1

(i)

$$\begin{aligned} D_f &= \mathbb{R} \setminus \{1, -1\} \\ D_g &= \{x \in \mathbb{R} : x \geq -1\} \end{aligned}$$

(ii)

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= \frac{1}{1 - (\sqrt{x+1})^2} \\ &= \frac{-1}{x} \end{aligned}$$

Since $g(x)$ is strictly increasing and $g(-1) = 0$,

$$D_{f \circ g} = \{x \in D_g : g(x) \in D_f\} = \{x \in \mathbb{R} : x \neq 0 \wedge x \geq -1\}$$

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) \\ &= \sqrt{\frac{1}{1 - x^2} + 1} \end{aligned}$$

Note that when $1 - x^2 < 0 \implies x^2 > 1$,

$$\frac{1}{1 - x^2} \geq -1 \implies 1 \leq x^2 - 1 \implies x^2 \geq 2$$

And when $1 - x^2 > 0 \implies x^2 < 1$,

$$\frac{1}{1 - x^2} \geq -1 \implies 1 \geq x^2 - 1 \implies x^2 \leq 2$$

Therefore

$$\begin{aligned} D_{g \circ f} &= \{x \in D_f : f(x) \in D_g\} \\ &= \{x \in \mathbb{R} : x^2 \neq 1 \wedge \frac{1}{1 - x^2} \geq -1\} \\ &= \{x \in \mathbb{R} : x^2 < 1 \vee x^2 \geq 2\} \\ &= \{x \in \mathbb{R} : -1 < x < 1 \vee x \leq -\sqrt{2} \vee x \geq \sqrt{2}\} \end{aligned}$$

2

(a)

$$\begin{aligned}
\lim_{x\to-2}\frac{x^4+3x^3+x^2+4}{x^4+4x^3+3x^2-4x-4} &= \lim_{x\to-2}\frac{(x^2-x+1)(x+2)^2}{(x^2-1)(x+2)^2} \\
&= \frac{(-2)^2-(-2)+1}{(-2)^2-1} \\
&= \frac{7}{3}
\end{aligned}$$

(b)

$$\begin{aligned}
\lim_{x\to2}\frac{\sqrt{x^2+7}-\sqrt{x^3+3}}{\sqrt{x+1}-\sqrt{2x-1}} &= \lim_{x\to2}\frac{(\sqrt{x^2+7}^2-\sqrt{x^3+3}^2)(\sqrt{x+1}+\sqrt{2x-1})}{(\sqrt{x+1}^2-\sqrt{2x-1}^2)(\sqrt{x^2+7}+\sqrt{x^3+3})} \\
&= \lim_{x\to2}\frac{[(x^2+7)-(x^3+3)](\sqrt{x+1}+\sqrt{2x-1})}{[(x+1)-(2x-1)](\sqrt{x^2+7}+\sqrt{x^3+3})} \\
&= \lim_{x\to2}\frac{-(x-2)(x^2+x+2)(\sqrt{x+1}+\sqrt{2x-1})}{-(x-2)(\sqrt{x^2+7}+\sqrt{x^3+3})} \\
&= \frac{(2^2+(2)+2)(\sqrt{2+1}+\sqrt{2(2)-1})}{\sqrt{2^2+7}+\sqrt{2^3+3}} \\
&= \frac{8}{11}\sqrt{33}
\end{aligned}$$

3

(a) Fix $\epsilon > 0$. Let $c = \frac{3}{\sqrt{2}}$ be constant. Let $\delta = \sqrt{c^2+\epsilon} - c > 0$. When $0 < \left|x+\sqrt{2}\right| < \delta$, we have

$$\begin{aligned}
\left|x^2-\sqrt{2}x-4\right| &= \left|x+\sqrt{2}\right|\left|x-2\sqrt{2}\right| \\
&< \delta\left|x+\sqrt{2}-3\sqrt{2}\right| \\
&\leq \delta(\delta+3\sqrt{2}) && \text{Triangle inequality} \\
&\leq (\delta+c)^2-c^2 \\
&= \sqrt{c^2+\epsilon}^2-c^2 \\
&= \epsilon
\end{aligned}$$

By definition of limit

$$\lim_{x\to-\sqrt{2}}(x^2-\sqrt{2}x) = 4$$

(b) Fix $\epsilon > 0$. Let $\delta = \sqrt{2\epsilon} > 0$. When $0 < |x-1| < \delta$, we have

$$\begin{aligned}\left|\frac{x}{x^2+1} - \frac{1}{2}\right| &= \left|\frac{2x - x^2 - 1}{2(x^2+1)}\right| \\ &= \frac{(x-1)^2}{2(x^2+1)} \qquad \text{Both terms are positive} \\ &< \frac{\delta^2}{2} \\ &= \epsilon\end{aligned}$$

By definition of limit

$$\lim_{x\to 1} \frac{x}{x^2+1} = \frac{1}{2}$$

4

Claim. When $a = 1$ and $b = -2021\sqrt{2}$, $\lim_{x\to\infty}(\sqrt{ax^2+1} - \sqrt{x^2+bx}) = 2021$.
Proof.

$$\begin{aligned}\lim_{x\to\infty}(\sqrt{ax^2+1} - \sqrt{x^2+bx}) &= \lim_{x\to\infty} \frac{(ax^2+1) - (x^2+bx)}{\sqrt{ax^2+1} + \sqrt{x^2+bx}} \\ &= \lim_{x\to\infty} \frac{1 - bx}{\sqrt{x^2+1} + \sqrt{x^2+bx}} \\ &= \lim_{x\to\infty} \frac{\frac{1}{x} - b}{\sqrt{1+\frac{1}{x^2}} + \sqrt{1+\frac{b}{x}}} \\ &= \frac{\lim_{x\to\infty} \frac{1}{x} - b}{\lim_{x\to\infty}(\sqrt{1+\frac{1}{x^2}} + \sqrt{1+\frac{b}{x}})} \\ &= \frac{-b}{\sqrt{2}} \\ &= 2021\end{aligned}$$

5

Claim. $(i) \implies (ii)$.
Proof. Fix $\epsilon > 0$. Let $M = \frac{1}{\epsilon} > 0$. By definition of infinite limit, there exists δ such that

$$0 < |x-a| < \delta \implies f(x) > M = \frac{1}{\epsilon} \implies \left|\frac{1}{f(x)}\right| < 0$$

Notice that $f(x) > M > 0\ \forall x \in (a-\delta, a+\delta) \setminus \{a\}$. Also, by definition of limit, $\lim_{x\to a} \frac{1}{f(x)} = 0$.

Claim. $(ii) \implies (i)$.
Proof. Let $f(x) > 0$ $\forall x \in (a - \delta_1, a + \delta_1) \setminus \{a\}$. Fix $M > 0$. Let $\epsilon = \frac{1}{M} > 0$. By definition of limit, there exists δ_2 such that $0 < |x - a| < \delta_2 \implies \left|\frac{1}{f(x)}\right| < \epsilon$. Let $\delta = \min(\delta_1, \delta_2)$. Then

$$
\begin{aligned}
0 < |x - a| < \delta &\implies \left|\frac{1}{f(x)}\right| < \epsilon && \text{since } \delta \leq \delta_2 \\
&\implies \frac{1}{\epsilon} < f(x) && f(x) > 0 \text{ as } \delta \leq \delta_1 \\
&\implies f(x) > M
\end{aligned}
$$

By definition of limit, $\lim_{x \to a} f(x) = \infty$.
Hence the two statements are equivalent.