SESSION 10

Brouwer's theorems

1. Balls, spheres, fixed points, and retractions

The Dutch mathematician L.E.J. Brouwer (1881–1966) proved some remarkable theorems about 'continuous' maps between familiar objects: circle, disk, solid ball, etc. The setting for these was the 'category of topological spaces and continuous maps.' For our purposes it is unnecessary to have any precise description of this category; we will instead eventually list certain facts which we will call 'axioms' and deduce conclusions from these axioms. Naturally, the axioms will not be selected at random, but will reflect our experience with 'cohesive sets' (sets in which it makes sense to speak of closeness of points) and 'continuous maps.' (Roughly, a map f is continuous if $f(p)$ doesn't instantaneously jump from one position to a far away position as we gradually move p. We met this concept in discussing Galileo's idea of a continuous motion of a particle, i.e. a continuous map from an interval of time into space.) There is even an advantage in not specifying our category precisely: our reasoning will apply to any category in which the axioms are true, and there are, in fact, many such categories ('topological spaces', 'smooth spaces', etc.).

We begin by stating Brouwer's theorems and by trying to see whether our intuition about continuous maps makes them seem plausible. First we describe the *Brouwer fixed point theorems.*

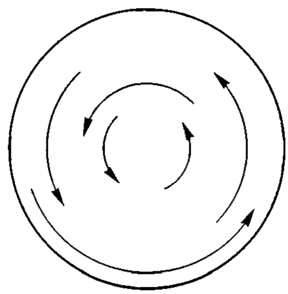
(1) *Let I be a line segment, including its endpoints (I for Interval) and suppose that $f\colon I \longrightarrow I$ is a continuous endomap. Then this map must have a fixed point: a point x in I for which $f(x) = x$.*

Example: Suppose that I is an interval of time, and that R is an interval of road, say the highway from Buffalo to Rochester. Suppose that two cars drive on this road. The first car drives at a constant speed from Buffalo to Rochester, so its motion is described by $I \xrightarrow{u} R$ (u for 'uniform' motion). Meanwhile, the second car starts anywhere along the road and just travels aimlessly along, perhaps occasionally parking for a while, then retracing its path for a while, and ending its journey at any point along the road. Let's denote the motion of this second car by $I \xrightarrow{m} R$. Now u is an invertible map, so we get $R \xrightarrow{u^{-1}} I$, and let $I \xrightarrow{f} I$ be the composite $f = u^{-1} \circ m$. Brouwer's theorem tells us that there must be some time t in I at which $f(t) = t$; that is, $u^{-1} m t = t$; so $m t = u t$, which says there is some time t at which the two cars are at the same point on the road. This seems not very surprising; if the first car drives from Buffalo to Rochester and the second car is always on the road, then of course the first car must at some time meet the second.

The next theorem is similar, but about a disk instead of an interval, and I find it much less obvious.

(2) *Let D be a closed disk (the plane figure consisting of all the points inside or on a circle), and f a continuous endomap of D. Then f has a fixed point.*

Example: Rotating the disk by a certain angle gives a continuous endomap of the disk; f could be the process 'turn 90 degrees.'



ALYSIA: What about the center?

Exactly! That is a fixed point. For this map it was easy to see that it has a fixed point, but for other maps it may not be so easy; yet the theorem says that as long as f is continuous, it will have at least one fixed point. This theorem seems to me much more surprising than the previous one.

Example: Suppose my disk is a portion of the Washington DC area, say the part inside or on the circular beltway. I also bring a map of the region, drawn on a piece of paper P. My map is thus a continuous map $D \xrightarrow{m} P$. If I am so callous as to crumple up the map and throw it out of the car window, so that it lands inside the beltway, I get an additional continuous map $P \xrightarrow{p} D$ (p for 'projection'), assigning to each point on the crumpled paper the point on the ground directly under it. Brouwer's theorem, applied to the map $f = p \circ m$: $D \longrightarrow D$, tells me that some point x inside the beltway is directly under the point $m(x)$ that represents x on the map. Do you find that surprising? I did when I first heard it. You can try the experiment, but please pick up the map afterward.

If it occurred to you that a perfect map would show every detail of the area, even including a picture of the discarded map, congratulate yourself. You have discovered the idea behind *Banach's fixed point theorem* for 'contraction' maps. You only have to go a step further: the discarded map has a small picture of the discarded map, and that picture has a smaller picture which has a smaller picture .... These pictures gradually close in on the one and only fixed point for our endomap. This beautifully simple idea only works for an endomap which shrinks distances, though. Brouwer's theorem applies to *every* continuous endomap of the disk.

Example: Here is a map to which Brouwer's theorem applies and Banach's doesn't. Suppose D is a disk-shaped room in a doll's house, and F is a larger-than-life floor plan of that room; we crumple F and discard it on D as before. The composite map $D \xrightarrow{m} F \xrightarrow{p} D$ this time will not shrink all distances, so the Banach idea doesn't apply. (In fact $p \circ m$ may have many fixed points, but they are not so easy to locate. It often happens that if a problem has only one solution, it's easy to find it; but if there are many solutions, it's hard to find even one of them.) The next theorem is about .... Any guess?

FATIMA: A ball?

Exactly! A solid ball. It says the following:

(3) *Any continuous endomap of a solid ball has a fixed point.*

To imagine an endomap, think of deforming the ball in any arbitrary way, but without tearing it.

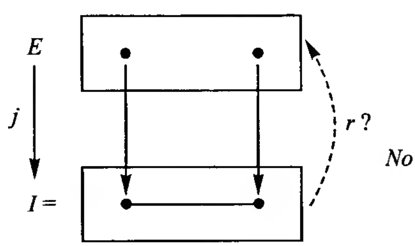
DANILO: Something like folding dough?

Yes, but without breaking it into separate pieces. I find it easier to imagine this endomap of the ball if I first have *two* 'objects,' a wad W of dough and a ball-shaped region B in space. Then I can use two maps from W to B: a 'uniform' placement $u : W \longrightarrow B$ in which the wad W exactly fills the region B, and the new placement after kneading the dough, $p : W \longrightarrow B$. Now u is invertible, and the endomap we want is pu^{-1}. It assigns to each point in the region the new location of the point in the dough that was originally there; it's a sort of 'change of address' map.

Now we describe the sequence of theorems known as *Brouwer retraction theorems.*

(I) *Consider the inclusion map $j : E \longrightarrow I$ of the two-point set E as boundary of the interval I. There is no continuous map which is a retraction for j.*

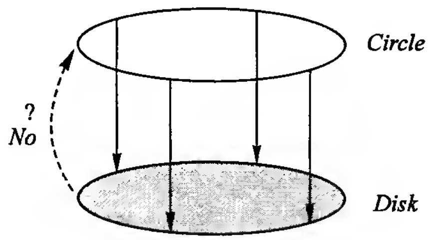
Recall that this means there is no continuous map $r : I \longrightarrow E$ such that $r \circ j = 1_E$.



In other words, it is not possible to map the interval continuously to its two endpoints and leave the endpoints in place. Isn't this reasonable? Isn't it pretty obvious that one cannot put one part of the interval on one of its endpoints and another part of it on the other without tearing it?

The next retraction theorem is about the disk and its boundary.

(II) *Consider the inclusion map $j : C \longrightarrow D$ of the circle C as boundary of the disk D into the disk. There is no continuous map which is a retraction for j.*



Again, this should seem quite reasonable. Suppose we have a drum made of a very flexible stretchable sheet. To get a retraction for the inclusion of the boundary we might imagine taking the sheet and squeezing it into the rim but without moving its boundary. One would think that this is not possible without puncturing or tearing the sheet. What this retraction theorem says is that this thought is correct.

The third retraction theorem is, as you can imagine, about the ball and its boundary (the sphere).

(III) *Consider the inclusion $j : S \longrightarrow B$ of the sphere S as boundary of the ball B into the ball. There is no continuous map which is a retraction for j.*

Now, here is the point about all these theorems: (1) and (I) are actually equivalent theorems, and so are the Theorems (2) and (II), and also Theorems (3) and (III). In other words, after proving the retraction theorems, which seem so reasonable, Brouwer could easily get as a consequence the fixed point theorems (which seem much less intuitive). We shall illustrate this by showing how Brouwer proved that (II) implies (2), and we'll leave the other cases for you to think about.

Let's write clearly what Brouwer promised to show:

If there is no continuous retraction of the disk to its boundary then every continuous map from the disk to itself has a fixed point.

However, Brouwer did not prove this directly. Instead of this he proved the following:

Given a continuous endomap of the disk with no fixed points, one can construct a continuous retraction of the disk to its boundary.

This is an example of the *contrapositive* form of a logical statement. The contrapositive form of 'A implies B' is '*not* B implies *not* A,' which conveys exactly the same information as 'A implies B,' just expressed in a different way. Below is an example of how it is used.

2. Digression on the contrapositive rule

A friend of mine, Meeghan, has many uncles. All of Meeghan's uncles are doctors. In Meeghan's world

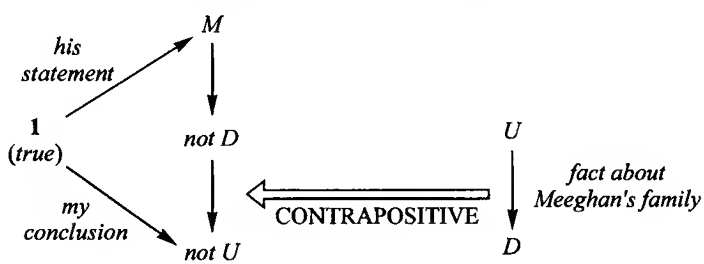
$$\textit{uncle} \xrightarrow{(\textit{implies})} \textit{doctor} \qquad\qquad \text{(PARTICULAR SITUATION)}$$

I went to her wedding and met some of them. There I had an interesting discussion with an intelligent man who I thought was another uncle, but in the course of the conversation he said that he was a mechanic. So I thought

$$\textit{mechanic} \xrightarrow{(\textit{implies})} \textit{not doctor} \qquad\qquad \text{(GENERAL KNOWLEDGE about our society)}$$

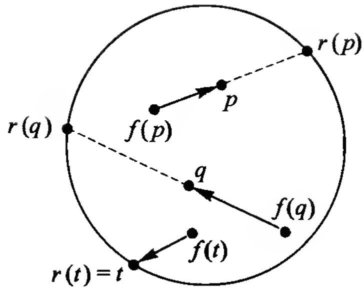
$$\textit{not doctor} \xrightarrow{(\textit{implies})} \textit{not Meeghan's uncle} \qquad\qquad \text{(CONTRAPOSITIVE of what is known of the particular situation)}$$

Therefore this man is not one of Meeghan's uncles.



3. Brouwer's proof

We return to Brouwer's theorems. To prove that the non-existence of a retraction implies that every continuous endomap has a fixed point, all we need to do is to assume that there is a continuous endomap of the disk which does not have any fixed point, and to build from it a continuous retraction for the inclusion of the circle into the disk.



So, let $j : C \longrightarrow D$ be the inclusion map of the circle into the disk as its boundary, and let's assume that we have an endomap of the disk, $f: D \longrightarrow D$, *which does not have any fixed point*. This means that for every point x in the disk D, $f(x) \neq x$.

From this we are going to build a retraction for j, i.e. a map $r: D \longrightarrow C$ such that $r \circ j$ is the identity on the circle. The key to the construction is the assumed property of f, namely that for every point x in the disk, $f(x)$ is different from x. Draw an arrow with its tail at $f(x)$ and its head at x. This arrow will 'point to' some point $r(x)$ on the boundary. When x was already a point on the boundary, $r(x)$ is x itself, so that r is a retraction for j, i.e. $rj = 1_C$.

Two things are worth noting: first, that sometimes something that looks impossible or hard to prove may be easily deduced from something that looks much more reasonable and is, in fact, easier to prove; and second, that to know that a map has no retraction often has very powerful consequences.

The reasoning leading to the proof of Brouwer's fixed point theorem can be summarized in the following diagram:

$$\begin{tikzcd}
\textit{f without fixed point} \arrow[r, Rightarrow, "\textit{proof}"] & \textit{retraction on the boundary} \arrow[d, Rightarrow, "\textit{contrapositive principle}"] \\
\substack{\textit{no retraction} \\ \text{(reasonable and true)}} \arrow[r, Rightarrow] & \substack{\textit{no f without fixed point} \\ \text{(surprising but now known to be true)}}
\end{tikzcd}$$

DANILO: Your conclusion sounds peculiar. Instead of 'every f has a fixed point,' you get 'there is no f without fixed point.'

You are right. We need to use another principle of logic, that *not*(*not* A) implies A, to reach 'every f has a fixed point.' Brouwer himself seriously questioned this rule of logic; and we will later see that there are examples of useful categories in whose 'internal' logic this rule does not hold. (This 'logical' difficulty turns out to be connected with the difficulty of actually locating a fixed point for f, if f is not a 'contraction map'.)

4. Relation between fixed point and retraction theorems

Exercise 1:
Let $j : C \longrightarrow D$ be, as before, the inclusion of the circle into the disk. Suppose that we have two continuous maps $D \underset{g}{\overset{f}{\rightrightarrows}} D$, and that g satisfies $g \circ j = j$. Use the retraction theorem to show that there must be a point x in the disk at which $f(x) = g(x)$. (Hint: The fixed point theorem is the special case $g = 1_D$, so try to generalize the argument we used in that special case.)

I mentioned earlier that each retraction theorem is equivalent to a fixed point theorem. That means that not only can we deduce the fixed point theorem from the retraction theorem, as we did, but we can also deduce the retraction theorem from the fixed point theorem. This is easier, and doesn't require a clever geometrical construction. Here is how it goes.

Exercise 2:
Suppose that A is a 'retract' of X, i.e. there are maps $A \underset{r}{\overset{s}{\rightleftarrows}} X$ with $r \circ s = 1_A$. Suppose also that X has the fixed point property for maps from T, i.e. for every endomap $X \xrightarrow{f} X$, there is a map $T \xrightarrow{x} X$ for which $fx = x$. Show that A also has the fixed point property for maps from T. (Hint: The proof should work in any category, so it should only use the algebra of composition of maps.)

Now you can apply Exercise 2 to the cases: T is $\mathbf{1}$ (any one-point space), X is the interval, the disk, or the ball, and A is its boundary (two points, circle, or sphere.) Notice that in each of these cases, there is an obvious 'antipodal' endomap a of A, sending each point to the diametrically opposite point; and a has no fixed point.

Exercise 3:
Use the result of the preceding exercise, and the fact that the antipodal map has no fixed point, to deduce each retraction theorem from the corresponding fixed point theorem.

In solving these exercises, you will notice that you have done more than was required. For example, from the fixed point theorem for the disk, you will have concluded not only that the inclusion map $C \longrightarrow D$ has no retraction, but also that C is not a retract of D (by *any* pair of maps.) In fact, the argument even shows that *none* of E, C, S, is a retract of *any* of I, D, B.

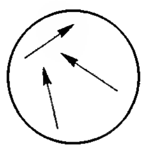
You will probably have noticed that the same reasoning is used in all dimensions; for instance, Exercise 1 applies to the interval or ball as well as the disk. In the next section we state things for the 'ball' case, but draw the pictures for the 'disk' case.

5. How to understand a proof: The objectification and 'mapification' of concepts

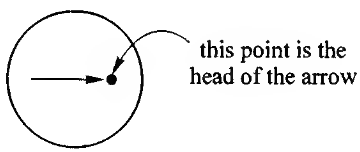
You may have felt that none of our reasoning about Brouwer's theorems was valid, since we still have no precise notion of 'continuous map.' What we wish to do next is to extract those properties which are needed for our reasoning, and see that our conclusions are valid in any category in which these properties (which we will call Axiom 1 and Axiom 2) hold.

Brouwer introduces in his proof, besides the sphere S and ball B and the inclusion map $S \xrightarrow{j} B$, several new *concepts*:

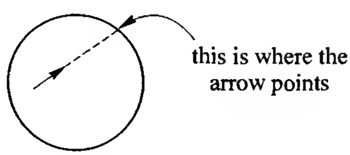
1. *arrows* in B:



2. each arrow has a *head*, in B:



3. each arrow in B *points* to a point in S:



To analyze his proof, then, we must bring these concepts into our category $\mathcal{C}$. This means that we will need:

1. an *object* A (whose points are the *arrows* in B);
2. a *map* $A \xrightarrow{h} B$ (assigning to each arrow its *head*); and
3. a *map* $A \xrightarrow{p} S$ (telling where each arrow *points*).

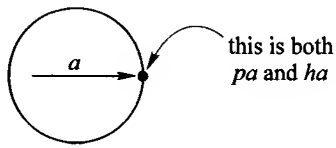
(Remember that a map in $\mathcal{C}$ *means* a 'continuous' map, so that any map obtained by composing maps in $\mathcal{C}$ will automatically be continuous.)

Now we have three objects and three maps:

$$\begin{tikzcd} & A \arrow[dl, "p"'] \arrow[d, "h"] \\ S \arrow[r, "j"'] & B \end{tikzcd}$$

and we can begin to ask: what special properties of these (now 'objectified') concepts are used in Brouwer's proof?

First, we observe that if an arrow has its head on the boundary, then its head *is* the place to which it points:



We will bring this into our category, by noting that a map $T \xrightarrow{a} A$ is a (smooth) 'listing' of arrows: $T \xrightarrow{a} A$.

Axiom 1: *If T is any object in $\mathcal{C}$, and $T \xrightarrow{a} A$ and $T \xrightarrow{s} S$ are maps satisfying $h\,a = j\,s$, then $p\,a = s$.*

The diagram below shows all the maps involved.

$$\begin{tikzcd} T \arrow[r, "a"] \arrow[d, "s"'] & A \arrow[dl, "p"'] \arrow[d, "h"] \\ S \arrow[r, "j"'] & B \end{tikzcd}$$

(Instead of just one arrow, we imagine a 'parameterized family' of arrows, one for each point in a 'parameter space' or 'test object' T; i.e. a map $T \xrightarrow{a} A$. The rest of the translation process leading to Axiom 1 just requires taking care to notice that p of an arrow is in S, while h of an arrow is in B; so to compare them we need to use the inclusion map $S \xrightarrow{j} B$.)

Already from Axiom 1, we can carry out part of Brouwer's argument:

Theorem 1: *If $B \xrightarrow{\alpha} A$ satisfies $h\alpha j = j$, then $p\alpha$ is a retraction for j.*

Proof: Put $T = S$, $s = 1_S$, and $a = \alpha j$ in Axiom 1.

Corollary: *If $h\alpha = 1_B$, then $p\alpha$ is a retraction for j.*

Second, we notice that if two points of B are *different*, there is an arrow from the first to the second; in fact each arrow in A should be thought of as having its head and tail *distinct*, otherwise it wouldn't 'point to' a definite place on the boundary S. We use the method of test objects again, with the idea that for each t, αt is the arrow from ft to gt.

Axiom 2: *If T is any object in $\mathcal{C}$, and $T \underset{g}{\overset{f}{\rightrightarrows}} B$ are any maps, then either there is a point $1 \xrightarrow{t} T$ with $ft = gt$, or there is a map $T \xrightarrow{\alpha} A$ with $h\alpha = g$.*

Now we can finish his argument:

Theorem 2: *Suppose we have maps*

$$B \underset{g}{\overset{f}{\rightrightarrows}} B$$

and $gj = j$, then either there is a point $\mathbf{1} \xrightarrow{b} B$ with $fb = gb$, or there is a retraction for $S \xrightarrow{j} B$.

Proof: Take $T = B$ in Axiom 2. We get: either there is a point $\mathbf{1} \xrightarrow{b} B$ with $fb = gb$, or there is a map $B \xrightarrow{\alpha} A$ with $h\alpha = g$; but then $h\alpha j = gj = j$, so Theorem 1 says that $p\alpha$ is a retraction for j.

If we take $g = 1_B$ in Theorem 2, we get a corollary.

Corollary: *If $B \xrightarrow{f} B$, then either there is a fixed point for f or there is a retraction for $S \xrightarrow{j} B$.*

(We gave, in Theorem 2, the more general version of Brouwer's theorem; the corollary is the original version.)

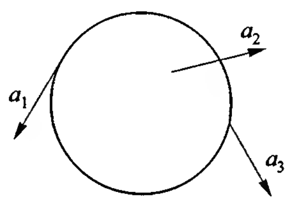
We will see later that in many categories $\mathcal{C}$, an object T may be large, and still have no 'points' $\mathbf{1} \xrightarrow{t} T$. In such a category, we should notice that we really didn't use the full strength of Axioms 1 and 2 in our proofs. It was enough to have Axiom 1 just for $T = S$, and Axiom 2 for $T = B$.

The main thing to study, though, is the way in which by objectifying certain concepts as maps in a category, the combining of concepts becomes *composition* of maps! Then we can condense a complicated argument into simple calculations using the associative law. Several hundred years ago, Hooke, Leibniz, and other great scientists foresaw the possibility of a 'philosophical algebra' which would have such features. This section has been quite condensed, and it may take effort to master it. You will need to go back to our previous discussion of Brouwer's proof, and carefully compare it with this version. Such a study will be helpful because this example is a model for the method of 'thinking categorically.'

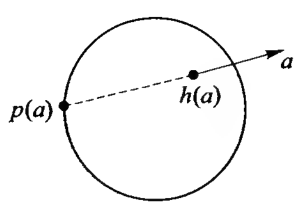
6. The eye of the storm

Imagine a fluid (liquid or gas) moving in a spherical container. (If you want a two-dimensional example, you can imagine water swirling in a teacup and observe the surface current, say by imagining tiny boats drifting.) Right now, each point in our ball is moving, and we draw an arrow with tail at that point to represent its velocity. That is, the length of the arrow is proportional to the speed of the point, and the arrow points in the direction of travel. Could it be that every point is moving with non-zero speed, or must there be at least one instantaneous 'eye of the storm?'

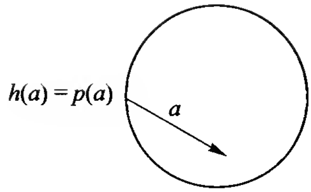
To answer this, we take a slightly different arrow-object A than we imagined before. Its points are to be the possible velocity arrows of particles moving in our ball with non-zero speed. These arrows are less constrained than in our previous arrow-object, since the head of the arrow may be outside the ball; the only restriction is that if a point is on the surface of the ball, its velocity arrow cannot 'point outward' – at worst it is tangent to the sphere. Here is a picture for dimension 2:



The arrows a_1 and a_2 are allowed as points in A, but a_3 is forbidden. Now we'll suppose that every point is moving, so we get a map $B \xrightarrow{\alpha} A$, assigning to each point of B the 'velocity arrow' at that point. For the map $A \xrightarrow{h} B$, we take the map assigning to each arrow its 'home.' (Remember that an arrow is supposed to represent the velocity of a moving point, so the tail of the arrow is the current home of the point.) Finally, for the map $A \xrightarrow{p} S$, we assign to each arrow its imaginary 'place of birth.' (It is customary to name winds in this way, as if a wind arriving from the north had always blown in one direction, and came from the farthest point that it could.)



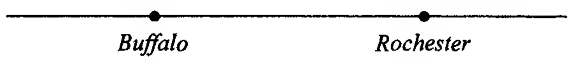
Axiom 1 says that if the moving point is on the sphere, then its 'place of birth' is its current location:



That is, the dot in the picture above is both $h(a)$ (as a point in the ball) and $p(a)$ (as a point on the sphere). Now you can work out for yourself that the corollary to Theorem 1 tells us that if there were a storm with no instantaneous 'eye,' there would be a retraction for the inclusion of the sphere into the ball.

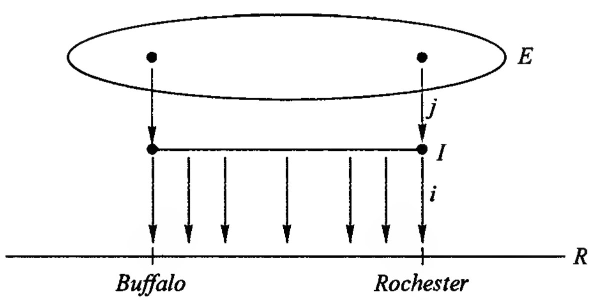
7. Using maps to formulate guesses

Let's return to the one-dimensional case, the two cars traveling on the highway.



Actually, the highway extends beyond these two cities. Suppose I drive along the road, starting in Buffalo and ending in Rochester; you start and finish at the same times, starting and finishing anywhere between Buffalo and Rochester. During our travels, we're allowed to go anywhere along the highway we want, even west of Buffalo or east of Rochester. Are you convinced that at some time we must meet? Why?

Notice that there are now three objects involved: I, an interval; E, its endpoints; and R, the long road. (You can imagine R as the whole line if you want.) We also have two 'inclusion maps':



My travel gives an additional map: $I \xrightarrow{m} R$, and your travel gives another: $I \xrightarrow{y} R$. The relations among these four maps are investigated in the exercises below.

Exercise 4:

(a) Express the restrictions given above on my travel and yours by equations involving composition of maps, introducing other objects and maps as needed.

(b) Formulate the conclusion that at some time we meet, in terms of composition of maps. (You will need to introduce the object **1**.)

(c) Guess a stronger version of Brouwer's fixed point theorem in two dimensions, by replacing E, I, and R by the circle, disk, and plane. (You can do it in three dimensions too, if you want.)

(d) Try to test your guess in (c); e.g. try to invent maps for which your conjectured theorem is not true.