MA2002 Assignment 2

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$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

Below I will quote this (Chapter 2, Theorem 7) without proof.

1

Note that

$$\lim_{x\to -1}\frac{\sin \pi x}{x+1}=-\lim_{x\to -1}\frac{\sin(\pi+\pi x)}{x+1}=-\pi\lim_{\pi(x+1)\to 0}\frac{\sin \pi(x+1)}{\pi(x+1)}=-\pi$$

$$\lim_{x\to -1}\frac{\sin\sin \pi x}{\sin \pi x}=\lim_{\sin \pi x\to 0}\frac{\sin\sin \pi x}{\sin \pi x}=1$$

Since both limits exist,

$$\lim_{x\to -1}\frac{\sin\sin \pi x}{x+1}=\lim_{x\to -1}\frac{\sin\sin \pi x}{\sin \pi x}\lim_{x\to -1}\frac{\sin \pi x}{x+1}=-\pi$$

Moreover,

$$\begin{aligned}
\lim_{x\to 2}\frac{\sqrt{x+3}-\sqrt{2x+1}}{\sqrt{3x-1}-\sqrt{4x-3}} &= \lim_{x\to 2}\frac{[(x+3)-(2x+1)](\sqrt{3x-1}+\sqrt{4x-3})}{[(3x-1)-(4x-3)](\sqrt{x+3}+\sqrt{2x+1})}\\
&= \lim_{x\to 2}\frac{(2-x)(\sqrt{3x-1}+\sqrt{4x-3})}{(2-x)(\sqrt{x+3}+\sqrt{2x+1})}\\
&= \frac{\sqrt{3(2)-1}+\sqrt{4(2)-3}}{\sqrt{(2)+3}+\sqrt{2(2)+1}}\\
&= 1
\end{aligned}$$

Since f is continuous, substituting we have

$$\begin{aligned}
-a+b&=-\pi\\
2a+b&=1
\end{aligned}$$

Solving,

$$a = \frac{(2a+b)-(-a+b)}{3} = \frac{1+\pi}{3}$$
$$b = 1 - \frac{2(1+\pi)}{3} = \frac{1-2\pi}{3}$$

2

Let

$$f(x) = (x-1)(x-3)(x-5)(x-7)(x-9)$$
$$g(x) = (x-2)(x-4)(x-6)(x-8)(x-10)$$
$$h(x) = f(x) - g(x)$$

be continuous functions (since they are polynomials).
Consider the equation

$$h(x) = 0$$

Note that

$$f(2) = (1)(-1)(-3)(-5)(-7) > 0$$
$$f(4) < 0 < f(6), f(8) < 0 < f(10)$$
$$g(2) = g(4) = g(6) = g(8) = g(10) = 0$$

Therefore, using intermediate value theorem,

$$\begin{aligned}
h(2) > 0 > h(4) &\implies \exists x \in (2,4) \quad h(x) = 0 \\
h(4) < 0 < h(6) &\implies \exists x \in (4,6) \quad h(x) = 0 \\
h(6) > 0 > h(8) &\implies \exists x \in (6,8) \quad h(x) = 0 \\
h(8) < 0 < h(10) &\implies \exists x \in (8,10) \quad h(x) = 0
\end{aligned}$$

Since the intervals are disjoint, the equation has at least four real roots.

3

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x} &= \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x + \Delta x - x}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})} \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})} \\
&= \frac{1}{2\sqrt{x}}
\end{aligned}$$

Note that

$$\lim_{\Delta x \to 0} 2x + \Delta x + \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} = 2x + \frac{1}{2\sqrt{x}}$$

$$\lim_{\Delta x \to 0} \sqrt{x^2 + \sqrt{x}} + \sqrt{(x+\Delta x)^2 + \sqrt{x + \Delta x}} = 2\sqrt{x^2 + \sqrt{x}}$$

Therefore since both limits exist,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x^2+\sqrt{x}} &= \lim_{\Delta x \to 0} \frac{\sqrt{(x+\Delta x)^2 + \sqrt{x+\Delta x}} - \sqrt{x^2+\sqrt{x}}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 + \sqrt{x+\Delta x} - (x^2 + \sqrt{x})}{\Delta x \left(\sqrt{(x+\Delta x)^2 + \sqrt{x+\Delta x}} + \sqrt{x^2+\sqrt{x}}\right)} \\
&= \lim_{\Delta x \to 0} \frac{2x + \Delta x + \frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x}}{\sqrt{(x+\Delta x)^2 + \sqrt{x+\Delta x}} + \sqrt{x^2+\sqrt{x}}} \\
&= \frac{\lim_{\Delta x \to 0} 2x + \Delta x + \frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x}}{\lim_{\Delta x \to 0} \sqrt{(x+\Delta x)^2 + \sqrt{x+\Delta x}} + \sqrt{x^2+\sqrt{x}}} \\
&= \frac{2x + \frac{1}{2\sqrt{x}}}{2\sqrt{x^2+\sqrt{x}}} \\
&= \frac{4x\sqrt{x}+1}{4\sqrt{x(x\sqrt{x}+1)}}
\end{aligned}$$

4

On C_1, differentiating both sides,

$$\frac{\mathrm{d}(y^2)}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x} 4a(a-x) \implies 2y\frac{\mathrm{d}y}{\mathrm{d}x} = -4a \implies \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2a}{y}$$

On C_2, differentiating both sides,

$$\frac{\mathrm{d}(y^2)}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x} 4b(b+x) \implies 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 4b \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2b}{y}$$

Clearly $P = (a-b, \pm 2\sqrt{ab})$ where $y^2 = 4ab$ satisfies

$$y^2 = 4a(a-x) = 4b(b+x)$$

Therefore at these intersections

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_{C_1} \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_{C_2} = \left(-\frac{2a}{y}\right)\left(\frac{2b}{y}\right) = -1$$

Since the product of slopes is -1, the tangents are perpendicular at P.

5

Let

$$g(x) = \frac{a - \cos x}{x} \quad (x \neq 0)$$
$$h(x) = bx + c$$

be differentiable functions (in their domain) with derivatives

$$\begin{aligned} g'(x) &= \frac{x(a-\cos x)' - (a - \cos x)x'}{x^2} \\ &= \frac{x \sin x + \cos x - a}{x^2} \\ &= \frac{\sin x}{x} + \frac{\cos x - a}{x^2} \quad (x \neq 0) \\ h'(x) &= b \end{aligned}$$

Since $\lim_{x\to 0} \frac{\sin x}{x}$ exists, $\lim_{x\to 0} g'(x)$ exists $\iff$ $\lim_{x\to 0} \frac{\cos x - a}{x^2}$ exists. In fact we can prove this explicitly: $(\Rightarrow)$ $\lim_{x\to 0} \frac{\cos x - a}{x^2} = \lim_{x\to 0} g'(x) - \lim_{x\to 0} \frac{\sin x}{x}$ exists. $(\Leftarrow)$ $\lim_{x\to 0} g'(x) = \lim_{x\to 0} \frac{\sin x}{x} + \lim_{x\to 0} \frac{\cos x - a}{x^2}$ exists. Now consider

$$\frac{\cos x - a}{x^2} = \frac{(\cos x - 1) + (1 - a)}{x^2} = -2\frac{\sin^2 \frac{x}{2}}{x^2} + \frac{1-a}{x^2} = \frac{-1}{2}\left(\frac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^2 + \frac{1-a}{x^2}$$

Clearly $\lim_{x\to 0} \frac{\cos x - a}{x^2}$ exists $\iff$ $\lim_{x\to 0} \frac{1-a}{x^2}$ exists $\iff$ $a = 1$. Therefore if $\lim_{x\to 0} g'(x)$ exists, it is

$$\lim_{x\to 0} g'(x) = \lim_{x\to 0} \frac{\sin x}{x} + \lim_{x\to 0} \frac{-1}{2}\left(\frac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^2 = 1 - \frac{1}{2}(1)^2 = \frac{1}{2}$$

Therefore since f is continuous (implied) and differentiable on $\mathbb{R}$,

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} \frac{1 - \cos x}{x} = \lim_{x\to 0} \frac{x}{2}\left(\frac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^2 = \left(\lim_{x\to 0} \frac{x}{2}\right)\left(\lim_{x\to 0} \frac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^2 = 0$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) \implies \lim_{x\to 0^-} g(x) = \lim_{x\to 0^+} h(x) \implies c = 0$$

$$\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^+} f'(x) \implies \lim_{x\to 0^-} g'(x) = \lim_{x\to 0^+} h'(x) \implies b = \frac{1}{2}$$

6

(a) Note that since $(x-2)^2 \geq 0$,

$$f(x) \begin{cases} > 0 & x > -1 \\ = 0 & x = -1 \\ < 0 & x < -1 \end{cases}$$

Therefore $\operatorname{argmin}_x f(x) \in [-2, -1)$. Since $\sqrt[5]{x+1}$ and $(x-2)^2$ are strictly increasing in the interval,

$$\min_x f(x) = f(-2) = \sqrt[5]{-2+1}(-2-2)^2 = -16$$

Similarly, $\operatorname{argmax}_x f(x) \in (-1, 3]$ where the derivative

$$f'(x) = \frac{1}{5}(x+1)^{\frac{-4}{5}}(x-2)^2 + 2(x-2)\sqrt[5]{x+1}$$

is well-defined. The stationary points where $f'(x) = 0$ are

$$\frac{1}{5}(x+1)^{\frac{-4}{5}}(x-2)^2 = -2(x-2)\sqrt[5]{x+1}$$

$$\begin{aligned}(x-2)^2 &= -10(x-2)(x+1) \\ x = 2 &\vee x - 2 = -10(x+1) \\ x &\in \{\frac{-8}{11}, 2\}\end{aligned}$$

Note that

$$\begin{aligned}
f(-1) &= 0 \\
f(\frac{-8}{11}) &= \sqrt[5]{\frac{-8}{11}+1}\left(\frac{-8}{11}-2\right)^2 = \sqrt[5]{\frac{3}{11}}\left(\frac{30}{11}\right)^2 > \sqrt[5]{\frac{1}{32}}(2)^2 = 2 \\
f(2) &= 0 \\
f(3) &= \sqrt[5]{3+1}(3-2)^2 < \sqrt[5]{32} = 2
\end{aligned}$$

Hence

$$\max_x f(x) = f(\frac{-8}{11}) = \sqrt[5]{\frac{3}{11}}\left(\frac{30}{11}\right)^2$$

(b) Given

$$g(x) = x - \cos 2x - 2\sin x + 2\cos x$$

Its derivative is

$$\begin{aligned}
g'(x) &= x' - (\cos 2x)' - (2\sin x)' + (2\cos x)' \\
&= 1 + 2\sin 2x - 2\cos x - 2\sin x \\
&= 4\sin x\cos x - 2\cos x - 2\sin x + 1 \\
&= (2\cos x - 1)(2\sin x - 1)
\end{aligned}$$

At the stationary points where $g'(x) = 0$, $\cos x = \frac{1}{2} \vee \sin x = \frac{1}{2} \implies x \in \{\frac{-\pi}{3}, \frac{\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{6}\}$.

Consider these stationary points along with the endpoints,

$$
\begin{aligned}
f(-\pi) &= -\pi - \cos(-2\pi) - 2\sin(-\pi) + 2\cos(-\pi) = -\pi - 3 \approx -6.14 \\
f(\frac{-\pi}{3}) &= \frac{-\pi}{3} - \cos\left(\frac{-2\pi}{3}\right) - 2\sin\left(\frac{-\pi}{3}\right) + 2\cos\left(\frac{-\pi}{3}\right) = \sqrt{3} + \frac{3}{2} - \frac{1}{3}\pi \approx 2.18 \\
f(\frac{\pi}{3}) &= \frac{\pi}{3} - \cos\left(\frac{2\pi}{3}\right) - 2\sin\left(\frac{\pi}{3}\right) + 2\cos\left(\frac{\pi}{3}\right) = \frac{1}{3}\pi - \sqrt{3} + \frac{3}{2} \approx 0.815 \\
f(\frac{\pi}{6}) &= \frac{\pi}{6} - \cos\frac{2\pi}{6} - 2\sin\left(\frac{\pi}{6}\right) + 2\cos\left(\frac{\pi}{6}\right) = \frac{1}{6}\pi + \sqrt{3} - \frac{3}{2} \approx 0.755 \\
f(\frac{5\pi}{6}) &= \frac{5\pi}{6} - \cos\frac{10\pi}{6} - 2\sin\left(\frac{5\pi}{6}\right) + 2\cos\left(\frac{5\pi}{6}\right) = \frac{5}{6}\pi - \sqrt{3} - \frac{3}{2} \approx -0.614 \\
f(\pi) &= \pi - \cos(2\pi) - 2\sin(\pi) + 2\cos(\pi) = \pi - 3 \approx 0.142
\end{aligned}
$$

Hence

$$\min_x f(x) = f(-\pi) = -\pi - 3$$

$$\max_x f(x) = f(\frac{-\pi}{3}) = \sqrt{3} + \frac{3}{2} - \frac{1}{3}\pi$$