The derivation of the Lindblad equation for the electron-electron scattering

I. LINDBLAD DYNAMICS

According to Eq. 4 and 5 of PHYSICAL REVIEW B 90, 125140 (2014) or Eq. 3.20 and 3.21 of Rosati's thesis, the scattering term has the form

$$\frac{d\hat{\rho}}{dt} = \hat{A}\hat{\rho}\hat{A}^\dagger - \frac{1}{2}\left\{\hat{A}^\dagger\hat{A},\hat{\rho}\right\},$$

where

$$\hat{A} = \sqrt{\frac{1}{\sqrt{2\pi}\bar{t}}}\int dt' e^{-\frac{\hat{H}_0 t'}{i\hbar}}\hat{H}e^{\frac{\hat{H}_0 t'}{i\hbar}}e^{-\frac{1}{4}\left(\frac{t'}{\bar{t}}\right)^2}. \tag{1}$$

Therefore,

$$\begin{aligned}
\frac{d\rho_{12}}{dt} =& \mathrm{Tr}\left(\hat{c}_2^\dagger\hat{c}_1\frac{d\hat{\rho}}{dt}\right)\\
=& \frac{1}{2}\mathrm{Tr}\left(\hat{c}_2^\dagger\hat{c}_1\hat{A}\hat{\rho}\hat{A}^\dagger + \hat{A}\hat{\rho}\hat{A}^\dagger\hat{c}_2^\dagger\hat{c}_1 - \hat{c}_2^\dagger\hat{c}_1\hat{A}^\dagger\hat{A}\hat{\rho} - \hat{\rho}\hat{A}^\dagger\hat{A}\hat{c}_2^\dagger\hat{c}_1\right)\\
=& \frac{1}{2}\mathrm{Tr}\left(\left[\hat{A}^\dagger,\hat{c}_2^\dagger\hat{c}_1\right]\hat{A}\hat{\rho} + \hat{\rho}\hat{A}^\dagger\left[\hat{c}_2^\dagger\hat{c}_1,\hat{A}\right]\right)\\
=& \frac{1}{2}\mathrm{Tr}\left(\left[\hat{A}^\dagger,\hat{c}_2^\dagger\hat{c}_1\right]\hat{A}\hat{\rho}\right) + H.C..
\end{aligned} \tag{2}$$

II. ELECTRON-ELECTRON

The electron-electron Hamiltonian reads (see Eq. 4 in PHYSICAL REVIEW B 92, 235423 (2015))

$$\begin{aligned}
\hat{H} =& \frac{1}{2}\sum_{1234} g_{1234}c_1^\dagger c_2^\dagger c_4 c_3,\\
g_{1234} =& \langle 1(r)|\langle 2(r')|V(r-r')|3(r)\rangle|4(r')\rangle\\
=& V(q_{13},\omega_{13})\,\delta_{k_1+k_2,k_3+k_4}\langle 1|3\rangle\langle 2|4\rangle,
\end{aligned}$$

where $q_{13} = k_1 - k_3$ and $\omega_{13} = \epsilon_1 - \epsilon_3$.
Insert it in 1,

$$\begin{aligned}
\hat{A} =& \frac{1}{2}\sum_{1234} A_{1234}c_1^\dagger c_2^\dagger c_4 c_3,\\
A_{1234} =& \sqrt{\frac{1}{\sqrt{2\pi}\bar{t}}}\int dt' e^{-\frac{(\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4)t'}{i\hbar}}g_{1234}e^{-\frac{1}{4}\left(\frac{t'}{\bar{t}}\right)^2}\\
=& \sqrt{\frac{2\pi}{\hbar}}g_{1234}\delta^{G,1/2}(\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4).
\end{aligned} \tag{3}$$

For later derivation, we define (we will use the symmetries of the matrix later)

$$\mathscr{A}_{1234} = \frac{1}{4}\left(A_{1234} - A_{2134} - A_{1243} - A_{2143}\right).$$

Since $V(q,\omega) = V(-q,-\omega)$, we have

$$\mathscr{A}_{1234} = \frac{1}{2}\left(A_{1234} - A_{1243}\right).$$

It satisfies (see Eq. 3.22 in Rosati's thesis)

$$\hat{A} = \frac{1}{2}\sum_{1234} \mathscr{A}_{1234} c_1^\dagger c_2^\dagger c_4 c_3.$$

Since $\hat{A}^\dagger = \frac{1}{2}\sum_{3456} \mathscr{A}^*_{3456} c_5^\dagger c_6^\dagger c_4 c_3$,

$$\begin{aligned}
\left[\hat{A}^\dagger, \hat{c}_2^\dagger \hat{c}_1\right] &= \begin{pmatrix} \frac{1}{2}\sum_{356} \mathscr{A}^*_{3256} c_5^\dagger c_6^\dagger c_1 c_3 + \frac{1}{2}\sum_{456} \mathscr{A}^*_{1456} c_5^\dagger c_6^\dagger c_4 c_2 \\ -\frac{1}{2}\sum_{346} \mathscr{A}^*_{3416} c_2^\dagger c_6^\dagger c_4 c_3 - \frac{1}{2}\sum_{345} \mathscr{A}^*_{3451} c_5^\dagger c_1^\dagger c_4 c_3 \end{pmatrix} \\
&= \sum_{345} \mathscr{A}^*_{3245} c_4^\dagger c_5^\dagger c_1 c_3 - \sum_{345} \mathscr{A}^*_{3415} c_2^\dagger c_5^\dagger c_4 c_3.
\end{aligned} \tag{4}$$

The above euqaiton is the same as Eq. A5 in PHYSICAL REVIEW B 90, 125140 (2014).
Insert Eq. 3 and 4 into Eq. 2,

$$\begin{aligned}
\frac{d\rho_{12}}{dt} &= \frac{1}{2}\mathrm{Tr}\left(\left[\hat{A}^\dagger, \hat{c}_2^\dagger \hat{c}_1\right] \hat{A}\hat{\rho}\right) + H.C. \\
&= \frac{1}{4}\mathrm{Tr}\left(\left(\sum_{345} \mathscr{A}^*_{3245} c_4^\dagger c_5^\dagger c_1 c_3 - \sum_{345} \mathscr{A}^*_{3415} c_2^\dagger c_5^\dagger c_4 c_3\right) \sum_{6789} \mathscr{A}_{6789} c_6^\dagger c_7^\dagger c_9 c_8 \hat{\rho}\right) + H.C. \\
&= \frac{1}{4}\sum_{3456789} \left(\mathscr{A}^*_{3245} k_{45136798} - \mathscr{A}^*_{3415} k_{25436798}\right) \mathscr{A}_{6789} + H.C., \\
k_{12345678} &= \mathrm{Tr}\left(c_1^\dagger c_2^\dagger c_3 c_4 c_5^\dagger c_6^\dagger c_7 c_8 \hat{\rho}\right).
\end{aligned} \tag{5}$$

To close the equation, we first approximate $k_{12345678}$,

$$k_{12345678} \approx \mathrm{Tr}\left(c_7 c_8 \hat{\rho} c_1^\dagger c_2^\dagger\right) \mathrm{Tr}\left(c_5^\dagger c_6^\dagger \hat{\rho} c_3 c_4\right).$$

In the mean-field approximation (or Hartree-Fock approximation, see Eq. 74 in Reviews of Modern Physics 74, 895 (2002)),

$$\begin{aligned}
\mathrm{Tr}\left(c_7 c_8 \hat{\rho} c_1^\dagger c_2^\dagger\right) &\approx \mathrm{Tr}\left(\hat{\rho} c_1^\dagger c_8\right) \mathrm{Tr}\left(\hat{\rho} c_2^\dagger c_7\right) - \mathrm{Tr}\left(\hat{\rho} c_2^\dagger c_8\right) \mathrm{Tr}\left(\hat{\rho} c_1^\dagger c_7\right) \\
&= \rho_{81}\rho_{72} - \rho_{82}\rho_{71}, \\
\mathrm{Tr}\left(c_5^\dagger c_6^\dagger \hat{\rho} c_3 c_4\right) &\approx \mathrm{Tr}\left(\hat{\rho} c_3 c_6^\dagger\right) \mathrm{Tr}\left(\hat{\rho} c_4 c_5^\dagger\right) - \mathrm{Tr}\left(\hat{\rho} c_4 c_6^\dagger\right) \mathrm{Tr}\left(\hat{\rho} c_3 c_5^\dagger\right) \\
&= (I-\rho)_{36}(I-\rho)_{45} - (I-\rho)_{46}(I-\rho)_{35},
\end{aligned}$$

using the relation $\hat{c}_1\hat{c}_2^\dagger = \delta_{12} - \hat{c}_2^\dagger \hat{c}_1$. Insert the above three equations in Eq. 5,

$$\frac{d\rho_{12}}{dt} = \frac{1}{4}\sum_{3456789} \begin{pmatrix} \mathscr{A}^*_{3245}\left[(I-\rho)_{17}(I-\rho)_{36} - (I-\rho)_{37}(I-\rho)_{16}\right]\left(\rho_{84}\rho_{95} - \rho_{85}\rho_{94}\right) \\ -\mathscr{A}^*_{3415}\left[(I-\rho)_{47}(I-\rho)_{36} - (I-\rho)_{37}(I-\rho)_{46}\right]\left(\rho_{82}\rho_{95} - \rho_{85}\rho_{92}\right) \end{pmatrix} \mathscr{A}_{6789} + H.C..$$

Use the relations $\mathscr{A}_{1234} = \mathscr{A}_{2143} = -\mathscr{A}_{2134} = -\mathscr{A}_{1243}$,

$$\frac{d\rho_{12}}{dt} = \frac{1}{4}\sum_{3456789} \left\{ \begin{array}{c} (I-\rho)_{17}(I-\rho)_{36}\mathscr{A}_{7698}\mathscr{A}^*_{2354}\rho_{84}\rho_{95} + (I-\rho)_{17}(I-\rho)_{36}\mathscr{A}_{7698}\mathscr{A}^*_{2345}\rho_{85}\rho_{94} \\ +(I-\rho)_{16}(I-\rho)_{37}\mathscr{A}_{6789}\mathscr{A}^*_{2345}\rho_{95}\rho_{84} + (I-\rho)_{16}(I-\rho)_{37}\mathscr{A}_{6789}\mathscr{A}^*_{2354}\rho_{85}\rho_{94} \\ -(I-\rho)_{36}\left[(I-\rho)_{74}\mathscr{A}_{3415}\mathscr{A}^*_{6789}\rho_{59}\right]^*\rho_{82} - (I-\rho)_{36}\left[(I-\rho)_{74}\mathscr{A}_{3415}\mathscr{A}^*_{6798}\rho_{58}\right]^*\rho_{92} \\ -(I-\rho)_{37}\left[(I-\rho)_{64}\mathscr{A}_{3415}\mathscr{A}^*_{7689}\rho_{59}\right]^*\rho_{82} - (I-\rho)_{37}\left[(I-\rho)_{64}\mathscr{A}_{3415}\mathscr{A}^*_{7698}\rho_{58}\right]^*\rho_{92} \end{array} \right\} + H.C..$$

Notice that in the bracket "{}" of the above equation, the first 4 terms are the same after sum and the last 4 terms are the same after sum, so that

$$\frac{d\rho_{12}}{dt} = \frac{1}{2}\sum_{3456789}\left\{\begin{array}{c}(I-\rho)_{17}\left[2\,(I-\rho)_{36}\,\mathscr{A}_{7698}\mathscr{A}^{*}_{2354}\rho_{84}\right]\rho_{95}\\ -\,(I-\rho)_{36}\left[2\,(I-\rho)_{74}\,\mathscr{A}_{3415}\mathscr{A}^{*}_{6789}\rho_{59}\right]^{*}\rho_{82}\end{array}\right\}+H.C..$$

Define

$$P_{1234} = 2\sum_{5678}(I-\rho)_{65}\,\mathscr{A}_{1537}\mathscr{A}^{*}_{2648}\rho_{78}.$$

We will have

$$\frac{d\rho_{12}}{dt} = \frac{1}{2}\sum_{795}(I-\rho)_{17}\,P_{7295}\rho_{95} - \sum_{368}(I-\rho)_{36}\,P^{*}_{3618}\rho_{82} + H.C..$$

This is the same as Eq. 21, 23, 24 in PHYSICAL REVIEW B 92, 235423 (2015).