NATIONAL UNIVERSITY OF SINGAPORE

SEMESTER 1, 2021/2022

MA2002 Calculus **Solution to Tutorial 9**

TUTORIAL PART I

1. (a) Take logarithmic function: $\ln|y| = 2\ln|x| + x\ln 2 - \frac{1}{3}\ln|\sin 3x|$.
   Differentiate with respect to x: $\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + \ln 2 - \frac{1}{3}\frac{3\cos 3x}{\sin 3x}$.
   Then $\frac{dy}{dx} = y\left(\frac{2}{x} + \ln 2 - \cot 3x\right) = \frac{x^2 2^x}{\sqrt[3]{\sin 3x}}\left(\frac{2}{x} + \ln 2 - \cot 3x\right)$.

   (b) Let $u = x^x$. Take logarithmic function: $\ln u = x\ln x$ $(x > 0)$.
   Differentiate with respect to x: $\frac{1}{u}\frac{du}{dx} = 1\cdot\ln x + x\cdot\frac{1}{x} = \ln x + 1$.
   Then $\frac{d}{dx}(x^x) = \frac{du}{dx} = u(\ln x + 1) = x^x(\ln x + 1)$.
   Now take logarithmic function to $y = x^{(x^x)}$: $\ln y = x^x \ln x$.
   Differentiate with respect to x: $\frac{1}{y}\frac{dy}{dx} = x^x(\ln x + 1)\ln x + x^x\cdot\frac{1}{x}$.
   Then $\frac{dy}{dx} = y(x^x(\ln x + 1)\ln x + x^{x-1}) = x^{(x^x)}(x^x(\ln x + 1)\ln x + x^{x-1})$.

2. (a) Recall that $\lim_{x\to\infty} e^x = \infty$. We can apply the l'Hôpital's rule repeatedly:
   $$\lim_{x\to\infty}\frac{e^x}{x^n} = \lim_{x\to\infty}\frac{(e^x)'}{(x^n)'} = \lim_{x\to\infty}\frac{e^x}{nx^{n-1}} = \lim_{x\to\infty}\frac{(e^x)'}{n(x^{n-1})'} = \lim_{x\to\infty}\frac{e^x}{n(n-1)x^{n-2}}$$
   $$= \cdots = \lim_{x\to\infty}\frac{e^x}{n(n-1)\cdots 2x} = \lim_{x\to\infty}\frac{(e^x)'}{n(n-1)\cdots 2(x)'} = \lim_{x\to\infty}\frac{e^x}{n(n-1)\cdots 2\cdot 1}$$
   $$= \lim_{x\to\infty}\frac{e^x}{n!} = \infty.$$

   (b) $$\lim_{x\to 0}(e^{2x} + 2x)^{1/x} = \lim_{x\to 0}\exp\left(\frac{1}{x}\ln(e^{2x} + 2x)\right) = \exp\left(\lim_{x\to 0}\frac{\ln(e^{2x} + 2x)}{x}\right)$$
   $$= \exp\left(\lim_{x\to 0}\frac{(2e^{2x} + 2)/(e^{2x} + 2x)}{1}\right) = \exp(4) = e^4.$$

   (c) $$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty}\exp\left(\frac{1}{x}\ln x\right) = \exp\left(\lim_{x\to\infty}\frac{\ln x}{x}\right) = \exp\left(\lim_{x\to\infty}\frac{1/x}{1}\right) = \exp(0) = 1.$$

(d) $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2} = \lim_{x\to 0}\exp\left(\frac{1}{x^2}\ln\frac{\tan x}{x}\right) = \exp\left(\lim_{x\to 0}\frac{\ln(\tan x/x)}{x^2}\right)$

$= \exp\left(\lim_{x\to 0}\frac{\frac{x\sec^2 x-\tan x}{x^2}\cdot\frac{x}{\tan x}}{2x}\right) = \exp\left(\lim_{x\to 0}\frac{x\sec^2 x-\tan x}{2x^3}\cdot\lim_{x\to 0}\frac{x}{\tan x}\right)$

$= \exp\left(\lim_{x\to 0}\frac{\sec^2 x + x\cdot 2\sec x\cdot\sec x\tan x-\sec^2 x}{6x^2}\right) = \exp\left(\lim_{x\to 0}\frac{\sec^2 x\tan x}{3x}\right)$

$= \exp\left(\lim_{x\to 0}\frac{1}{3\cos^3 x}\frac{\sin x}{x}\right) = \exp\left(\frac{1}{3}\cdot 1\right) = \sqrt[3]{e}.$

3. (i) Let $y = \sinh x = \dfrac{e^x - e^{-x}}{2}$.

Set $X = e^x$. Then $y = \dfrac{X - X^{-1}}{2}$, or equivalently $X^2 - 2yX - 1 = 0$.

Solve the equation in X, we obtain $X = \dfrac{2y \pm \sqrt{(2y)^2-(-4)}}{2} = y \pm \sqrt{y^2+1}$.

Note that $X = e^x > 0$. So $X = y + \sqrt{y^2+1}$, i.e., $x = \ln X = \ln(y+\sqrt{y^2+1})$.

Therefore, $\sinh^{-1} x = \ln(x+\sqrt{x^2+1})$, $x \in \mathbb{R}$.

(ii) $\dfrac{d}{dx}\sinh^{-1} x = \dfrac{d}{dx}\ln(x+\sqrt{x^2+1}) = \dfrac{1+\frac{x}{\sqrt{x^2+1}}}{x+\sqrt{x^2+1}} = \dfrac{\frac{1}{\sqrt{x^2+1}}(\sqrt{x^2+1}+x)}{x+\sqrt{x^2+1}}$

$= \dfrac{1}{\sqrt{x^2+1}}.$

4. (a) Let $x = 2\tan t$, $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then $\sqrt{x^2+4} = 2\sec t$ and $\dfrac{dx}{dt} = 2\sec^2 t$.

$\displaystyle\int \frac{dx}{x^2\sqrt{x^2+4}} = \int \frac{1}{(2\tan t)^2 2\sec t}\cdot 2\sec^2 t\,dt = \frac{1}{4}\int\frac{\cos t}{\sin^2 t}\,dt = \frac{1}{4}\int\frac{1}{\sin^2 t}\,d(\sin t)$

$= -\dfrac{1}{4\sin t} + C = -\dfrac{1}{4}\dfrac{\sec t}{\tan t} + C = -\dfrac{\sqrt{x^2+4}}{4x} + C.$

(b) Note that $6x - x^2 = 3^2 - (x-3)^2$. Let $x - 3 = 3\sin t$, $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Then $x = 3(1+\sin t)$, $\sqrt{6x-x^2} = 3\cos t$ and $\dfrac{dx}{dt} = 3\cos t$.

$\displaystyle\int \frac{x^2}{\sqrt{6x-x^2}}\,dx = \int\frac{3^2(1+\sin t)^2}{3\cos t}\cdot 3\cos t\,dt = 9\int(1+2\sin t+\sin^2 t)\,dt$

$\displaystyle= 9\int\left(1+2\sin t+\frac{1}{2}(1-\cos 2t)\right)dt = 9\int\left(\frac{3}{2}+2\sin t-\frac{1}{2}\cos 2t\right)dt$

$= 9\left(\dfrac{3}{2}t - 2\cos t - \dfrac{1}{4}\sin 2t\right) + C = 9\left(\dfrac{3}{2}t - 2\cos t - \dfrac{1}{2}\sin t\cos t\right) + C$

$= 9\left(\dfrac{3}{2}\sin^{-1}\dfrac{x-3}{3} - 2\dfrac{\sqrt{6x-x^2}}{3} - \dfrac{1}{2}\dfrac{x-3}{3}\dfrac{\sqrt{6x-x^2}}{3}\right) + C$

$= \dfrac{27}{2}\sin^{-1}\dfrac{x-3}{3} - \dfrac{1}{2}x\sqrt{6x-x^2} - \dfrac{9}{2}\sqrt{6x-x^2} + C.$

(c) $\int \frac{\ln x}{x^2}\,dx = -\int \ln x\, d(1/x) = -\left(\ln x \cdot \frac{1}{x} - \int \frac{1}{x}\cdot (\ln x)'\,dx\right)$
$= -\frac{\ln x}{x} + \int \frac{1}{x}\cdot\frac{1}{x}\,dx = -\frac{\ln x}{x} - \frac{1}{x} + C.$

(d) $\int \tan^{-1}\left(\frac{1}{x}\right)dx = x\tan^{-1}\left(\frac{1}{x}\right) - \int x\cdot\left(\tan^{-1}\left(\frac{1}{x}\right)\right)'dx$
$= x\tan^{-1}\left(\frac{1}{x}\right) - \int x\cdot \frac{-\frac{1}{x^2}}{1+(\frac{1}{x})^2}\,dx = x\tan^{-1}\left(\frac{1}{x}\right) + \int \frac{x}{1+x^2}\,dx$
$= x\tan^{-1}\left(\frac{1}{x}\right) + \frac{1}{2}\ln(1+x^2) + C.$

(e) $\int \cos(\ln x)\,dx = x\cos(\ln x) - \int x(\cos(\ln x))'\,dx = x\cos(\ln x) + \int x\sin(\ln x)\cdot\frac{1}{x}\,dx$
$= x\cos(\ln x) + \int \sin(\ln x)\,dx.$
$\int \sin(\ln x)\,dx = x\sin(\ln x) - \int x(\sin(\ln x))'\,dx = x\sin(\ln x) - \int x\cos(\ln x)\cdot\frac{1}{x}\,dx$
$= x\sin(\ln x) - \int \cos(\ln x)\,dx.$
Then we can solve that $\int \cos(\ln x)\,dx = \frac{1}{2}x\sin(\ln x) + \frac{1}{2}x\cos(\ln x) + C.$

(f) Let $t = \sqrt{x}$. Then $x = t^2$ $(t \geq 0)$, and $\frac{dx}{dt} = 2t$. Therefore,
$\int e^{\sqrt{x}}\,dx = \int e^t\cdot 2t\,dt = 2\int t\,d(e^t) = 2\left(te^t - \int e^t\cdot(t)'\,dt\right) = 2(te^t - e^t) + C$
$= 2e^t(t-1) + C = 2e^{\sqrt{x}}(\sqrt{x}-1) + C.$

(g) We first convert the rational function into its partial fraction form. Suppose

$$\frac{4(x+1)}{x^2(x^2+4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+4} = \frac{Ax(x^2+4)+B(x^2+4)+(Cx+D)x^2}{x^2(x^2+4)}$$
$$= \frac{(A+C)x^3+(B+D)x^2+4Ax+4B}{x^2(x^2+4)}.$$

Compare the coefficients:

$$A+C=0,\quad B+D=0,\quad 4A=4 \quad\text{and}\quad 4B=4.$$

Solve the system, we have $A=1, B=1, C=-1$ and $D=-1$. Therefore,

$$\int \frac{4(x+1)}{x^2(x^2+4)}\,dx = \int\left(\frac{1}{x}+\frac{1}{x^2}-\frac{1}{x^2+4}-\frac{x}{x^2+4}\right)dx$$
$$= \ln|x| - \frac{1}{x} - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) - \frac{1}{2}\ln(x^2+4) + C.$$

(h) Let $u = \cos\theta$. Then $\frac{du}{d\theta} = -\sin\theta$.
$\int \frac{\sin\theta}{\cos^2\theta+\cos\theta-2}\,d\theta = -\int \frac{du}{(u-1)(u+2)}\,du = -\frac{1}{3}\int\left(\frac{1}{u-1}-\frac{1}{u+2}\right)du$

$$= -\frac{1}{3}(\ln|u-1| - \ln|u+2|) + C = \frac{1}{3}\ln(2+\cos\theta) - \frac{1}{3}\ln(1-\cos\theta) + C.$$

(i) Let $x = \dfrac{1}{t}$. Then $t = \dfrac{1}{x}$ and $\dfrac{dx}{dt} = -\dfrac{1}{t^2}$.

$$\int \frac{1}{x^{2002}-x}\,dx = \int \frac{1}{\frac{1}{t^{2002}} - \frac{1}{t}} \cdot \frac{-1}{t^2}\,dt = \int \frac{t^{2000}}{t^{2001}-1}\,dt = \frac{1}{2001}\ln|t^{2001}-1| + C$$

$$= \frac{1}{2001}\ln\left|\frac{1}{x^{2001}} - 1\right| + C.$$

Tutorial Part II

1. (i) Let $f(x) = e^x - (1+x)$. Then $f'(x) = e^x - 1$. Recall that e^x is increasing.

   If $x > 0$, then $f'(x) > e^0 - 1 = 0$; so f is increasing on $[0,\infty)$.

   If $x < 0$, then $f'(x) < e^0 - 1 = 0$; so f is decreasing on $(-\infty, 0]$.

   Therefore, f attains the absolute minimum at $x = 0$.

   So for all $x \in \mathbb{R}$, $f(x) \geq f(0) = 0$, i.e., $e^x \geq 1 + x$.

   (ii) Let $M > 0$. Take $N = M - 1$. Then by (i),

   $$x > N \Rightarrow e^x \geq x + 1 > N + 1 = M.$$

   It follows that $\lim\limits_{x\to\infty} e^x = \infty$.

   (iii) Let $y = -x$. Then $x \to -\infty \Leftrightarrow y \to \infty$. Then by (ii),

   $$\lim_{x\to-\infty} e^x = \lim_{y\to\infty} e^{-y} = \lim_{y\to\infty} \frac{1}{e^y} = 0.$$

2. If $a_1 = \cdots = a_k = 1$, then it is trivial that

   $$\sqrt[m]{a_1} + \cdots + \sqrt[m]{a_k} = \sqrt[n]{a_1} + \cdots + \sqrt[n]{a_k} = k.$$

   Suppose not all $a_1, \ldots, a_k$ are equal to 1. Define $f(x) = a_1^x + \cdots + a_k^x$. Then

   $$f'(x) = a_1^x \ln a_1 + \cdots + a_k^x \ln a_k,$$
   $$f''(x) = a_1^x (\ln a_1)^2 + \cdots + a_k^x (\ln a_k)^2.$$

   Note that $f''(x) > 0$ for all $x \in \mathbb{R}$. It follows that f' is increasing on $\mathbb{R}$. In particular, for all $x > 0$,

   $$f'(x) > f'(0) = \ln a_1 + \cdots + \ln a_k = \ln(a_1 \cdots a_k) = \ln 1 = 0.$$

   Therefore, f is increasing on $[0,\infty)$.

   If $m > n > 0$, then $0 < 1/m < 1/n$, and thus $f(1/m) < f(1/n)$. That is,

   $$\sqrt[m]{a_1} + \cdots + \sqrt[m]{a_k} < \sqrt[n]{a_1} + \cdots + \sqrt[n]{a_k}.$$

3. If we substitute $x = 2\tan^{-1} t$, where $t \in (-\pi, \pi)$, then

$$\int f(\sin x, \cos x)\, dx = \int f\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right) \frac{2}{1+t^2}\, dt.$$

Therefore,

$$\begin{aligned}
\int \frac{dx}{2+\sin x - 2\cos x} &= \int \frac{1}{2 + \frac{2t}{1+t^2} - 2\frac{1-t^2}{1+t^2}} \frac{2}{1+t^2}\, dt \\
&= \int \frac{2\, dt}{2(1+t^2) + 2t - 2(1-t^2)} \\
&= \int \frac{dt}{t(2t+1)} = \int \left(\frac{1}{t} - \frac{2}{2t+1}\right) dt \\
&= \ln|t| - \ln|2t+1| + C \\
&= \ln\left|\tan\frac{x}{2}\right| - \ln\left|2\tan\frac{x}{2} + 1\right| + C.
\end{aligned}$$