Limits

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Limits

- The limit (L) of $f(x)$ as $x \to a$ is the value that $f(x)$ approaches as x gets arbitrarily close to a

$$\lim_{x \to a} f(x) = L$$

- Recall our example of e:

$$\lim_{n \to \infty} (1 + \frac{1}{n})^n = 2.7183... = e$$

Limit example I

Sometimes finding the limit is straightforward. How would we approach this problem?

Let $f(x)$ be $x+3$. Find:

$$\lim_{x\to 3} f(x)$$

Limit example I

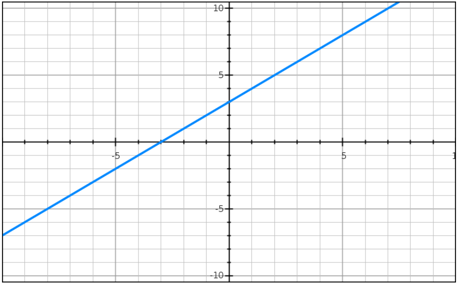
In this case, we can evaluate the function at 3

$$f(x) = (x + 3)$$

$$f(3) = (3 + 3) = 6$$

Limit example I

If we graph this function, we see that as x approaches 3 (from either direction), the value of y approaches 6. The limit of $f(x) = (x + 3)$ as x approaches 3 is 6.



Limit example II

Consider the following function:

$$f(x) = \frac{x^2 - 1}{x - 1}$$

What is $f(1)$?

$f(1)$ is undefined

If we were to graph this function, we would find a hole at $x = 1$. The function is not continuous - if we drew the graph, we would have to lift our pencil here.

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So, what is $\lim_{x \to 1} f(x)$?

What is $\lim_{x \to \infty} f(x)$?

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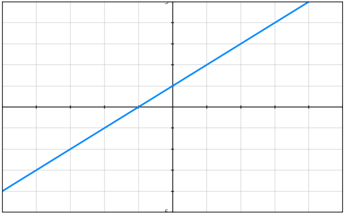
Limit example II

We can see that as x approaches 1, $f(x)$ approaches 2.
$\lim_{x\to 1} f(x) = 2$

As x approaches infinity, $f(x)$ also approaches infinity.
$\lim_{x\to\infty} f(x) = \infty$



Caution: Graphing programs like this can be misleading. We should see a hole at $(1, 2)$, because we know the function is undefined at this point.

Limit example II

We can also find $\lim_{x \to 1} f(x)$ by factoring.

$$f(x) = \frac{x^2 - 1}{x - 1}$$

Factor (in this case, just the numerator) and cancel like terms

$$\frac{(x+1)(x-1)}{(x-1)}$$

$$\frac{x+1}{1}$$

Now we can substitute in

$$\frac{1+1}{1} = 2$$

So whenever $x \neq 1$, $\lim_{x \to 1} f(x)$ is 2

Sided Limits

Limits can differ as you approach from different sides. Consider:

$$f(x) = \begin{cases} x^2, & \text{if } x < 2 \\ 2x - 6, & \text{if } x \geq 2 \end{cases}$$

This is a piecewise function. What would its graph look like?

What is $\lim_{x \to 2} f(x)$?

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(Blank frame for graphing)

Sided Limits

Our answers differ as we approach from the left and from the right

We wind up with

- a limit as x approaches 2 from the right, or from above: $\lim_{x \to 2^+} f(x)$
- a limit as x approaches 2 from the left, or from below: $\lim_{x \to 2^-} f(x)$

Properties of limits

$$\lim_{x \to a} c = c$$

$$\lim_{x \to a} x = a$$

$$\lim_{x \to a} (f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$$

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$$\lim_{x\to a}(cf(x)) = c\lim_{x\to a} f(x)$$

$$\lim_{x\to a}(f(x)^n) = \left(\lim_{x\to a} f(x)\right)^n \qquad \text{iff } \lim_{x\to a} f(x) > 0$$

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Practice

$$\lim_{x \to 5} \frac{1}{3}$$

$$\lim_{x \to 5} \frac{1}{x}$$

$$\lim_{x \to 5} \frac{(x-5)(x+3)}{x-5}$$

$$\lim_{x \to -2^-} \frac{4}{x+2}$$

$$\lim_{x \to -2^+} \frac{4}{x+2}$$