MA2002 Assignment 4

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Problem 1

(a) Given $\frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x$,

$$1 = e^0 = \lim_{x\to 0}\frac{e^x - e^0}{x - 0} = \lim_{x\to 0}\frac{e^x - 1}{x}$$

Then

$$\lim_{n\to\infty} n(4^{1/n} - 1) = \lim_{x\to 0}\frac{(e^{x\ln 4} - 1)\ln 4}{x\ln 4} = 2\ln 2$$

$$\begin{aligned}\int_1^3 2^x\,\mathrm{d}x &= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{2^{1+2(i\,-\,1)/n}}{n}\\ &= \lim_{n\to\infty}\frac{2(2^{2n/n} - 1)}{n(2^{2/n} - 1)}\\ &= \frac{3}{\ln 2}\end{aligned}$$

(b)

$$\begin{aligned}\int_0^6 \cos 3x\,\mathrm{d}x &= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{6}{n}\cos 3\left(\frac{6i}{n}\right)\\ &= \lim_{n\to\infty}\frac{6}{n}\left(\sum_{i=0}^{n}\cos i\left(\frac{18}{n}\right) - 1\right)\\ &= \lim_{n\to\infty}\frac{6}{n}\left(\frac{\sin\frac{9(2n+1)}{n}}{2\sin\frac{9}{n}} - \frac{1}{2}\right)\\ &= \lim_{n\to\infty}\frac{1}{3}\left(\frac{\frac{9}{n}\sin\left(18 + \frac{1}{n}\right)}{\sin\frac{9}{n}}\right)\\ &= \frac{\sin 18}{3}\end{aligned}$$

Problem 2

(a) Let $t = 1 - x^2$ and (informally) $\mathrm{d}t = -2x\,\mathrm{d}x$. Then

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n^5}\sum_{i=1}^{n} i^3\sqrt{n^2-i^2} &= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left(\frac{i}{n}\right)^3\sqrt{1-\left(\frac{i}{n}\right)^2} \\
&= \int_0^1 x^3\sqrt{1-x^2}\,\mathrm{d}x \\
&= \int_1^0 \frac{-1}{2}(1-t)\sqrt{t}\,\mathrm{d}t \\
&= \frac{1}{2}\left(\int_0^1 \sqrt{t}\,\mathrm{d}t - \int_0^1 t^{3/2}\,\mathrm{d}t\right) \\
&= \frac{1}{2}\left(\frac{2}{3}1^{3/2} - \frac{2}{5}1^{5/2}\right) \\
&= \frac{2}{15}
\end{aligned}$$

(b) Let $t = \frac{x}{2}$ and (informally) $\mathrm{d}t = \frac{1}{2}\,\mathrm{d}x$. Then

$$\begin{aligned}
\lim_{n\to\infty}\sum_{i=1}^{n}\frac{n^3}{16n^4-i^4} &= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left(16-\left(\frac{i}{n}\right)^4\right)^{-1} \\
&= \int_0^1 \frac{1}{16-x^4}\,\mathrm{d}x \\
&= \frac{1}{16}\int_0^{\frac{1}{2}} \frac{2}{1-t^4}\,\mathrm{d}t \\
&= \frac{1}{16}\left(\int_0^{\frac{1}{2}}\frac{1}{1-t^2}\,\mathrm{d}t + \int_0^{\frac{1}{2}}\frac{1}{1+t^2}\,\mathrm{d}t\right) \\
&= \frac{1}{16}\left(\frac{1}{2}\ln\frac{1+x}{1-x}\Big|_0^{\frac{1}{2}} + \arctan x\Big|_0^{\frac{1}{2}}\right) \\
&= \frac{1}{16}\left(\frac{\ln 3}{2} + \arctan\frac{1}{2}\right)
\end{aligned}$$

Problem 3

(a) Let $f(t) = \left(\frac{1}{1-4t}\right)^{\frac{1}{t}}$ and $g(x) = \sin^2 x$. Then

$$\lim_{t\to 0} f(t) = \lim_{t\to 0}\left(\frac{1}{1-4t}\right)^{\frac{1}{t}} = \left(\lim_{t\to 0}(1-4t)^{\frac{1}{-4t}}\right)^4 = e^4, \lim_{x\to 0} g(x) = 0$$

Since there exists $|x| < \delta$ where $g(x) \neq 0 \quad \forall \delta > 0$,

$$\begin{aligned}\lim_{x\to 0}\left(\frac{\cos x}{\cos 3x}\right)^{\csc^2 x} &= \lim_{x\to 0}\left(\frac{1}{4\cos^2 x - 3}\right)^{\csc^2 x}\\ &= \lim_{x\to 0}\left(\frac{1}{1-4\sin^2 x}\right)^{1/\sin^2 x}\\ &= \lim_{x\to 0} f(g(x))\\ &= e^4\end{aligned}$$

(b) Note that

$$\left(1+\frac{2}{x^{3/2}}\right)^{x^{3/2}} < \left(1+\frac{2}{x^{3/2}-1+x^{-1/2}}\right)^{x^{3/2}} = \left(\frac{x^2+\sqrt{x}+1}{x^2-\sqrt{x}+1}\right)^{x^{3/2}} < \left(1+\frac{2}{x^{3/2}-1}\right)^{x^{3/2}}$$

Since

$$\lim_{x\to\infty}\left(1+\frac{2}{x^{3/2}}\right)^{x^{3/2}} = \left(\lim_{x\to\infty}\left(1+\frac{2}{x^{3/2}}\right)^{\frac{x^{3/2}}{2}}\right)^2 = e^2$$

$$\lim_{x\to\infty}\left(1+\frac{2}{x^{3/2}-1}\right)^{x^{3/2}} = \left(\lim_{x\to\infty}\left(1+\frac{2}{x^{3/2}-1}\right)^{\frac{x^{3/2}-1}{2}}\right)^2\left(\lim_{x\to\infty}\left(1+\frac{2}{x^{3/2}-1}\right)\right) = e^2$$

By squeeze theorem

$$\lim_{x\to\infty}\left(\frac{x^2+\sqrt{x}+1}{x^2-\sqrt{x}+1}\right)^{x^{3/2}} = e^2$$

Problem 4

(a) Let $x = 1 + 4\tan^2 t$ and $\mathrm{d}x = 8\sec^2 t \tan t \,\mathrm{d}t$. Let $\theta = \arctan\frac{1}{\sqrt{2}}$ and $\sin\theta = \frac{1}{\sqrt{3}}$. Then

$$\begin{aligned}
\int_1^3 \frac{\mathrm{d}x}{(x+3)^2\sqrt{x^2+2x-3}} &= \int_1^3 (x+3)^{\frac{-5}{2}}(x-1)^{\frac{-1}{2}}\,\mathrm{d}x \\
&= \int_0^\theta (4\sec^2 t)^{-5/2}(4\tan^2 t)^{-1/2} 8\sec^2 t \tan t \,\mathrm{d}t \\
&= \frac{1}{8}\int_0^\theta \cos^3 t \,\mathrm{d}t \\
&= \frac{1}{32}\left(\int_0^\theta \cos 3t \,\mathrm{d}t + \int_0^\theta 3\cos t \,\mathrm{d}t\right) \\
&= \frac{1}{32}\left(\frac{\sin 3\theta}{3} + 3\sin\theta\right) \\
&= \frac{1}{32}\left(\frac{3\sin\theta - 4\sin^3\theta}{3} + 3\sin\theta\right) \\
&= \frac{1}{32}\left(4\sin\theta - \frac{4\sin^3\theta}{3}\right) \\
&= \frac{1}{8}\left(\frac{1}{\sqrt{3}}\right)\left(1 - \frac{1}{3}\left(\frac{1}{\sqrt{3}}\right)^2\right) \\
&= \frac{\sqrt{3}}{27}
\end{aligned}$$

(b) Informally, let $u = v = \arcsin x$ and $\mathrm{d}u = \mathrm{d}v = \frac{\mathrm{d}x}{\sqrt{1-x^2}}$. Then using integration by parts,

$$\int_0^1 \frac{\arcsin x}{\sqrt{1-x^2}}\,\mathrm{d}x = (\arcsin x)^2\Big|_0^1 - \int_0^1 \frac{\arcsin x}{\sqrt{1-x^2}}\,\mathrm{d}x$$

$$\int_0^1 \frac{\arcsin x}{\sqrt{1-x^2}}\,\mathrm{d}x = \frac{1}{2}\left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{8}$$

Informally, let $x = \sin t$ and $\mathrm{d}x = \cos t\,\mathrm{d}t$. Then for some constant C,

$$\begin{aligned}
\int \sqrt{1-x^2}\,\mathrm{d}x &= \int \cos^2 t \,\mathrm{d}t \\
&= \int \frac{1+\cos 2t}{2}\,\mathrm{d}t \\
&= \frac{1}{2}\left(t + \frac{1}{2}\sin 2t\right) + C \\
&= \frac{1}{2}\left(x\sqrt{1-x^2} + \arcsin x\right) + C
\end{aligned}$$

Informally, let

$$\begin{aligned} u &= \arcsin x \\ \mathrm{d}u &= \frac{\mathrm{d}x}{\sqrt{1-x^2}} \\ v &= \frac{1}{2}\left(x\sqrt{1-x^2} + \arcsin x\right) \\ \mathrm{d}v &= \sqrt{1-x^2}\,\mathrm{d}x \end{aligned}$$

Then using integration by parts,

$$\begin{aligned} \int_0^1 \sqrt{1-x^2}\arcsin x\,\mathrm{d}x &= \left[\frac{1}{2}\left(x\sqrt{1-x^2} + \arcsin x\right)\arcsin x\right]\Bigg|_0^1 \\ &\quad - \int_0^1 \frac{1}{2}\left(x\sqrt{1-x^2} + \arcsin x\right)\frac{\mathrm{d}x}{\sqrt{1-x^2}} \\ &= \frac{1}{2}\left(\arcsin x\right)^2\Big|_0^1 - \frac{1}{2}\left(\int_0^1 x\,\mathrm{d}x + \frac{\pi^2}{8}\right) \\ &= \frac{\pi^2}{8} - \frac{1}{2}\left(\frac{1}{2} + \frac{\pi^2}{8}\right) \\ &= \frac{1}{16}(\pi^2 - 4) \end{aligned}$$

Problem 5

Lecture notes provide so few properties...

Lemma. *Given $n, N \in \mathbb{N}$ there exists $t \geq N$ such that $e^t \geq t^n$.*

Proof. From Bernoulli we have

$$e^x = \lim_{y\to 0}\left(1+\frac{1}{y}\right)^{xy} > \lim_{y\to 0}\left(1+\frac{xy}{y}\right) > x$$

Since $e^x > 0$, $e^{4n} - 4n^2 = (e^{2n} - 2n)(e^{2n} + 2n) > 0$. Let $m = \max\{N, n\}$ and $t = e^{4m} > N$. Then

$$e^t = e^{e^{4m}} > e^{4m^2} > (e^{4m})^n = t^n$$

□

Lemma. $\lim_{t\to\infty} {}^{t^n}/_{e^t} = 0 \quad \forall n \in \mathbb{N}$.

Proof. From above,

$$0 < \frac{t^n}{e^t} < \frac{t^n}{t^{n+1}} = \frac{1}{t}$$

Then the result follows immediately from squeeze theorem. □

Lemma. $\lim_{x\to 0^+}(\ln x)^n x^m = 0 \quad \forall m,n \in \mathbb{N}$.

Proof. From above,

$$\lim_{x\to 0^+}(\ln x)^n x^m = (-1)^n \lim_{t\to\infty} t^n e^{mt} = 0$$

□

Consider

$$\begin{aligned} u &= (\ln x)^n \\ \mathrm{d}u &= \frac{n}{x}(\ln x)^{n-1}\,\mathrm{d}x \\ v &= \frac{x^{m+1}}{m+1} \\ \mathrm{d}v &= x^m\,\mathrm{d}x \end{aligned}$$

With some abuse of notation, note that

$$\begin{aligned} I(m,n) &= (\ln x)^n \left(\frac{x^{m+1}}{m+1}\right)\Big|_{0^+}^{1} - \int_0^1 \frac{n}{x}(\ln x)^{n-1}\frac{x^{m+1}}{m+1}\,\mathrm{d}x \\ &= 0 - \lim_{x\to 0^+}(\ln x)^n\frac{x^{m+1}}{m+1} - \frac{n}{m+1}I(m,n-1) \\ &= -\frac{n}{m+1}I(m,n-1) \end{aligned}$$

Since

$$I(m,0) = \int_0^1 x^m\,\mathrm{d}x = \frac{1}{m+1}$$

And for $n > 0$,

$$\frac{(-1)^n n!}{(m+1)^{n+1}} = \left(-\frac{n}{m+1}\right)\frac{(-1)^{n-1}(n-1)!}{(m+1)^n}$$

By induction on n we have

$$I(m,n) = \frac{(-1)^n n!}{(m+1)^{n+1}}$$

Problem 6

Suppose there exists $c \in [a,b]$ where $f(c) \neq 0$. Since f is continuous, there exists $\delta > 0$ such that for all $|x-c| < \delta$,

$$\left|f(x) - f(c)\right| < \epsilon = \left|\frac{f(c)}{2}\right| \implies \left|f(x)\right| > \left|\frac{f(c)}{2}\right|$$

By $\left[f(x)\right]^2 \geq 0$ and max-min inequality,

$$\int_a^b \left[f(x)\right]^2 \mathrm{d}x \geq \int_{\max a, c-\delta}^{\min b, c+\delta} \left[f(x)\right]^2 \mathrm{d}x > \left[\frac{f(c)}{2}\right]^2 \delta > 0$$

which contradicts. As such $f(x) = 0 \quad \forall x \in [a, b]$.

Problem 7

By mean value theorem for definite integral, there exists $c \in [0, 1]$ such that

$$g(c) = \int_0^1 g(x)\,\mathrm{d}x$$

By monotonicity of f and g,

$$\left[f(x) - f(c)\right]\left[g(x) - g(c)\right] = \left(\frac{f(x) - f(c)}{x - c}\right)\left(\frac{g(x) - g(c)}{x - c}\right)(x - c)^2 \geq 0$$

Then by max-min inequality,

$$\int_0^1 \left[f(x) - f(c)\right]\left[g(x) - g(c)\right] \mathrm{d}x \geq 0$$

Hence

$$\begin{aligned}
\int_0^1 f(x)g(x)\,\mathrm{d}x &\geq \int_0^1 f(c)\left[g(x) - g(c)\right] \mathrm{d}x + \int_0^1 f(x)g(c)\,\mathrm{d}x \\
&= f(c)\left[\int_0^1 g(x)\,\mathrm{d}x - g(c)\right] + g(c)\int_0^1 f(x)\,\mathrm{d}x \\
&= \left(\int_0^1 f(x)\,\mathrm{d}x\right)\left(\int_0^1 g(x)\,\mathrm{d}x\right)
\end{aligned}$$