NATIONAL UNIVERSITY OF SINGAPORE

SEMESTER 1, 2021/2022

MA2002 Calculus **Tutorial 5** (27th September – 1st October)

TUTORIAL PART I

This part consists of relatively basic questions which cover the course materials. The solutions to these questions will be recorded.

1. Show that the function $f(x) = 2x - \sin x$ has *exactly one* zero in $(-\infty, \infty)$.

2. Show that the equation $x^4 - 4x + 1 = 0$ has *exactly two* real roots.

3. It is known that a quadratic equation has at most two real roots. How many real roots can a cubic equation $x^3 + bx^2 + cx + d = 0$ have? Justify your answer, and give examples to illustrate *all* possibilities.

4. A number c is said to be a **fixed point** of a function f if $f(c) = c$.

   Suppose that f is differentiable on $\mathbb{R}$ and that $f'(x) \neq 1$ for all $x \in \mathbb{R}$. Prove that f has *at most one* fixed point.

5. Suppose that f is a function such that $f'(x) = 1/x$ for all $x > 0$. Prove that if $f(1) = 0$, then $f(xy) = f(x) + f(y)$ for all $x > 0$ and $y > 0$.

6. (a) Let f be a function whose derivative is given by $f'(x) = (x-1)(x+2)(x-3)$.

   (b) Let $g(x) = x^{1/5}(x+6)$.

   For each of f and g, find the following:

   (i) Critical points.

   (ii) Open intervals on which the function is increasing and decreasing.

   (iii) Points where the function assumes local maximum and minimum values.

7. Prove that the function $g(x) = x^{101} + x^{51} + x + 1$ has neither a local maximum nor a local minimum.

8. Show that if $x > 0$ then
$$\sqrt{1+x} < 1 + \frac{x}{2}.$$

TUTORIAL PART II

This part consists of relatively difficult questions to promote independent learning and inculcate critical thinking abilities. The solutions will not be recorded. You may attempt them after you have gained a good understanding of the questions in Part I. The complete solution of this part is provided.

1. Suppose that f'' exists throughout $(-\infty,\infty)$ and that f has three zeros in $(-\infty,\infty)$. Show that f'' has at least one zero in $(-\infty,\infty)$.
2. Suppose that f' exists and is continuous on $[a,b]$ and f'' exists on (a,b), where $a<b$. Show that there exists a number $c\in(a,b)$ such that
$$f(b)=f(a)+(b-a)f'(a)+\frac{(b-a)^2}{2}f''(c).$$
3. Suppose that the graph of a function f is concave up on an open interval I. Show that for any $a,b\in I$, where $a<b$, and $0<\lambda<1$,
$$(1-\lambda)f(a)+\lambda f(b)>f((1-\lambda)a+\lambda b).$$

Answers to Part I:

6. (a) (i) $x=-2,1,3$, (ii) increasing on $(-2,1)$ and on $(3,\infty)$, decreasing on $(-\infty,-2)$ and on $(1,3)$, (iii) local minimum at $x=-2,3$, local maximum at $x=1$.
   (b) (i) $x=-1,0$, (ii) increasing on $(-1,0)$ and on $(0,\infty)$, decreasing on $(-\infty,-1)$, (iii) local minimum at $x=-1$.