NATIONAL UNIVERSITY OF SINGAPORE

SEMESTER 1, 2021/2022

MA2002 Calculus **Solution to Tutorial 6**

TUTORIAL PART I

1. (a) $f'(x) = 3-3x^2$. Then $f'(x) = 0 \Rightarrow x = \pm 1$.

	$(-\infty,-1)$	$(-1,1)$	$(1,\infty)$
$f'(x)$	$-$	$+$	$-$
$f(x)$	$\searrow$	$\nearrow$	$\searrow$

Then f is increasing on $(-1,1)$, and decreasing on $(-\infty,-1)$ and on $(1,\infty)$.

f has a local minimum 0 at -1, and a local maximum 4 at 1.

$f''(x) = -6x$. Then $f''(x) = 0 \Rightarrow x = 0$.

	$(-\infty,0)$	$(0,\infty)$
$f''(x)$	$+$	$-$
$f(x)$	concave up	concave down

The graph of f is concave up on $(-\infty,0)$, and concave down on $(0,\infty)$.

The point $(0,2)$ is the inflection point of f.

(b) $g'(x) = 4-\sec^2 x$. Then $g'(x) = 0 \Rightarrow \cos x = \pm\frac{1}{2} \Rightarrow x = \pm\frac{\pi}{3}$ $(x \in (-\frac{\pi}{2},\frac{\pi}{2}))$.

	$(-\frac{\pi}{2},-\frac{\pi}{3})$	$(-\frac{\pi}{3},\frac{\pi}{3})$	$(\frac{\pi}{3},\frac{\pi}{2})$
$f'(x)$	$-$	$+$	$-$
$f(x)$	$\searrow$	$\nearrow$	$\searrow$

Then g is increasing on $(-\frac{\pi}{3},\frac{\pi}{3})$, and decreasing on $(-\frac{\pi}{2},-\frac{\pi}{3})$ and on $(\frac{\pi}{3},\frac{\pi}{2})$.

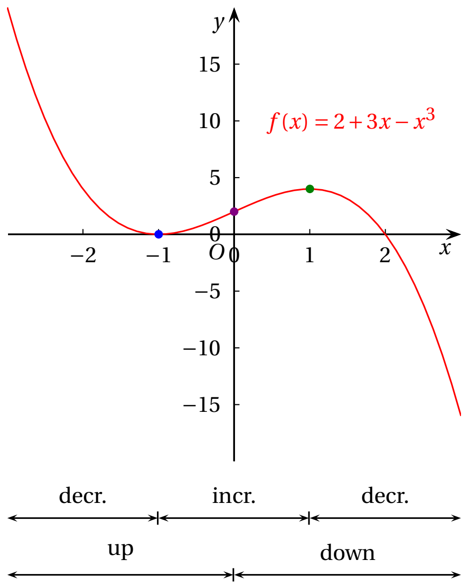
g has a local minimum $\sqrt{3}-\frac{4\pi}{3}$ at $-\frac{\pi}{3}$, and a local maximum $\frac{4\pi}{3}-\sqrt{3}$ at $\frac{\pi}{3}$.

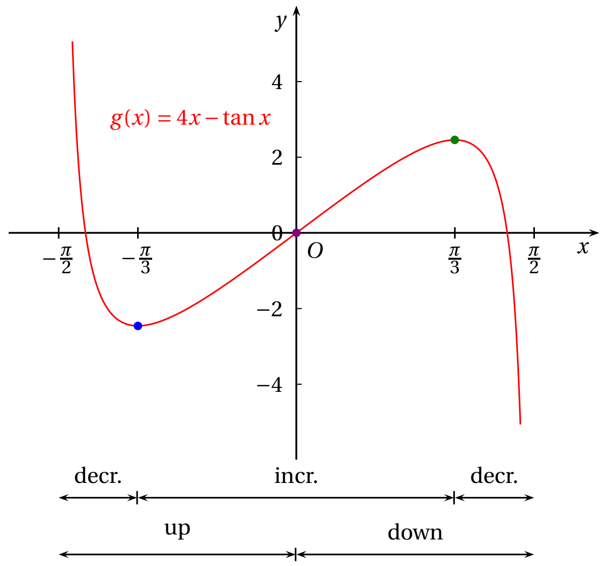
$g''(x) = -2\sec^2 x\tan x$. Then $g''(x) = 0 \Rightarrow \tan x = 0 \Rightarrow x = 0$.

	$(-\frac{\pi}{2},0)$	$(0,\frac{\pi}{2})$
$f''(x)$	$+$	$-$
$f(x)$	concave up	concave down

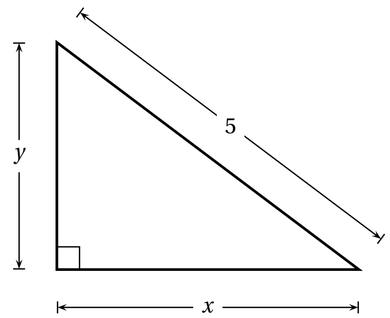
The graph of g is concave up on $(-\frac{\pi}{2},0)$, and concave down on $(0,\frac{\pi}{2})$.

The point $(0,0)$ is the inflection point of g.





2. Let the length of the two legs of the right triangle be x and y respectively.



Then by Pythagorean's Theorem, $x^2 + y^2 = 5^2 = 25$, i.e., $y = \sqrt{25 - x^2}$.

The area of the right triangle is thus given by

$$A(x) = \frac{1}{2} x \sqrt{25 - x^2}, \quad 0 \le x \le 5.$$

Then $A'(x) = \dfrac{25 - 2x^2}{2\sqrt{25 - x^2}}$. Let $A'(x) = 0$. We have $x = \dfrac{5}{\sqrt{2}}$.

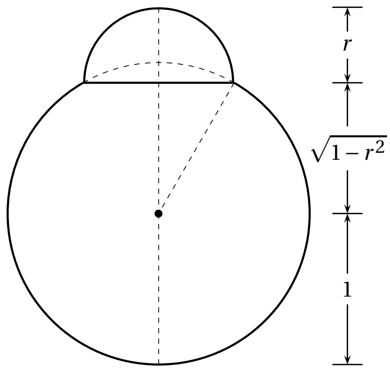
Comparing the values of A at the end points $0, 5$, and at the critical number $\dfrac{5}{\sqrt{2}}$:

$$A(0) = 0, \quad A(5) = 0, \quad A\left(\frac{5}{\sqrt{2}}\right) = \frac{25}{4},$$

we see that A attains the maximum $\dfrac{25}{4}$ when $x = \dfrac{5}{\sqrt{2}}$.

Therefore, the right triangle attains the maximum area $\dfrac{25}{4}\,\text{cm}^2$ when it is the isosceles right triangle.

3. Let r be the radius of the hemisphere bubble.



Then the height of the bubble tower is given by

$$h(r) = r + \sqrt{1-r^2} + 1, \quad 0 \le r \le 1.$$

Then $h'(r) = 1 - \dfrac{r}{\sqrt{1-r^2}}$. Let $h'(r) = 0$. We have $r = \dfrac{1}{\sqrt{2}}$.

Comparing the values of h at the end points 0, 1, and at the critical number $\dfrac{1}{\sqrt{2}}$:

$$h(0) = 2, \quad h(1) = 2, \quad h\Big(\frac{1}{\sqrt{2}}\Big) = \sqrt{2} + 1 \approx 2.414,$$

we see that h attains the maximum $\sqrt{2} + 1$ at $r = \dfrac{1}{\sqrt{2}}$.

Therefore, the bubble tower has the maximum height $\sqrt{2}+1$ when the radius of the hemisphere bubble is $\dfrac{1}{\sqrt{2}}$.

4. Let $\theta = \angle BAC$. Then $\angle BOC = 2\theta$. So the distance from A to B is $|\overline{AB}| = 4\cos\theta$, and the arc length from B to C is $|\overset{\frown}{BC}| = 2 \cdot 2\theta = 4\theta$. Then the time spent from A to C is given by

$$T(\theta) = \frac{|\overline{AB}|}{2} + \frac{|\overset{\frown}{BC}|}{4} = \frac{4\cos\theta}{2} + \frac{4\theta}{4} = 2\cos\theta + \theta, \quad 0 \le \theta \le \frac{\pi}{2}.$$

Then $T'(\theta) = 1 - 2\sin\theta$. Let $T'(\theta) = 0$. We have $\theta = \dfrac{\pi}{6}$.

Comparing the values of T at the end points $0, \dfrac{\pi}{2}$, and at the critical number $\dfrac{\pi}{6}$:

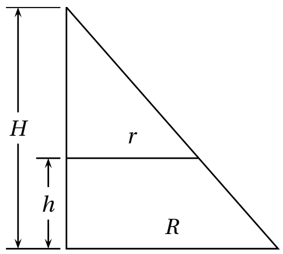
$$T(0) = 2, \quad T\Big(\frac{\pi}{2}\Big) = \frac{\pi}{2} \approx 1.57, \quad T\Big(\frac{\pi}{6}\Big) = \sqrt{3} + \frac{\pi}{6} \approx 2.26,$$

we see that T attains the minimum $\dfrac{\pi}{2}$ at $r = \dfrac{\pi}{2}$.

Therefore, in order to arrive the point C as soon as possible, she should walk around the lake.

5. Since the cylinder is inscribed in the cone, by similar triangles, we have

$$\frac{r}{R} = \frac{H-h}{H}, \quad \text{i.e., } r = \frac{R}{H}(H-h).$$



Then the volume of the cylinder is given by

$$V(h) = \pi r^2 h = \frac{\pi R^2}{H^2}(H-h)^2 h, \quad 0 \le h \le H.$$

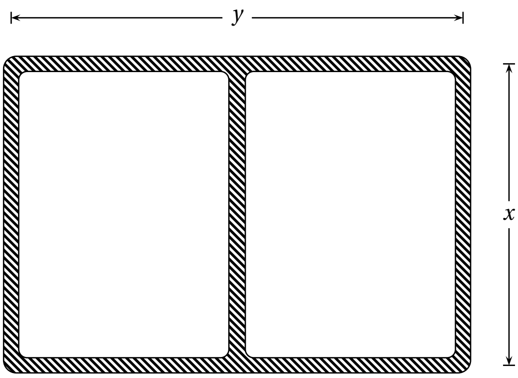
$V'(h) = \dfrac{\pi R^2}{H^2}(H-h)(H-3h)$. Let $V'(h) = 0$. Then $h = \dfrac{H}{3}$.

Comparing the values of V at the end points $0, H$, and at the critical number $\dfrac{H}{3}$:

$$V(0) = 0, \quad V(H) = 0, \quad T\left(\frac{H}{3}\right) = \frac{4\pi}{27}R^2 H,$$

we see that V attains the maximum $\dfrac{4\pi}{27}R^2 H$ at $h = \dfrac{H}{3}$ and $r = \dfrac{2R}{3}$.

6. Let x and y be the length and the height of the rectangular plot respectively. Then $xy = 216$, i.e., $y = \dfrac{216}{x}$.



The length of the fence is thus given by

$$L(x) = 3x + 2y = 3x + \frac{432}{x}, \quad x > 0.$$

$L'(x) = 3 - \dfrac{432}{x^2}$. Let $L'(x) = 0$. Then $x = 12$.

If $0 < x < 12$, $L'(x) < 0$; if $x > 12$, $L'(x) > 0$. By Increasing/Decreasing Test,

$$L \text{ is decreasing on } (0, 12], \text{ and it is increasing on } [12, \infty).$$

So L attains the minimum 72 at $x = 12$. Moreover, if $x = 12$, then $y = 18$.

Therefore, the fence has the smallest length 72 m when the rectangle is 12 m × 18 m.

7. At the corner, the pipe has to be turned through the angles θ, where $0 < \theta < \frac{\pi}{2}$. At these angles, the length of available room for turning the pipe is given by

$$L(\theta) = \frac{9}{\sin\theta} + \frac{6}{\cos\theta}, \quad 0 < \theta < \frac{\pi}{2}.$$

Then $L'(\theta) = -\dfrac{9\cos\theta}{\sin^2\theta} + \dfrac{6\sin\theta}{\cos^2\theta}$. Let $L'(\theta) = 0$. We have $\theta = \tan^{-1}\sqrt[3]{\dfrac{3}{2}}$.
Note that $\sin\theta$ is increasing and $\cos\theta$ is decreasing on $(0, \frac{\pi}{2})$. So $L'(\theta)$ is a increasing on $(0, \frac{\pi}{2})$. In other words, the graph of L is concave up on $(0, \frac{\pi}{2})$.

In particular, the graph of L lies above the horizonal tangent line at $\theta = \theta_0 = \tan^{-1}\sqrt[3]{\dfrac{3}{2}}$. So L has the absolute minimum $L(\theta_0) \approx 21.07$ ft.

A pipe of length ℓ can be carried horizontally around the corner if and only if

$$\ell \le L(\theta) \quad \text{for all } \theta \in (0, \tfrac{\pi}{2}).$$

In other words, $\ell \le \min\left\{L(\theta) \mid \theta \in (0, \frac{\pi}{2})\right\} \approx 21$.

So the longest pipe which can be carried horizontally around the corner is 21 m.

Tutorial Part II

1. $f'(x) = 1 - \sin x$. Solving $f'(x) = 0$ on $(-2\pi, 2\pi)$, we have $x = -\frac{3\pi}{2}$ and $\frac{\pi}{2}$, which are the critical numbers of f on $(-2\pi, 2\pi)$.

	$(-2\pi, -\frac{3\pi}{2})$	$(-\frac{3\pi}{2}, \frac{\pi}{2})$	$(\frac{\pi}{2}, 2\pi)$
$f'(x)$	+	+	+
$f(x)$	$\nearrow$	$\nearrow$	$\nearrow$

By Increasing/Decreasing Test, f is increasing on $[-2\pi, -\frac{3\pi}{2}]$, on $[-\frac{3\pi}{2}, \frac{\pi}{2}]$ and on $[\frac{\pi}{2}, 2\pi]$. So f is increasing on $[-2\pi, 2\pi]$, and thus it has no local maximum or local minimum on $(-2\pi, 2\pi)$.

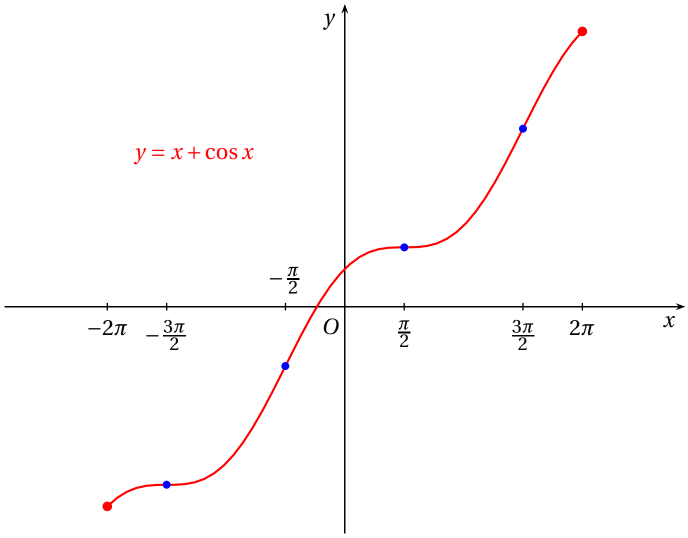
$f''(x) = -\cos x$. Solving $f''(x) = 0$ on $(-2\pi, 2\pi)$, we have $x = \pm\frac{\pi}{2}$ and $\pm\frac{3\pi}{2}$.

	$(-2\pi, -\frac{3\pi}{2})$	$(-\frac{3\pi}{2}, -\frac{\pi}{2})$	$(-\frac{\pi}{2}, \frac{\pi}{2})$	$(\frac{\pi}{2}, \frac{3\pi}{2})$	$(\frac{3\pi}{2}, 2\pi)$
$f''(x)$	−	+	−	+	−
$f(x)$	Down	Up	Down	UP	Down

By Concavity Test, the graph of $f(x)$ is concave up on $(-\frac{3\pi}{2}, -\frac{\pi}{2})$ and on $(\frac{\pi}{2}, \frac{3\pi}{2})$, and it is concave down on $(-2\pi, -\frac{3\pi}{2})$, on $(-\frac{\pi}{2}, \frac{\pi}{2})$ and on $(\frac{3\pi}{2}, 2\pi)$.

So on the interval $(-2\pi, 2\pi)$, f has four inflection points:

$$(-\tfrac{3\pi}{2}, -\tfrac{3\pi}{2}), \quad (-\tfrac{\pi}{2}, -\tfrac{\pi}{2}), \quad (\tfrac{\pi}{2}, \tfrac{\pi}{2}) \quad \text{and} \quad (\tfrac{3\pi}{2}, \tfrac{3\pi}{2}).$$



2. The cross-section of the rain gutter is a trapezium with base 10 and 10+20 $\cos\theta$ and height $10\sin\theta$. We shall maximize its area

$$A(\theta)=\frac{1}{2}(10+10+20\cos\theta)\cdot 10\sin\theta=100(1+\cos\theta)\sin\theta,\quad 0\le\theta\le\frac{\pi}{2}.$$

$A'(\theta)=100(2\cos^2\theta+\cos\theta-1)=100(1+\cos\theta)(2\cos\theta-1)$. Solving $A'(\theta)=0$ on $(0,\frac{\pi}{2})$, we have $\theta=\frac{\pi}{3}$, which is the critical number of $A(\theta)$ on $(0,\frac{\pi}{2})$.

Comparing the values of $A(\theta)$ at end points $0,\frac{\pi}{2}$, and at the critical number $\frac{\pi}{3}$:

$$A(0)=0,\quad A\Big(\frac{\pi}{2}\Big)=100\quad\text{and}\quad A\Big(\frac{\pi}{3}\Big)=75\sqrt{3}\approx 129.9,$$

we see that $A(\theta)$ obtains the absolute maximum $75\sqrt{3}$ at $\theta=\frac{\pi}{3}$.

Therefore, the gutter could carry the maximum amount of water when $\theta=\frac{\pi}{3}$.

3. Note that $\theta=\alpha-\beta$, where $\tan\alpha=\dfrac{18+32}{h}=\dfrac{50}{h}$ and $\tan\beta=\dfrac{32}{h}$. Then

$$\tan\theta=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}=\frac{\frac{50}{h}-\frac{32}{h}}{1+\frac{50}{h}\frac{32}{h}}=\frac{18h}{h^2+1600}.$$

Since $\tan\theta$ is increasing on $(0,\frac{\pi}{2})$, maximizing θ is equivalent to maximizing $\tan\theta$.

Define $f(h)=\dfrac{18h}{h^2+1600}$ $(h>0)$. Then $f'(h)=\dfrac{18(1600-h^2)}{(h^2+1600)^2}$.

Solving $f'(h)=0$ on $h>0$, we have $h=40$, which is the critical number of $f(h)$.

If $0<h<40$, then $f'(h)>0$. So f is increasing on $(0,40]$.

If $h>40$, then $f'(h)<0$. So f is decreasing on $[40,\infty)$.

Therefore, f has the absolute maximum at $h=40$. Equivalently, the kicker has the largest angle if he is 40 ft away from the goal post line.