NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 1 EXAMINATION (2010–2011)

MA1102R Calculus

November 2010 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains a total of **EIGHT (8)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** questions. The marks for questions are not necessarily the same; marks for each question are indicated at the beginning of the question.
3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Question 1 [13 marks]

(a) Prove the limit $\lim_{x \to -2} x^3 = -8$ using the ϵ, δ-definition.

(b) Find the values of real numbers a and b for which the function

$$f(x) = \begin{cases} \dfrac{(\sin x - a)(\cos x - b)}{e^x - 1}, & \text{if } x \neq 0, \\ 5, & \text{if } x = 0, \end{cases}$$

is continuous at $x = 0$.

Question 2 [12 marks]

Find the following limits.

(a) $\lim_{x \to \infty} x^{\frac{1}{\ln(x^3+1)}}$.

(b) $\lim_{x \to 0} \left(\dfrac{2 + e^{1/x}}{1 + e^{4/x}} + \dfrac{\sin x}{|x|} \right)$.

Question 3 [16 marks]

(a) Find the coordinates of the inflection points, if any, of the function

$$f(x) = x^{8/3}(11 - x).$$

(b) Suppose that the function g is defined on $\mathbb{R}$ and satisfies

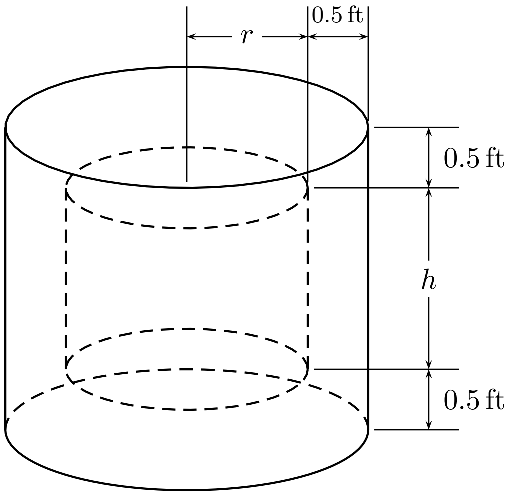
$$|g(x) - g(y)| \leq (x - y)^2 \quad \text{for all } x, y \in \mathbb{R}.$$

Show that g is a constant function.

(c) Evaluate $\dfrac{d}{dx} \displaystyle\int_0^{3x} \sin((x - t)^2)\, dt$.

Question 4 [8 marks]

A cylindrical container for storing radioactive waste is to be constructed from lead and have a thickness of 0.5 ft (see the figure). If the volume of the outside cylinder is to be $16\pi\,\text{ft}^3$, find the radius and the height of the inside cylinder that will result in a container of maximum storage capacity.



Question 5 [12 marks]

Evaluate the following integrals.

(a) $\displaystyle\int \frac{\sqrt{\tan(\ln x)}\,\sec^4(\ln x)}{x}\,dx$

(b) $\displaystyle\int_0^\infty \frac{3^x}{1+3^{2x}}\,dx.$

Question 6 [13 marks]

(a) Find the area of the surface generated by revolving the curve

$$y = \cosh x, \quad 0 \le x \le \ln 2,$$

about the x-axis.

(b) Let V_1 and V_2 be the volumes of the solids generated when the region enclosed by the curves $y = kx^2$ and $y = k(4x - 3x^2)$ (where $k > 0$) is revolved about the x-axis and the y-axis, respectively. Find the value of k for which V_1 is half of V_2.

Question 7 [16 marks]

(a) Solve the initial value problem

$$\frac{dy}{dx} + (\cos x)y = 2x\, e^{-\sin x}, \quad y(\pi) = 0.$$

(b) Two chemicals Y and Z react to form the substance X according to the differential equation

$$\frac{dQ}{dt} = k(50 - Q)(100 - Q),$$

where $Q = Q(t)$ denotes the amount of substance X per unit volume at time t, and k is a positive constant. Initially, no amount of X is present.

(i) Derive an expression for the amount of X per unit volume at time t.

(ii) What is the amount of X per unit volume when it is forming at the slowest rate?

Question 8 [10 marks]

Let f be a twice differentiable function defined on $\mathbb{R}$. Suppose that f'' is continuous on $\mathbb{R}$ and $f(0) = f(1) = 0$. Prove that there exists a number $c \in [0, 1]$ such that

$$\int_0^1 f(x)\, dx = -\frac{1}{12} f''(c).$$

[END OF PAPER]