NATIONAL UNIVERSITY OF SINGAPORE

SEMESTER 1, 2021/2022

MA2002 Calculus **Solution to Tutorial 10**

TUTORIAL PART I

1. Solving $x^4 = 8x$, we have $x = 0$ and $x = 2$. Note that $8x \geq x^4$ on $[0,2]$. Then the area enclosed by $y = x^4$ and $y = 8x$ is given by

$$A = \int_0^2 (8x - x^4)\,dx = \left[4x^2 - \frac{x^5}{5}\right]_{x=0}^{x=2} = \frac{48}{5}.$$

2. (i) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ has two branches $y_1 = \dfrac{b}{a}\sqrt{a^2 - x^2}$ and $y_2 = -\dfrac{b}{a}\sqrt{a^2 - x^2}$.
Note that $-a \leq x \leq a$. Then the area of the ellipse is given by

$$\int_{-a}^{a} (y_1 - y_2)\,dx = \frac{2b}{a}\int_{-a}^{a}\sqrt{a^2 - x^2}\,dx = \frac{2b}{a}\cdot\frac{\pi a^2}{2} = \pi ab.$$

(ii) Let z be fixed. Then the cross-section of the ellipsoid cut by the plane perpendicular to the z-axis at z is the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 - \dfrac{z^2}{c^2}$. That is,

$$\frac{x^2}{\left(a\sqrt{1 - \frac{z^2}{c^2}}\right)^2} + \frac{y^2}{\left(b\sqrt{1 - \frac{z^2}{c^2}}\right)^2} = 1.$$

Let $A(z)$ denote the area of the cross-section at z. Then by (i),

$$A(z) = \pi\left(a\sqrt{1 - \frac{z^2}{c^2}}\right)\left(b\sqrt{1 - \frac{z^2}{c^2}}\right) = \pi ab\left(1 - \frac{z^2}{c^2}\right).$$

Note that $-c \leq z \leq c$. Then the volume of the ellipsoid is given by

$$\int_{-c}^{c} A(z)\,dz = \pi ab\left[z - \frac{z^3}{3c^2}\right]_{z=-c}^{z=c} = \frac{4\pi}{3}abc.$$

3. (a) Washer: $V = \displaystyle\int_0^2 \pi(x^3)^2\,dx = \left.\frac{\pi x^7}{7}\right|_{x=0}^{x=2} = \frac{128\pi}{7}.$

Cylindrical Shell: $y = x^3 \Rightarrow x = \sqrt[3]{y}$ and $0 \leq x \leq 2 \Rightarrow 0 \leq y \leq 8$. Then

$$V = \int_0^8 2\pi y(2 - \sqrt[3]{y})\,dy = \pi\left[2y^2 - \frac{6}{7}y^{7/3}\right]_{y=0}^{y=8} = \frac{128\pi}{7}.$$

(b) Washer: $V = \int_0^{\pi/2} \pi \left(\sqrt{2\sin 2y}\right)^2 \, dy = -\pi \cos 2y \Big|_{y=0}^{y=\pi/2} = 2\pi.$

Cylindrical Shell: $0 \le y \le \dfrac{\pi}{2} \Rightarrow 0 \le x \le \sqrt{2}.$

$x = \sqrt{2\sin 2y}$ has two branches $y_1 = \dfrac{1}{2}\sin^{-1}\dfrac{x^2}{2}$ and $y_2 = \dfrac{\pi}{2} - \dfrac{1}{2}\sin^{-1}\dfrac{x^2}{2}$.

$$\begin{aligned} V &= \int_0^{\sqrt{2}} 2\pi x (y_2 - y_1)\, dx = \int_0^{\sqrt{2}} 2\pi x \left(\frac{\pi}{2} - \sin^{-1}\frac{x^2}{2}\right) dx \\ &= \pi \left[\frac{\pi x^2}{2} - x^2 \sin^{-1}\frac{x^2}{2} - \sqrt{4 - x^4}\right]_{x=0}^{x=\sqrt{2}} = 2\pi. \end{aligned}$$

(c) Washer: Solving $4 - x^2 = 2 - x$, we have $x = -1$ and $x = 2$. Since $4 - x^2 \ge 2 - x$ on $[-1, 2]$, the volume is given by

$$V = \int_{-1}^{2} \pi [(4 - x^2)^2 - (2 - x)^2]\, dx = \pi \left[12x + 2x^2 - 3x^3 + \frac{x^5}{5}\right]_{x=-1}^{x=2} = \frac{108\pi}{5}.$$

Cylindrical Shell: At $x = -1$, $y = 3$ and at $x = 2$, $y = 0$.

$y = 4 - x^2 \Rightarrow x = \pm\sqrt{4 - y}$, and $y = 2 - x \Rightarrow x = 2 - y$.

$$\begin{aligned} &\int_0^3 2\pi y \left(\sqrt{4 - y} - (2 - y)\right) dy \\ &\qquad = 2\pi \left[\frac{2}{5}(4 - y)^{5/2} - \frac{8}{3}(4 - y)^{3/2} - \left(y^2 - \frac{y^3}{3}\right)\right]_{y=0}^{y=3} = \frac{188\pi}{15}. \end{aligned}$$

$$\int_3^4 2\pi y \cdot 2\sqrt{4 - y}\, dx = 8\pi \left[\frac{1}{5}(4 - y)^{5/2} - \frac{4}{3}(4 - y)^{3/2}\right]_{y=3}^{y=4} = \frac{136\pi}{15}.$$

Therefore, the volume is given by

$$V = \frac{188\pi}{15} + \frac{136\pi}{15} = \frac{108\pi}{5}.$$

(d) Washer: $y = x^2 \Rightarrow x = \sqrt{y}$, and $0 \le x \le 1 \Rightarrow 0 \le y \le 1$.

The cross-section is an annulus of outer radius $1 + 1 = 2$ and inner radius $\sqrt{y} + 1$.

Then the volume is given by

$$V = \int_0^1 \pi [2^2 - (\sqrt{y} + 1)^2]\, dy = \pi \left[3y - \frac{y^2}{2} - \frac{4}{3}y^{3/2}\right]_{y=0}^{y=1} = \frac{7\pi}{6}.$$

Cylindrical Shell: $V = \int_0^1 2\pi (x + 1) x^2\, dx = 2\pi \left[\dfrac{x^4}{4} + \dfrac{x^3}{3}\right]_{x=0}^{x=1} = \dfrac{7\pi}{6}.$

(e) Washer: $y = \sin x$ has two branches $x = \sin^{-1} y$ and $x = \pi - \sin^{-1} y$.

The cross-section is an annulus of outer radius $\pi - \sin^{-1} y$ and inner radius $\sin^{-1} y$.

Note that $0 \le x \le \pi \Rightarrow 0 \le y \le 1$. Then the volume is given by

$$V = \int_0^1 \pi[(\pi - \sin^{-1} y)^2 - (\sin^{-1} y)^2]\,dy = \int_0^1 2\pi^2 \cos^{-1} y\,dy$$
$$= 2\pi^2 \left[y\cos^{-1} y - \sqrt{1-y^2} \right]_{y=0}^{y=1} = 2\pi^2.$$

Cylindrical Shell: $V = \displaystyle\int_0^{\pi} 2\pi x \sin x\,dx = 2\pi(-x\cos x + \sin x)\Big|_{x=0}^{x=\pi} = 2\pi^2.$

(f) Washer: $x = (y-3)^2$ has two branches $y_1 = 3 + \sqrt{x}$ and $y_2 = 3 - \sqrt{x}$.
The cross-section is an annulus of outer radius $y_1 - 1 = 2 + \sqrt{x}$ and inner radius $y_2 - 1 = 2 - \sqrt{x}$. Then the volume is given by

$$V = \int_0^4 \pi[(2+\sqrt{x})^2 - (2-\sqrt{x})^2]\,dx = \frac{16\pi}{3} x^{3/2}\Big|_{x=0}^{x=4} = \frac{128\pi}{3}.$$

Cylindrical Shell: Solving $4 = (y-3)^2$, we have $y = 1$ and $y = 5$. Then

$$V = \int_1^5 2\pi(y-1)[4-(y-3)^2]\,dy = 2\pi\left[\frac{4}{3}(y-1)^3 - \frac{1}{4}(y-1)^4\right]_{y=1}^{y=5} = \frac{128\pi}{3}.$$

4. (a) $y = \sqrt{2-x^2}$. Then $\dfrac{dy}{dx} = -\dfrac{x}{\sqrt{2-x^2}}$, and $\sqrt{1+\left(\dfrac{dy}{dx}\right)^2} = \sqrt{\dfrac{2}{2-x^2}}$. Then

$$L = \int_0^1 \sqrt{\frac{2}{2-x^2}}\,dx = \sqrt{2}\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)\Big|_{x=0}^{x=1} = \frac{\sqrt{2}\pi}{4}.$$

(b) $y = \ln(\cos x)$. Then $\dfrac{dy}{dx} = -\tan x$, and $\sqrt{1+\left(\dfrac{dy}{dx}\right)^2} = \sqrt{1+\tan^2 x} = \sec x \ge 0$.

$$L = \int_0^{\pi/3} \sec x\,dx = \ln(\sec x + \tan x)\Big|_{x=0}^{x=\pi/3} = \ln(2+\sqrt{3}).$$

(c) $x = \dfrac{y^3}{6} + \dfrac{1}{2y}$. Then $\dfrac{dx}{dy} = \dfrac{y^2}{2} - \dfrac{1}{2y^2}$, and

$$\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+\left(\frac{y^2}{2} - \frac{1}{2y^2}\right)^2} = \frac{y^2}{2} + \frac{1}{2y^2}.$$

Therefore,

$$L = \int_2^3 \left(\frac{y^2}{2} + \frac{1}{2y^2}\right)dy = \left[\frac{y^3}{6} - \frac{1}{2y}\right]_{y=2}^{y=3} = \frac{13}{4}.$$

5. (a) $y = \sqrt{2x - x^2}$. Then $\dfrac{dy}{dx} = \dfrac{1-x}{\sqrt{2x-x^2}}$, and

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{(1-x)^2}{2x - x^2}} = \frac{1}{\sqrt{2x - x^2}}.$$

Then the surface of revolution is

$$A = \int_{1/2}^{3/2} 2\pi \sqrt{2x - x^2} \frac{1}{\sqrt{2x - x^2}} \, dx = \int_{1/2}^{3/2} 2\pi \, dx = 2\pi.$$

(b) $x = 2\sqrt{4 - y}$. Then $\dfrac{dx}{dy} = -\dfrac{1}{\sqrt{4-y}}$, and

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + \frac{1}{4-y}} = \sqrt{\frac{5-y}{4-y}}.$$

Then the surface of revolution is

$$\begin{aligned}\int_0^{15/4} 2\pi \cdot 2\sqrt{4-y} \sqrt{\frac{5-y}{4-y}} \, dy &= \int_0^{15/4} 4\pi \sqrt{5-y} \, dy \\ &= -\frac{8\pi}{3}(5-y)^{3/2} \Big|_{y=0}^{y=15/4} = \frac{35\sqrt{5}\pi}{3}.\end{aligned}$$

(c) $x^{2/3} + y^{2/3} = 1$. Then $\dfrac{2}{3}x^{-1/3} + \dfrac{2}{3}y^{-1/3}\dfrac{dy}{dx} = 0$, and then $\dfrac{dy}{dx} = -\dfrac{y^{1/3}}{x^{1/3}}$.
Note that the curve is not differentiable at $x = 0$. By symmetry, we only need to consider the revolution of the curve in the first quadrant, $0 \le x \le 1$, $0 \le y \le 1$.

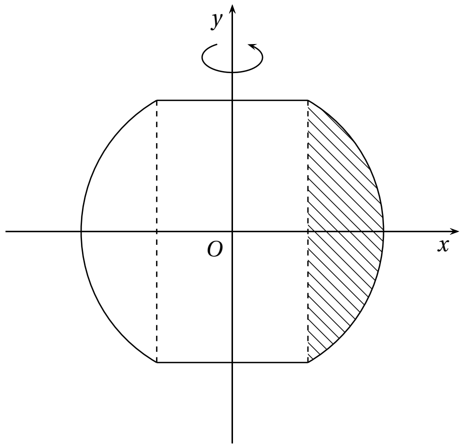
$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{y^{2/3}}{x^{2/3}}} = \frac{1}{x^{1/3}}.$$

Then the surface area of revolution is given by

$$\begin{aligned}A &= 2\int_0^1 2\pi \frac{y}{x^{1/3}} \, dx = -\int_0^1 4\pi y^{2/3} \left(-\frac{y^{1/3}}{x^{1/3}}\right) dx \\ &= -\int_1^0 4\pi y^{2/3} \, dy = \frac{12\pi}{5} y^{5/3} \Big|_{y=0}^{y=1} = \frac{12\pi}{5}.\end{aligned}$$

Tutorial Part II

1. The solid is formed by rotating the region enclosed by $x^2 + y^2 = R^2$ ($x > 0$) and $x = r$ about the y-axis.



Washer Method:

Suppose the solid is cut by the plane perpendicular to the y-axis at point y. The cross-section is an annulus of inner radius $r_1 = r$ and outer radius $r_2 = \sqrt{R^2 - y^2}$, and its area is given by

$$A(y) = \pi(r_2^2 - r_1^2) = \pi(R^2 - y^2 - r^2).$$

Solving $x^2 + y^2 = R^2$ and $x = r$, we have $y = \pm\sqrt{R^2 - r^2}$. Therefore, the volume of the solid is

$$\begin{aligned} V &= \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} \pi(R^2 - y^2 - r^2)\,dy \\ &= \pi\left[(R^2 - r^2)y - \frac{y^3}{3}\right]_{y=-\sqrt{R^2-r^2}}^{y=\sqrt{R^2-r^2}} = \frac{4\pi}{3}(R^2 - r^2)^{3/2}. \end{aligned}$$

Cylindrical Shell Method:

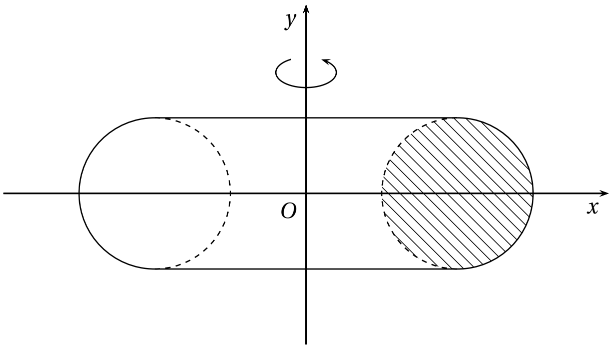
We cut the solid using right circular cylinder of radius x. Then $y = \pm\sqrt{R^2 - x^2}$, and its height is $2\sqrt{R^2 - x^2}$. The side area of the cylinder is

$$A(x) = 2\pi x \cdot 2\sqrt{R^2 - x^2} = 4\pi x\sqrt{R^2 - x^2}.$$

Note that $r \le x \le R$. So the volume of the solid is given by

$$V = \int_r^R 4\pi x\sqrt{R^2 - x^2}\,dx = -\frac{4\pi}{3}(R^2 - x^2)^{3/2}\bigg|_{x=r}^{x=R} = \frac{4\pi}{3}(R^2 - r^2)^{3/2}.$$

2. The torus is formed by rotating the circle $(x - R)^2 + y^2 = r^2$ about the y-axis.



Washer Method:

Suppose the torus is cut by the plane perpendicular to the y-axis at point y.

Since $(x-R)^2+y^2=r^2$ implies $x=R\pm\sqrt{r^2-y^2}$, the cross-section is an annulus of inner radius $r_1=R-\sqrt{r^2-y^2}$ and outer radius $r_2=R+\sqrt{r^2-y^2}$. Its area is

$$A(y)=\pi(r_2^2-r_1^2)=4\pi R\sqrt{r^2-y^2}.$$

Note that $-r\le y\le r$. The volume of the torus is given by

$$\begin{aligned}V&=\int_{-r}^{r}4\pi R\sqrt{r^2-y^2}\,dy\\&=4\pi R\int_{-r}^{r}\sqrt{r^2-y^2}\,dy=4\pi R\cdot\frac{\pi r^2}{2}=2\pi^2Rr^2.\end{aligned}$$

Cylindrical Shell Method:

Suppose the torus is cut by the right circular cylinder of radius x. Then $y=\pm\sqrt{r^2-(x-R)^2}$. So the height of the cylinder is $2\sqrt{r^2-(x-R)^2}$ and its side area is

$$A(x)=2\pi x\cdot2\sqrt{r^2-(x-R)^2}=4\pi x\sqrt{r^2-(x-R)^2}.$$

Note that $(x-R)^2\le r^2$, i.e., $R-r\le x\le R+r$. The volume of the torus is given by

$$\begin{aligned}V&=\int_{R-r}^{R+r}4\pi x\sqrt{r^2-(x-R)^2}\,dx=\int_{-r}^{r}4\pi(R+t)\sqrt{r^2-t^2}\,dt\\&=4\pi R\int_{-r}^{r}\sqrt{r^2-t^2}\,dt+4\pi\int_{-r}^{r}t\sqrt{r^2-t^2}\,dt\\&=4\pi R\cdot\frac{\pi r^2}{2}+0=2\pi^2Rr^2.\end{aligned}$$

Surface Area of Torus:

The circle $(x-R)^2+y^2=r^2$ has two branches:

$$x_1=R-\sqrt{r^2-y^2}\quad\text{and}\quad x_2=R+\sqrt{r^2-y^2}.$$

For each $i=1,2$,

$$\sqrt{1+\left(\frac{dx_i}{dy}\right)^2}=\sqrt{1+\left(\frac{\pm y}{\sqrt{r^2-y^2}}\right)^2}=\frac{r}{\sqrt{r^2-y^2}}.$$

Note that $-r \le y \le r$. The surface area of the torus is given by

$$\begin{aligned}
A &= \int_{-r}^{r} 2\pi x_1 \cdot \frac{r}{\sqrt{r^2-y^2}}\,dy + \int_{-r}^{r} 2\pi x_2 \cdot \frac{r}{\sqrt{r^2-y^2}}\,dy \\
&= \int_{-r}^{r} 2\pi (x_1+x_2) \cdot \frac{r}{\sqrt{r^2-y^2}}\,dy \\
&= \int_{-r}^{r} \frac{4\pi R r}{\sqrt{r^2-y^2}}\,dy = 4\pi R r \sin^{-1}\left(\frac{y}{r}\right)\Big|_{y=-r}^{y=r} = 4\pi^2 R r.
\end{aligned}$$

3. (i) Note that the surface $z = e^{-(x^2+y^2)}$ is formed by rotating the curve $z = e^{-x^2}$ $(x \ge 0)$ about the z-axis.

We can apply the cylindrical shell method to evaluate the volume:

$$\begin{aligned}
V &= \int_0^{\infty} 2\pi x e^{-x^2}\,dx = \lim_{b\to\infty} \int_0^{b} 2\pi x e^{-x^2}\,dx \\
&= \lim_{b\to\infty} (-\pi e^{-x^2})\Big|_{x=0}^{x=b} = \lim_{b\to\infty} \pi(1-e^{-b^2}) = \pi.
\end{aligned}$$

(ii) Suppose the solid is cut using the plane perpendicular to the y-axis at point y. Then the area of the cross-section is

$$A(y) = \int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx = e^{-y^2} \int_{-\infty}^{\infty} e^{-x^2}\,dx.$$

Therefore, the volume of the solid is given by

$$\begin{aligned}
V &= \int_{-\infty}^{\infty} A(y)\,dy = \int_{-\infty}^{\infty} \left[e^{-y^2} \int_{-\infty}^{\infty} e^{-x^2}\,dx \right] dy \\
&= \int_{-\infty}^{\infty} e^{-y^2}\,dy \cdot \int_{-\infty}^{\infty} e^{-x^2}\,dx = \left[\int_{-\infty}^{\infty} e^{-x^2}\,dx \right]^2.
\end{aligned}$$

Both methods compute the same volume. Then

$$\left[\int_{-\infty}^{\infty} e^{-x^2}\,dx \right]^2 = \pi.$$

It follows that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.