NATIONAL UNIVERSITY OF SINGAPORE

SEMESTER 1, 2021/2022

MA2002 Calculus **Tutorial 1** (30^{th} August – 3^{rd} September)

TUTORIAL PART I

This part consists of relatively basic questions which cover the course materials. These questions will be discussed during tutorial sessions.

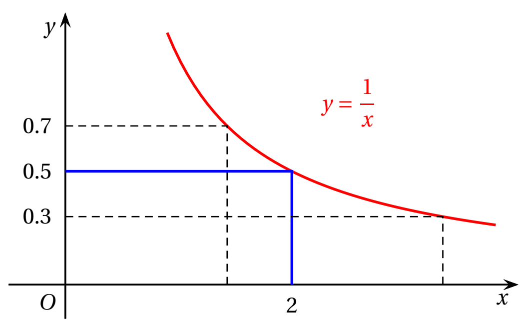
1. Find the following limits.

   (a) $\lim_{x\to 1}\left(\dfrac{1}{x-1}+\dfrac{1}{x^2-3x+2}\right)$, (b) $\lim_{x\to 9^-}\left(\sqrt{9-x}+\lfloor x+1\rfloor\right)$.

   (The floor function $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x.)

2. Use the given graph of $f(x)=1/x$ to find a number $\delta>0$ such that

$$0<|x-2|<\delta \Rightarrow \left|\frac{1}{x}-0.5\right|<0.2.$$



3. Prove the following limits using the ϵ,δ-definition.

   (a) $\lim_{x\to 3}\left(\dfrac{4}{3}x-2\right)=2$, (b) $\lim_{x\to -1}\left(2x^2-x-1\right)=2$,

   (c) $\lim_{x\to 1}x^3=1$, (d) $\lim_{x\to 1}\dfrac{2x^2+3x-2}{x+2}=1$.

4. Use the ϵ,δ-definition of limit and one-sided limits to show that

$$\lim_{x\to a}f(x)=L \quad \text{if and only if} \quad \lim_{x\to a^-}f(x)=\lim_{x\to a^+}f(x)=L.$$

5. Suppose that $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = c$, where c is a real number. Using the precise definition of limit and infinite limit, prove that

$$\lim_{x\to a}\big(f(x)+g(x)\big) = \infty.$$

6. **Definition of Limit at Infinity**. We write $\lim_{x\to\infty} f(x) = L$ if $f(x)$ is arbitrarily close to L by taking x sufficiently large. Precisely, it means that for every $\epsilon > 0$ there is a number N such that

$$x > N \Rightarrow |f(x) - L| < \epsilon.$$

One can verify that the limit laws for $x \to a$ still hold for $x \to \infty$. From now on, we will use the limit laws at infinity without proof unless stated otherwise.

(a) Write down the precise definitions of

(i) $\lim_{x\to-\infty} f(x) = L$, (ii) $\lim_{x\to\infty} f(x) = \infty$.

(b) Use the precise definition to show that $\lim_{x\to\infty} \dfrac{x}{2x+1} = \dfrac{1}{2}$.

Tutorial Part II

This part consists of relatively difficult questions to promote independent learning and inculcate critical thinking abilities. The solutions will not be discussed during the tutorial session. You may attempt them after you have gained a good understanding of the questions in Part I. The complete solution of this part is provided.

1. Find the limit $\lim_{x\to\infty}\left(\sqrt{x^2+x} - \sqrt{x^2-x}\right)$.

2. (i) Prove using the ϵ, δ-definition that for $a > 0$,

$$\lim_{x\to a} \sqrt[3]{x} = \sqrt[3]{a}.$$

[*Hint*: Recall that $b^3 - c^3 = (b-c)(b^2+bc+c^2)$.]

(ii) Based on the idea in part (i), prove that for $a > 0$, $n \in \mathbb{Z}^+$,

$$\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}.$$

3. Suppose that $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = c$, where c is a positive number. Prove by the precise definition that

$$\lim_{x\to a} f(x)g(x) = \infty.$$

Answers to Part I:

1. (a) -1, (b) 9.

2. $4/7$.