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Securitization of Longevity Risk: Pricing Survivor Bonds With Wang Transform in the Lee-Carter Framework

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Abstract

Longevity risk is a major issue for insurers and pension funds, especially in the selling of annuity products. In that respect, securitization of this risk could offer great opportunities for hedging. This article proposes to design survivor bonds which could be issued directly by insurers. In order to guaranty some transparency in the product, the survivor bond is based on a public mortality index. The classical Lee-Carter model for mortality forecasting is used to price a risky coupon survivor bond based on this index.

Introduction and Motivation

Longevity Risk

During the 20th century, the human mortality globally declined. Record life expectancy has been rising over the last century at a remarkably regular pace. In most industrialized countries, mortality at adult and old ages reveal decreasing annual death probabilities (see, e.g., McDonald, Cairns, Gwilt, and Miller, 1998). Since 1970, the main factor driving continued gains in life expectancy in industrialized countries is a reduction of death rates among the elderly.

These mortality improvements pose a challenge for the planning of public retirement systems as well as for the private life annuities business. When long-term living

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benefits are concerned, the calculation of expected present values (for pricing or reserving) requires an appropriate mortality projection in order to avoid underestimation of future costs. Actuaries have therefore to resort to lifetables including a forecast of the future trends of mortality (the so-called projected tables). A new risk thus emerges: the risk that the mortality projections turn out to be erroneous, and that the annuitants live longer than predicted by the projected lifetables. This is the so-called longevity risk.

Mortality Projection Methods

Different approaches for building projected lifetables have been developed so far. Elementary aproaches simply extrapolate observed trends in the sequence of annual death probabilities at each age. The exponential model used by the Mortality Investigation Committee of the United Kingdom Institute of Actuaries falls in this category. Since Cramér and Wold (1935), the evolution over time of graduated mortality curves is popular for the purpose of extrapolation. One classical procedure is based on the projection of parameters; see e.g., Felipe, Guillén, and Perez-Marin (2002) for a recent application. This approach of course heavily relies on the appropriateness of the retained parametric models. To avoid this drawback, Lee and Carter (1992) proposed a simple model for describing the secular change in mortality as a function of a single time index. The main statistical tool of Lee and Carter (1992) is least-squares estimation via singular value decomposition of the matrix of the log age-specific observed forces of mortality together with Box-Jenkins modeling for time series. For a review of recent applications of the Lee-Carter methodology, we refer the interested readers to Lee (2000). For a general account of projecting mortality methods, see Pitacco (2004) and Wong-Fupuy and Haberman (2004).

Voluntary Annuity Markets

In a seminal article, Yaari (1965) considered individuals' decisions to purchase annuities. Relying on a series of strong assumption (namely that consumers are Von Neumann-Morgenstern expected-utility maximizers, that their preferences are time independent, that the only risk they face is longevity risk, that complete insurance for this type of risk is available through annuities, that annuities are actuarially fair, that there is only one asset which pays a given interest rate, and that consumers can borrow and lend at the same rate), the prediction of Yaari's model is that full annuitization is optimal in the absence of a bequest motive (see also Davidoff, Brown, and Diamond, 2003).

Empirical studies seem, however, to contradict the prediction that consumers should annuitize all their wealth. Annuity markets, especially voluntary ones, appear to be very small in most countries. Several arguments can be raised to explain this distortion, namely bequest motives, underestimation of the average remaining life expectancy at age 65 by individuals, adverse selection and actuarially unfair premiums. Another reason why so few people avoid themselves of the private market for annuities is that most individuals benefit from life annuities offered by public Social Security regimes and, for a significant fraction of the labor force, employer-sponsored pension plans. According to Impavido, Thorburn, and Wadsworth (2003), the observed discrepancy between the prediction of Yaari's model and the observed small size of voluntary annuity markets are justified if either consumers have strong bequest motives or if providers charge too high costs for insuring against longevity risk. In the presence of bequest motives, the individuals would be better off by holding a portfolio of annuitized and traditional assets.

The excessive risk borne by insurers offering annuities explains the high premiums charged for this product. Predicting future mortality improvements remains problematic, especially at most advanced ages. Unfortunately, for annuity providers, the experience in both developed and developing countries has been that the rate of improvement is most often found to have been underestimated. Underestimation of improvements means that companies have to increase provisions. Capital and reinsurance are classical protection against this risk but turn out to be rather expensive.

In light of a baby-boom cohort near retirement, of possible reforms of public pension regimes and the shift from defined benefit to defined contribution private pension plans, an increased interest in individual annuity products can be expected in the future. Several European governments also envisage shifting (at least partially) from a pure pay-as-you-go to funding methods for public pensions. This will make very attractive hedging mechanisms for longevity risk, as those described in this article.

Securitization of Longevity Risk

Since no one can accurately predict the future, risk management of mortality and longevity is an indispensable part in the annuity providers' operations. Life annuity contracts typically run for several decades so that a lifetable which may seem to be on the safe side at the beginning of the contract might well turn out not to be so. Moreover, contrarily to financial assets (which can be very volatile), changes in forces of mortality occur slowly and pose a long-term, but permanent, problem.

Reinsurance treaties covering longevity risk are usually expensive and many life insurance companies are reluctant to buy long-term reinsurance coverage (because of substantial credit risk). Blake and Burrows (2001) suggested that the governments should help insurance companies hedge their mortality risks by issuing survivor bonds whose coupon payments depend on the proportion of the population surviving to particular ages. The survivor bonds provide a very good hedge against mortality improvement risk; if annuitants live longer, the insurance companies would then make annuity payments for longer periods, but they would also receive greater offsetting coupon payments on their survivor bonds asset positions. See also Dowd (2003), and the reply by Blake (2003).

Securitization offers an interesting alternative to reinsurance. We refer the reader to the recent article by Cowley and Cummins (2005) for an overview about securitization of life insurance assets and liabilities. In comparison with reinsurance treaties, mortality contracts sold on financial markets only depend on the general development of mortality and are not tailored to the insurer's portfolio. As pointed out by Lin and Cox (2005), the potential for greater underwriting capacity, innovative long term contracting, and lower costs make securitization worth investigating as a supplement to traditional reinsurance.

With the expected expansion of the private life annuity markets, insurers will have to manage their risk in issuing new annuity policies. In that respect, securitization of longevity risk appears especially promising. In the last few decades, catastrophic risk has been successfully passed to financial markets using the so-called CAT-bonds; see e.g., Cox and Pedersen (2000), Cox, Pedersen, and Fairchild (2000), and Cummins, Lalonde, and Philips (2004). In contrast with CAT-bonds, mortality-based financial instruments may not be zero-beta assets due to the link between stock prices and demographic changes. Let us briefly discuss this controversial issue.

According to the life-cycle theory, workers first save for retirement (accumulation phase) and start a decumulation phase when they retire, spending the accumulated savings. The equilibrium asset prices in the economy are thus influenced by the demographically-driven supply and demand. This is particularly appealing as the baby-boom cohort (those born roughly in the two decades after World War II) in the European Union and United States nears and moves to retirement.

In the United States, much has been written about the relationship between the baby-boom generation retirement savings and the phenomenal increase in financial asset prices which characterized the period from 1990 through 1999. This generation entered its peak savings years and several authors argue that asset prices are likely to fall as baby boomers retire: when agents retire, they dis-save and sell the assets not bequeathed to the next generation to fund their consumption. Ang and Maddaloni (2003) conducted an empirical study for the G5 countries. They demonstrated that the most powerful predictive variable for international excess returns is the change in the proportion of retired people as a fraction of of the adult population. A growing proportion of retired people forecasts decreases in the equity premium. Jamal and Quayes (2004) investigated the relationship between the demographic structure and stock prices, incorporating demand and supply factors. They found that the proportion of the population in the prime earning age has a direct influence on stock prices.

Of course, this discussion ignores the fact that in a global economy the assets being sold might be purchased by the pension funds of young workers in Asia, which would at least partially counter the shock.

To sum up, the increase in longevity does not directly influence financial asset prices. More subtly, the decrease observed for annual death probabilities modifies the demographic structure, which in turn has some effect on stock prices. But, even if the longevity risk has a positive or negative correlation with the market, some investors may buy mortality-based bonds as a diversification. We will come back to these issues later in the article.

Agenda

Our aim is to develop an index of mortality and to describe how bonds could base their coupons on that index. The mortality index will be based on the time index resulting from the fit of Lee-Carter model.

Section Stochastic Modeling for Dynamic Mortality introduces the notation and assumption used thoughout this article, as well as the Lee-Carter modeling for dynamic mortality. Section Pricing Survivor Bonds With Wang Distorted Risk Measures describes the pricing of the survivor bonds with the help of Wang risk measure. Section Bounds on Certainty Equivalents proposes several bounds on the price of the survivor bonds. Section Numerical Illustration offers a numercial illustration using Belgian data. The final section concludes.

STOCHASTIC MODELING FOR DYNAMIC MORTALITY

Notation

We analyze the changes in mortality as a function of both age x and time t. This "period analysis" is known to be more appropriate than a "cohort analysis;" we refer the interested reader, e.g., to Tuljapurkar and Boe (1998) for more details. Henceforth,

- $T_x(t)$ is the remaining lifetime of an individual aged x on January the first of year t; this individual will die at age $x + T_x(t)$ in year $t + T_x(t)$.
- $q_x(t)$ is the probability that an x-aged individual in calendar year t dies before reaching age $x + 1$, i.e., $q_x(t) = \Pr[T_x(t) \leq 1]$.
- $p_x(t) = 1 - q_x(t)$ is the probability that an x-aged individual in calendar year t reaches age $x + 1$, i.e., $p_x(t) = \Pr[T_x(t) > 1]$.
- $\mu_x(t)$ is the mortality force at age x during calendar year t, that is,

$$\mu_x(t) = \lim_{\Delta \to 0} \frac{\Pr[x < T_0(t-x) \leq x + \Delta \mid T_0(t-x) > x]}{\Delta}.$$

As pointed out by Dahl (2004), actuaries have traditionally been calculating premiums and reserves using a deterministic mortality intensity. Here, as in the article by Dahl (2004), the mortality force at age x, $\mu_x(t)$, is a stochastic process. We will make this point clear further in the article, when the Lee-Carter model will be described.

Assumption

In this article, we assume that the age-specific mortality rates are constant within bands of age and time, but allowed to vary from one band to the next. Specifically, given any integer age x and calendar year t, it is supposed that

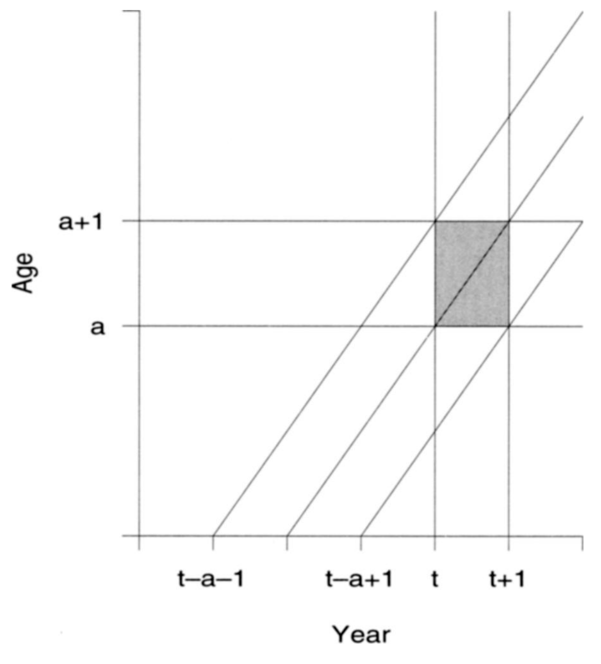
$$\mu_{x+\xi}(t + \tau) = \mu_x(t) \quad \text{for} \quad 0 \leq \xi, \quad \tau < 1. \tag{1}$$

Under (1), we have for integer age x and calendar year t that

$$p_x(t) = \exp(-\mu_x(t)). \tag{2}$$

Assumption (1) is best illustrated with the aid of a coordinate system that has calendar time as abscissa and age as coordinate. Such a representation is called a Lexis diagram after the German demographer who introduced it. Both time scales are divided into yearly bands, which partition the Lexis plane into squares, as shown in Figure 1. Equation (1) assumes that the mortality rate is constant within each shaded square corresponding to integer age a and calendar year t, but allows it to vary from one square to another.

FIGURE 1
Lexis Diagram



Lee-Carter Model

Let us recall the basic features of the classical Lee-Carter approach. The latter is in essence a relational model assuming that

$$\ln \mu_x(t) = \alpha_x + \beta_x \kappa_t, \tag{3}$$

where the parameters β_x and κ_t are subject to the constraints

$$\sum_t \kappa_t = 0 \quad \text{and} \quad \sum_x \beta_x = 1 \tag{4}$$

ensuring model identification.

Interpretation of the parameters involved in model (3) is quite simple. The value of α_x exactly equals the average of $\ln \mu_x(t)$ over time t so that $\exp \alpha_x$ is the general shape of the mortality schedule. The actual forces of mortality change according to an overall mortality index κ_t modulated by an age response β_x. The shape of the β_x profile tells which rates decline rapidly and which slowly over time in response to change in κ_t. The time factor κ_t is intrinsically viewed as a stochastic process and Box-Jenkins techniques are then used to model and forecast κ_t.

Time Index

To forecast, Lee and Carter (1992) assume that the α_x's and β_x's remain constant over time and forecast future values of κ_t using a standard univariate time series model. After testing several specifications, they found that a random walk with drift was the most appropriate model for their data. They made clear that other ARIMA models might be preferable for different data sets, but in practice the random walk with drift model for κ_t is used almost exclusively. According to this model, the κ_t's obey to

$$\kappa_t = \kappa_{t-1} + \theta + \xi_t \text{ with iid } \xi_t \sim \mathcal{N}or(0, \sigma^2), \tag{5}$$

where θ is known as the drift parameter and $\mathcal{N}or(0, \sigma^2)$ stands for the Normal distribution with mean 0 and variance σ^2. We will retain the model (5) throughout this article. Note that since the κ_t's obey to the dynamics of Equation (5), the $\mu_x(t)$'s are not constant but develop over time following a stochatic process.

We will assume in the remainder of this article that the values $\kappa_1, \ldots, \kappa_{t_0}$ are known but that the κ_{t_0+k}'s, $k = 1, 2, \ldots,$ are unknown and have to be projected from (5). To forecast the time index at time $t_0, + k$ with all data available up to t_0, we use

$$\kappa_{t_0+k} = \kappa_{t_0} + k\theta + \sum_{j=1}^{k} \xi_{t_0+j}.$$

The point estimate of the stochastic forecast is thus

$$\mathbb{E}[\kappa_{t_0+k} \mid \kappa_1, \ldots, \kappa_{t_0}] = \kappa_{t_0} + k\theta$$

which follows a straight line as a function of the forecast horizon k, with slope θ. The conditional variance of the forecast is

$$\mathbb{V}[\kappa_{t_0+k} \mid \kappa_1, \ldots, \kappa_{t_0}] = k\sigma^2.$$

Therefore, the conditional standard errors for the forecast increase with the square root of the distance to the forecast horizon k.

PRICING SURVIVOR BONDS WITH WANG DISTORTED RISK MEASURES

Hedging Longevity Risk

If L_{x_0} is the total number of annuities issued to the initial cohort made up of individuals aged x_0 at time t_0, then the insurer's payout at time t seen from time t_0, denoted as CF_t, is given by

$$CF_t = L_{x_0\,t}p_{x_0}^{\text{prosp}},$$

where

$$
\begin{aligned}
{}_t p_{x_0}^{\text{prosp}} &= \prod_{j=0}^{t-1} p_{x_0+j}(t_0+j) \\
&= \exp\left(-\sum_{j=0}^{t-1} \mu_{x_0+j}(t_0+j)\right) \\
&= \exp\left(-\sum_{j=0}^{t-1} \exp(\alpha_{x_0+j} + \beta_{x_0+j}\kappa_{t_0+j})\right).
\end{aligned}
$$

Note that the ${}_t p_{x_0}^{\text{prosp}}$'s are random since they involve the κ_{t_0+j}'s obeying to (5). On the other hand, the initial single premium paid by each individual of the cohort is based on an expected cash flow given by

$$
CF_t^* = L_{x_0}\, {}_t p_{x_0}^{\text{ref}},
$$

where the t-year survival probability ${}_t p_{x_0}^{\text{ref}}$ is computed based on the reference lifetable (prospective or not) used by the insurer for pricing and reserving. Throughout this article, we will assume that the insurer uses the pointwise projections obtained from the Lee-Carter model, that is,

$$
{}_t p_{x_0}^{\text{ref}} = \exp\left(-\sum_{j=0}^{t-1} \exp(\alpha_{x_0+j} + \beta_{x_0+j}(\kappa_{t_0} + j\theta))\right). \tag{6}
$$

The possible loss experienced by the insurer is then given by

$$
S_t = CF_t - CF_t^* = L_{x_0}\left({}_t p_{x_0}^{\text{prosp}} - {}_t p_{x_0}^{\text{ref}}\right).
$$

Suppose now that the insurer issues at time t_0 an index-linked bond of floating coupon denoted K_t; let N_{x_0} be the total nominal value of the bonds issued. At the same time he uses the amount collected to buy fixed-rate bonds of coupon k.

The two bonds are supposed to have the same maturity n. At time t, $0 < t \le n$, the difference betwen received coupons (fixed leg) and coupons to pay (floating leg) is given by

$$
\Delta_t = N_{x_0}(k - H_t).
$$

The longevity risk is perfectly hedged if

$$
\begin{aligned}
\Delta_t = S_t \Leftrightarrow N_{x_0}(k - H_t) &= L_{x_0}\left({}_t p_{x_0}^{\text{prosp}} - {}_t p_{x_0}^{\text{ref}}\right) \\
\Leftrightarrow H_t &= k + \frac{L_{x_0}}{N_{x_0}}\left({}_t p_{x_0}^{\text{prosp}} - {}_t p_{x_0}^{\text{ref}}\right).
\end{aligned}
$$

If we chose the number of bonds to issue such that

$$\frac{L_{x_0}}{N_{x_0}} = k \Leftrightarrow N_{x_0} = \frac{L_{x_0}}{k}$$

then one has a form of floating rate

$$H_t = k\big(1 + {}_t p_{x_0}^{\text{ref}} - {}_t p_{x_0}^{\text{prosp}}\big).$$

This motivates us to build a floating bond with coupons given by $K_t = H_t + k^* = kC_t + k^*$ where $C_t = 1 + {}_t p_{x_0}^{\text{ref}} - {}_t p_{x_0}^{\text{prosp}}$ is the index, k^* is an additive margin and k is the usual fixed coupon.

Design of Survivor Bonds

Let t_0 be the calendar year during which a group of annuity contracts are issued to a cohort of annuitants aged x_0. For instance, x_0 may be the retirement age, and we follow the cohort of individuals retiring in year t_0 and buying an annuity when they stop working.

The coupon paid by the survivor bond should be related to the deviation of the actual survival probability ${}_t p_{x_0}^{\text{prosp}}$ compared to the survival probability ${}_t p_{x_0}^{\text{ref}}$ coming from the reference lifetable (used to price the annuities). Specifically, the general form of the coupon of the index-linked bond paid at time t is

$$K_t = k\big(1 + {}_t p_{x_0}^{\text{ref}} - {}_t p_{x_0}^{\text{prosp}}\big) + k^*,$$

where k is the standard coupon and k^* is an additive margin of the bond (risk premium paid by the investor who will assume the longevity risk, i.e., the risk that ${}_t p_{x_0}^{\text{ref}} < {}_t p_{x_0}^{\text{prosp}}$). The way to define this margin will be discussed in Section Valuation of Longevity Risk.

It is clear from the formula defining the coupon K_t that the risk passed to the financial market is not the increase in longevity itself, but rather that future survival probabilities exceed those corresponding to a reference lifetable. In that respect, a survivor bond with coupon K_t could attract any investor interested in diversification, who will be rewarded by the risk premium. We will come back to this issue in the conclusion.

Mortality Index in the Lee-Carter Framework

For the cohort aged x_0 in year t_0, we define a mortality index I_t, $t = 1, 2, \ldots,$ as

$$I_t = {}_t p_{x_0}^{\text{prosp}} = \exp(-S_t) \tag{7}$$

with

$$S_t = \sum_{j=0}^{t-1} \exp\big(\alpha_{x_0+j} + \beta_{x_0+j}\kappa_{t_0+j}\big) = \sum_{j=0}^{t-1} \delta_j \exp(X_j),$$

where $\delta_j = \exp(\alpha_{x_0+j}) > 0$ and $X_j = \beta_{x_0+j}\kappa_{t_0+j}$. Conditional upon κ_{t_0}, we have that $X_j \sim \mathcal{N}or(\mu_j, \sigma_j^2)$ with

$$\mu_j = \beta_{x_0+j}(\kappa_{t_0} + j\theta) \quad \text{and} \quad \sigma_j^2 = (\beta_{x_0+j})^2 j\sigma^2, \tag{8}$$

with the convention that a normally distributed random variable with zero variance is constantly equal to the mean. Multiplied by the number of individuals initially present in the cohort, I_t gives the expected number of survivors after t years.

Pricing via Certainty Equivalence

To make the pricing of survivor bonds operational, we need a dynamic mortality model, such as presented in Section Stochastic Modeling for Dynamic Mortality, and we need a term structure of interest rates.

The pricing methodology used below can be developed for very general term structures. The main assumption is the independence between mortality uncertainty and financial risks.

As usual, let us denote as $P(s, t)$ the price of a zero-coupon bond issued at time s and paying \$1 at time t, $s \leq t$. We define also the risk-free rate $r(t)$. Under the risk neutral measure $\mathbb{Q}$, we can write

$$P(s,t) = \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_s^t r(u)\,du\right) \Big| \mathcal{F}_s\right].$$

We assume that the risk-free rate process and $\{\kappa_{t_0+k}, k = 1, 2, \ldots\}$ are independent.

A possible classical model for the risk-free rate is given by the Hull and White model, according to which the risk-free rate process obeys to the equation

$$dr(t) = A(m(t) - r(t))\,dt + \sigma\,dW(t),$$

where $\{W(t), t \geq 0\}$ is a Browian motion under $\mathbb{Q}$ and m is a deterministic function of time. Other structures can of course be considered.

Then, we derive the price of the survivor bond from the following indifference equation:

$$\sum_{t=1}^{n} kP(0,t) + P(0,n) = \sum_{t=1}^{n} \varrho[K_t]P(0,t) + P(0,n), \tag{9}$$

where $\varrho\,[K_t]$ is the certainty equivalent to the future random cash flow K_t. The left-hand side is the present value of an obligation paying constant annual coupon k whereas the right-hand side is the present value of the survivor bond. To write the indifference Equation (9), we have assumed that mortality risk is independent from financial risk. Coming back to the introduction, we agree that the validity of this assumption can be questioned. Nevertheless, we retain it (at least as a working hypothesis) essentially because we cannot infer the nature of the eventual dependence between mortality risk and financial risk.

Valuation of Longevity Risk

Now, let ϱ be a Wang distorted risk measure, i.e., ϱ can be cast into the form

$$\varrho[K_t] = \int_0^{+\infty} g(\Pr[K_t > \xi])\, d\xi,$$

where the distortion function g is assumed to be nondecreasing and concave, with $g(0) = 0$ and $g(1) = 1$. Then, ϱ is known to be coherent in the sense of Artzner et al. (1999). This allows us to rewrite the indifference Equation (9) as

$$0 = \sum_{t=1}^{n} P(0,t)\big(k - k\big(1 + {}_t p_{x_0}^{\mathrm{ref}} - \varrho[I_t]\big) - k^*\big)$$

$$\Leftrightarrow \tilde{k} = \frac{k^*}{k} = \frac{\sum_{t=1}^{n} P(0,t)\big(\varrho[I_t] - {}_t p_{x_0}^{\mathrm{ref}}\big)}{\sum_{t=1}^{n} P(0,t)}, \tag{10}$$

where $\tilde{k}$ is the relative additive margin of the survivor bond.

Formula (10) gives us a way to define the risk premium k^* in relation with the risk behavior of the investor (modeled by the Wang distorted risk measure ϱ).

Wang Risk Transform

Definition. Wang (2000) recently proposed to price risks using the transform that now bears his name, that is the so-called Wang transform. This transform relies on a single parameter, called the market price of risk (extending the well-known Sharpe ratio in CAPM or the Black-Scholes pricing approach). Lin and Cox (2005) successfully applied Wang transform to price mortality risk bonds. We will use this risk measure throughout this work.

Given a distribution function F, its Wang transform F_λ^* is defined as

$$F_\lambda^*(x) = \Phi(\Phi^{-1}(F(x)) + \lambda), \quad x \in \mathbb{R}, \tag{11}$$

where Φ is the distribution function corresponding to the standard Normal distribution and λ is a parameter called the market price of risk.

Given a risk with distribution function F, the corresponding Wang risk measure is the expected value associated with the distribution function F_λ^*, i.e.

$$\varrho_\lambda^*[X] = \int_0^{+\infty} \bar{F}_\lambda^*(t)\, dt, \tag{12}$$

where $\bar{F}_\lambda^* = 1 - F_\lambda^*$ is the survival function corresponding to the distribution function F_λ^*.

Application to Survivor Bonds

We have now to determine $\varrho_\lambda^*[I_t]$, where ϱ_λ^* is the Wang risk measure (12). In this case, the distribution function F_t of I_t is given by

$$F_t(p) = \Pr[I_t \leq p] = \Pr[S_t \geq -\ln p], \quad 0 \leq p \leq 1,$$

where S_t is a linear combination of correlated LogNormal random variables. Then,

$$\varrho_\lambda^*[I_t] = \int_0^1 \big(1 - \Phi\big(\Phi^{-1}(F_t(p)) + \lambda\big)\big)\, dp. \tag{13}$$

The analytical computation of $\varrho^*[I_t]$ via (13) is difficult and numerical alternatives must be contemplated. A convenient procedure consists in simulating the X_j's (from the dynamics (5) for the κ_t's) to approximate the distribution function of the S_t's. In the next section, we derive bounds on $\varrho_\lambda^*[I_t]$. These bounds will be seen to be accurate enough for practical purposes.

BOUNDS ON CERTAINTY EQUIVALENTS

A Useful Property

Wang transforms enjoy the following nice property:

Property 1: *Let X and X_λ^* have distribution function F and F_λ^* given in (11), respectively. Then, given $Z \sim \mathcal{N}or(\mu, \sigma^2)$ and an increasing continuous function h, $X = h(Z) \Rightarrow X_\lambda^*$ is distributed as $h(Z_\lambda^*)$ where $Z_\lambda^* \sim \mathcal{N}or(\mu - \lambda\sigma, \sigma^2)$.*

Proof. Clearly, for any real x,

$$F(x) = \Pr[Z \leq h^{-1}(x)] = \Phi\left(\frac{h^{-1}(x) - \mu}{\sigma}\right)$$

so that the Wang transform F_λ^* is given by

$$\begin{aligned} F_\lambda^*(x) &= \Phi\left(\frac{h^{-1}(x) - \mu}{\sigma} + \lambda\right) \\ &= \Phi\left(\frac{h^{-1}(x) - (\mu - \lambda\sigma)}{\sigma}\right) \\ &= \Pr\left[h\left(Z_\lambda^*\right) \leq x\right] \end{aligned}$$

as announced. Q.E.D.

In practice, we can use Property 1 as follows. Assume one is interested in $\varrho_\lambda^*[X]$. Then, it suffices to compute the expectation of X_λ^*, that is,

$$\varrho_\lambda^*[X] = \mathbb{E}\big[X_\lambda^*\big] = \mathbb{E}\big[h\big(Z_\lambda^*\big)\big].$$

The Strategy

Recall from (7) that $\varrho_\lambda^*[I_t] = \varrho_\lambda^*[\exp(-S_t)]$. Let us denote as $\preceq_{\text{cx}}$ the convex order (i.e., $X \preceq_{\text{cx}} Y$ if, and only if, $\mathbb{E}[v(X)] \leq \mathbb{E}[v(Y)]$ for all the convex functions v such that the expectations exist). From Theorem 1 in Dhaene et al. (2002), we know that there exist two functions $h_1^{(t)}$ and $h_2^{(t)}$ such that the stochastic inequalities

$$h_1^{(t)}(Z) \preceq_{\text{cx}} S_t \preceq_{\text{cx}} h_2^{(t)}(Z) \Leftrightarrow -h_1^{(t)}(Z) \preceq_{\text{cx}} -S_t \preceq_{\text{cx}} -h_2^{(t)}(Z)$$

hold true for each t, where $Z \sim \mathcal{N}or(0,1)$. This in turn implies

$$\exp\left(-h_1^{(t)}(Z)\right) \preceq_{\text{icx}} \exp(-S_t) \preceq_{\text{icx}} \exp\left(-h_2^{(t)}(Z)\right),$$

where $\preceq_{\text{icx}}$ denotes the increasing convex order, also called stop-loss order in actuarial circles (i.e., $X \preceq_{\text{icx}} Y$ if, and only if, $\mathbb{E}[v(X)] \leq \mathbb{E}[v(Y)]$ for all the nondecreasing convex functions v such that the expectations exist).

Now, any coherent Wang distorted risk measure is in agreement with a comparison in the $\preceq_{\text{icx}}$ sense, so that with ϱ_λ^* as in (12), we have

$$\varrho_\lambda^*\left[\exp\left(-h_1^{(t)}(Z)\right)\right] \leq \varrho_\lambda^*[I_t] \leq \varrho_\lambda^*\left[\exp\left(-h_2^{(t)}(Z)\right)\right]. \tag{14}$$

The computations on the bounds in (14) are easy considering Property 1. We indeed have

$$\varrho_\lambda^*\left[\exp\left(-h_1^{(t)}(Z)\right)\right] = \mathbb{E}\left[\exp\left(-h_1^{(t)}(Z_\lambda^*)\right)\right]$$

and

$$\varrho_\lambda^*\left[\exp\left(-h_2^{(t)}(Z)\right)\right] = \mathbb{E}\left[\exp\left(-h_2^{(t)}(Z_\lambda^*)\right)\right],$$

where Z and Z_λ^* are as in Property 1.

Derivation of the Upper Bound on $\varrho_\lambda^*[I_t]$

An upper bound S_t^u in the convex sense on S_t is given by

$$S_t \preceq_{\text{cx}} S_t^u = \sum_{j=0}^{t-1} \delta_j \exp(\mu_j + \sigma_j Z), \quad \text{with} \quad Z \sim \mathcal{N}or(0,1),$$

where μ_j and σ_j are given in (8). The upper bound S_t is of the form $h_2^{(t)}(z)$, with the function $h_2^{(t)}$ defined as

$$h_2^{(t)}(z) = \sum_{j=0}^{t-1} \delta_j \exp(\mu_j + \sigma_j z).$$

Now, the upper bound on $\varrho^*_\lambda[I_t]$ is given by

$$\begin{aligned}
\mathbb{E}\big[\exp\big(-h_2^{(t)}(Z^*_\lambda)\big)\big] &= \int_0^1 \Pr\big[\exp\big(-h_2^{(t)}(Z^*_\lambda)\big) > x\big]\,dx \\
&= \int_0^1 \Pr\big[h_2^{(t)}(Z^*_\lambda) < -\ln x\big]\,dx \\
&= \int_0^1 \Pr\left[\sum_{j=0}^{t-1} \delta_j \exp(\mu_j - \lambda\sigma_j + \sigma_j Z) < -\ln x\right] dx \\
&= \int_0^1 G_2^{(t)}(-\ln x)\,dx,
\end{aligned}$$

where $G_2^{(t)}(-\ln x) = \Phi(\nu_x)$ with ν_x given by the solution of the equation

$$\sum_{j=0}^{t-1} \delta_j \; \exp(\mu_j - \lambda\sigma_j + \sigma_j \nu_x) = -\ln x.$$

Derivation of the Lower Bound on $\varrho[I_t]$

A lower bound S_t^l in the convex sense on S_t is obtained by conditioning S_t on some random variable T (since we know from Strassen's theorem that $\mathbb{E}[S_t \mid T] \preceq_{cx} S_t$). The problem is then to choose an appropriate conditioning variable T (making $\mathbb{E}[S_t \mid T]$ and S_t "as close as possible"). Following Kaas, Dhaene, and Goovaerts (2000), we determine T by a first-order approximation to S_t, i.e.,

$$\begin{aligned}
S_t &= \sum_{j=0}^{t-1} \delta_j \exp(X_j) \\
&= \sum_{j=0}^{t-1} \delta_j \exp(\mu_j + (X_j - \mu_j)) \\
&\approx \sum_{j=0}^{t-1} \delta_j \exp(\mu_j)(1 + (X_j - \mu_j)) \\
&= c + \sum_{j=0}^{t-1} \delta_j \exp(\mu_j) X_j,
\end{aligned}$$

where c is the appropriate constant. So, we take

$$T = \sum_{j=0}^{t-1} \delta_j \exp(\mu_j) X_j.$$

The lower bound $\mathbb{E}[S_t \mid T]$ is then given by

$$S_t^l = \sum_{j=0}^{t-1} \delta_j \exp\left(\mu_j + \rho_j \sigma_j Z + \frac{1}{2}(1-\rho_j^2)\sigma_j^2\right) \preceq_{\mathrm{cx}} S_t,$$

where ρ_i, $i = 0, \ldots, t-1$, is the correlation coefficient between T and X_i, that is,

$$\rho_i = \frac{\mathbb{C}\mathrm{ov}[X_i, T]}{\sigma_i \sigma_T} = \frac{\sum_{j=0}^{t-1} \delta_j \exp(\mu_j) \mathbb{C}\mathrm{ov}[X_i, X_j]}{\sigma_i \sqrt{\sum_{j=0}^{t-1}\sum_{k=0}^{t-1} \delta_j \delta_k \exp(\mu_j + \mu_k) \beta_{x_0+j} \beta_{x_0+k} \min\{j,k\} \sigma^2}},$$

where

$$\mathbb{C}\mathrm{ov}[X_i, X_j] = \beta_{x_0+i} \beta_{x_0+j} \min\{i, j\} \sigma^2.$$

In the application we have in mind, β_{x_0+i} and β_{x_0+j} typically have the same sign so that all the ρ_i's are nonnegative. Then, S_t^l is the sum of t comonotonic random variables, which makes the derivation of its distribution function easy.

First, note that $S_t^l = h_1^{(t)}(Z)$ with

$$h_1^{(t)}(z) = \sum_{j=0}^{t-1} \delta_j \exp\left(\mu_j + \rho_j \sigma_j z + \frac{1}{2}(1-\rho_j^2)\sigma_j^2\right).$$

The function $h_1^{(t)}$ defined above is nondecreasing as $\rho_j \geq 0$ holds true for all j (which will be the case in the applications). The lower bound on $\varrho_\lambda^*[I_t]$ is then obtained as follows:

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(-h_1^{(t)}(Z_\lambda^*)\right)\right] \\
&= \int_0^1 \Pr\left[\exp\left(-h_1^{(t)}(Z_\lambda^*)\right) > x\right] dx \\
&= \int_0^1 \Pr\left[h_1^{(t)}(Z_\lambda^*) < -\ln x\right] dx \\
&= \int_0^1 \Pr\left[\sum_{j=0}^{t-1} \delta_j \exp\left(\mu_j - \lambda\rho_j\sigma_j + \rho_j \sigma_j Z + \frac{1}{2}(1-\rho_j^2)\sigma_j^2\right) < -\ln x\right] dx \\
&= \int_0^1 G_1^{(t)}(-\ln x)\, dx,
\end{aligned}$$

where $G_1^{(t)}(-\ln x) = \Phi(\nu_x)$ with ν_x given by the solution of the equation

$$\sum_{j=0}^{t-1} \delta_j \exp\left(\mu_j - \lambda\rho_j\sigma_j + \rho_j\sigma_j\nu_x + \frac{1}{2}(1-\rho_j^2)\sigma_j^2\right) = -\ln x.$$

NUMERICAL ILLUSTRATION

Description of the Data Set

The data used to illustrate this article relate to the Belgian population, males and females separately. They concern death probabilities computed by the BfP (for "Bureau fédéral du Plan," a federal agency based in Brussels) based on the number of deaths by gender, age, and year of birth, as well as the corresponding initial exposure-to-risk (on January the first of each year). Here we use data relating to calendar years $t = 1970, \ldots, 2001$ and ages $x = 50, \ldots, 99$. No preliminary smoothing procedure is applied but the lifetables are closed until 125 years of age as described in Denuit and Goderniaux (2005).

The data are displayed in Figure 2 for Belgian males and in Figure 3 for Belgian females. The conventional way to examine mortality is to plot the logarithm of the mortality rates against age. As calendar time also enters the problem, we plot here the mortality surface $(x, t) \mapsto \widehat{q_x(t)}$. The mortality surface on the log-scale shows the classical pattern of mortality: severe mortality at young ages, then accident hump and finally considerable variations at advanced ages. Moreover, it portrays the development of death probabilities over calendar time.

Lee-Carter Projection Model

Formula (2) allows us to compute the forces of mortality as

$$\widehat{\mu_x(t)} = -\ln(1 - \widehat{q_x(t)}) \text{ provided } \widehat{q_x(t)} < 1.$$

The $\widehat{\mu_x(t)}$'s will be the target for modeling. We fit the Lee-Carter model and project the κ_t's beyond the observation horizon. The results are displayed in Figure 4. We obtained $\hat{\theta} = -0.8318936$ and $\hat{\sigma}^2 = 0.8112362$ for men and $\hat{\theta} = -1.048672$ and $\hat{\sigma}^2 = 1.390124$ for women.

Estimation of the Market Price of Risk

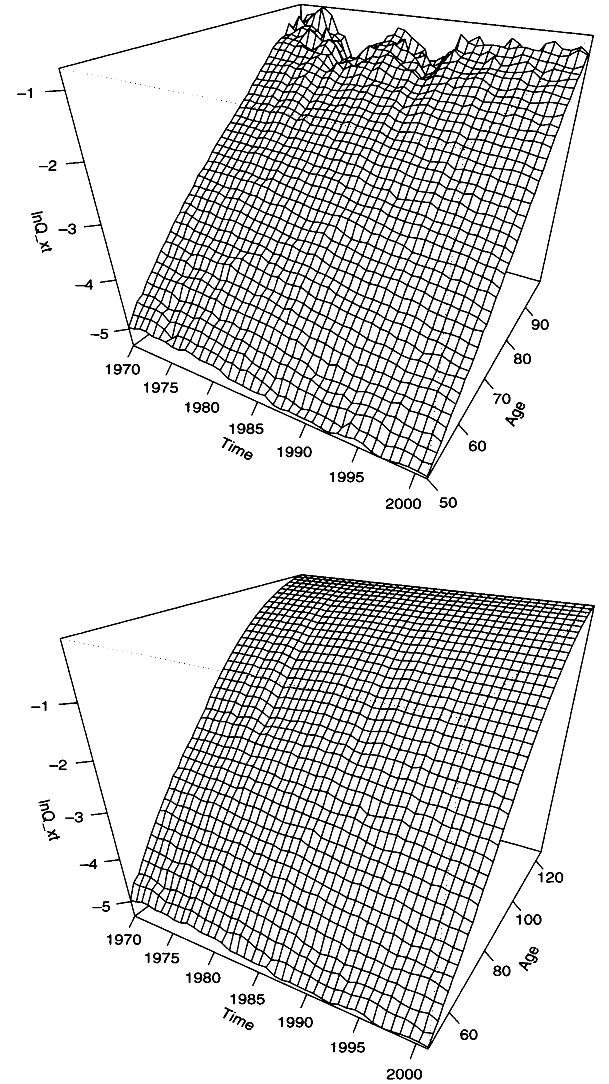
We consider $x_0 = 65$ and $t_0 = 2005$, i.e., the cohort retiring in year 2005. Let us estimate the market price of risk $\lambda_{65}(2005)$ for this cohort. To this end, we compute p_{65+k}^{market} according to the rules recommended by the Belgian regulatory authorities (CBFA), that is

$$p_{65+k}^{\text{market}} = p_{65+k-5}^{\text{MR}},$$

where p_{65+k-5}^{MR} is the one-year survival probability at age $65 + k - 5$ (so that the individual is made 5 years younger to take into account the adverse selection present in the annuity market), computed according to the Makeham model with parameters fixed by Royal Decree. Specifically,

$$p_x^{\text{MR}} = sg^{c^x(c-1)} \quad \text{where} \quad 0 < s \leq 1, \quad 0 < g < 1, \quad \text{and} \quad c > 1.$$

FIGURE 2
Rough Death Probabilities (Top) and Completed Ones (Bottom) on the Log-scale for Belgian Males

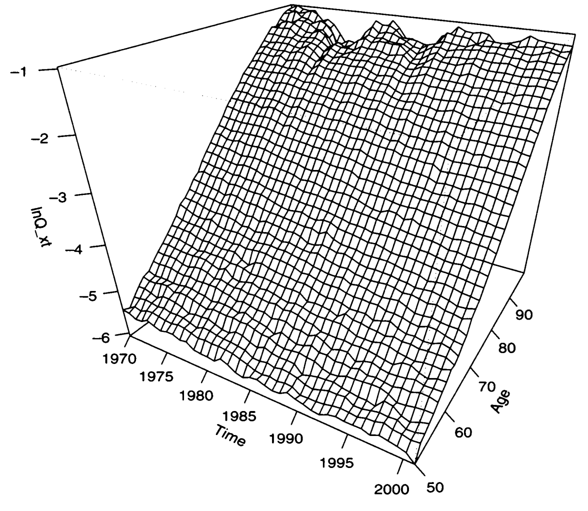


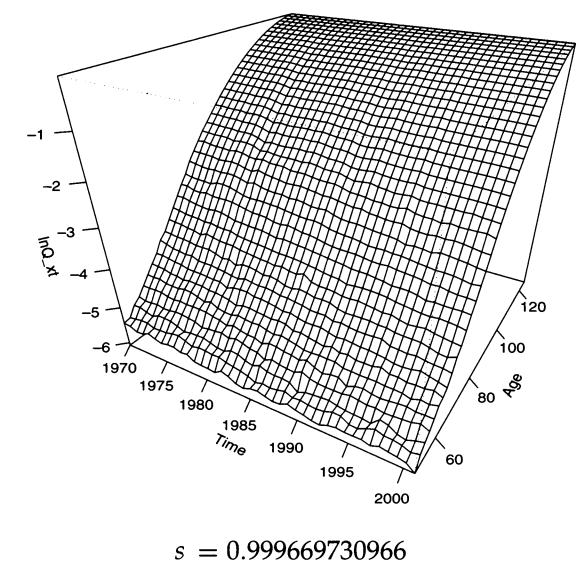
The parameters are

$$s = 0.999441703848$$
$$c = 1.101077536030$$
$$g = 0.999733441115$$

for men and

FIGURE 3
Rough Death Probabilities (Top) and Completed Ones (Bottom) on the Log-scale for Belgian Females





$$s = 0.999669730966$$

$$c = 1.116792453830$$

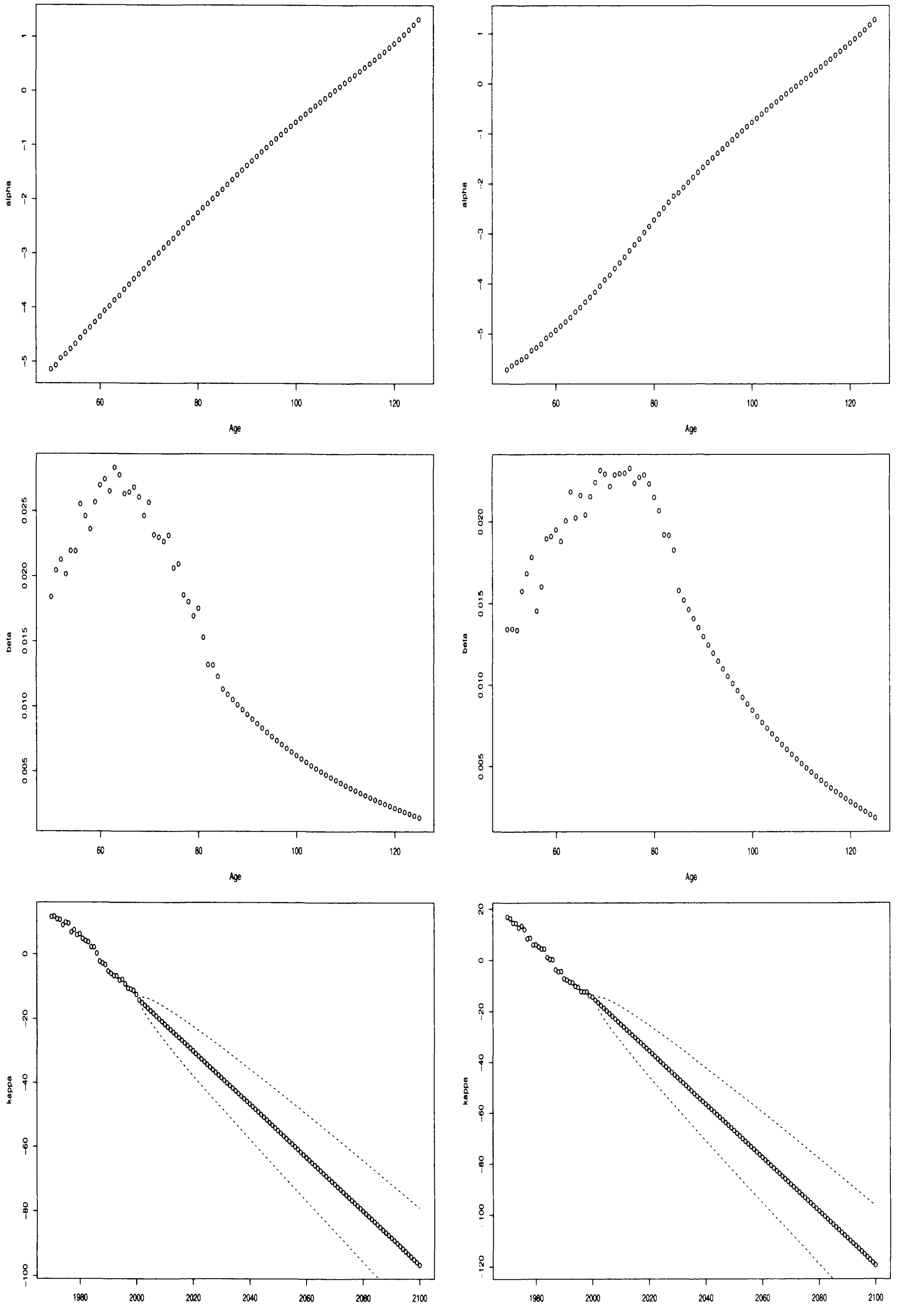
$$g = 0.999951440172$$

for women.

Using the p_{65+k}^{market}'s, we can compute the value of $a_{65}^{\text{market}}(2005)$. To this end, we use an interest rate equal to 3.25 percent, as adopted by the vast majority of insurance companies operating in Belgium. This gave 14.52168 for men and 16.38532 for women.

FIGURE 4

Values of α_x (Top), β_x (Middle) and κ_t (Bottom); Men Are on the Left, and Women on the Right



The same values computed with the probabilities (6) are 12.18551 for men and 15.06734 for women. The difference between these two life annuity premiums is attributable to the risk premium.

In Belgium, $a_{65}^{\text{market}}(2005)$ is the appropriate proxy for the market price of an annuity sold to an 65-year-old individual. This is due to the specific rules applying to Belgian insurance companies and pension funds. In order to encourage individuals to annuitize their benefits in the second pillar, the Belgian law specifies that the periodic payments provided by the annuity cannot be smaller than the ones obtained with the p_{65+k}^{market} probabilities. In practice, all the companies have adopted this minimum annuitization rule.

The Belgian situation is typical of heavily regulated countries. A similar approach could be followed in France, where the official TPRV lifetables (TPRV is the acronym for the French *Table Prospective pour Rentes Viagères*, which can be translated as projected lifetable for life annuities) is used by the majority of the French insurance companies. In this case, $a_{65}^{\text{market}}(2005)$ could be computed based on this table.

We think of the projected lifetable as the physical distribution, which requires a distortion to obtain market price, that is, $\lambda_{65}(2005)$ is the solution of

$$a_{65}^{\text{market}}(2005) = \sum_{t\geq 1}(1+i)^{-t}\left(1 - \Phi\left(\Phi^{-1}\left({}_tq_{65}^{\text{ref}}\right) + \lambda_{65}(2005)\right)\right),$$

where the discount factors are computed with $i = 3.25\%$, and ${}_tq_{65}^{\text{ref}}$ is the probability that a 65-year-old annuitant does not reach age $65 + t$, according to the reference projected lifetable (6). The probability ${}_tq_{65}^{\text{ref}}$ plays the role of the distribution function $F(\cdot)$ in (11). Since the right-hand side of this equation is monotone in λ, the solution is unique. We get $\lambda_{65}(2005) = -0.4722883$ for men and -0.2966378 for women (as expected, $\lambda_{65}(2005)$ is negative, expressing the fact that $a_{65}^{\text{market}}(2005)$ exceeds the corresponding premium computed with the survival probabilities (6)).

Remark 1: One of the referees pointed out that the estimated market price of risk seems very high, especially for men, compared to property catastrophic bonds (where the value for λ is about -0.45). It is true that the longevity risk is supposed to be less risky than the property cat risk since the processes of longevity risk are much smoother than property cat risks. Nevertheless, the coupon is related to the difference between ${}_tp_{x_0}^{\text{ref}}$ and ${}_tp_{x_0}^{\text{prosp}}$. In our case, ${}_tp_{x_0}^{\text{ref}}$ is taken as the point prediction deduced from the average trajectory of the random walk with drift model governing the evolution of the κ_t's. No safety loadings are included, which explains the high market price for risk. The reason is that the probability that the floating coupon will be reduced is much higher than with CAT bonds. The value of λ could be lowered if a more conservative reference lifetable was used by the insurer.

As explained above, the rules imposed on the Belgian insurers have been inspired mostly by social reasons: the government wanted to offer incentives for the annuitization in the second pillar, and the p_{65+k}^{market} probabilities do not result from an actuarial or demographic analysis.

Remark 2: The flat discount rate of 3.25 percent has been used in the computation of $a_{65}^{\text{market}}(2005)$ because it has been adopted by the vast majority of the companies operating in Belgium. For the sake of comparison, we have computed the market price of risk obtained with other interest rates, ranging from 2 to 4 percent, or using the zero-coupon price structure of the Belgian market depicted in Figure 6. This gave the results contained in the following table:

	$\lambda_{65}(2005)$	
Interest	Men	Women
Flat rate of 2%	−0.4900724	−0.3080401
Flat rate of 2.5%	−0.482867	−0.3034523
Flat rate of 3%	−0.4757752	−0.2988992
Flat rate of 3.50%	−0.4688345	−0.2944244
Flat rate of 4%	−0.4620456	−0.2900492
Zero-coupon	−0.4449350	−0.2795467

Computations of the $\varrho_\lambda^*[I_t]$'s

Let us now compute the value of $\varrho_\lambda^*[I_t]$. Figure 5 displays the lower and upper bounds on $\varrho_\lambda^*[I_t]$ for the years 2006–2065 (continuous lines). We see that the bounds are almost indistinguishable, making the "exact" computation by simulation useless for practical purposes. The broken line in the same figure represents the survival probabilities (6). For each t, we have $\varrho_\lambda^*[I_t] \geq {}_tp_{65}^{\text{ref}}$, as expected.

Determination of the Relative Additive Margin $\tilde{k}$

Let us now compute the relative additive margin according to the formula (10). A typical example of zero-coupon price structure (Belgian market, end of 2004) is depicted in Figure 6.

The values of the relative additive margin $\tilde{k}$ are displayed in Figure 7 in function of calendar time (the maturity n is the difference between calendar time and the issuance date 2005).

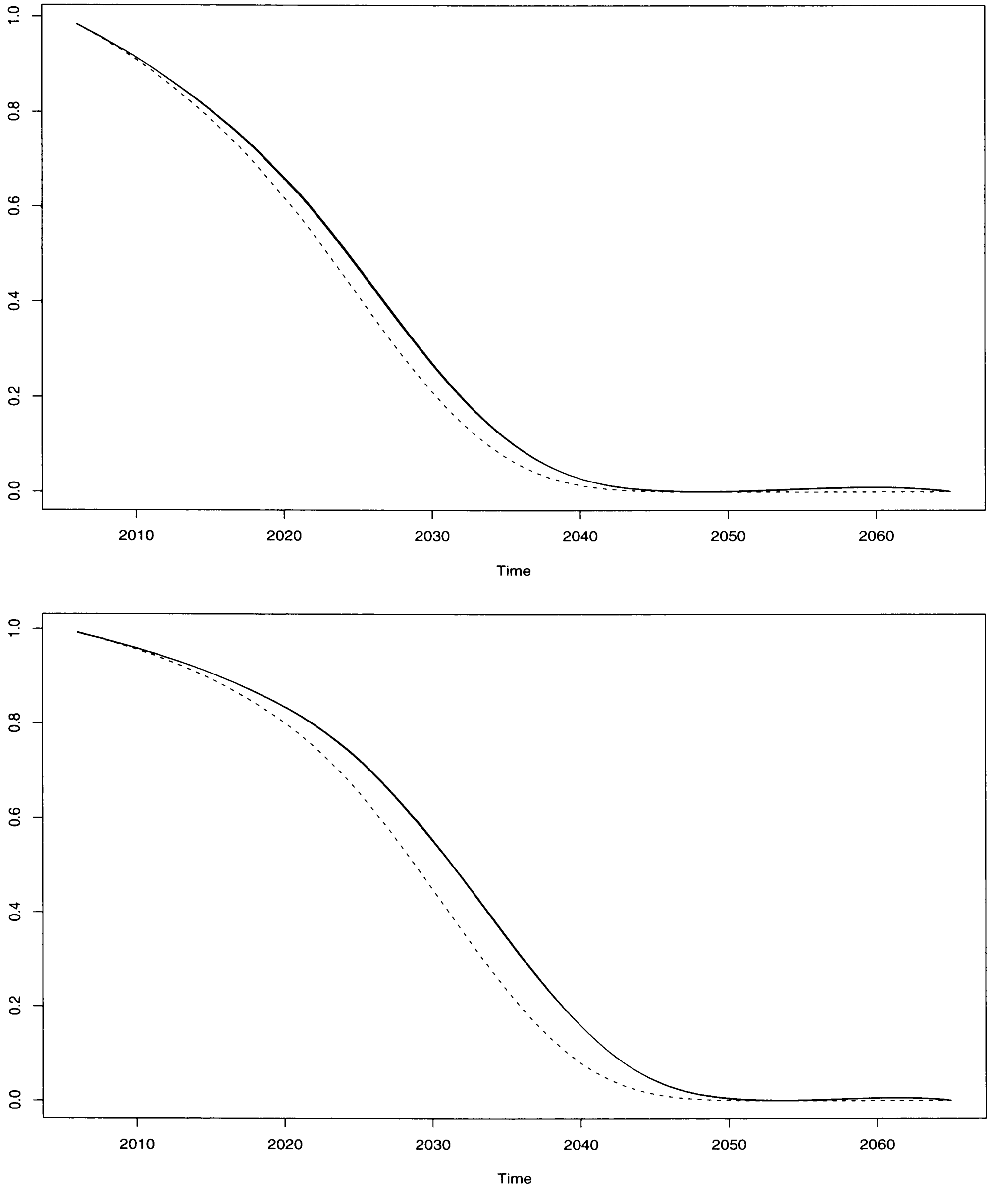
As expected, the relative additive margin is generally increasing with the maturity. Facing bigger longevity risk for longer maturities leads the investor to ask a more important risk loading. Nevertheless, the relative additive margin shows a light decreasing for very long maturity. The level of the relative additive margin is quite reasonable (less than 5 percent in all cases), suggesting in this example that the cost of issuance of such a product would not be too expensive for the insurer.

CONCLUSION

The Ageing Challenge for Insurers

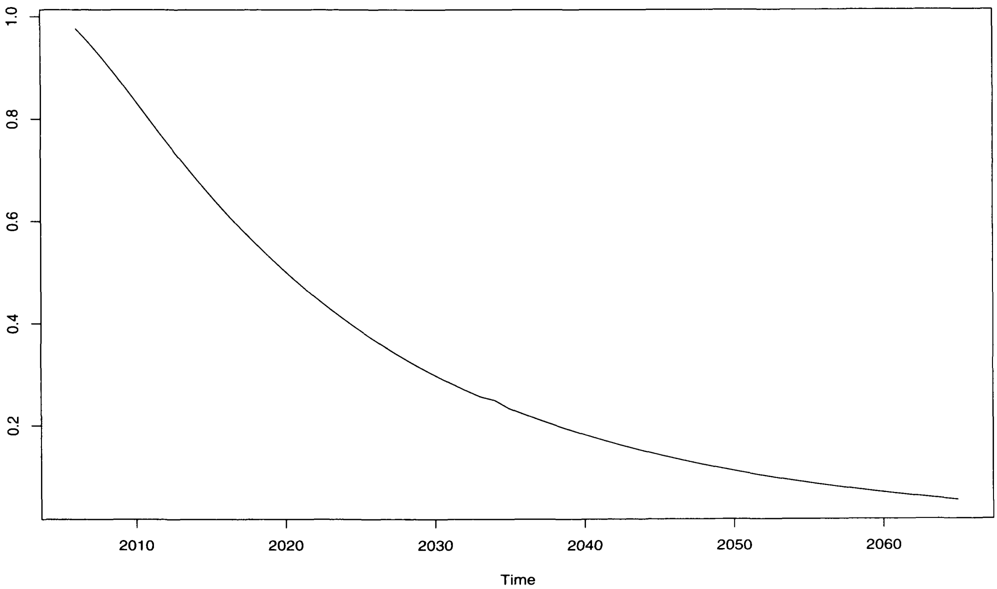
In Organisation for Economic Co-operation and Development countries, the financial burden of public pensions represents about one-tenth of the Gross Domestic Product, makes up more than three-quarters of all social insurance, and contributed to a quarter of the growth in total public expenditures since 1960. The elderly dependence ratio (i.e., the ratio of people over 65 years of age to those of

FIGURE 5
Bounds on $\varrho^*_\lambda[I_t]$ for the Years 2006 to 2065 (Continuous Lines) and Survival Probabilities (6) (Broken Line); Separately for Men (Top) and Women (Bottom)



15–64 years) is expected to increase rapidly. Central forecasts by the European Commission show that the European Union's elderly dependency ratio will reach 48 in 2040. Governments will face serious problems in financing state pensions under pay-as-you-go schemes, and the market of private annuities will certainly play an increasingly important role in the future.

FIGURE 6
Prices of the Zero-Coupon Bonds



As pointed out, e.g., by Dunnewijk (2002), this evolution poses numerous challenges to private insurance companies. Managing longevity risk is one of them. In this article, we addressed the important issue of the securitization of longevity risk. Specifically, we have designed survivor bonds that can be issued by pools of insurance companies. These bonds are based on a mortality index, derived in the framework of the Lee-Carter model for mortality forecasting.

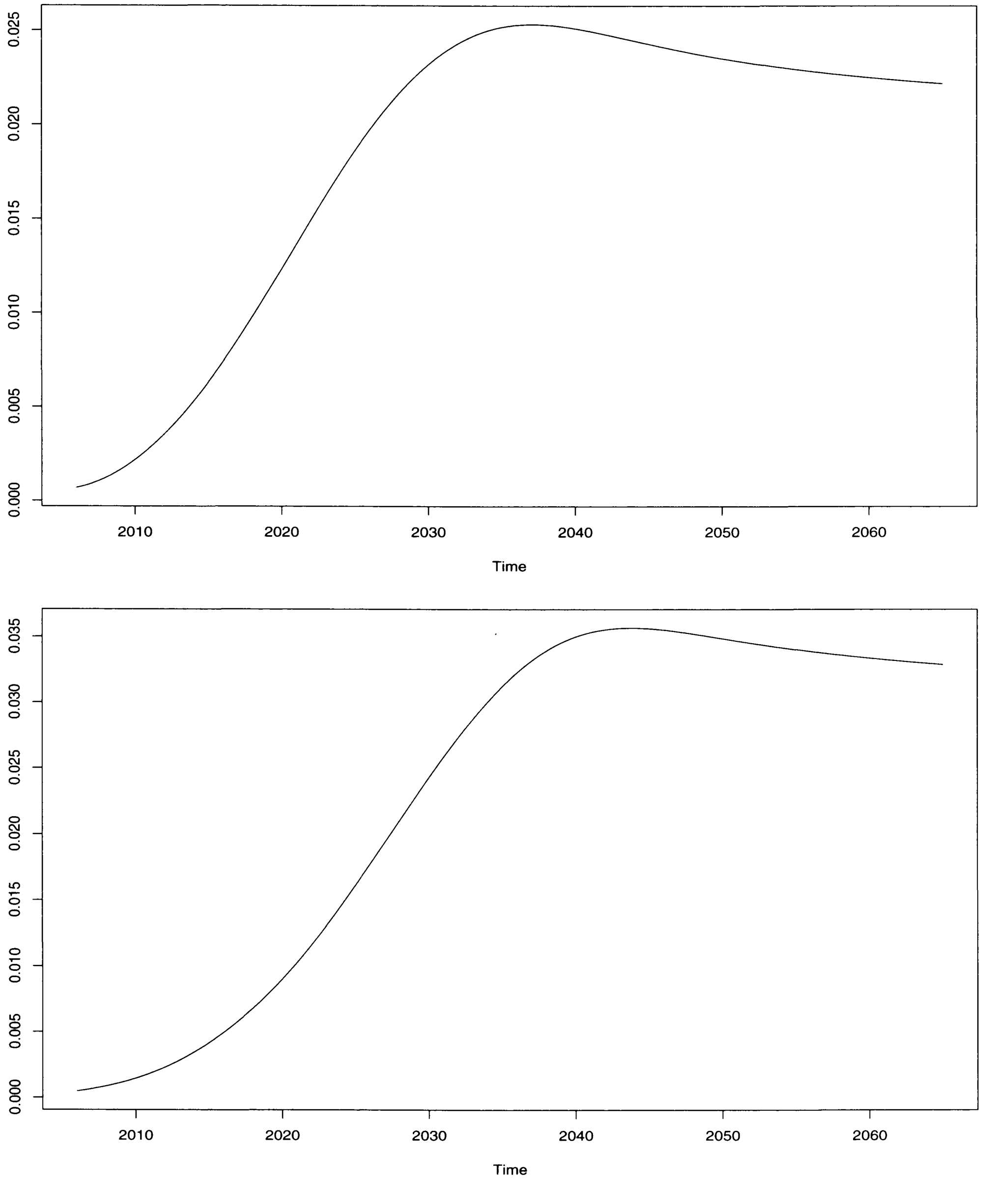
Note that the coupon is based on the deviation of future survival probabilities from a reference (projected) lifetable. Therefore, it is not the increase in longevity that is passed to the financial market, but only the possible adverse deviation from this reference lifetable. This makes the product attractive for any investor interested in diversification (since there is no correlation between the return of the survivor bond designed in this article and the financial markets, as explained below).

Is the Mortality Index Correlated With Asset Returns?

As already mentioned in the introduction, there may be some correlation between the ageing population phenomenon and asset returns. Because of life-cycle theory, as an ageing population retires, assets are sold to finance consumption. This could generate negative shocks to asset prices which impact on term structure.

Note however that this phenomenon does not apply *stricto sensu* to the mortality index defined in this article. Contrarily to Blake and Burrows (2001), the index used in this article is not based on the demographic structure of the population but on a survival probability. Therefore, the life-cycle theory does not indicate a direct influence of the mortality index defined in this article on asset returns (because the mortality index does not give the proportion of the population at retirement age).

FIGURE 7
Relative Additive Margin $\bar{k}$; Separately for Men (Top) and Women (Bottom)



As pointed out by a referee, in a global economy, the assets being sold by pension funds in countries with ageing population might be purchased by the pension funds of young workers in Asia, which could at least counter the shock. Even if the correlation between mortality and asset returns remains problematic, it is not critical for the mortality index used in the present work.

Let us also mention that Milevsky (2005) developed an index for tracking the dynamic behavior of life annuity payouts over time, based on the concept of self-annuitization. More precisely, this index is defined as the internal rate of return over a fixed deferral period that an individual would have to earn on the nonannuitized portfolio in order to replicate the income payout of the annuity and still be able to acquire the same income pattern assuming current pricing remains unchanged. On the basis of weekly Canadian life annuity quotes, this index was highly correlated with long-term Government of Canada bonds. This shows, again, that the choice of the appropriate index is crucial for life insurance securitization.

Possible Extensions

Let us mention that the variability inherent to the estimation of the α_x's, β_x's, and κ_t's has not been accounted for. The analytical derivation of credible intervals is out of reach, because all the actuarial quantities of interest are complicated nonlinear functions of the Lee-Carter parameters. Therefore, a bootstrap procedure seems to be the appropriate compromise, as described in Brouhns, Denuit, and Vankeilegom (2005).

The idea is to generate bootstrapped values for the α_x's, β_x's, and κ_t's as well as for the projected κ_t's. Finally, a bootstrapped value for the risk premium can be computed. Note that the uncertainty about the term structure can also be integrated in the bootstrap procedure.

The assumption of a random walk with drift process for the κ_t's is not crucial for the analysis carried out in this article. The results can be easily extended to any process $\{\kappa_t, t \in \mathbb{N}\}$ with Gaussian marginals.

Throughout the article, we assumed that the survivor bond is the only asset of this type available in the market. In case several bonds are issued, no-arbitrage reasonings must be applied.

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