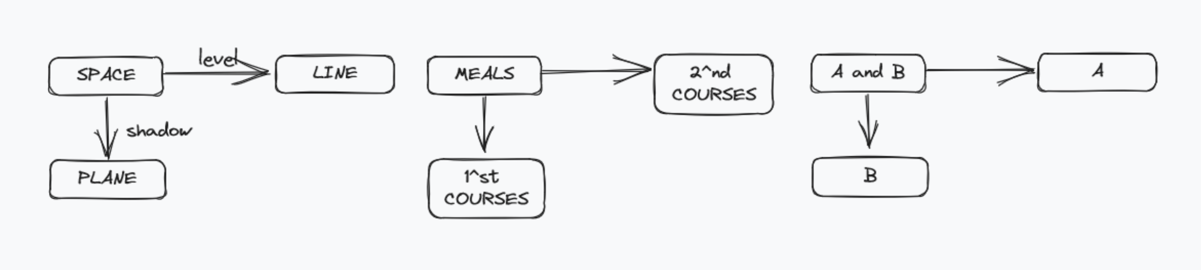
Conceptual Mathematics Note

Session 1 Multiplications of Objects



PART 1 The Category of Sets

Article 1 Sets, maps, composition

An example of category

Category of finite sets and maps:

- Object: one finite set or collection
- Map: consists of a set A, a set B and a rule $b = f(a)$
  - dot in domain has one out, dot in codomain(target) has any number of in.
  - If A = B, called **endomap**

Notion:

> internal diagram: draw arrow from each dot to each target dot

> external diagram: draw one arrow from set A to B

Composition of Maps

$$A \to B \to C$$

Rules

> identity laws: composition of f and I equals to f

> associative law: h(gf) = (hg)f

Point

singleton set: a set with only 1 element, called as ‘1

> Definition: A point of a set X is a map 1 → X

💡 Point is a map and it picks out one element in X
Composing it with another map also gets a point

Session 2 Review

$(x+1)^2 = x^2 + 2x + 1$ are different rules, on natural numbers they always provide the same result for the same input, so the maps of the two rules f, g are the same map.

in a category, maps are same if:

> f, g have the same domain and codomain
> each point 1 → A, f a = g a

PART 2 The algebra of composition

Article 2 Isomorphisms

1. Isomorphisms

> inverse map: if $g \circ f = I_A, f \circ g = I_B$

> Definitions: A map f: A→ B is called a **isomorphism,** or **invertible map,** if there is a map g: B → A for which g is the inverse map of f.
> A and B are said to be **isomorphic** if there is a isomorphism A→B

Properties:

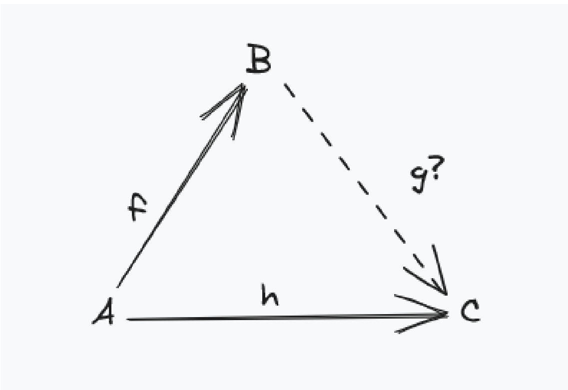
- Reflexive: A is isomorphic to A
- Symmetric: A is isomorphic to B, then B is isomorphic to A
- Transitive: A is isomorphic to B, B is ~ to C, then A is ~ to C

💡 “Division” is related to finding a inverse

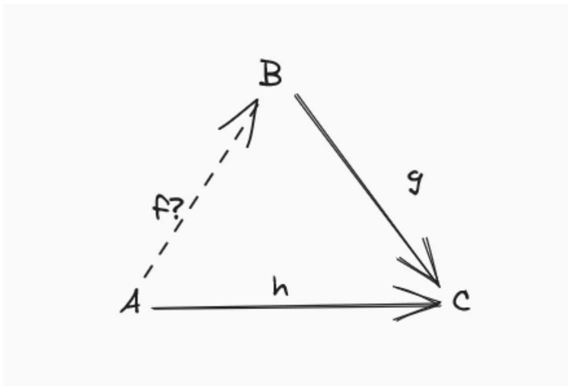
2. General division problems: Determination and choice

Two sorts of division problems for maps:

Determination



Choice



Determination

💡 Determination stands for to determine the h with f: if there is a solution g, then **h is determined by f**, or **h depends only on f**.

Example: |B| = 1, which is b

$$h(x) = (g \circ f)(x) = g(b)$$

h is called constant for it's the same value whatever x in A takes

Choice

💡 Choice means choose a element z for f(x) so that z mapped to h(x) by g

for any x in A, h(x) is given, then f(x) must be in the set of z that g(z) = h(x)

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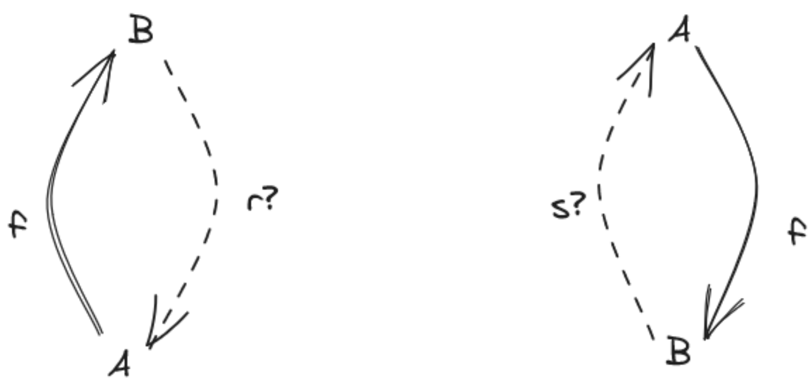
3. Retractions, sections and idempotents

Special cases for determination and choice, when $h = I_A$:

- determination → retraction
- choice → section



Propositions 1

> If a map f: A→ B has a section, then for any T and for any map y: T→B, there exists a map x: T→ A, for which $f \circ x = y$

if the single choice problem has a solution, then every choice problem with the same f has a solution.

f is said to be **surjective（满射）**

💡 If B has some element which is not f(x) from x in A, then f cannot have section.

Similar for determination, omit the proposition here.

Proposition 2

> If f: A→B has a retraction, Then for any set T and for any pair of maps x1: T→A, x2: T→A,
> if $f \circ x_1 = f \circ x_2 \ \ then \ \ x_1 = x_2$

If 2 satisfies, then f is said to be **injective for maps from T（单射）**

If for every T, then f is said to be **injective or is a monomorphism（单态）**

💡 If there are two elements x1,x2 of A that $x_1 \neq x_2 \ \ yet f(x_1) = f(x_2)$, then f cannot have retraction.

Proposition 2*

> If f: A→B has a section, Then for ant set T and any pair t1: B→ T, t2: B→ T of maps from B to T, if $t_1 \circ f = t_2 \circ f$ then t1 = t2

> Definition: A map f if $t_1 \circ f = t_2 \circ f \ \ then \ \ t_1 = t_2$ for every T, is called an **epimorphism （满同态）**

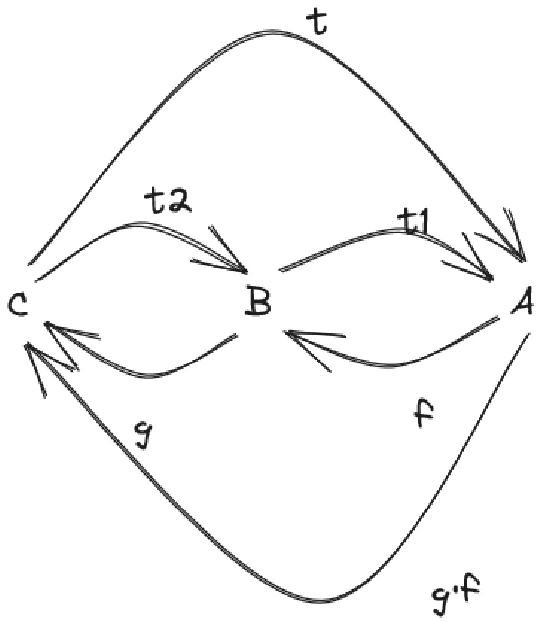
Proposition 3

> If f: A→B has retraction and g: B→C has retraction, then $g \circ f$: A→C has retraction.

Exercise 8:

Prove that the composite of two maps, each having sections, has itself a section.

Let $f \circ t_1 = I_A$, $g \circ t_2 = I_B$, then $t = t_1 \circ t_2$



Theorem (uniqueness of inverses)

If f has both retraction and section then they are the same.

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4. Isomorphisms and automorphisms

Definitions:

> A map f is called an **isomorphism（同构）** if there exists another map f^{-1} which is both a retraction and a section for f

> An isomorphism which is f: A→A is called a **automorphism（自同构）**

A and B have the same number of elements ↔ A and B are **isomorphic** ↔ There exists an isomorphism from A to B in the category

#Aut(A) = #Isom(A, B)

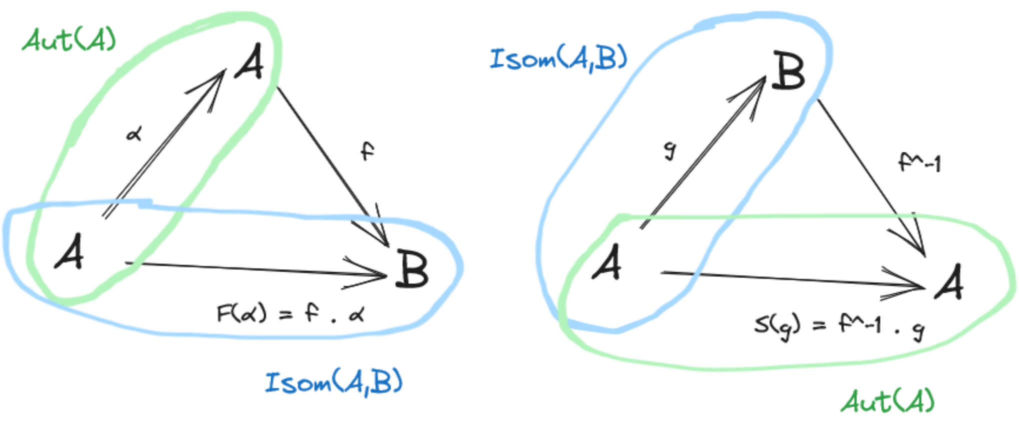
Aut(A) is the set of automorphisms of A, Isom(A, B) is the set of isomorphism from A→B

A map F: Aut(A) → Isom(A, B)

To choose a isomorphism f: A→B, then $F(\alpha) = f \circ \alpha$, which is a isomorphism

To prove that F itself is a isomorphism:

- Construct S: Isom(A,B) → Aut(A)
- Define $S(g) = f^{-1} \circ g$, for each g in Isom(A, B)



Automorphism is also called **permutation**

The previous categories are categories of **sets**, now change to categories of **permutations**

- object: A and an automorphism of it

- map: f: A→B is map of sets A→B, but also $f \circ \alpha = \beta \circ f$

Session 4

Examples of Isomorphisms

algebraic category

- object: A set A together with a combining rule
  - (R, +), (R>0, x)
- map: f: (A, *) → (A’, *’) needs to
  - f(a * b) = f(a) *’ f(b)
  - exp: (R, +) → (R>0, *), for exp(a+b) = exp(a) * exp(b)
  - Abstract examples:
    - odd + even = odd, even + even = even …
    - positive * neg = neg, neg * positive = neg …

Exercise 2:
Find an isomorphism

$$(\{odd, even\},\ +) \xrightarrow{f} (\{positive,\ negative\}, \times)$$

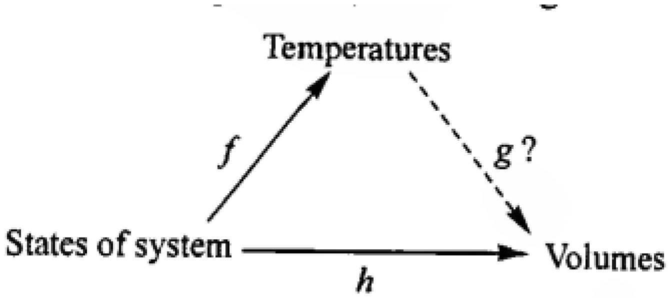
Hint: There are only two invertible maps of sets from $\{odd, even\}$ to $\{pos., neg.\}$. One of them ‘respects the combining rules’, but the other doesn’t.

odd → negative, even → positive

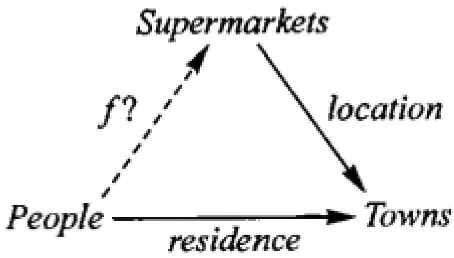
If not, f(odd + odd) = f(odd) * f(odd) = positive * positive = positive is not f(even) = negative

Session 5 Division of maps: Sections and retractions

Determination problem means if one attribute of a system: by map the status with f, can determine another attribute which is mapped from the status with h



Choice problem is to choose for a of A an element b that g(b) = h(a)



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Session 6 Two general aspects or uses of maps

1. Sorting of the domain by a property

g: X→ B can sort X as the **sort** b of B

X is divided into B fibers

2. Naming or sampling of the codomain

A map f: A→X is sampling |A| elements from X

3. Philosophical explanation of the two aspects

Session 7 Isomorphisms and coordinates

Session 9 Retracts and idempotents

1. Retracts and comparisons

A **is at most as big as** B:

Definition: *$A \trianglelefteq B$ means that there is at least one map from A to B.*

properties:

- reflextive
- transitive

💡 This definition only tells non-empty sets from empty sets.

A is retract of B: A→B→A

Definition: *A is a* **retract of** *B means that there are maps $A \xrightarrow{s} B \xrightarrow{r} A$ with $rs = 1_A$. (We write this as $A \leq_R B$.)*

💡 This definition says **A has at most as many points as B**

2. Idempotents as records of retracts

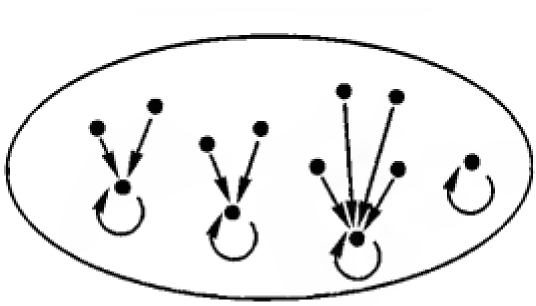
For rs = I, then the other order e = sr is **idempotent:**

$$ee = (sr)(sr) = s(rs)r = sr = e$$

fixed points of endomap e are those chosen by s from B as the representation of B (Note that A has at most as many points as B), because s will map r(fixed points) to the fixed points themselves.

The subset of fixed points in B is **isomorphic** to the set of A.

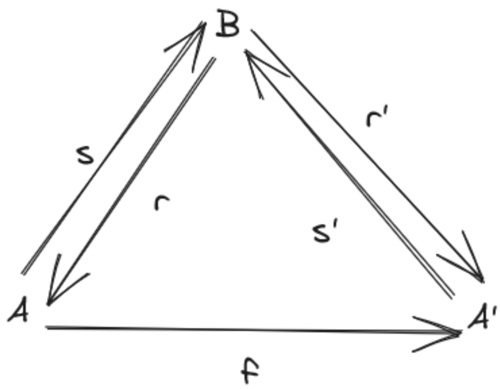
Each point mapped by an **idempotent endomap** will ended looping at a fix point:



B can be divided into small sets by e, define a **splitting of e** as:

Definition: *(In any category) If $B \xrightarrow{e} B$ is an idempotent map, a* **splitting of** *e consists of an object A together with two maps $A \underset{r}{\overset{s}{\rightleftarrows}} B$ with $rs = 1_A$ and $sr = e$.*

Exercise 3:
(In any category) Suppose that both $A \underset{r}{\overset{s}{\rightleftarrows}} B$ and $A' \underset{r'}{\overset{s'}{\rightleftarrows}} B$ split the same idempotent $B \xrightarrow{e} B$. Use these maps to construct an isomorphism $A \xrightarrow{f} A'$.



$e = sr = s'r', rs = I_A, r's' = I_{A'}$

$f = r's, f^{-1} = rs'$

$ff^{-1} = r'srs' = r's'r's' = II = I$, the same for f^-1 f

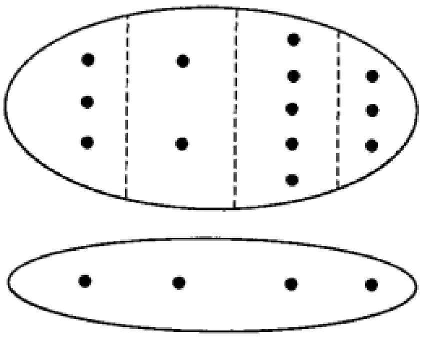
3. A puzzle

B(a larger set than A) and its map endomap e can be used to represent A if:

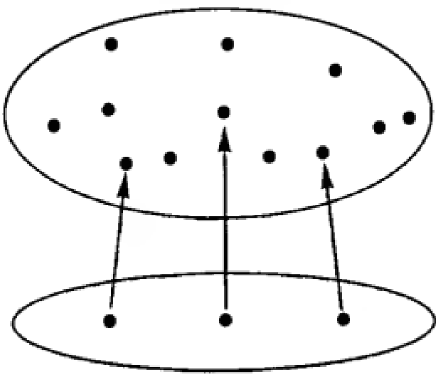
- B is structurally simpler than A

4. Three kinds of retract problems

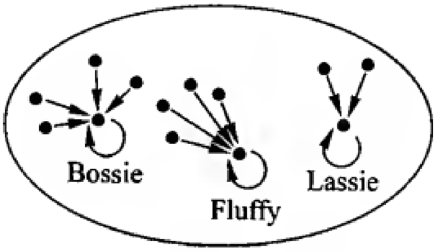
Museum director's problem: Given $B \xrightarrow{r} A$, choose $A \xrightarrow{s} B$ satisfying $rs = 1_A$. Mental picture: View r as sorting B into A sorts:



Bird-watcher's problem: Given $A \xrightarrow{s} B$, choose $B \xrightarrow{r} A$ satisfying $rs = 1_A$. Mental picture: View s as a sampling of B by A:



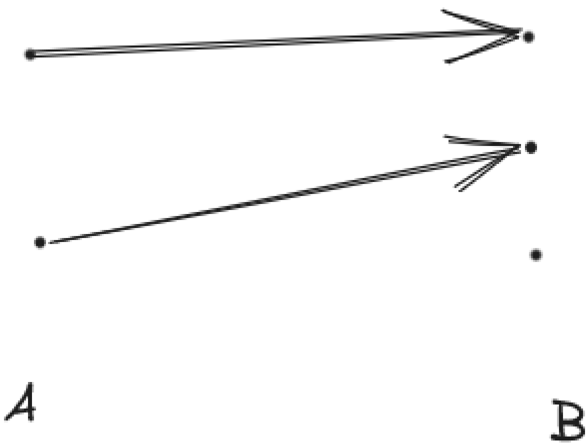
Child's problem: Given B, choose a map $B \xrightarrow{e} B$ satisfying $ee = e$. Having watched children for years, I remain as puzzled as ever about the selection of the idempotent endomap e associating to each animal the most familiar animal it resembles. After that's done, though, the rest of the job (splitting the idempotent) is easy:



5. Comparing infinite sets

Quiz

1



2

✔️

2*

q' = qpq

1*

A = {2, 4, 6…} B = {1, 2, 3…} f = 1/2 x

Composition of opposed maps

Session 10 Brouwer's theorems

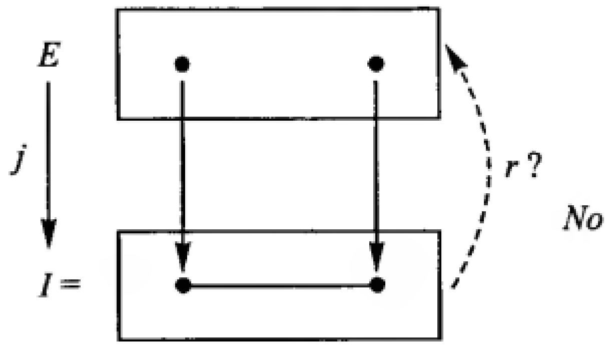
1. Balls, spheres, fixed points and retractions

Brouwer fixed point theorems:

- Let I be a line segment, including its endpoints, and suppose that f: I→I is a continuous endomap. Then this map must have a fixed point: x in I which f(x) = x.
- Let D be a closed disk, and f a continuous endomap of D. Then f has a fixed point.
- Any continuous endomap of a solid ball has a fixed point.

Brouwer retraction theorems:

- Consider the inclusion map j: E→I of the two-point set E as boundary of the interval I. There is no continuous map which is a retraction for j.



- Consider the inclusion map j: C→D of the circle C as the boundary of the disk D into the disk. There is no continuous map which is a retraction for j.
- Consider the inclusion j: S→B of the sphere S as boundary of the ball B into the ball. There is no continuous map which is a retraction of j.

> 💡 To prove the retraction theorems(easier) is equivalent to proving the corresponding fix point theorems(harder).

3. Brouwer’s proof

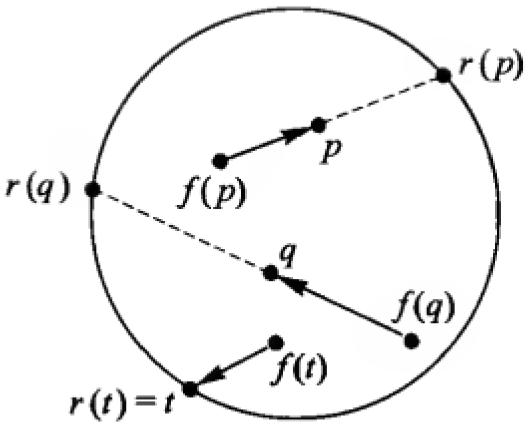
to prove:

> No continuous retraction of the disk to its boundary → every continuous map from the disk to itself has a fixed point.

→ to prove:

> Any continuous endomap of the disk with no fixed points → exists a continuous retraction of the disk to its boundary.

construct the retraction r as below:



4. Relation between fixed point and retraction theorems

Exer2 shows that the fixed point property is inherited by **retract**

> 💡 Fixed point property means **every endomap** has fixed points(maybe be the same)

If X has fixed points, and A is a retract of X, then A has fixed points.

So to prove A is not a retract → to prove A has no fixed points but X has → to find **one** endomap without fixed points → antipodal endomap

5. How to understand a proof

Sets of dots:

- B: a ball
- S: A sphere

Elements of the category:

- object: A, a set of arrows in B
- map:
  - h: A→B, to get head
  - p: A→S, to get point on the sphere

We will bring this into our category, by noting that a map $T \xrightarrow{a} A$ is a (smooth) 'listing' of arrows: $T \xrightarrow{a} A$.

Axiom 1: *If T is any object in $\mathcal{C}$, and $T \xrightarrow{a} A$ and $T \xrightarrow{s} S$ are maps satisfying $ha = js$, then $pa = s$.*

For Axiom1:

- T is a topological space, like a disk or interval. It contains a collection of points.
- The map a: T -> A assigns an arrow in A to each point t in T.
- So as t varies continuously over T, it traces out a continuous family of arrows in A.

💡 T is called **parameter space or test object** here.

Theorem 1: *If $B \xrightarrow{\alpha} A$ satisfies $h\alpha j = j$, then $p\alpha$ is a retraction for j.*

Proof: Put $T = S$, $s = 1_S$, and $a = \alpha j$ in Axiom 1.

Corollary: *If $h\alpha = 1_B$, then $p\alpha$ is a retraction for j.*

Describe:

- j is the map from S to B, we want to find a retraction for j, which goes back from the ball to the sphere.
- α maps from the ball B to a set of arrows A, equal to find the arrow of f(x) , g(x), because these two dots can determine an arrow
- h finds the head of that arrow
- so hαj is to map the dots in the ball back to the same dots on the sphere they are mapped from by j, using an arrow's head
- The theorem says if the arrow's head is the same as the dot in the ball, even the point of the arrow should be the same as the original point on the sphere

Axiom 2: *If T is any object in $\mathcal{C}$, and $T \underset{g}{\overset{f}{\rightrightarrows}} B$ are any maps, then either there is a point $1 \xrightarrow{t} T$ with $ft = gt$, or there is a map $T \xrightarrow{\alpha} A$ with $h\alpha = g$.*

Now we can finish his argument:

Theorem 2: *Suppose we have maps*

$$B \underset{g}{\overset{f}{\rightrightarrows}} B$$

and $gj = j$, then either there is a point $\mathbf{1} \xrightarrow{b} B$ with $fb = gb$, or there is a retraction for $S \xrightarrow{j} B$.

Describe:

- Axiom 2: either there is a dot gets no arrow in T, which start = end, or there can find a map from the dots to g = the heads of the arrows of the dots
- Theorem 2: to take T = B(treat all set as the test set), either there is dot fb = gb, or hαj = gj = j, which satisfies Theorem1, so pα is a retraction for j.

To get the fixed point Theorem, let g = I then gj = j is always true, so here is a **Corollary**

- either fb = b or j: S→B has a retraction

6. The eye of the storm

7. Using maps to formulate guesses