NATIONAL UNIVERSITY OF SINGAPORE

SEMESTER 1, 2021/2022

MA2002 Calculus **Solution to Tutorial 3**

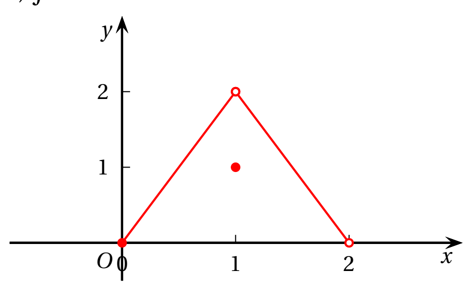
TUTORIAL PART I

1. (i) It is given by definition that $f(1) = 1$. Since

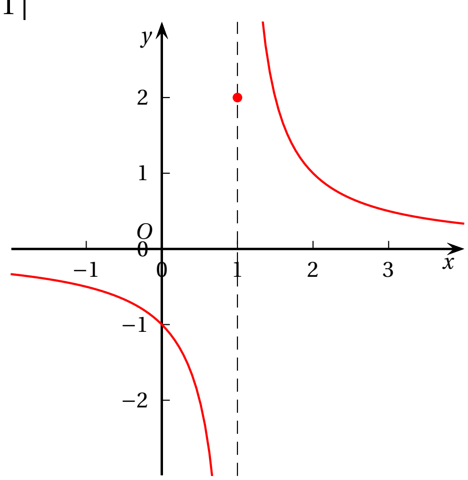
$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2x = 2 \quad \text{and} \quad \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (-2x+4) = 2,$$

$\lim_{x\to 1} f(x)$ exists and equals 2. Since $\lim_{x\to 1} f(x) \neq f(1)$, f is discontinuous at $x = 1$.

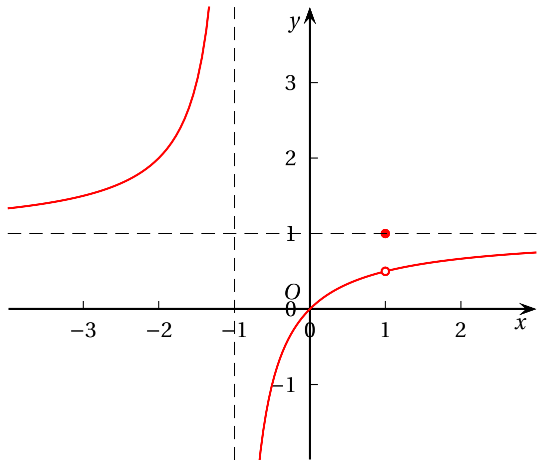
(ii) f is undefined at $x = 2$, so $f(2)$ does not exist. $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (-2x+4) = 0$. Since $f(2)$ does not exist, f is discontinuous at $x = 2$.



2. (a) $\lim_{x\to 1} f(x) = \lim_{x\to 1} \dfrac{1}{x-1}$ does not exist, so f is discontinuous at $x = 1$.
Since $\lim_{x\to 1} |f(x)| = \lim_{x\to 1} \left|\dfrac{1}{x-1}\right| = \infty$, f has an infinite discontinuity at $x = 1$.



(b) $\lim_{x\to 1} f(x) = \lim_{x\to 1} \dfrac{x^2-x}{x^2-1} = \lim_{x\to 1} \dfrac{x(x-1)}{(x+1)(x-1)} = \lim_{x\to 1} \dfrac{x}{x+1} = \dfrac{1}{2} \neq 1 = f(1)$, so f is discontinuous at $x=1$. Since $\lim_{x\to 1} f(x)$ exists, f has a removable discontinuity at $x=1$.



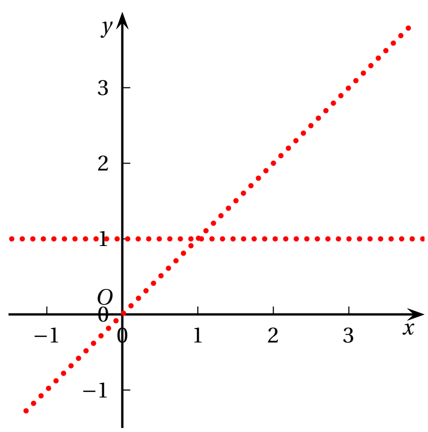
(c) Let $x > 1$. If x is rational, then $1 < f(x) = x$; if x is irrational, then $1 = f(x) < x$. So $1 \le f(x) \le x$ for all real numbers $x > 1$.

Since $\lim_{x\to 1^+} 1 = \lim_{x\to 1^+} x = 1$, by squeeze theorem, $\lim_{x\to 1^+} f(x) = 1$.

Let $x < 1$. If x is rational, then $x = f(x) < 1$; if x is irrational, then $x < f(x) = 1$. So $x \le f(x) \le 1$ for all real numbers $x < 1$.

Since $\lim_{x\to 1^-} x = \lim_{x\to 1^-} 1 = 1$, by squeeze theorem, $\lim_{x\to 1^-} f(x) = 1$.

Then $\lim_{x\to 1} f(x) = 1 = f(1)$. So f is continuous at $x=1$.



3. (a) Let $f(x) = \sin x + x + 1$. Then

$$f(-\pi) = 1 - \pi < 0 \quad \text{and} \quad f(0) = 1 > 0.$$

Since f is continuous on $[-\pi, 0]$, by intermediate value theorem, there exists a number $c \in (-\pi, 0)$ such that $f(c) = 0$.

Therefore, $\sin x + x + 1 = 0$ has at least one real solution $x = c$.

(b) Let $f(x) = \sqrt{x-3} - \dfrac{10}{x-5}$. Then

$$f(6) = \sqrt{3} - 10 < 0 \quad \text{and} \quad f(10) = \sqrt{7} - 2 > 0.$$

Since f is continuous on $[6, 10]$, by intermediate value theorem, there exists a number $c \in (6, 10)$ such that $f(c) = 0$.
Therefore, $\sqrt{x-3} = \dfrac{10}{x-5}$ has at least one real solution $x = c$.

4. (a) The slope of the tangent line of $y = f(x)$ at $(-1, 3)$ is

$$\begin{aligned} m = f'(-1) &= \lim_{x \to -1} \frac{f(x) - f(-1)}{x - (-1)} = \lim_{x \to -1} \frac{(4 - x^2) - 3}{x + 1} \\ &= \lim_{x \to -1} \frac{(1 + x)(1 - x)}{1 + x} = \lim_{x \to -1} (1 - x) = 2. \end{aligned}$$

Then the tangent line of $y = f(x)$ passing through $(-1, 3)$ is given by

$$y - 3 = 2(x + 1); \quad \text{that is, } y = 2x + 5.$$

(b) The slope of the tangent line of $y = f(x)$ at $(2, 8)$ is

$$\begin{aligned} m = f'(2) &= \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{x^3 - 8}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x^2 + 2x + 4)}{x - 2} \\ &= \lim_{x \to 2} (x^2 + 2x + 4) = 12. \end{aligned}$$

Then the tangent line of $y = f(x)$ passing through $(2, 8)$ is given by

$$y - 8 = 12(x - 2); \quad \text{that is, } y = 12x - 16.$$

5. i) Suppose the line ℓ is tangent to $y = x^2$ at $x = a$. Since $y' = 2x$ and $y'\big|_{x=a} = 2a$, the equation of ℓ can be written as $y - a^2 = 2a(x - a)$; that is, $y = 2ax - a^2$.

ii) Suppose the line ℓ is also tangent to $y = x^2 - 2x + 2$ at $x = b$. Since $y' = 2x - 2$ and $y'\big|_{x=b} = 2b - 2$, the equation of ℓ can also be written as $y - (b^2 - 2b + 2) = (2b - 2)(x - b)$; that is, $y = (2b - 2)x - b^2 + 2$.

iii) Note that the slope-intercept form of ℓ is unique. Then

$$2b - 2 = 2a \quad \text{and} \quad -b^2 + 2 = -a^2.$$

Solving the simultaneous equations, we have $a = 1/2$ and $b = 3/2$.
Then the equation of ℓ is $y = x - 1/4$.

6. $\lim\limits_{x\to -2^-} \dfrac{f(x)-f(-2)}{x-(-2)} = \lim\limits_{x\to -2^-} \dfrac{|x+2|}{x+2} = \lim\limits_{x\to -2^-} \dfrac{-(x+2)}{x+2} = \lim\limits_{x\to -2^-} (-1) = -1.$

   $\lim\limits_{x\to -2^+} \dfrac{f(x)-f(-2)}{x-(-2)} = \lim\limits_{x\to -2^+} \dfrac{|x+2|}{x+2} = \lim\limits_{x\to -2^+} \dfrac{x+2}{x+2} = \lim\limits_{x\to -2^+} 1 = 1.$

   Then $f'(-2) = \lim\limits_{x\to -2} \dfrac{f(x)-f(-2)}{x-(-2)}$ does not exist. Hence, f is not differentiable at $x=-2$.

7. Clearly f is differentiable on $\mathbb{R}\setminus\{2\}$. Suppose f is differentiable at $x=2$. In particular, f is continuous at $x=2$. Then

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2} f(x) = f(2).$$

   That is, $2m+b=4$.

   $\lim\limits_{x\to 2^-} \dfrac{f(x)-f(2)}{x-2} = \lim\limits_{x\to 2^-} \dfrac{x^2-4}{x-2} = \lim\limits_{x\to 2^-} \dfrac{(x-2)(x+2)}{x-2} = \lim\limits_{x\to 2^-} (x+2) = 4.$

   $\lim\limits_{x\to 2^+} \dfrac{f(x)-f(2)}{x-2} = \lim\limits_{x\to 2^+} \dfrac{(mx+b)-4}{x-2} = \lim\limits_{x\to 2^+} \dfrac{mx-2m}{x-2} = \lim\limits_{x\to 2^+} m = m.$

   Since $f'(2) = \lim\limits_{x\to 2} \dfrac{f(x)-f(2)}{x-2}$ exists, we must have $m=4$. So $b = 4-2m = -4$.

8. (a) $\dfrac{dy}{dx} = (x^2+1)'\left(x+5+\dfrac{1}{x}\right) + (x^2+1)\left(x+5+\dfrac{1}{x}\right)'$

   $= 2x\left(x+5+\dfrac{1}{x}\right) + (x^2+1)\left(1-\dfrac{1}{x^2}\right) = (2x^2+10x+2) + \left(x^2-\dfrac{1}{x^2}\right)$

   $= 3x^2+10x+2-\dfrac{1}{x^2}.$

   (b) $g'(x) = \dfrac{(x^2-4)'(x+0.5)-(x^2-4)(x+0.5)'}{(x+0.5)^2} = \dfrac{2x(x+0.5)-(x^2-4)}{(x+0.5)^2} = \dfrac{x^2+x+4}{(x+0.5)^2}.$

   (c) $\dfrac{dv}{dx} = \left(\dfrac{1}{x}+1-\dfrac{4}{\sqrt{x}}\right)' = -\dfrac{1}{x^2} - 4\left(-\dfrac{1}{2}\right)x^{-3/2} = -\dfrac{1}{x^2} + 2x^{-3/2}.$

   (d) $f'(x) = \dfrac{(x^3+x)'(x^4-2)-(x^3+x)(x^4-2)'}{(x^4-2)^2} = \dfrac{(3x^2+1)(x^4-2)-(x^3+x)4x^3}{(x^4-2)^2}$

   $= \dfrac{(3x^6+x^4-6x^2-2)-(4x^6+4x^4)}{(x^4-2)^2} = -\dfrac{x^6+3x^4+6x^2+2}{(x^4-2)^2}.$

TUTORIAL PART II

1. Define $f(x) = a(x^3+x-2)+b(x^3+2x^2-1)$. Then

$$f(-1) = -4a < 0, \quad f(1) = 2b > 0, \quad \text{and} \quad f \text{ is continuous on } [-1,1].$$

   By Intermediate Value Theorem, there exists a number $c \in (-1,1)$ such that $f(c)=0$.

   We shall verify that c is a solution to neither $x^3+x-2=0$ nor $x^3+2x^2-1=0$.

   i) $x^3+x-2 = (x-1)(x^2+x+2) \neq 0$ for all $x \in (-1,1)$. In particular, c is not a solution to $x^3+x-2=0$.

ii) Since $f(c) = 0$, $b(c^3+2c^2-1) = -a(c^3+c-2) \neq 0$. So c is also not a solution to $x^3+2x^2-1 = 0$.

Therefore, $c \in (-1,1)$ is a solution to $\dfrac{a}{x^3+2x^2-1} + \dfrac{b}{x^3+x-2} = 0$.

2. For f to be continuous at $x = 3$, we must have $\lim_{x\to 3} f(x) = f(3)$. In particular,
$$\lim_{x\to 3^-} f(x) = f(3).$$
That is, $\lim_{x\to 3^-}(x^2-1) = 2a\cdot 3$, which implies that $8 = 6a$, i.e., $a = 4/3$. Thus
$$f(x) = \begin{cases} x^2-1, & \text{if } x < 3,\\ 8x/3, & \text{if } x \geq 3.\end{cases}$$
Now
$$\lim_{h\to 0^-}\frac{f(3+h)-f(3)}{h} = \lim_{h\to 0^-}\frac{((3+h)^2-1)-8}{h} = \lim_{h\to 0^-}(h+6) = 6,$$
$$\lim_{h\to 0^+}\frac{f(3+h)-f(3)}{h} = \lim_{h\to 0^+}\frac{\frac{8(3+h)}{3}-8}{h} = \lim_{h\to 0^+}\frac{8}{3} = \frac{8}{3}.$$
Since $\lim_{h\to 0^-}\dfrac{f(3+h)-f(3)}{h} \neq \lim_{h\to 0^+}\dfrac{f(3+h)-f(3)}{h}$, f is not differentiable at $x = 3$.

3. (i) It is given that $|g(x)| \leq x^2$ for all $-1 \leq x \leq 1$. In particular, $|g(0)| \leq 0$. So we must have $g(0) = 0$. Therefore,
$$\lim_{x\to 0}\frac{g(x)-g(0)}{x-0} = \lim_{x\to 0}\frac{g(x)}{x}.$$
If $-1 \leq x \leq 1$ and $x \neq 0$, then $\left|\dfrac{g(x)}{x}\right| \leq |x|$; that is, $-|x| \leq \dfrac{g(x)}{x} \leq |x|$.
As $\lim_{x\to 0}(-|x|) = \lim_{x\to 0}|x| = 0$, by Squeeze Theorem, $\lim_{x\to 0}\dfrac{g(x)}{x}$ exists and equals 0.
Therefore, g is differentiable at $x = 0$ and $g'(0) = 0$.

(ii) If $x \neq 0$, then $|g(x)| = |x^2\sin(1/x)| \leq x^2$; if $x = 0$, then $|g(0)| = 0 \leq 0^2$. So $|g(x)| \leq x^2$ for all $x \in \mathbb{R}$.
Therefore, by (i) g is differentiable at $x = 0$ and $g'(0) = 0$.