MA2001 Assignment 4

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Problem 1

(a) (i) By Theorem 5.3.10, $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \iff \boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^T\boldsymbol{b}$. Then

$$\left(\begin{array}{ccc|c} 2 & -4 & 2 & 0 \\ -4 & 14 & -10 & 4 \\ 2 & -10 & 8 & -4 \end{array}\right) \xrightarrow{rref} \left(\begin{array}{ccc|c} 1 & 0 & -1 & \frac{4}{3} \\ 0 & 1 & -1 & \frac{2}{3} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The solution set is

$$\left\{ \begin{pmatrix} \frac{4}{3} \\ \frac{2}{3} \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \middle| \, t \in \mathbb{R} \right\}$$

(ii) By above, the projection is

$$\boldsymbol{w} = \begin{pmatrix} 0 & 2 \\ 1 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} \frac{4}{3} \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{4}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$$

(iii) By above,

$$\boldsymbol{A} \left(\boldsymbol{x} - \begin{pmatrix} \frac{4}{3} \\ \frac{2}{3} \\ 0 \end{pmatrix} \right) = \boldsymbol{0}$$

The solution set is

$$\left\{ \begin{pmatrix} \frac{4}{3} \\ \frac{2}{3} \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \middle| \, t \in \mathbb{R} \right\}$$

(iv)

$$\left(\begin{array}{ccc|c} 4 & 3 & -3 & \frac{4}{3} \\ 3 & 1 & -1 & \frac{2}{3} \\ 2 & -1 & 1 & 0 \end{array}\right) \xrightarrow{rref} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{2}{15} \\ 0 & 1 & -1 & \frac{4}{15} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Therefore $\left(\frac{4}{3}, \frac{2}{3}, 0\right)^T$ is a least squares solution of $\boldsymbol{Ax} = \boldsymbol{b}$ that is in the column space of $\boldsymbol{B}$.

(b) Since projections are least squares solutions,

$$\|\boldsymbol{f} - (\boldsymbol{CD})\boldsymbol{v}\| = \|\boldsymbol{f} - \boldsymbol{C}(\boldsymbol{Dv})\| \leq \|\boldsymbol{f} - \boldsymbol{Cx}\| \leq \|\boldsymbol{f} - (\boldsymbol{CD})\boldsymbol{x}\|$$

Therefore the projection of $\boldsymbol{f}$ onto the column space of $\boldsymbol{CD}$ is $\boldsymbol{CDv}$.

Problem 2

(a) (i) Using Gram-Schmidt,

$$\begin{aligned}
\boldsymbol{v}_1 &= \boldsymbol{a}_1 = (1,1,2,1) \\
\boldsymbol{v}_2 &= \boldsymbol{a}_2 - \frac{\boldsymbol{a_2} \cdot \boldsymbol{v_1}}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1}\boldsymbol{v}_1 \\
&= (1,0,1,0) - \frac{3}{7}(1,1,2,1) \\
&= \frac{1}{7}(4,-3,1,-3) \\
\boldsymbol{v}_3 &= \boldsymbol{a}_3 - \frac{\boldsymbol{a_3} \cdot \boldsymbol{v_1}}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1}\boldsymbol{v}_1 - \frac{\boldsymbol{a_3} \cdot \boldsymbol{v_2}}{\boldsymbol{v}_2 \cdot \boldsymbol{v}_2}\boldsymbol{v}_2 \\
&= (1,2,1,1) - \frac{6}{7}(1,1,2,1) - \frac{-\,4/7}{35/49}\frac{1}{7}(4,-3,1,-3) \\
&= (1,2,1,1) - \frac{6}{7}(1,1,2,1) + \frac{4}{35}(4,-3,1,-3) \\
&= \frac{1}{5}(3,4,-3,-1) \\
\boldsymbol{v}_4 &= \boldsymbol{a}_4 - \frac{\boldsymbol{a_4} \cdot \boldsymbol{v_1}}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1}\boldsymbol{v}_1 - \frac{\boldsymbol{a_4} \cdot \boldsymbol{v_2}}{\boldsymbol{v}_2 \cdot \boldsymbol{v}_2}\boldsymbol{v}_2 - \frac{\boldsymbol{a_4} \cdot \boldsymbol{v_3}}{\boldsymbol{v}_3 \cdot \boldsymbol{v}_3}\boldsymbol{v}_3 \\
&= (0,1,0,1) - \frac{2}{7}(1,1,2,1) - \frac{-\,6/7}{35/49}\frac{1}{7}(4,-3,1,-3) - \frac{3/5}{35/25}\frac{1}{5}(3,4,-3,-1) \\
&= (0,1,0,1) - \frac{2}{7}(1,1,2,1) + \frac{6}{35}(4,-3,1,-3) - \frac{3}{35}(3,4,-3,-1) \\
&= \frac{1}{7}(1,-1,-1,2)
\end{aligned}$$

Normalizing we have

$$T = \left\{ \frac{1}{\sqrt{7}} \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \frac{1}{7\sqrt{35}} \begin{pmatrix} 4 \\ -3 \\ 1 \\ -3 \end{pmatrix}, \frac{1}{5\sqrt{35}} \begin{pmatrix} 3 \\ 4 \\ -3 \\ -1 \end{pmatrix}, \frac{1}{7\sqrt{7}} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 2 \end{pmatrix} \right\}$$

(ii)

$$[\boldsymbol{a}_1]_T = \begin{pmatrix} \sqrt{7} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$[\boldsymbol{a}_2]_T = \begin{pmatrix} \frac{3}{\sqrt{7}} \\ \sqrt{35} \\ 0 \\ 0 \end{pmatrix}$$

$$[\boldsymbol{a}_3]_T = \begin{pmatrix} \frac{6}{\sqrt{7}} \\ \frac{-4}{5}\sqrt{35} \\ \sqrt{35} \\ 0 \end{pmatrix}$$

$$[\boldsymbol{a}_4]_T = \begin{pmatrix} \frac{2}{\sqrt{7}} \\ \frac{-6}{5}\sqrt{35} \\ \frac{3}{7}\sqrt{35} \\ \sqrt{7} \end{pmatrix}$$

(iii) Using $[\boldsymbol{a}_i]_T$ as coefficients for linear combinations of $\boldsymbol{t}_j$,

$$\begin{pmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 \end{pmatrix} = \begin{pmatrix} \boldsymbol{t}_1 & \boldsymbol{t}_2 & \boldsymbol{t}_3 & \boldsymbol{t}_4 \end{pmatrix} \begin{pmatrix} [\boldsymbol{a}_1]_T & [\boldsymbol{a}_2]_T & [\boldsymbol{a}_3]_T & [\boldsymbol{a}_4]_T \end{pmatrix}$$

$$\boldsymbol{A} = \begin{pmatrix} \frac{\sqrt{7}}{7} & \frac{4\sqrt{35}}{245} & \frac{3\sqrt{35}}{175} & \frac{\sqrt{7}}{49} \\ \frac{\sqrt{7}}{7} & -\frac{3\sqrt{35}}{245} & \frac{4\sqrt{35}}{175} & -\frac{\sqrt{7}}{49} \\ \frac{2\sqrt{7}}{7} & \frac{\sqrt{35}}{245} & -\frac{3\sqrt{35}}{175} & -\frac{\sqrt{7}}{49} \\ \frac{\sqrt{7}}{7} & -\frac{3\sqrt{35}}{245} & -\frac{\sqrt{35}}{175} & \frac{2\sqrt{7}}{49} \end{pmatrix} \begin{pmatrix} \sqrt{7} & \frac{3\sqrt{7}}{7} & \frac{6\sqrt{7}}{7} & \frac{2\sqrt{7}}{7} \\ 0 & \sqrt{35} & -\frac{4\sqrt{5}\sqrt{7}}{5} & -\frac{6\sqrt{5}\sqrt{7}}{5} \\ 0 & 0 & \sqrt{35} & \frac{3\sqrt{5}\sqrt{7}}{7} \\ 0 & 0 & 0 & \sqrt{7} \end{pmatrix}$$

(b) (i) Let $\boldsymbol{A}$ be orthogonal and λ be its eigenvalue. Then

$$\lambda^2 \boldsymbol{x}^T\boldsymbol{x} = (\boldsymbol{A}\boldsymbol{x})^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}^T(\boldsymbol{A}^T\boldsymbol{A})\boldsymbol{x} = \boldsymbol{x}^T\boldsymbol{x} \implies \lambda^2 = 1$$

Therefore the eigenvalues of an orthogonal matrix are ± 1.

(ii) Since $\boldsymbol{A}$ is diagonalizable, there exists invertible $\boldsymbol{P}$ and diagonal $\boldsymbol{D}$ such that $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$. By above, $\boldsymbol{D}^2 = \boldsymbol{I}$. Then

$$\boldsymbol{A}^2 = (\boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1})(\boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}) = \boldsymbol{P}\boldsymbol{D}^2\boldsymbol{P}^{-1} = \boldsymbol{P}\boldsymbol{P}^{-1} = \boldsymbol{I}$$

Problem 3

(a) (i) Expanding, we have

$$\begin{pmatrix} a_n \\ a_{n+1} \\ a_{n+2} \end{pmatrix} = \begin{pmatrix} a_n \\ a_{n+1} \\ \frac{1}{2}a_n + \frac{1}{2}a_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} a_{n-1} \\ a_n \\ a_{n+1} \end{pmatrix}$$

where

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Then

$$\begin{pmatrix} a_5 \\ * \\ * \end{pmatrix} = \boldsymbol{A}^5 \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{3}{8} & \frac{5}{8} \\ 0 & \frac{5}{16} & \frac{11}{16} \\ 0 & \frac{11}{32} & \frac{21}{32} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{5}{16} \\ * \\ * \end{pmatrix}$$

$$a_5 = \frac{5}{16}$$

(ii) Using MATLAB,

```
>> x = sym('x');
>> factor(det(A - x * eye(3)), x, 'FactorMode', 'full')

ans =

[-2, x, 2*x + 1, x - 1]
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$$\det(\boldsymbol{A} - x\boldsymbol{I}) = -2x(2x+1)(x-1)$$

The eigenvalues are $\left\{ 0, \frac{-1}{2}, 1 \right\}$.

When $x = 0$,

$$\boldsymbol{A} - x\boldsymbol{I} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

By inspection $(1, 0, 0)^T$ is a corresponding eigenvector.
When $x = \frac{-1}{2}$,

$$\boldsymbol{A} - x\boldsymbol{I} = \begin{pmatrix} \frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & \frac{1}{2} & 1 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

By inspection $(-4, 2, -1)^T$ is a corresponding eigenvector.
When $x = 1$,

$$\boldsymbol{A} - x\boldsymbol{I} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

By inspection $(1, 1, 1)^T$ is a corresponding eigenvector.
Therefore

$$\boldsymbol{A} = \begin{pmatrix} 1 & -4 & 1 \\ 0 & 2 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -4 & 1 \\ 0 & 2 & 1 \\ 0 & -1 & 1 \end{pmatrix}^{-1}$$

Then

$$
\begin{aligned}
\begin{pmatrix} a_n \\ * \\ * \end{pmatrix} &= \boldsymbol{A}^n \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & -4 & 1 \\ 0 & 2 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}^n \begin{pmatrix} 1 & -4 & 1 \\ 0 & 2 & 1 \\ 0 & -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix} \\
&= \begin{pmatrix} 1 & -4 & 1 \\ 0 & 2 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \left(-\frac{1}{2}\right)^n & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -2 \\ 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix} \\
&= \begin{pmatrix} 1 & -4 & 1 \\ 0 & 2 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \left(-\frac{1}{2}\right)^n & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -\frac{1}{6} \\ \frac{1}{3} \end{pmatrix} \\
&= \begin{pmatrix} 1 & -4 & 1 \\ 0 & 2 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -\frac{1}{6}\left(-\frac{1}{2}\right)^n \\ \frac{1}{3} \end{pmatrix} \\
&= \begin{pmatrix} \frac{2}{3}\left(-\frac{1}{2}\right)^n + \frac{1}{3} \\ * \\ * \end{pmatrix}
\end{aligned}
$$

Hence

$$a_n = \frac{2}{3}\left(-\frac{1}{2}\right)^n + \frac{1}{3}$$

(b)

$$(\boldsymbol{I} + \boldsymbol{D} + \boldsymbol{D}^2 + \dots)_{i,i} = \sum_{j=0}^{\infty} \lambda_i^j = \frac{1}{1 - \lambda_i}$$

where $\boldsymbol{D}$ is a diagonal matrix with entries $|\lambda_i| < 1$.

Since $\boldsymbol{I}+\boldsymbol{D}+\boldsymbol{D}^2+\dots$ is absolutely convergent,

$$\boldsymbol{I}+\boldsymbol{A}+\boldsymbol{A}^2+\dots=\boldsymbol{P}(\boldsymbol{I}+\boldsymbol{D}+\boldsymbol{D}^2+\dots)\boldsymbol{P}^{-1}=\boldsymbol{P}\begin{pmatrix} \frac{1}{1-\lambda_1} & 0 & \dots & 0 \\ 0 & \frac{1}{1-\lambda_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{1-\lambda_n} \end{pmatrix}\boldsymbol{P}^{-1}$$

Problem 4

(i) Using MATLAB,

$$\det(\boldsymbol{A}-x\boldsymbol{I})=-(x-1)(2x-1)^2$$

The eigenvalues are $\left\{1,\frac{1}{2}\right\}$.

When $x=1$,

$$\boldsymbol{A}-x\boldsymbol{I}=\begin{pmatrix} -\frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & -\frac{1}{3} \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

By inspection $(1,1,1)^T$ is a corresponding eigenvector. Then the orthonormal basis of E_1 is

$$\left\{\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$$

When $x=\frac{1}{2}$,

$$\boldsymbol{A}-x\boldsymbol{I}=\begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

By inspection $(1,-1,0)^T$ and $(1,0,-1)^T$ are corresponding eigenvectors.

Using Gram-Schmidt,

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}-\frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}=\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

Then the orthonormal basis of $E_{\frac{1}{2}}$ is

$$\left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \right\}$$

Therefore,

$$\boldsymbol{P}^T\boldsymbol{A}\boldsymbol{P} = \boldsymbol{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

where

$$\boldsymbol{P} = \begin{pmatrix} \frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} \\ \frac{\sqrt{3}}{3} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} \\ \frac{\sqrt{3}}{3} & 0 & -\frac{\sqrt{2}\sqrt{3}}{3} \end{pmatrix}$$

(ii) Using $\boldsymbol{P}^T = \boldsymbol{P}^{-1}$,

$$\boldsymbol{A}^n = \boldsymbol{P}\boldsymbol{D}^n\boldsymbol{P}^T$$

Then

$$\begin{aligned}
\lim_{n\to\infty} \boldsymbol{A}^n &= \boldsymbol{P}\left(\lim_{n\to\infty}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2^n} & 0 \\ 0 & 0 & \frac{1}{2^n} \end{pmatrix}\right)\boldsymbol{P}^T \\
&= \begin{pmatrix} \frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} \\ \frac{\sqrt{3}}{3} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} \\ \frac{\sqrt{3}}{3} & 0 & -\frac{\sqrt{2}\sqrt{3}}{3} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} & -\frac{\sqrt{2}\sqrt{3}}{3} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} \\ \frac{\sqrt{3}}{3} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} \\ \frac{\sqrt{3}}{3} & 0 & -\frac{\sqrt{2}\sqrt{3}}{3} \end{pmatrix}\begin{pmatrix} \frac{\sqrt{3}}{3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}
\end{aligned}$$

(iii) Yes. By inspection,

$$\lim_{n\to\infty} \boldsymbol{A}^n\boldsymbol{v} = \left(\lim_{n\to\infty} \boldsymbol{A}^n\right)\boldsymbol{v} \in \operatorname{ran}\left(\lim_{n\to\infty} \boldsymbol{A}^n\right) = E_1$$

Problem 5

(i) Since B is a basis, $\begin{pmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 & \boldsymbol{u}_4 \end{pmatrix}$ is invertible. Then

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{u}_2 & \boldsymbol{u}_1 & \boldsymbol{u}_3 & (\boldsymbol{u}_1 + \boldsymbol{u}_2) \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 & \boldsymbol{u}_4 \end{pmatrix}^{-1}$$

(ii) By linearity,

$$\begin{aligned} T(\boldsymbol{u}_1 + \boldsymbol{u}_2) &= \boldsymbol{u}_1 + \boldsymbol{u}_2 \\ T(\boldsymbol{u}_1 - \boldsymbol{u}_2) &= -(\boldsymbol{u}_1 - \boldsymbol{u}_2) \\ T(\boldsymbol{u}_3) &= \boldsymbol{u}_3 \\ T(\boldsymbol{u}_4 - \boldsymbol{u}_1 - \boldsymbol{u}_2) &= \boldsymbol{0} \end{aligned}$$

Since the union of their eigenspaces spans $\mathbb{R}^4$ and distinct eigenvalues correspond to distinct eigenvectors (Remark 6.2.5.3), the eigenvalues of $\boldsymbol{A}$ are $\{ 1, -1, 0 \}$ only.

(iii) Since B is a basis, $T(B) = \{ \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_1 + \boldsymbol{u}_2 \}$ spans the range of T. By the linear independence of B its basis is $\{ \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3 \}$.

(iv) By above, $\operatorname{span}\{ \boldsymbol{u}_4 - \boldsymbol{u}_1 - \boldsymbol{u}_2 \} \subseteq \ker T$. By rank-nullity and above, $\dim \ker T = 1$. By Definition 3.6.3 its basis is $\{ \boldsymbol{u}_4 - \boldsymbol{u}_1 - \boldsymbol{u}_2 \}$.

(v) By linearity,

$$\begin{aligned} T \circ S(\boldsymbol{v}) &= T(v_1\boldsymbol{u}_1 + v_2\boldsymbol{u}_2 + v_3\boldsymbol{u}_3 + v_4\boldsymbol{u}_4) \\ &= v_1T(\boldsymbol{u}_1) + v_2T(\boldsymbol{u}_2) + v_3T(\boldsymbol{u}_3) + v_4T(\boldsymbol{u}_4) \\ &= (v_2 + v_4)\boldsymbol{u}_1 + (v_1 + v_4)\boldsymbol{u}_2 + v_3\boldsymbol{u}_3 \end{aligned}$$