NATIONAL UNIVERSITY OF SINGAPORE

SEMESTER 1, 2021/2022

MA2002 Calculus **Solution to Tutorial 1**

TUTORIAL PART I

1. The domain of f is $\mathbb{R}$ and the domain of g is $\{x \mid x \neq 0\} = \mathbb{R} \setminus \{0\}$.
   (i) $f \circ g(x) = f(g(x)) = f(1/x) = 1 - 1/x^3$, and the domain of $f \circ g$ is given by
   $$\{x \mid x \neq 0 \text{ and } 1/x \in \mathbb{R}\} = \mathbb{R} \setminus \{0\}.$$
   (ii) $g \circ f(x) = g(f(x)) = g(1 - x^3) = 1/(1 - x^3)$, and the domain of $g \circ f$ is given by
   $$\{x \mid x \in \mathbb{R} \text{ and } 1 - x^3 \neq 0\} = \mathbb{R} \setminus \{1\}.$$
   (iii) $f \circ f(x) = f(f(x)) = f(1 - x^3) = 1 - (1 - x^3)^3$, and the domain of $f \circ f$ is given by
   $$\{x \mid x \in \mathbb{R} \text{ and } 1 - x^3 \in \mathbb{R}\} = \mathbb{R}.$$
   (iv) $g \circ g(x) = g(g(x)) = g(1/x) = 1/(1/x) = x$, and the domain of $g \circ g$ is given by
   $$\{x \mid x \neq 0 \text{ and } 1/x \neq 0\} = \mathbb{R} \setminus \{0\}.$$

2. $g \circ h(x) = g(h(x)) = g(\sqrt{x+3}) = \cos\sqrt{x+3}$, and
   $$f \circ g \circ h(x) = f(g \circ h(x)) = f(\cos\sqrt{x+3}) = \frac{2}{\cos\sqrt{x+3} + 1}.$$

3. (a) $f(x) = x^{-3}$ is odd, for $f(-x) = (-x)^{-3} = -(x^{-3}) = -f(x)$.
   (b) $f(x) = |\sin x| - 4x^3$ is even, for $f(-x) = |\sin(-x)| - 4(-x)^2 = |\sin x| - 4x^2 = f(x)$.
   (c) $f(x) = 3x^3 + 2x^2 + 1$ is neither odd nor even. For instance, $f(1) = 6$ and $f(-1) = 0$.

4. (a) $\lim_{x \to -4} (x+3)^{2021} = (-4+3)^{2021} = (-1)^{2021} = -1$.
   (b) $\lim_{u \to 1} \frac{u^4 - 1}{u^3 - 1} = \lim_{u \to 1} \frac{(u-1)(u^3 + u^2 + u + 1)}{(u-1)(u^2 + u + 1)} = \lim_{u \to 1} \frac{u^3 + u^2 + u + 1}{u^2 + u + 1} = \frac{1^3 + 1^2 + 1 + 1}{1^2 + 1 + 1} = \frac{4}{3}$.

(c) $\lim_{x\to 1} \frac{x-1}{\sqrt{x+3}-2} = \lim_{x\to 1} \frac{(x-1)(\sqrt{x+3}+2)}{(\sqrt{x+3}-2)(\sqrt{x+3}+2)} = \lim_{x\to 1} \frac{(x-1)(\sqrt{x+3}+2)}{(x+3)-2^2}$
$= \lim_{x\to 1}(\sqrt{x+3}+2) = \sqrt{1+3}+2 = 4.$

(d) $\lim_{x\to 4} \frac{x^2-4x}{x^2-3x-4} = \lim_{x\to 4} \frac{x(x-4)}{(x+1)(x-4)} = \lim_{x\to 4} \frac{x}{x+1} = \frac{4}{4+1} = \frac{4}{5}.$

(e) $\lim_{x\to -1} \frac{x^2-4x}{x^2-3x-4}$ does not exist, because

$$\lim_{x\to -1}(x^2-3x-4) = (-1)^2-3(-1)-4 = 0,$$

but

$$\lim_{x\to -1}(x^2-4x) = (-1)^2-4(-1) = 5 \neq 0.$$

In general, if $\lim_{x\to a} \frac{f(x)}{g(x)}$ exists and equals $L \in \mathbb{R}$, and $\lim_{x\to a} g(x) = 0$, we must have

$$\lim_{x\to a} f(x) = \lim_{x\to a} \frac{f(x)}{g(x)} \cdot \lim_{x\to a} g(x) = L \cdot 0 = 0.$$

(f) $\lim_{h\to 0} \frac{\sqrt{1+h}-1}{h} = \lim_{h\to 0} \frac{(\sqrt{1+h}-1)(\sqrt{1+h}+1)}{h(\sqrt{1+h}+1)} = \lim_{h\to 0} \frac{(1+h)-1^2}{h(\sqrt{1+h}+1)}$
$= \lim_{h\to 0} \frac{1}{\sqrt{1+h}+1} = \frac{1}{\sqrt{1+0}+1} = \frac{1}{2}.$

(g) Note that for all $x > 0$, $-1 \le \sin(1/x) \le 1$. Then

$$-\sqrt{x} \le \sqrt{x}\sin(1/x) \le \sqrt{x}.$$

Since $\lim_{x\to 0^+}(-\sqrt{x}) = \lim_{x\to 0^+}\sqrt{x} = 0$, by squeeze theorem $\lim_{x\to 0^+}\sqrt{x}\sin(1/x)$ exists and equals 0.

(h) $\lim_{x\to 1} \frac{\sqrt{x}-x^2}{1-\sqrt{x}} = \lim_{x\to 1} \frac{(\sqrt{x}-x^2)(1+\sqrt{x})}{(1-\sqrt{x})(1+\sqrt{x})} = \lim_{x\to 1} \frac{\sqrt{x}+x-x^2-x^2\sqrt{x}}{1-x}$
$= \lim_{x\to 1} \frac{x-x^2+\sqrt{x}(1-x^2)}{1-x} = \lim_{x\to 1}(x+\sqrt{x}(1+x)) = 1+\sqrt{1}(1+1) = 3.$

5. (a) If $x \to 5^-$, then $6 \to 6 \neq 0$ and $x-5 \to 0$, so $\left|\frac{6}{x-5}\right| \to \infty$. Moreover, if $x < 5$, then $\frac{6}{x-5} < 0$. Therefore, $\lim_{x\to 5^-} \frac{6}{x-5} = -\infty$.

(b) If $x \to 0$, then $x-1 \to -1 \neq 0$ and $x^2(x+2) \to 0$, so $\left|\frac{x-1}{x^2(x+2)}\right| \to \infty$. Moreover, if $0 < |x| < 1$, then $\frac{x-1}{x^2(x+2)} < 0$. Therefore, $\lim_{x\to 0} \frac{x-1}{x^2(x+2)} = -\infty$.

(c) If $x \to \pi^-$, then $1 \to 1 \neq 0$ and $\sin x \to 0$, so $|\csc x| = \left|\frac{1}{\sin x}\right| \to \infty$. Moreover, if $0 < x < \pi$, then $\csc x > 0$. Therefore, $\lim_{x\to \pi^-} \csc x = \infty$.

(d) If $x \to 1^+$, then $x+1 \to 2 \neq 0$ and $x\sin\pi x \to 0$, so $\left|\dfrac{x+1}{x\sin\pi x}\right| \to \infty$. Moreover, if $1 < x < 2$, then $\dfrac{x+1}{x\sin\pi x} < 0$, then $\displaystyle\lim_{x\to 1^+} \frac{x+1}{x\sin\pi x} = -\infty$.

6. Recall the result in 4(e): If $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)}$ exists and $\displaystyle\lim_{x\to a} g(x) = 0$, then $\displaystyle\lim_{x\to a} f(x) = 0$.

   Suppose $\displaystyle\lim_{x\to 1} \frac{ax^2+a^2x-2}{x^3-3x+2}$ exists. Since $\displaystyle\lim_{x\to 1}(x^3-3x+2) = 1^3 - 3\cdot 1 + 2 = 0$, we must have $\displaystyle\lim_{x\to 1}(ax^2+a^2x-2) = 0$, that is, $a + a^2 - 2 = 0$. Therefore, $a = 1$ or -2.

   If $a = 1$, then $\displaystyle\lim_{x\to 1} \frac{ax^2+a^2x-2}{x^3-3x+2} = \lim_{x\to 1} \frac{x^2+x-2}{x^3-3x+2} = \lim_{x\to 1} \frac{(x-1)(x+2)}{(x-1)^2(x+2)} = \lim_{x\to 1} \frac{1}{x-1}$ which does not exist.

   If $a = -2$, then $\displaystyle\lim_{x\to 1} \frac{ax^2+a^2x-2}{x^3-3x+2} = \lim_{x\to 1} \frac{-2x^2+4x-2}{x^3-3x+2} = \lim_{x\to 1} \frac{-2(x-1)^2}{(x-1)^2(x+2)}$ $\displaystyle= \lim_{x\to 1} \frac{-2}{x+2} = \frac{-2}{1+2} = -\frac{2}{3}$.

7. If x is rational, then $0 \le f(x) = x^2$; if x is irrational, then $0 = f(x) \le x^2$. So for any real number x, we have $0 \le f(x) \le x^2$.

   Since $\displaystyle\lim_{x\to 0} 0 = \lim_{x\to 0} x^2 = 0$, by squeeze theorem $\displaystyle\lim_{x\to 0} f(x)$ exists and equals 0.

Tutorial Part II

1. Observe that inductively,

$$f_n(x) = f_0(f_{n-1}(x)) = f_{n-1}(x)^2 = (f_{n-2}(x)^2)^2 = f_{n-2}(x)^{2^2} = \cdots = f_0(x)^{2^n}.$$

   This holds for $n = 1, 2, \ldots$. Since $f_0(x) = x^2$, we have $f_n(x) = (x^2)^{2^n} = x^{2^{n+1}}$.

2. For $x \neq 1$, as $0 \le \cos^2\left(\dfrac{2\pi}{x-1}\right) \le 1$, we have $4 \le 4 + \cos^2\left(\dfrac{2\pi}{x-1}\right) \le 5$ and so

$$4(x-1)^2 \le (x-1)^2\left(4 + \cos^2\left(\frac{2\pi}{x-1}\right)\right) \le 5(x-1)^2.$$

   Since $\displaystyle\lim_{x\to 1} 4(x-1)^2 = \lim_{x\to 1} 5(x-1)^2 = 0$, by Squeeze Theorem,

$$\lim_{x\to 1}(x-1)^2\left(4 + \cos^2\left(\frac{2\pi}{x-1}\right)\right) = 0.$$

3. For example, let $f(x) = g(x) = \dfrac{|x|}{x} = \begin{cases} 1, & \text{if } x > 0, \\ -1, & \text{if } x < 0. \end{cases}$

   Since $\lim\limits_{x\to 0^+} \dfrac{|x|}{x} = 1$ and $\lim\limits_{x\to 0^-} \dfrac{|x|}{x} = -1$ are not equal, $\lim\limits_{x\to 0} \dfrac{|x|}{x}$ does not exist.

   On the other hand, $\dfrac{|x|}{x}\dfrac{|x|}{x} = 1$ for all $x \in \mathbb{R}\setminus\{0\}$, so $\lim\limits_{x\to 0}\left(\dfrac{|x|}{x}\dfrac{|x|}{x}\right) = \lim\limits_{x\to 0} 1 = 1$.

4. i) Find the coordinates of Q:

   Solving simultaneous equations $\begin{cases} (x-1)^2 + y^2 = 1, \\ x^2 + y^2 = r^2, \end{cases}$ we have $x = \dfrac{r^2}{2}$.

   This is the x-coordinate of the intersection Q. So the y-coordinates of Q is

$$\sqrt{r^2 - x^2} = r\sqrt{1 - \left(\frac{r}{2}\right)^2}.$$

   ii) Find the equation of PQ:

   The equation of the straight line passing through $P(0, r)$ and Q is given by

$$\frac{y-r}{x-0} = \frac{r\sqrt{1-(\frac{r}{2})^2} - r}{\frac{r^2}{2} - 0} = \frac{\sqrt{1-(\frac{r}{2})^2} - 1}{\frac{r}{2}}.$$

   iii) Find the x-coordinate of R:

   Note that R is on the x-axis. Let $y = 0$ in the equation of PQ. Then the x-coordinate of R is given by $x = \dfrac{r^2}{2\left(1 - \sqrt{1-(\frac{r}{2})^2}\right)}$.

   iv) Find the limiting position of R:

   We can evaluate the limit

$$\begin{aligned}
\lim_{r\to 0^+} \frac{r^2}{2\left(1-\sqrt{1-(\frac{r}{2})^2}\right)} &= \lim_{r\to 0^+} \frac{r^2\left(1+\sqrt{1-(\frac{r}{2})^2}\right)}{2\left(1-\sqrt{1-(\frac{r}{2})^2}\right)\left(1+\sqrt{1-(\frac{r}{2})^2}\right)} \\
&= \lim_{r\to 0^+} \frac{r^2\left(1+\sqrt{1-(\frac{r}{2})^2}\right)}{2\cdot(\frac{r}{2})^2} \\
&= \lim_{r\to 0^+} 2\left(1+\sqrt{1-\left(\frac{r}{2}\right)^2}\right) \\
&= 4.
\end{aligned}$$

   Therefore, as $r \to 0^+$, $R \to (4, 0)$ along the x-axis.