NATIONAL UNIVERSITY OF SINGAPORE

SEMESTER 1, 2021/2022

MA2002 Calculus **Solution to Tutorial 8**

TUTORIAL PART I

1. (a) Let $u = \frac{\pi}{x}$. Then $\frac{du}{dx} = -\frac{\pi}{x^2}$. So

$$\int \frac{\cos(\pi/x)}{x^2}\,dx = -\frac{1}{\pi}\int \cos u\,du = -\frac{1}{\pi}\sin u + C = -\frac{1}{\pi}\sin\left(\frac{\pi}{x}\right) + C.$$

(b) $\int (2+\tan^2\theta)\,d\theta = \int (1+\sec^2\theta)\,d\theta = \theta + \tan\theta + C.$

(c) $\int \cos\theta(\tan\theta + \sec\theta)\,d\theta = \int (\sin\theta + 1)\,d\theta = -\cos\theta + \theta + C.$

(d) Let $u = 1+\sqrt{x}$. Then $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$. So

$$\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx = 2\int \frac{1}{u^2}\,du = -\frac{2}{u} + C = -\frac{2}{1+\sqrt{x}} + C.$$

(e) Let $u = \tan y$. Then $\frac{du}{dy} = \sec^2 y$. So

$$\int \frac{\sec^2 y}{\sqrt{1-\tan^2 y}}\,dy = \int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1} u + C = \sin^{-1}(\tan y) + C.$$

(f) Let $u = \csc x + \cot x$. Then $\frac{du}{dx} = -\csc x\cot x - \csc^2 x = -\csc x(\cot x + \csc x)$. So

$$\begin{aligned}\int \csc x\,dx &= \int \frac{\csc x(\csc x + \cot x)}{\csc x + \cot x}\,dx = \int \frac{-1}{u}\,du = -\ln|u| + C\\ &= -\ln|\csc x + \cot x| + C.\end{aligned}$$

(g) Let $u = 1+\sin^2(x-1)$. Then $\frac{du}{dx} = 2\sin(x-1)\cos(x-1)$. So

$$\begin{aligned}\int \sqrt{1+\sin^2(x-1)}\,\sin(x-1)\cos(x-1)\,dx &= \frac{1}{2}\int \sqrt{u}\,du = \frac{1}{2}\frac{u^{3/2}}{3/2} + C\\ &= \frac{1}{3}(1+\sin^2(x-1))^{3/2} + C.\end{aligned}$$

2. (a) $\int_0^1 x^2(x+1)^2\,dx = \int_0^1 (x^4+2x^3+x^2)\,dx = \left[\frac{x^5}{5} + \frac{x^4}{2} + \frac{x^3}{3}\right]_{x=0}^{x=1} = \frac{31}{30}.$

(b) $\int_0^4 |\sqrt{x}-1|\,dx = \int_0^1 (1-\sqrt{x})\,dx + \int_1^4 (\sqrt{x}-1)\,dx = \left[x - \frac{x^{3/2}}{3/2}\right]_{x=0}^{x=1} + \left[\frac{x^{3/2}}{3/2} - x\right]_{x=1}^{x=4} = 2.$

(c) Let $u = 1 + \frac{1}{t}$. Then $\frac{du}{dt} = -\frac{1}{t^2}$. So

$$\int_{-1}^{-1/2} t^{-2}\sin^2\left(1+\frac{1}{t}\right)dt = -\int_0^{-1}\sin^2 u\,du = \int_{-1}^0 \frac{1-\cos 2u}{2}\,du$$
$$= \left[\frac{u}{2} - \frac{\sin 2u}{4}\right]_{u=-1}^{u=0} = \frac{1}{2} - \frac{\sin 2}{4}.$$

(d) Let $u = \ln x$. Then $\frac{du}{dx} = \frac{1}{x}$. So $\int_2^4 \frac{dx}{x(\ln x)^2} = \int_{\ln 2}^{\ln 4}\frac{1}{u^2}\,du = \left.\frac{-1}{u}\right|_{u=\ln 2}^{u=\ln 4} = \frac{1}{2\ln 2}.$

3. (a) Let $y = x^3 - 3x^2 + 2x = x(x-1)(x-2)$. Then $y \ge 0$ on $[0,1]$ and $y \le 0$ on $[1,2]$.

$$\int y\,dx = \int (x^3 - 3x^2 + 2x)\,dx = \frac{x^4}{4} - x^3 + x^2 + C.$$

$$\int_0^2 |y|\,dx = \int_0^1 y\,dx + \int_1^2 (-y)\,dx = \left[\frac{x^4}{4} - x^3 + x^2\right]_{x=0}^{x=1} - \left[\frac{x^4}{4} - x^3 + x^2\right]_{x=1}^{x=2}$$
$$= \frac{1}{4} - \frac{-1}{4} = \frac{1}{2}.$$

(b) Let $u = 4 - x^2$. Then $\frac{du}{dx} = -2x$. So

$$\int y\,dx = \int x\sqrt{4-x^2}\,dx = -\frac{1}{2}\int\sqrt{u}\,du = -\frac{1}{2}\frac{u^{3/2}}{3/2} + C = -\frac{1}{3}(4-x^2)^{3/2} + C.$$

Note that $y \le 0$ on $[-2,0]$ and $y \ge 0$ on $[0,2]$. Then

$$\int_{-2}^2 |y|\,dx = \int_{-2}^0 (-y)\,dx + \int_0^2 y\,dx = \left.\frac{1}{3}(4-x^2)^{3/2}\right|_{x=-2}^{x=0} - \left.\frac{1}{3}(4-x^2)^{3/2}\right|_{x=0}^{x=2}$$
$$= \frac{8}{3} - \frac{-8}{3} = \frac{16}{3}.$$

4. (a) Let $u = x^2 + 1$. Then $\frac{du}{dx} = 2x$. So

$$\int \frac{2x}{(x^2+1)^2}\,dx = \int \frac{1}{u^2}\,du = -\frac{1}{u} + C = -\frac{1}{x^2+1} + C.$$

$$\int_{-\infty}^{\infty} \frac{2x}{(x^2+1)^2}\,dx = \int_{-\infty}^0 \frac{2x}{(x^2+1)^2}\,dx + \int_0^{\infty}\frac{2x}{(x^2+1)^2}\,dx$$
$$= \lim_{a\to-\infty}\left(\left[-\frac{1}{x^2+1}\right]_{x=a}^{x=0}\right) + \left(\lim_{b\to\infty}\left[-\frac{1}{x^2+1}\right]_{x=0}^{x=b}\right)$$
$$= \lim_{a\to-\infty}\left(-1 + \frac{1}{1+a^2}\right) + \lim_{b\to\infty}\left(1 - \frac{1}{1+b^2}\right) = (-1) + 1 = 0.$$

(b) Let $u = \tan^{-1} x$. Then $\dfrac{du}{dx} = \dfrac{1}{1+x^2}$. So

$$\int \frac{16\tan^{-1} x}{1+x^2}\,dx = 16\int u\,du = 8u^2 + C = 8(\tan^{-1} x)^2 + C.$$

$$\int_0^\infty \frac{16\tan^{-1} x}{1+x^2}\,dx = \left(\lim_{b\to\infty} 8(\tan^{-1} x)^2\Big|_{x=0}^{x=b}\right) = 8\cdot\left(\frac{\pi}{2}\right)^2 = 2\pi^2.$$

(c) $\displaystyle\int \frac{1}{\sqrt[5]{x}}\,dx = \frac{x^{4/5}}{4/5} + C.$

$$\int_0^a \frac{1}{\sqrt[5]{x}}\,dx = \lim_{t\to0^+}\int_t^a \frac{1}{\sqrt[5]{x}}\,dx = \lim_{t\to0^+}\left(\frac{5}{4}x^{4/5}\Big|_{x=t}^{x=a}\right) = \lim_{t\to0^+}\frac{5}{4}(a^{4/5} - t^{4/5}) = \frac{5}{4}a^{4/5}.$$

5. (i) Let $u = \pi - x$. Then $\dfrac{du}{dx} = -1$. We have

$$\begin{aligned}\int_0^\pi x f(\sin x)\,dx &= \int_\pi^0 (\pi - u) f(\sin(\pi - u))\,(-1)\,du\\ &= \int_0^\pi (\pi - u) f(\sin u)\,du\\ &= \pi\int_0^\pi f(\sin u)\,du - \int_0^\pi u f(\sin u)\,du\end{aligned}$$

Therefore,

$$\int_0^\pi x f(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx.$$

(ii) Let $f(x) = \dfrac{x}{2-x^2}$. It is continuous on $[0,1]$. Then

$$\begin{aligned}\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx &= \int_0^\pi x f(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx\\ &= \frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}\,dx = -\frac{\pi}{2}\int_0^\pi \frac{(\cos x)'}{1+\cos^2 x}\,dx\\ &= -\frac{\pi}{2}\tan^{-1}(\cos x)\Big|_{x=0}^{x=\pi} = \frac{\pi^2}{4}.\end{aligned}$$

6. (a) Note that $f(1) = 2$. $f'(x) = 5x^4 - 3x^2 + 2$. Then $f'(1) = 4$. So

$$(f^{-1})'(2) = \frac{1}{f'(1)} = \frac{1}{4}.$$

(b) Note that $f(1) = 2$. $f'(x) = \dfrac{3x^2 + 2x + 1}{2\sqrt{x^3 + x^2 + x + 1}}$. Then $f'(1) = \dfrac{3}{2}$. So

$$(f^{-1})'(2) = \frac{1}{f'(1)} = \frac{2}{3}.$$

7. (a) i) $\ln|y| = \dfrac{1}{3}\left[\ln|x| + \ln|x-2| - \ln(x^2+1)\right].$

ii) $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{3}\left(\dfrac{1}{x} + \dfrac{1}{x-2} - \dfrac{2x}{x^2+1}\right).$

iii) $\dfrac{dy}{dx} = \dfrac{y}{3}\left(\dfrac{1}{x} + \dfrac{1}{x-2} - \dfrac{2x}{x^2+1}\right) = \dfrac{1}{3}\sqrt[3]{\dfrac{x(x-2)}{x^2+1}}\left(\dfrac{1}{x} + \dfrac{1}{x-2} - \dfrac{2x}{x^2+1}\right).$

(b) i) $\ln|y| = \ln|x| + \ln|\sin x| - \dfrac{1}{2}\ln|\sec x|.$

ii) $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{x} + \dfrac{\cos x}{\sin x} - \dfrac{1}{2}\dfrac{\sec x \tan x}{\sec x} = \dfrac{1}{x} + \cot x - \dfrac{1}{2}\tan x.$

iii) $\dfrac{dy}{dx} = y\left(\dfrac{1}{x} + \cot x - \dfrac{1}{2}\tan x\right) = \dfrac{x\sin x}{\sqrt{\sec x}}\left(\dfrac{1}{x} + \cot x - \dfrac{1}{2}\tan x\right).$

Tutorial Part II

1. Let $f(x) = 2 + x - x^2 = (2-x)(1+x)$. Define

$$g(x) = \begin{cases} f(x), & \text{if } -1 \le x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

Then $g(x) \ge f(x)$ and $g(x) \ge 0$ for all $x \in \mathbb{R}$. Therefore,

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx \le \int_{-\infty}^{\infty} g(x)\,dx = \int_{-1}^{2} g(x)\,dx = \int_{-1}^{2} f(x)\,dx.$$

It follows that $\displaystyle\int_a^b f(x)\,dx$ is maximized when $a = -1$ and $b = 2$.

2. (i) Let $t = a - x$. Then $x = a - t$ and $\dfrac{dx}{dt} = -1$. Therefore,

$$\int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx = \int_a^0 \frac{f(a-t)}{f(a-t)+f(t)}(-1)\,dt = \int_0^a \frac{f(a-x)}{f(a-x)+f(x)}\,dx.$$

(ii) It follows from (i) that

$$\begin{aligned} 2\int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx &= \int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx + \int_0^a \frac{f(a-x)}{f(x)+f(a-x)}\,dx \\ &= \int_0^a \left[\frac{f(x)}{f(x)+f(a-x)} + \frac{f(a-x)}{f(x)+f(a-x)}\right] dx \\ &= \int_0^a 1\,dx = x\Big|_{x=0}^{x=a} = a. \end{aligned}$$

Using $f(x) = x^4$ and $a = 1$, we can compute that $\displaystyle\int_0^1 \frac{x^4}{x^4+(1-x)^4}\,dx = \frac{1}{2}$.

3. (i) Recall that $\ln x$ is increasing. So $\ln 2 > \ln 1 = 0$.

Let $M > 0$. Choose a rational number $r > \dfrac{M}{\ln 2}$. Take $c = 2^r$. Then $c > 1$ and

$$\ln c = \ln 2^r = r\ln 2 > \frac{M}{\ln 2}\cdot \ln 2 = M.$$

(ii) Let $M > 0$. Using (i), there is a number $c > 1$ such that $\ln c > M$. Note also that $\ln 1 = 0 < M$. Then by applying the Intermediate Value Theorem to $\ln x$ on $[1, c]$, there exists a number $x_0 \in (1, c)$ such that $\ln x_0 = M$.

(iii) Let $M < 0$. Then $-M > 0$. By (ii), there exists a number $x_1 > 1$ such that $\ln x_1 = -M$. Let $x_0 = 1/x_1$. Then $\ln x_0 = -\ln x_1 = -(-M) = M$.

(iv) It follows from (ii), (iii) and the fact $\ln 1 = 0$ that the range of $\ln x$ is $\mathbb{R}$.

Let $M > 0$. By (i), there exists a number $c > 1$ such that $\ln c > M$. Then

$$x > c \Rightarrow \ln x > \ln c > M.$$

By definition $\lim_{x\to\infty} \ln x = \infty$.

Let $y = 1/x$. Then $x \to 0^+$ if and only if $y \to \infty$. So

$$\lim_{x\to 0^+} \ln x = \lim_{y\to\infty} \ln(1/y) = \lim_{y\to\infty} (-\ln y) = -\infty.$$