1

(i)

$$\left(\begin{array}{ccccc|c} 0 & 1 & 2 & -1 & 0 & a \\ 0 & 0 & 0 & 1 & 1 & b \\ 1 & -1 & 2 & 0 & 1 & c \\ 0 & -1 & -2 & 1 & 0 & d \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 1 & c \\ 0 & 0 & 0 & 1 & 1 & b \\ 0 & 1 & 2 & -1 & 0 & a \\ 0 & -1 & -2 & 1 & 0 & d \end{array}\right)$$

$$\xrightarrow{R_4 + R_3} \left(\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 1 & c \\ 0 & 0 & 0 & 1 & 1 & b \\ 0 & 1 & 2 & -1 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & a+d \end{array}\right)$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 1 & c \\ 0 & 1 & 2 & -1 & 0 & a \\ 0 & 0 & 0 & 1 & 1 & b \\ 0 & 0 & 0 & 0 & 0 & a+d \end{array}\right)$$

$$\xrightarrow{R_2 + R_3} \left(\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 1 & c \\ 0 & 1 & 2 & 0 & 1 & a+b \\ 0 & 0 & 0 & 1 & 1 & b \\ 0 & 0 & 0 & 0 & 0 & a+d \end{array}\right)$$

$$\xrightarrow{R_1 + R_2} \left(\begin{array}{ccccc|c} 1 & 0 & 4 & 0 & 2 & a+b+c \\ 0 & 1 & 2 & 0 & 1 & a+b \\ 0 & 0 & 0 & 1 & 1 & b \\ 0 & 0 & 0 & 0 & 0 & a+d \end{array}\right)$$

(ii) (a) No solution if $a + d \neq 0$.

(b) The system does not have unique solution, as the matrix is of rank 3 but the system has 5 unknowns.

(c) Infinitely many solution exists if $a + d = 0$.

(iii) Reading off the matrix above or otherwise, we have

$$\begin{aligned} x_1 &= -4u + -2v \\ x_2 &= -2u - v \\ x_3 &= u \\ x_4 &= -v \\ x_5 &= v \end{aligned}$$

where u, v are arbitrary parameters.

(iv) By above or otherwise, we have

$$
\begin{aligned}
x_1 &= -4x_3 + -2x_5 + a + b + c \\
x_2 &= -2x_3 - x_5 + a + b \\
x_3 &= x_3 \\
x_4 &= -x_5 + b \\
x_5 &= x_5
\end{aligned}
$$

Substituting $x_3 = x_5 = 0$, the particular solution is

$$
\begin{aligned}
x_1 &= a + b + c \\
x_2 &= a + b \\
x_3 &= 0 \\
x_4 &= b \\
x_5 &= 0
\end{aligned}
$$

2

(i) Using MATLAB command, the reduced row echelon form of the augmented matrix is

$$
\left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)
$$

The system is consistent.

(ii)

$$
\begin{pmatrix} 1 & 2 \\ -2 & 1 \\ 1 & -5 \\ 5 & 1 \end{pmatrix} \boldsymbol{x} = \begin{pmatrix} 0 \\ -5 \\ 7 \\ 9 \end{pmatrix}
$$

(iii) Using MATLAB command or otherwise, we have

$$
\boldsymbol{A}^T \boldsymbol{A} = \begin{pmatrix} 1 & -2 & 1 & 5 \\ -2 & 1 & -5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -2 & 1 \\ 1 & -5 \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} 31 & 0 \\ 0 & 31 \end{pmatrix}
$$

$$
\boldsymbol{A}^T \boldsymbol{b} = \begin{pmatrix} 1 & -2 & 1 & 5 \\ -2 & 1 & -5 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -5 \\ 7 \\ 9 \end{pmatrix} = \begin{pmatrix} 62 \\ -31 \end{pmatrix}
$$

(iv) Note that $\boldsymbol{A}^T\boldsymbol{A}$ is invertible since it is a scalar matrix.

$$\begin{aligned}
\boldsymbol{x} &= (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T\boldsymbol{b} \\
&= \begin{pmatrix} 31 & 0 \\ 0 & 31 \end{pmatrix}^{-1} \begin{pmatrix} 62 \\ -31 \end{pmatrix} \\
&= (31\boldsymbol{I})^{-1} \begin{pmatrix} 62 \\ -31 \end{pmatrix} \\
&= \frac{1}{31} \begin{pmatrix} 62 \\ -31 \end{pmatrix} \\
&= \begin{pmatrix} 2 \\ -1 \end{pmatrix}
\end{aligned}$$

(v) Note that the converse is true (by substitution) i.e. every solution of (*) satisfies equation in (iv). From above, this equation has a unique solution. Therefore (*) has at most one solution. But since it is consistent, (*) has at least one solution. Therefore (*) has a unique solution which also (uniquely) satisfies (iv). Therefore the solution in (iv) satisfies (*).

3

(i) Using MATLAB command or otherwise, we have

$$\begin{aligned}
\det \boldsymbol{A} &= (a+b)[(a)(2b) - (a+3b)(2b)] \\
&\quad - a[(a+b)(2b) - (a+3b)(a+b)] \\
&\quad + b[(a+b)(2b) - (a)(a+b)] \\
&= a^3 + a^2b - 4ab^2 - 4b^3 \\
&= a(a^2 - 4b^2) + b(a^2 - 4b^2) \\
&= (a+b)(a-2b)(a+2b)
\end{aligned}$$

For $\boldsymbol{A}$ to be invertible, $\det \boldsymbol{A} \neq 0$. Conditions are $a \neq -b$ and $a \neq \pm 2b$

(ii) When $\boldsymbol{A}$ is singular, $\det \boldsymbol{A} = 0$. Therefore exactly one of $a + b = 0, a - 2b = 0, a + 2b = 0$ holds since $a, b \neq 0$.

- Let $a + b = 0$ or $b = -a$.

$$\begin{aligned}
\boldsymbol{A} &= \begin{pmatrix} a+b & a & b \\ a+b & a & a+3b \\ a+b & 2b & 2b \end{pmatrix} \\
&= \begin{pmatrix} 0 & a & -a \\ 0 & a & -2a \\ 0 & -2a & -2a \end{pmatrix} \\
&\xrightarrow{R_1/a}\xrightarrow{R_2/a}\xrightarrow{R_3/a} \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & -2 \\ 0 & -2 & -2 \end{pmatrix} \\
&\xrightarrow{R_3/-2}\xrightarrow{R_2-R_1}\xrightarrow{R_3-R_1} \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 2 \end{pmatrix} \\
&\xrightarrow{R_2/-1}\xrightarrow{R_3-2R_2}\xrightarrow{R_1+R_2} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

- Let $a - 2b = 0$ or $a = 2b$.

$$\begin{aligned}
\boldsymbol{A} &= \begin{pmatrix} a+b & a & b \\ a+b & a & a+3b \\ a+b & 2b & 2b \end{pmatrix} \\
&= \begin{pmatrix} 3b & 2b & b \\ 3b & 2b & 5b \\ 3b & 2b & 2b \end{pmatrix} \\
&\xrightarrow{R_1/b}\xrightarrow{R_2/b}\xrightarrow{R_3/b} \begin{pmatrix} 3 & 2 & 1 \\ 3 & 2 & 5 \\ 3 & 2 & 2 \end{pmatrix} \\
&\xrightarrow{R_2-R_1}\xrightarrow{R_3-R_1} \begin{pmatrix} 3 & 2 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & 1 \end{pmatrix} \\
&\xrightarrow{R_1/3}\xrightarrow{R_2-4R_3}\xrightarrow{R_2\leftrightarrow R_3} \begin{pmatrix} 1 & \frac{2}{3} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

- Let $a + 2b = 0$ or $a = -2b$.

$$\begin{aligned}
\boldsymbol{A} &= \begin{pmatrix} a+b & a & b \\ a+b & a & a+3b \\ a+b & 2b & 2b \end{pmatrix} \\
&= \begin{pmatrix} -b & -2b & b \\ -b & -2b & b \\ -b & 2b & 2b \end{pmatrix} \\
&\xrightarrow{R_1/-b}\xrightarrow{R_2/-b}\xrightarrow{R_3/-b} \begin{pmatrix} 1 & 2 & -1 \\ 1 & 2 & -1 \\ 1 & -2 & -2 \end{pmatrix} \\
&\xrightarrow{R_2-R_1}\xrightarrow{R_3-R_1}\xrightarrow{R_2\leftrightarrow R_3} \begin{pmatrix} 1 & 2 & -1 \\ 0 & -4 & -1 \\ 0 & 0 & 0 \end{pmatrix} \\
&\xrightarrow{R_2/-4}\xrightarrow{R_1-2R_2} \begin{pmatrix} 1 & 0 & \frac{-3}{2} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

4

(i) The statement is false as

$$\left(\begin{array}{ccc|c} a & b & c & d \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

is not consistent.

(ii) $h \neq 0$ since $i \neq 0$ and the system is consistent. Therefore a, e are leading entries and so $a \neq 0$.

(iii) If the system represents three planes that intersect at a line, then the general solution of the system has exactly one parameter. Then the matrix is of rank 2. Therefore a, e are leading entries and h is not, and so $a, e \neq 0$.

(iv) (Similarly) If the general solution of the system has exactly one parameter, then the matrix is of rank 2. Therefore a, e are leading entries and h is not, and so $h = 0$.

5

(i) From EROs we have

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \boldsymbol{A} = \boldsymbol{B} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 0 & 1 \\ -2 & 1 & 1 \end{pmatrix} \boldsymbol{A}$$

(ii) Note that

$$\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & \frac{1}{2}\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 1\end{pmatrix}\boldsymbol{B} = \begin{pmatrix}1 & 1 & 0\\0 & 0 & 1\\0 & 0 & 0\end{pmatrix} = \boldsymbol{R}$$

Therefore

$$E_1 = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & \frac{1}{2}\end{pmatrix}$$
$$E_2 = \begin{pmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 1\end{pmatrix}$$
$$E_3 = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 1 & 1\end{pmatrix}$$
$$E_4 = \begin{pmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{pmatrix}$$
$$E_5 = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\-2 & 0 & 1\end{pmatrix}$$

(iii) All of the listed matrices are row equivalent to $\boldsymbol{A}$ since their reduced row echelon forms ($\boldsymbol{R}$) are equal to each other since it is invariant under multiplication with matrices representing elementary row operations.

(iv) Note that $\det \boldsymbol{A} = c \det \boldsymbol{R} = 0 (c \neq 0)$ since $\boldsymbol{R}$ contains one row with all zeros. Then $\det \boldsymbol{B} = \det \boldsymbol{C} \det \boldsymbol{A} = 0$. But

$$\det \boldsymbol{C} = \det \begin{pmatrix}1 & 0 & 0\\-2 & 0 & 1\\-2 & 1 & 1\end{pmatrix} = 1 \neq \det \boldsymbol{B}$$

So $\boldsymbol{B}$ and $\boldsymbol{C}$ are not row equivalent.

6

(i) From $\boldsymbol{AB}$ is symmetric we have

$$\begin{aligned}\boldsymbol{AB} &= \begin{pmatrix}b-2a+2c & a+4b-c & 2a-b-2c\\d-2b+2e & b+4d-e & 2b-d-2e\\e-2c+2f & c+4e-f & 2c-e-2f\end{pmatrix}\\&= \begin{pmatrix}b-2a+2c & d-2b+2e & e-2c+2f\\a+4b-c & b+4d-e & c+4e-f\\2a-b-2c & 2b-d-2e & 2c-e-2f\end{pmatrix} = \boldsymbol{BA}\end{aligned}$$

Comparing terms we have

$$\begin{aligned} a+4b-c &= d-2b+2e \\ 2a-b-2c &= e-2c+2f \\ 2b-d-2e &= c+4e-f \end{aligned}$$

Therefore

$$\begin{pmatrix} 1 & 6 & -1 & -1 & -2 & 0 \\ 2 & -1 & 0 & 0 & -1 & -2 \\ 0 & 2 & -1 & -1 & -6 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

(ii) By MATLAB, the general solution of the system is

$$\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} = d' \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + e' \begin{pmatrix} 0 \\ -1 \\ -8 \\ 0 \\ 1 \\ 0 \end{pmatrix} + f' \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

where d', e', f' are arbitrary.

(iii) By above, we have

$$\boldsymbol{A} = \begin{pmatrix} f & -e & -d-8e+f \\ -e & d & e \\ -d-8e+f & e & f \end{pmatrix}$$

7

(a) (i) Since $(\boldsymbol{BA})\boldsymbol{x} = \boldsymbol{B}(\boldsymbol{Ax}) = \boldsymbol{0}$ has a non trivial solution $\boldsymbol{x} \neq \boldsymbol{0}$, $\boldsymbol{BA}$ is not invertible. (Theorem 2.4.7 from text)

(ii) Consider $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\boldsymbol{B} = \boldsymbol{A}^T$. Note that

$$\boldsymbol{AB} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} = \boldsymbol{I}$$

Therefore $\boldsymbol{AB}$ can be invertible.

(iii) The row vectors of $\boldsymbol{A}$ are linearly dependent when $m > n$. Therefore $\boldsymbol{A}$ is row equivalent to another matrix containing one row of all zeros instead of the identity matrix. Therefore $\boldsymbol{AB}$ is not invertible. (Theorem 2.4.7 from text)

(b) (i) Take $\boldsymbol{A} = \boldsymbol{I}$ to be the identity matrix and $\boldsymbol{B} = \boldsymbol{0}$ to be the zero matrix. Clearly $\boldsymbol{AB}$ is not invertible.

(ii) Same construction as above.

8

Let $\boldsymbol{P} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B} \end{pmatrix}$.

(i) Note that

$$(\boldsymbol{A}_U \boldsymbol{B}_L)_{i,j} = \sum_{k=1}^{2n} \begin{pmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{pmatrix}_{i,k} \begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B} \end{pmatrix}_{k,j}$$

By property of identity matrix,

(a) When both $i, j <= n$,

$$(\boldsymbol{A}_U \boldsymbol{B}_L)_{i,j} = \sum_{k'=1}^{n} \boldsymbol{A}_{i,k'} \boldsymbol{I}_{k',j} = \sum_{k'=j} \boldsymbol{A}_{i,k'} \boldsymbol{I}_{k',j} = \boldsymbol{A}_{i,j} = \boldsymbol{P}_{i,j}$$

(b) Likewise when both $i, j >= n$,

$$(\boldsymbol{A}_U \boldsymbol{B}_L)_{i,j} = \sum_{k''=1}^{n} \boldsymbol{I}_{i-n,k''} \boldsymbol{B}_{k'',j-n} = \sum_{k''=i-n} \boldsymbol{I}_{i-n,k''} \boldsymbol{B}_{k'',j-n} = \boldsymbol{B}_{i-n,j-n} = \boldsymbol{P}_{i,j}$$

Otherwise, one of the (matrix) operands in the product is $\boldsymbol{0}$. In which case

$$(\boldsymbol{A}_U \boldsymbol{B}_L)_{i,j} = 0 = \boldsymbol{P}_{i,j}$$

Therefore,

$$\boldsymbol{A}_U \boldsymbol{B}_L = \boldsymbol{P} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B} \end{pmatrix}$$

(ii) Let $P(n)$ be the proposition that $\det \boldsymbol{A}_U = \det \boldsymbol{A}$ where $\boldsymbol{A}$ is an $n \times n$ matrix. The base case is trivially true, as for all a

$$\det \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = a$$

Assuming $P(n)$, consider $P(n+1)$. Let $\boldsymbol{M}_{i,j}$ and $\boldsymbol{M}'_{i,j}$ be the submatrix obtained from $\boldsymbol{A}$ and $\boldsymbol{A}_U$ by removing its ith row and jth column. Then by exchanging the (ordered) row containing $2n$ zeros and one 1 with the row above which (bubbling up), we have

$$\begin{aligned}
\det \boldsymbol{M}'_{1,j} &= \det \begin{pmatrix} \boldsymbol{M}_{1,j} & \boldsymbol{0}_{2n} \\ \boldsymbol{0}_{2n} & 1 \end{pmatrix} \\
&= (-1)^{2n} \det \begin{pmatrix} \boldsymbol{0}_{2n} & 1 \\ \boldsymbol{M}_{1,j} & \boldsymbol{0}_{2n} \end{pmatrix} && \text{Exchanging rows } 2n \text{ times} \\
&= (-1)^{4n} \det \boldsymbol{M}_{1,j} && \text{Expands } (2n+1)\text{th cofactor} \\
&= \det \boldsymbol{M}_{1,j}
\end{aligned}$$

Therefore $P(n+1)$ is true, since

$$\det \boldsymbol{A}_U = (\boldsymbol{A}_U)_{1,1} \det \boldsymbol{M}'_{1,j} = \boldsymbol{A}_{1,1} \det \boldsymbol{M}_{1,j} = \det \boldsymbol{A}$$

Hence $\det \boldsymbol{A}_U = \det \boldsymbol{A}$ by induction on n.
(Similarly) Let $P(n)$ be the proposition that $\det \boldsymbol{B}_L = \det \boldsymbol{B}$ where $\boldsymbol{B}$ is an $n \times n$ matrix. The base case is trivially true, as for all b

$$\det \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} = b$$

Assuming $P(n)$, consider $P(n+1)$. Let $\boldsymbol{M}_{i,j}$ and $\boldsymbol{M}'_{i,j}$ be the submatrix obtained from $\boldsymbol{B}$ and $\boldsymbol{B}_L$ by removing its ith row and jth column. Then

$$\det \boldsymbol{M}'_{1,j} = \det \begin{pmatrix} 1 & \boldsymbol{0}_{2n} \\ \boldsymbol{0}_{2n} & \boldsymbol{M}_{1,j} \end{pmatrix} = \det \boldsymbol{M}_{1,j}$$

Therefore $P(n+1)$ is true, since

$$\det \boldsymbol{B}_L = (\boldsymbol{B}_L)_{1,1} \det \boldsymbol{M}'_{1,j} = \det \boldsymbol{M}_{1,j} = \det \boldsymbol{B}$$

Hence $\det \boldsymbol{B}_L = \det \boldsymbol{B}$ by induction on n.

$$\det \begin{pmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B} \end{pmatrix} = \det \boldsymbol{A}_U \det \boldsymbol{B}_L = \det \boldsymbol{A} \det \boldsymbol{B}$$

(iii) Suppose $\boldsymbol{A}$ is invertible. Let $\boldsymbol{M} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{C} \\ \boldsymbol{0} & \boldsymbol{B} \end{pmatrix}$ and $\boldsymbol{M}' = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{A}^{-1}\boldsymbol{C} \\ \boldsymbol{0} & \boldsymbol{I} \end{pmatrix}$ be an upper triangular matrix. Clearly $\det \boldsymbol{M}' = 1$ since all the diagonal entries of $\boldsymbol{M}'$ is 1. Note that, ignoring zero terms, we have

$$(\boldsymbol{P}\boldsymbol{M}')_{i,j} = \begin{cases} \sum_{k=1}^{n} \boldsymbol{A}_{i,k} \boldsymbol{I}_{k,j} = \boldsymbol{A}_{i,j} & i, j \leq n \\ \sum_{k=1}^{n} \boldsymbol{A}_{i,k} (\boldsymbol{A}^{-1}\boldsymbol{C})_{k,j-n} = \boldsymbol{C}_{i,j-n} & i \leq n, j > n \\ 0 & i > n, j \leq n \\ \sum_{k=1}^{n} \boldsymbol{B}_{i-n,k} \boldsymbol{I}_{k,j-n} = \boldsymbol{B}_{i-n,j-n} & i, j > n \end{cases}$$

Hence $\boldsymbol{PM'} = \boldsymbol{M}$. Then

$$\det \boldsymbol{M} = \det \boldsymbol{P} \det \boldsymbol{M'} = \det \boldsymbol{A} \det \boldsymbol{B}$$

Otherwise, when $\det \boldsymbol{A} = 0$ the reduced row echelon form of $\boldsymbol{M}$ has a column that has no pivot since the reduced row echelon form of $\boldsymbol{A}$ contains one row of all zeros, and therefore $\det \boldsymbol{M} = 0$.

Hence $\det \boldsymbol{M} = \det \boldsymbol{A} \det \boldsymbol{B}$ is independent of $\boldsymbol{C}$.