National University of Singapore

Semester 1, 2021/2022 **MA2001** **Homework Assignment 1**

(a) Use A4 size paper and pen (blue or black ink) to write your answers. (Students may also type out the answers or write the answers electronically using their devices.)

(b) Write down your student number and full name clearly on the top left of every page of the answer scripts.

(c) Write the page number on the top right corner of each page of answer scripts.

(d) This assignment consists of 4 pages and 8 questions. Total mark is 80 marks.

(e) To submit your answer scripts, do the following:

(i) Scan or take pictures of your work (make sure the images can be read clearly).

(ii) Merge all your answers into one pdf file. Arrange them in order of the questions.

(iii) Name the pdf file by **StudentNo HW1** (e.g. **A123456R HW1**).

(iv) Upload your pdf into the LumiNUS folder Homework 1 submission.

(f) Deadline for submission is 10 September, 2021 by 11.59pm. **Late submission will not be accepted.**

1. The following augmented matrix belongs to some linear system with variables x_1, x_2, x_3, x_4, x_5:
$$\left(\begin{array}{ccccc|c} 0 & 1 & 2 & -1 & 0 & a \\ 0 & 0 & 0 & 1 & 1 & b \\ 1 & -1 & 2 & 0 & 1 & c \\ 0 & -1 & -2 & 1 & 0 & d \end{array}\right)$$
where a, b, c, d are some real numbers.

   (i) [3 marks] Reduce the augmented matrix to a row echelon form using three elementary row operations (do not use MATLAB. Indicate the three e.r.o.).

   (ii) [3 marks] Write down conditions in terms of a, b, c, d for the linear system to have (a) no solution; (b) exactly one solution; (c) infinitely many solutions.

   (iii) [3 marks] If the system is homogeneous, write down a general solution of the system.

   (iv) [3 marks] If the system is non-homogeneous with a particular solution
$$x_1 = \square, x_2 = \square, x_3 = 0, x_4 = \square, x_5 = 0,$$
find this particular solution in terms of a, b, c, d.

2. Consider the following linear system :
$$\left\{\begin{array}{rcrcr} x & + & 2y & = & 0 \\ -2x & + & y & = & -5 \\ x & - & 5y & = & 7 \\ 5x & + & y & = & 9 \end{array}\right. \qquad (*)$$

   (i) [2 marks] Use MATLAB command to write down the reduced row echelon form of the augmented matrix and conclude whether the system $(*)$ is consistent(without finding its solutions, if any).

   (ii) [2 marks] Write the linear system $(*)$ in matrix equation form $\boldsymbol{Ax} = \boldsymbol{b}$.

   (iii) [2 marks] Compute $\boldsymbol{A}^T\boldsymbol{A}$ and $\boldsymbol{A}^T\boldsymbol{b}$ for $\boldsymbol{A}$ and $\boldsymbol{b}$ in part (ii).

   (iv) [2 marks] Find the solution of the linear system $\boldsymbol{A}^T\boldsymbol{Ax} = \boldsymbol{A}^T\boldsymbol{b}$ without using MATLAB. Show your working.

   (v) [2 marks] Without substituting the solution in (iv) into the original system $(*)$, can you tell whether this solution satisfies $(*)$? Briefly explain your answer.

3. $\boldsymbol{A} = \begin{pmatrix} a+b & a & b \\ a+b & a & a+3b \\ a+b & 2b & 2b \end{pmatrix}$ where a, b are non-zero.

   (i) [4 marks] Find all the conditions in terms of a and b such that $\boldsymbol{A}$ is invertible. Show your working.

(ii) [6 marks] When $\boldsymbol{A}$ is not invertible, write down all the possible reduced row echelon forms of $\boldsymbol{A}$. Show clearly how your answers are derived.

4. The following augmented matrix is in row echelon form and belongs to some non-homogeneous linear system:

$$\left(\begin{array}{ccc|c} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \end{array}\right)$$

Determine whether the following statements are true or false. Explain carefully how you derive your answers.
(Note: The entries a, e, h need not be leading entries of the row echelon form.)

(i) [2 marks] If $i = 0$, then the system is consistent.

(ii) [2 marks] If the system is consistent and $i \neq 0$, then $a \neq 0$.

(iii) [2 marks] If the system represents three planes that intersect at a line, then a and e are not zero.

(iv) [2 marks] If the system is consistent and its general solution has exactly one parameter, then $h = 0$.

5. Suppose

$$\boldsymbol{A} \xrightarrow{R_3-2R_1} \xrightarrow{R_2 \leftrightarrow R_3} \xrightarrow{R_3+R_2} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & 2 \end{pmatrix} = \boldsymbol{B}.$$

(i) [3 marks] Find an invertible matrix $\boldsymbol{C}$ such that $\boldsymbol{B} = \boldsymbol{C}\boldsymbol{A}$.

(ii) [4 marks] Write $\boldsymbol{A} = \boldsymbol{E}_1\boldsymbol{E}_2\boldsymbol{E}_3\boldsymbol{E}_4\boldsymbol{E}_5\boldsymbol{R}$ where $\boldsymbol{R}$ is the reduced row echelon form of $\boldsymbol{A}$ and $\boldsymbol{E}_i$ are elementary row operations. (i.e. Write down the matrices $\boldsymbol{E}_i$ and $\boldsymbol{R}$ explicitly.)

(iii) [3 marks] Which of the following matrices are row equivalent to $\boldsymbol{A}$?

$$\boldsymbol{R}, \quad \boldsymbol{E}_5\boldsymbol{R}, \quad \boldsymbol{E}_4\boldsymbol{E}_5\boldsymbol{R}, \quad \boldsymbol{E}_3\boldsymbol{E}_4\boldsymbol{E}_5\boldsymbol{R}, \quad \boldsymbol{E}_2\boldsymbol{E}_3\boldsymbol{E}_4\boldsymbol{E}_5\boldsymbol{R}$$

Justify your answer.
(Hint: If two matrices are row equivalent, what can we say about their reduced row echelon forms?)

(iv) [2 marks] Are $\boldsymbol{B}$ and $\boldsymbol{C}$ in (i) row equivalent to each other? Justify your answer.

6. Let $\boldsymbol{A} = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} -2 & 1 & 2 \\ 1 & 4 & -1 \\ 2 & -1 & -2 \end{pmatrix}$. Suppose $\boldsymbol{AB}$ is a symmetric matrix.

(i) [3 marks] Use the above information to set up a linear system with three equations and variables a, b, c, d, e, f.

(ii) [3 marks] Find the general solution of the system in (i). (You may use MATLAB)

(iii) [2 marks] Use (ii) to express the matrix $\boldsymbol{A}$ in terms of d, e, f only.

7. Let $\boldsymbol{A}$ be an $m \times n$ matrix and $\boldsymbol{B}$ be an $n \times m$ matrix.

   (a) Suppose $\boldsymbol{Ax} = \mathbf{0}$ has non-trivial solutions.

      (i) [2 marks] Can $\boldsymbol{BA}$ be invertible?

      (ii) [2 marks] Can $\boldsymbol{AB}$ be invertible if $m < n$?

      (iii) [2 marks] Can $\boldsymbol{AB}$ be invertible if $m > n$?

   (b) Suppose $\boldsymbol{Ax} = \mathbf{0}$ has only the trivial solution.

      (i) [2 marks] Must $\boldsymbol{AB}$ be invertible?

      (ii) [2 marks] Must $\boldsymbol{BA}$ be invertible?

   Justify all your answers. (For each part in (a), if your answer is yes, you have to give an example to support the statement. For each part in (b), if your answer is no, you have to give a counter-example.)

8. Let $\boldsymbol{A}$ be an $n \times n$ matrix. We denote by

$$\boldsymbol{A}_U = \begin{pmatrix} \boldsymbol{A} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I} \end{pmatrix} \quad \text{and} \quad \boldsymbol{A}_L = \begin{pmatrix} \boldsymbol{I} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{A} \end{pmatrix}$$

   the block matrices of size $2n \times 2n$.

   For example, if $\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, then $\boldsymbol{A}_U = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $\boldsymbol{A}_L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \end{pmatrix}$.

   (i) [3 marks] Show that $\boldsymbol{A}_U\boldsymbol{B}_L = \begin{pmatrix} \boldsymbol{A} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{B} \end{pmatrix}$ for any $n \times n$ matrices $\boldsymbol{A}$ and $\boldsymbol{B}$.

   (ii) [4 marks] Show that $\det \boldsymbol{A}_U = \det \boldsymbol{A}$ and $\det \boldsymbol{B}_L = \det \boldsymbol{B}$, and find $\begin{vmatrix} \boldsymbol{A} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{B} \end{vmatrix}$.

   (iii) [3 marks] What is $\begin{vmatrix} \boldsymbol{A} & \boldsymbol{C} \\ \mathbf{0} & \boldsymbol{B} \end{vmatrix}$ where $\boldsymbol{C}$ is some $n \times n$ matrix? Explain how you obtain your answer.