NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2012-2013

MA1102R Calculus

May 2013 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains a total of **EIGHT (8)** questions and comprises **FOUR (4)** printed pages.

2. Answer **ALL** questions. The marks for the questions are not necessarily the same; marks for each question are indicated at the beginning of the question. The total is 120 marks.

3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Question 1 [15 marks]

It is known that for any x, the following limit exists:

$$\lim_{y\to 0} \frac{(x+y)^3 - x^3}{y}.$$

(a) Find, for each given x, a candidate $a(x)$ of the limit of the quotient $\dfrac{(x+y)^3 - x^3}{y}$ as y approaches to 0.

(b) Show that, for any given x, the candidate $a(x)$ found above is actually the limit of the quotient $\dfrac{(x+y)^3 - x^3}{y}$ as y approaches to 0, using the ϵ–δ–definition of limits.

(c) Explain what the limit means geometrically.

Question 2 [15 marks]

Evaluate the following limits:

(a) $\displaystyle\lim_{x\to 2} \frac{x^m - 2^m}{\sin(x^n - 2^n)}$ (m, n are positive integers).

(b) $\displaystyle\lim_{y\to 0} \frac{\sin(y) - \sin(y)\cos(y)}{y - \sin(y)}$.

(c) $\displaystyle\lim_{x\to 3} \frac{x^2 - 3x + 1}{x^3 + 2x - 27}$.

Question 3 [20 marks]

(a) Evaluate the derivative $\displaystyle\frac{d}{dx}\int_0^x t^2\cos(x^3 - t^3)dt$.

(b) Evaluate the definite integral $\displaystyle\int_0^1 \frac{x^2}{x^2 + (1-x)^2}dx$.

(c) Find $\dfrac{d^2y}{dx^2}$ for the function $y = e^{\cos(x^2)}$.

(d) Find $\dfrac{dy}{dx}$ where $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$.

Question 4 [15 marks]

Solve the following differential equations:

(a) $y' + (2x+1)y = x + \dfrac{1}{2}$.

(b) $\dfrac{1}{x}\dfrac{dy}{dx} + y = ye^{x^2}$.

(c) $\dfrac{dv}{dt} = -0.01(v^2 - 900)$ with the initial condition that when $t = 0$, $v = 0$.

Question 5 [12 marks]

(a) Show that the equation $16x^4 - 40x^2 + 9 = 0$ has exactly two solutions in the closed interval $[-1, 1]$.

(b) Find the volume of the solid generated by revolving the region enclosed by the curve $y = 9 - x^2$ and the line $y = 3 - x$ about the x-axis.

Question 6 [15 marks]

(a) Let g be a continuous function satisfying the following condition:

$$\text{for all } x, \text{ if } -1 \le x \le 1, \text{ then } \ (|g(1+x) - 2| \le x^2).$$

(i) Find the value of g at $x = 1$.

(ii) Show that g is differentiable at $x = 1$.

(iii) Find $g'(1)$.

(b) A rectangular open tank of volume 64 m^3 is to have a square base. The cost per square meter for the bottom is \$8 and for the sides is \$4. Find the dimensions of the tank for the cost of the material to be the least.

Question 7 [10 marks]

Let i and n be two integers such that $1 \le i \le n$. Let $g_{i,n}(x) = (1-x)^n(1+ix)$ be a function defined on the unit interval $[0, 1]$. Show that for all positive integers $i \le n$, the function $g_{i,n}$ is a one–to–one function and the range of $g_{i,n}$ is also the unit interval $[0, 1]$.

Question 8 [18 marks]

Let n be a positive integer. Let $f_n(x) = (n+1)x^n - nx^{n+1}$, for real number x.

(a) For each positive integer n,

(i) find the open intervals on which f_n is either increasing or decreasing.

(ii) find the coordinates of local maximum points or local minimum points of f_n, if any.

(iii) find the open intervals on which f_n is either concave up or concave down.

(iv) find the coordinates of the inflection points of f_n, if any.

(b) Show that

(i) if n is an odd positive integer, then f_n has an absolute maximum on the whole real line and that f_n attains its absolute maximum value at exactly one point.

(ii) if n is an even positive integer, then f_n has an absolute maximum on the open interval $(0, +\infty)$ and that f_n attains its absolute maximum value at exactly one point.

END OF PAPER