NATIONAL UNIVERSITY OF SINGAPORE

MA1102R — CALCULUS

SEMESTER 2: AY 2019/2020

2 May 2020   09:00 – 11:30

INSTRUCTIONS TO CANDIDATES

1. **Get ready a signed copy of the Exam declaration form for this exam**.
2. Use A4 size paper and pen (blue or black ink) to write your answers.
3. Write down your student number clearly on the top left of every page of the answers. **Do not write your name.**
4. Write on **one side of the paper only**. Write the question number and page number on the top right corner of each page (e.g. Q1P1, Q1P2, . . ., Q2P1, . . .).
5. This examination paper contains **EIGHT (8)** questions and comprises **FIVE (5)** pages. Answer **ALL** questions.
6. The total mark for this paper is **ONE HUNDRED (100)**.
7. This is an **OPEN BOOK** examination.
8. You may use any calculator. However, you should lay out systematically the various steps in the calculations.
9. **Join the Zoom conference** and turn on the video setting at all time during the exam. Adjust your camera such that your face and upper body including your hands are captured on Zoom.
10. You may go for a short toilet break (not more than 5 minutes) during the exam.
11. At the end of the exam,
    - scan or take pictures of your work (make sure the images can be read clearly) together with the declaration form;
    - merge all your images into one pdf file (arrange them in the order: Declaration form, Q1 to Q8 in their page sequence);
    - name the pdf file by Matric No_MA1102R (e.g. A123456B_MA1102R);
    - upload your pdf into the LumiNUS folder “Exam Submission”.
12. The Exam Submission folder will close at **11:30 hr (including preparing and uploading answers)**. After the folder is closed, exam answers that are not submitted will not be accepted, unless there is a valid reason.

Question 1 [15 marks]

(a) Let $f(x) = x^3 - x^2 + x - 2$.

(i) Prove that $f(x) = 0$ has a solution between -1 and 2.

(ii) Prove that $f(x) = 0$ has exactly one solution on $\mathbb{R}$.

(b) Let $g(x) = \dfrac{\sqrt{x}}{\sqrt{x} - 3}$.

(i) Prove that g is one-to-one.

(ii) Find g^{-1}.

(iii) Identify the domain and range of g^{-1}.

(c) Find $\displaystyle\int x \sec^2 x \, dx$.

Question 2 [11 marks]

(a) Evaluate $\displaystyle\lim_{x\to 0} \left(\frac{1}{\sin^2 x} - \frac{1}{x^2} \right)$.

(b) Use **only** the ϵ, δ-definition to show that $\displaystyle\lim_{x\to 1} \left(x + \frac{1}{x^2+1} \right) = \frac{3}{2}$.

Question 3 [11 marks]

Let $f(x) = \begin{cases} (\sin x)^{\sin x} & \text{if } 0 < x < \pi, \\ \lim\limits_{x\to 0^+} (\sin x)^{\sin x} & \text{if } x = 0, \\ \lim\limits_{x\to \pi^-} (\sin x)^{\sin x} & \text{if } x = \pi. \end{cases}$

In this question, give your answers in **simplified exact** form.

(i) Find $f(0)$ and $f(\pi)$.

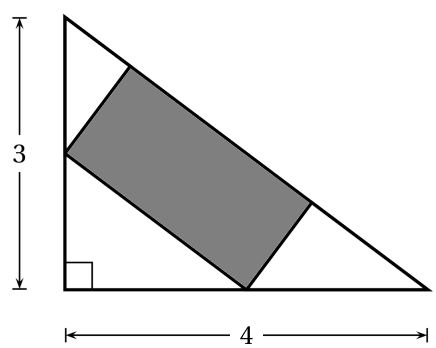
(ii) Find the open intervals on which f is increasing.

(iii) Find the open intervals on which f is decreasing.

(iv) Find the coordinates of the absolute maximum and minimum points of f on the interval $[0, \pi]$.

Question 4 [11 marks]

Determine **dimensions and area** of the rectangle of the largest area that can be inscribed in the right-angled triangle with base length 4 and height length 3 as shown in the following figure.



Question 5 [12 marks]

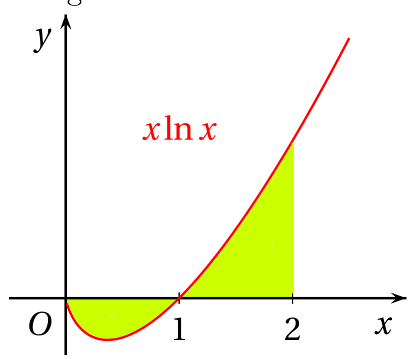
(a) Find the length of the curve

$$y = \int_0^x \sqrt{\cos 2t}\, dt, \quad 0 \le x \le \pi/4.$$

(b) The region between the x-axis and the curve

$$y = f(x) = \begin{cases} 0, & \text{if } x = 0, \\ x \ln x, & \text{if } 0 < x \le 2, \end{cases}$$

is revolved about the x-axis to generate the solid.



(i) Prove that f is continuous from right at $x = 0$.

(ii) Find the volume of the solid. Give your answer in simplified exact form.

Question 6 [16 marks]

(a) Use Riemann sums to evaluate

$$\lim_{n\to\infty} \sum_{k=1}^{n} \ln \sqrt[n]{1 + (k/n)}$$

(b) Find

$$\int \frac{dx}{x(x+1)(x+2)\cdots(x+m)}.$$

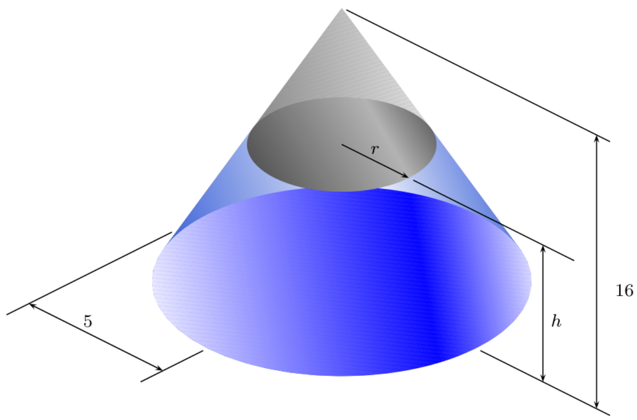
Here, m is a positive integer. Give your answer in terms of x and m.

Question 7 [16 marks]

(a) Solve the initial value problem:

$$\frac{dy}{dx} = -\frac{y}{e^y + x}, \quad y(0) = 1.$$

(b) A right circular **cone** tank has radius 5 ft and height 16 ft. The tank is being drained at $0.5\sqrt{h}\,\text{ft}^3/\text{min}$ when the height of water in the tank is h ft. Suppose at $t = 0$, the tank is full.



(i) Derive a formula for the height h ft of the water at time t min. Give your answer in terms of $t = f(h)$, where f is a function of h.

(ii) How long does it take to empty the tank? Give your answer in simplified exact form.

Question 8 [8 marks]

Suppose f is a continuous decreasing function on $[0, \infty)$ and $\displaystyle\int_0^\infty f(t)\,dt$ converges. Prove that

$$\lim_{x \to \infty} x f(x) = 0.$$

[End of Paper]