NATIONAL UNIVERSITY OF SINGAPORE

SEMESTER 1, 2021/2022

MA2002 Calculus **Tutorial 9** (26^{th} October – 29^{th} October)

TUTORIAL PART I

This part consists of relatively basic questions which cover the course materials. The solutions to these questions will be recorded.

1. Use logarithmic differentiation to find the derivative of y with respect to x.

   (a) $y = \dfrac{x^2\,2^x}{\sqrt[3]{\sin 3x}}$, (b) $y = x^{(x^x)}$, $x > 0$.

2. Find the following limits.

   (a) $\displaystyle\lim_{x\to\infty} \frac{e^x}{x^n}$ $(n \in \mathbb{Z}^+)$, (b) $\displaystyle\lim_{x\to 0}(e^{2x} + 2x)^{1/x}$,

   (c) $\displaystyle\lim_{x\to\infty} x^{1/x}$, (d) $\displaystyle\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2}$.

3. (i) Show that the inverse hyperbolic sine function $\sinh^{-1} x$ is given by

$$\sinh^{-1} x = \ln(x + \sqrt{x^2+1}), \quad x \in \mathbb{R}.$$

   (ii) Verify that the derivative of $\sinh^{-1} x$ is given by

$$\frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}, \quad x \in \mathbb{R}.$$

4. Evaluate the following indefinite integrals.

   (a) $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+4}}$, (b) $\displaystyle\int \frac{x^2}{\sqrt{6x-x^2}}\,dx$,

   (c) $\displaystyle\int \frac{\ln x}{x^2}\,dx$, (d) $\displaystyle\int \tan^{-1}\left(\frac{1}{x}\right)dx$,

   (e) $\displaystyle\int \cos(\ln x)\,dx$, (f) $\displaystyle\int e^{\sqrt{x}}\,dx$,

   (g) $\displaystyle\int \frac{4(x+1)}{x^2(x^2+4)}\,dx$, (h) $\displaystyle\int \frac{\sin\theta}{\cos^2\theta+\cos\theta-2}\,d\theta$,

   (i) $\displaystyle\int \frac{1}{x^{2002}-x}\,dx$.

TUTORIAL PART II

This part consists of relatively difficult questions to promote independent learning and inculcate critical thinking abilities. The solutions will not be recorded. You may attempt them after you have gained a good understanding of the questions in Part I. The complete solution of this part is provided.

1. (i) Show that $e^x \geq 1 + x$ for all $x \in \mathbb{R}$.

   (ii) Using part (i), show that $\lim_{x\to\infty} e^x = \infty$.

   (iii) Deduce from part (ii) that $\lim_{x\to-\infty} e^x = 0$.

2. Let $a_1, a_2, \ldots, a_k$ be positive numbers such that $a_1 a_2 \cdots a_k = 1$. By considering the function $f(x) = a_1^x + \cdots + a_k^x$, prove that for any positive integers m and n, where $m > n$,

$$\sqrt[m]{a_1} + \cdots + \sqrt[m]{a_k} \leq \sqrt[n]{a_1} + \cdots + \sqrt[n]{a_k}.$$

3. Evaluate the integral

$$\int \frac{dx}{2 + \sin x - 2\cos x}.$$

Answers to Part I:

1. (a) $\dfrac{x^2\, 2^x}{\sqrt[3]{\sin 3x}} \left(\dfrac{2}{x} + \ln 2 - \cot 3x \right)$, (b) $x^x x^{(x^x)} \left((\ln x)^2 + \ln x + \dfrac{1}{x} \right)$.

2. (a) ∞, (b) e^4, (c) 1, (d) $e^{1/3}$.

4. (a) $-\dfrac{\sqrt{4+x^2}}{4x} + C$, (b) $\dfrac{27}{2}\sin^{-1}\left(\dfrac{x-3}{3}\right) - \dfrac{9}{2}\sqrt{6x-x^2} - \dfrac{x}{2}\sqrt{6x-x^2} + C$,

   (c) $-\dfrac{\ln x}{x} - \dfrac{1}{x} + C$, (d) $x\tan^{-1}\left(\dfrac{1}{x}\right) + \dfrac{1}{2}\ln(1+x^2) + C$,

   (e) $\dfrac{1}{2}x\sin(\ln x) + \dfrac{1}{2}x\cos(\ln x) + C$, (f) $2e^{\sqrt{x}}(\sqrt{x} - 1) + C$,

   (g) $\ln|x| - \dfrac{1}{x} - \dfrac{1}{2}\ln(x^2+4) - \dfrac{1}{2}\tan^{-1}\left(\dfrac{x}{2}\right) + C$,

   (h) $\dfrac{1}{3}\ln|\cos\theta + 2| - \dfrac{1}{3}\ln|\cos\theta - 1| + C$, (i) $\dfrac{1}{2001}\ln|1 - x^{-2001}| + C$.