National University of Singapore

Semester 1, 2021/2022 MA2001 Homework Assignment 3

(a) Use A4 size paper and pen (blue or black ink) to write your answers. (Students may also type out the answers or write the answers electronically using their devices.)

(b) Write down your student number and full name clearly on the top left of every page of the answer scripts.

(c) Write the page number on the top right corner of each page of answer scripts.

(d) This assignment consists of 4 pages and 7 questions. Total mark is 80 marks.

(e) To submit your answer scripts, do the following:

  (i) Scan or take pictures of your work (make sure the images can be read clearly).

  (ii) Merge all your answers into one pdf file. Arrange them in order of the questions.

  (iii) Name the pdf file by **StudentNo HW3** (e.g. **A123456R HW3**).

  (iv) Upload your pdf into the LumiNUS folder Homework 3 submission.

(f) Deadline for submission is 22 October, 2021 by 11.59pm. **Late submission will not be accepted.**

1. Let $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 2 & 3 & 3 \\ 2 & 4 & 1 & 3 & 2 \\ 1 & 2 & -4 & -3 & -5 \\ 3 & 6 & 3 & 6 & 1 \end{pmatrix}$.

   (i) [3 marks] Find a basis for the row space of $\boldsymbol{A}$ such that the vectors in the basis are the rows in $\boldsymbol{A}$.

   (ii) [2 marks] Extend the basis in (i) to a basis for $\mathbb{R}^5$ with some standard basis vectors. Show how you obtain your answer.

   (iii) [3 marks] Let $S = \left\{ \begin{pmatrix} 2 \\ 4 \\ 2 \\ 6 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ -3 \\ 6 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ -5 \\ 1 \end{pmatrix} \right\}$. Show that S is a basis for the column space of $\boldsymbol{A}$.

   (iv) [2 marks] Is it possible to find a matrix in row echelon form whose column space is the same as $\boldsymbol{A}$? Justify your answer.

2. Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ be a basis for a vector space V.
   For parts (i) to (iii) below, give your examples of subspaces in terms of the vectors in S.

   (i) [3 marks] Find two subspaces U_1 and U_2 of V such that

   $$\dim U_1 = 3, \quad \dim U_2 = 2, \quad \dim U_1 \cap U_2 = 1.$$

   (ii) [3 marks] Find a subspace W of V such that $\dim W = 2$ and W does not contain any vector in S.

   (iii) [3 marks] If X_1 and X_2 are two different subspaces of V such that $\dim X_1 = \dim X_2 = 3$, what are the possible dimensions of $X_1 \cap X_2$? Given examples of X_1 and X_2 for each such possible dimension.

   (iv) [3 marks] True or false: If T is a proper subset of V that contains S, then T cannot be a subspace of V. Justify your answer.

3. (a) [4 marks] Let $\boldsymbol{D} = \begin{pmatrix} x+1 & 1 & 1 & 1 \\ 0 & x-2 & -3 & -3 \\ 0 & 0 & x^2-x-2 & x+1 \\ 0 & 0 & x-2 & 1 \end{pmatrix}$.
   Find all values of x such that rank $\boldsymbol{D}$ = 1, 2, 3, 4.

   (b) Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $m \times n$ and $n \times k$ matrices respectively.

      (i) [4 marks] Show that, if $\boldsymbol{A}$ is full rank, then rank $\boldsymbol{AB}$ = rank $\boldsymbol{B}$.

      (ii) [2 marks] Is it true that, if $\boldsymbol{B}$ is full rank, then rank $\boldsymbol{AB}$ = rank $\boldsymbol{A}$? Justify your answer.

4. Given a homogeneous system $\boldsymbol{Ax} = \boldsymbol{0}$ with three equations and six variables. Suppose a basis for the nullspace of $\boldsymbol{A}$ is given by

$$\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

(i) [4 marks] Find the reduced row echelon form of $\boldsymbol{A}$.

(ii) [2 marks] Suppose $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ is a solution of $\boldsymbol{Ax} = \boldsymbol{b}$ for some $\boldsymbol{b} \in \mathbb{R}^3$. Write down a general solution of $\boldsymbol{Ax} = \boldsymbol{b}$.

(iii) [3 marks] True or false: The non-homogeneous system $\boldsymbol{Ax} = \boldsymbol{c}$ is consistent for every $\boldsymbol{c} \in \mathbb{R}^3$. Justify your answer.

(iv) [3 marks] If $\boldsymbol{B}$ is a 3×3 invertible matrix, find a basis for the nullspace of $\boldsymbol{BA}$. Explain how you derive the answer.

5. $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ and $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ be two ordered bases for a subspace W in $\mathbb{R}^n$. Suppose $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3 \in W$ such that

$$\begin{array}{rcrcrcr} \boldsymbol{w}_1 &=& -\boldsymbol{u}_1 &+& \boldsymbol{u}_2 &+& \boldsymbol{u}_3 \\ \boldsymbol{w}_2 &=& -2\boldsymbol{u}_1 &+& \boldsymbol{u}_2 && \\ \boldsymbol{w}_3 &=& -2\boldsymbol{u}_1 && &-& \boldsymbol{u}_3 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcr} \boldsymbol{w}_1 &=& 2\boldsymbol{v}_1 &+& \boldsymbol{v}_2 &+& 3\boldsymbol{v}_3 \\ \boldsymbol{w}_2 &=& \boldsymbol{v}_1 &-& 2\boldsymbol{v}_2 &-& 2\boldsymbol{v}_3 \\ \boldsymbol{w}_3 &=& 3\boldsymbol{v}_1 && &+& 3\boldsymbol{v}_3 \end{array}$$

(i) [3 marks] Find $[\boldsymbol{w}_i]_S$ and $[\boldsymbol{w}_i]_T$ for $i = 1, 2, 3$ without using transition matrix.

(ii) [3 marks] Use (i) to find the transition matrix from S to T.

(iii) [2 marks] Use (ii) to find $[\boldsymbol{v}_i]_S$ for $i = 1, 2, 3$.

(iv) [3 marks] Define the 3×3 matrices $\boldsymbol{A} = (\boldsymbol{u}_1 \ \ \boldsymbol{u}_2 \ \ \boldsymbol{u}_3)$ and $\boldsymbol{B} = (\boldsymbol{v}_1 \ \ \boldsymbol{v}_2 \ \ \boldsymbol{v}_3)$. Find a matrix $\boldsymbol{C}$ such that $\boldsymbol{A} = \boldsymbol{BC}$.
(Hint: Consider the augmented matrix $(\boldsymbol{B} \mid \boldsymbol{A})$.)

(v) [3 marks] Let $\boldsymbol{A}$ be as in (iv) such that $\boldsymbol{A}^T\boldsymbol{A} = \begin{pmatrix} 2 & -1 & -2 \\ -1 & 6 & 3 \\ -2 & 3 & 5 \end{pmatrix}$.

Find $\boldsymbol{w}_i \cdot \boldsymbol{w}_j$, for $i, j = 1, 2, 3$.

6. Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ be an ordered basis for a subspace $V \in \mathbb{R}^4$ with

$$\boldsymbol{u}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{u}_2 = \begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{u}_3 = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}.$$

(i) [3 marks] Apply Gram-Schmidt process to convert S into an orthonormal basis for V. Show all your workings clearly (follow the algorithm in theorem 5.2.19 exactly).

(ii) [3 marks] Let $\boldsymbol{w} = \begin{pmatrix} 1 \\ 5 \\ -1 \\ 2 \end{pmatrix}$. Find the projection of $\boldsymbol{w}$ onto the subspace V.

(iii) [2 marks] Find a unit vector $\boldsymbol{n}$ such that for all $\boldsymbol{u} \in V$, $\boldsymbol{n} \cdot \boldsymbol{u} = 0$.

(iv) [4 marks] Is it possible to find a nonzero vector $\boldsymbol{n}'$ such that

(a) for all $\boldsymbol{u} \in V$, $\boldsymbol{n}' \cdot \boldsymbol{u} = 0$, and

(b) $\{\boldsymbol{n}, \boldsymbol{n}'\}$ is linearly independent, where $\boldsymbol{n}$ is the unit vector in (iii)?

7. Let W be a subspace of $\mathbb{R}^n$. Recall (from Exercise 5 Q7) that

$$W^{\perp} = \{\boldsymbol{u} \in \mathbb{R}^n \mid \boldsymbol{u} \text{ is orthogonal to } W\}$$

is also a subspace of $\mathbb{R}^n$.

(i) [2 marks] Show that $W \cap W^{\perp} = \{\boldsymbol{0}\}$.

(ii) [4 marks] Given a basis $S = \{\boldsymbol{w}_1, \boldsymbol{w}_2, \cdots, \boldsymbol{w}_m\}$ for W. Suppose we
(`I`) extend S to a basis for $\mathbb{R}^n$:

$$T = \{\boldsymbol{w}_1, \boldsymbol{w}_2, \cdots, \boldsymbol{w}_m, \boldsymbol{w}_{m+1} \cdots, \boldsymbol{w}_n\}$$

and
(`II`) apply Gram Schmidt to T to get an orthogonal basis

$$T' = \{\boldsymbol{w}'_1, \boldsymbol{w}'_2, \cdots, \boldsymbol{w}'_m, \boldsymbol{w}'_{m+1}, \cdots, \boldsymbol{w}'_n\}.$$

Show that $W = \text{span}\{\boldsymbol{w}'_1, \boldsymbol{w}'_2, \cdots, \boldsymbol{w}'_m\}$ and $W^{\perp} = \text{span}\{\boldsymbol{w}'_{m+1}, \boldsymbol{w}'_{m+2}, \cdots, \boldsymbol{w}'_n\}$.
Hint: Use Exercise 3 Q43. Also note that if $U = \text{span}(X)$ and $V = \text{span}(Y)$, then $U + V = \text{span}(X \cup Y)$.

(iii) [4 marks] Let $\boldsymbol{A}$ be an $n \times k$ matrix with W as the column space. Show that the solution space of $\boldsymbol{A}\boldsymbol{A}^T\boldsymbol{x} = \boldsymbol{0}$ is given by $W^{\perp}$.