NATIONAL UNIVERSITY OF SINGAPORE

SEMESTER 1, 2021/2022

MA2002 Calculus **Solution to Tutorial 4**

TUTORIAL PART I

1. (a) $\displaystyle\lim_{x\to 1}\frac{\sin(x-1)}{x^2+x-2}=\lim_{x\to 1}\frac{\sin(x-1)}{(x-1)(x+2)}=\lim_{x\to 1}\frac{\sin(x-1)}{x-1}\cdot\lim_{x\to 1}\frac{1}{x+2}=1\cdot\frac{1}{1+2}=\frac{1}{3}.$

   (b) $\displaystyle\lim_{x\to 0}\frac{\sin ax}{\sin bx}=\lim_{x\to 0}\left(\frac{\sin ax}{ax}\frac{bx}{\sin bx}\frac{a}{b}\right)=\lim_{x\to 0}\frac{\sin ax}{ax}\cdot\lim_{x\to 0}\frac{bx}{\sin bx}\cdot\frac{a}{b}=1\cdot 1\cdot\frac{a}{b}=\frac{a}{b}.$

2. (a) $\displaystyle f'(x)=\left[\frac{(x-1)^4}{(x^2+2x)^5}\right]'=\frac{[(x-1)^4]'\cdot(x^2+2x)^5-(x-1)^4\cdot[(x^2+2x)^5]'}{[(x^2+2x)^5]^2}$

   $\displaystyle=\frac{4(x-1)^3\cdot(x^2+2x)^5-(x-1)^4\cdot 5(x^2+2x)^4(2x+2)}{(x^2+2x)^{10}}=-\frac{2(x-1)^3(3x^2-4x-5)}{(x^2+2x)^6}.$

   (b) $\displaystyle f'(x)=-\frac{\left[\left(x+\frac{1}{x}\right)^2\right]'}{\left[\left(x+\frac{1}{x}\right)^2\right]^2}=-\frac{\left(x^2+2+\frac{1}{x^2}\right)'}{\left(x+\frac{1}{x}\right)^4}=-\frac{2x+0-\frac{2}{x^3}}{\left(x+\frac{1}{x}\right)^4}$

   $\displaystyle=-\frac{\frac{1}{x^3}2(x^4-1)}{\frac{1}{x^4}(x^2+1)^4}=-\frac{2x(x^4-1)}{(x^2+1)^4}=-\frac{2x(x^2-1)}{(x^2+1)^3}.$

   (c) $f'(x)=\cos(\sin(\sin x))\cdot\cos(\sin x)\cdot\cos x.$

3. $y=\sin(\cos x)$. Then $\dfrac{dy}{dx}=\cos(\cos x)\cdot(\cos x)'=-\cos(\cos x)\cdot\sin x$, and

$$\begin{aligned}\frac{d^2y}{dx^2}&=-[\cos(\cos x)]'\cdot\sin x-\cos(\cos x)\cdot(\sin x)'\\&=-(-\sin(\cos x))\cdot(-\sin x)\cdot\sin x-\cos(\cos x)\cdot\cos x\\&=-\sin(\cos x)\cdot\sin^2 x-\cos(\cos x)\cdot\cos x.\end{aligned}$$

4. (a) Differentiate the equation with respect to x:

$$\frac{d}{dx}(\sin x+\cos y)=\frac{d}{dx}(\sin x\cos y).$$

   That is,

$$\cos x-\sin y\cdot\frac{dy}{dx}=\cos x\cdot\cos y+\sin x\cdot(-\sin y)\cdot\frac{dy}{dx}.$$

   Solving for $\dfrac{dy}{dx}$, we have $(\sin y-\sin x\sin y)\dfrac{dy}{dx}=\cos x-\cos x\cos y$. Therefore,

$$\frac{dy}{dx}=\frac{\cos x(1-\cos y)}{\sin y(1-\sin x)}.$$

(b) Differentiate the equation with respect to x:

$$\frac{d}{dx}\tan(x-y) = \frac{d}{dx}\left(\frac{y}{1+x^2}\right).$$

That is,

$$\sec^2(x-y)\cdot\left(1-\frac{dy}{dx}\right) = \frac{\frac{dy}{dx}(1+x^2)-y\cdot 2x}{(1+x^2)^2}.$$

Solving for $\frac{dy}{dx}$, we have $\frac{dy}{dx}\left[\frac{1}{1+x^2}+\sec^2(x-y)\right] = \sec^2(x-y)+\frac{2xy}{(1+x^2)^2}$.
Therefore,

$$\frac{dy}{dx} = \frac{(1+x^2)^2\sec^2(x-y)+2xy}{(1+x^2)[1+(1+x^2)\sec^2(x-y)]}.$$

5. Differentiating the equation of the curve with respect to x,

$$\frac{d}{dx}(x^2+2xy-y^2+x) = 2,$$

we obtain $2x+2y+2x\cdot\frac{dy}{dx}-2y\cdot\frac{dy}{dx}+1=0$.
Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = -\frac{\frac{1}{2}+x+y}{x-y}.$$

Then $\left.\frac{dy}{dx}\right|_{(1,2)} = \frac{7}{2}$, and the equation of the tangent line at $(1,2)$ is given by

$$y-2 = \frac{7}{2}(x-1); \quad \text{that is, } 7x-2y=3.$$

6. If $y_0=0$, then $x_0=\pm a$, and the tangent line is $x=\pm a$.
Suppose $y_0\neq 0$. Differentiating the equation of the ellipse with respect to x, we have

$$\frac{2x}{a^2}+\frac{2y}{b^2}\cdot\frac{dy}{dx}=0.$$

Then $\frac{dy}{dx} = -\frac{2x/a^2}{2y/b^2} = -\frac{x/a^2}{y/b^2}$ and $\left.\frac{dy}{dx}\right|_{(x_0,y_0)} = -\frac{x_0/a^2}{y_0/b^2}$.
Then the equation of the tangent line at (x_0,y_0) is given by

$$\frac{y-y_0}{x-x_0} = -\frac{x_0/a^2}{y_0/b^2}.$$

That is,

$$\frac{x_0}{a^2}(x-x_0)+\frac{y_0}{b^2}(y-y_0)=0.$$

Note that (x_0,y_0) satisfies the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. We have

$$\frac{x_0x}{a^2}+\frac{y_0y}{b^2} = \frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}=1.$$

7. (a) $f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3)$. Let $f'(x) = 0$. Then $x = 1$ and $x = 3$ are the critical numbers of f on $(-1, 4)$.
   Comparing the values at critical numbers $f(1) = 6$ and $f(3) = 2$, and the values at end points $f(-1) = -14$ and $f(4) = 6$, we conclude that f has the absolute maximum 6 at $x = 1$ and $x = 4$ and the absolute minimum -14 at $x = -1$.
   (b) $f'(x) = \dfrac{4(2-x)}{3x^{2/3}}$. Then $f'(x)$ does not exist at $x = 0$, and $f'(x) = 0$ when $x = 2$. So $x = 0$ and $x = 2$ are the critical numbers of f on $(-1, 8)$.
   Comparing the values at critical numbers $f(0) = 0$ and $f(2) = 6\sqrt[3]{2}$, and the values at end points $f(-1) = -9$ and $f(8) = 0$, we conclude that f has the absolute maximum $6\sqrt[3]{2}$ at $x = 2$ and the absolute minimum -9 at $x = -1$.

8. Define $f(x) = x^r + (1-x)^r$. Then $f'(x) = r(x^{r-1} - (1-x)^{r-1})$.
   Let $f'(x) = 0$. Then $x^{r-1} = (1-x)^{r-1}$ implies that $(x^{r-1})^{\frac{1}{r-1}} = ((1-x)^{r-1})^{\frac{1}{r-1}}$, i.e., $x = 1 - x$.
   So $x = 1/2$ is the only critical number of f on $(0, 1)$.
   If $r > 1$, then $2^{r-1} > 2^0 = 1$, and thus $\dfrac{1}{2^{r-1}} < 1$.
   Comparing $f(1/2) = \dfrac{1}{2^{r-1}}$ and the values at end points $f(0) = f(1) = 1$, we see that f has the absolute maximum 1 at $x = 0$ and $x = 1$, and the absolute minimum $\dfrac{1}{2^{r-1}}$ at $x = 1/2$.
   Therefore, for any $0 \le x \le 1$, we have
   $$\frac{1}{2^{r-1}} \le x^r + (1-x)^r \le 1.$$

9. Suppose f has a local extreme value at $x = x_0$. Since f is a polynomial, it is differentiable at $x = x_0$. By Fermat's Theorem, we must have $f'(x_0) = 0$.
   This means that the quadratic equation $f'(x) = 3x^2 + 2bx + c = 0$ has a real root. So we must have $(2b)^2 - 4(3c) = 4(b^2 - 3c) \ge 0$. However, this contradicts the assumption that $b^2 < 3c$.
   Therefore, f has no local extreme values.

TUTORIAL PART II

1. (i) By the definition of derivative,
$$\begin{aligned}\frac{d}{dx}(x^{\frac{1}{n}}) = \lim_{y\to x}\frac{y^{\frac{1}{n}} - x^{\frac{1}{n}}}{y-x} &= \lim_{y\to x}\frac{y^{\frac{1}{n}} - x^{\frac{1}{n}}}{(y^{\frac{1}{n}} - x^{\frac{1}{n}})(y^{\frac{n-1}{n}} + y^{\frac{n-2}{n}}x^{\frac{1}{n}} + \cdots + x^{\frac{n-1}{n}})}\\ &= \lim_{y\to x}\frac{1}{y^{\frac{n-1}{n}} + y^{\frac{n-2}{n}}x^{\frac{1}{n}} + \cdots + x^{\frac{n-1}{n}}}\\ &= \frac{1}{x^{\frac{n-1}{n}} + x^{\frac{n-2}{n}}x^{\frac{1}{n}} + \cdots + x^{\frac{n-1}{n}}}\\ &= \frac{1}{nx^{\frac{n-1}{n}}} = \frac{1}{n}x^{\frac{1}{n}-1}.\end{aligned}$$

(ii) Let $r \in \mathbb{Q}$. We can write $r = m/n$, where $m \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$.
Let $u = x^{\frac{1}{n}}$. Then by (i), $\dfrac{du}{dx} = \dfrac{1}{n} x^{\frac{1}{n}-1}$. Consequently,

$$\begin{aligned}\frac{d}{dx}x^r &= \frac{d}{dx}x^{\frac{m}{n}} = \frac{d}{dx}u^m = \frac{du}{dx}\cdot\frac{d}{du}u^m = \frac{1}{n}x^{\frac{1}{n}-1}\cdot m u^{m-1}\\ &= \frac{1}{n}x^{\frac{1}{n}-1}\cdot m(x^{\frac{1}{n}})^{m-1} = \frac{m}{n}x^{\frac{m}{n}-1} = r x^{r-1}.\end{aligned}$$

2. For the curve $y = \sin(x - \sin x)$ to have a horizontal tangent,

$$\frac{dy}{dx} = \cos(x - \sin x)\cdot(1 - \cos x) = 0.$$

Note that at the x-axis, $y = \sin(x - \sin x) = 0$. So

$$\cos^2(x - \sin x) = 1 - \sin^2(x - \sin x) = 1 \neq 0.$$

Therefore, we must have $1 - \cos x = 0$. This implies that $x = 2n\pi$, $n \in \mathbb{Z}$.
Hence the points required are $(2n\pi, 0)$ where $n \in \mathbb{Z}$.

3. Let (x_0, y_0) be a point on $x^{2/3} + y^{2/3} = a^{2/3}$, which is not on the coordinate axes. Differentiating the equation with respect to x, we have

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0.$$

Therefore, $\dfrac{dy}{dx} = -\dfrac{x^{-1/3}}{y^{-1/3}} = -\dfrac{y^{1/3}}{x^{1/3}}$ and $\left.\dfrac{dy}{dx}\right|_{(x_0,y_0)} = -\dfrac{y_0^{1/3}}{x_0^{1/3}}$. The equation of the tangent line passing through (x_0, y_0) can be written as

$$\frac{y - y_0}{x - x_0} = -\frac{y_0^{1/3}}{x_0^{1/3}}.$$

That is, $\dfrac{x - x_0}{x_0^{1/3}} + \dfrac{y - y_0}{y_0^{1/3}} = 0$; or equivalently,

$$\frac{x}{x_0^{1/3}} + \frac{y}{y_0^{1/3}} = x_0^{2/3} + y_0^{2/3} = a^{2/3}.$$

We see that the x- and y-intercepts of the tangent line are $x_0^{1/3}a^{2/3}$ and $y_0^{1/3}a^{2/3}$ respectively. Then the length of the portion cut of by the coordinate axes is

$$\sqrt{\left(x_0^{1/3}a^{2/3}\right)^2 + \left(y_0^{1/3}a^{2/3}\right)^2} = a^{2/3}\sqrt{x_0^{2/3} + y_0^{2/3}} = a^{2/3}\sqrt{a^{2/3}} = a,$$

which is a constant.

4. If $f(x) = x^a(1-x)^b$, then

$$f'(x) = ax^{a-1}(1-x)^b - x^a b(1-x)^{b-1} = x^{a-1}(1-x)^{b-1}(a(1-x) - bx).$$

Let $f'(x) = 0$. We have $x = \dfrac{a}{a+b}$, which is the only critical number on $(0,1)$, and

$$f\left(\frac{a}{a+b}\right) = \left(\frac{a}{a+b}\right)^a\left(\frac{b}{a+b}\right)^b = \frac{a^a b^b}{(a+b)^{a+b}}.$$

Comparing with the values at end points $f(0) = 0$ and $f(1) = 0$, we conclude that f has the absolute maximum $\dfrac{a^a b^b}{(a+b)^{a+b}}$.