BRÄNDÉN-KRASIKOV-SHAPIRO TYPE THEOREM FOR LINEAR DIFFERENCE OPERATOR

Abstract. In this paper, we give a complete characterisation of the entire function f which both f and its general linear difference Lf omit zero. It can be seen as a difference version of the Frank-Hellerstein theorem, which can be regarded as a dual version of (localized) Hermite-Poulain theorem in some sense. With this characterisation in hand, we establish a version of Brändén-Krasikov-Shapiro theorem for linear difference operator preserving $\Pi_{\mathcal{E}}(\emptyset)$. Our method can be extended partially to a meromorphic function with finitely many zeros and poles.

The study of linear operators that preserve certain nice value distribution properties of some class of entire functions has a long history, which can be dated back to Hermite, Laguerre, Polya and Schur at the beginning of 20 century.

According to Boreck and Brändén [3], one of the outstanding problems in this area can be formulated in the following: let Ω be a set in $\mathbb{C}$ with considerable interest in some sense, and let $\Pi(\Omega)$ be the set of (real or complex) entire functions with all zeros in Ω.

Problem 1. *Try to characterize all linear transformation* $T : \Pi(\Omega) \longrightarrow \Pi(\Omega) \cup \{0\}$.

This problem has its root in number theory, combinatorics, random matrix theory, complex analysis and many other branches in mathematics [1, 2]. One may think the corresponding $\Pi(\emptyset)$ is not an interesting object at first glance. However, one may deform some functions in $\Pi(\Omega)$ to get a suitable limit in $\Pi(\emptyset)$, where Ω is non-empty. Later, we will see more subtle philosophy and principle in $\Pi(\emptyset)$ from Nevanlinna theory. It is reasonable to consider the corresponding question, especially for $\Omega = \emptyset$. This is our motivation in this paper.

Almost in the same period in the twentieth century, Rolf Nevanlinna created what is now called Nevanlinna theory, which was described as a great achievement in twentieth–century mathematics by Herman Weyl. The topic of value distribution between f and its differential is a very important area in complex analysis, especially after the birth of Nevanlinna theory. Although much progress has been made, the whole picture is unclear and very mysterious. In 1926, Saxer [20] proved that if a transcendental entire function f satisfy $ff'f'' \neq 0$, then f is exactly of the form

2020 *Mathematics Subject Classification.* 30C15, 30D15, 30D35, 47B38.

Key words and phrases. Frank-Hellerstein Theorem, Hermite-Poulain Theorem, Navanlinna Theory, Linear Difference Operator.

12 Dec 2024 21:16:57 PST Analysis
241212-Zhang Version 1 - Submitted to Bull. London Math. Soc.

e^{az+b}, where $a(\neq 0), b$ are constants. Subsequently, this result was generalized by Csillag [8] to the case $ff^{(m)}f^{(n)} \neq 0$ where $0 < m < n$.

Later, Hayman [15] (for $n = 2$) and Clunie [7] (for $n \geq 3$) improved the Csillag Theorem as follows.

Theorem A (Hayman-Clunie theorem)**.** *Let f be a meromorphic function in the complex plane, and let $n(\geq 2)$ be a positive integer. If f and $f^{(n)}$ have only a finite number of zeros and poles, then*

$$f(z) = R(z)e^{P(z)},$$

where $R(z)$ is a rational function and $P(z)$ is a polynomial.

Further extension of the Hayman-Clunie theorem to general meromorphic function with only finitely many zeros (without the assumption on the poles) was done by Frank [10] and Langley [19]. It is of great interest to extend the Hayman-Clunie theorem to general linear (or even non-linear) differential operators. This problem was addressed in [17, Problem 1.42].

Problem 2. *Let $\hat{L}$ be a linear differential operator with $\hat{L}f = \sum_{k=0}^{n} a_k f^{(k)}$ where the coefficients a_k are polynomials. Try to characterize all meromorphic functions such that f and $\hat{L}f$ have no zeros.*

Problem 2 arises some attempts to understand the linear differential operators preserving the set of all meromorphic functions without zeros. In some sense, this relates to $T : \Pi(\emptyset) \longmapsto \Pi(\emptyset)$ in Problem 1.

Remark 1. *It seems that the linear operators $T : \Pi(\emptyset) \longrightarrow \Pi(\emptyset)$ may be confusing at the first glance. And it is not evident for the connection with Problem 1. While, in the field of Nevanlinna theory, there is a general philosophy that one may first prove a theorem for the meromorphic function without zeros, and then one can expect a theorem for the meromorphic function with multiple zeros.*

Here we give an example to illustrate this beautiful phenomenon. In 1959, Hayman [15] *proved a remarkable result, which was a cornerstone in studying value distribution between f and $f^{(k)}$,*

"If f is a transcendental meromorphic function without zeros, then $f^{(k)}$ takes every non-zero value infinitely often for $k = 1, 2, \ldots$. "

Fang and Wang [9] *proved the following result based on a landmark result of Yamanoi* [21].

"If f is a transcendental meromorphic function with only multiple zeros, then $f^{(k)}$ takes every non-zero value infinitely often for $k = 1, 2, \ldots$. "

Frank and Hellerstein [11] solved Problem 2 under the setting that f is an entire function and proved the following results.

Theorem B (Frank-Hellerstein theorem)**.** *Let f be a transcendental entire function. Assume that f and $\hat{L}(f) := \sum\limits_{k=0}^{n} a_k f^{(k)}, (a_n \neq 0)$ have no zero, then*

$$f(z) = \exp(az + b + \exp(cz + d)),$$

for some $a, b, c, d \in \mathbb{C}$.

We refer the readers to [17] for further progress for the general linear differential operators.

Recently, there has been a great trend in studying the difference version of Nevanlinna theory [5, 6, 12, 13, 14]. Let $\mathcal{M}$ be the space of all meromorphic functions. We define the shifting difference operator E from $\mathcal{M}$ to itself,

$$Ef(z) = f(z + 1)$$

and the forward difference operator by,

$$\Delta f(z) = (E - I)f(z) = f(z + 1) - f(z),$$

where I represents the identity operator. The standard n-th order difference operator Δ^n can be defined by $\Delta^n = (E - I)^n$. Let $P(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial of degree k. We introduce the linear difference operator L on $\mathcal{M}$ induced by P,

$$Lf(z) = P(E)f(z) = \sum_{k=0}^{n} a_k E^k f = \sum_{k=0}^{n} a_k f(z + k).$$

Define the degree of L by $\deg L = \deg P$. A difference operator $L = P(E)$ is called a nuclear difference operator if the corresponding polynomial P is monomial.

Inspired by Problem 2, we propose the following question.

Question 1. *Let f be a transcendental meromorphic function and $L = \sum_{k=0}^{n} a_k E^k$ be a difference operator, where a_k are some polynomials. Try to characterize all functions f such that Lf and f have no zeros.*

If the operator L is of nuclear difference type, this question is completely trivial. To the best of our knowledge, this question is widely open.

Without exceptional emphasis, the difference operator is induced by some polynomials, that is, the coefficients of polynomial are constants rather than functions.

In this paper, we establish a difference version of the Frank-Hellerstein theorem, which is an analogue of the celebrated Hermite-Poulain theorem.

Theorem C (Hermite-Poulain theorem)**.** *Let $\mathcal{HP}$ be the set of all hyperbolic polynomials H, that is H is real and all roots of H lie in $\mathbb{R}$. Assume that*

$$\hat{L}H(z) = \sum_{k=0}^{n} a_k H^{(k)}(z)$$

maps $\mathcal{HP}$ to $\mathcal{HP}\bigcup\{0\}$, then its characteristic polynomial

$$g(z)=\sum_{k=0}^{n}a_kz^k$$

is a hyperbolic polynomial.

Now, we state two main theorems in this paper, one is a version of localized Hermite-Poulain theorem and another is a version of the Frank-Hellerstein theorem. Here, "localized" means that we fix f, and try to find a possible version of the Hermite-Poulain theorem. Let $\Pi_{\mathcal{E}}(\emptyset)$ be the set of all transcendental entire functions without zeros.

Theorem 1 (Localized Hermite-Poulain theorem for $\Pi_{\mathcal{E}}(\emptyset)$)**.** *Let f be a function in $\Pi_{\mathcal{E}}(\emptyset)$. Assume that $Lf=P(E)f=\sum_{k=0}^{n}a_kE^kf$ does also lie in $\Pi_{\mathcal{E}}(\emptyset)\bigcup\{0\}$. Then its characteristic polynomial P satisfies the following condition: there exist s and j with $0\leq s\leq j-1\leq n-1$, and a non-zero constant λ such that λ is one common root of*

$$P\left(e_j^l\lambda\right)-e_j^{ls}P(\lambda)=0$$

for each $0\leq l\leq j-1$, where $e_j=\exp\left(\frac{2\pi i}{j}\right)$.

Indeed Theorem 1 can be deduced from the following theorem.

Theorem 2 (Difference Version of Frank-Hellerstein theorem)**.** *Let f be a transcendental entire function from $\mathbb{C}$ to $\mathbb{C}\backslash\{0\}$ and L be a non-nuclear difference operator such that $Lf=P(E)f=\sum_{k=0}^{n}a_kE^kf\not\equiv0$. Then Lf omits zero if and only if there exist $0\leq s\leq j-1\leq n-1$, and non-zero λ such that j is minimal for the following equation*

$$E^jf=\lambda^jf, \tag{1}$$

and λ is a common root of

$$P\left(e_j^k\lambda\right)-e_j^{ks}P(\lambda)=0$$

for any $0\leq k\leq j-1$, where $e_j=\exp\left(\frac{2\pi i}{j}\right)$.

Theorem 2 gives a complete answer to Question 1 for entire functions. Brändén, Krasihov and Shapiro [4] established the following result by using Brändén-Borcea's characterisation of linear operators on $\mathbb{C}[z]$ preserving $\Pi(\mathbb{R})$.

Theorem D (Brändén-Krasikov-Shapiro)**.** *Let T be a linear operator from $\mathbb{C}[z]$ to $\mathbb{C}[z]$ for any $p\in\mathbb{C}[z]$,*

$$Tp(z)=\sum_{j=0}^{k}q_j(z)p(z-j),$$

where $q_0, \ldots, q_k$ *are polynomials. Then* $T(\mathcal{HP}) \subset \mathcal{HP} \bigcup \{0\}$ *if and only if* $q_j \not\equiv 0$ *for at most one* j.

Theorem D can be seen as a difference analogue of the famous Hermite-Poulain theorem, which characterizes all linear differential operators acting on $\mathbb{C}[z]$ preserving $\Pi(\mathbb{R})$ by the corresponding characteristic polynomials.

Now, we establish a version of Brändén-Krasikov-Shapiro for $\Pi_{\mathcal{E}}(\emptyset)$.

Theorem 3. *Let* $L = \sum_{k=0}^{n} a_k E^k$ *be a difference operator. Then* L *maps* $\Pi_{\mathcal{E}}(\emptyset)$ *to* $\Pi_{\mathcal{E}}(\emptyset) \cup \{0\}$ *if and only if* $a_i \neq 0$ *for at most one* i.

Proof. The sufficient condition is obvious, we only need to prove the necessary part. We assume that there exists a non-nuclear difference operator $L = \sum_{k=0}^{n} a_k E^k \not\equiv 0$ such that L maps $\Pi_{\mathcal{E}}(\emptyset)$ to $\Pi_{\mathcal{E}}(\emptyset) \cup \{0\}$. It is obvious that $\ker L \neq \Pi_{\mathcal{E}}(\emptyset)$. Thus, by Theorem 2, for any $f \in \Pi_{\mathcal{E}}(\emptyset) \setminus \ker L$, there exists $1 \leq j \leq n$ and non-zero constant λ such that $E^j f = \lambda^j f$. Let $g(z) = \exp(z^2)$. It is easy to see that $g \in \Pi_{\mathcal{E}}(\emptyset) \setminus \ker L$ and $E^j g(z) = \exp(j^2 + 2jz) g(z)$. Thus, there exists no λ such that $E^j g = \lambda^j g$ for some $1 \leq j \leq n$. We get the contradiction. This proves that L has to be either a nuclear difference type or zero. □

Next, we prove that $j = 1$ or 2 for the generic linear difference operator. Denote the space of all linear operators with $\deg L \leq n$ by

$$V = \left\{ L : L = \sum_{k=0}^{n} a_k E^k : a_k \in \mathbb{C}, 0 \leq k \leq n \right\}.$$

Theorem 4. *There exists an open and dense set* W *in* V *such that for any transcendental entire function* f *and* $L \in W$, *if* f *and* Lf *both omit zero, then*

$$f(z) = \exp(h(z) + Cz),$$

for some constant C, *where* h *is an entire function with period* 1 *or* 2.

The characterisation of f in Theorem 4 is just a reformulation of (1) for $j = 1, 2$, see also Lemma 1 in Section 1.

Proof. Let $\tilde{P}(z) = \sum\limits_{l=0}^{n} u_l z^l$ for some $u_0, \ldots u_n \in \mathbb{C}$ be a ploynomial of degree no more than n. Inspired by Theorem 2 we define

$$Q^s_{k,m}(\lambda) = \tilde{P}(e^k_m \lambda) - e^{ks}_m \tilde{P}(\lambda)$$

and

$$V_{s,m} := \{(u_0, \ldots u_n) : Res(Q^s_{1,m}, Q^s_{2,m}) = 0\},$$

where $e_m = \exp\left(\frac{2\pi i}{m}\right)$ and

$$Res(Q^s_{1,m}, Q^s_{2,m}) = \begin{vmatrix} a_0 & 0 & \cdots & 0 & b_0 & 0 & \dots & 0 \\ a_1 & a_0 & \dots & 0 & b_1 & b_0 & \dots & 0 \\ a_2 & a_1 & \ddots & 0 & b_2 & b_1 & \ddots & 0 \\ \vdots & \vdots & \ddots & a_0 & \vdots & \vdots & \ddots & b_0 \\ a_n & a_{n-1} & \cdots & \vdots & b_n & b_{n-1} & \dots & \vdots \\ 0 & a_n & \ddots & \vdots & 0 & b_n & \ddots & \vdots \\ \vdots & \vdots & \ddots & a_{n-1} & \vdots & \vdots & \ddots & b_{n-1} \\ 0 & 0 & \cdots & a_n & 0 & 0 & \cdots & b_n \end{vmatrix}_{(2n+2)\times(2n+2)}.$$

Here, $a_l = u_l(e_m^{s+l} - 1)$, $b_l = u_l(e_m^{2s+2l} - 1)$ for $0 \leq \; l \leq n$. If necessary, we add zero terms to preserve the "degree" of $Q^s_{k,m}(\lambda)$ is n. It is easy to see that $V_{s,m}$ is a subvariety in $\mathbb{C}^{n+1}$ when $e_m \neq \pm 1$. Otherwise, the above determinant will reduce to zero trivially. It implies the set $V_{s,m}$ is whole space V. In the absence of nuclear type in Theorem 2, for $0 \leq j \leq n$, we define

$$V_j = \{(u_0, \dots, u_n) : u_i \neq 0 \text{ only for } i = j\}.$$

Denote Φ be the linear isomorphism from $\mathbb{C}^{n+1}$ to V,

$$\Phi : (u_0, \dots, u_n) \mapsto \sum_{l=0}^{n} u_l E^l.$$

Let

$$W = V \backslash \left(\bigcup_{m=3}^{n} \bigcup_{s=0}^{m-1} \Phi(V_{s,m}) \bigcup \bigcup_{j=0}^{n} \Phi(V_j) \right).$$

It is evident that this set W is an open and dense set in V. In order to prove Theorem 4, we only need to prove $j \leq 2$ by Theorem 2. Otherwise, if there exists some $j \geq 3$ such that $E^j f = \lambda^j f$ for $\lambda \in \mathbb{C}\backslash 0$ and λ is the common root of

$$P\left(e_j^k \lambda\right) - e_j^{ks} P(\lambda) = 0$$

for $k = 1, \dots, j-1$ and $0 \leq s \leq j-1$. We know that λ is the common non-zero root of $Q^s_{1,j}$ and $Q^s_{2,j}$. Theofore, the resultant of $Q^s_{1,j}$ and $Q^s_{2,j}$ is zero. This implies that the vector corresponding to L, that is,

$$(c_0, \dots, c_n)$$

lies in the variety of $V_{s,j}$. It contradicts our assumption that $L \in W$. Now, we complete this proof by Lemma 1.

□

12 Dec 2024 21:16:57 PST Analysis
241212-Zhang Version 1 - Submitted to Bull. London Math. Soc.

1. Preliminaries

In this section, we will collect some basic results used in this paper.

Lemma 1. *Let g be an entire function. Assume that $E^j g = \mu + g$ for some constant μ and positive integer j, then*

$$g(z) = h(z) + \frac{\mu z}{j},$$

where $h(z)$ is an entire function with period j.

Let us recall some basic concepts in Nevanlinna theory. Let f be a non-constant meromorphic function, the Nevanlinna characteristic is defined by

$$T(r,f) = m(r,f) + N(r,f),$$

where

$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta$$

and

$$N(r,f) := \int_0^r \frac{n(t,f) - n(0,f)}{t} dt + n(0,f)\log r.$$

Here, $n(t,f)$ is the number of poles of f in $\{z : |z| < t\}$ counting multiplicities.

Lemma 2 (Borel theorem, [22])**.** *Let p_1, $p_2, \ldots,$ p_n $(n \geq 2)$ be meromorphic functions and q_1, $q_2, \ldots,$ q_n be entire functions such that*

(i) $\sum_{j=1}^{n} p_j e^{q_j} \equiv 0$;
(ii) *for $1 \leq s < t \leq n$, $q_s - q_t$ is not a constant;*
(iii) *for $1 \leq j \leq n$, $1 \leq s < t \leq n$, $T(r,p_j) = o\{T(r, e^{q_s - q_t})\}$ $(r \to \infty, r \notin E)$, where E is of finite linear measure or finite logarithmic measure.*

Then $p_j \equiv 0$, $j = 1, \ldots, n$.

Lemma 3. *Let R be a rational function with at least one finite pole. And let L be a difference operator of degree n, then*

$$L(R) \not\equiv 0.$$

Proof. Assume that $L(R) \equiv 0$. Let $L = \sum_{j=0}^{n} a_j E^j$, where $a_n \neq 0$. It is clear that L can not be a monomial. Thus, there is at least one term E^j except for E^n with a nonzero coefficient. Let X be the set of all finite poles of R. Now, we choose $z_0 \in X$ such that

$$\Re z_0 = \min\{\Re z : z \in X\}.$$

It is easy to see that $z_0 - n$ is a pole of $R(z+n)$. Since $LR \equiv 0$ by assumption, there exist at least one term with $E^m R(z_0 - n) = \infty$ where $m < n$. This implies $z_0 - n + m \in X$ which contradicts the definition of z_0. $\square$

Lemma 4. *Let H be a polynomial of degree $n \geq 2$ and $L \neq 0$ be a difference operator of degree l with $l \leq n$. Then*

$$L(H) \not\equiv 0.$$

Proof. It is enough to prove that $H(z), H(z+1), \ldots, H(z+l)$ are linear independent over $\mathbb{C}$. This is equivalent to show that $H, \Delta H, \ldots, \Delta^l H$ are linear indpendent over $\mathbb{C}$. While the statement is clear for the latter formulation since these $l+1$ polynomials have different degrees. $\square$

2. Proof of Theorem 1 and Theorem 2

Proof of Theorem 2. Firstly, we prove the "only if" part. Assuming that f and Lf omit zero, there exist entire functions φ and ψ such that

$$\begin{cases} f = e^{\varphi}, \\ Lf = e^{\psi}. \end{cases}$$

Then

$$a_0 e^{\varphi} + a_1 e^{E\varphi} + \ldots + a_n e^{E^n \varphi} + (-1) e^{\psi} \equiv 0.$$

By Lemma 2, there exist $0 \leq k_1 < k_2 \leq n$ such that

$$E^{k_1}\varphi - E^{k_2}\varphi \equiv \text{constant} \Leftrightarrow \varphi - E^{k_2 - k_1}\varphi \equiv \text{constant}.$$

Or there exists $0 \leq k \leq n$ such that

$$E^k \varphi - \psi \equiv \text{constant}.$$

The latter condition can be reduced to the former by Lemma 2 to

$$E^k \varphi - \psi \equiv c \Rightarrow \sum_{j \neq k} a_j e^{E^j \varphi} + (a_k - e^{-c}) e^{E^k \varphi} \equiv 0,$$

where c is the constant. Therefore we can define a non-empty set

$$\mathscr{F} = \{0 < k \leq n : \varphi - E^k \varphi \equiv c_k \text{ for some constant } c_k\}.$$

Let $j = \min \mathscr{F}$. We claim that for any $k \in \mathscr{F}$, $\frac{k}{j} \in \mathbb{N}$. Otherwise, set $k = jl + r$, where $l, r \in \mathbb{N}$, $1 \leq r \leq j-1$. Then let

$$\varphi - E^j \varphi \equiv c_1, \varphi - E^k \varphi \equiv c_2 \Rightarrow \varphi - E^{jl}\varphi \equiv lc_1 \Rightarrow \varphi - E^{k-jl}\varphi \equiv c_2 - lc_1.$$

that means $r = k - jl \in \mathscr{F}$. But $r < j$, which contradicts the definition of j. Taking λ satisfies $\lambda^j = e^{-c_1}$, then we have

$$E^j f = \lambda^j f.$$

Now we consider Lf.

$$\begin{aligned} Lf &= \sum_{k=0}^{n} a_k E^k f = \sum_{l=0}^{j-1} \sum_{k\equiv l(\bmod j)} a_k E^k f = \sum_{l=0}^{j-1} \sum_{k\equiv l(\bmod j)} a_k \lambda^{k-l} E^l f \\ &= \sum_{l=0}^{j-1} \lambda^{-l} E^l f \left(\sum_{k\equiv l(\bmod j)} a_k \lambda^k \right) = \sum_{l=0}^{j-1} \lambda^{-l} A_l E^l f, \end{aligned}$$

where $A_l = \sum_{k\equiv l(\bmod j)} a_k \lambda^k$. Then we claim that there exists $0 \le s \le j-1$ such that for $0 \le l \le j-1$, the nonzero term A_l occurs only when $l = s$. From $Lf \not\equiv 0$, we have at least one term $A_s \neq 0$. If there is another $A_r \neq 0$, then we can use Lemma 2 again. It will contradict the minimality of j.

Therefore we may observe that

$$P(\lambda) = \sum_{l=0}^{j-1} A_l = A_s.$$

Let $e_j = \exp\left(\frac{2\pi i}{j}\right)$ and apply the same technique (the computation is held for all $m \in \{0, \dots, j-1\}$), we get

$$\begin{aligned} P\left(e_j^m \cdot \lambda\right) &= \sum_{k=0}^{n} a_k e_j^{mk} \lambda^k = \sum_{l=0}^{j-1} e_j^{ml} \left(\sum_{k\equiv l(\bmod j)} a_k \lambda^k \right) \\ &= \sum_{l=0}^{j-1} e_j^{ml} A_l = e_j^{ms} A_s = e_j^{ms} P(\lambda). \end{aligned}$$

Then we prove the "if" part. Notice that the case $j = 1$ is easy to get the conclusion. We can assume $j > 1$ to make sure the computation below is meaningful. For convenience, let $w = e_j$ and the notation A_l represent the same meaning as above. As assumptions, the following condition,

$$P\left(w^m \cdot \lambda\right) = \sum_{l=0}^{j-1} w^{ml} A_l = w^{ms} P(\lambda).$$

for $m = 0, 1, , \dots, j-1$ can be written as linear equations,

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ w^{-s} & w^{1-s} & \dots & w^{j-1-s} \\ \dots & \dots & \dots & \dots \\ w^{-s(j-1)} & w^{(1-s)(j-1)} & \dots & w^{(j-1-s)(j-1)} \end{pmatrix} \begin{pmatrix} A_0 \\ A_1 \\ \dots \\ A_{j-1} \end{pmatrix} = \begin{pmatrix} P(\lambda) \\ P(\lambda) \\ \dots \\ P(\lambda) \end{pmatrix}.$$

Note that $A_s \neq 0$, $A_l = 0$ for $l \neq s$ and $0 \leq l \leq j-1$ is a solution to the above linear equations. Moreover, the coefficient matrix is of full rank, which guarantees the solution is unique. Therefore, the focus has another formulation,

$$Lf = \sum_{l=0}^{j-1} \lambda^{-l} A_l E^l f = \lambda^{-s} A_s E^s f$$

which omits zero due to $\lambda^{-s} A_s \neq 0$ and the non-vanishing of $E^s f$. □

Proof of Theorem 1. In fact, the proof of Theorem 2 have dealt with the linear difference operator $L : \Pi_{\mathcal{E}}(\emptyset) \to \Pi_{\mathcal{E}}(\emptyset)$. Therefore, the last case that we need to take care of is $L : \Pi_{\mathcal{E}}(\emptyset) \to \{0\}$. Recalling the previous proof, we just change the assumption $Lf = e^{\psi}$ to be $Lf = 0$ here. And then the usage of Lemma 2 is easier due to the vanishing of Lf.

It is also effective for the rest until we claim "the nonzero term A_l only occurs when $l = s$". Under this circumstance, this will be modified as "$A_l = 0$ for $0 \leq l \leq j-1$". Then we can continue our proof with $A_s = 0$ in proof of Theorem 2. It has no influence on the accessibility to the conclusion.

For the "if part", the most important point is the uniqueness of the solution of linear equations while this fact is still correct here. We just modify the conclusion $Lf \neq 0$ to be $Lf \equiv 0$. The corresponding solution would be zero solution. And the proof is completed. □

Remark 2. *Based on a well-known fact in Nevalinna theory, the coefficients in the linear difference operator can be extended to rational functions in Theorem 1 and 2.*

Here, we state a further refinement of the results by restricting the growth condition on f.

Theorem 5. *Let f be a transcendental entire function and L be a non-nuclear difference operator such that*

$$\liminf_{r\to\infty} \frac{\log^+ T(r,f)}{r} = 0. \tag{2}$$

Assume that f and Lf omit zero in the complex plane $\mathbb{C}$, then

$$f(z) = \exp(az + b)$$

for $a \in \mathbb{C}\backslash\{0\}$, $b \in \mathbb{C}$.

Proof. Since f omits zero, we can write $f = \exp h$ for some non-constant entire function h. By Lemma 1 and Theorem 2, we know that h can be written by the following form

$$h(z) = h_1(z) + cz$$

where c is some constant and h_1 is a periodic function. By the known estimate [16, Theorem 1.6]

$$T(r,f) \leq \log^+ M(r,f) \leq 3T(2r,f)$$

and (2), we have

$$\liminf_{r\to\infty} \frac{\log^+ \log^+ M(r,f)}{r} = 0.$$

This implies the following estimate directly,

$$\liminf_{r\to\infty} \frac{\log^+ A(r,h)}{r} = 0,$$

where $A(r,h) = \max\{\Re h(z); |z| = r\}$. Recall the Poisson-Jensen's formula [18, Theorem 1.1]

$$h(z) = \frac{1}{2\pi}\int_0^{2\pi} \Re h(Re^{i\theta})\frac{Re^{i\theta}+z}{Re^{i\theta}-z}d\theta + i\Im h(0).$$

for any $z \in \{z : |z| < R\}$, we have

$$\liminf_{r\to\infty} \frac{T(r,h)}{r} = \liminf_{r\to\infty} \frac{\log^+ M(r,h)}{r} = 0.$$

Since $T(r,h) = T(r,h_1) + O(\log r)$, we get

$$\liminf_{r\to\infty} \frac{T(r,h_1)}{r} = 0.$$

While for a non-constant periodic entire function h_1, it is well known [22, Lemma 5.1] that

$$\liminf_{r\to\infty} \frac{T(r,h_1)}{r} > 0,$$

which leads to a contradiction. □

3. Difference version of Frank-Hellerstein theorem for meromorphic functions

Motivated by Theorem A, it is natural to extend Theorem 2 to meromorphic functions with finitely many zeros and poles. Indeed, the strategy in the proof of Theorem 2, completely works in this setting. We state it in the following theorem without proof.

Theorem 6. *Let f be a transcendental meromorphic function and L be a non-nuclear difference operator. Suppose that f and $Lf = \sum_{k=0}^{n} a_k E^k f$ have only at most finite number of zeros and poles, then there exist $0 \leq s \leq j-1 \leq n-1$, $\lambda \in \mathbb{C}$ such that*

$$f(z) = Q(z)h(z),$$

where Q is a rational function, h is a transcendental entire function omitting zero, $E^j h = \lambda^j h$, λ and Q satisfies the following conditions,

$$P(e_j^k \lambda) - e_j^{ks} P(\lambda) = 0,$$

and

$$[P(e_j^k \lambda E) - e_j^{ks} P(\lambda E)] Q \equiv 0,$$

for any $k = 0, \ldots, j-1$.

Remark 3. *When we allow the meromorphic function to have zeros or poles, our method reveals that the number of zeros and poles is of great importance in the structure of f for discussing Frank-Hellerstin's type result.*

We say a polynomial P with degree n is of pseudo-symmetric type, if there exists $j > 1$ and s such that

$$P(e_j z) \equiv e_j^s P(z),$$

where $e_j = \exp\left(\frac{2\pi i}{j}\right)$. It is not difficult to see that $j \leq n$ indeed.

Theorem 7. *Let f be a transcendental entire function with at least n zeros counting multiplicities and L be a non-nuclear difference operator of degree no more than n. Suppose that f and $Lf = \sum_{k=0}^{n} a_k E^k f$ be a non-constant entire function have only a finite number of zeros, and the polynomial P corresponding to L is not pseudo-symmetric type, then*

$$f(z) = H(z) \exp\left(\tilde{h}(z) + Cz\right),$$

where $H(z)$ is a polynomial, $C \in \mathbb{C}$ and $\tilde{h}(z)$ is an entire function with period 1.

Proof. Since f has finite many zeros, we can write $f = Qh$, where Q is a polynomial with degree no less than n and h is an entire function from $\mathbb{C}$ to $\mathbb{C}\backslash\{0\}$. By Theorem 6, we know that

$$E^j h = \lambda^j h$$

for some j and non-zero λ.

We only need to prove that $j = 1$ in this setting by Lemma 1. Otherwise, we assume that $j \geq 2$. For $0 \leq k \leq j-1$, we define

$$L_k(E) = P\left(e_j^k \lambda E\right) - e_j^{ks} P\left(\lambda E\right).$$

We know that $L_k(E)Q = 0$ for each k. Since P is not pseudo-symmetric type, it means that $L_1(E) \neq 0$. By Lemma 4, we have a contradiction.

□

When we replace Lemma 4 with Lemma 3 above, we will have a rational version of Theorem 7 with the assumption that f has at least one pole.

Acknowledgements. We extend our sincere gratitude to Professor MingLing Fang for his interesting discussions during the initial phases of this work. We also thank Dr. JiaXing Huang for his insighful discussion during the preparation of our preprint.

References

[1] J. Borcea and P. Brändén, The Lee-Yang and Pólya-Schur programs. I. Linear operators preserving stability. *Invent. Math.* **177** (2009), no. 3, 541–569.

[2] J. Borcea and P. Brändén, The Lee-Yang and Pólya-Schur programs. II. Theory of stable polynomials and applications. *Comm. Pure Appl. Math.* **62** (2009), no. 12, 1595–1631.

[3] J. Borcea and P. Brändén, Pólya-Schur master theorems for circular domains and their boundaries. *Ann. of Math. (2)* **170** (2009), no. 1, 465–492.

[4] P. Brändén, I. Krasikov, and B. Shapiro, Elements of Pólya-Schur theory in the finite difference setting. *Proc. Amer. Math. Soc.* **144** (2016), no. 11, 4831–4843.

[5] Y. M. Chiang and S. J. Feng, On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane. *Ramanujan J.* **16** (2008), no. 1, 105–129.

[6] Y. M. Chiang and S. J. Feng, On the growth of logarithmic differences, difference quotients and logarithmic derivatives of meromorphic functions. *Trans. Amer. Math. Soc.* **361** (2009), no. 7, 3767–3791.

[7] J. Clunie, On integral and meromorphic functions. *J. London Math. Soc.* **37** (1962), 17–27.

[8] P. Csillag, über ganze Funktionen, welche drei nicht verschwindende Ableitungen besitzen. *Math. Ann.* **110** (1935), no. 1, 745–752.

[9] M. Fang and Y. Wang, A note on the conjectures of Hayman, Mues and Gol'dberg. *Comput. Methods Funct. Theory* **13** (2013), no. 1, 533–543.

[10] G. Frank, Eine Vermutung von Hayman über Nullstellen meromorpher Funktionen. *Math. Z.* **149** (1976), no. 1, 29–36.

[11] G. Frank and S. Hellerstein, On the meromorphic solutions of nonhomogeneous linear differential equations with polynomial coefficients. *Proc. Lond. Math. Soc. (3)* **53** (1986), no. 3, 407–428.

[12] R. G. Halburd and R. J. Korhonen, Difference analogue of the lemma on the logarithmic derivative with applications to difference equations. *J. Math. Anal. Appl.* **314** (2006), no. 2, 477–487.

[13] R. G. Halburd and R. J. Korhonen, Nevanlinna theory for the difference operator. *Ann. Acad. Sci. Fenn. Math.* **31** (2006), no. 2, 463–478.

[14] R. G. Halburd, R. J. Korhonen, and K. Tohge, Holomorphic curves with shift-invariant hyperplane preimages. *Trans. Amer. Math. Soc.* **366** (2014), no. 8, 4267–4298.

[15] W. K. Hayman, Picard values of meromorphic functions and their derivatives. *Ann. of Math. (2)* **70** (1959), 9–42.

[16] W. K. Hayman, *Meromorphic functions*. Oxford Math. Monographs, Clarendon Press, Oxford, 1964.

[17] W. K. Hayman and E. F. Lingham, *Research problems in function theory*, Fiftieth anniversary edition, Problem Books in Mathematics, Springer, Cham, 2019.

[18] S. Lang and W. Cherry, *Topics in Nevanlinna theory*, Lecture Notes in Mathematics, vol. 1433, Springer-Verlag, Berlin, 1990.

[19] J. K. Langley, Proof of a conjecture of Hayman concerning f and f''. *J. Lond. Math. Soc. (2)* **48** (1993), no. 3, 500–514.

[20] W Saxer, Sur les valeurs exceptionelles des dérivées successives des fonctions méromorphes. *C. R. Acad. Sci. Paris* **182** (1926), 831–833.

[21] K. Yamanoi, Zeros of higher derivatives of meromorphic functions in the complex plane. *Proc. Lond. Math. Soc. (3)* **106** (2013), no. 4, 703–780.

[22] C. C. Yang and H. X. Yi, *Uniqueness theory of meromorphic functions*, Kluwer Academic Publishers, Dordrecht, 2003.