NATIONAL UNIVERSITY OF SINGAPORE

SEMESTER 1, 2021/2022

MA2002 Calculus **Solution to Tutorial 2**

REVIEW

Recall the *intuitive definition* of **limit**: We write $\lim_{x\to\infty} f(x) = L$, or

$$\underset{(x\neq a)}{x\to a} \quad \Rightarrow \quad f(x)\to L$$

if by taking x sufficiently close to a (but not equal to a), the value of $f(x)$ is arbitrarily close to L.

Mathematically, two numbers α and β are close if their distance $|\alpha-\beta|$ is small. So we have the following observation:

$$0<|x-a|<\delta \Rightarrow |f(x)-L|<\epsilon.$$

We shall still consider the meaning of "$f(x)$ is arbitrarily close to L". This means that $|f(x)-L|$ can be as small as possible, in the sense that it can be smaller than *any* given positive number whenever $|x-a|$ ($\neq 0$) is small. Therefore, the *precise definition* is:

For any $\epsilon>0$, there exists a number $\delta>0$ such that

$$0<|x-a|<\delta \Rightarrow |f(x)-L|<\epsilon.$$

In the proof using the precise definition, the key point is to find a proper δ for the given ϵ. We shall note the following:

(i) The choice of δ is an expression in terms of ϵ, but not in terms of x. For example, $\delta=\epsilon+x^2$ must be wrong.

(ii) δ represents the distance, so it must be always >0. For example, we cannot take $\delta=\epsilon-1$.

(iii) Unless $y=f(x)$ is a constant function, when $\epsilon\to 0$ we must have $\delta\to 0$ as well (cf. Q2). For example, one cannot take $\delta=\epsilon+1$.

(iv) The choice of δ is not unique. For example, in Q3(b), one may use $\delta=\min\{1,\epsilon/7\}$ or $\delta=\min\{2,\epsilon/9\}$, etc..

For the infinite limit $\lim_{x\to a} f(x) = \infty$, or

$$\underset{(x\neq a)}{x \to a} \quad \Rightarrow \quad f(x) \to \infty,$$

it means that by taking x sufficiently close to a, the value of $f(x)$ is arbitrarily large. We have understood the meaning of "x sufficiently close to a". Then what is "$f(x)$ is arbitrarily large"?

This is simply to say $f(x)$ is very large, in the sense that $f(x)$ can be larger than any given number M whenever $|x-a|$ $(x \neq a)$ is small. So the *precise definition of infinite limit* is:

For any number M, there is a number $\delta > 0$ such that

$$0 < |x-a| < \delta \Rightarrow f(x) > M.$$

In the textbook, it only considers when $M > 0$; but this is not essential: If the statement holds for all $M \in \mathbb{R}$, then in particular it holds for all $M \in \mathbb{R}^+$. Conversely, if the statement holds for all $M \in \mathbb{R}^+$, i.e., we can choose x close enough to a so that $f(x) > M$, then for any $M' \le 0$, $f(x) > M > M'$; so the statement holds for nonpositive numbers as well. Our definition just follows the one from the textbook.

From the precise definitions, we see that although the notations for $\lim_{x\to a} f(x) = L$ (where L is a number) and $\lim_{x\to a} f(x) = \infty$ are similar, they are different by considering the behavior of $f(x)$ when x is near a. In particular, an *infinite limit* is not a (*finite*) *limit*.

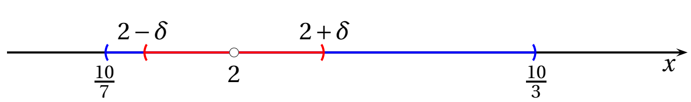
Solution to Part I

1. (a) $\lim_{x\to 1}\left(\frac{1}{x-1} + \frac{1}{x^2-3x+2}\right) = \lim_{x\to 1}\left(\frac{1}{x-1} + \frac{1}{(x-1)(x-2)}\right) = \lim_{x\to 1}\frac{(x-2)+1}{(x-1)(x-2)}$
$= \lim_{x\to 1}\frac{x-1}{(x-1)(x-2)} = \lim_{x\to 1}\frac{1}{x-2} = \frac{1}{1-2} = -1.$

(b) If $x \to 9^-$ then $x+1 \to 10$, and if $8 < x < 9$ then $x+1 < 9+1 = 10$; so $\lim_{x\to 9^-} \lfloor x+1 \rfloor = 9$. Then

$$\lim_{x\to 9^-}\left(\sqrt{9-x} + \lfloor x+1 \rfloor\right) = \lim_{x\to 9^-}\sqrt{9-x} + \lim_{x\to 9^-}\lfloor x+1 \rfloor = \sqrt{9-9} + 9 = 9.$$

2. $\left|\frac{1}{x} - 0.5\right| < 0.2 \Leftrightarrow -0.2 < \frac{1}{x} - 0.5 < 0.2 \Leftrightarrow 0.3 < \frac{1}{x} < 0.7 \Leftrightarrow \frac{10}{7} < x < \frac{10}{3}.$
So we shall find a proper $\delta > 0$ such that $(2-\delta, 2+\delta)\setminus\{2\}$ is a subset of the interval $(10/7, 10/3)$.



It follows that $10/7 \le 2-\delta$ and $2+\delta \le 10/3$. We have $\delta \le 4/7$ and $\delta \le 4/3$. So $\delta \le 4/7$. In particular, we may take $\delta = 4/7$ (or any positive number $\le 4/7$).

3. (a) *Idea*: Let $\epsilon > 0$ be given. Our aim is to find a proper $\delta > 0$ such that

$$0 < |x-3| < \delta \Rightarrow \left|\left(\frac{4}{3}x-2\right)-2\right| < \epsilon.$$

The restriction $0 < |x-3| < \delta$ is simple enough. We shall simplify the right-hand side: $\left|\left(\frac{4}{3}x-2\right)-2\right| = \frac{4}{3}|x-3| < \frac{4}{3}\delta.$

Comparing this with the expectation: $\left|\left(\frac{4}{3}x-2\right)-2\right| < \epsilon$, we see that it suffices to take a $\delta > 0$ such that $\frac{4}{3}\delta \le \epsilon$.

Proof: For $\epsilon > 0$, choose $\delta = 3\epsilon/4$. Then whenever $0 < |x-3| < \delta$, we have

$$\left|\left(\frac{4}{3}x-2\right)-2\right| = \frac{4}{3}|x-3| < \frac{4}{3}\delta = \epsilon.$$

(b) *Idea*: Again, let $\epsilon > 0$, we want to find a proper $\delta > 0$ so that

$$0 < |x+1| < \delta \Rightarrow |(2x^2-x-1)-2| < \epsilon.$$

Suppose $0 < |x+1| < \delta$. We hope to link $|(2x^2-x-1)-2|$ to $|x+1|$:

$$|(2x^2-x-1)-2| = |2x^2-x-3| = |2x-3|\,|x+1| = |2(x+1)-5|\,|x+1|.$$

Well, the restriction is for $|x+1|$, not for $x+1$. In order to link to $|x+1|$, we may try the triangle inequality to get a slightly bigger number:

$$|2(x+1)-5|\,|x+1| \le (2|x+1|+5)\,|x+1|.$$

Now, we can apply the restriction to continue:

$$(2|x+1|+5)\,|x+1| < (2\delta+5)\,\delta.$$

Comparing with the expectation: $|(2x^2-x-1)-2| < \epsilon$, it suffices to choose a $\delta > 0$ such that $(2\delta+5)\delta \le \epsilon$. Then how to make the choice?

As mentioned earlier, when $\epsilon \to 0$, we must have $\delta \to 0$. So $2\delta+5 \to 5$, which is "almost" a positive constant. We would like to try

$$(2\delta+5)\delta \le C\delta \le \epsilon \quad (C > 0).$$

The first inequality holds if and only if $2\delta+5 \le C$, and the second holds if and only if $\delta \le \epsilon/C$. Recall that $\delta > 0$. So $C \ge 2\delta+5 > 5$.

For instance, if we use $C = 7$, then $2\delta+5 \le 7 \Leftrightarrow \delta \le 1$. Thus, it suffices to choose $\delta > 0$ such that $\delta \le 1$ and $\delta \le \epsilon/7$, i.e., $\delta \le \min\{1, \epsilon/7\}$. If we use $C = 9$, then it requires $\delta \le \min\{2, \epsilon/9\}$.

Proof: For $\epsilon > 0$, choose $\delta = \min\{1, \epsilon/7\}$. Then whenever $0 < |x+1| < \delta$,

$$\begin{aligned}
|(2x^2 - x - 1) - 2| = |2x^2 - x - 3| = |2x-3|\,|x+1| \\
&= |2(x+1) - 5|\,|x+1| \le (2|x+1| + 5)|x+1| \\
&< (2\delta + 5)\delta \le (2 \cdot 1 + 5)\delta \\
&= 7\delta \le 7(\epsilon/7) = \epsilon.
\end{aligned}$$

(c) *Proof*: For $\epsilon > 0$, choose $\delta = \min\{1, \epsilon/7\}$. Then whenever $0 < |x-1| < \delta$,

$$\begin{aligned}
|x^3 - 1| &= |x-1|\,|x^2 + x + 1| = |x-1|\,|(x-1)^2 + 3(x-1) + 3| \\
&\le |x-1|\,(|x-1|^2 + 3|x-1| + 3) \\
&< \delta(\delta^2 + 3\delta + 3) \le \delta(1^2 + 3 \cdot 1 + 3) = \delta \cdot 7 \le (\epsilon/7) \cdot 7 = \epsilon.
\end{aligned}$$

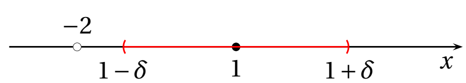
(d) *Proof*: For $\epsilon > 0$, choose $\delta = \min\{3, \epsilon/2\}$. Then whenever $0 < |x-1| < \delta$,

$$x + 2 = (x-1) + 3 > -\delta + 3 \ge -3 + 3 = 0,$$

and then

$$\begin{aligned}
\left|\frac{2x^2 + 3x - 2}{x+2} - 1\right| = \left|\frac{(x+2)(2x-1)}{x+2} - 1\right| &= |(2x-1) - 1| \\
&= 2|x-1| < 2\delta \le 2(\epsilon/2) = \epsilon.
\end{aligned}$$

Remark: If the question is $\lim_{x \to 1}(2x-1) = 1$, then we can simply use $\delta = \epsilon/2$. But the function is $\dfrac{2x^2 + 3x - 2}{x+2}$, which is undefined at $x = -2$:



So we must have make sure that $-2 \notin (1-\delta, 1+\delta) \backslash \{1\}$: $-2 \le 1 - \delta$, i.e., $\delta \le 3$.

4. ($\Rightarrow$): Suppose $\lim_{x \to a} f(x) = L$. Then for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon.$$

Therefore, if $0 < a - x < \delta$, then $|f(x) - L| < \epsilon$; that is, $\lim_{x \to a^-} f(x) = L$.
If $0 < x - a < \delta$, then $|f(x) - L| < \epsilon$; that is, $\lim_{x \to a^+} f(x) = L$.

($\Leftarrow$): Suppose $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$.
For $\epsilon > 0$, there exists a $\delta_1 > 0$ such that

$$0 < a - x < \delta_1 \Rightarrow |f(x) - L| < \epsilon,$$

and there exists a $\delta_2 > 0$ such that

$$0 < x - a < \delta_2 \Rightarrow |f(x) - L| < \epsilon.$$

Choose $\delta = \min\{\delta_1, \delta_2\}$. Then whenever $0 < |x-a| < \delta$, we have $|f(x) - L| < \epsilon$. That is, $\lim_{x\to a} f(x) = L$.

5. *Idea*: Let $M > 0$ be given. Our expectation is to find a proper $\delta > 0$ such that

$$0 < |x-a| < \delta \Rightarrow f(x) + g(x) > M.$$

Note that f and g are independent to each other. So we need to find lower bounds for f and g respectively: $f(x) > A$ and $g(x) > B$ with $A + B \geq M$.

We are more familiar with limit. So let us try $\lim_{x\to a} g(x) = c$ first:

For any $\epsilon > 0$, there exists a number $\delta_1 > 0$ such that

$$0 < |x-a| < \delta_1 \Rightarrow |g(x) - c| < \epsilon.$$

Since we need a lower bound for $g(x)$, we need to further expand $|g(x) - c| < \epsilon$ as $-\epsilon < g(x) - c < \epsilon$, and thus obtain $g(x) > c - \epsilon$.

Consider $\lim_{x\to a} f(x) = \infty$: For any $M_2 > 0$, there exists a number $\delta_2 > 0$ such that

$$0 < |x-a| < \delta_2 \Rightarrow f(x) > M_2.$$

Now the situation is very clear: We just need to find proper $\epsilon > 0$ and $M_2 > 0$ so that $(c-\epsilon) + M_2 \geq M$. In fact, ϵ can be any positive number, and M_2 is positive such that $M_2 \geq M - (c - \epsilon)$, for example, $M_2 = \max\{1, M - (c-\epsilon)\}$.

Proof: Fix an $\epsilon > 0$. Since $\lim_{x\to a} g(x) = c$, there exists a $\delta_1 > 0$ such that

$$0 < |x-a| < \delta_1 \Rightarrow |g(x) - c| < \epsilon \Rightarrow g(x) > c - \epsilon.$$

Let $M > 0$ be given. Since $\lim_{x\to a} f(x) = \infty$, there exists a $\delta_2 > 0$ such that

$$0 < |x-a| < \delta_2 \Rightarrow f(x) > \max\{1, M - (c-\epsilon)\}.$$

Choose $\delta = \min\{\delta_1, \delta_2\}$. Then whenever $0 < |x-a| < \delta$, we have

$$\begin{aligned} f(x) + g(x) &> \max\{1, M - (c-\epsilon)\} + (c - \epsilon) \\ &\geq M - (c-\epsilon) + (c-\epsilon) = M. \end{aligned}$$

By definition, $\lim_{x\to a}(f(x) + g(x)) = \infty$.

6. (a) (i) $\lim_{x\to -\infty} f(x) = L$: For every $\epsilon > 0$, there exists a number N such that

$$x < N \Rightarrow |f(x) - L| < \epsilon.$$

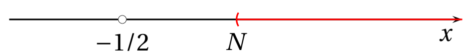
(ii) $\lim_{x\to\infty} f(x) = \infty$: For every $M > 0$, there exists a number N such that

$$x > N \Rightarrow f(x) > M.$$

(b) *Idea*: Let $\epsilon > 0$ be given. Our aim is to find a proper N such that

$$x > N \Rightarrow \left|\frac{x}{2x+1} - \frac{1}{2}\right| < \epsilon.$$

First of all, we need to make sure that for all $x > N$, the function $\frac{2}{2x+1}$ is well-defined, i.e, $x \neq -1/2$.



Therefore, N shall be chosen so that $N \geq -1/2$. Now let us simplify

$$\left|\frac{x}{2x+1} - \frac{1}{2}\right| = \left|-\frac{1}{2(2x+1)}\right|.$$

Since $x > N \geq -1/2$, we have $2x+1 > 0$, and thus

$$\left|-\frac{1}{2(2x+1)}\right| = \frac{1}{2(2x+1)} < \frac{1}{2(2N+1)}.$$

It is clear that we shall take N so that $\frac{1}{2(2N+1)} \leq \epsilon$, i.e., $N \geq \frac{1}{4\epsilon} - \frac{1}{2}$.

Note that $N \geq -1/2$ is absorbed by the restriction $N \geq \frac{1}{4\epsilon} - \frac{1}{2}$.

Proof: For $\epsilon > 0$, choose $N = \frac{1}{4\epsilon} - \frac{1}{2}$. Then whenever $x > N$, we have

$$\left|\frac{x}{2x+1} - \frac{1}{2}\right| = \left|-\frac{1}{2(2x+1)}\right| = \frac{1}{2(2x+1)} < \frac{1}{2(2N+1)} = \epsilon.$$

Solution to Part II

1. $$\lim_{x\to\infty}\left(\sqrt{x^2+x} - \sqrt{x^2-x}\right) = \lim_{x\to\infty} \frac{(\sqrt{x^2+x} - \sqrt{x^2-x})(\sqrt{x^2+x} + \sqrt{x^2-x})}{\sqrt{x^2+x} + \sqrt{x^2-x}}$$
$$= \lim_{x\to\infty} \frac{(x^2+x) - (x^2-x)}{\sqrt{x^2+x} + \sqrt{x^2-x}} = \lim_{x\to\infty} \frac{2x}{\sqrt{x^2+x} + \sqrt{x^2-x}}$$
$$= \lim_{x\to\infty} \frac{2}{\sqrt{1+\frac{1}{x}} + \sqrt{1-\frac{1}{x}}} = \frac{2}{\sqrt{1+0} + \sqrt{1-0}} = 1.$$

2. (a) For $\epsilon > 0$, choose $\delta = \min\{a, \epsilon\sqrt[3]{a^2}\}$. Then whenever $0 < |x-a| < \delta$, we have

$$x > a - \delta \geq \delta - \delta = 0,$$

and then

$$|\sqrt[3]{x}-\sqrt[3]{a}| = \frac{|x-a|}{\sqrt[3]{x^2}+\sqrt[3]{x}\sqrt[3]{a}+\sqrt[3]{a^2}} < \frac{|x-a|}{\sqrt[3]{a^2}} < \frac{\delta}{\sqrt[3]{a^2}} \leq \epsilon.$$

By definition, $\lim_{x\to a} \sqrt[3]{x} = \sqrt[3]{a}$.

(b) For $\epsilon > 0$, choose $\delta = \min\{a, \epsilon \sqrt[n]{a^{n-1}}\}$. Then whenever $0 < |x-a| < \delta$, we have

$$x > a - \delta \geq \delta - \delta = 0,$$

and then

$$\begin{aligned} |\sqrt[n]{x}-\sqrt[n]{a}| &= \frac{|x-a|}{\sqrt[n]{x^{n-1}}+\sqrt[n]{x^{n-2}}\sqrt[n]{a}+\cdots+\sqrt[n]{a^{n-1}}} \\ &< \frac{|x-a|}{\sqrt[n]{a^{n-1}}} < \frac{\delta}{\sqrt[n]{a^{n-1}}} \leq \epsilon. \end{aligned}$$

By definition, $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$.

3. Let $\epsilon = c/2 > 0$. Since $\lim_{x\to a} g(x) = c$, there exists a $\delta_1 > 0$ such that

$$0 < |x-a| < \delta_1 \Rightarrow |g(x)-c| < \epsilon \Rightarrow g(x) > c - \epsilon = c/2.$$

For every $M > 0$, since $\lim_{x\to a} f(x) = \infty$, there exists a $\delta_2 > 0$ such that

$$0 < |x-a| < \delta_2 \Rightarrow f(x) > 2M/c.$$

Choose $\delta = \min\{\delta_1, \delta_2\}$. Then whenever $0 < |x-a| < \delta$, we have

$$f(x)g(x) > (2M/c)\cdot(c/2) = M.$$

Therefore, by definition, $\lim_{x\to a} f(x)g(x) = \infty$.