NATIONAL UNIVERSITY OF SINGAPORE

SEMESTER 1, 2021/2022

MA2002 Calculus **Tutorial 8** (18^{th} October – 22^{nd} October)

TUTORIAL PART I

This part consists of relatively basic questions which cover the course materials. The solutions to these questions will be recorded.

1. Evaluate the following indefinite integrals.

   (a) $\displaystyle\int \frac{\cos(\pi/x)}{x^2}\,dx$,

   (b) $\displaystyle\int (2+\tan^2\theta)\,d\theta$,

   (c) $\displaystyle\int \cos\theta\,(\tan\theta+\sec\theta)\,d\theta$,

   (d) $\displaystyle\int \frac{1}{\sqrt{x}\,(1+\sqrt{x})^2}\,dx$,

   (e) $\displaystyle\int \frac{\sec^2 y}{\sqrt{1-\tan^2 y}}\,dy$,

   (f) $\displaystyle\int \csc x\,dx$,

   (g) $\displaystyle\int \sqrt{1+\sin^2(x-1)}\,\sin(x-1)\,\cos(x-1)\,dx$.

2. Evaluate the following definite integrals.

   (a) $\displaystyle\int_0^1 x^2(x+1)^2\,dx$,

   (b) $\displaystyle\int_0^4 \left|\sqrt{x}-1\right|\,dx$,

   (c) $\displaystyle\int_{-1}^{-1/2} t^{-2}\sin^2\left(1+\frac{1}{t}\right)dt$,

   (d) $\displaystyle\int_2^4 \frac{dx}{x(\ln x)^2}$.

3. For each of the following functions, find the area of the region between the graph of the function and the x-axis.

   (a) $y = x^3-3x^2+2x$, $x\in[0,2]$.

   (b) $y = x\sqrt{4-x^2}$, $x\in[-2,2]$.

4. Evaluate the following improper integrals.

   (a) $\displaystyle\int_{-\infty}^{\infty} \frac{2x}{(x^2+1)^2}\,dx$,

   (b) $\displaystyle\int_0^{\infty} \frac{16\tan^{-1}x}{1+x^2}\,dx$,

   (c) $\displaystyle\int_0^a \frac{1}{\sqrt[5]{x}}\,dx \quad (a>0)$.

5. Let f be a continuous function on $[0,1]$. Show that

$$\int_0^\pi x f(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx.$$

Hence, evaluate the integral $\displaystyle\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx$.

6. Evaluate the derivative of f^{-1} at 2. (You may assume that the function is one-to-one in its domain without proof.)

(a) $f(x) = x^5 - x^3 + 2x$, (b) $f(x) = \sqrt{x^3 + x^2 + x + 1}$.

7. Use logarithmic differentiation to find the derivative of y with respect to x.

(a) $y = \sqrt[3]{\dfrac{x(x-2)}{x^2+1}}$, (b) $y = \dfrac{x\sin x}{\sqrt{\sec x}}$.

Tutorial Part II

This part consists of relatively difficult questions to promote independent learning and inculcate critical thinking abilities. The solutions will not be recorded. You may attempt them after you have gained a good understanding of the questions in Part I. The complete solution of this part is provided.

1. Find the interval $[a,b]$ for which the value of the integral

$$\int_a^b (2 + x - x^2)\,dx$$

is a maximum.

2. Let $a > 0$, and let f be continuous on $[0,a]$.

(i) Show that

$$\int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx = \int_0^a \frac{f(a-x)}{f(x)+f(a-x)}\,dx.$$

(ii) Using part (i), show that

$$\int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx = \frac{a}{2}.$$

Hence, evaluate the definite integral $\displaystyle\int_0^1 \frac{x^4}{x^4+(1-x)^4}\,dx$.

3. (i) Let $M > 0$. Use the identity $\ln x^r = r\ln x$ $(x > 0, r \in \mathbb{Q})$ to find a number $c > 0$ such that $\ln c > M$.

(ii) Let $M > 0$. Use part (i) and the Intermediate Value Theorem to show that there exists $x_0 > 0$ such that $\ln x_0 = M$.

(iii) Let $M < 0$. Use part (ii) to show that there exists $x_0 > 0$ such that $\ln x_0 = M$.

(iv) Use the above to conclude that the range of $\ln x$ is $\mathbb{R}$,

$$\lim_{x\to\infty} \ln x = \infty \quad \text{and} \quad \lim_{x\to 0^+} \ln x = -\infty.$$

Answers to Part I:

1. (a) $-\frac{1}{\pi}\sin\left(\frac{\pi}{x}\right) + C$, (b) $\theta + \tan\theta + C$, (c) $\theta - \cos\theta + C$, (d) $-\frac{2}{1+\sqrt{x}} + C$,
   (e) $\sin^{-1}(\tan y) + C$, (f) $-\ln|\csc x + \cot x| + C$, (g) $\frac{1}{3}\left(1+\sin^2(x-1)\right)^{3/2} + C$.
2. (a) $\frac{31}{30}$, (b) 2, (c) $\frac{1}{2} - \frac{1}{4}\sin 2$, (d) $\frac{1}{2\ln 2}$.
3. (a) $\frac{1}{2}$, (b) $\frac{16}{3}$.
4. (a) 0, (b) $2\pi^2$, (c) $\frac{5}{4}a^{4/5}$.
5. $\frac{\pi^2}{4}$.
6. (a) $\frac{1}{4}$, (b) $\frac{2}{3}$.
7. (a) $\frac{1}{3}\sqrt[3]{\frac{x(x-2)}{x^2+1}}\left(\frac{1}{x} + \frac{1}{x-2} - \frac{2x}{x^2+1}\right)$, (b) $\frac{x\sin x}{\sqrt{\sec x}}\left(\frac{1}{x} + \cot x - \frac{1}{2}\tan x\right)$.