 3 \to -1 \end{pmatrix} \in \widetilde{S}_{\{1,2,3\}}$.

Then $\widetilde{S}_{E(\Gamma)}$ acts on $\overline{C}_2(M,\infty)^{E(\Gamma)}$ by permuting factors and composing with
$$\begin{aligned} \overline{C}_2(M,\infty) &\longrightarrow \overline{C}_2(M,\infty) \\ (x,y) &\longrightarrow (y,x) \end{aligned}.$$

<u>Def</u> $X_\Gamma := \bigcup_{\sigma \in \widetilde{S}_{E(\Gamma)}} \sigma \cdot \overline{C}_\Gamma(M) \subseteq \overline{C}_2(M,\infty)^{E(\Gamma)}$



- X_Γ is compact, metrisable, and admits maps $f_e : X_\Gamma \hookrightarrow \overline{C}_2(M,\infty)^{E(\Gamma)} \xrightarrow{\pi_e} \overline{C}_2(M,\infty)$
- X_Γ is defined only using the topology of $\overline{C}_2(M,\infty) \rightrightarrows M$.
- After removing a subset $T_2 \subseteq X_\Gamma$ of codim ≥ 2,

  X_Γ is a topological mfld with boundary and "bindings" $S := \bigcup_{\sigma, A} \sigma\cdot \overline{S}^\Gamma_A$, A of type 3

  s.t.

  at each binding component, signed count of pages $= 0$.

$$\Rightarrow H_c^{d|V(\Gamma)|}(X_\Gamma - T_2 - S; R) \twoheadrightarrow R$$

$$\wr\wr$$

$$H^{d|V(\Gamma)|}(X_\Gamma, T_2 \cup S; R)$$

$$\wr\wr$$

$$H^{d|V(\Gamma)|}(X_\Gamma, S; R)$$

Bundle version of everything :

$$\begin{tikzcd} \overline{C}_\Gamma(M) \arrow[r, squiggly] & \overline{C}_\Gamma(\pi) \arrow[d] \\ & B \end{tikzcd} , \quad \begin{tikzcd} X_\Gamma \arrow[r, squiggly] & X_\Gamma(\pi) \arrow[d] \\ & B \end{tikzcd} , \quad \begin{tikzcd} S_A^\Gamma \arrow[r, squiggly] & S_A^\Gamma(\pi) \arrow[d] \\ & B \end{tikzcd} ,$$

$$\begin{tikzcd} S \arrow[r, squiggly] & S(\pi) \arrow[d] \\ & B \end{tikzcd} , \quad \begin{tikzcd} T_2 \arrow[r, squiggly] & T_2(\pi) \arrow[d] \\ & B \end{tikzcd} .$$

(Well-defined for any $\mathcal{G}$-bundle and are natural w.r.t. $\mathcal{G}$-bundle maps)

To deal with type 3 strata

Some technical lemmas (4.6 & 4.7)

$\rightsquigarrow$ The cup product

$$\begin{tikzcd} H^{d-1}(\overline{C}_2(\pi)/\sim_F) \otimes \cdots \otimes H^{d-1}(\overline{C}_2(\pi)/\sim_F) \arrow[r, "\cup"] \arrow[dr] & H^{|E(\Gamma)|(d-1)}(X_\Gamma(\pi)) \\ & H^{|E(\Gamma)|(d-1)}(X_\Gamma(\pi), S(\pi)) \arrow[u] \end{tikzcd}$$

$$\omega_{e_1} \otimes \cdots \otimes \omega_{e_{|E|}} \longmapsto f_{e_1}^* \omega_{e_1} \cup \cdots \cup f_{e_{|E|}}^* \omega_{e_{|E|}}$$

factors through naturally.

$\Rightarrow$ Def $\Omega_\Gamma(\pi) := \bigcup_{e \in E(\Gamma)} f_e^*(\Omega) \in H^{|E(\Gamma)|(d-1)}(X_\Gamma(\pi), S(\pi))$.

Cohomology pushforward

$$\pi_* : H^{|E(\Gamma)|(d-1)}(X_\Gamma(\pi), S(\pi)) \longrightarrow H^{|E(\Gamma)|(d-1)-|V(\Gamma)|d}\Big(B;\ \underbrace{H^{d|V(\Gamma)|}(X_\Gamma, S)}_{\text{local coefficient on } B}\Big)$$

$$\longrightarrow H^{\cdots}(B; R)$$

<u>Def</u>. $K_{\pi,F,\Gamma} := \pi_*(\Omega_\Gamma(\pi)) \in H^{|E(\Gamma)|(d-1)-|V(\Gamma)|d}(B; R)$.

As promised: all definitions are made only using the topological bundle structure of $\overline{C}_2(\pi) \underset{f_2}{\overset{f_1}{\rightrightarrows}} E \xrightarrow{\pi} B$ and the framing map $F : \partial\overline{C}_2(\pi) \to S^{d-1}$.

Another way to formulate the theorem:

<u>Def</u> A homeomorphism between open subsets $U, V \subseteq \mathbb{R}^d$, $f: U \to V$ is <u>almost smooth</u> if $(f,f): U\times U \to V \times V$ lifts to a homeomorphism $\tilde{f}: Bl_\Delta(U\times U) \to Bl_\Delta(V\times V)$.

$\mathbb{R}^d \to \mathbb{R}^d$

$x \to -\frac{1}{2}\log|x| \cdot x$?

<u>Def</u>. <u>Almost smooth mfld</u> ...

<u>Prop</u> Let $f: U \to V$ be almost smooth, then $\forall x \in U$, $\tilde{f}: ST_xU \to ST_{f(x)}V$ is induced by a linear map $T_xU \to T_{f(x)}V$.

So, we can make sense of the tangent bundle (defined modulo a positive scaling function) on an almost smooth mfld.

<u>Def</u>. A <u>framing</u> on an almost smooth mfld M^d is a map $F: STM \to S^{d-1}$ s.t. is a linear homeomorphism on each fiber.

The theorem can (probably) be rephrased as :

Kontsevich's invariants can be defined for fiber bundles with fiber almost smooth mfld (with structure group almost smooth automorphisms), trivialized near a section S_∞ and has a vertical framing standard near S_∞.

(Fundamental Thm of smoothing theory + Prop above

$\Rightarrow$ almost smooth $=$ smooth in dim ≥ 5)

(it seems..)