Calculus

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August 2022

Fundamentals of calculus

- Calculus is about dealing with infinitesimal values
- For our purposes, two big ideas
  - Derivatives
  - Integrals

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Fond memories

- Who enjoyed their last calculus class?
- We need calculus for formal theory and statistics
- We won't dig into the Fundamental Theorem of Calculus or proofs here; you just need to be able to apply it

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Derivative

- "Derivative" is just a fancy term for slope
- Slope is the rate of change $\frac{\delta y}{\delta x}$ or $\frac{dy}{dx}$
- Specifically, the derivative is the *instantaneous* rate of change
- We need slope for our statistics, which are all about fitting lines
- We also need slope for taking maxima and minima
- The equation for a line is $y = mx + b$ What is its slope?

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Derivatives

- What is slope of quadratic function?
- How would we find it?

(board examples of line and quadratic function)

Finding the derivative

- Slope is rise over run, which is $\dfrac{f(x+\Delta x)-f(x)}{\Delta x}$
- To see why, consider the slope of a line connecting two points

$$m=\frac{f(x_2)-f(x_1)}{x_2-x_1}$$

- We can define $x_2=x_1+\Delta x$ (or equivalently $\Delta x=x_2-x_1$)
- This gives us

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Finding the derivative

As we've seen, for a curve, we need to be infinitely close for our line's defining points, yielding

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{1}$$

This gives us this instantaneous slope (rate of change) of a function at every point on its domain. Equation (1) is the definition of the derivative.

Notation

Leibniz:

- $\frac{d}{dx}f(x)$ is "The derivative of f x with respect to x"
  - (Or, "The instantaneous rate of change in f of x with respect to x")
- If $y = f(x)$, $\frac{dy}{dx}$ is "The derivative of y with respect to x"
  - Warning: Do not try to cancel out the d's, no matter how tempting it is
- Advantage of always specifying the variable with respect to which we're differentiating (it's the one in the denominator)

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Lagrange's prime notation

- $f'(x)$ (read: "f prime x") is the derivative of $f(x)$
- This is useful when it's clear which variable we mean (e.g. when there's only one)

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Notation summary

$$\frac{dy}{dx} = \frac{d}{dx} f(x) = f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Power rule

- Power rule is important for differentiation by hand
- It tells us how to take the derivative of a polynomial
- $\dfrac{d(ax^n)}{dx} = anx^{n-1}$
- Notation reminder: $\frac{dy}{dx} = f'(x)$

Application of power rule

What is $\dfrac{d(x^2)}{dx}$?

- x^2
- $2x^{2-1}$
- $2x$

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Another application of power rule

What is $\frac{d(4x^3)}{dx}$?

- $4x^3$
- $4 * 3x^{3-1}$
- $12x^2$

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Practice power rule

Take the derivative of each of these

$$x^3$$

$$3x^2$$

$$60x^{11}$$

$$x$$

$$\frac{4}{x^2}$$

$$9\sqrt{x}$$

$$6x^{5/2}$$

$$11,596,232$$

Practice power rule

Evaluate the derivatives at x=2 and x=-1

$$x^3$$

$$3x^2$$

$$60x^{11}$$

$$x$$

$$\frac{4}{x^2}$$

$$9\sqrt{x}$$

$$6x^{5/2}$$

$$11,596,232$$

Special functions

- A few functions have particular rules
- $\frac{d(ln(x))}{dx} = \frac{1}{x}$
- $$\frac{d(log_b(x))}{dx} = \frac{1}{x * ln(b)}$$
- $\frac{d(e^x)}{dx} = e^x$
- $\frac{d(a^x)}{dx} = a^x ln(a)$
- $\frac{dy}{dx} c = 0$
- $\frac{d(x^x)}{dx} = x^x(1 + ln(x))$

Derivatives of addition and subtraction

- Easiest rule to remember
- $\frac{d(f(x) \pm g(x))}{dx} = f'(x) \pm g'(x)$

Practice

Take the derivative of each of these

$$x^2 + x + 5$$

$$x^4 - 4x^3 + 5x^2 + 8x - 6$$

$$3x^5 - 6x^2$$

$$5x^2 + 8\sqrt{x} - \frac{1}{x}$$

$$ln(x) + 5e^x - 4x^3$$

Product rule

- A little more complicated
- $\frac{d(f(x)\times g(x))}{dx} = f'(x)g(x) + g'(x)f(x)$
- Example: $2x \times 3x$

Practice

Take the derivative of each of these

Remember, $\frac{d(f(x)*g(x))}{dx} = f'(x)g(x) + g'(x)f(x)$

$$x^3 * x$$

$$e^x * x^2$$

$$ln(x) * x^{-3}$$

Quotient rule

$$\frac{d\frac{f(x)}{g(x)}}{dx} = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$$

If you're having trouble with this, just apply the product rule to

$$\frac{d[f(x) * g^{-1}(x)]}{dx}$$

Practice

Take the derivative of each of these

Remember, $\frac{d\frac{f(x)}{g(x)}}{dx} = \frac{f'(x)g(x)-g'(x)f(x)}{[g(x)]^2}$

$$\frac{x^5}{x^2}$$

$$\frac{e^x}{x^3}$$

Chain rule

$$\frac{d[f(g(x))]}{dx} = f'(g(x)) * g'(x)$$

Application

$$\frac{d(f(g(x)))}{dx} = f'(g(x)) * g'(x)$$

Let's take the derivative of a function of a function

$$\frac{d[ln(x^2)]}{dx}$$

$$f(x) = ln(x), g(x) = x^2$$

$$f'(x) = \frac{1}{x}, g'(x) = 2x$$

$$\frac{1}{x^2} * 2x = \frac{2}{x}$$

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Practice

Take the derivative of each of these
Remember, $\frac{d(f(g(x)))}{dx} = f'(g(x)) * g'(x)$

$$(3x^4 - 8)^2$$

$$e^{x^2}$$

Second derivative

- Same process as taking single derivative, except input for second derivative is output from first
- Second derivative tells us whether the slope of a function is increasing, decreasing, or staying the same at any point x on the function's domain.
- Ex: driving a car.
  - $f(x) =$ distance traveled at time x
  - $f'(x) =$ speed at time x
  - $f''(x) =$ acceleration at time x

Blank slide for graphing

Graph $f(x) = x^2$, $f'(x)$, and $f''(x)$

Application

- $\frac{d^2(x^4)}{dx^2} = f''(x^4)$
- First, we take the first derivative
- $f'(x^4) = 4x^3$
- Then we use that output to take the second derivative
- $f''(x^4) = f'(4x^3) =$
- $f''(x^4) = f'(4x^3) = 12x^2$

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Practice

Take the second derivative of the following functions

$$x^5$$

$$6x^2$$

$$4ln(x)$$

$$3x$$

$$4x^{3/2}$$