National University of Singapore

Semester 1, 2021/2022 **MA2001** **Homework Assignment 4**

(a) Use A4 size paper and pen (blue or black ink) to write your answers. (Students may also type out the answers or write the answers electronically using their devices.)

(b) Write down your student number and full name clearly on the top left of every page of the answer scripts.

(c) Write the page number on the top right corner of each page of answer scripts.

(d) This assignment consists of 4 pages and 5 questions. Total mark is 60 marks. Some hints are given at the end of this homework set.

(e) To submit your answer scripts, do the following:

(i) Scan or take pictures of your work (make sure the images can be read clearly).

(ii) Merge all your answers into one pdf file. Arrange them in order of the questions.

(iii) Name the pdf file by **StudentNo HW4** (e.g. **A123456R HW4**).

(iv) Upload your pdf into the LumiNUS folder Homework 4 submission.

(f) Deadline for submission is 13 November, 2021 by 11.59pm. **Late submission will not be accepted.**

1. (a) Let $\boldsymbol{A} = \begin{pmatrix} 0 & 2 & -2 \\ 1 & -1 & 0 \\ -1 & 3 & -2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

   (i) [3 marks] Find all least squares solutions to the equation $\boldsymbol{Ax} = \boldsymbol{b}$.

   (ii) [2 marks] Find the projection of $\boldsymbol{b}$ onto the subspace

$$V = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \right\}.$$

   (iii) [2 marks] Let $\boldsymbol{w}$ be the projection of $\boldsymbol{b}$ onto V found in (ii). Find, without working or using MATLAB, the solution set to $\boldsymbol{Ax} = \boldsymbol{w}$. Explain how your answer is derived.

   (iv) [3 marks] Let $\boldsymbol{B} = \begin{pmatrix} 4 & 3 & -3 \\ 3 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix}$.

   Find a least squares solution $\boldsymbol{u}$ of $\boldsymbol{Ax} = \boldsymbol{b}$ such that $\boldsymbol{u}$ is in the column space of $\boldsymbol{B}$.

   (b) [3 marks] Let $\boldsymbol{C}$ be an $m \times k$ matrix, $\boldsymbol{D}$ a $k \times n$ matrix, and $\boldsymbol{f}$ a vector in $\mathbb{R}^m$. Suppose $\boldsymbol{v} \in \mathbb{R}^n$ is such that $\boldsymbol{Dv}$ is a least squares solution to $\boldsymbol{Cx} = \boldsymbol{f}$. Show that the projection of $\boldsymbol{f}$ onto the columns space of $\boldsymbol{CD}$ is $\boldsymbol{CDv}$.

2. (a) Let $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}$.

   (i) [3 marks] Use Gram-Schmidt process to convert the columns of $\boldsymbol{A}$ to an orthonormal basis T for $\mathbb{R}^4$. Show your working and give the exact values for the entries of the basis. Do not use MATLAB.

   (ii) [2 marks] Find the coordinate vectors of the columns of $\boldsymbol{A}$ with respect to the basis T. Write them as column vectors.

   (iii) [3 marks] Use (i) and (ii) above to express the matrix as $\boldsymbol{A} = \boldsymbol{QU}$, where $\boldsymbol{Q}$ is an orthogonal matrix and $\boldsymbol{U}$ is an upper triangular matrix. Briefly explain how you obtain the answer.

   (b) Prove the following statements.

   (i) [3 marks] The eigenvalues of an orthogonal matrix are ± 1.

   (ii) [3 marks] Suppose $\boldsymbol{A}$ is an diagonalizable orthogonal matrix, then $\boldsymbol{A}^2 = \boldsymbol{I}$.

3. (a) Consider the following recursion formula

$$a_n = \frac{1}{2}(a_{n-1} + a_{n-2}), \quad n \geq 0$$

with $a_0 = 1$, and $a_1 = 0$.

(i) [3 marks] Find the recurrence matrix $\boldsymbol{A}$ and use it to find a_5. Give your answers in fraction form.

(ii) [3 marks] By diagonalizing $\boldsymbol{A}$, find the general formula for a_n. Show all your workings clearly, however, you do not need to show your elementary row operations.

(b) [3 marks] Suppose $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$ where $\boldsymbol{P}$ is an invertible matrix and $\boldsymbol{D} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$ is a diagonal matrix such that $|\lambda_i| < 1$ for all $i = 1, ..., .n$. Find

$$\boldsymbol{I} + \boldsymbol{A} + \boldsymbol{A}^2 + \boldsymbol{A}^3 + \cdots .$$

Leave your answer in terms of $\boldsymbol{P}$ and λ_i, $i = 1, ..., n$.

4. Let

$$\boldsymbol{A} = \begin{pmatrix} 2/3 & 1/6 & 1/6 \\ 1/6 & 2/3 & 1/6 \\ 1/6 & 1/6 & 2/3 \end{pmatrix}.$$

(i) [5 marks] Orthogonally diagonalize $\boldsymbol{A}$, that is, find an orthogonal matrix $\boldsymbol{P}$ such that $\boldsymbol{P}^T\boldsymbol{A}\boldsymbol{P}$ is a diagonal matrix. Give the exact values for the entries of $\boldsymbol{P}$. Include the steps of finding the eigenvalues and eigenvectors. However, you do not need to show your elementary row operations.

(ii) [3 marks] Use (i) to find the limit of $\boldsymbol{A}^n$ as $n \to \infty$.

(iii) [3 marks] Is it true that for any vector $\boldsymbol{v} \in \mathbb{R}^3$, the limit of $\boldsymbol{A}^n\boldsymbol{v}$ is a vector in the eigenspace of $\boldsymbol{A}$ associated to 1? Justify your answer.

5. Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation with $\boldsymbol{A}$ as the standard matrix, and $B = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\}$ be a basis for $\mathbb{R}^4$ such that

$$T(\boldsymbol{u}_1) = \boldsymbol{u}_2, \quad T(\boldsymbol{u}_2) = \boldsymbol{u}_1, \quad T(\boldsymbol{u}_3) = \boldsymbol{u}_3, \quad T(\boldsymbol{u}_4) = \boldsymbol{u}_1 + \boldsymbol{u}_2.$$

(i) [2 marks] Find $\boldsymbol{A}$ in terms of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4$.

(ii) [4 marks] Find all the eigenvalues of $\boldsymbol{A}$.

(iii) [2 marks] Find a basis for the range of T in terms of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4$. Justify your answer.

(iv) [2 marks] Find a basis for the kernel of T in terms of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4$. Justify your answer.

(v) [3 marks] Let $S : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation defined as follow: For every $\boldsymbol{v} \in \mathbb{R}^4$, write $\boldsymbol{v} = (v_1, v_2, v_3, v_4)$. Then

$$S(\boldsymbol{v}) = v_1\boldsymbol{u}_1 + v_2\boldsymbol{u}_2 + v_3\boldsymbol{u}_3 + v_4\boldsymbol{u}_4.$$

Compute $T \circ S(\boldsymbol{v})$ in terms of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4$.

Hints

(I) Q1(b) For $\boldsymbol{Dv}$ to be a least square solution to $\boldsymbol{Cx} = \boldsymbol{f}$, what is the linear system in terms of $\boldsymbol{C}$ and $\boldsymbol{f}$ of which $\boldsymbol{Dv}$ is a solution? Deduce from there that $\boldsymbol{v}$ is a least squares solution of $\boldsymbol{CDx} = \boldsymbol{f}$.

(II) Q2(a)(iii) What is the transition matrix from the basis T to the basis formed by the columns of $\boldsymbol{A}$?

(III) Q2(b)(i) You may apply Ex 5 Q32(a). Q2(b)(ii) Diagonalize $\boldsymbol{A}$.

(IV) Q3(b) Recall that if $|x| < 1$, then $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$.

(V) Q4(iii) Express $\boldsymbol{v}$ as a linear combination of a basis consisting of eigenvectors of $\boldsymbol{A}$.

(VI) Q5

- (i) Use "stacking" method.
- (ii) Try to form eigenvectors as linear combinations of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4$.
- (iii) How can we express $R(T)$ in terms of a basis for the domain?
- (iv) How is $\ker(T)$ related to $\boldsymbol{A}$?
- (v) Express $S(\boldsymbol{v})$ as a product of a square matrix and a column vector.