Bayesian Data Analysis, class 3b

Andrew Gelman

Chapter 3: Introduction to multiparameter models (part 2)

- ► Theory problem
- Computing problem
- Applied problem

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Theory problem

▶ Prove that a posterior density is proper by bounding the integral over the range $(\alpha, \beta) \in (-\infty, \infty) \times (-\infty, \infty)$

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Computing problem

► Compute posterior density on a grid

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- Paired experiment on chicken brains
- How does the effect depend on frequency? Is there evidence from the data that the effect is not constant across frequencies?
- Should we use the sham data in the estimates?

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- Logistic regression
- Computation for a two-parameter model
- Inference for a ratio
- Elementary modeling and computation

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Dose, x_i (log g/ml)	Number of animals, n_i	Number of deaths, y_i
-0.86	5	0
-0.30	5	1
-0.05	5	3
0.73	5	5

► Modeling the outcomes:

y_i|∅_i ~ Binomial(n_i, 0_i), for i = 1,..., 4
 List all the assumptions in that model

Prior distribution

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 - $\theta_i = \operatorname{logit}^{-1}(\alpha + \beta x_i), \text{ for } i = 1, \dots, 4$
 - $ho(\alpha,\beta)\propto 1$
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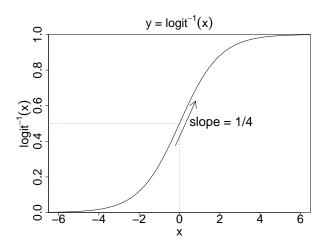


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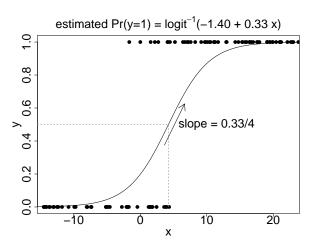
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Logistic regression



A clean example



Posterior computation

Unnormalized posterior density:

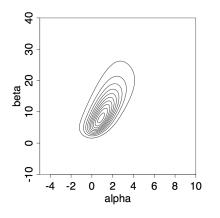
$$p(\alpha, \beta|y) \propto p(\alpha, \beta) \prod_{i=1}^{k} p(y_i|\alpha, \beta, n_i, x_i)$$

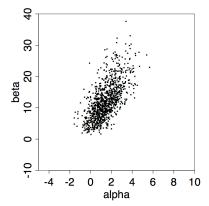
$$= 1 \cdot [\mathsf{logit}^{-1}(\alpha + \beta x_i)]^{y_i} [1 - \mathsf{logit}^{-1}(\alpha + \beta x_i)]^{n_i - y_i}$$

- Different ways to write the R function:
 - post <- function (a,b,y,n,x){
 prod((invlogit(a+b*x))^y*(1-invlogit(a+b*x))^(n-y))}</pre>
 - post <- function (a,b,y,n,x){
 prod (dbinom (y, n, invlogit(a+b*x)))}</pre>
 - ▶ log_post <- function (a,b,y,n,x){
 sum (dbinom (y, n, invlogit(a+b*x), log.p=TRUE))}</pre>
- Contour plot (need crude estimate)
- Posterior simulation on grid

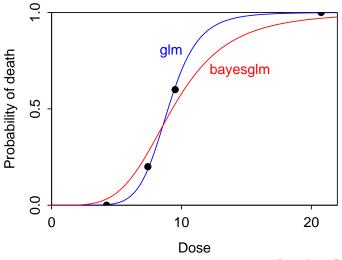


Contour plot and posterior simulation

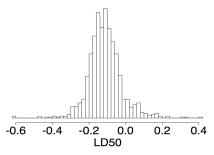




Bayesian estimates given noninformative and weakly informative priors



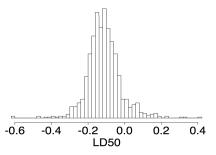
- ► A qoi!
- Lethal dose 50%: p = 0.5, so $logit^{-1}(\alpha + \beta x) = 0.5$, that is $x = -\alpha/\beta$
- ► From posterior simulations of (α, β) , get simulations of LD50 = $-\alpha/\beta$:





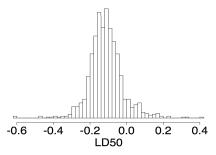
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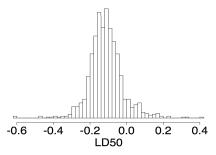


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But LD50 is not such a great qoi

- ▶ LD50 = $-\alpha/\beta$
- ► The x-value where the probability curve crosses 50%
- ▶ What if β is near 0?
- ▶ What is β could be positive or negative?
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- ► LD50
- Ratio of regression coefficients
- Incremental cost-effectiveness ratio
- Instrumental variables
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Ratio of regression coefficients

The quote:

"Economic freedom is almost 50 times more effective than democracy in restraining nations from going to war. When measures of both economic freedom and democracy are included in a statistical study, economic freedom is about 50 times more effective than democracy in diminishing violent conflict. The impact of economic freedom on whether states fight or have a military dispute is highly significant while democracy is not a statistically significant predictor of conflict."

Table 2.1: Effect of Economic Freedom on Militarized Interstate Disputes (MIDs)

Variable	Coefficient	(Standard Error)
Economic Freedom	-0.567**	(0.179)
Capabilities	2.777	(8.491)
Population	2.08×10^{-6} *	(8.18×10^{-7})
Major Power?	0.853	(1.133)
Democracy Score	-0.011	(0.065)
Defense Pact?	-0.628	(0.482)
GDP per Capita	8.01×10^{-6}	(8.04×10^{-5})
Trade Openness	1.57×10^{-7}	(1.50×10^{-6})
_spline1	$6.08 \times 10^{-4**}$	(2.32 × 10 ⁻⁴)
_spline2	-4.53×10 ⁻⁴ *	(1.76 × 10 ⁻⁴)
_spline3	1.19×10^{-4}	(4.87×10^{-5})
Intercept	-0.381	(1.210)
N	2519	
Log-likelihood	-161.719	
X ² ₍₁₁₎	160.564	
Significance levels:	†: 10% *: 5%	**: 1%

^{► &}quot;Economic freedom is about 50 times more effective than democracy in diminishing violent conflict"?

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"Economic freedom is about 50 times more effective than democracy in diminishing violent conflict"?

- Compare two medical treatments
 - ightharpoonup Old treatment has cost C_1 and efficacy E_1
 - New treatment has cost C₂ and efficacy E₂
- ▶ Incremental cost-effectiveness ratio, $\frac{C_2 C_1}{E_2 E_1}$
- Estimated ratio, $\frac{\widehat{C}_2 \widehat{C}_1}{\widehat{E}_2 \widehat{E}_1}$
- Now throw in some posterior uncertainty
- Four quadrants

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- ▶ Goal: inference about θ_y/θ_x
- Notoriously difficult to get an interval estimate with clsoe to 95% (or any specified) coverage
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The folk theorem and the Pinocchio principle

- When you have computational problems, often theres a problem with your model
- The Pinocchio Principle: A model that is created solely for computational reasons can take on a life of its own

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3.8. Elementary modeling and computation

- ▶ Write the likelihood
- Set down a weakly-informative prior density
- Write the unnormalized posterior density as an R function
- Lay out a grid based on crude estimate and standard error
- Posterior simulations
- Qoi's and predictions

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 - Theory problem: normal appxoximation to a simple posterior density
 - Computing problems simulate data and then fit the model
 Applied problems set the joint prior distribution for two horses are set on.

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