MA2002 Notes

Contents

1 Functions

1.1 Commonly Used Sets

$$\varnothing = \text{empty set} = \{\}$$
$$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$$
$$\mathbb{Q} = \text{set of rational numbers} = \{m/n \mid m,\, n \in \mathbb{Z},\, n \neq 0\}$$
$$\mathbb{R} = \text{set of real numbers}$$

$\mathbb{X}^+ = \{x \mid x \in \mathbb{X} \text{ and } x > 0\}$ and $\mathbb{X}^- = \{x \mid x \in \mathbb{X} \text{ and } x < 0\}$ for $\mathbb{X} = \mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$. For example,

$$\mathbb{N} = \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$$

1.2 Composite Functions

Let f and g be functions with domain A and B respectively. Then the domain of $g \circ f$ is

$$\{x \mid x \in A \text{ and } f(x) \in B\}$$

1.3 Even/Odd Functions

1.3.1 Even Function

A function f with domain A is an **even function** if

$$\forall x \in A \quad f(-x) = f(x)$$

1.3.2 Odd Function

A function f with domain A is an **odd function** if

$$\forall x \in A \quad f(-x) = -f(x)$$

2 Limits

2.1 Precise Definition of Limits

Side(s)	Value	Notation	For	there exists	such that	implies
Both sides	Finite	$\lim\limits_{x \to a} f(x) = L$	$\epsilon > 0$	$\delta > 0$	$0 < \lvert x - a \rvert < \delta$	$\lvert f(x) - L \rvert < \epsilon$
Right hand		$\lim\limits_{x \to a^+} f(x) = L$			$0 < x - a < \delta$	
Left hand		$\lim\limits_{x \to a^-} f(x) = L$			$0 < a - x < \delta$	
Both sides	Infinite	$\lim\limits_{x \to a} f(x) = \infty$	$M > 0$		$0 < \lvert x - a \rvert < \delta$	$f(x) > M$
Right hand		$\lim\limits_{x \to a^+} f(x) = \infty$			$0 < x - a < \delta$	
Left hand		$\lim\limits_{x \to a^-} f(x) = \infty$			$0 < a - x < \delta$	
Both sides	−Infinite	$\lim\limits_{x \to a} f(x) = -\infty$	$M < 0$		$0 < \lvert x - a \rvert < \delta$	$f(x) < M$
Right hand		$\lim\limits_{x \to a^+} f(x) = -\infty$			$0 < x - a < \delta$	
Left hand		$\lim\limits_{x \to a^-} f(x) = -\infty$			$0 < a - x < \delta$	
Infinity	Finite	$\lim\limits_{x \to \infty} f(x) = L$	$\epsilon > 0$	a number N	$x > N$	$\lvert f(x) - L \rvert < \epsilon$
	Infinite	$\lim\limits_{x \to \infty} f(x) = \infty$	$M > 0$			$f(x) > M$
	−Infinite	$\lim\limits_{x \to \infty} f(x) = -\infty$	$M < 0$			$f(x) < M$
−Infinity	Finite	$\lim\limits_{x \to -\infty} f(x) = L$	$\epsilon > 0$	a number N	$x < N$	$\lvert f(x) - L \rvert < \epsilon$
	Infinite	$\lim\limits_{x \to -\infty} f(x) = \infty$	$M > 0$			$f(x) > M$
	−Infinite	$\lim\limits_{x \to -\infty} f(x) = -\infty$	$M < 0$			$f(x) < M$

2.2 Limit Laws

Let a, $c \in \mathbb{R}$ and $\lim\limits_{x \to a} f(x)$, $\lim\limits_{x \to a} g(x) \in \mathbb{R}$, then,

2.2.1 Constant Function

$$\lim_{x \to a} c = c$$

2.2.2 Identity Function

$$\lim_{x \to a} x = a$$

2.2.3 Constant Multiple Rule

$$\lim_{x \to a}(c \cdot f(x)) = c \cdot \lim_{x \to a} f(x)$$

2.2.4 Sum Rule

$$\lim_{x \to a}(f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

2.2.5 Difference Rule

$$\lim_{x \to a}(f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

2.2.6 Product Rule

$$\lim_{x \to a}(f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

2.2.7 Quotient Rule

If $\lim_{x \to a} g(x) \neq 0$, then,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

2.2.8 Power Rule

$$\lim_{x \to a}[f(x)]^n = \left(\lim_{x \to a} f(x)\right)^n, \text{ where } n \in \mathbb{Z}$$

2.2.9 Root Rule

If n is odd or $\left(\text{n is even and } \lim_{x \to a} f(x) \geq 0\right)$, then,

$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$$

2.3 Intuitive Conclusion on Limits

If $f(x) = g(x)$ for all x in an open interval containing a, except at a, then

$$\text{If } \lim_{x \to a} f(x) = L, \text{ then } \lim_{x \to a} g(x) = L$$

2.4 Inequality on Limits

2.4.1 Lemma

If $f(x) \geq 0$ for all x in an open interval containing a, except at a, then

$$\text{If } \lim_{x \to a} f(x) = L, \text{ then } L \geq 0$$

2.4.2 Theorem

If $f(x) \geq g(x)$ for all x in an open interval containing a, except at a, then if

1. $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then
   (a) $L \geq M$

2.5 Squeeze Theorem

If $f(x) \leq g(x) \leq h(x)$ for all x in an open interval containing a, except at a, then

$$\text{If } \lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L, \text{ then } \lim_{x \to a} g(x) = L$$

2.6 Other Lemmas

If $\lim_{x \to a} f(x)$ exists and is positive, then:

$$f(x) > 0 \text{ for all } x \text{ in an open interval containing } a, \text{ except at } a$$

If $\lim_{x \to a} f(x)$ exists and is negative, then:

$$f(x) < 0 \text{ for all } x \text{ in an open interval containing } a, \text{ except at } a$$

2.7 Other Limits

$$\lim_{h \to 0} \frac{\sin h}{h} = 1$$
$$\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$$

3 Continuity

3.1 Definition of Continuity

A function f is **continuous** at a if

$$\lim_{x \to a} f(x) = f(a)$$

3.2 Definition of Discontinuity

A function f is **discontinuous** at a if it is not continuous at a.

3.3 Types of Discontinuity

3.3.1 Removable Discontinuity and Continuous Extension

A function f has a **removable discontinuity** at a if

1. $\lim_{x \to a} f(x)$ exists, and
2. $f(a)$ is undefined or $\lim_{x \to a} f(x) \neq f(a)$

A function f_1 is the **continuous extension** of f at a if

$$f_1(x) = \begin{cases} f(x) & x \neq a \\ \lim_{x \to a} f(x) & x = a \end{cases}$$

3.3.2 Infinite Discontinuity

A function f has an **infinite discontinuity** at a if

$$\lim_{x \to a^+} f(x) = \pm\infty \text{ or } \lim_{x \to a^-} f(x) = \pm\infty$$

3.3.3 Jump Discontinuity

A function f has a **jump discontinuity** at a if

1. $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exists, and
2. $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$

3.4 One-Sided Continuity

A function f is **continuous** from the **left** at a if

$$\lim_{x \to a^-} f(x) = f(a)$$

and f is **continuous** from the **right** at a if

$$\lim_{x \to a^+} f(x) = f(a)$$

3.5 Continuity on Intervals

A function f is continuous on a closed interval $[a, b]$ if f is

1. continuous at every $x \in (a, b)$,
2. continuous from the right at a, and
3. continuous from the left at b

3.6 Properties of Continuous Functions

Let f and g be continuous functions at a, then

3.6.1 Constant Multiples

cf is continuous at a, where $c \in \mathbb{R}$

3.6.2 Sums

$f + g$ is continuous at a

3.6.3 Differences

$f - g$ is continuous at a

3.6.4 Products

fg is continuous at a

3.6.5 Quotients

If $g(a) \neq 0$, then f/g is continuous at a

3.6.6 Powers

f^n is continuous at a, where $n \in \mathbb{Z}$

3.7 Substitution in Limits

3.7.1 Main Theorem

Let f and g be functions, if

1. $\lim\limits_{x \to a} f(x) = b$ and $\lim\limits_{y \to b} g(y) = c$, and
2. $f(x) \neq b$ for all x in an open interval containing a except at a

Then $\lim\limits_{x \to a} g(f(x)) = c = \lim\limits_{y \to b} g(y)$

3.7.2 Further Results

$$\lim_{x \to a} f(x) = \lim_{h \to 0} f(a + h)$$

3.7.3 Alternative Definition of Continuity

f is continuous at $a \iff \lim\limits_{h \to 0} f(a + h) = f(a)$

3.8 Composite Functions

3.8.1 Limit Operator Commutes With Continuous Function

Let f and g be functions, if

1. $\lim_{x \to a} f(x) = b$, and
2. g is continuous at b

Then $\lim_{x \to a} g(f(x)) = g(b) = g\left(\lim_{x \to a}(f(x)\right)$

3.8.2 Composite of Continuous Functions

If f is continuous at a and g is continuous at $f(a)$, then

$$g \circ f \text{ is continuous at } a$$

3.9 Continuous Functions

3.9.1 Constant Functions

$$\forall c \in \mathbb{R} \quad f(x) = c \text{ is continuous on } \mathbb{R}$$

3.9.2 Identity Function

$$f(x) = x \text{ is continuous on } \mathbb{R}$$

3.9.3 Integer Power Functions

$$\forall n \in \mathbb{N} \quad f(x) = x^n \text{ is continuous on } \mathbb{R}$$

3.9.4 Monomials

$$f(x) = cx^n \text{ is continuous on } \mathbb{R}$$

3.9.5 Polynomials

$$f(x) = c_n x^n + c_{n-1} x^{n-1} + \ldots + c_1 x + c_0 \text{ is continuous on } \mathbb{R}$$

3.9.6 Rational Functions

A function f is a **rational function** if

$$f(x) = P(x)/Q(x), \text{ where } P(x) \text{ and } Q(x) \text{ are polynomials}$$

f is continuous on its domain $\{x \mid Q(x) \neq 0\}$

3.9.7 Root Functions

$$f(x) = \sqrt[n]{x} \text{ is continuous on its domain} = \begin{cases} \mathbb{R} & n \text{ is odd} \\ [0, \infty) & n \text{ is even} \end{cases}$$

3.9.8 Rational Power Functions

$$\forall r \in \mathbb{Q} \quad f(x) = x^r \text{ is continuous on its domain}$$

3.9.9 Trigonometric Functions

1. $\sin x$ and $\cos x$ are continuous on $\mathbb{R}$
2. $\tan x$ and $\sec x$ are continuous on their domain $= \{x \mid \cos x \neq 0\} = \mathbb{R} \setminus \{\frac{\pi}{2} + k\pi \mid k \in \mathbb{Z}\}$
3. $\cot x$ and $\csc x$ are continuous on their domain $= \{x \mid \sin x \neq 0\} = \mathbb{R} \setminus \{k\pi \mid k \in \mathbb{Z}\}$

3.10 Intermediate Value Theorem (Simple Version)

Suppose

1. f is continuous on a finite closed interval $[a, b]$, and
2. $f(a) < 0$ and $f(b) > 0$; or $f(a) > 0$ and $f(b) < 0$

Then there exists $c \in (a, b)$ such that $f(c) = 0$

3.11 Intermediate Value Theorem (General Version)

Suppose

1. f is continuous on a finite closed interval $[a, b]$, and
2. $f(a) \neq f(b)$ and N is between $f(a)$ and $f(b)$

Then there exists $c \in (a, b)$ such that $f(c) = N$

4 Derivatives

4.1 Definition of Derivative at a point

The **derivative** of a function f at a is the limit

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

If $f'(a)$ exists, then f is **differentiable** at a.

4.2 Derivative as a Function

The **derivative** of a function f is the function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}, \text{ if the limit exists}$$

4.3 Differentiability on an open interval

A function f is differentiable on an open interval I if

f is differentiable at every point in I

4.4 Differentiability implies Continuity

If a function f is differentiable at a, then f is continuous at a

4.5 Differentiation Formulas

Suppose f and g are differentiable at x and $c \in \mathbb{R}$

4.5.1 Constant Functions

$$\frac{d}{dx}(c) = 0$$

4.5.2 Constant Multiple

$$(cf)'(x) = cf'(x)$$

4.5.3 Sum

$$(f+g)'(x) = f'(x) + g'(x)$$

4.5.4 Difference

$$(f-g)'(x) = f'(x) - g'(x)$$

4.5.5 Product

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

4.5.6 Quotient

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

4.5.7 Integer Power

$$\forall n \in \mathbb{Z} \quad \frac{d}{dx}x^n = nx^{n-1}$$

4.6 Differentiable Functions

4.6.1 Polynomials

Every polynomial is differentiable on $\mathbb{R}$

4.6.2 Rational Functions

Every rational function is differentiable on its domain

4.6.3 Trigonometric Functions

Trigonometric Functions are differentiable on their domain

$$\frac{d}{dx}\sin x = \cos x$$
$$\frac{d}{dx}\cos x = -\sin x$$
$$\frac{d}{dx}\tan x = \sec^2 x$$
$$\frac{d}{dx}\cot x = -\csc^2 x$$
$$\frac{d}{dx}\sec x = \sec x \tan x$$
$$\frac{d}{dx}\csc x = -\csc x \cot x$$

4.7 Chain Rule

If f is differentiable at x and g is differentiable at $f(x)$, then

$$(g \circ f)'(x) = g'(f(x))f'(x)$$

4.8 Implicit Differentiation

4.8.1 Implicit Function

Let $f(x, y) = 0$ be an equation in x and y. If y can be expressed in x near a point on $f(x, y) = 0$, then

y is an **implicit function** of x near the point

4.8.2 Implicit Differentiation

Furthermore, if y is an implicit function of x such that $\frac{dy}{dx}$ exists, then $\frac{dy}{dx}$ can be evaluated by implicit differentiation as follows:

1. Differentiate $f(x, y) = 0$ w.r.t. x, regarding y as a differentiable function in x
2. Solve for $\frac{dy}{dx}$ in terms of x and y

4.9 Higher Derivatives

Let f be a function

1. The **zeroth derivative** of f is $f = f^{(0)}$
2. For $n \in \mathbb{N}$, the $\boldsymbol{n^{th}}$ **derivative** of f is $f^{(n)} = (f^{(n-1)})'$
   (a) f is $\boldsymbol{n}$ **times differentiable** if $f^{(n)}$ exists
3. Let $y = f(x)$. $f^{(n)}(x) = \frac{d^n y}{dx^n}$
   (a) This is the n^{th} derivative of y w.r.t. x

5 Applications of Derivatives

5.1 Extreme Values

Let f be a function with domain D

5.1.1 Absolute Maximum

f has an **absolute/global maximum** value at $c \in D$ if

$$\forall x \in D \quad f(c) \geq f(x)$$

5.1.2 Absolute Minimum

f has an **absolute/global minimum** value at $c \in D$ if

$$\forall x \in D \quad f(c) \leq f(x)$$

5.2 Local Extreme Values

Let f be a function with domain D

5.2.1 Local Maximum

f has an **relative/local maximum** value at $c \in D$ if

For all x in an open interval containing c, $f(c) \geq f(x)$

5.2.2 Local Minimum

f has an **relative/local minimum** value at $c \in D$ if

For all x in an open interval containing c, $f(c) \leq f(x)$

5.3 Extreme Value Theorem

If f is continuous on a finite closed interval $[a, b]$. Then

f attains the extreme values on $[a, b]$

Precisely, there exist $c, d \in [a, b]$ such that

$\forall x \in [a, b] \quad f(c) \leq f(x) \leq f(d)$

5.4 Extreme Value Problem

If f is continuous on a finite closed interval $[a, b]$

1. Evaluate the values of f at endpoints: $f(a)$ and $f(b)$
2. Find local extreme values of f on (a, b)
3. Compare the values obtained in Steps 1 and 2:
   (a) The largest value is the absolute maxmimum value
   (b) The smallest value is the absolute minimum value

5.5 Fermat's Theorem

Let f be a function such that

1. f has a local extreme value at c, and
2. f is differentiable at c

Then $f'(c) = 0$

5.5.1 Remarks

Equivalently, if f has a local extreme value at c, then either

1. f is not differentiable at c, or
2. $f'(c) = 0$

5.6 Critical Point

Let c be an interior point of the domain of f. c is a critical point of f if either

1. $f'(c)$ does not exist, or
2. $f'(c) = 0$

5.7 Stationary Point

c is a stationary point of f if $f'(c) = 0$

5.8 Closed Interval Method

Let f be continuous on $[a, b]$

1. Evaluate the values of f at endpoints: $f(a)$ and $f(b)$
2. Evaluate the values of f at critical points on (a, b)
3. Compare the values obtained in Steps 1 and 2:
   (a) The largest value is the absolute maxmimum value
   (b) The smallest value is the absolute minimum value

5.9 Rolle's Theorem

If f is a function such that

1. it is continuous on $[a, b]$
2. it is differentiable on (a, b), and
3. $f(a) = f(b)$

Then there exists a number $c \in (a, b)$ such that $f'(c) = 0$

5.10 Mean Value Theorem

If f is a function such that

1. it is continuous on $[a, b]$
2. it is differentiable on (a, b)

Then there exists a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

5.11 Derivative Zero Implies Constant Function

If f is a function such that

1. it is continuous on an interval I
2. it is differentiable on I' = the interior of I, and
3. $\forall x \in I' \quad f'(x) = 0$

Then there exists a constant $C \in \mathbb{R}$ such that

$$\forall x \in I \quad f(x) = C$$

5.12 Functions with Same Derivative Differ By A Constant

If f and g are functions such that

1. they are continuous on an interval I
2. they are differentiable on I' = the interior of I, and
3. $\forall x \in I' \quad f'(x) = g'(x)$

Then there exists a constant $C \in \mathbb{R}$ such that

$$\forall x \in I \quad f(x) = g(x) + C$$

5.13 Increasing Function

A function f is increasing on a set I if

$$\text{For any } a, b \in I \quad a < b \implies f(a) < f(b)$$

5.14 Decreasing Function

A function f is decreasing on a set I if

$$\text{For any } a, b \in I \quad a < b \implies f(a) > f(b)$$

5.15 Increasing Test

If f is a function such that

1. it is continuous on an interval I
2. it is differentiable on I' = the interior of I, and
3. $\forall x \in I' \quad f'(x) > 0$

Then f is increasing on I

5.16 Decreasing Test

If f is a function such that

1. it is continuous on an interval I
2. it is differentiable on $I' =$ the interior of I, and
3. $\forall x \in I' \quad f'(x) < 0$

Then f is decreasing on I

5.17 Increasing Differentiable Functions

If f is differentiable on an open interval I and if f is increasing on I, then

$$\forall x \in I \quad f'(x) \geq 0$$

5.18 Decreasing Differentiable Functions

If f is differentiable on an open interval I and if f is decreasing on I, then

$$\forall x \in I \quad f'(x) \leq 0$$

5.19 First Derivative Test

If f is a function such that

1. it is continuous at a critical point c, and
2. it is differentiable on an open interval containing c, except at c

Then if

1. f' changes from negative to positive at c
   (a) then f has a local minimum value at c
2. f' changes from positive to negative at c
   (a) then f has a local maximum value at c
3. f' does not change sign at c
   (a) then f does not have a local extreme value at c

5.20 Second Derivative Test

If $f'(c) = 0$, then if

1. $f''(c) > 0$, then f has a local minimum value at c
2. $f''(c) < 0$, then f has a local maximum value at c

5.21 Concavity

5.21.1 Concave Up

If f is differentiable on an open interval I. f is **concave up** if

1. the graph of f lies above all its tangent lines on I

More precisely,

$$\text{For all } a, b \in I, a \neq b \quad f(b) - f(a) > f'(a)(b - a)$$

5.21.2 Concave Down

If f is differentiable on an open interval I. f is **concave down** if

1. the graph of f lies below all its tangent lines on I

More precisely,

$$\text{For all } a, b \in I, a \neq b \quad f(b) - f(a) < f'(a)(b - a)$$

5.22 First Derivative and Concavity

If f is a differentiable function on an open interval I

5.22.1 Concave Up

$$f' \text{ is increasing on I} \iff f \text{ is concave up on } I$$

5.22.2 Concave Down

$$f' \text{ is decreasing on I} \iff f \text{ is concave down on } I$$

5.23 Concavity Test

If f is twice differentiable on an open interval I

5.23.1 Concave Up

$$(\forall x \in I \quad f''(x) > 0) \implies f \text{ is concave up on } I$$

5.23.2 Concave Down

$$(\forall x \in I \quad f''(x) < 0) \implies f \text{ is concave down on } I$$

5.24 Inflection Point

f has an **inflection point** at c if

1. f is continuous at c, and
2. f changes concavity at c

5.25 Twice Differentiable Inflection Point

If a function f

1. has an inflection point at c, and
2. is twice differentiable at c

Then $f''(c) = 0$

5.26 L'Hôpital's Rule (Baby Version)

If f and g are functions differentiable at a such that

- $f(a) = g(a) = 0$ and $g'(a) \neq 0$

Then $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$

5.27 Cauchy's Mean Value Theorem

If f and g are functions such that

1. they are continuous on $[a, b]$
2. they are diferentiable on (a, b), and
3. $\forall x \in (a, b) \quad g'(x) \neq 0$

Then there exists $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

5.28 L'Hôpital's Rule (0/0 Version)

If f and g are functions such that

1. $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$, and
2. $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists or equals $\pm\infty$
   (a) $\frac{f'(x)}{g'(x)}$ is defined on an open interval I containing a, except at a

i. f and g are differentiable on $I \setminus \{a\}$

ii. $\forall x \in I \setminus \{a\} \quad g'(x) \neq 0$

Then $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$

5.28.1 Remarks

1. a may be finite or infinite

2. $x \to a$ may be replaced with one-sided limits

5.29 L'Hôpital's Rule (∞/∞ Version)

If f and g are functions such that

1. $\lim\limits_{x \to a} |f(x)| = \lim\limits_{x \to a} |g(x)| = \infty$, and

2. $\lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$ exists or equals $\pm\infty$

   (a) $\frac{f'(x)}{g'(x)}$ is defined on an open interval I containing a, except at a

      i. f and g are differentiable on $I \setminus \{a\}$

      ii. $\forall x \in I \setminus \{a\} \quad g'(x) \neq 0$

Then $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$

5.29.1 Remarks

1. a may be finite or infinite

2. $x \to a$ may be replaced with one-sided limits

3. The condition $\lim\limits_{x \to a} |f(x)|$ is unnecessary

6 Integrals

6.1 Limit of a Sequence

Let $\{a_n\}$ be a sequence. Then $\lim\limits_{n \to \infty} a_n = L \in \mathbb{R}$ means for $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$n > N \implies |a_n - L| < \epsilon$$

6.2 Limit of a Sequence and Function

Let f be a function such that

1. $\forall n \in \mathbb{N} \quad a_n = f(n)$, and
2. $\lim\limits_{x \to \infty} f(x) = L$

Then $\lim\limits_{n \to \infty} a_n = L$

6.3 Definite Integral

Let f be a continuous function on $[a, b]$

1. Divide $[a, b]$ into n equal subintervals, each of length $\Delta x = \frac{b-a}{n}$
   - $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n] \qquad x_i = a + i\Delta x$
2. Choose **sample points** $x_1^*, x_2^*, \ldots, x_n^*$ from these subintervals:
   - $x_1^* \in [x_0, x_1], x_2^* \in [x_1, x_2], \ldots, x_n^* \in [x_{n-1}, x_n]$
3. Compute the **Riemann sum**:
   - $f(x_1^*)\Delta x + \ldots + f(x_n^*)\Delta x = \sum\limits_{i=1}^{n} f(x_i^*)\Delta x$
4. The **definite integral** of f from a to b:
   - $\lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\, dx$

6.4 Geometric Properties of the Integral

6.4.1 Integral of the Negative of a Function

$$\int_a^b (-f(x))\, dx = -\int_a^b f(x)\, dx$$

6.4.2 Integral of Constant Functions

$$\int_a^b c\, dx = c(b-a)$$

6.4.3 Monotonicity

Let f and g be continuous functions on $[a, b]$ and if

- $\forall x \in [a, b] \quad f(x) \geq g(x)$

Then $\int_a^b f(x)\, dx \geq \int_a^b g(x)\, dx$

6.4.4 Max-Min Inequality

Let f be a continuous function on $[a,b]$, and m and M are the minimum and maximum values of f on $[a,b]$ respectively. Then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

6.4.5 Additivity

Let f be a continuous function on $[a,b]$. Then $\forall c \in (a,b)$,

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

6.4.6 Order of Integration

Definition. If $a > b$ and f is a function continuous on $[b,a]$. Then define

$$\int_a^b f(c)\,dx = -\int_b^a f(x)\,dx$$

6.4.7 Zero Width Interval

Definition. If f is defined at a, then define

$$\int_a^a f(x)\,dx = 0$$

6.5 Properties of the Integral

Let f and g be a continuous function on an interval I and $k \in \mathbb{R}$

6.5.1 Additivity

For any $a,b,c \in I$,

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

6.5.2 Constant Multiple

For any $a,b \in I$,

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$

6.5.3 Sum and Difference

For any $a,b \in I$,

$$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

6.6 Fundamental Theorem of Calculus (Part I)

If f is continuous on $[a, b]$. Let $g(x) = \int_a^x f(t)\, dt$. Then

1. g is continuous on $[a, b]$
2. g is differentiable on (a, b), and
3. $\forall x \in (a, b) \quad g'(x) = f(x)$

6.6.1 Remarks

Let $c \in [a, b]$. Then

- $\int_a^x f(t)\, dt - \int_c^x f(t)\, dt = \int_a^c f(t)\, dt$ is a constant
  1. $\int_c^x f(t)\, dt$ is continuous on $[a, b]$,
  2. differentiable on (a, b) with derivative $f(x)$

6.7 Mean Value Theorem for Definite Integrals

If f is continuous on $[a, b]$, then there exists $c \in (a, b)$ such that

$$\int_a^b f(x)\, dx = (b - a) f(c)$$

6.8 Fundamental Theorem of Calculus (Part II)

If f is a function continuous on $[a, b]$ and if F is a function such that

1. it is continuous on $[a, b]$,
2. differentiable on (a, b), and
3. $\forall x \in (a, b) \quad F'(x) = f(x)$

Then

$$\int_a^b f(x)\, dx = F(b) - F(a) = F(x)\Big|_{x=a}^{x=b}$$

6.9 Basic Integration Formulae

If f and g are continuous functions and $k \in \mathbb{R}$. Then

6.9.1 Constant Multiple Rule

$$\int k f(x)\, dx = k \int f(x)\, dx$$

6.9.2 Sum and Difference Rule

$$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

6.9.3 Power Rule

$$\forall r \in \mathbb{Q} \setminus -1 \qquad \int x^r\,dx = \frac{x^{r+1}}{r+1} + C$$

6.9.4 Trigonometric Functions

$$\int \cos x\,dx = \sin x + C$$
$$\int \sin x\,dx = -\cos x + C$$
$$\int \sec^2 x\,dx = \tan x + C$$
$$\int \csc^2 x\,dx = -\cot x + C$$
$$\int \sec x \tan x\,dx = \sec x + C$$
$$\int \csc x \cot x\,dx = -\csc x + C$$

6.10 Substitution Rule (Indefinite Integral)

If

1. $u = g(x)$ is differentiable and its range is an interval I
2. g' is continuous and f is continuous on I

Then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

6.11 Substitution Rule (Definite Integral)

If

1. g' is continuous on $[a, b]$, and
2. f is continuous on the range of g

Then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

6.12 Integral of Odd/Even Functions

Let f be a function continuous on $[-a, a]$

6.12.1 Odd

If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$

6.12.2 Even

If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

6.13 Improper Integral

Definition. Let f be continuous on $[a, b)$, discontinuous at b from the left. Then the **improper integral**

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

Let f be continuous on $(a, b]$, discontinuous at a from the right. Then the **improper integral**

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

6.13.1 Additivity

Definition. If f is discontinuous at $c \in (a, b)$. The **improper integral**

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

LHS exists $\iff$ both integrals on RHS exists

6.14 Improper Integral and Continuous Extension

6.14.1 Continuous Extension to Endpoints

Let f be a function such that

1. it is continuous on (a, b), and

2. $\lim\limits_{x \to a^+} f(x)$ and $\lim\limits_{x \to b^-} f(x)$ exists

Let f_1 be the continuous extension of f. Then

$$\int_a^b f(x)\,dx = \int_a^b f_1(x)\,dx$$

6.14.2 Removable Discontinuities

Let f be a function such that

1. it is continuous on $[a, b]$ except at a finite number of points at which f has removable discontinuities

Let f_1 be the continuous extension of f. Then

$$\int_a^b f(x)\,dx = \int_a^b f_1(x)\,dx$$

6.15 Improper Integral to Infinity

Definition. Let f be a function such that

6.15.1 Improper Integral to ∞

$\int_a^t f(x)\,dx$ exists for every $t \geq a$. Then the **improper integral**

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

6.15.2 Improper Integral to $-\infty$

$\int_t^b f(x)\,dx$ exists for every $t \leq b$. Then the **improper integral**

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

6.16 Improper Integral on $(-\infty, \infty)$

Definition. Let $a \in \mathbb{R}$, the **improper integral**

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

6.16.1 Remarks

- LHS is convergent $\iff$ both integrals on RHS are convergent
- If $\int_{-\infty}^a f(x)\,dx$ and $\int_a^\infty f(x)\,dx$ are convergent, then $\forall b \in \mathbb{R}$,
    1. $\int_{-\infty}^b f(x)\,dx$ and $\int_b^\infty f(x)\,dx$ are convergent, and
    2. $\int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx = \int_{-\infty}^b f(x)\,dx + \int_b^\infty f(x)\,dx$

7 Inverse Functions and Transcendental Functions

7.1 One-to-One Functions

Definition. Let f be a function with domain D. f is **one-to-one** if for any $a, b \in D$,

$$a \neq b \implies f(a) \neq f(b)$$

Equivalently, for any $a, b \in D$,

$$f(a) = f(b) \implies a = b$$

7.2 Inverse Function

Definition. Let f be a one-to-one function with domain A and range B

- Then $\forall y \in B$, there is a unique $x \in A$ such that $f(x) = y$

The **inverse function** of f, denoted by f^{-1}, is defined by

- For any $x \in A$ and $y \in B$ $\quad f^{-1}(y) = x \iff y = f(x)$
    1. f^{-1} has domain B and range A

7.3 Inverse of Inverse Function

$$(f^{-1})^{-1} = f$$

7.4 Composite of Function and its Inverse is an Identity Function

If for any $x \in A$ and $y \in B$ $\quad y = f(x) \iff x = f^{-1}(y)$, then

$$\forall x \in A \quad f^{-1}(f(x)) = x$$
$$\forall y \in B \quad f(f^{-1}(y)) = y$$

7.5 Equation for Inverse Functions

Let f be a one-to-one function. Then f has an inverse function, say f^{-1}. To find an equation for f^{-1},

1. Let $y = f(x)$
2. Solve for x in terms of y: $\quad x = f^{-1}(y)$
3. Interchange x and y: $\quad y = f^{-1}(x)$

7.6 Geometric Meaning of Interchanging x and y

1. In the Cartesian plane $\mathbb{R}^2$, interchanging x and y has the same effect as reflecting the graph w.r.t. the straight line $y = x$

2. Thus, the graph of f and f^{-1} are symmetric w.r.t. the line $y = x$

7.7 Continuous Functions are one-to-one iff monotonic

Theorem. Let f be a function continuous on an interval I, then

- f is one-to-one $\iff$ f is monotonic
  - Monotonic means either increasing or decreasing

7.8 Inverse of Continuous Function is Continuous

If f is a function one-to-one and continuous on an interval I. Then the inverse function f^{-1} is also continuous

7.9 Inverse of Differentiable Function is Differentiable

If f is a function one-to-one and continuous on an interval I. If f is differentiable at an interior point a of I, $f'(a) \neq 0$, and $b = f(a)$, then

$$(f^{-1})'(b) = \frac{1}{f'(a)}$$

7.10 Inverse Trigonometric Functions

Function	Domain	Range	Derivative
$\sin^{-1} x$	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$	$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$	$\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\mathbb{R}$	$(-\frac{\pi}{2}, \frac{\pi}{2})$	$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$
$\cot^{-1} x$	$\mathbb{R}$	$(0, \pi)$	$\frac{d}{dx} \cot^{-1} x = -\frac{1}{1+x^2}$
$\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$	$\frac{d}{dx} \sec^{-1} x = \frac{1}{x\sqrt{x^2-1}}$
$\csc^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$(0, \frac{\pi}{2}] \cup (\pi, \frac{3\pi}{2}]$	$\frac{d}{dx} \csc^{-1} x = -\frac{1}{x\sqrt{x^2-1}}$

7.11 Inverse Trigonometric Identities

$$\forall x \in [-1, 1] \quad \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$
$$\forall x \in \mathbb{R} \quad \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$$
$$\sec^{-1} x + \csc^{-1} x = \begin{cases} \frac{\pi}{2} & x \geq 1 \\ \frac{5\pi}{2} & x \leq -1 \end{cases}$$

7.12 Logarithmic Function

Definition. The **natural logarithmic function** is defined by

$$\text{For } x > 0 \quad \ln x = \int_1^x \frac{1}{t}\, dt$$

7.13 Properties of $\ln x$

- $\ln 1 = 0$
- $\ln x$ is continuous and differentiable on $\mathbb{R}^+$
- $\forall x > 0 \quad \frac{d}{dx} \ln x = \frac{1}{x}$ and $\frac{d^2}{dx^2} \ln x = -\frac{1}{x^2}$
  - $\ln x$ is increasing and concave down on $\mathbb{R}^+$
- $\lim_{x \to 0^+} \ln x = -\infty$, $\lim_{x \to \infty} \ln x = \infty$, and range of $\ln x$ is $\mathbb{R}$

7.14 Logarithmic Laws

$$\text{For } x > 0,\ a > 0 \quad \ln(ax) = \ln a + \ln x$$
$$\text{For } x > 0,\ r \in \mathbb{Q} \quad \ln(x^r) = r \ln x$$

7.15 Integral of $\frac{1}{x}$

Theorem. $\forall x \neq 0$

$$\frac{d}{dx} \ln|x| = \frac{1}{x}$$

7.16 Integral of Rational Trigonometric Functions

$$\int \sec x\, dx = \ln|\sec x + \tan x| + C$$
$$\int \csc x\, dx = -\ln|\csc x + \cot x| + C$$

$$\int \tan x \, dx = -\ln|\cos x| + C$$
$$\int \cot x \, dx = \ln|\sin x| + C$$

7.17 Logarithmic Differentiation

If $y = [f_1(x)]^{r_1} \times \ldots \times [f_n(x)]^{r_n}$, where

1. $r_1, \ldots, r_n \in \mathbb{Q}$, and
2. $f_1, \ldots, f_n$ are non-zero differentiable functions

Then **logarithmic differentiation** may be applied as follows:

1. Take the absolute value:
   - $|y| = |f_1(x)|^{r_1} \times \ldots \times |f_n(x)|^{r_n}$
2. Take natural logarithm:
   - $\ln|y| = r_1 \ln|f_1(x)| + \ldots + r_n \ln|f_n(x)|$
3. Differentiate w.r.t. x:
   - $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{r_1 f_1'(x)}{f_1(x)} + \ldots + \dfrac{r_n f_n'(x)}{f_n(x)}$

7.17.1 Remarks

Logarithmic differentiation is not applicable if $y = 0$

7.18 Euler's Number e

Definition. The **Euler's number e** is the number such that $\ln e = 1$

7.19 Exponential Function $\exp x$

Definition. Let $\exp x = e^x$ be the inverse function of $\ln x$. Then $\exp x$ has domain $\mathbb{R}$ and range $\mathbb{R}^+$

7.20 Properties of $\exp x$

1. $\lim\limits_{x \to -\infty} \exp x = 0$ and $\lim\limits_{x \to \infty} \exp x = \infty$

7.21 Derivative of $\exp x$

$$\frac{d}{dx} \exp x = \exp x$$

7.22 Exponential Function a^x

Definition. The exponential function of base $a > 0$ is defined by

$$\forall x \in \mathbb{R} \quad a^x = \exp(x \ln a)$$

7.23 Properties of a^x

For all $a > 0$ and $x, y \in \mathbb{R}$

$$\begin{aligned}
&\ln(a^x) = x \ln a \\
&a^x a^y = a^{x+y} \\
&a^{-x} = 1/a^x \\
&(a^x)^y = a^{xy} \\
&\frac{d}{dx} a^x = a^x \ln a
\end{aligned}$$

7.24 Real Power Rule

Theorem. $\forall a \in \mathbb{R} \quad \forall x > 0$,

$$\frac{d}{dx} x^a = a x^{a-1}$$

7.25 Derivative of x^x

$$\forall x > 0 \quad \frac{d}{dx} x^x = (\ln x + 1) x^x$$

7.26 Derivative of $f(x)^{g(x)}$

Derivative of $f(x)^{g(x)}$ can be found by logarithmic differentiation

7.27 e as a limit

Theorem.

$$e = \lim_{x \to 0} (1 + x)^{1/x}$$

7.28 Limits of $f(x)^{g(x)}$

To find $\lim\limits_{x \to a} f(x)^{g(x)}$, where $f(x) > 0$,

1. Express $f(x)^{g(x)} = \exp[g(x) \ln f(x)]$
2. Interchange lim operator and exp function

7.29 Hyperbolic Trigonometric Functions

7.29.1 Hyperbolic Sine Function

Definition. The **hyperbolic sine function** is defined by

$$\sinh x = \frac{\exp x - \exp(-x)}{2}$$

$\sinh x$ is increasing on its domain $\mathbb{R}$ and has range $\mathbb{R}$

7.29.2 Hyperbolic Cosine Function

Definition. The **hyperbolic cosine function** is defined by

$$\cosh x = \frac{\exp x + \exp(-x)}{2}$$

$\sinh x$ has domain $\mathbb{R}$, it is increasing on $[0, \infty)$

7.30 Hyperbolic Trigonometric Identities

$$\cosh^2 t - \sinh^2 t = 1$$
$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$
$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$

7.31 Derivatives of Hyperbolic Trigonometric Functions

$$\frac{d}{dx} \sinh x = \cosh x$$
$$\frac{d}{dx} \cosh x = \sinh x$$

7.32 Inverse Hyperbolic Trigonometric Functions

Function	Domain	Range	Derivative
$\sinh^{-1} x$	$\mathbb{R}$	$\mathbb{R}$	$\frac{d}{dx} \sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}$
$\cosh^{-1} x$	$[1, \infty)$	$[0, \infty)$	$\frac{d}{dx} \sinh^{-1} x = \frac{1}{\sqrt{x^2-1}}$

8 Techniques of Integration

8.1 Inverse Substitution Rule

Let f be a continuous function. If $x = g(t)$ is

1. one-to-one, and
2. g' is continuous

Then

$$\int f(x)\,dx = \int f(g(t))g'(t)\,dt$$

8.2 Integration by Parts

Let u and v be functions that are differentiable with continuous derivatives. Then

$$\int \left(u\frac{dv}{dx}\right)\,dx = uv - \int \left(\frac{du}{dx}v\right)\,dx$$

Or in differential forms,

$$\int u\,dv = uv - \int v\,du$$

8.3 Trigonometric Substitution

If the integrand contains the square root of quadratic functions, one may try the **trigonometric substitution** method as follows:

1. Complete the square to obtain the following forms:

Form	Substitution	Domain	Simplified Form
$\sqrt{a^2 - x^2} \quad (a > 0)$	$x = a\sin t$	$t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$	$a\cos t$
$\sqrt{a^2 + x^2} \quad (a > 0)$	$x = a\tan t$	$t \in (-\frac{\pi}{2}, \frac{\pi}{2})$	$a\sec t$
$\sqrt{x^2 - a^2} \quad (a > 0)$	$x = a\sec t$	$t \in [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$	$a\tan t$

8.4 Unique Factorisation of Polynomial

Theorem. Every non-constant single-variable polynomial with real coefficients can be uniquely factorised as the product of

1. real linear factors
   - For all $a \in \mathbb{R},\ r \in \mathbb{N} \quad (x+a)^r$
2. real irreducible quadratic factors
   - For all $b, c \in \mathbb{R},\ s \in \mathbb{N},\ b^2 < 4c \quad (x^2 + bx + c)^s$

8.5 Proper Rational Function

Definition. If $f(x) = \frac{A(x)}{B(x)}$ is a rational function where $A(x)$ and $B(x)$ are polynomials. $f(x)$ is a **proper rational function** if

$$\deg A(x) < \deg B(x)$$

8.6 Converting from Improper to Proper Rational Function

If $f(x) = \frac{A(x)}{B(x)}$ where $A(x)$ and $B(x)$ are polynomials and $\deg A(x) \geq \deg B(x)$. Then the following method converts $f(x)$ into a proper rational function:

1. Use long division to get
   - $A(x) = B(x)Q(x) + R(x)$, where $\deg R(x) < \deg B(x)$
2. Then $f(x) = \frac{A(x)}{B(x)} = Q(x) + \frac{R(x)}{B(x)}$

8.7 Decomposing Proper Rational Function into Partial Fractions

Theorem. Every proper rational function can be uniquely expressed as the sum of **partial fractions**. Let $f(x) = \frac{A(x)}{B(x)}$ be a proper rational function. Then $f(x)$ is the sum of the following partial fractions:

1. If $x + a$ is a linear factor of $B(x)$ with multiplicity r
   - $\dfrac{A_1}{x+a} + \dfrac{A_2}{(x+a)^2} + \ldots + \dfrac{A_r}{(x+a)^r}$
2. If $x^2 + bx + c$ is an irreducible factor of $B(x)$ with multiplicity s
   - $\dfrac{B_1x + C_1}{x^2+bx+c} + \dfrac{B_2x + C_2}{(x^2+bx+c)^2} + \ldots + \dfrac{B_sx + C_s}{(x^2+bx+c)^s}$

8.8 Integration of Rational Functions

8.8.1 Integral of $\frac{1}{(x+a)^k}$

To evaluate $\displaystyle\int \frac{dx}{(x+a)^k}$, use the substitution $u = x + a$

8.8.2 Integral of $\frac{x}{(x^2+bx+c)^k}$

To evaluate $\displaystyle\int \frac{x}{(x^2+bx+c)^k}\,dx$, use the substitution $u = x^2 + bx + c$. Then

$$\int \frac{x}{(x^2+bx+c)^k}\,dx = \frac{1}{2}\int \frac{du}{u^k} - \frac{b}{2}\int \frac{dx}{(x^2+bx+c)^k}$$

8.8.3 Integral of $\frac{1}{(x^2+bx+c)^k}$

1. $x^2 + bx + c = (x + \frac{b}{2})^2 + (c - \frac{b^2}{4})$
2. Let $u = x + \frac{b}{2}$, $\alpha = \sqrt{c - \frac{b^2}{4}}$. Then $x^2 + bx + c = u^2 + \alpha^2$
   - $\displaystyle\int \frac{dx}{(x^2+bx+c)^k} = \int \frac{du}{(u^2+\alpha^2)^k}$

3. Let $u = \alpha v$. Then $u^2 + \alpha^2 = \alpha^2(v^2+1)$

   - $\displaystyle\int \frac{du}{(u^2+\alpha^2)^k} = \int \frac{\alpha\, dv}{(\alpha^2(1+v^2))^k} = \frac{1}{\alpha^{2k-1}} \int \frac{dv}{(1+v^2)^k}$

8.9 Universal Trigonometric Substitution

Let f be a rational expression in two variables. Then

$$\int f(\sin x, \cos x)\, dx = \int f\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right) \frac{2}{1+t^2}\, dt$$

8.10 More Integral Formulae

8.10.1 Integral of Integer Power of $\frac{1}{1+x^2}$

Let $x = \tan t,\ t \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$$\forall n \in \mathbb{N} \quad \int \frac{dx}{(1+x^2)^n} = \int (\cos t)^{2n-2}\, dt$$

8.10.2 Integral of Non-Zero Power of $\cos x$

$\forall n \neq 0,$

$$\int (\cos x)^n\, dx = \frac{1}{n}(\cos x)^{n-1} \sin x + \frac{n-1}{n} \int (\cos x)^{n-2}\, dx$$

8.10.3 Integral of $\ln x$

$$\int \ln x\, dx = x \ln x - x + C$$

8.10.4 Integral of $\sin^{-1} x$

$$\int \sin^{-1} x\, dx = x \sin^{-1} x + \sqrt{1-x^2} + C$$

9 Applications of Definite Integrals

9.1 Area Under Function

Let f be a nonnegative function continuous on $[a,b]$. Then the area of the region between $y = f(x)$ and the x-axis on $[a,b]$ is

$$\int_a^b f(x)\, dx$$

9.2 Area Between Two Functions

Let f and g be functions such that

1. they are continuous on $[a, b]$, and
2. $\forall x \in [a, b] \quad f \geq g$

Then the area of the region between $y = f(x)$ and $y = g(x)$ on $[a, b]$ is

$$\int_a^b [f(x) - g(x)]\, dx$$

9.3 Area of Plane Region

If R is a plane region in the xy-coordinate system.

1. If R is placed along the x-axis on $[a, b]$, then
   (a) Cut R using vertical line segment at $x \in [a, b]$, then
      - Length $\ell(x) =$ upper endpoint $-$ lower endpoint

   (b) Area of $R = \int_a^b \ell(x)\, dx$
2. If R is placed along the y-axis on $[c, d]$, then
   (a) Cut R using horizontal line segment at $y \in [c, d]$, then
      - Length $L(y) =$ right endpoint $-$ left endpoint

   (b) Area of $R = \int_c^d L(y)\, dy$

9.4 Volume of Solid

If a solid is placed along the x-axis on $[a, b]$, then

1. Cut the solid using planes perpendicular to the x-axis at $x \in [a, b]$, then
   - Area $A(x) =$ area of cross-section at x
2. The volume of the solid $= \int_a^b A(x)\, dx$

If a solid is placed along the y-axis on $[c, d]$, then

1. Cut the solid using planes perpendicular to the y-axis at $y \in [c, d]$, then
   - Area $A(y) =$ area of cross-section at y
2. The volume of the solid $= \int_c^d A(y)\, dy$

9.5 Disk Method

Let f be a continuous function. The volume of the solid formed by rotating the region between $y = f(x)$ and the x-axis on $[a, b]$ about the x-axis is

$$\pi \int_a^b [f(x)]^2 \, dx$$

9.6 Washer Method

Let f and g be continuous functions such that $\forall x \in [a, b] \quad f(x) \geq g(x) \geq 0$. The volume of the solid formed by rotating the region between $y = f(x)$ and $y = g(x)$ on $[a, b]$ about the x-axis is

$$\pi \int_a^b ([f(x)]^2 - [g(x)]^2) \, dx$$

9.7 Cylindrical Shell Method

Let f be continuous and non-negative on $[a, b]$, where either $a \geq 0$ or $b \leq 0$. The volume of the solid formed by rotating the region between $y = f(x)$ and the $x - axis$ on $[a, b]$ about the y-axis is

$$\begin{cases} 2\pi \int_a^b x f(x) \, dx & a \geq 0 \\ -2\pi \int_a^b x f(x) \, dx & b \leq 0 \end{cases}$$

9.8 Arc Length

If f is continuous on $[a, b]$. The **arc length** of the curve $y = f(x)$ on $[a, b]$ is defined by

$$\int_a^b \sqrt{1 + [f'(x)]^2} \, dx$$

9.9 Surface Area of Revolution

If f is nonnegative and continuous on $[a, b]$. The area of the surface formed by rotating the curve $y = f(x)$ on $[a, b]$ about the x-axis is

$$2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2} \, dx = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

10 First Order Ordinary Differential Equations

10.1 Ordinary Differential Equation

Definition. An **ordinary differential equation** (abbr. **ODE**) is an equation of the form

$$F(x, y, \frac{dy}{dx}, \ldots, \frac{d^n y}{dx^n}) = 0$$

where y is an implicit function in variable x

10.1.1 Degree of ODE

The highest order of the derivative is the **degree** of the ODE

10.2 First Order ODE

A **first order ODE** has the form

$$\frac{dy}{dx} = F(x, y)$$

10.3 First Order ODE in only x

10.3.1 Method to Solve

$\dfrac{dy}{dx} = f(x)$ can be solved by simply integrating both sides w.r.t. x. Then

$$y = \int f(x)\, dx$$

10.4 First Order ODE in only y

10.4.1 Method to Solve

$\dfrac{dy}{dx} = g(y) \implies \dfrac{dx}{dy} = \dfrac{1}{g(y)}$ (for $g(y) \neq 0$). Then,

$$x = \int \frac{1}{g(y)}\, dy$$

10.5 Separable First Order ODE

Definition. A first order ODE $\dfrac{dy}{dx} = F(x, y)$ is separable if

$$F(x, y) = f(x)g(y)$$

10.5.1 Method to Solve

$\frac{dy}{dx} = f(x)g(y) \implies \frac{1}{g(y)}\frac{dy}{dx} = f(x)$ (for $g(y) \neq 0$). Then,

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$

10.6 Homogeneous Functions

Definition. Let $F(x_1, \ldots, x_m)$ be a function in m variables. F is **homogeneous of degree n** if $\forall t \in \mathbb{R} \setminus \{0\}$

$$F(tx_1, \ldots, tx_m) = t^n F(x_1, \ldots, x_m)$$

10.7 Homogeneous First Order ODE

Definition. A first order ODE $\frac{dy}{dx} = F(x, y)$ is **homogeneous** if

1. $F(x, y)$ is homogeneous of degree zero
   - $\forall t \in \mathbb{R} \setminus \{0\} \quad F(tx, ty) = F(x, y)$
   - **Remarks.** The term $\frac{y}{x}$ or $\frac{x}{y}$ should appear in the function

10.7.1 Method to Solve

1. Convert equation to standard form $\frac{dy}{dx} = F(x, y)$
2. Let $z = y/x$
   - $y = xz$ and $\frac{dy}{dx} = z + x\frac{dz}{dx}$
   - $\forall x \neq 0 \quad F(x, y) = F(x, xz) = F(1, z)$
3. The ODE becomes $z + x\frac{dz}{dx} = F(1, z)$
   - This is separable in variables x and z

10.8 First Order Linear ODE

Definition. A first order ODE $\frac{dy}{dx} = F(x, y)$ is **linear** if

$$F(x, y) = f(x)y + g(x)$$

The **standard form** of a first order linear ODE is

$$\frac{dy}{dx} + p(x)y = q(x)$$

10.8.1 Method to Solve

- If $\dfrac{dy}{dx} + p(x)y = 0$. Then it is a separable equation.
- If $\dfrac{dy}{dx} + p(x)y = q(x)$, proceed as follows:
  1. Evaluate $\int p(x)\,dx = P(x) + C$
  2. Evaluate an **integrating factor** $v(x) = e^{P(x)}$
  3. $y = \dfrac{1}{v(x)} \displaystyle\int v(x)q(x)\,dx$
  4. **Remark.** Different integrating factors differ by a constant multiple and produce the same solution

10.9 Bernoulli's Equation

Definition. A **Bernoulli's differential equation** has the form

- $\dfrac{dy}{dx} + p(x)y = q(x)y^n$, where $n \in \mathbb{R}$

10.9.1 Method to Solve

- If $n = 0$, it is a first order linear ODE
- If $n = 1$, it is a first order linear and separable ODE
- If $n \neq 0, 1$, proceed as follows:
  1. Let $z = y^{1-n}$
  2. $\dfrac{dz}{dx} + (1-n)p(x)z = (1-n)q(x)$, which is a linear equation

10.10 Exponential Growth and Decay

Theorem. The general solution to $\dfrac{dy}{dt} = ky$ is

$$y = C\exp(kt)$$

10.11 Continuously Compounded Interest

Theorem. If an amount A_0 is invested at r interest. If the interest is compounded continuously, the value of the investment at time t is

$$A(t) = A_0 \exp(rt)$$

10.12 Exponential Decay

Let $\frac{dm}{dt} = km$, where $k < 0$. Then

$$m(t) = m(0)\exp(kt)$$

10.12.1 Half-Life

half-life $\boldsymbol{t_{1/2}}$ is the time required for half of the quantity to decay. Then

$$k = -\frac{\ln 2}{t_{1/2}}$$

10.13 Logistic Population Growth

Definition. The **logistic growth** model with $M > 0$ as the **limiting population** or **carrying capacity** and $r > 0$ is given by

$$\frac{dP}{dt} = r(M - P)P$$

10.13.1 Solving the Logistic Growth Model

This is a Bernoulli's differential equation with the following general solution which is also known as the **logistic function**:

$$P(t) = \frac{M}{1 + C\exp(-Mrt)}$$

10.14 Newton's Law of Cooling

Let $T(t)$ be the temperature of an object at time t and T_S be the surrounding temperature. Then

- $\frac{dT}{dt} = -r(T - T_S)$, where $r > 0$ is a constant

Then

$$T(t) = T_S + (T_0 - T_S)\exp(-rt)$$

11 Other Results

11.1 Geometry

11.1.1 Slope of Perpendicular Lines

Given a pair of perpendicular lines ℓ_1 and ℓ_2 with slopes m_1 and m_2 respectively, then

$$m_1 \cdot m_2 = -1$$

11.1.2 Circle

The equation of a circle with radius r and centre (a, b) is

$$(x-a)^2 + (y-b)^2 = r^2$$

11.1.3 Ellipse

The equation of an ellipse with width $2a$, height $2b$, and center (x_0, y_0) is

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$$

11.1.4 Hyperbola

The equation of a hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

11.1.5 Astroid

The equation of an astroid with radius of the fixed circle a is

$$x^{2/3} + y^{2/3} = a^{2/3}$$

11.1.6 Sphere

11.1.6.1 Volume The volume V of a sphere with radius r is

$$V = \frac{4}{3}\pi r^3$$

11.1.6.2 Surface Area The surface area A of a sphere with radius r is

$$A = 4\pi r^2$$

11.1.7 Right Cylinder

11.1.7.1 Volume The volumne V of a right cylinder with base area A and height H is

$$V = AH$$

11.1.8 Cone

11.1.8.1 Volume The volume V of a cone with base area A and height H is

$$V = \frac{1}{3}AH$$

11.1.9 Right Circular Cone

11.1.9.1 Volume The volume V of a right circular cone with radius r and height h is

$$V = \frac{1}{3}\pi r^2 h$$

11.1.9.2 Surface Area A right circular cone with radius r and height h has slant height $l = \sqrt{r^2 + h^2}$. Its surface area A is

$$A = \pi r^2 + \pi r l$$

11.2 Triangle Inequality

For any a, $b \in \mathbb{R}$,

$$|a| - |b| \leq |a + b| \leq |a| + |b|$$

11.3 Trigonometric Identities

11.3.1 Sum and Difference Formula

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

11.3.2 Double-Angle Formulae

$$\sin(2\theta) = 2\sin\theta\cos\theta = \frac{2\tan\theta}{1 + \tan^2\theta}$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \frac{1 - \tan^2\theta}{1 + \tan^2\theta}$$
$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$$
$$\cot(2\theta) = \frac{\cot^2\theta - 1}{2\cot\theta}$$
$$\sec(2\theta) = \frac{\sec^2\theta}{2 - \sec^2\theta}$$
$$\csc(2\theta) = \frac{\sec\theta\csc\theta}{2}$$

11.3.3 Others

$\frac{\sin x}{x} < 1$ and $|\sin x| \leq |x|$

For $x \in (0, \frac{\pi}{2})$:

$$x < \tan x$$

11.4 Algebraic Identities

11.4.1 Difference of Cubes Formula

$$b^3 - c^3 = (b - c)(b^2 + bc + c^2)$$

11.4.2 Difference of Nth Power Formula

$$b^n - c^n = (b - c)(b^{n-1} + b^{n-2}c + b^{n-3}c^2 + \ldots + bc^{n-2} + c^{n-1})$$

11.5 Binomial Theorem

For $n \in \mathbb{Z}, n \geq 0$,

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

where $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$