MA2002 Assignment 3

Xu Junheng Marcus

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Problem 1

Given $f''(x) + f(x) = 0 \quad \forall x$ and $f(0) = f'(0) = 1$,

(i) Let

$$g(x) = f(x)\sin x + f'(x)\cos x$$

Then

$$\begin{aligned} g'(x) &= f(x)\cos x + f'(x)\sin x + f''(x)\cos x - f'(x)\sin x \\ &= (f(x) + f''(x))\cos x \\ &= 0 \end{aligned}$$

$g(x)$ is constant, so $g(x) = g(0) = 1$. In other words,

$$f(x)\sin x + f'(x)\cos x = 1$$

(ii) Let

$$h(x) = f(x)\cos x - f'(x)\sin x$$

Then

$$\begin{aligned} h'(x) &= f'(x)\cos x - f(x)\sin x - f''(x)\sin x + f'(x)\cos x \\ &= -(f(x) + f''(x))\sin x \\ &= 0 \end{aligned}$$

$h(x)$ is constant, so $h(x) = h(0) = 1$. In other words,

$$f(x)\cos x - f'(x)\sin x = 1$$

(iii) Using (i) and (ii),

$$\begin{aligned} \sin x(f(x)\sin x + f'(x)\cos x) + \\ \cos x(f(x)\cos x - f'(x)\sin x) &= \sin x + \cos x \\ f(x)(\sin^2 x + \cos^2 x) &= \sin x + \cos x \\ f(x) &= \sin x + \cos x \end{aligned}$$

Problem 2

$$f(x) = (x+1)^{4/7}(x-1)^{3/7}$$

(i)

$$\begin{aligned}
f'(x) &= \frac{4}{7}(x+1)^{-3/7}(x-1)^{3/7} + \frac{3}{7}(x-1)^{-4/7}(x+1)^{4/7} \\
&= \frac{4}{7}\left(\frac{x-1}{x+1}\right)^{3/7} + \frac{3}{7}\left(\frac{x+1}{x-1}\right)^{4/7} \\
&= \frac{4}{7}t^{3/7} + \frac{3}{7}t^{-4/7}
\end{aligned}$$

where $t = \frac{x-1}{x+1}$. When $t < 0$ and $f'(x) \leq 0$,

$$\begin{aligned}
\frac{4}{7}t^{3/7} &\leq \frac{-3}{7}t^{-4/7} \\
\frac{x-1}{x+1} = t &\leq \frac{-3}{7}\cdot\frac{7}{4} = \frac{-3}{4} \\
x &\leq \frac{1}{7}
\end{aligned}$$

Note that $t > 0 \implies f'(x) > 0$. Then by continuity of f',

$$f'(x)\begin{cases} > 0 & x \in (-\infty, -1) \\ < 0 & x \in (-1, 1/7) \\ = 0 & x = 1/7 \\ > 0 & x \in (1/7, 1) \\ > 0 & x \in (1, \infty) \end{cases}$$

f is increasing in $(-\infty, -1)$ and $(1/7, \infty)$ and decreasing in $(-1, 1/7)$.

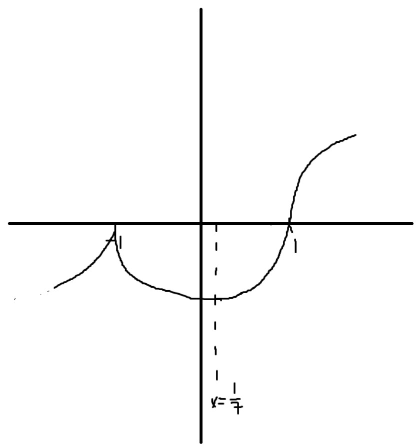
(ii)

$$\begin{aligned}
f''(x) &= \left(\frac{12}{49}t^{-4/7} - \frac{12}{49}t^{-11/7}\right)\frac{\mathrm{d}t}{\mathrm{d}x} \\
&= \frac{12}{49}t^{-4/7}\left(1 - \frac{1}{t}\right)\frac{2}{(x+1)^2}
\end{aligned}$$

The local minimum of f is at $x = \frac{1}{7}$ where $f(\frac{1}{7}) = \left(\frac{8}{7}\right)^{4/7}\left(\frac{-6}{7}\right)^{3/7} = \frac{-4}{7}\sqrt[7]{54}$.

(iii) $\operatorname{sgn} f''(x) = \operatorname{sgn}(1 - \frac{1}{t}) = \operatorname{sgn}\frac{-2}{x-1}$. Therefore f concaves up in $(-\infty, 1)$ and concaves down in $(1, \infty)$.

(iv) Note that $t \neq 1$. The only inflection point where $f''(x) = 0$ is $(1, 0)$ where $t = 0$.



(v)

Problem 3

(a) (i)

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(h)-f(0)}{h} \\ &= \lim_{h\to 0} 1 + 3h\sin\left(\frac{1}{h}\right) \\ &= 1 \end{aligned}$$

since h dominates $\sin\left(\frac{1}{h}\right)$.

(ii)

$$\begin{aligned} f'(x) &= 1 + (3x^2)\cos\left(\frac{1}{x}\right)\frac{-1}{x^2} + 6x\sin\left(\frac{1}{x}\right) \\ &= 1 - 3\cos\left(\frac{1}{x}\right) + 6x\sin\left(\frac{1}{x}\right) \end{aligned}$$

for $x \neq 0$. Using Archimedean property of $\mathbb{R}$ we can choose a positive integer $n > \frac{1}{2\pi\delta}$. Then $-\delta < c = \frac{1}{2n\pi} < \delta$ and $f'(c) = 1 - 3\cos 2n\pi < 0$.

(b) (i) Note that $1 \leq 2 + \sin\left(\frac{1}{x}\right) \leq 3$.

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(h)-f(0)}{h} \\ &= \lim_{h\to 0} h[2 + \sin\left(\frac{1}{h}\right)] \\ &= 0 \end{aligned}$$

since h dominates the bounded term $2 + \sin\left(\frac{1}{h}\right)$. Moreover, since $x \neq 0 \implies x^2 > 0$, $f(x) > f(0) = 0$. Therefore f has a local minimum at 0.

(ii)

$$\begin{aligned} f'(x) &= x^2 \cos\left(\frac{1}{x}\right)\frac{-1}{x^2} + 2x[2 + \sin\left(\frac{1}{x}\right)] \\ &= 2x[2 + \sin\left(\frac{1}{x}\right)] - \cos\left(\frac{1}{x}\right) \end{aligned}$$

for $x \neq 0$. Using Archimedean property of $\mathbb{R}$ we can choose $n > \frac{(1/\pi\delta)-1}{2}$. Then choose $-\delta < c_1 = \frac{-1}{(2n+1)\pi} < 0$ then $f'(c_1) = \frac{-2}{(2n+1)\pi}(2 + \sin[-(2n+1)\pi)] + 1 = 1 - \frac{4}{(2n+1)\pi} > 0$. Likewise we can choose $n > \frac{1}{2\pi\delta}$. Then $\delta > c_2 = \frac{1}{2n\pi} > 0$ then $f'(c_2) = \frac{2}{n\pi} - 1 < 0$.

Problem 4

The revenue is given by

$$r(p) = \begin{cases} 0 & p \geq 31 \\ (11000 + 1000(20 - p))p & 16 < p < 31 \\ 15000p & p \leq 16 \end{cases}$$

where p is the ticket price. When $p \leq 16$, maximum revenue is achieved when $p = 16$. When $16 < p < 31$, the revenue

$$r(p) = 1000p(31 - p) = 1000[(\frac{31}{2})^2 - (p - \frac{31}{2})^2]$$

strictly increases when p decreases. Therefore, the owners of the team should set the ticket price to $p = 16$ to maximize their revenue.

Problem 5

The cost of the material is given by

$$c(l) = 24l^2 + 4(12)\frac{125}{l} = 24(l^2 + \frac{250}{l})$$

where l is the side length of the square base. The derivative is

$$\frac{\mathrm{d}c}{\mathrm{d}l} = 24(2l - \frac{250}{l^2}) = 48(l - \frac{125}{l^2})$$

Then $c'(l) = 0$ when $l = \sqrt[3]{125} = 5$. By second derivative test,

$$\frac{\mathrm{d}}{\mathrm{d}l}\left(l - \frac{125}{l^2}\right) = 1 + \frac{250}{l^3} > 0$$

the cost of the material is least when side length is 5m and height is 5m.

Problem 6

Let ρ be the ratio between the arc AB after the sector is cut out and the circumference of the original piece of paper. Clearly $\rho \in [0, 1]$. Let $r_0 = 5$ be the original radius. Then the radius of the base of the cup is ρr_0. The capacity of the cup is given by

$$V(\rho) = \frac{\pi}{3}(\rho r_0)^2\sqrt{r_0^2 - (\rho r_0)^2} = \frac{\pi r_0^3}{3}\rho^2\sqrt{1-\rho^2}$$

The derivative is

$$\frac{\mathrm{d}V}{\mathrm{d}\rho} = \frac{\pi r_0^3}{3}\left(2\rho\sqrt{1-\rho^2} + \rho^2\frac{-\rho}{\sqrt{1-\rho^2}}\right)$$

When $V'(\rho) = 0$, the unique stationary point is where

$$\begin{aligned}
2\sqrt{1-\rho^2} &= \frac{\rho^2}{\sqrt{1-\rho^2}} \\
2(1-\rho^2) &= \rho^2 \\
\rho^2 &= \frac{2}{3}
\end{aligned}$$

Since $V(\rho) > 0$ except that $V(0) = V(1) = 0$, $V(\rho)$ has an absolute maximum by extreme value theorem. Since $\rho = \sqrt{\frac{2}{3}}$ is the unique stationary point of $V(\rho)$, it is the absolute maximum of V. Then the maximum capacity of the cup is

$$V = \frac{\pi r_0^3}{3}\left(\frac{2}{3}\sqrt{1-\frac{2}{3}}\right) = \frac{250\pi}{9\sqrt{3}}$$

Problem 7

From Tutorial 5 Part II, if f is twice differentiable, there exists $c \in (a, b)$ such that

$$f(b) = f(a) + (b-a)f'(a) + \frac{f''(c)}{2}(b-a)^2$$

Let $g(x) = f(a + b - x)$. Then g is twice differentiable, $g'(x) = -f'(x)$ and $g''(x) = f''(x)$. Applying above, there exists $c \in (a, b)$ such that

$$\begin{aligned}
f(a) = g(b) &= g(a) + (b-a)g'(a) + \frac{g''(c)}{2}(b-a)^2 \\
&= f(b) + (a-b)f'(a) + \frac{f''(c)}{2}(a-b)^2
\end{aligned}$$

Therefore there exists $u \in (0, \frac{1}{2})$ and $v \in (\frac{1}{2}, 1)$ such that

$$\begin{aligned} f(1) - f(0) &= (f(\frac{1}{2}) - f(0)) - (f(\frac{1}{2}) - f(1)) \\ &= (\frac{1}{2}f'(0) + \frac{f''(u)}{2}\left(\frac{1}{2}\right)^2) - (\frac{1}{2}f'(1) + \frac{f''(v)}{2}\left(\frac{1}{2}\right)^2) \\ &= \frac{f''(u) - f''(v)}{8} \end{aligned}$$

Then $c = \arg\max_{\{u,v\}} \left|f''(x)\right| \in (0,1)$ satisfies

$$\left|f''(c)\right| \geq \frac{\left|f''(u)\right| + \left|f''(v)\right|}{2} \geq \left|\frac{f''(u) - f''(v)}{2}\right| = 4\left|f(1) - f(0)\right|$$