Arithmetic and Algebra

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August 2021

Properties of Arithmetic

Commutative properties

- $a + b = b + a$
- $a * b = b * a$

Associative properties

- $(a + b) + c = a + (b + c)$
- $(a * b) * c = a * (b * c)$

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Properties of Arithmetic

Distributive property

- $a(b+c) = ab + ac$
- $\dfrac{a+b}{c} = \dfrac{a}{c} + \dfrac{b}{c}$

Identity properties – there exist values that leave x unchanged

- 0 for addition and subtraction
- 1 for multiplication and division

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- 0 for addition and subtraction
- 1 for multiplication and division

Friendly Reminder

Division by zero is undefined

- There is no number which, when multiplied by zero, gives us x (assuming $x \neq 0$)
- Don't break math

Properties of Equalities – Relationships that hold with (Real) Numbers

Symmetric relationships

- $a = b \leftrightarrow b = a$

Transitive relationships

- $a = b$ and $b = c \quad \Rightarrow \quad a = c$
- $a > b$ and $b > c \quad \Rightarrow \quad a > c$

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Order of Operations – PEMDAS

- Parentheses
- Exponents
- Multiplication and Division (left to right)
- Addition and Subtraction (left to right)

$$(12 \div 3 + 4) - (4^2 - 6 * 2) = ?$$

$$(10 - 48 \div 12 * 2)^2 + 3^2 * (8 - 6) = ?$$

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Fractions

Consist of

$$\frac{\text{numerator}}{\text{denominator}}$$

or

$$\frac{\text{how many we have}}{\text{total}}$$

Fractions

Addition and subtraction require a common denominator

Add or subtract across the top (numerators) but *not* across the bottom (denominators)

- $\frac{4}{6} + \frac{1}{6} = \frac{5}{6}$
- $\frac{4}{6} - \frac{1}{6} = \frac{3}{6}$
  - Can reduce to $\frac{1}{2}$

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Fractions

- $\frac{4}{6} + \frac{1}{5} \neq \frac{5}{11}$
  - Would need to get to a common denominator $\frac{4}{6} * \frac{5}{5} = \frac{20}{30}$ and $\frac{1}{5} * \frac{6}{6} = \frac{6}{30}$, so $\frac{20}{30} + \frac{6}{30} = \frac{26}{30}$
  - Then reduce, $\frac{26}{30} = \frac{13}{15}$

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Fractions

Multiplication and division do *not* require a common denominator

Multiply across the top (numerators) and bottom (denominators)

- $\frac{4}{6} * \frac{1}{6} = \frac{4}{36}$
  - Reduces to $\frac{1}{9}$
- $\frac{4}{6} * \frac{1}{5} = \frac{4}{30}$
  - Reduces to $\frac{2}{15}$

Fractions

To divide, flip the number to the *right* of the sign, then multiply

- $\frac{4}{6} \div \frac{1}{6}$
  - $\frac{4}{6} * \frac{6}{1} = \frac{24}{6} = 4$
- $\frac{4}{6} \div \frac{1}{5}$
  - $\frac{4}{6} * \frac{5}{1} = \frac{20}{6} = \frac{10}{3} = 3\frac{1}{3}$

Negative Fractions

A negative sign in front of a fraction can be in front of the numerator, in front of the denominator, or in front of the whole fraction

$$\frac{-1}{2} = \frac{1}{-2} = -\frac{1}{2} = -0.5$$

A fraction with a negative numerator *and* a negative denominator is positive (the negatives cancel out)

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Exponents

Product rule

- When multiplying two powers with the same base, add the exponents
- $x^m * x^n = x^{m+n}$
- $4^2 * 4^3 = (4 * 4)(4 * 4 * 4) = 4^5$

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Power rule

- To raise a power to a power, multiply the exponents
- $(x^m)^n = x^{mn}$
- $(5^2)^3 = (5 * 5)^3 = (5 * 5)(5 * 5)(5 * 5) = 5^6$

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Quotient rule

- When dividing two powers with the same base, subtract the exponents
- $4^5 \div 4^2 = \dfrac{4*4*4*4*4}{4*4} = 4^3$

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Negative exponents

- Any nonzero number raised to a negative power equals its reciprocal raised to the opposite positive power
  - $4^{-2} = \frac{1}{4^2} = \frac{1}{16}$

Fractional Exponents

- $x^{\frac{1}{2}} = \sqrt{x}, \quad x^{\frac{1}{3}} = \sqrt[3]{x}, \quad x^{\frac{2}{3}} = \sqrt[3]{x^2} \ldots$
- More generally, $x^{\frac{m}{n}} = \sqrt[n]{x^m}$.
- $2^{\frac{2}{3}} = \sqrt[3]{2^2} = \sqrt[3]{4}$

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Exponents

- Rules of 1
  - Any number raised to the power of 1 is itself
    - $x^1 = x$
    - $3^1 = 3$
  - 1 raised to any power is 1
- Zero rule
  - Any nonzero number raised to the power of zero is 1

Exponents - Common Errors

- An exponent *only* applies to the number *unless* there are parentheses that indicate otherwise.
  - $-4^2 \neq (-4)^2$
- To use the product rule, two numbers *must* have the same base
  - $4^2 * 2^3 \neq 8^{2+3}$
- The product rule does *not* apply to the sum of two numbers
  - $2^2 + 2^3 \neq 2^{2+3}$

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Practice

Simplify as much as possible

$$2 - 3 \div 1$$

$$3 * -2 \div 3$$

$$2 + 4 \div 2$$

$$\frac{-1}{4} * \frac{-4}{5}$$

$$\frac{-1}{4} \div \frac{-4}{5}$$

$$9(3^2)$$

Summation

Sometimes we want to take the sum of a sequence of numbers. Rather than writing them all out, we can use summation notation.

$$a_m + a_{m+1} + a_{m+2} \ldots + a_n = \sum_{i=m}^{n} a_i$$

- i is the index of summation
- m is the lower bound of summation
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Summation

$$1 + 2 + 3 + 4 = \sum_{i=1}^{4} i = 10$$

$$\sum_{i=2}^{5} i^2 = 2^2 + 3^2 + 4^2 + 5^2 = 48$$

Summation Properties

$$\sum_{i=1}^{n} c = cn$$

$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

$$\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$$

Product Notation

To multiply a sequence of numbers, we use product notation, or pi notation, in the same way:

$$1 \times 2 \times 3 \times 4 = \prod_{i=1}^{4} i = 24$$

Factorials

Exclamation points are called "factorials" in mathematics.

$$n! = n \times (n-1) \times (n-2) \times \dots [n-(n-1)] = \prod_{i=1}^{n} i$$

$$5! = \prod_{i=1}^{5} i = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

Factorials are important in the study of probability. You'll learn much more about them in Stats 1.

Inequalities

Mathematical expression which shows that two quantities are not equal

- $>$ Greater than
- $<$ Lesser than
- $\geq$ Greater than or equal to
- $\leq$ Lesser than or equal to

The “solution” to an inequality is a value that makes the inequality true

Rules for Inequalities

You can add any positive or negative number *to both sides* (i.e. add or subtract)

- $x + 1 > 4 \quad \Rightarrow \quad x > 3$

You can multiply or divide *both sides* by any positive number

- $40x > 10 \quad \Rightarrow \quad x > \frac{10}{40} \quad (x > \frac{1}{4})$

You can multiply or divide *both sides* by a *negative* number, *but* you must flip the sign!

- $-4x > 24 \quad \Rightarrow \quad x < -6$

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Simplifying and Evaluating Expressions

Simplifying expressions

- Clear parentheses
- Combine like terms
- Combine constants

Evaluating expressions

- Substitute a specific value for each variable and then perform operations

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$$6x + -2x - 7$$

$$y^7 y^5 y^6 y^4$$

$$x + 4x(y - 3) + zx$$

$$(2a^2)(4a^4)$$

$$4ab^2 + 5ab^2 - (6a^2)^3$$

Quadratic Formula

We've done some problems where we want to solve for x but have a squared term in our equation

$$x^2 - 4x - 12 = 0$$
$$(x+2)(x-6) = 0$$

And we solve for $x = -2$ and $x = 6$ by factoring.

Quadratic Formula

Sometimes this is easy, but sometimes it's not.

$$2x^2 + 5x + 3 = 0$$

It can be particularly tricky if we have multiple variables involved.

$$12x^2 + \alpha x - 3 = 0$$

Quadratic Formula

Generally speaking, if we have an equation of the format

$$ax^2 + bx + c = 0$$

and we want to solve for x, we can use the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Quadratic Formula

In our first example from above

$$2x^2 + 5x + 3 = 0$$

$$a = 2$$

$$b = 5$$

$$c = 3$$

When plugged into the quadratic formula, this gives us

$$x = \frac{-5 \pm \sqrt{5^2 - 4(2)(3)}}{2(2)}$$

which we can then solve down to find the value(s) of x.

Quadratic Formula

Similarly in our second example from above

$$12x^2 + \alpha x - 3 = 0$$

$$a = 12$$

$$b = \alpha$$

$$c = -3$$

When plugged into the quadratic formula, this gives us

$$x = \frac{-\alpha \pm \sqrt{\alpha^2 - 4(12)(-3)}}{2(12)}$$

which we can then solve down to find the value(s) of x (in terms of α).

Results from the quadratic formula

Sometimes although we wind up with both a positive and a negative value of x, only one of them (usually the positive one) is substantively interesting for the question at hand or makes sense for the theoretical question we're addressing.

Practice

$$x^2 + x - 4 = 0$$

$$x^2 - 3x - 4 = 0$$

$$x^2 - 4 = 0$$

$$6x^2 + 11x - 35 = 0$$

$$8x^2 = 15 - 14x$$

System of Linear Equations

A collection of two or more linear equations involving the same set of variables.

$$y = 2x + 4$$

$$3x + y = 9$$

System of Linear Equations

To solve them, we're looking for the values of each variable that make all of the equations simultaneously true.

However, we can't solve for more variables than we have equations. We just don't have enough information in that case.

There are different ways to solve these, but the easiest is to use the substitution method, or by elimination of variables.

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Solving Systems of Linear Equations

From the first equation in our example, we know that $y = 2x + 4$. So we can substitute that value into the second equation anywhere we see a y.

$$3x + y = 9$$
$$3x + (2x + 4) = 9$$
$$5x + 4 = 9$$
$$5x = 5$$
$$x = 1$$

Solving Systems of Linear Equations

Now that we have a value $x = 1$, we can substitute that back into the first equation to find a value of y.

$$y = 2x + 4$$
$$y = 2(1) + 4$$
$$y = 6$$

So we've found that when $x = 1$ and $y = 6$, both of these equations are true.

Solving Systems of Linear Equations

We can also do this with more than two equations, and more than two variables, again provided we have enough equations for the number of variables.

Let's work through an example

$$x + 3y - 2z = 5$$

$$3x + 5y + 6z = 7$$

$$2x + 4y + 3z = 8$$

Solve for x, y, and z.

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Greek Alphabet

A	α	Alpha	N	ν	Nu
B	β	Beta	Ξ	ξ	Xi
Γ	γ	Gamma	O	o	Omicron
Δ	δ	Delta	Π	π	Pi
E	ϵ	Epsilon	P	ρ	Rho
Z	ζ	Zeta	Σ	σ	Sigma
H	η	Eta	T	τ	Tau
Θ	θ	Theta	Υ	υ	Upsilon
I	ι	Iota	Φ	ϕ	Phi
K	κ	Kappa	X	χ	Chi
Λ	λ	Lambda	Ψ	ψ	Psi
M	μ	Mu	Ω	ω	Omega