NATIONAL UNIVERSITY OF SINGAPORE

SEMESTER 1, 2021/2022

MA2002 Calculus **Solution to Tutorial 11**

TUTORIAL PART I

1. (a) Suppose $2\sqrt{xy}\,\dfrac{dy}{dx} = 1$, $x, y > 0$. Then $\sqrt{y}\,\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}$. Integrate with respect to x:

$$\int \sqrt{y}\,dy = \int \frac{1}{2\sqrt{x}}\,dx \Rightarrow \frac{2}{3}y^{3/2} = \sqrt{x} + C.$$

That is, $y = \left[\dfrac{3}{2}(\sqrt{x}+C)\right]^{2/3} = \left(\dfrac{3}{2}\right)^{2/3}(\sqrt{x}+C)^{2/3}$.

(b) Suppose $\sqrt{x}\,\dfrac{dy}{dx} = e^{y+\sqrt{x}}$, $x > 0$. Then $e^{-y}\,\dfrac{dy}{dx} = \dfrac{e^{\sqrt{x}}}{\sqrt{x}}$. Integrate with respect to x:

$$\int e^{-y}\,dy = \int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx \Rightarrow -e^{-y} = 2e^{\sqrt{x}} + C.$$

That is, $y = -\ln(-2e^{\sqrt{x}} - C)$.

(c) Suppose $\dfrac{dy}{dx} = \dfrac{x^2e^{y/x}+y^2}{xy}$. Let $z = \dfrac{y}{x}$. Then $y = zx$ and $\dfrac{dy}{dx} = x\dfrac{dz}{dx} + z$. The equation becomes

$$x\frac{dz}{dx} + z = \frac{e^{y/x}+(y/x)^2}{y/x} = \frac{e^z+z^2}{z} = \frac{e^z}{z} + z.$$

Then $x\dfrac{dz}{dx} = \dfrac{e^z}{z}$, i.e., $ze^{-z}\dfrac{dz}{dx} = \dfrac{1}{x}$. Integrate with respect to x:

$$\int ze^{-z}\,dz = \int \frac{1}{x}\,dx \Rightarrow -e^{-z}(z+1) = \ln|x| + C.$$

That is, $-e^{-y/x}(y/x+1) = \ln|x| + C$; or equivalently, $y + x = -xe^{y/x}(\ln|x|+C)$.

(d) Suppose $x\dfrac{dy}{dx} + 3y = \dfrac{\sin x}{x^2}$, $x > 0$. Then $\dfrac{dy}{dx} + \dfrac{3}{x}\cdot y = \dfrac{\sin x}{x^3}$.

Since $\displaystyle\int \frac{3}{x}\,dx = 3\ln x + c$, an integrating factor is $v(x) = e^{3\ln x} = x^3$.

Therefore, the general solution is given by

$$y = \frac{1}{x^3}\int x^3\cdot\frac{\sin x}{x^3}\,dx = \frac{1}{x^3}\int \sin x\,dx = \frac{1}{x^3}(C - \cos x).$$

(e) Suppose $(t-1)^3\dfrac{ds}{dt}+4(t-1)^2s=t+1$, $t>1$. Then $\dfrac{ds}{dt}+\dfrac{4}{t-1}\cdot s=\dfrac{t+1}{(t-1)^3}$.
Since $\displaystyle\int \frac{4}{t-1}\,dt=4\ln(t-1)+c$, an integrating factor is $v(t)=e^{4\ln(t-1)}=(t-1)^4$.
Therefore, the general solution is given by

$$\begin{aligned} s&=\frac{1}{(t-1)^4}\int (t-1)^4\cdot\frac{t+1}{(t-1)^3}\,dt=\frac{1}{(t-1)^4}\int (t^2-1)\,dt\\ &=\frac{1}{(t-1)^4}\left(\frac{t^3}{3}-t+C\right)=\frac{t^3}{3(t-1)^4}-\frac{t}{(t-1)^4}+\frac{C}{(t-1)^4}. \end{aligned}$$

(f) Suppose $\dfrac{dy}{dx}-y=-y^2$. Let $z=y^{1-2}=y^{-1}$. Then $\dfrac{dz}{dx}=-y^{-2}\dfrac{dy}{dx}$.
Multiply $-y^{-2}$ to the equation:

$$(-y^{-2})\frac{dy}{dx}+(-y^{-2})(-y)=(-y^{-2})(-y^2).$$

That is, $\dfrac{dz}{dx}+z=1$. Since $\displaystyle\int 1\,dx=x+c$, an integrating factor is $v(x)=e^x$.
Therefore, the general solution is given by

$$z=\frac{1}{e^x}\int e^x\cdot 1\,dx=\frac{1}{e^x}(e^x+C)=1+Ce^{-x}.$$

Hence, $y=\dfrac{1}{z}=\dfrac{1}{1+Ce^{-x}}$.

2. (a) Suppose $\dfrac{dy}{dt}=e^t\sin(e^t-2)$. Integrate with respect to x:

$$y=\int e^t\sin(e^t-2)\,dt=-\cos(e^t-2)+C.$$

Then $0=y(\ln 2)=-\cos(e^{\ln 2}-2)+C=-1+C$. It follows that $C=1$, and the solution is given by $y=1-\cos(e^t-2)$.

(b) Suppose $x\dfrac{dy}{dx}=y+x^2\sin x$, $x>0$. Then $\dfrac{dy}{dx}-\dfrac{1}{x}\cdot y=x\sin x$.
Since $\displaystyle\int\left(-\frac{1}{x}\right)dx=-\ln x+c$, an integrating factor is $v(x)=e^{-\ln x}=x^{-1}$. Therefore,

$$y=\frac{1}{x^{-1}}\int x^{-1}\cdot x\sin x\,dx=x\int \sin x\,dx=x(C-\cos x).$$

Then $0=y(\pi)=\pi(C-\cos\pi)=\pi(C+1)$. It follows that $C+1=0$, and the solution is given by $y=x(-1-\cos x)$.

(c) Suppose $(x+1)\dfrac{dy}{dx}-2(x^2+x)y=\dfrac{e^{x^2}}{x+1}$, $x>-1$. Then $\dfrac{dy}{dx}-2xy=\dfrac{e^{x^2}}{(x+1)^2}$.
Since $\displaystyle\int(-2x)\,dx=-x^2+c$, an integrating factor is $v(x)=e^{-x^2}$. Therefore,

$$y=\frac{1}{e^{-x^2}}\int e^{-x^2}\cdot\frac{e^{x^2}}{(x+1)^2}\,dx=e^{x^2}\int\frac{1}{(x+1)^2}\,dx=e^{x^2}\left(C-\frac{1}{x+1}\right).$$

Then $5 = y(0) = e^0(C-1)$. It follows that $C = 6$, and the solution is given by $y = e^{x^2}\left(6 - \dfrac{1}{x+1}\right)$.

3. Suppose $\dfrac{dy}{dx} = 4x^3 y$. Then $\dfrac{1}{y}\dfrac{dy}{dx} = 4x^3$. Integrate with respect to x:

$$\int \frac{1}{y}\, dy = \int 4x^3\, dx \Rightarrow \ln|y| = x^4 + c.$$

That is, $y = \pm e^c e^{x^4} = Ce^{x^4}$. It is given that the curve passes through the point $(0,7)$. Then $7 = Ce^0$ implies $C = 7$. Therefore, the equation of the curve is $y = 7e^{x^4}$.

4. Let $y = y(t)$ be the weight of δ-glucono lactone present at time t. Then $\dfrac{dy}{dt} = -0.6y$.
Solve the equation:

$$\begin{aligned}\frac{1}{y}\frac{dy}{dt} = -0.6 &\Rightarrow \ln|y| = -0.6t + c\\ &\Rightarrow y = \pm e^c e^{-0.6t} = Ce^{-0.6t}.\end{aligned}$$

The initial condition is given by $y(0) = 100$. Then $100 = Ce^0$ implies $C = 100$. So $y = 100e^{-0.6t}$. After one hour, there are $y(1) = 100e^{-0.6} \approx 54.88$ grams left.

5. Let $L(x)$ be the intensity of light x feet deep. Then $\dfrac{dL}{dx} = -kL$ for some constant k.
Solve the equation:

$$\begin{aligned}\frac{1}{L}\frac{dL}{dx} = -k &\Rightarrow \ln|L| = -kx + c\\ &\Rightarrow L = \pm e^c e^{-kx} = Ce^{-kx}, \quad (C \neq 0).\end{aligned}$$

It is given that $L(18) = \dfrac{1}{2}L(0)$, i.e., $Ce^{-k\cdot 18} = \dfrac{1}{2}Ce^{-k\cdot 0}$. Then $k = \dfrac{\ln 2}{18}$.
Solve $L(x) = \dfrac{1}{10}L(0)$, i.e., $Ce^{-kx} = \dfrac{1}{10}C$. Then $x = \dfrac{\ln 10}{k} = \dfrac{18\ln 10}{\ln 2} \approx 59.8$.
So the diver can work at most 59.8 ft without artificial light.

6. (i) Let $P = P(t)$ be the population at time t. Then $\dfrac{dP}{dt} = 0.0015P(150-P)$.
Solve the equation:

$$\begin{aligned}\frac{1}{P(150-P)}\frac{dP}{dt} = 0.0015 &\Rightarrow \int \frac{1}{P(150-P)}\, dP = 0.0015\int 1\, dt\\ &\Rightarrow \frac{1}{150}\int\left(\frac{1}{P} + \frac{1}{150-P}\right) dP = 0.0015\int 1\, dt\\ &\Rightarrow \ln\left|\frac{P}{150-P}\right| = 0.225t + c.\end{aligned}$$

We can solve P explicitly:

$$\ln\left|\frac{150-P}{P}\right| = -0.225t - c \Rightarrow \frac{150-P}{P} = \pm e^{-c}e^{-0.225t} = Ce^{-0.225t}$$
$$\Rightarrow P = \frac{150}{1+Ce^{-0.225t}}.$$

The initial condition is $P(0) = 6$. Then $6 = \dfrac{150}{1+Ce^0}$. So $C = \dfrac{150}{6} - 1 = 24$.

Therefore, $P = \dfrac{150}{1+24e^{-0.225t}}$.

(ii) We can solve that $t = \dfrac{1}{0.225}\ln\left|\dfrac{24P}{150-P}\right|$.

If $P = 100$, $t = \dfrac{\ln 48}{0.225} \approx 17.21$ weeks; if $P = 125$, $t = \dfrac{\ln 120}{0.225} \approx 21.28$ weeks.

7. Let $T = T(t)$ be the temperature of the water at time t, and T_s be the temperature of the refrigerator. Then $\dfrac{dT}{dt} = -r(T - T_s)$. Solve the equation:

$$\frac{1}{T-T_s}\frac{dT}{dt} = -r \Rightarrow \int \frac{1}{T-T_s}\,dT = -r\int 1\,dt$$
$$\Rightarrow \ln|T - T_s| = -rt + c.$$

That is, $T = \pm e^c e^{-rt} + T_s = Ce^{-rt} + T_s$.

The initial condition is $T(0) = 46$. Then $C = 46 - T_s$.

It is also given that $T(10) = 39$ and $T(20) = 33$. Then

$$(46 - T_s)e^{-10r} + T_s = 39,$$
$$(46 - T_s)e^{-20r} + T_s = 33.$$

It follows that $\left(\dfrac{39-T_s}{46-T_s}\right)^2 = e^{-20r} = \dfrac{33-T_s}{46-T_s}$. We can solve that $T_s = -3°$C.

Tutorial Part II

1. Differentiate the identity with respect to x. We have

$$2f(x)f'(x) = [f(x)]^2 + [f'(x)]^2.$$

That is, $[f(x) - f'(x)]^2 = 0$. It follows that $f'(x) - f(x) = 0$ for all $x \in \mathbb{R}$.

Since $\int (-1)\,dx = -x + c$, an integrating factor is given by e^{-x}. Then

$$f(x) = \frac{1}{e^{-x}}\int e^{-x}\cdot 0\,dx = Ce^x.$$

To determine the value of C, let $x = 0$ in the identity. We have $[f(0)]^2 = 100$, i.e., $f(0) = \pm 10$. On the other hand, $f(0) = Ce^0 = C$. So $C = \pm 10$.

Therefore, $f(x) = 10e^x$ or $f(x) = -10e^x$.

2. Let $z = \ln y$. Then $y = e^z$ and $\dfrac{dy}{dx} = e^z \dfrac{dz}{dx}$. Then the equation becomes
$$e^z \frac{dz}{dx} - \frac{e^z}{x} z = xe^z.$$
Or equivalently, $\dfrac{dz}{dx} - \dfrac{z}{x} = x$. Since $\displaystyle\int \left(-\frac{1}{x}\right) dx = -\ln x + c$, an integrating factor is $e^{-\ln x} = x^{-1}$. Then the general solution is given by
$$z = \frac{1}{x^{-1}} \int x^{-1} \cdot x \, dx = x(x + C) = x^2 + Cx.$$
Therefore, $y = e^z = \exp(x^2 + Cx)$.

3. Let $S = S(t)$ be the weight of salt left at time t. Then the rate at which the salt enters the tank is
$$0.03\,\mathrm{kg/L} \times 25\,\mathrm{L/min} = 0.75\,\mathrm{kg/min},$$
and the rate at which the salt leaves the tank is
$$\frac{S\,\mathrm{kg}}{5000\,\mathrm{L}} \times 25\,\mathrm{L/min} = \frac{S}{200}\,\mathrm{kg/min}.$$
Therefore, we have the following ordinary differential equation:
$$\frac{dS}{dt} = 0.75 - \frac{S}{200}, \quad S(0) = 20.$$
Since $\displaystyle\int \frac{1}{200} dt = \frac{t}{200} + c$, an integrating factor is $v(t) = e^{\frac{t}{200}}$. Then
$$S = \frac{1}{e^{\frac{t}{200}}} \int e^{\frac{t}{200}} \cdot 0.75 \, dt = e^{-\frac{t}{200}} (150e^{\frac{t}{200}} + C) = 150 + Ce^{-\frac{t}{200}}.$$
Let $t = 0$. Then $20 = S(0) = 150 + Ce^0$. So $C = -130$. Therefore, $S = 150 - 130e^{-\frac{t}{200}}$.
After half an hour, there are $S(30) = 150 - 130e^{-\frac{30}{200}} \approx 38.1$ kg of salt left in the tank.