MA2001 Assignment 2

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(i) Using Example 3.1.8 from text,

$$U = \{(3t - 2s, s, t) \mid s, t \in \mathbb{R}\}$$
$$V = \{(2t - 3s, s, t) \mid s, t \in \mathbb{R}\}$$

(ii) Implicit notation

$$U \cap V = \{(x, y, z) \mid x + 2y - 3z = x + 3y - 2z = 0\}$$

Since

$$\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix}$$

Explicit notation

$$U \cap V = \{(5t, -t, t) \mid t \in \mathbb{R}\}$$

(iii) Let $w = (t' - 2, t' + 1, t') \in W$ be arbitrary. Since

$$(t' - 2) + 2(t' + 1) - 3t' = t'(1 + 2 - 3) - 2 + 2 = 0$$

we have $w \in U$. Since the choice of w is arbitrary, $W \subseteq U$.

(iv)

$$(t - 2) + 3(t + 1) - 2t = 0 \implies 2t + 1 = 0 \implies t = \frac{-1}{2}$$

$$\begin{aligned} U \cap V \cap W &= V \cap W && \text{as } W \subseteq U \\ &= \{(t-2, t+1, t) \mid t \in \mathbb{R}, (t-2) + 3(t+1) - 2t = 0\} \\ &= \{(t-2, t+1, t) \mid t = \frac{-1}{2}\} \\ &= \{(\frac{-1}{2} - 2, \frac{-1}{2} + 1, \frac{-1}{2})\} \\ &= \{(\frac{-5}{2}, \frac{1}{2}, \frac{-1}{2})\} \end{aligned}$$

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(i)

$$\left(\begin{array}{cccc|c|c|c} 1 & -1 & 0 & 0 & 1 & 0 & 1 \\ 2 & 1 & 1 & 2 & 0 & 2 & 1 \\ 0 & -3 & 1 & -4 & 1 & -1 & 2 \\ -1 & 1 & 0 & 0 & -1 & 0 & -1 \end{array}\right) \xrightarrow{rref} \left(\begin{array}{cccc|c|c|c} 1 & 0 & 0 & 1 & 1/2 & 1/2 & 1/2 \\ 0 & 1 & 0 & 1 & -1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 & -1 & -1/2 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Since the system is consistent, $S_1 \subseteq \text{span}(S_2)$. By Theorem 3.2.10, $\text{span}(S_1) \subseteq \text{span}(S_2)$.

(ii) From above, the system S_2 represents is not always consistent since its row-echelon form contains a zero row. Therefore $\operatorname{span}(S_2) \neq \mathbb{R}^4$. (Alternatively, it is easy to show that $(0,0,0,1) \notin \operatorname{span}(S_2)$.)

(iii)
$$\left(\begin{array}{ccc|c|c|c|c} 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{rref} \left(\begin{array}{ccc|c|c|c|c} 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 1 & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$(a,b,c,d) \notin \operatorname{span}(S_1) \iff$ the system is not consistent $\iff a + d \neq 0$.

(iv) Let
$$\boldsymbol{A} = \begin{pmatrix} (S_2)_1 & (S_2)_2 & (S_2)_3 & (S_2)_4 \end{pmatrix}$$
be a square matrix. By above $\operatorname{rank}(A) = 3$. By rank-nullity theorem we can write $\operatorname{span}(\{\boldsymbol{n}\}) = \operatorname{null}(\boldsymbol{A^T})$ for some $\boldsymbol{n} \in \mathbb{R}^4$. Now
$$\operatorname{span}(S_2) = \{\boldsymbol{A}\boldsymbol{v} \mid \boldsymbol{v} \in \mathbb{R}^4\} = \boldsymbol{C}(\boldsymbol{A}) = \boldsymbol{R}(\boldsymbol{A}^T) = \{\boldsymbol{v} \mid \boldsymbol{v} \cdot \boldsymbol{n} = 0\}$$
Therefore such equation exists. In fact it is $x + w = 0$.

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(i)
$$\begin{aligned} V_1 &= \{(t-2s, s+3t, 3s, t) \mid s, t \in \mathbb{R}\} \\ &= \{(-2s, s, 3s, 0) + (t, 3t, 0, t) \mid s, t \in \mathbb{R}\} \\ &= \operatorname{span}(\{(-2,1,3,0), (1,3,0,1)\}) \end{aligned}$$
is a subspace of $\mathbb{R}^4$.

(ii) Let
$$W = \operatorname{span}(\{(1,10,3,3)\})$$
Clearly W contains a vector of form $(*,*,3,3)$ and is a subspace of $\mathbb{R}^4$. Now since
$$\begin{pmatrix} 1 \\ 10 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 3 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$
but $(-2,1,3,0) \notin W$ we have $W \subset V$.

(iii) Note that $(0,1,0,-1) \in V_2$ and $(-1,1,-1,1) \in V_2$ but $(0,1,0,-1)+(-1,1,-1,1) = (-1,2,-1,0) \notin V_2$. Therefore V_2 is not a subspace of $\mathbb{R}^4$.

(iv) $\{(0,0,0,0)\} \subseteq V_2$ since $\mathbf{0} \in V_2$. Clearly the zero space is closed under vector addition and scalar multiplication.

4

(i) Since $S \subseteq \operatorname{span}(S) = V$, $\boldsymbol{x} = a\boldsymbol{u} + b\boldsymbol{v} + c\boldsymbol{w}$ for some $a, b, c \in \mathbb{R}$ as $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is a basis for V. Then $\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w} + \boldsymbol{x} = (a+1)\boldsymbol{u} + (b+1)\boldsymbol{v} + (c+1)\boldsymbol{w}$ is a linear combination of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$.

(ii) If $\boldsymbol{x} = \boldsymbol{u}$ then $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{x}\}$ are not linearly independent.

(iii) If $\boldsymbol{x} = \boldsymbol{u}$ then $\mathbf{0} \in \{\boldsymbol{u} - \boldsymbol{x}, \boldsymbol{v} - \boldsymbol{x}, \boldsymbol{w} - \boldsymbol{x}\}$ are not linearly independent and therefore not a basis for V.

(iv) $\{\boldsymbol{v},\boldsymbol{w}\} \subset \{\boldsymbol{v},\boldsymbol{w},\boldsymbol{x}\} \implies \text{span}(\{\boldsymbol{v},\boldsymbol{w}\}) \subset \text{span}(\{\boldsymbol{v},\boldsymbol{w},\boldsymbol{x}\})$. Let $\boldsymbol{y} \in \text{span}(\{\boldsymbol{v},\boldsymbol{w},\boldsymbol{x}\}) \setminus \text{span}(\{\boldsymbol{v},\boldsymbol{w}\})$. Then $a \neq 0$ since for any $g,h \in \mathbb{R}$,

$$\begin{aligned}\boldsymbol{y} &= a\boldsymbol{x} + b\boldsymbol{v} + c\boldsymbol{w}\\ &= a(d\boldsymbol{u} + e\boldsymbol{v} + f\boldsymbol{w}) + b\boldsymbol{v} + c\boldsymbol{w}\\ &= (ad)\boldsymbol{u} + (ae+b)\boldsymbol{v} + (af+c)\boldsymbol{w}\\ &\neq g\boldsymbol{v} + h\boldsymbol{w}\end{aligned}$$

Since $a \neq 0$ we have $\text{span}(\{\boldsymbol{v},\boldsymbol{w},\boldsymbol{x}\}) = \text{span}(\{\boldsymbol{v},\boldsymbol{w},a\boldsymbol{x}+b\boldsymbol{v}+c\boldsymbol{w}\}) = \text{span}(\{\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\}) = V$.

(v) Let $\boldsymbol{a},\boldsymbol{b} \in V$ be arbitrary, and $U = \{\boldsymbol{y}+\boldsymbol{z} \mid \boldsymbol{z} \in V\}$. Since $\boldsymbol{a}+\boldsymbol{b} \in V$ and $\boldsymbol{y} = (\boldsymbol{y}+\boldsymbol{a}+\boldsymbol{b}) - (\boldsymbol{a}+\boldsymbol{b}) \notin V$, $(\boldsymbol{y}+\boldsymbol{a}+\boldsymbol{b}) \notin V$. Since $(\boldsymbol{y}+\boldsymbol{a}),(\boldsymbol{y}+\boldsymbol{b}) \in U$ but $(\boldsymbol{y}+\boldsymbol{a})+(\boldsymbol{y}+\boldsymbol{b}) = \boldsymbol{y}+(\boldsymbol{y}+\boldsymbol{a}+\boldsymbol{b}) \notin U$, U is not a subspace of $\mathbb{R}^n$.

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(i) Since S spans V it suffices to prove that vectors in S are linearly independent. Now

$$\begin{pmatrix}1&1&2&3&4\\1&2&2&3&3\\1&1&2&2&3\end{pmatrix} \xrightarrow{rref} \begin{pmatrix}1&0&2&0&2\\0&1&0&0&-1\\0&0&0&1&1\end{pmatrix}$$

Since after Gauss-Jordan elimination there are no zero rows, vectors in S are linearly independent and therefore S is a basis for V.

(ii) Since

$$\left(\begin{array}{ccc|c}1&1&1&0\\1&2&1&-5\\2&2&2&0\\3&3&2&-3\\4&3&3&2\end{array}\right) \xrightarrow{rref} \left(\begin{array}{ccc|c}1&0&0&2\\0&1&0&-5\\0&0&1&3\\0&0&0&0\\0&0&0&0\end{array}\right)$$

$\boldsymbol{v} = (0,-5,0,-3,2) \in V$ and its coordinate vector is $(2,-5,3)$ with respect to S.

(iii) Since

$$\left(\begin{array}{ccc|c|c|c}3&2&2&1&1&1\\3&3&3&1&2&1\\6&4&4&2&2&2\\7&5&6&3&3&2\\10&6&7&4&3&3\end{array}\right) \xrightarrow{rref} \left(\begin{array}{ccc|c|c|c}1&0&0&1/3&-1/3&1/3\\0&1&0&-2/3&2/3&1/3\\0&0&1&2/3&1/3&-1/3\\0&0&0&0&0&0\\0&0&0&0&0&0\end{array}\right)$$

$S \subseteq \text{span}(T) \implies \text{span}(S) \subseteq \text{span}(T)$. Since

$$\left(\begin{array}{ccc|c|c|c}1&1&1&3&2&2\\1&2&1&3&3&3\\2&2&2&6&4&4\\3&3&2&7&5&6\\4&3&3&10&6&7\end{array}\right) \xrightarrow{rref} \left(\begin{array}{ccc|c|c|c}1&0&0&1&0&1\\0&1&0&0&1&1\\0&0&1&2&1&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{array}\right)$$

$T \subseteq \text{span}(S) \implies \text{span}(T) \subseteq \text{span}(S)$. Therefore $\text{span}(S) = \text{span}(T)$. The coordinate vectors of T with respect to S are $\{(1,0,2),(0,1,1),(1,1,0)\}$. Since

$$\begin{pmatrix}1&0&2\\0&1&1\\1&1&0\end{pmatrix} \xrightarrow{rref} \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} = \boldsymbol{I}$$

The set of vectors T is linearly independent. Therefore T is also a basis for V.

(iv)

$$\boldsymbol{w} = (3,3,6,7,10) + (2,3,4,5,6) - (2,3,4,6,7) = (3,3,6,6,9)$$

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(i) For $\boldsymbol{A}$,

$$\begin{pmatrix} 1 & 3 & 1 & 3 \\ 3 & -1 & 3 & -1 \\ 2 & 1 & 2 & 1 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Basis of null space is of form

$$\begin{pmatrix} x_1 \\ x_2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \\ 0 \\ 1 \end{pmatrix}$$

Reading off rows with pivots,

$$S = \{s \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \mid s, t \in \mathbb{R}\}$$

Likewise for $\boldsymbol{B}$,

$$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 2 & 1 & 1 & 2 \\ 0 & 1 & 3 & 2 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence,

$$T = \{s \begin{pmatrix} 1 \\ -3 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \end{pmatrix} \mid s, t \in \mathbb{R}\}$$

(ii) Basis of S is

$$\{\boldsymbol{B}(S) = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}\}$$

Basis of T is

$$\{\boldsymbol{B}(T) = \begin{pmatrix} 1 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \end{pmatrix}\}$$

(iii)

$$\begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & -3 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \boldsymbol{I}$$

Therefore $\boldsymbol{B}(S) \cup \boldsymbol{B}(T)$ spans $\mathbb{R}^4$ and is a basis of $\mathbb{R}^4$. Then $\boldsymbol{v} = (a\boldsymbol{s_1} + b\boldsymbol{s_2}) + (c\boldsymbol{t_1} + d\boldsymbol{t_2}) = \boldsymbol{s} + \boldsymbol{t}$ for some $a, b, c, d \in \mathbb{R}$. These constants are uniquely determined, otherwise the basis vectors are linearly dependent, which contradicts. Hence here $\boldsymbol{s}, \boldsymbol{t}$ are uniquely determined.

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(i) Since

$$\left(\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 0 & 3 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & -3 & 1 \end{array}\right) \xrightarrow{rref} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

The system is inconsistent. Therefore $S \nsubseteq \{ax + by + cz = 1 \mid a, b, c \in \mathbb{R}\}$. Hence no such plane exists.

(ii)

$$\begin{aligned} U &= \{\boldsymbol{u_1} + s(\boldsymbol{u_1} - \boldsymbol{u_3}) + t(\boldsymbol{u_1} - \boldsymbol{u_4}) \mid s, t \in \mathbb{R}\} \\ &= \{\boldsymbol{x} \mid \boldsymbol{x} \cdot ((\boldsymbol{u_1} - \boldsymbol{u_3}) \times (\boldsymbol{u_1} - \boldsymbol{u_4})) = \boldsymbol{u_1} \cdot ((\boldsymbol{u_1} - \boldsymbol{u_3}) \times (\boldsymbol{u_1} - \boldsymbol{u_4}))\} \\ &= \{\boldsymbol{x} \mid \boldsymbol{x} \cdot 14(1,0,0) = \boldsymbol{u_1} \cdot 14(1,0,0)\} \\ &= \{(x, y, z) \mid x = 1\} \end{aligned}$$

$$\begin{aligned} V &= \{\boldsymbol{u_2} + s(\boldsymbol{u_2} - \boldsymbol{u_3}) + t(\boldsymbol{u_2} - \boldsymbol{u_4}) \mid s, t \in \mathbb{R}\} \\ &= \{\boldsymbol{x} \mid \boldsymbol{x} \cdot ((\boldsymbol{u_2} - \boldsymbol{u_3}) \times (\boldsymbol{u_2} - \boldsymbol{u_4})) = \boldsymbol{u_2} \cdot ((\boldsymbol{u_2} - \boldsymbol{u_3}) \times (\boldsymbol{u_2} - \boldsymbol{u_4}))\} \\ &= \{\boldsymbol{x} \mid \boldsymbol{x} \cdot 2(4,2,1) = \boldsymbol{u_2} \cdot 2(4,2,1)\} \\ &= \{(x, y, z) \mid 4x + 2y + z = 7\} \end{aligned}$$

The system is

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 4 & 2 & 1 & 7 \end{array}\right)$$

(iii) Choose the three points to be $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$. Then

$$\begin{pmatrix} 1 & 4 & 2 \\ 0 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & 2/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{pmatrix}$$

Since the null space is $\text{span}((-2, -1, 3))$, the equation of plane P is $-2x - y + 3z = 0$ which is a subspace of $\mathbb{R}^3$.

(iv) Three points lie on a plane that corresponds to a subspace if and only if they lie on a plane that contains the origin. ($\Leftarrow$): Let $\boldsymbol{u} = (a, b, c), \boldsymbol{v} = (d, e, f) \in P$. Then $pa + qb + rc = pd + qe + rf = 0$ for some p, q, r determined by P. Then $p(a + d) + q(b + e) + r(c + f) = 0 \implies \boldsymbol{u} + \boldsymbol{v} \in P$. Since the origin acts as additive identity and points on P are closed under addition and scalar multiplication, P corresponds to a subspace. ($\Rightarrow$): A subspace contains an additive identity which in this case corresponds to the origin.