NATIONAL UNIVERSITY OF SINGAPORE

SEMESTER 1, 2021/2022

MA2002 Calculus **Tutorial 3** (6^{th} September – 10^{th} September)

TUTORIAL PART I

This part consists of relatively basic questions which cover the course materials. These questions will be discussed during tutorial sessions.

1. Consider the function $f(x) = \begin{cases} 2x, & \text{if } 0 \le x < 1, \\ 1, & \text{if } x = 1, \\ -2x+4, & \text{if } 1 < x < 2. \end{cases}$

   (i) Determine whether $f(1)$ and $\lim_{x\to 1} f(x)$ exist. Is f continuous at $x = 1$?

   (ii) Determine whether $f(2)$ and $\lim_{x\to 2^-} f(x)$ exist. Is f continuous at $x = 2$?

2. For each of the following functions, determine whether it is continuous at $x = 1$. Give a proof if it is continuous, and state the type of discontinuity otherwise.

   (a) $f(x) = \begin{cases} \dfrac{1}{x-1}, & \text{if } x \ne 1, \\ 2, & \text{if } x = 1. \end{cases}$

   (b) $f(x) = \begin{cases} \dfrac{x^2-x}{x^2-1}, & \text{if } x \ne 1, \\ 1, & \text{if } x = 1. \end{cases}$

   (c) $f(x) = \begin{cases} x, & \text{if } x \text{ is rational}, \\ 1, & \text{if } x \text{ is irrational}. \end{cases}$

3. Show that each of the following equations has at least one real root.

   (a) $\sin x + x + 1 = 0$.

   (b) $\sqrt{x-3} = \dfrac{10}{x-5}$.

4. For each of the following functions, use the definition of derivative to find the slope of the graph at the given point. Then find an equation for the tangent line of the graph there.

   (a) $f(x) = 4 - x^2$, at $(-1, 3)$.

   (b) $f(x) = x^3$, at $(2, 8)$.

5. Is there a straight line which is tangent to both the curves $y = x^2$ and $y = x^2 - 2x + 2$? If so, find its equation. If not, why not?

6. Show that the function $f(x) = |x+2|$ is not differentiable at $x = -2$.

7. Let
$$f(x)=\begin{cases} x^2, & \text{if } x\le 2,\\ mx+b, & \text{if } x>2.\end{cases}$$
Find the values of m and b that make f differentiable everywhere.

8. Find the derivatives of the following functions.
   (a) $y=(x^2+1)\left(x+5+\dfrac{1}{x}\right)$,
   (b) $g(x)=\dfrac{x^2-4}{x+0.5}$,
   (c) $v=\dfrac{1+x-4\sqrt{x}}{x}$,
   (d) $f(x)=\dfrac{x^3+x}{x^4-2}$.

TUTORIAL PART II

This part consists of relatively difficult questions to promote independent learning and inculcate critical thinking abilities. The solutions will not be discussed during the tutorial session. You may attempt them after you have gained a good understanding of the questions in Part I. The complete solution of this part is provided.

1. If a and b are positive numbers, prove that the equation
$$\frac{a}{x^3+2x^2-1}+\frac{b}{x^3+x-2}=0$$
has at least one solution in the interval $(-1,1)$.

2. Find the value of a for which the function
$$f(x)=\begin{cases} x^2-1, & \text{if } x<3,\\ 2ax, & \text{if } x\ge 3,\end{cases}$$
is continuous at $x=3$. For this value of a, is f differentiable at $x=3$? Justify your answer.

3. (i) Let g be a function satisfying $|g(x)|\le x^2$ for $-1\le x\le 1$. Show that g is differentiable at $x=0$ and find $g'(0)$.

   (ii) Show that the function
$$g(x)=\begin{cases} x^2\sin(1/x), & \text{if } x\ne 0,\\ 0, & \text{if } x=0,\end{cases}$$
is differentiable at $x=0$ and find $g'(0)$.

Answers to Part I:

1. (a) yes, yes, no, (b) no, yes, no.

2. (a) infinite discontinuity, (b) removable discontinuity, (c) continuous.

4. (a) $m = 2,\ y = 2x + 5$, (b) $m = 12,\ y = 12x - 16$.

5. yes, $y = x - \frac{1}{4}$.

7. $m = 4$, $b = -4$.

8. (a) $3x^2 + 10x + 2 - \frac{1}{x^2}$, (b) $\frac{x^2 + x + 4}{(x + 0.5)^2}$, (c) $-\frac{1}{x^2} + 2x^{-\frac{3}{2}}$, (d) $-\frac{x^6 + 3x^4 + 6x^2 + 2}{(x^4 - 2)^2}$.