MA2001 Assignment 3

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Problem 1

(i) Apply Gauss-Jordan elimination on $\boldsymbol{A}^T$,

$$\begin{pmatrix} 1 & 2 & 1 & 3 \\ 2 & 4 & 2 & 6 \\ 2 & 1 & -4 & 3 \\ 3 & 3 & -3 & 6 \\ 3 & 2 & -5 & 1 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore

$$\{ (1,2,2,3,3), (2,4,1,3,2), (3,6,3,6,1) \}$$

is a basis for $\boldsymbol{R}(\boldsymbol{A})$ since it is a linearly independent subset of row vectors of $\boldsymbol{A}$ that spans $\boldsymbol{R}(\boldsymbol{A})$.

(ii) Apply Gauss-Jordan elimination on $\boldsymbol{A}$,

$$\begin{pmatrix} 1 & 2 & 2 & 3 & 3 \\ 2 & 4 & 1 & 3 & 2 \\ 1 & 2 & -4 & -3 & -5 \\ 3 & 6 & 3 & 6 & 1 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since columns 2 and 4 are non-pivot, the extended basis for $\mathbb{R}^5$ is

$$\{ (1,2,2,3,3), (2,4,1,3,2), (3,6,3,6,1), (0,1,0,0,0), (0,0,0,1,0) \}$$

(iii) From above, S corresponds to columns 2, 4, 5. Since

$$\begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 \\ 4 \\ 2 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -4 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -3 \\ 6 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 2 \\ 4 \\ 2 \\ 6 \end{pmatrix}$$

S spans $\boldsymbol{C}(\boldsymbol{A})$. Moreover $\operatorname{rank}(\boldsymbol{A}) = 3$, so S is a basis for $\boldsymbol{C}(\boldsymbol{A})$.

(iv) No. Let $\boldsymbol{M}$ be a matrix in row echelon form. Suppose $\boldsymbol{C}(\boldsymbol{M}) = \boldsymbol{C}(\boldsymbol{A})$. Then $\operatorname{rank}(\boldsymbol{M}) = 3$. Note that

$$\begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 3 \\ 2 \\ -5 \\ 1 \end{pmatrix} \notin \boldsymbol{C}(\boldsymbol{M})$$

since basis vectors of $\boldsymbol{C}(\boldsymbol{M})$ are standard vectors.

Problem 2

(i) $U_1 = \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ and $U_2 = \operatorname{span}\{\boldsymbol{v}_3, \boldsymbol{v}_4\}$. Sets $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ and $\{\boldsymbol{v}_3, \boldsymbol{v}_4\}$ are subsets of S, hence linearly independent. Therefore $\dim U_1 = 3$ and $\dim U_2 = 2$. Consider $\boldsymbol{u} = a\boldsymbol{v}_1 + b\boldsymbol{v}_2 + c\boldsymbol{v}_3 = d\boldsymbol{v}_3 + e\boldsymbol{v}_4 \in U_1 \cap U_2$ for some $a, b, c, d, e \in \mathbb{R}$. Then $a\boldsymbol{v}_1 + b\boldsymbol{v}_2 + (c-d)\boldsymbol{v}_3 - e\boldsymbol{v}_4 = \mathbf{0}$. Since S is linearly independent we have $a = b = e = 0$ and $c = d$. So $\dim U_1 \cap U_2 = \dim \operatorname{span} \boldsymbol{v_3} = 1$.

(ii) $W = \operatorname{span}\{\boldsymbol{v}_1 + \boldsymbol{v}_2, \boldsymbol{v}_3 + \boldsymbol{v}_4\}$. The set $\{\boldsymbol{v}_1 + \boldsymbol{v}_2, \boldsymbol{v}_3 + \boldsymbol{v}_4\}$ is linearly independent, since $a(\boldsymbol{v}_1 + \boldsymbol{v}_2) + b(\boldsymbol{v}_3 + \boldsymbol{v}_4) = \mathbf{0} \iff a = b = 0$ by linear independence of S. Therefore $\dim W = 2$. WLOG suppose $\boldsymbol{v_1} = a(\boldsymbol{v}_1 + \boldsymbol{v}_2) + b(\boldsymbol{v}_3 + \boldsymbol{v}_4) \in W$ for some $a, b \in \mathbb{R}$. Then $(a-1)\boldsymbol{v}_1 + a\boldsymbol{v}_2 + b\boldsymbol{v}_3 + b\boldsymbol{v}_4 = \mathbf{0}$ contradicts linearly independence of the set $\{\boldsymbol{v}_1 + \boldsymbol{v}_2, \boldsymbol{v}_3 + \boldsymbol{v}_4\}$.

(iii) $\dim X_1 + \dim X_2 - \dim X_1 \cap X_2 = \dim X_1 + X_2 \leq \dim V = \dim \operatorname{span} S = 4$. Then $2 \leq \dim X_1 \cap X_2 \leq \min\{\dim X_1, \dim X_2\} = 3$. Take $X_1 = X_2$. Then $\dim X_1 \cap X_2 = \dim X_1 = \dim X_2 = 3$. Take $X_1 = \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ and $X_2 = \operatorname{span}\{\boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$. Then $\dim X_1 \cap X_2 = \dim \operatorname{span} v_2, v_3 = 2$ by above.

(iv) No. Suppose T is a subspace of V and $S \subseteq T \subset V$. Then $V = \operatorname{span} S \subseteq T \subset V$ contradicts.

Problem 3

(a)

$$D = \begin{pmatrix} x+1 & 1 & 1 & 1 \\ 0 & x-2 & -3 & -3 \\ 0 & 0 & (x+1)(x-2) & x+1 \\ 0 & 0 & x-2 & 1 \end{pmatrix}$$

It suffices to consider $x = -1, 2$ and otherwise.

When $x = -1$, $\operatorname{rank}(\boldsymbol{D}) = 2$.

$$D = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & -3 & -3 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 0 & 1 & 0 & 4/3 \\ 0 & 0 & 1 & -1/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

When $x = 2$, $\operatorname{rank}(\boldsymbol{D}) = 3$.

$$D = \begin{pmatrix} 3 & 1 & 1 & 1 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Otherwise, third and fourth row are multiples of each other, so $\operatorname{rank}(\boldsymbol{D}) = 3$. Therefore $\operatorname{rank}(\boldsymbol{D}) = 2$ if $x = -1$ and $\operatorname{rank}(\boldsymbol{D}) = 3$ otherwise.

(b) (i) If $\boldsymbol{A}$ is of full column rank, then $\operatorname{null}(\boldsymbol{A}) = \{\mathbf{0}\}$ by rank-nullity theorem. Let T_A be the injective linear map from $\boldsymbol{C}(\boldsymbol{B})$ to $\boldsymbol{C}(\boldsymbol{AB})$. Let S be a basis for $\boldsymbol{C}(\boldsymbol{B})$ and $T_A(S) = \{T_A(\boldsymbol{v}) \mid \boldsymbol{v} \in S\}$. Then $\sum_{\boldsymbol{v} \in S} \alpha_i T_A(\boldsymbol{v}) = T_A(\sum_{\boldsymbol{v} \in S} \alpha_i \boldsymbol{v}) = \mathbf{0} \iff \sum_{\boldsymbol{v} \in S} \alpha_i \boldsymbol{v} = \mathbf{0} \iff \alpha_i = 0$ and $\boldsymbol{x} = T_A(\sum_{\boldsymbol{v} \in S} \alpha_i \boldsymbol{v}) \in \boldsymbol{C}(\boldsymbol{AB}) \iff \boldsymbol{x} = \sum_{\boldsymbol{v} \in S} \alpha_i T_A(\boldsymbol{v}) \in \operatorname{span} T_A(S)$. Since $T_A(S)$ is linearly independent and it spans $\boldsymbol{C}(\boldsymbol{AB})$, $T_A(S)$ is a basis for $\boldsymbol{C}(\boldsymbol{AB})$. Since T_A is injective, $\operatorname{rank} \boldsymbol{AB} = \dim \boldsymbol{C}(\boldsymbol{AB}) = |T_A(S)| = |S| = \dim \boldsymbol{C}(\boldsymbol{B}) = \operatorname{rank} \boldsymbol{B}$.

(ii) $\forall \boldsymbol{M} \operatorname{rank} \boldsymbol{M} = \operatorname{rank} \boldsymbol{M}^T$ since column and row ranks are equal. Then $\operatorname{rank} \boldsymbol{AB} = \operatorname{rank} (\boldsymbol{AB})^T = \operatorname{rank} \boldsymbol{B}^T \boldsymbol{A}^T = \operatorname{rank} \boldsymbol{A}^T = \operatorname{rank} \boldsymbol{A}$ by above.

Problem 4

(i) Let

$$\boldsymbol{N} = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ -1 & 2 & 0 & -1 & 0 & 1 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 & -1/2 & 1/2 \\ 0 & 0 & 1 & 0 & -1/2 & -1/2 \end{pmatrix}$$

Since $\boldsymbol{N}^T\boldsymbol{A}^T = (\boldsymbol{A}\boldsymbol{N})^T = \boldsymbol{0}$,

$$\begin{pmatrix} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 & -1/2 & 1/2 \\ 0 & 0 & 1 & 0 & -1/2 & -1/2 \end{pmatrix} \boldsymbol{A}^T = \boldsymbol{0}$$

Then

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1/2 & -1/2 \\ 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1/2 & 1/2 & 0 & 1 & 0 \\ 0 & -1/2 & 1/2 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{pmatrix}$$

(ii) The general solution is

$$\boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + a \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} -1 \\ 2 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

where $a, b, c \in \mathbb{R}$

(iii) True. $\dim \boldsymbol{C}(\boldsymbol{A}) = \operatorname{rank} \boldsymbol{A} = 6 - \dim \operatorname{null} A = 3 = \dim \mathbb{R}^3$ and $\boldsymbol{C}(\boldsymbol{A}) \subseteq \mathbb{R}^3$. Let S be a basis for $\boldsymbol{C}(\boldsymbol{A})$. Suppose $\exists \boldsymbol{v} \in \mathbb{R}^3 \setminus \boldsymbol{C}(\boldsymbol{A})$. Then $\boldsymbol{v} \notin S$. Then $S \cup \{\boldsymbol{v}\}$ is a set of 4 linearly independent vectors, which contradicts. Therefore $\boldsymbol{C}(\boldsymbol{A}) = \mathbb{R}^3$ i.e. $\forall \boldsymbol{c} \in \mathbb{R}^3 \quad \exists \boldsymbol{x} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{c}$.

(iv) Since $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0} \implies \boldsymbol{B}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$, $\operatorname{null}(\boldsymbol{A}) \subseteq \operatorname{null}(\boldsymbol{B}\boldsymbol{A})$ in general. Moreover $\boldsymbol{B}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0} \implies \boldsymbol{A}\boldsymbol{x} = \boldsymbol{B}^{-1}\boldsymbol{0} = \boldsymbol{0}$, so $\operatorname{null}(\boldsymbol{B}\boldsymbol{A}) \subseteq \operatorname{null}(\boldsymbol{A})$. Therefore $\operatorname{null}(\boldsymbol{B}\boldsymbol{A}) = \operatorname{null}(\boldsymbol{A})$. The basis for the null space of $\boldsymbol{B}\boldsymbol{A}$ is

$$\{ \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \}$$

Problem 5

$$\begin{pmatrix} \boldsymbol{w}_1 & \boldsymbol{w}_2 & \boldsymbol{w}_3 \end{pmatrix} = \begin{pmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 \end{pmatrix} \begin{pmatrix} -1 & -2 & -2 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{pmatrix} \begin{pmatrix} 2 & 1 & 3 \\ 1 & -2 & 0 \\ 3 & -2 & 3 \end{pmatrix}$$

(i)

$$[\boldsymbol{w}_1]_S = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, [\boldsymbol{w}_2]_S = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, [\boldsymbol{w}_3]_S = \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix}$$

$$[\boldsymbol{w}_1]_T = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, [\boldsymbol{w}_2]_T = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}, [\boldsymbol{w}_3]_T = \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}$$

(ii) The transition matrix is

$$\begin{pmatrix} 2 & 1 & 3 \\ 1 & -2 & 0 \\ 3 & -2 & 3 \end{pmatrix} \begin{pmatrix} -1 & -2 & -2 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} -4 & -7 & 5 \\ -3 & -8 & 6 \\ -8 & -18 & 13 \end{pmatrix}$$

(iii) The matrix inverse is

$$\begin{pmatrix} -1 & -2 & -2 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 3 \\ 1 & -2 & 0 \\ 3 & -2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -4/3 & -1/3 & 2/3 \\ 3 & 4 & -3 \\ 10/3 & 16/3 & -11/3 \end{pmatrix}$$

Hence

$$[\boldsymbol{v}_1]_S = \begin{pmatrix} -4/3 \\ 3 \\ 10/3 \end{pmatrix}, [\boldsymbol{v}_2]_S = \begin{pmatrix} -1/3 \\ 4 \\ 16/3 \end{pmatrix}, [\boldsymbol{v}_3]_S = \begin{pmatrix} 2/3 \\ -3 \\ -11/3 \end{pmatrix}$$

(iv) From above,

$$\boldsymbol{C} = \begin{pmatrix} -4 & -7 & 5 \\ -3 & -8 & 6 \\ -8 & -18 & 13 \end{pmatrix}$$

(v)

$$\begin{aligned} \begin{pmatrix} \boldsymbol{w}_1^T \\ \boldsymbol{w}_2^T \\ \boldsymbol{w}_3 \end{pmatrix} \begin{pmatrix} \boldsymbol{w}_1 & \boldsymbol{w}_2 & \boldsymbol{w}_3 \end{pmatrix} &= \begin{pmatrix} -1 & -2 & -2 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}^T \begin{pmatrix} \boldsymbol{u}_1^T \\ \boldsymbol{u}_2^T \\ \boldsymbol{u}_3 \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 \end{pmatrix} \begin{pmatrix} -1 & -2 & -2 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 1 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & -1 & -2 \\ -1 & 6 & 3 \\ -2 & 3 & 5 \end{pmatrix} \begin{pmatrix} -1 & -2 & -2 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 25 & 20 & 0 \\ 20 & 18 & 3 \\ 0 & 3 & 5 \end{pmatrix} \end{aligned}$$

We have $\boldsymbol{w}_1 \cdot \boldsymbol{w}_1 = 25$, $\boldsymbol{w}_2 \cdot \boldsymbol{w}_2 = 18$, $\boldsymbol{w}_3 \cdot \boldsymbol{w}_3 = 5$, $\boldsymbol{w}_1 \cdot \boldsymbol{w}_2 = \boldsymbol{w}_2 \cdot \boldsymbol{w}_1 = 20$, $\boldsymbol{w}_1 \cdot \boldsymbol{w}_3 = \boldsymbol{w}_3 \cdot \boldsymbol{w}_1 = 0$, $\boldsymbol{w}_2 \cdot \boldsymbol{w}_3 = \boldsymbol{w}_3 \cdot \boldsymbol{w}_2 = 3$.

Problem 6

(i)

$$\begin{aligned}
\boldsymbol{v}_1 = \boldsymbol{u}_1 &= \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} \\
\boldsymbol{v}_2 = \boldsymbol{u}_2 &- \frac{\boldsymbol{u}_2 \cdot \boldsymbol{v}_1}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1}\boldsymbol{v}_1 \\
&= \begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \end{pmatrix} - \frac{3}{5}\begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\,{}^{8}\!/_{5} \\ {}^{4}\!/_{5} \\ 0 \\ 0 \end{pmatrix} \\
\boldsymbol{v}_3 = \boldsymbol{u}_3 &- \frac{\boldsymbol{u}_3 \cdot \boldsymbol{v}_1}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1}\boldsymbol{v}_1 - \frac{\boldsymbol{u}_3 \cdot \boldsymbol{v}_2}{\boldsymbol{v}_2 \cdot \boldsymbol{v}_2}\boldsymbol{v}_2 \\
&= \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} - \frac{1}{5}\begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} - \frac{-\,{}^{8}\!/_{5}}{{}^{16}\!/_{5}}\begin{pmatrix} -{}^{8}\!/_{5} \\ {}^{4}\!/_{5} \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} {}^{4}\!/_{5} \\ -\,{}^{2}\!/_{5} \\ 2 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -{}^{8}\!/_{5} \\ {}^{4}\!/_{5} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}
\end{aligned}$$

(ii)

$$\begin{aligned}
\mathrm{proj}_V(\boldsymbol{w}) &= \frac{\boldsymbol{w} \cdot \boldsymbol{u}_1}{\boldsymbol{u}_1 \cdot \boldsymbol{u}_1}\boldsymbol{u}_1 + \frac{\boldsymbol{w} \cdot \boldsymbol{u}_2}{\boldsymbol{u}_2 \cdot \boldsymbol{u}_2}\boldsymbol{u}_2 + \frac{\boldsymbol{w} \cdot \boldsymbol{u}_3}{\boldsymbol{u}_3 \cdot \boldsymbol{u}_3}\boldsymbol{u}_3 \\
&= \frac{11}{5}\begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} + \frac{9}{5}\begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} {}^{17}\!/_{30} \\ 8 \\ {}^{1}\!/_{3} \\ {}^{1}\!/_{6} \end{pmatrix}
\end{aligned}$$

(iii)

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & {}^{1}\!/_{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Normalizing, we have

$$\boldsymbol{n} = \frac{\sqrt{5}}{2}\begin{pmatrix} 0 \\ 0 \\ -{}^{1}\!/_{2} \\ 1 \end{pmatrix}$$

(iv) No. Let $\{\, \boldsymbol{n}, \boldsymbol{n}' \,\}$ be linearly independent. $S \cup \{\, \boldsymbol{n}, \boldsymbol{n}' \,\} \subseteq \mathbb{R}^4$ is linearly dependent. Then we can write $\boldsymbol{n}' = a\boldsymbol{n} + \boldsymbol{u}$ where $\boldsymbol{u} = b\boldsymbol{u}_2 + c\boldsymbol{u}_2 + d\boldsymbol{u}_3 \in V$ for some $a, b, c, d \in \mathbb{R}$. Then

$$\boldsymbol{n}' \cdot \boldsymbol{u} = a(\boldsymbol{n} \cdot \boldsymbol{u}) + \boldsymbol{u} \cdot \boldsymbol{u} = \boldsymbol{u} \cdot \boldsymbol{u} \geq 0$$

By (a) equality holds, so $\boldsymbol{u} = \boldsymbol{0}$. Then $\boldsymbol{n}' = \boldsymbol{n}$ contradicts (b).

Problem 7

(i) $\boldsymbol{v} \in W \cap W^\perp \iff \boldsymbol{v} \cdot \boldsymbol{v} = 0 \iff \boldsymbol{v} = \boldsymbol{0}$. Therefore $W \cap W^\perp = \{ \boldsymbol{0} \}$.

(ii) Take T to be an ordered basis extended from S for $\mathbb{R}^n$ and T' to be an orthogonal basis derived from T using Gram Schmidt. By definition $\boldsymbol{w}'_k = \boldsymbol{w}_k - \sum_{j<k} \operatorname{proj}_{\boldsymbol{w}'_j}(\boldsymbol{w}_k)$. Then

$$\boldsymbol{w}_k = \boldsymbol{w}'_k + \sum_{j<k} \alpha_j \boldsymbol{w}'_j \in \operatorname{span}\{ \boldsymbol{w}'_1, \boldsymbol{w}'_2, \ldots, \boldsymbol{w}'_m \}$$

Taking $k = m$ we have

$$W = \operatorname{span}\{ \boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_m \} \subseteq \operatorname{span}\{ \boldsymbol{w}'_1, \boldsymbol{w}'_2, \ldots, \boldsymbol{w}'_m \}$$

Lemma. $\boldsymbol{w}'_k \in \operatorname{span}\{ \boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_k \}$

Proof. We proceed by induction on k. Trivially $\boldsymbol{w}'_1 = \boldsymbol{w}_1 \in \operatorname{span}\{ \boldsymbol{w}_1 \}$. Suppose the lemma stands for all $j < k$. Then there exists a sequence α_j such that

$$\boldsymbol{w}'_k - \boldsymbol{w}_k = \sum_{j<k} \operatorname{proj}_{\boldsymbol{w}'_j}(\boldsymbol{w}_k) = \sum_{j<k} \alpha_j \boldsymbol{w}'_j \in \bigcup_{j<k} \operatorname{span}\{ \boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_j \} = \operatorname{span}\{ \boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_{k-1} \}$$

Then $\boldsymbol{w}'_k = \boldsymbol{w}_k + (\boldsymbol{w}'_k - \boldsymbol{w}_k) \in \operatorname{span}\{ \boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_k \}$. □

Therefore $\operatorname{span}\{ \boldsymbol{w}'_1, \boldsymbol{w}'_2, \ldots, \boldsymbol{w}'_m \} \subseteq \operatorname{span}\{ \boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_m \} = W$. Then $W = \operatorname{span}\{ \boldsymbol{w}'_1, \boldsymbol{w}'_2, \ldots, \boldsymbol{w}'_m \}$. Since T' is orthogonal, for all sequence α_j

$$(\sum_{j=1}^{n-m} \alpha_j \boldsymbol{w}'_{m+j}) \cdot \boldsymbol{w}_i = \sum_{j=1}^{n-m} \alpha_j (\boldsymbol{w}'_{m+j} \cdot \boldsymbol{w}_i) = 0$$

Therefore $\operatorname{span}\{ \boldsymbol{w}'_{m+1}, \boldsymbol{w}'_{m+1}, \ldots, \boldsymbol{w}'_n \} \subseteq W^\perp$ and $\dim W^\perp \geq n - m$. From hint,

$$\dim W + \dim W^\perp - \dim W \cap W^\perp = \dim W + W^\perp \leq \dim \mathbb{R}^n$$

$$\dim W^\perp = \dim W + W^\perp + \dim W \cap W^\perp - \dim W \leq n - m$$

Therefore $\dim W^\perp = n - m$. Now suppose $\exists \boldsymbol{v} \in W^\perp \setminus \operatorname{span}\{ \boldsymbol{w}'_{m+1}, \boldsymbol{w}'_{m+1}, \ldots, \boldsymbol{w}'_n \}$. Then the set $\{ \boldsymbol{v}, \boldsymbol{w}'_{m+1}, \boldsymbol{w}'_{m+1}, \ldots, \boldsymbol{w}'_n \}$ is linearly independent, so $\operatorname{span}\{ \boldsymbol{v}, \boldsymbol{w}'_{m+1}, \boldsymbol{w}'_{m+1}, \ldots, \boldsymbol{w}'_n \}$ is a subspace of $W^\perp$ with dimension $n - m + 1$, which contradicts. Therefore $W^\perp \subseteq \operatorname{span}\{ \boldsymbol{w}'_{m+1}, \boldsymbol{w}'_{m+1}, \ldots, \boldsymbol{w}'_n \}$. Using above we have $W^\perp = \operatorname{span}\{ \boldsymbol{w}'_{m+1}, \boldsymbol{w}'_{m+1}, \ldots, \boldsymbol{w}'_n \}$.

(iii) Let $\boldsymbol{a}_i$ be the ith column vector of $\boldsymbol{A}$. Suppose $\boldsymbol{A}\boldsymbol{A}^T\boldsymbol{x} = \boldsymbol{0}$. Then $\boldsymbol{A}^T\boldsymbol{x} \cdot \boldsymbol{a}_i^T = 0 \quad \forall i$. Therefore $\boldsymbol{A}^T\boldsymbol{x} \in \boldsymbol{R}(\boldsymbol{A})^\perp$. But $\boldsymbol{A}^T\boldsymbol{x} \in \boldsymbol{C}(\boldsymbol{A}^T) = \boldsymbol{R}(\boldsymbol{A})$. By above $\boldsymbol{A}^T\boldsymbol{x} = \boldsymbol{0} \iff \boldsymbol{x} \cdot \boldsymbol{a}_i = 0 \quad \forall i \iff \boldsymbol{x} \in W^\perp$. So $\operatorname{null} \boldsymbol{A}\boldsymbol{A}^T \subseteq W^\perp$. Moreover $\boldsymbol{x} \in W^\perp \iff \boldsymbol{A}^T\boldsymbol{x} = \boldsymbol{0} \implies \boldsymbol{A}\boldsymbol{A}^T\boldsymbol{x} = \boldsymbol{0}$, so $W^\perp \subseteq \operatorname{null} \boldsymbol{A}\boldsymbol{A}^T$. Then the solution space of $\boldsymbol{A}\boldsymbol{A}^T\boldsymbol{x} = \boldsymbol{0}$ is $W^\perp$.