NATIONAL UNIVERSITY OF SINGAPORE

SEMESTER 1, 2021/2022

MA2002 Calculus **Solution to Tutorial 5**

TUTORIAL PART I (PARTIAL)

1. i) Existence: $x = 0$ is a zero to $f(x) = 2x - \sin x$. So $f(x)$ has *at least one* zero.

   ii) Uniqueness: Suppose f has two zeros at $x_1 < x_2$.
   Since $f(x_1) = f(x_2) = 0$ and f is differentiable, by Rolle's Theorem, there exists a number $c \in (x_1, x_2)$ such that $f'(c) = 0$.
   On the other hand, $f'(x) = 2 - \cos x \geq 1 > 0$ for all $x \in \mathbb{R}$, a contradiction.
   So $f(x)$ has *at most one* zero in $\mathbb{R}$.

   Therefore, $f(x)$ has *exactly one* zero in $\mathbb{R}$.

2. Let $f(x) = x^4 - 4x + 1$. Then f is differentiable on $\mathbb{R}$.

   i) Existence: $f(0) = 1 > 0$ and $f(1) = -2 < 0$. Then by Intermediate Value Theorem, there exists $c_1 \in (0,1)$ such that $f(c_1) = 0$.

   $f(1) = -2 < 0$ and $f(2) = 9 > 0$. Then by Intermediate Value Theorem, there exists $c_2 \in (1,2)$ such that $f(c_2) = 0$.

   Since $c_1 < 1 < c_2$, $f(x) = 0$ has *at least two* real roots.

   ii) Uniqueness: Assume that $f(x) = 0$ has three real roots $x_1 < x_2 < x_3$. Then $f(x_1) = f(x_2) = f(x_3) = 0$. By Rolle's Theorem, there exist $d_1 \in (x_1, x_2)$ and $d_2 \in (x_2, x_3)$ such that $f'(d_1) = f'(d_2) = 0$.

   However, $f'(x) = 4x^3 - 4$. We would have $d_1 = d_2 = 1$, a contradiction. So $f(x) = 0$ has *at most two* real roots.

   Therefore, $f(x) = 0$ has *exactly two* real roots.

3. i) First of all, we show that $f(x) = 0$ has at least one real root. Note that

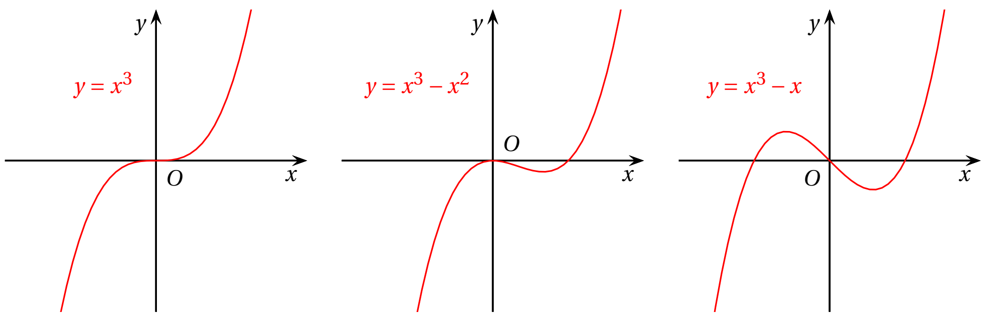
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} (x^3 + bx^2 + cx + d) = \lim_{x\to\infty} x^3 \left(1 + \frac{b}{x} + \frac{c}{x^2} + \frac{d}{x^3}\right) = \infty.$$

In particular, there exists a number $M > 0$ such that $x > M \Rightarrow f(x) > 0$. Take $x_2 = M + 1$. Then $f(x_2) > 0$.
Similarly, noting that $\lim_{x\to-\infty} f(x) = -\infty$, we have some $x_1 < 0$ such that $f(x_1) < 0$.
Since f is continuous on $[x_1, x_2]$, by Intermediate Value Theorem, there exists a number $x_0 \in (x_1, x_2)$ such that $f(x_0) = 0$.

ii) We now prove that $f(x) = 0$ has at most 3 real roots.

Suppose it has four real roots $c_1 < c_2 < c_3 < c_4$. Since f is differentiable, by Rolle's Theorem there exist numbers $d_1 \in (c_1, c_2)$, $d_2 \in (c_2, c_3)$ and $d_3 \in (c_3, c_4)$ such that $f'(d_1) = f'(d_2) = f'(d_3) = 0$.
However, $f'(x) = 3x^2 + 2bx + c = 0$ is a quadratic equation, which has at most 2 real roots. So the equation $f(x) = 0$ should have at most 3 real roots.

iii) Therefore, a cubic equation could have 1, 2 or 3 real roots. The examples are illustrated below:



4. Suppose f has two fixed points $c_1 < c_2$. Then $f(c_1) = c_1$ and $f(c_2) = c_2$.
Since f is continuous on $[c_1, c_2]$ and differentiable on (c_1, c_2), by Mean Value Theorem, there would exist a number $c \in (c_1, c_2)$ such that

$$f'(c) = \frac{f(c_2) - f(c_1)}{c_2 - c_1} = \frac{c_2 - c_1}{c_2 - c_1} = 1.$$

However, this contradicts the assumption that $f'(x) \neq 1$ for all $x \in \mathbb{R}$.
Therefore, f has *at most* one fixed point.

5. Let $y > 0$ be a constant. Then for all $x > 0$,

$$\frac{d}{dx} f(xy) = \frac{1}{xy} \cdot y = \frac{1}{x} = \frac{d}{dx} f(x).$$

Then there exists a constant C such that $f(xy) = f(x) + C$.

Let $x = 1$. Then $f(y) = f(1) + C = C$. We conclude that for all $x > 0$ and $y > 0$,

$$f(xy) = f(x) + f(y).$$

6. (a) i) It is known that $f'(x) = (x-1)(x+2)(x-3)$. Solve $f'(x) = 0$. Then $x = -2$, $x = 1$ and $x = 3$ are the critical numbers of f.

ii) $f'(x) < 0$ on $(-\infty, -2)$ and $(1, 3)$, and $f'(x) > 0$ on $(-2, 1)$ and $(3, \infty)$.
By Increasing/Decreasing Test, f is decreasing on $(-\infty, -2)$ and on $(1, 3)$; and it is increasing on $(-2, 1)$ and on $(3, \infty)$.

iii) From the table

	$(-\infty, -2)$	$(-2, 1)$	$(1, 3)$	$(3, \infty)$
f'	$-$	$+$	$-$	$+$
f	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$

we see that f has a local minimum at $x = -2$ and $x = 3$, and a local maximum at $x = 1$.

7. Assume that g has a local extreme value at c. Since g is differentiable, by Fermat's Theorem, we must have $g'(c) = 0$.

However, $g'(x) = 101x^{100} + 51x^{50} + 1 \geq 1 > 0$ for all $x \in \mathbb{R}$, which is a contradiction.

Therefore, g has neither a local maximum nor a local minimum.

8. Define $f(x) = \sqrt{1+x} - 1 - \frac{x}{2}$. Then for all $x > 0$,

$$f'(x) = \frac{1}{2\sqrt{x+1}} - \frac{1}{2} < \frac{1}{2\sqrt{0+1}} - \frac{1}{2} = 0.$$

So f is decreasing on $[0, \infty)$.

Therefore, for all $x > 0$, we have $f(x) < f(0) = \sqrt{1+0} - 1 - \frac{0}{2} = 0$. That is,

$$\sqrt{1+x} < 1 + \frac{x}{2}.$$

TUTORIAL PART II

1. Suppose that f has three zeros at $a_1 < a_2 < a_3$. Then $f(a_1) = f(a_2) = f(a_3) = 0$.
Since f'' exists on $\mathbb{R}$, both f' and f are continuous and differentiable on $\mathbb{R}$.
By Rolle's Theorem applied to f on $[a_1, a_2]$ and $[a_2, a_3]$ respectively, there exist numbers $b_1 \in (a_1, a_2)$ and $b_2 \in (a_2, a_3)$ such that $f'(b_1) = f'(b_2) = 0$.
Note that $b_1 < b_2$. By Rolle's Theorem applied to f' on $[b_1, b_2]$, there exists at least one point $c \in (b_1, b_2)$ at which $f''(c) = 0$.

2. Define $g(x) = f(x) - f(a) - (x-a)f'(a) - M(x-a)^2$, where M is the number such that $f(b) = f(a) + (b-a)f'(a) + M(b-a)^2$. Then

$$g'(x) = f'(x) - f'(a) - 2M(x-a) \quad \text{and} \quad g''(x) = f''(x) - 2M.$$

We see that $g(a) = 0$, and by the choice of M we have $g(b) = 0$. By Rolle's Theorem applied to g on $[a, b]$, there exists a number $x_0 \in (a, b)$ such that $g'(x_0) = 0$.
Note that $g'(a) = 0$. Apply Rolle's Theorem to g' on $[a, x_0]$, there exists a number $c \in (a, x_0)$ such that $g''(c) = 0$.
Therefore, $f''(c) = 2M$. It follows from $g(b) = 0$ that

$$f(b) = f(a) + (b-a)f'(a) + \frac{f''(c)}{2}(b-a)^2.$$

3. Let $x_0 = (1-\lambda)a + \lambda b$. Then $a < x_0 < b$.
Apply Mean Value Theorem to f on $[a, x_0]$, there is a number $c_1 \in (a, x_0)$ such that

$$f'(c_1) = \frac{f(x_0) - f(a)}{x_0 - a} = \frac{f(x_0) - f(a)}{\lambda(b-a)}.$$

Apply Mean Value Theorem to f on $[x_0, b]$, there is a number $c_2 \in (x_0, b)$ such that

$$f'(c_2) = \frac{f(b) - f(x_0)}{b - x_0} = \frac{f(b) - f(x_0)}{(1-\lambda)(b-a)}.$$

Recall that the graph of f is concave up. So f' is increasing.
Since $c_1 < x_0 < c_2$, we have $f'(c_1) < f'(c_2)$. Then

$$\frac{f(x_0) - f(a)}{\lambda} < \frac{f(b) - f(x_0)}{1-\lambda}.$$

Therefore,

$$(1-\lambda)f(a) + \lambda f(b) > (1-\lambda)f(x_0) + \lambda f(x_0) = f(x_0).$$