Introduction to Complex Dynamics in One Dimension

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Week 2 Mon. (Feb.24): Page 1-6.
Week 2 Wed. (Feb.26): Page 7-9.
Week 3 Mon. (Mar.3): Page 10-21.
Week 4 Mon. (Mar.10): Page 22-28.
Week 4 Wed. (Mar.12): Page 28-34.

Contents

1 Introduction and Basic Concepts

1.1 Attracting Fixed Point

Let P be a complex polynomial of degree $d > 2$. Let z_0 be a fixed point, i.e. $P(z_0) = z_0$. We may assume that $P(z_0) = z_0 = 0$. Let $\lambda = P'(z_0)$.

We call Q conjugate to P, if $\exists\phi$ 1-1 analytic, defined in a neighborhood of z_0, $\phi(z_0) = 0$, $Q = \phi \circ P \circ \phi^{-1}$.

Note that P^n means iteration, instead of power.

Thm 1.2 (Koenigs)

Assume $0 < |\lambda| < 1$. Then we can find a Q conjugate to P, $Q = \lambda z$.

Proof Define $\phi_n = \dfrac{P^n}{\lambda^n}$, $\lim\limits_{n\to\infty} \phi^n = \phi$. Since $\phi_n \circ P = \dfrac{P^{n+1}}{\lambda^n} = \lambda\phi^{n+1}$, by taking limit we have $\phi \circ P = \lambda\phi$. $\phi^n(z_0) = 0$ so $\phi(z_0) = 0$.

So we only need to show ϕ_n converges on a neighborhood of 0. $P'(0) = \lambda$ means that we can choose δ, C such that for $|z| < \delta$, $|P(z) - \lambda z| < C|z|^2$. On the one hand,

$$|P(z)| < |\lambda||z| + C|z|^2 < (|\lambda| + C\delta)|z| \Rightarrow |P^n(z)| < (|\lambda| + C\delta)^n|z|.$$

So by making $|\lambda| + C\delta < 1$, $|P^n(z)| < |z| < \delta$ .On the other hand,

$$|\phi_{n+1} - \phi_n| = \left|\frac{P(P^n) - \lambda P^n}{\lambda^{n+1}}\right| < \frac{C|P^n|^2}{|\lambda|^{n+1}} < \frac{C((|\lambda| + C\delta)^n|z|)^2}{|\lambda|^{n+1}} < \frac{C\delta^2}{|\lambda|}\left(\frac{(|\lambda| + C\delta)^2}{|\lambda|}\right)^n.$$

By making $\dfrac{(|\lambda| + C\delta)^2}{|\lambda|} < 1$, we prove the convergency.

Note Konigs 译名为科尼格斯.

证明中 ϕ 的具体形式一般无法得知. ϕ_n 的构造思路如下. 令 $\phi_1 = Id$, 通过 $\phi_1 \circ P = \lambda\phi_2$ 得到 $\phi_2 = P/\lambda$. 由此递归可以依次得到 $\phi_n = P^n/\lambda^n$, 随后证明是 *Cauchy* 列.

Thm 1.3 (Boettcher)

Assume $\lambda = 0$. Denote $w = \phi(z)$, then we can find a Q conjugate to P, $Q(w) = w^p$.

Note P not necessarily a polynomial here. w is called a Boettcher coordinate(波切尔坐标). p is the "degree" of P, i.e.$P(z) = a_p z^p + a_{p+1} z^{p+1} + \cdots$.

Proof We may assume $P(z) = z^p + \cdots$. Because by $z = cz'$, we can conjugate $P(z) = a_p z^p + \cdots$ with $P'(z') = (z')^p + \cdots$. The conjugation is unique up to roots of unity.

Define $\phi_n = (P^n)^{\frac{1}{p^n}}$, $\lim\limits_{n\to\infty} \phi_n = \phi$. Since $\phi_n \circ P = (P^{n+1})^{\frac{1}{p^n}} = \phi_{n+1}^p$, we will have $\phi \circ P = \phi^p = Q \circ \phi$. Here ϕ_n is well defined, because $P^n(z) = z^{p^n} e^{\alpha_n(z)}$, with $\alpha_n(0) = 0$, then $\phi_n(z) = ze^{\alpha_n(z)/p^n}$.

Similarly, we need to prove the convergence on a neiborhood of 0. The progress is analogous to which in Thm 1.2, so we omit it.

Note 这里将 0 作为超吸性不动点, 习惯上也会选择 ∞ 作为超吸性不动点.

Exercise

Basin of attraction of z_0 is $U := \{z | P^n(z) \to z_0\}$ for z_0 attracting. Immediate basin of attraction is the connected component of U which contains z_0.

1.4 $U = D$, $U^I = D$.

1.5 $Q(w) = \dfrac{w^2}{w^2+1}$, $Q'(w) = \dfrac{2w}{(w^2+1)^2}$. When $w \to \infty$, $Q'(w) \to 0$.

1.6 z_f are the roots of $z^2 - z + c = 0$, $P'(z_f) = 2z_f$. $z_f^1 + z_f^2 = 1$, so unless $z_f^1 = z_f^2 = 1/2, c = 1/4$, one of $|P'(z_f)|$ will stricly greater than 1.

Thm 1.7

There is a critical point in the immediate basin of attraction of an attracting fixed point.

Proof If $\lambda = 0$, then the fixed point z_0 is critical itself, so we assume $0 < |\lambda| < 1$. Since P has at most $d-1$ critical points, we can find a neiborhood of z_0 that contains no critical point. Also because basin of attraction is open, the neiborhood can be chosen inside the basin. We may assume $z_0 = 0$, the neiborhood is $U_0 = D$.

Let $U_1 = f(U_0)$, where f is the inverse of P in U_0. Then U_1 is simply connected with 0 inside, and lies inside the immediate basin. If U_2 contains a critical point, then we come to the end. Otherwise, let $U_2 = f(U_1)$. If all U_n contains no critical point, then $f^n : U_0 \to U_n$ is 1-1. However, $(f^n)'(0) = 1/\lambda^n \to \infty$, so U_n will not tends to any D-type area. If U_n tends to $\tilde{\mathbb{C}}$-type, then P will have no critical point. If U_n tends to $\mathbb{C}$-type, then P has only one critical point, which means $P(z) = (z-a)^d$. But then $a = 0$ and $\lambda = 0$. Therefore, there must be a moment when U_n contains a critical point.

Note 讲义中少提了 U_0 属于 immediate basin 的条件. 另外, 吸引域显然是在 F 里的.

这个证明不需要 P 是多项式. 但拟正则动力系统中可能没有建立这个.

关于临界点. 严格来说 d 阶多项式的临界点有 $2d-2$ 个, 其中 $d-1$ 个有限, ∞ 算作一个 $d-1$ 重的临界点.

关于吸引域的连通性. 对多项式的有限不动点来说直接吸引域是单连通的, ∞ 处的直接吸引域是多连通的. *(单连通可以通过最大模原理证明, 具体不会写.)*

对证明过程的补充. f^n 在 U_0 上单叶, $f^n(0) = 0, (f^n)'(0) = 1/\lambda^n$, 则由 *Koebe 1/4* 定理, $U_n = f^n(U_0)$ 包含一个半径为 $(f^n)'(0)/4$ 的圆. 这表明 U_n 包含半径任意大的圆, 因此会扩张到全平面. 另一个说法是, U_n 作为直接吸引域总是双曲的 *(单位圆盘)*, 因此不可能覆盖抛物的面 *(去点平面)*. *(但是原证明中提到 U_n 是递增的, 不知道怎么验证.)*

1.2 Julia Set

Basic Properties of F and J

Fatou set $F := \{p : \exists U(p) \text{ open}, P^n|_{U(p)} \text{ is a normal family in } \mathbb{C}\}$. Julia set $J := \tilde{\mathbb{C}} \backslash F$. Normal family: for any subsequence $\{f^{n_k}\}$, any compact $K \subset U(p)$, there exists $\{f^{n_{k_j}}\}$ converges uniformly on K. F is open, J is compact, $\infty \in F$.

Completely invariant: $P(E) = E, P^{-1}(E) = E$. Actually $P^{-1}(E) = E$ can guarantee completely invatiant, but $P(E) = E$ cannot.

Note 尽管对于多项式, ∞ 由于总是被映射到自身, 因此属于 F. 但对于超越整函数, ∞ 作为本性奇点会被放进 J, 此时 J 可以说是紧的. 如果 ∞ 不被放入 J, 那么很难说 J 应该算紧还是不紧. *(大致是这个意思.)*

Thm 1.9 *J is nonempty.*

Proof If J is empty, $\{P^n\}$ is a normal family on $\tilde{\mathbb{C}}$. Let $\{P^{n_k}\}$ converge uniformly to f, then f is surjective, thus not constant. Each P^{n_k} has d^{n_k} zeros, so the zeros of f cannot be decided.

Note 这个结论对其他函数也成立, 但证明是不同的. F 可以是空集, 例如指数函数. *(甚至能构造有理函数 $F = \emptyset$ 的例子, 但比较复杂.)*

证明中提到 f 只有有限个零点, 但 P^{n_k} 有 d^{n_k} 个零点, 因此需要通过 *Rouche's thm* 来说明矛盾. *(但鲁歇定理需要一条简单闭曲线, 需要把全平面割成两块来说明吗?)*

Thm 1.12 *F and J are both completely invariant.*

Thm 1.13 *$F(P) = F(P^N), J(P) = J(P^N)$ for integer $N \geq 2$.*

Note 事实上, 取自然数的任意子列 a_n, 都可以证明 $J(P) = J(P^{a_n})$, 但这个证明并不容易.

这个定理的另一个证明方法要通过 F 的另一种定义: 存在 z 的邻域 U, $P^n|_U$ 等度连续. 然后就可以通过 P 的一致连续性完成证明. *(大致是这个意思.)*

Exceptional Set

Exceptional set E_z: $z \in J, U = U(z), E_z := \tilde{\mathbb{C}} \backslash \cup_n P^n(U)$. Since P^n is not normal family here, E_z contains at most two elements by Montel's theorem.

Note 例外点的另一个定义: $\#\{P^{-n}(z)\}$ 有限的点.

例外点与 z 的选取无关. 对于多项式而言, $P^{-1}(\infty) = (\infty)$, 所以 ∞ 总是属于例外点.

例外点属于周期点, 周期点不一定是例外点. 例外点属于 F.

Lemma 1.14 If E_z has two points $\{0, \infty\}$, then $P(z) = a(z-z_0)^d + z_0$. Otherwise, E_z consists of ∞ only.

Exercise

1.19 The exceptional set of P and P^N are the same.

Other Properties of J

Thm 1.15 If z is a nonexceptional point, then $J \subset \overline{\cup P^{-n}(z)}$.

Proof $\forall w \in J, U = U(w)$, by definition of nonexceptional, $z \in P^n(U)$ for some n, i.e. $\exists \eta \in U$ such that $\eta \in P^{-n}(z)$. So $w \in \overline{\cup P^{-n}(z)}$.

Thm 1.16 If $z \in J$, then $J = \overline{\cup P^{-n}(z)}$.

Proof We only need $\overline{\cup P^{-n}(z)} \subset J$. Since $P^{-n}(J) = J$, $z \in J$, we have $P^{-n}(z) \in J$. $\cup_n P^{-n}(z)$ is an increasing series, so the closure also lies inside J.

Thm 1.17 If $U \subset J$ is nonempty and completely invariant, then $\overline{U} = J$.

Proof $\forall z \in U, P^{-n}(z) \subset P^{-n}(U) = U$, then $J = \overline{\cup P^{-n}(z)} \subset \overline{U}$. Obviously $\overline{U} \subset J$.

Thm 1.21 Suppose that U is a union of connected components of F and suppose that U is completely invariant. Then $J = \partial U$.

Proof 1. ∂U is contained in J.

2. ∂U is nonempty (by assumption $F \neq \emptyset$).

3. ∂U is completely invariant. $P^{-1}(U) = U \Rightarrow \overline{P^{-1}(U)} = \overline{U}$, by the continuity of P^{-1}, $\overline{P^{-1}(U)} = P^{-1}(\overline{U})$.

Then use Thm 1.17.

Note 这说明 J 没有内点, $\partial F = J$.

U 只需要是 F 的一部分, 但满足完全不变是一个比较高的要求. 比如 z^2 的 J 是单位圆周, 取圆周内作为 U 或者圆周外作为 U, 都可以得到 $\partial U = J$. 另外, 由于用到了 P^{-1} 的连续性, 这个方法不能直接推广到其他函数.

Thm 1.22 J contains no isolated point.

Proof Suppose $z_0 \in J$ is isolated. Consider $z_1 \in P^{-1}(z_0)$, then $z_1 \in J$. By Thm 1.16, $\exists n, w \in P^{-n}(z_1), w \in U(z_0)$. But z_0 is the only point in $U(z_0)$ that belongs to J, so $w = z_0$. Then $z_0 = P^{n+1}(z_0)$. We may assume that $z_0 = P(z_0)$, since $J(P) = J(P^{n+1})$. If $P^{-1}(z_0) \neq \{z_0\}$, we can find $z_1 \in P^{-1}(z_0)$, then by the same reason we get conflict. If $P^{-1}(z_0) = \{z_0\}$, then $P'(z_0) = 0$, so $z_0 \in F$.

Note $P(z_1) = P(z_0) = z_0$ 的情况需要额外说明, $P^{-1}(z_0) = \{z_0\}$ 是唯一的轨道, 产生矛盾.

Periodic Orbits

Define periodic point $P^m(z_0) = z_0$, periodic orbit $\{z_0, P(z_0), ..., P^{m-1}(z_0)\}$. Similarly let $\lambda = (f^m)'(z_0)$, we call $|\lambda| > 1$ repelling, $|\lambda| = 1$ neutral, $|\lambda| < 1$ attracting.

Number of Attracting Orbits An attracting point of P of order m in equivalent to an attracting point of P^m, to which there is a critical point of P^m related by Thm 1.7. But number of critical point of P and P^m are always the same. P has $d-1$ critical point, so P has $d-1$ attracting orbits.

Note 这个应该不包含 ∞ 处.

Number of Neutral Orbits 这一段无法理解. 大致思想是扰动多项式, 使过半的中性点变成吸引点. 而吸引点有 $d-1$ 个, 故中性点不超过 $2d-2$ 个. (有一段笔记在 iPad 上.)

Let $P(z) = z^d + ... + a_0$, $R(z,t) = (1-t)P + tz^d$, $0 \leq t \leq 1$. We denote the solution set of $R^m(z,t) = z$ by Z_m, with d^m points inside for each t. Also denote the solution of $\dfrac{\partial}{\partial z} R^m(z,t) - 1 = 0$ by X_m, with $d^m - 1$ elements. For $t = 1$, $Z_m \cap X_m = \emptyset$.

这个方法由 *Fatou* 提出, 在 *Milnor* 书 *P153,* 吕以辇书 *P31* 中也有讲到.

这个方法在其他地方几乎没有使用, 可能反全纯动力系统中会用, 或者在解 $P(z) = \overline{z}$ 时.

Thm J is the closure of the repelling periodic orbits.

Proof Let $z \in J, U = U(z)$, f_1, f_2 be two inverses of P on U. Define

$$g_n(w) = (\frac{P^n(w) - f_1(w)}{P^n(w) - f_2(w)})(\frac{w - f_2(w)}{w - f_1(w)}).$$

Then g_n omits $0, 1, \infty$, therefore normal. Then we can derive that P^n is normal.

Note 这只是一个大致证明思路, 技巧性较强.

2 Invariant Measures

2.1 Introduction

Def Invariant probability measure μ on $\tilde{\mathbb{C}}$: for any Borel set B, $0 \leq \mu(B) \leq 1 = \mu(\tilde{\mathbb{C}})$, $\mu(P^{-1}(B)) = \mu(B)$.

Def Equilibrium measure for f: $\mu_f = \lim_{n\to\infty} \frac{1}{d^n} \sum_{f^n(z)=w} \delta_z$, where w is a nonexceptional point. This measure satisfies: f-invariant, maximal entropy, is supported on Julia set, and can be obtained as the Laplacian of the Green function.

Note 这一段是根据 chatgpt 补充的. $\mu_f(A)$ 的含义是 $P^{-n}(w)$ 有多少概率落在 A 里. 但这个定义有个问题, 无法验证一个测度是不是平衡测度, 和 Example 2 中提到的性质也没有直接关联. 这和后续给的定义也不同.

Def Let $|A|$ denote the area of a set A, δ_{z_0} denote the Dirac mass at z_0.

Example 1 Let z_0 be a fixed point, then δ_{z_0} is an invariant measure.

Example 2 $d\theta/(2\pi)$ on unit circle with $P(z) = z^2$. Note that it is also an equilibrium measure, namely for a small arc, the two preimage have same length.

Example 3 δ_∞ is an invariant equilibrium measure. For $P(z) = z^d$, δ_0 is also an invariant equilibrium measure. Note that 0 and ∞ are two exceptional points in this condition.

Thm 2.1 (Target of the Section) Let P be a polynomial of degree $d \geq 2$. Then there is a unique equilibrium measure μ which gives no mass to the exceptional set.

Thm 2.2 (Koebe distortion theorem) $f : D \to D$ 1-1 holomorphic function, $f(0) = 0$, $0 < s < 1$. Then $\exists C$ such that $\sup_{|z|=s} |f(z)|^2 \leq C|f(D)|$.

Note 证明比较巧妙, 几乎只用到了 Cauchy Schwartz 不等式. 结论主要是为 Lyubich 的引理做铺垫.

Thm 2.11 (Koebe distortion theorem) $f : D \to \mathbb{C}$ 1-1 holomorphic function, $0 < s < 1$. Then $\exists C$ such that $\sup_{|z|,|w|\leq s} |f(z) - f(w)| \leq C\sqrt{|f(D)|}$.

Exercise 2.5 Let $\phi \circ P = Q \circ \phi$. If z_0 is a fixed point for P, then $\phi(z_0)$ is a fixed point for Q and $Q'(w_0) = P'(z_0)$.

2.2 Convergence

Def C: set of critical points.

$V_l := \cup_{q=1}^{l} P^q(C)$ is called the postcritical set. Especially, $V = P(C)$ and $V_\infty = \cup_{l\geq 1} V_l$.

Lemma 2.6 (Lyubich) Let $\epsilon > 0$. There exists $l > 0$ (relies on ϵ) that for any topological discs $D \subset\subset \tilde{D} \subset\subset \mathbb{C}\backslash V_l$, P^n has at least $(1-\epsilon)d^n$ inverses branches $g_{i,n}$ (n relies on $\epsilon, D, \tilde{D}$, sufficiently large), such that $g_{i,n}(D)$ has diameter at most $cd^{-n/2}$ (c is independent of n).

Note $\subset\subset$ 的意思是紧包含, 即闭包也包含在其中. 证明的主要思想是大部分原像圆盘都很小, 少部分会比较大, 但这少部分的数量是可控的. 主要的方法是 Thm 2.11. 另外, 在超越函数的领域中, $J \subset \overline{V_\infty}$ 是一个很难处理的情况, Lyubich 的这个估计目前还不是很能改进.

Def $\mu_{n,x} = \dfrac{1}{d^n} \displaystyle\sum_{z, P^n(z)=x} \delta_z, x \in \tilde{\mathbb{C}}, n \geq 1.$

Def Let λ_n be finite measures, $C(\tilde{\mathbb{C}})$ be space of continuous functions. If $\forall \phi \in C(\tilde{\mathbb{C}})$, $\displaystyle\int \phi d\lambda_n \to \int \phi d\lambda$, we say λ_n converge weakly to λ.

Lemma 2.7 If $\mu_{n,x}(y) \to 0$, then $\mu_{n,x}(P(y)) \to 0$.

Corollary 2.8 If $\mu_{n,x}(C) \to 0$, then $\forall l, \mu_{n,x}(V_l) \to 0$ as $n \to \infty$.

Thm 2.9 Suppose $\mu_{n,x}(C), \mu_{n,y}(C)$ converge to 0. Then $\mu_{n,x} - \mu_{n,y}$ converges weakly to 0.

Note 这个定理的思想是, 因为想将 $\mu_{n,x}$ 变为 μ, 就要考虑 $\mu_{n,x}$ 与 $\mu_{n,y}$ 差了多少. 如果 x, y 始终有单叶原像, 那根据 Koebe 定理它们最终一定会很接近. 但如果 x, y 遇到了临界点, 可能就会失控. 因此要说明这些控制不了的点很少.

这个证明好像能推广到有理函数上.

Exercise 2.12 $\{\delta_{1/n} - \delta_{-1/n}\}_n$ converges weakly to 0.

2.3 Equilibrium Measure

Lemma 2.14 $x \notin E$, then $\mu_{n,x}(C) \to 0$.

Note The Exceptional set of a polynomial is the largest finite set which is completely invariant.

Cor 2.15 $x, y \notin E$, then $\mu_{n,x} - \mu_{n,y} \to 0$ weakly.

Def $\mu_{n,x}$ are pull-backs of measures: $\mu_{n,x} = (P^n)^*(\delta_x)/d^n$. Generally, let ν be a measure on $\tilde{\mathbb{C}}$, we define the pullback measure: $P^*(\nu)(U) = \nu(P(U))$ for U satisfing $P : U \to P(U)$ 1-1, and $P^*(\nu)(c) = m\nu(P(c))$ for c a critical point of multiplicity m.

Prop By definition, $P^*(\delta_x)(y) = m\delta_x(P(y))$.

Lemma 2.16 $P^*(\mu_{n,x}) = d \cdot \mu_{n+1,x}$.

Def An invariant probability measure λ is an equilibrium measure if $P^*(\lambda) = d\lambda$.

Lemma 2.18 Let $\lambda_n(x) = \dfrac{1}{n}\displaystyle\sum_{j=1}^{n} \mu_{j,x}$. The measures $\lambda_{n+1,x} - \lambda_{n,x}$ and $\lambda_{n+1} - P^*\lambda_n/d$ have mass at most $2/(n+1)$.

Def $\phi : \tilde{\mathbb{C}} \to \mathbb{C}$ continuous, $P_*(\phi)(z) = \displaystyle\sum_{P(w)=z} \phi(w)$ is also continuous, called the push-forward of ϕ.

Lemma 2.20

$$\int_{\tilde{\mathbb{C}}} \phi P^*\nu = \int_{\tilde{\mathbb{C}}} P_*(\phi)\nu.$$

Lemma 2.21 There exists a probability measure μ so that $\mu = P^*(\mu)/d$. Moreover the measure has no mass on the exceptional set.

Thm 2.22 The μ above is unique. Moreover for any probablity measure ν with no mass on E, $\dfrac{(P^n)^*\nu}{d^n} \to \mu$. In particular, $\mu_{n,x} = \dfrac{(P^n)^*\delta_x}{d^n}$ converges to μ if and only if $x \notin E$.

Thm 2.17 (2.1) Let P be a polynomial of degree $d \geq 2$. Then there is a unique equilibrium measure μ which gives no mass to the exceptional set.

2.4 Ergodic and Mixing

Def ν a probability measure. ν is mixing if $\forall E, F$, $\nu(E \cap P^{-n}(F)) \to \nu(E)\nu(F)$.

Def ν is ergodic for P if for every invariant Borel set E, $\mu(E) = 0$ or ∞.

Thm μ is mixing and ergodic.

证明暂略.

3 Topics on Fatou Sets

3.1 Neutral Fixed Point

We consider $P(0) = 0$, $\lambda = P'(0)$, $|\lambda| = 1$. Then

Case I: λ is a rational rotation. (Thm 3.5: the iterates converge uniformly to 0, $0 \in J$.)

Case I.1: $\lambda = 1$.

Case I.2: λ is a root of unity.

Case II: λ is irrational but diophantine. (Thm 3.1: P conjugate to λz.)

Case III: λ is irrational and not diophantine. (Thm 3.2, Lemma 3.3: P may not conjugate to λz.)

We also show that almost all λ are diophantine. (Lemma 3.4)

Def (Diophantine) $|\lambda| = 1$ is diophantine if $\exists c > 0, \mu > 1$, such that $\forall n \geq 1$, $|\lambda^n - 1| \geq \dfrac{c}{n^\mu}$.

Thm 3.1 (Siegel) $P(0) = 0, P'(0) = \lambda$, λ is diophantine. Then $\exists \phi$ holomorphic conjugation, $\phi(0) = 0, \phi'(0) = 1$, so that $\phi(P(z)) = \lambda\phi(z)$ on a neiborhood of 0.

由于实在不感兴趣, 这一节的证明暂略.

3.2 Denjoy-Wolff Theorem

Thm 3.6 Let f be an analytic function, $f(D) \subset D$. Then the only cases are:

Case I: $f \in Aut(D)$ fixes a point $p \in D$.

Case II: $f \in Aut(D)$ has no fix point in D, $\exists \alpha \in \partial D$, $f^n(z)$ converges uniformly to α on compact subsets.

Case III: $f \notin Aut(D)$, $\exists \alpha \in D$, $f^n(z)$ converges uniformly to α on compact subsets.

Case IV: $f \notin Aut(D)$, $\exists \alpha \in \partial D$, $f^n(z)$ converges uniformly to α on compact subsets.

Proof (by Beardon) Case I is usual. If we solve $f(p) = e^{i\theta}\dfrac{p-a}{1-\bar{a}p} = p$, the two roots will satisfy $p_1p_2 = \dfrac{-e^{i\theta}a}{\bar{a}} \Rightarrow |p_1p_2| = 1$. Unless $|p_1| = |p_2| = 1$, there is a fixed point inside D.

For Case II, we can think of f as a biholomorphic map on the upper half plane, then extend by reflection to the lower half plane. It will be of the form $\dfrac{az+b}{cz+d}$ (real parameter), with two fixed points conjugate. Since there are no solution in H^+, the two fixed points are real. We may assume they are ∞, then $f(z) = az + b$. If $a > 1$, then ∞ is an attracting fixed point. If $a < 1$, then the solution of $az + b = z$ is an attracting fixed point. If $a = 1$, then $b \neq 0$, $f^n(z) = z + nb$, all points converge to ∞.

For Case III and IV, the proof uses the fact that f is strictly contracting under Poincare distance, i.e.$\rho(f(z), f(w)) < \rho(z, w)$.

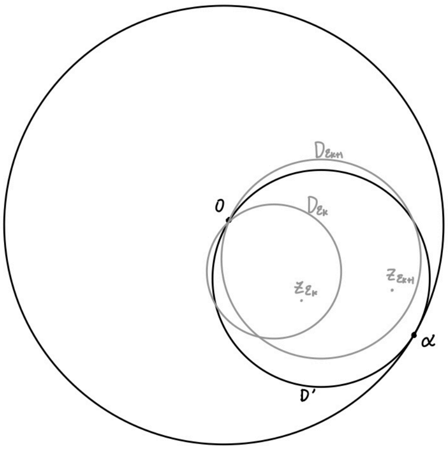
If $\exists q \in D$, $|f^n(q)|$ does not converge to 1, then $\exists r < 1$ and $|f^{n_k}(q)| < r$, then we can choose $r < s < 1$ such that $|f^{n_k+1}(q)| < s$. So $\exists \sigma < 1$, such that $\rho(f^{n_k+1}(q), f^{n_k+2}(q)) \leq \sigma\rho(f^{n_k}(q), f^{n_k+1}(q))$. By induction we have $\rho(f^{n_k+1}(q), f^{n_k+2}(q)) \leq \sigma^k\rho(f^{n_1}(q), f^{n_1+1}(q))$. (这里的严格压缩性只在该子列中成立, 其他迭代步骤仅能保证非扩展. 然而在轨道迭代中总是会无数次经过子列中的元素, 每次经过就得到严格压缩, 最终得到了整体压缩的结果.)

Therefore $\rho(f^{n_k+1}(q), f^{n_k+2}(q)) \to 0$. So we can assume that $f^{n_k+1}(q) \to p$ with $p \in D$. But $\rho(p, f(p)) = 0$, p will be a fixed point. Since $f \notin Aut(D)$, p will be an attracting fixed point by Schwartz lemma, and $\forall z \in D, f^n(z) \to p$.

The complementary condition is $\forall z \in D, |f^n(z)| \to 1$. Let $f_\epsilon(z) = (1-\epsilon)f(z)$, then f_ϵ will has an attracting fixed point z_ϵ. If we choose $\epsilon_k \to 0$, then $z_\epsilon \to 1$.

Let $T_\epsilon = \dfrac{z - z_\epsilon}{1 - \overline{z_\epsilon}z}$, $D_\epsilon = \{|T_\epsilon| < |z_\epsilon|\}$. Then $f_\epsilon(D_\epsilon) \subset D_\epsilon$. Let $D' = \lim D_{\epsilon_k}$, then $0 \in \partial D'$. Here we choose 0 as a boundary point so that D_ϵ is not only a disc under Poincare distance, but also under Euclidean distance. This is important, otherwise D' will be too complex to analysis.

(后面部分难以描述, 大致意思见下图.)



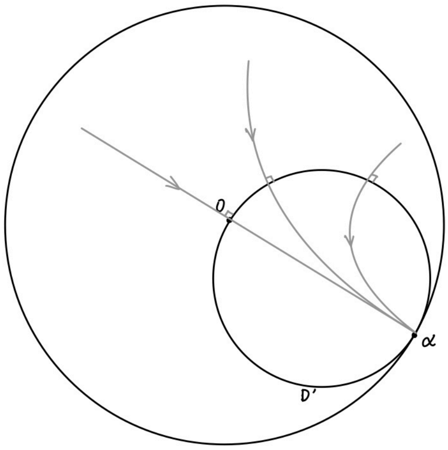


图 1: How D_{ϵ_k} converge and how orbits attracted to α

3.3 Snail Lemma

Thm 3.9 Let P be a polynomial with $\deg P \geq 2, P(0) = 0, P'(0) = e^{2\pi i\theta}$, θ irrational. Then there is no Fatou component U that $P^{-1}(U) = U$, $P^n(z) \to 0$ uniformly on compact subsets.

(此处 P 为多项式是不必要的, 此结论对解析的 f 亦成立. 这块内容实际上是 Fatou 分支的分类中的重要结果.)

Proof We prove by contradiction. Let U be an invariant Fatou component, $P^n \to 0$ uniformly on compact subsets.

Pick $z_0 \in U$, denote its orbit by $\{z_n\}$. Let $V_0 \subset\subset U$ be a connected open set containing z_0, z_1, this is possible because U is invariant. Define $V_n = P^n(V_0)$, then $z_n, z_{n+1} \in V_n$. We call $\cup V_n$ a snail, and the orbit $\{x_n\}$ is contained in the snail.

Since $P'(0) \neq 0$, P and P^n are univalent in a neighborhood of 0. Note that V_n converge to 0, so $\{V_n\}_{n \geq N}$ is in the neighborhood of 0. We may assume that P is univalent in all V_n, and let V_0 be a topological disc. (Here we should require $0 \notin V_0$.)

Define $\phi_n(z) = \dfrac{P^n(z)}{P^n(z_0)}$. Then ϕ_n are univalent on V_0, $\phi_n(z_0) = 1$, $0 \notin \phi_n(V_0)$. (This is because univalent, 0 itself is the only preimage.)

Define $\psi : D \to V_0, 0 \mapsto z_0$ be biholomorphic. Define $h_n(z) = \phi_n(\psi(z)) - 1$, then h_n is univalent on disc, $h_n(0) = 0, h_n(z) \neq -1$. By Koebe 1/4 theorem, $D(0, \dfrac{|h_n'(0)|}{4}) \subset h_n(D)$, but $-1 \notin h_n(D)$, so $|h_n'(0)| \leq 4$.

Next we will show that $\{h_n\}$ is a normal family. Here we investigate $g_n = \dfrac{h_n(z)}{h_n'(0)}$. This family satisfies ① univalent on D ② $g_n(0) = 0$ ③ $g_n'(0) = 1$, so it is normal. This normality uses Montel's theorem, g_n map punctured disc to an area omitting at least two points, 0 and another point a_n on the unit circle. However, traditional Montel's theorem requires the two points to be specific, here a_n is not fixed. Nonetheless, a corollary says that if the two points have positive spherical(?) distance, normality is also true. The positive distance is guaranteed by $|h_n'(0)| \leq 4 \Rightarrow |a_n| > 1/4$. Now we have normality on punctured disc, we still need to add 0. If 0 is a pole, then the image of a small neighborhood of 0 will cover a really big area, containing a_n, this is impossible. If 0 is an essential point, then g_n cannot be univalent near 0. Therefore we can add 0 to the domain of normality.

(*An example when a family is normal on punctured disc but not on disc:* $f_n(z) = nz$.)

Since h_n is normal, ϕ_n is also normal on V_0. Then we are going to prove that V_n won't be too small, so that we can do the cover below. We want $D(z_n, \sigma|z_n|) \subset V_n$ for a $\sigma > 0$, which equals $D(1, \sigma) \subset \phi_n(V_0)$, and by Koebe-1/4 theorem, this equals that for all limit functions of ϕ_n, the derivatives at z_0 have a uniform lower bound σ.

If not, we can find a subsequence ϕ_{n_k}, with $\phi_{n_k}'(z_0) \to 0$. Then for a large n_k, $\phi_{n_k}(V_0)$ can be small enough around 1, so that there is a small enough $\Delta\alpha$, $\phi_{n_k}(V_0) \subset \{z : Arg(z) \in (\alpha_n, \alpha_n + \Delta\alpha)\}$. Note that $V_n = \dfrac{\phi_{n_k}(V_0)}{z_n}$, so $V_n \subset \{z : Arg(z) \in (\alpha_n', \alpha_n' + \Delta\alpha)\}$. Since $P(z) = e^{2\pi i\theta}z + O(z^2)$, $V_{n+1} \subset \{z : Arg(z) \in (\alpha_n' + \theta, \alpha_n' + \theta + \Delta\alpha)\}$. But let $\Delta\alpha < \theta$, we have $V_n \cap V_{n+1} = \emptyset$, this is impossible.

Now we can choose N, so that $V_n, \cdots, V_{n+N}$ can cover an annulus. On the one hand, if we look at the arguments, since θ is irrational, N will only related to σ so that $\exists m < N, Arg(z_{n+m}) - Arg(z_n) = m\theta (\mod 2\pi) < c\sigma$. On the other hand, $|z_n|, \cdots, |z_{n+N}|$ can be really close to each other when N is not too large.

Then $\cup_{i=n}^{n+N} V_i$ covers an annulus, $\cup_{i=n+N+1}^{n+2N} V_i$ covers another but smaller. So on $\cup_{i=1}^{\infty}$ covers a punctured disc. But this means 0 is attracting (maybe by Schwartz lemma), conflicting with $|P'(0)| = 1$.

Another Proof 这是一个更加偏分析的证明, 前面部分相同, 在构造 ϕ 后采用了不同方法导出矛盾.

Let $\phi_n(z) = \dfrac{P^n(z)}{P^n(z_0)}$. Note that

$$\phi_n(P(z)) = \frac{P^{n+1}(z)}{P^n(z_0)} = \frac{P^n(z)}{P^n(z_0)} \cdot \frac{P(P^n(z)) - P(P^n(0))}{P^n(z) - P^n(0)} \to \phi_n(z) \cdot P'(0).$$

Since $\{\phi_n\}$ is normal, let $\phi_{n_k} \to g$, then $g \circ P = e^{2\pi i\theta} g$. g is nonconstant, therefore univalent. But $|g \circ P^n(z)| = |g(z)|$, $P^n(z) \to 0$, so $|g(z)| = g(0)$, conflict.

Snail Lemma (more general) $P(U) = U, 0 \in \partial U, P(0) = 0$, then $P'(0) = 1$.

Proof Let $\lambda = P'(0)$. If λ is irrational, then by Denjoy-Wolff theorem, 0 is an attracting point. If $\lambda^n = 1$, then by the same process above, $g \circ P^{nk+j}(z) = \lambda^j g(z)$, no matter k. But g is univalent, so P has periodic points everywhere, conflict. If $\lambda = 1$, then g is constant, we avoid the conflict.

(此证明存疑, 参考书上似乎有误, 这里只是大致思想, 实际处理时可能遇到 g 在 0 处无定义等问题. 另外, Milnor 书 16.2 也有 Snail Lemma 的证明, Beardon 书上也有, 可参考.)