NATIONAL UNIVERSITY OF SINGAPORE

SEMESTER 1, 2021/2022

MA2002 Calculus **Solution to Tutorial 7**

TUTORIAL PART I

1. (a) $\displaystyle\lim_{t\to-3}\frac{t^3-4t+15}{t^2-t-12}=\lim_{t\to-3}\frac{(t^3-4t+15)'}{(t^2-t-12)'}=\lim_{t\to-3}\frac{3t^2-4}{2t-1}=\frac{3(-3)^2-4}{2(-3)-1}=-\frac{23}{7}.$

   (b) $\displaystyle\lim_{x\to0}\frac{8x^2}{\cos x-1}=\lim_{x\to0}\frac{(8x^2)'}{(\cos x-1)'}=\lim_{x\to0}\frac{16x}{-\sin x}=-16\cdot\lim_{x\to0}\frac{x}{\sin x}=-16.$

   (c) $\displaystyle\lim_{t\to0}\frac{t(1-\cos t)}{t-\sin t}=\lim_{t\to0}\frac{[t(1-\cos t)]'}{(t-\sin t)'}=\lim_{t\to0}\frac{1-\cos t+t\sin t}{1-\cos t}=\lim_{t\to0}\frac{(1-\cos t+t\sin t)'}{(1-\cos t)'}$
   $\displaystyle=\lim_{t\to0}\frac{2\sin t+t\cos t}{\sin t}=2+\lim_{t\to0}\cos t\cdot\lim_{t\to0}\frac{t}{\sin t}=2+1\cdot1=3.$

   (d) $\displaystyle\lim_{x\to\infty}\frac{\sqrt{9x+1}}{\sqrt{x+1}}=\lim_{x\to\infty}\frac{\sqrt{9+\frac{1}{x}}}{\sqrt{1+\frac{1}{x}}}=\frac{\sqrt{9+0}}{\sqrt{1+0}}=3.$

   (e) $\displaystyle\lim_{x\to(\pi/2)^-}\frac{\sec x}{\tan x}=\lim_{x\to(\pi/2)^-}\frac{1/\cos x}{\sin x/\cos x}=\lim_{x\to(\pi/2)^-}\frac{1}{\sin x}=\frac{1}{\sin(\pi/2)}=1.$

   (f) $\displaystyle\lim_{x\to1}\frac{\sqrt{3-x}-\sqrt{1+x}}{x^2+x-2}=\lim_{x\to1}\frac{(\sqrt{3-x}-\sqrt{1+x})'}{(x^2+x-2)'}=\lim_{x\to1}\frac{\frac{-1}{2\sqrt{3-x}}-\frac{1}{2\sqrt{1+x}}}{2x+1}$
   $\displaystyle=\frac{\frac{-1}{2\sqrt{3-1}}-\frac{1}{2\sqrt{1+1}}}{2\cdot1+1}=-\frac{1}{3\sqrt{2}}=-\frac{\sqrt{2}}{6}.$

   (g) $\displaystyle\lim_{x\to a}\frac{x^m-a^m}{x^n-a^n}=\lim_{x\to a}\frac{(x^m-a^m)'}{(x^n-a^n)'}=\lim_{x\to a}\frac{mx^{m-1}}{nx^{n-1}}=\frac{ma^{m-1}}{na^{n-1}}=\frac{m}{n}\,a^{m-n}.$

   (h) $\displaystyle\lim_{x\to\pi/2}(\sec x-\tan x)=\lim_{x\to\pi/2}\left(\frac{1}{\cos x}-\frac{\sin x}{\cos x}\right)=\lim_{x\to\pi/2}\frac{(1-\sin x)'}{(\cos x)'}=\lim_{x\to\pi/2}\frac{-\cos x}{-\sin x}$
   $\displaystyle=\frac{-\cos(\pi/2)}{-\sin(\pi/2)}=0.$

2. Divide $[0,a]$ into n equal subintervals: $[x_0,x_1],[x_1,x_2],\ldots,[x_{n-1},x_n]$.

   The length of each subinterval is $\Delta x=\dfrac{a}{n}$, and $x_i=i\cdot\Delta x=\dfrac{ia}{n}$ for $i=0,1,2,\ldots,n$.

Let $f(x) = x^3$. The Riemann sum is given by

$$S_n = [f(x_1) + f(x_2) + \cdots + f(x_n)]\Delta x = \left[\left(\frac{a}{n}\right)^3 + \left(\frac{2a}{n}\right)^3 + \cdots + \left(\frac{na}{n}\right)^3\right]\frac{a}{n}$$
$$= \frac{a^4}{n^4}(1^3 + 2^3 + \cdots + n^3) = \frac{a^4}{n^4} \cdot \left[\frac{n(n+1)}{2}\right]^2 = \frac{a^4(n+1)^2}{4n^2}.$$

Therefore,

$$\int_0^a x^3\,dx = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a^4}{4}\left(1 + \frac{1}{n}\right)^2 = \frac{a^4}{4}.$$

3. $\displaystyle\int_3^1 h(r)\,dr = \int_{-1}^1 h(r)\,dr - \int_{-1}^3 h(r)\,dr = 0 - 6 = -6.$

4. (a) Let $u = x^2$. Then $\dfrac{du}{dx} = 2x$, and

$$\frac{dy}{dx} = \frac{d}{dx}\int_0^{x^2} \cos(t^{1/3})\,dt = \frac{du}{dx}\frac{d}{du}\int_0^u \cos(t^{1/3})\,dt$$
$$= 2x \cdot \cos(u^{1/3}) = 2x\cos(x^{2/3}).$$

(b) Let $u = \sqrt{x}$. Then $\dfrac{du}{dx} = \dfrac{1}{2\sqrt{x}}$, and

$$\frac{dy}{dx} = \frac{d}{dx}\int_\pi^{\sqrt{x}} \sin t\,dt = \frac{du}{dx}\frac{d}{du}\int_\pi^u \sin t\,dt$$
$$= \frac{1}{2\sqrt{x}}\sin u = \frac{\sin\sqrt{x}}{2\sqrt{x}}.$$

(c) Let $u = \tan x$. Then $\dfrac{du}{dx} = \sec^2 x$, and

$$\frac{dy}{dx} = -\frac{d}{dx}\int_0^{\tan x} \frac{dt}{(1+t^2)^2} = -\frac{du}{dx}\frac{d}{du}\int_0^u \frac{dt}{(1+t^2)^2}$$
$$= -\sec^2 x \cdot \frac{1}{(1+u^2)^2} = -\sec^2 x \cdot \frac{1}{(1+\tan^2 x)^2}$$
$$= -\sec^2 x \cdot \frac{1}{\sec^4 x} = -\cos^2 x.$$

5. Let $u = g(x)$ and $v = h(x)$. Then for any $a \in \mathbb{R}$,

$$\frac{d}{dx}F(x) = \frac{d}{dx}\left(\int_a^v f(t)\,dt - \int_a^u f(t)\,dt\right)$$
$$= \frac{dv}{dx}\frac{d}{dv}\int_a^v f(t)\,dt - \frac{du}{dx}\frac{d}{du}\int_a^u f(t)\,dt$$
$$= h'(x)f(v) - g'(x)f(u)$$
$$= h'(x)f(h(x)) - g'(x)f(g(x)).$$

Then

$$\begin{aligned}\frac{d}{dx}\int_{\cos x}^{5x}\cos(t^2)\,dt &= (5x)'\cos((5x)^2)-(\cos x)'\cos((\cos x)^2)\\ &= 5\cos(25x^2)+\sin x\cos(\cos^2 x).\end{aligned}$$

6. $F(x) = x\int_a^x f(t)\,dt - \int_a^x tf(t)\,dt$. Then

$$\begin{aligned}\frac{d}{dx}F(x) &= \frac{d}{dx}\left(x\int_a^x f(t)\,dt\right)-\frac{d}{dx}\int_a^x tf(t)\,dt\\ &= \int_a^x f(t)\,dt + x\cdot f(x) - xf(x) = \int_a^x f(t)\,dt.\end{aligned}$$

Therefore,

$$F''(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

TUTORIAL PART II

1. (a) $\displaystyle\lim_{x\to 0}\frac{\tan x - x - x^3/3}{\sin^5 x} = \lim_{x\to 0}\frac{x^5}{\sin^5 x}\lim_{x\to 0}\frac{\tan x - x - x^3/3}{x^5}$

$$\begin{aligned}&= \left(\lim_{x\to 0}\frac{x}{\sin x}\right)^5\lim_{x\to 0}\frac{(\tan x - x - x^3/3)'}{(x^5)'} = 1^5\cdot\lim_{x\to 0}\frac{\sec^2 x - 1 - x^2}{5x^4}\\ &= \frac{1}{5}\lim_{x\to 0}\frac{\tan^2 x - x^2}{x^4} = \frac{1}{5}\lim_{x\to 0}\frac{\tan x - x}{x^3}\lim_{x\to 0}\frac{\tan x + x}{x}\\ &= \frac{1}{5}\lim_{x\to 0}\frac{(\tan x - x)'}{(x^3)'}\lim_{x\to 0}\left(\frac{\tan x}{x}+1\right) = \frac{1}{5}\lim_{x\to 0}\frac{\sec^2 x - 1}{3x^2}\cdot(1+1)\\ &= \frac{2}{15}\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^2 = \frac{2}{15}\cdot 1^2 = \frac{2}{15}.\end{aligned}$$

(b) $\displaystyle\lim_{x\to\infty}(\sqrt{x^2+x+1}-\sqrt{x^2-x}) = \lim_{x\to\infty}\frac{(\sqrt{x^2+x+1}-\sqrt{x^2-x})(\sqrt{x^2+x+1}+\sqrt{x^2-x})}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}$

$$\begin{aligned}&= \lim_{x\to\infty}\frac{(x^2+x+1)-(x^2-x)}{\sqrt{x^2+x+1}+\sqrt{x^2-x}} = \lim_{x\to\infty}\frac{2x+1}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}\\ &= \lim_{x\to\infty}\frac{2+\frac{1}{x}}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+\sqrt{1-\frac{1}{x}}} = \frac{2+0}{\sqrt{1+0+0}+\sqrt{1-0}} = 1.\end{aligned}$$

2. i) Divide the interval $[a,b]$ into n equal subintervals

$$[x_0,x_1],[x_1,x_2],\ldots,[x_{n-1},x_n].$$

The length of each subinterval is $\Delta x = x_i - x_{i-1}$, $i=1,\ldots,n$, $x_0 = a$ and $x_n = b$.

ii) Choose the sample points $x_i^* = \sqrt{x_{i-1}\cdot x_i}\in[x_{i-1},x_i]$, $i=1,\ldots,n$.

iii) Then the Riemann sum is given by

$$\begin{aligned} S_n &= \sum_{n=1}^{n} (x_i^*)^{-2} \Delta x = \sum_{i=1}^{n} (\sqrt{x_{i-1}x_i})^{-2}(x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} \frac{x_i - x_{i-1}}{x_{i-1}x_i} = \sum_{i=1}^{n} \left(\frac{1}{x_{i-1}} - \frac{1}{x_i} \right) = \frac{1}{x_0} - \frac{1}{x_n} = \frac{1}{a} - \frac{1}{b}. \end{aligned}$$

iv) Therefore, the definite integral can be evaluated as

$$\int_a^b x^{-2}\, dx = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{1}{a} - \frac{1}{b} \right) = \frac{1}{a} - \frac{1}{b}.$$

3. Define $F(x) = \displaystyle\int_x^{x+T} f(t)\, dt$. Then $\dfrac{d}{dx}F(x) = f(x+T) - f(x) = 0$ for all $x \in \mathbb{R}$.

It follows that F is constant on $\mathbb{R}$. So $F(a) = F(0)$ for all $a \in \mathbb{R}$. Therefore,

$$\int_a^{a+T} f(t)\, dt = \int_0^T f(t)\, dt, \quad \text{for all } a \in \mathbb{R}.$$