**NATIONAL UNIVERSITY OF SINGAPORE**

SEMESTER 1, 2021/2022

**MA2002 Calculus** **Tutorial 4** (13$^{\text{th}}$ September – 17$^{\text{th}}$ September)

TUTORIAL PART I

This part consists of relatively basic questions which cover the course materials. The solutions to these questions will be recorded.

1. Using $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$, find the following limits.

   (a) $\lim_{x\to 1} \frac{\sin(x-1)}{x^2+x-2}$,

   (b) $\lim_{x\to 0} \frac{\sin ax}{\sin bx}$ $\quad (a \neq 0,\ b \neq 0)$.

2. Find the derivatives of the functions using differentiation formulas.

   (a) $f(x) = \frac{(x-1)^4}{(x^2+2x)^5}$,

   (b) $f(x) = \frac{1}{(x+1/x)^2}$,

   (c) $f(x) = \sin(\sin(\sin x))$.

3. Find $\frac{d^2y}{dx^2}$ for the function $y = \sin(\cos x)$.

4. Find $\frac{dy}{dx}$ by implicit differentiation:

   (a) $\sin x + \cos y = \sin x \cos y$,

   (b) $\tan(x-y) = \frac{y}{1+x^2}$.

5. Find an equation of the tangent line to the curve $x^2 + 2xy - y^2 + x = 2$ at the point $(1, 2)$.

6. Let $(x_0, y_0)$, where $y_0 \neq 0$, be a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Show that the tangent line to the ellipse passing through $(x_0, y_0)$ is
$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1.$$

7. Use the closed interval method to find the absolute maximum and minimum values of each function $f$ on the given interval and the values of $x$ where they occur.

   (a) $f(x) = x^3 - 6x^2 + 9x + 2, \qquad [-1, 4]$,

   (b) $f(x) = \sqrt[3]{x}\,(8-x), \qquad [-1, 8]$.

8. Let $r > 1$ be a rational number. Prove that for any $x \in [0,1]$,
$$\frac{1}{2^{r-1}} \le x^r + (1-x)^r \le 1.$$

9. Let $f(x) = x^3 + bx^2 + cx + d$ be a cubic function. Show that if $b^2 < 3c$, then $f$ has no local extreme values.

## TUTORIAL PART II

This part consists of relatively difficult questions to promote independent learning and inculcate critical thinking abilities. The solutions will not be recorded. You may attempt them after you have gained a good understanding of the questions in Part I. The complete solution of this part is provided.

1. (i) Using the definition of derivative, show that for any $n \in \mathbb{Z}^+$,
$$\frac{d}{dx}(x^{1/n}) = \frac{1}{n}x^{1/n-1}, \quad x > 0.$$
   (ii) Deduce the Power Rule for rational numbers, that is, for any $r \in \mathbb{Q}$,
$$\frac{d}{dx}x^r = rx^{r-1}, \quad x > 0.$$

2. Find all the points, if any, such that the curve $y = \sin(x - \sin x)$ has horizontal tangents at the $x$-axis.

3. Let $a > 0$. Show that the length of the portion of any tangent line to the astroid
$$x^{2/3} + y^{2/3} = a^{2/3}$$
cut off by the coordinate axes is constant.

4. If $a$ and $b$ are positive real numbers, find the absolute maximum value of
$$f(x) = x^a(1-x)^b, \quad 0 \le x \le 1.$$

**Answers to Part I:**

1. (a) $1/3$, (b) $a/b$.
2. (a) $-\frac{2(x-1)^3(3x^2-4x-5)}{(x^2+2x)^6}$, (b) $-\frac{2x(x^2-1)}{(x^2+1)^3}$, (c) $\cos(\sin(\sin x))\cos(\sin x)\cos x$.
3. $-\cos x\cos(\cos x) - \sin^2 x\sin(\cos x)$.
4. (a) $\frac{\cos x\,(1-\cos y)}{\sin y\,(1-\sin x)}$, (b) $\frac{(1+x^2)^2\sec^2(x-y)+2xy}{(1+x^2)(1+(1+x^2)\sec^2(x-y))}$.
5. $7x - 2y = 3$.

7. (a) max $= 6$ at $x = 1$ and $x = 4$, min $= -14$ at $x = -1$,

   (b) max $= 6\sqrt[3]{2}$ at $x = 2$, min $= -9$ at $x = -1$.