NATIONAL UNIVERSITY OF SINGAPORE

SEMESTER 1, 2021/2022

MA2002 Calculus **Tutorial 11** (8^{th} November – 12^{th} November)

TUTORIAL PART I

This part consists of relatively basic questions which cover the course materials. The solutions to these questions will be recorded.

1. Solve the following differential equations.

   (a) $2\sqrt{xy}\,\dfrac{dy}{dx} = 1, \quad x, y > 0.$

   (b) $\sqrt{x}\,\dfrac{dy}{dx} = e^{y+\sqrt{x}}, \quad x > 0.$

   (c) $\dfrac{dy}{dx} = \dfrac{x^2(e^y)^{1/x} + y^2}{xy}.$

   (d) $x\dfrac{dy}{dx} + 3y = \dfrac{\sin x}{x^2}, \quad x > 0.$

   (e) $(t-1)^3\dfrac{ds}{dt} + 4(t-1)^2 s = t + 1, \quad t > 1.$

   (f) $\dfrac{dy}{dx} - y = -y^2.$

2. Solve the following initial value problems.

   (a) $\dfrac{dy}{dt} = e^t \sin(e^t - 2), \qquad y(\ln 2) = 0.$

   (b) $x\dfrac{dy}{dx} = y + x^2 \sin x, \quad x > 0, \qquad y(\pi) = 0.$

   (c) $(x+1)\dfrac{dy}{dx} - 2(x^2 + x)y = \dfrac{e^{x^2}}{x+1}, \quad x > -1, \qquad y(0) = 5.$

3. Find an equation of the curve that satisfies $\dfrac{dy}{dx} = 4x^3 y$ and whose y-intercept is 7.

4. In some chemical reactions, the rate at which the amount of a substance changes with time is proportional to the amount of the substance present. For the change of δ-glucono lactone to gluconic acid, for example,

$$\frac{dy}{dt} = -0.6y$$

when t is measured in hours. If there are 100 grams of δ-glucono lactone present when $t = 0$, how many grams will be left after the first hour?

5. The intensity $L(x)$ of light x feet beneath the surface of the ocean satisfies the differential equation
$$\frac{dL}{dx} = -kL.$$
A diver knows from experience that diving to 18 ft in the Caribbean Sea cuts the intensity in half. He cannot work without artificial light when the intensity falls below one-tenth of the surface value. About how deep can he expect to work without artificial light?

6. A 2000-gal tank can support no more than 150 guppies. Six guppies are introduced into the tank. Assume that the rate of growth of the population is
$$\frac{dP}{dt} = 0.0015\,P(150 - P),$$
where time t is in weeks.
   (i) Find a formula for the guppy population in terms of t.
   (ii) How long will it take for the guppy population to be 100? 125?

7. A pan of warm water (46°C) was put in a refrigerator. Ten minutes later, the water's temperature was 39°C; ten minutes after that, it was 33°C. Use Newton's Law of Cooling to estimate how cold the refrigerator was.

TUTORIAL PART II

This part consists of relatively difficult questions to promote independent learning and inculcate critical thinking abilities. The solutions will not be recorded. You may attempt them after you have gained a good understanding of the questions in Part I. The complete solution of this part is provided.

1. Find all functions f such that f' is continuous and
$$[f(x)]^2 = 100 + \int_0^x \left\{[f(t)]^2 + [f'(t)]^2\right\} dt \qquad \text{for all } x \in \mathbb{R}.$$

2. Using an appropriate change of variable, solve the differential equation
$$\frac{dy}{dx} - \frac{y}{x}\ln y = xy, \quad x > 0.$$

3. A tank contains 20 kg of salt dissolved in 5000 L of water. Brine that contains 0.03 kg of salt per liter of water enters the tank at a rate of 25 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt remains in the tank after half an hour?

Answers to Part I:

1. (a) $y = \left(\frac{3}{2}\right)^{2/3}(x^{1/2}+C)^{2/3}$, (b) $y = -\ln(-2e^{\sqrt{x}} - C)$,

   (c) $y + x = -xe^{y/x}(\ln|x| + C)$, (d) $y = \dfrac{C - \cos x}{x^3}$,

   (e) $s = \dfrac{t^3}{3(t-1)^4} - \dfrac{t}{(t-1)^4} + \dfrac{C}{(t-1)^4}$, (f) $y = \dfrac{1}{1+Ce^{-x}}$.

2. (a) $y = 1 - \cos(e^t - 2)$, (b) $y = -x\cos x - x$, (c) $y = 6e^{x^2} - \dfrac{e^{x^2}}{x+1}$.

3. $y = 7e^{x^4}$.

4. 54.88 grams.

5. 59.8 ft.

6. (i) $P(t) = \dfrac{150}{1+24e^{-0.225t}}$, (ii) 17.21 weeks, 21.28 weeks.

7. $-3°$C.