NATIONAL UNIVERSITY OF SINGAPORE

SEMESTER 1, 2021/2022

MA2002 Calculus **Homework Assignment 3**

IMPORTANT:

(i) Write your name and student number on every page of your answer scripts.

(ii) Scan your scripts as a single PDF document. Other formats are not acceptable.

(iii) Rename your PDF document by student number. For example, A1234567X.pdf.

(iv) Log in to LumiNUS, and upload your PDF document to one of the subfolders in Files — Student Submission — Homework Assignment 3, depending on the first letter of your name on your student card.

(v) Submit by 18$^{\text{th}}$ October 2021 (Monday) 23:59.

1. Let f be a function such that $f''(x) + f(x) = 0$ for all $x \in \mathbb{R}$ such that $f(0) = f'(0) = 1$. [10]

   (i) Prove that $f(x)\sin x + f'(x)\cos x = 1$.

   (ii) Prove that $f(x)\cos x - f'(x)\sin x = 1$.

   (iii) Using (i) and (ii) derive that $f(x) = \sin x + \cos x$.

2. Let $f(x) = (x+1)^{4/7}(x-1)^{3/7}$. [10]

   (i) Find the open intervals on which f is increasing and decreasing.

   (ii) Find all local maximum and minimum points of f.

   (iii) Find the open intervals on which f is concave up and concave down.

   (iv) Find all inflection points of f.

   (v) Use the information from (i)–(iv) to sketch the proof of f.

3. Recall that if a differentiable function f is increasing on an open interval I, then $f'(x) \geq 0$ for all $x \in I$, and if f is decreasing on I, then $f'(x) \leq 0$ for $x \in I$. [10]

   Exercise (a) shows that even if a differentiable function f has $f'(c) > 0$, it may not be true that f is increasing near c (precisely, in an open interval containing c).

   Exercise (b) shows that even if a differentiable function f has a local minimum value at c, it may not be true that f is decreasing on the left of c and increasing on the right of c.

   (a) Let $f(x) = \begin{cases} x + 3x^2 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$

(i) Find the value of $f'(0)$.

(ii) For any $\delta > 0$, find a number $c \in (-\delta, \delta)$ such that $f'(c) < 0$.

(b) Let $f(x) = \begin{cases} x^2[2 + \sin(1/x)] & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$

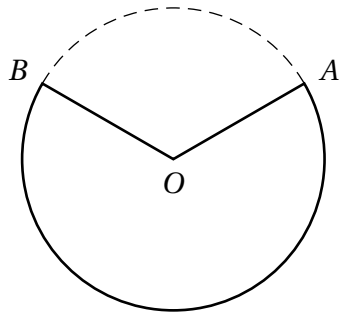
(i) Find the value of $f'(0)$ and show that f has a local minimum value at 0.

(ii) For any $\delta > 0$, find a number $c_1 \in (-\delta, 0)$ such that $f'(c_1) > 0$ and a number $c_2 \in (0, \delta)$ such that $f'(c_2) < 0$.

4. A hockey team plays in an arena with a seating capacity of 15 000 spectators. With the ticket price set at \$20, average attendance at a game has been 11 000. A market survey indicates that for each dollar the ticket price is lowered, average attendance will increase by 1 000. How should the owners of the team set the ticket price to maximize their revenue from ticket sales? [8]

5. A rectangular open tank of volume $125\,\text{m}^3$ is to have a square base. The cost per square metre for the bottom is \$24 and for the sides is \$12. Find the dimensions of the tank for the cost of the material to be the least. [8]

6. A cone-shaped drinking cup is made from a circular piece of paper of radius 5 cm by cutting out a sector and joining the edges OA and OB. Find the maximum capacity of such a cup. [8]



7. Let f be a twice differentiable function on $\mathbb{R}$. If $f'(0) = f'(1) = 0$, prove that there exists $c \in (0,1)$ such that $|f''(c)| \geq 4|f(1) - f(0)|$. [6]

Hint: Tutorial 5 Part II.