NATIONAL UNIVERSITY OF SINGAPORE

MA1102R — CALCULUS

2020 – 2021 SEMESTER I

28 November 2020, 1:00 pm – 3:00 pm

INSTRUCTIONS TO CANDIDATES

1. Use A4 size paper and pen (blue or black ink) to write your answers.
2. Write down your student number clearly on the top left of every page of the answers.
3. Write on one side of the paper only. Start a new question in a new page. Write the question number and page number on the top right corner of each page (e.g., Q1P1, Q1P2, . . . , Q2P1).
4. This test paper contains **SEVEN (7)** questions. Answer **ALL** questions.
5. The total mark for this paper is **ONE HUNDRED (100)**.
6. This is an **OPEN BOOK** examination.
7. You may use non-graphing calculators. However, you should lay out systematically the various steps in the calculations.
8. Join the Zoom conference and turn on the video setting at all time during the exam. Adjust your camera such that your face and upper body including your hands are captured on Zoom.
9. You may go for a short toilet break (not more than 5 minutes) during the test.
10. At the end of the test,
    (i) Scan or take pictures of your work (make sure the images can be read clearly).
    (ii) Merge all your images into one PDF file in correct order.
    (iii) Name the PDF file by "MA1102R_Student Number" (e.g., MA1102R_A1234567X.pdf).
    (iv) Upload your PDF into the LumiNUS Folder "Exam – Exam Submission".
    (v) Review your submission to ensure that it is successful.
    (vi) The Exam Submission folder will close on 28 November 2020, 3:30 pm. After the folder is closed, exam answers that are not submitted will not be accepted.

Question 1 [10 marks]

Let $f(x) = x^3 e^{-x^2}$.

(i) Find the open intervals on which f is increasing and decreasing.

(ii) Find the x-coordinates of the local maximum and minimum points of f.

(iii) Find the open intervals on which f is concave up and concave down.

(iv) Find the x-coordinates of the inflection points of f.

Question 2 [18 marks]

(a) Using only the ϵ, δ-definition of limit, prove that

$$\lim_{x \to 1} \frac{1}{\sqrt{5 - x^2}} = \frac{1}{2}.$$

(b) Find the following limit:

$$\lim_{x \to 0} \left(\frac{1 + \sin x}{1 + x} \right)^{1/x^3}.$$

(c) Using Riemann sums, express the following limit

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{i}{2n^2 + 5in + 2i^2}$$

as a definite integral. Hence, find the limit.

Question 3 [16 marks]

(a) Let f be a continuous function on $\mathbb{R}$ and a be a constant. Find $F'''(x)$, where

$$F(x) = \int_a^x (x - t)^2 f(t)\, dt.$$

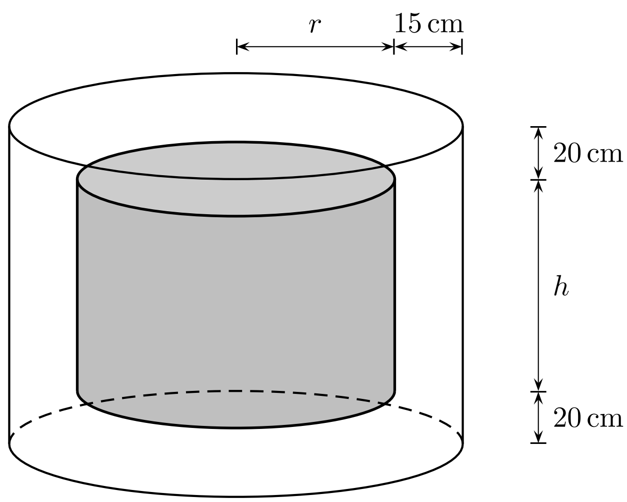
(b) Determine the value of r for which the integral

$$\int_1^{\infty} \left(\frac{r}{x + 1} - \frac{3x}{2x^2 + r} \right) dx$$

is convergent, and evaluate the integral for this value of r.

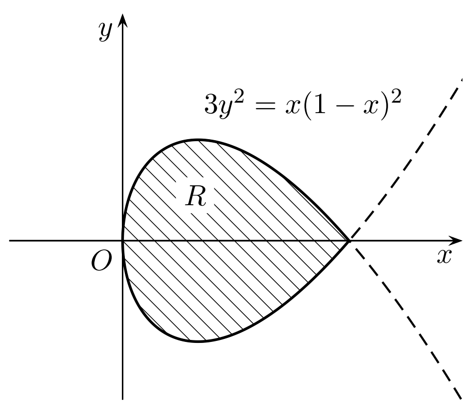
Question 4 [10 marks]

A right cylindrical container for storing radioactive waste is to be constructed from lead and has a thickness of 15 cm on the side and 20 cm on the top and the bottom. Suppose the volume of the outside cylinder is $9\pi\,\mathrm{m}^3$. Prove that when the container has the maximum storage capacity, the radius and the height of the inside cylinder are 135 cm and 360 cm respectively.



Question 5 [18 marks]

Let R be the region enclosed by the loop of the curve C defined by $3y^2 = x(1-x)^2$.



(i) Find the volume of the solid formed by rotating R about the x-axis.

(ii) Find the volume of the solid formed by rotating R about the y-axis.

(iii) Find the arc length of the loop of C.

(iv) Find the area of the surface formed by rotating the loop of C about the x-axis.

Question 6 [18 marks]

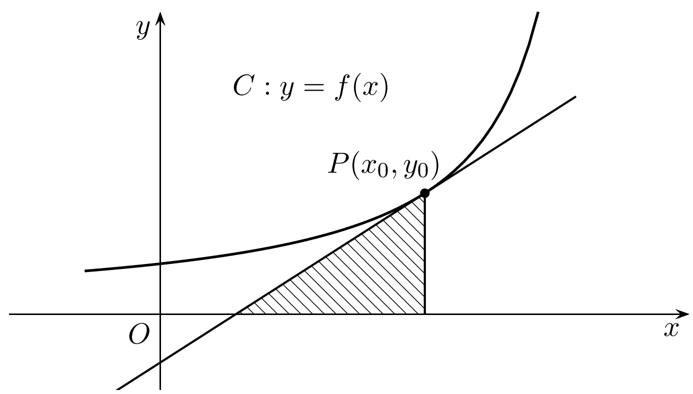
(a) Consider the following initial value problem:

$$\frac{dy}{dx} = x^3 + \frac{2y}{x} - \frac{y^2}{x}, \quad \text{where } y = 0 \text{ at } x = 1 \quad (x > 0).$$

(i) Use the substitution $y = 1/z - x^2$ to convert the differential equation of the given initial value problem into a first order linear equation in x and z.

(ii) Solve the differential equation obtained in (i). Hence, solve the initial value problem.

(b) Let C be the graph of a differentiable increasing function f. Suppose that for every point P of C, the area of the triangle bounded by the tangent line of C at P, the altitude from P to the x-axis, and the x-axis has a constant value 1102.



(i) Prove that

$$[f(x)]^2 = Kf'(x)$$

for some constant K. Find the value of the constant K.

(ii) Suppose that $f(1) = 1$. Find the value of c such that $f(c) = 2$.

Question 7 [10 marks]

(a) Let f be a differentiable function on $\mathbb{R}$ such that

$$\lim_{x\to-\infty} f(x) = 1 \quad \text{and} \quad \lim_{x\to\infty} f(x) = 1.$$

Prove that there exists $c \in \mathbb{R}$ such that $f'(c) = 0$.

(b) Let f be a continuous function on $\mathbb{R}$ such that $f(x + \pi) = f(x)$ for all $x \in \mathbb{R}$. Prove that if function g is continuous on $[0, \pi]$, then

$$\lim_{n\to\infty} \int_0^{\pi} \pi f(nx)g(x)\, dx = \int_0^{\pi} f(x)\, dx \int_0^{\pi} g(x)\, dx.$$