National University of Singapore

Semester 1, 2021/2022 **MA2001** **Homework Assignment 2**

(a) Use A4 size paper and pen (blue or black ink) to write your answers. (Students may also type out the answers or write the answers electronically using their devices.)

(b) Write down your student number and full name clearly on the top left of every page of the answer scripts.

(c) Write the page number on the top right corner of each page of answer scripts.

(d) This assignment consists of 4 pages and 7 questions. Total mark is 80 marks.

(e) To submit your answer scripts, do the following:

  (i) Scan or take pictures of your work (make sure the images can be read clearly).

  (ii) Merge all your answers into one pdf file. Arrange them in order of the questions.

  (iii) Name the pdf file by **StudentNo HW2** (e.g. **A123456R HW2**).

  (iv) Upload your pdf into the LumiNUS folder Homework 2 submission.

(f) Deadline for submission is 1 October, 2021 by 11.59pm. **Late submission will not be accepted.**

1. Let $U = \{(x, y, z) \mid x + 2y - 3z = 0\}$ and $V = \{(x, y, z) \mid x + 3y - 2z = 0\}$.

   (i) [4 marks] Write down an explicit set notation for each of U and V.

   (ii) [3 marks] Write down both an implicit set notation and an explicit set notation for $U \cap V$.

   (iii) [2 marks] Is $W = \{(t - 2, t + 1, t) \mid t \in \mathbb{R}\}$ a subset of U? Justify your answer.

   (iv) [3 marks] Find $U \cap V \cap W$.

2. Let $S_1 = \{(1, 0, 1, -1), (0, 2, -1, 0), (1, 1, 2, -1)\}$ and $S_2 = \{(1, 2, 0, -1), (-1, 1, -3, 1), (0, 1, 1, 0), (0, 2, -4, 0)\}$.

   (i) [2 marks] Is $\text{span}(S_1) \subseteq \text{span}(S_2)$?

   (ii) [2 marks] Is $\text{span}(S_2) = \mathbb{R}^4$?

   (iii) [3 marks] Find a necessary and sufficient condition on $a, b, c, d \in \mathbb{R}$ such that $(a, b, c, d) \notin \text{span}(S_1)$.

   (iv) [3 marks] Is it possible to find a single linear equation $px+qy+rz+sw = 0$ such that

   $$\text{span}(S_2) = \{(x, y, z, w) \mid px + qy + rz + sw = 0\}?$$

   Justify all your answers.

3. Let $V_1 = \{(t - 2s, s + 3t, 3s, t) \mid s, t \in \mathbb{R}\}$ and $V_2 = \{(x, y, z, w) \mid x + y + z + w = 0 \text{ and } xy - zw = 0\}$.

   (i) [2 marks] Show that V_1 is a subspace of $\mathbb{R}^4$ by expressing V_1 as a linear span.

   (ii) [3 marks] Find a proper subset of V_1 which is a subspace of $\mathbb{R}^4$ and contains a vector of the form $(*, *, 3, 3)$.

   (iii) [2 marks] Show that V_2 is not a subspace of $\mathbb{R}^4$.

   (iv) [3 marks] Is it possible to find a subset of V_2 which satisfies the closure properties under vector addition and scalar multiplication? Justify your answer.

4. Let $S = \{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{x}\} \subseteq \mathbb{R}^n$ and $V = \mathrm{span}(S)$. Suppose $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is a basis for V and $\boldsymbol{x}$ is not the zero vector.

   Determine whether the following statements are true or false. Justify your answers.

   (i) [2 marks] $\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w} + \boldsymbol{x}$ is a linear combination of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$.

   (ii) [2 marks] Any three vectors in S are linearly independent.

   (iii) [2 marks] $\{\boldsymbol{u} - \boldsymbol{x}, \boldsymbol{v} - \boldsymbol{x}, \boldsymbol{w} - \boldsymbol{x}\}$ is a basis for V.

   (iv) [3 marks] If $\mathrm{span}\{\boldsymbol{v}, \boldsymbol{w}\} \neq \mathrm{span}\{\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{x}\}$, then $V = \mathrm{span}\{\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{x}\}$.

   (v) [3 marks] If $\boldsymbol{y} \notin V$, then $\{\boldsymbol{y} + \boldsymbol{z} \mid \boldsymbol{z} \in V\}$ cannot be a subspace of $\mathbb{R}^n$.

5. Let $S = \{(1,1,2,3,4), (1,2,2,3,3), (1,1,2,2,3)\}$ and $V = \mathrm{span}(S)$.
   Also let $T = \{(3,3,6,7,10), (2,3,4,5,6), (2,3,4,6,7)\}$.

   (i) [3 marks] Show that S is a basis for V.

   (ii) [3 marks] Show that $\boldsymbol{v} = (0,-5,0,-3,2)$ belong t V and find the coordinate vector of $\boldsymbol{v}$ with respect to S.

   (iii) [4 marks] Show that T is also a basis for V.

   (iv) [2 marks] Suppose $\boldsymbol{w} \in V$ such that $(\boldsymbol{w})_T = (1,1,-1)$. Find $\boldsymbol{w}$.

6. Let $\boldsymbol{A} = \begin{pmatrix} 1 & 3 & 1 & 3 \\ 3 & -1 & 3 & -1 \\ 2 & 1 & 2 & 1 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 2 & 1 & 1 & 2 \\ 0 & 1 & 3 & 2 \end{pmatrix}$.

   (i) [4 marks] Find the solution spaces S of $\boldsymbol{Ax} = \boldsymbol{0}$ and T of $\boldsymbol{Bx} = \boldsymbol{0}$. Give your answers in explicit set notation.

   (ii) [2 marks] Find a basis for each of S and T in (i).

   (iii) [4 marks] Show that every vector $\boldsymbol{v}$ in $\mathbb{R}^4$ can be expressed as $\boldsymbol{v} = \boldsymbol{s} + \boldsymbol{t}$ in a unique way where $\boldsymbol{s} \in S$ and $\boldsymbol{t} \in T$.

7. Let $\boldsymbol{u}_1 = (1, 4, 2)$, $\boldsymbol{u}_2 = (0, 3, 1)$, $\boldsymbol{u}_3 = (1, 1, 1)$, $\boldsymbol{u}_4 = (1, 3, -3)$ represent four points in the xyz-space, and $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\}$.

   (i) [3 marks] Show that we cannot find a plane that contains all the four points in S.

   (ii) [4 marks] Find a linear system with 2 equations that represent two planes U and V such that $\boldsymbol{u}_1$ lies on U and $\boldsymbol{u}_2$ lies on V, and $\boldsymbol{u}_3, \boldsymbol{u}_4$ are solutions of the system.

   (iii) [3 marks] Find the equation of a plane P that contains three points in S such that P is a subspace of $\mathbb{R}^3$.

   (iv) [4 marks] Write down a necessary and sufficient condition on any three points in $\mathbb{R}^3$ such that the three points lie on a plane that corresponds to a subspace of $\mathbb{R}^3$. Justify your answer.