NATIONAL UNIVERSITY OF SINGAPORE

SEMESTER 1, 2021/2022

MA2002 Calculus **Tutorial 1** (23^{rd} August – 27^{th} August)

TUTORIAL PART I

This part consists of relatively basic questions which cover the course materials. These questions will be discussed during tutorial sessions.

1. For the functions $f(x) = 1 - x^3$ and $g(x) = \dfrac{1}{x}$, find following functions and their domains.

   (i) $f \circ g$, (ii) $g \circ f$, (iii) $f \circ f$, (iv) $g \circ g$.

2. For the functions $f(x) = \dfrac{2}{x+1}$, $g(x) = \cos x$ and $h(x) = \sqrt{x+3}$, find $f \circ g \circ h$.

3. For each of the following functions, determine whether it is even, odd, or neither.

   (a) $f(x) = x^{-3}$, (b) $f(x) = |\sin x| - 4x^2$, (c) $f(x) = 3x^3 + 2x^2 + 1$.

4. For each of the following, evaluate the limit, if it exists.

   (a) $\lim\limits_{x \to -4} (x+3)^{2021}$, (b) $\lim\limits_{u \to 1} \dfrac{u^4 - 1}{u^3 - 1}$, (c) $\lim\limits_{x \to 1} \dfrac{x-1}{\sqrt{x+3} - 2}$,

   (d) $\lim\limits_{x \to 4} \dfrac{x^2 - 4x}{x^2 - 3x - 4}$, (e) $\lim\limits_{x \to -1} \dfrac{x^2 - 4x}{x^2 - 3x - 4}$, (f) $\lim\limits_{h \to 0} \dfrac{\sqrt{1+h} - 1}{h}$,

   (g) $\lim\limits_{x \to 0^+} \sqrt{x} \sin(1/x)$, (h) $\lim\limits_{x \to 1} \dfrac{\sqrt{x} - x^2}{1 - \sqrt{x}}$.

5. Determine the following infinite limits.

   (a) $\lim\limits_{x \to 5^-} \dfrac{6}{x-5}$, (b) $\lim\limits_{x \to 0} \dfrac{x-1}{x^2(x+2)}$, (c) $\lim\limits_{x \to \pi^-} \csc x$, (d) $\lim\limits_{x \to 1^+} \dfrac{x+1}{x \sin \pi x}$.

6. Is there a real number a such that

$$\lim_{x \to 1} \frac{ax^2 + a^2 x - 2}{x^3 - 3x + 2}$$

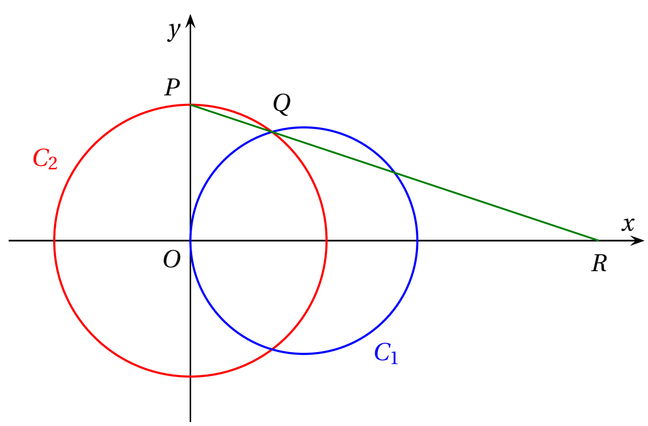
   exists? If so, find the value of a and the value of the limit.

7. Let
$$f(x) = \begin{cases} x^2, & \text{if } x \text{ is rational,} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$
Prove that $\lim_{x\to 0} f(x) = 0$.

Tutorial Part II

This part consists of relatively difficult questions to promote independent learning and inculcate critical thinking abilities. The solutions will not be discussed during the tutorial session. You may attempt them after you have gained a good understanding of the questions in Part I. The complete solution of this part is provided.

1. If $f_0(x) = x^2$ and $f_n(x) = f_0(f_{n-1}(x))$ for $n = 1, 2, \ldots$, find a formula for $f_n(x)$, $n = 1, 2, \ldots$.
2. Evaluate the limit
$$\lim_{x\to 1}(x-1)^2\left(4+\cos^2\left(\frac{2\pi}{x-1}\right)\right).$$
3. Give examples of functions f and g such that $\lim_{x\to a} f(x)g(x)$ exists even though neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists.
4. The figure shows a fixed circle C_1 with equation $(x-1)^2+y^2 = 1$ and a shrinking circle C_2 with radius r and center the origin O. Suppose that P is the point $(0, r)$, Q is the upper point of intersection of the two circles, and R is the point of intersection of the line PQ and the x-axis. What happens to R as C_2 shrinks, that is, as $r \to 0^+$?



Answers to Part I:

1. (i) $f\circ g(x) = 1 - \dfrac{1}{x^3}$, domain $= \mathbb{R}\setminus\{0\}$, (ii) $g\circ f(x) = \dfrac{1}{1-x^3}$, domain $= \mathbb{R}\setminus\{1\}$,
   (iii) $f\circ f(x) = 1-(1-x^3)^3$, domain $= \mathbb{R}$, (iv) $g\circ g(x) = x$, domain $= \mathbb{R}\setminus\{0\}$.
2. $f\circ g\circ h(x) = \dfrac{2}{\cos(\sqrt{x+3})+1}$.
3. (a) odd, (b) even, (c) neither.
4. (a) -1, (b) $4/3$, (c) 4, (d) $4/5$, (e) does not exist, (f) $1/2$, (g) 0, (h) 3.
5. (a) $-\infty$, (b) $-\infty$, (c) ∞, (d) $-\infty$.
6. yes, -2, $-2/3$.