# Matrices

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# Introduction

# Scalars

- Let's start with something familiar, with a new word
- One number (12, for example) is referred to as a scalar
- This can be thought of as a $1x1$ matrix

# Vectors

- We can put several scalars together to make a vector
- An example is

$$\begin{bmatrix} 12 \\ 14 \\ 15 \end{bmatrix} = b \tag{1}$$

- Since this is a column of numbers, we cleverly refer to it as a column vector

## Row Vectors

- If we take $b$ and arrange it so that it it a row of numbers instead of a column, we refer to it as a row vector:

$$\begin{bmatrix} 12 & 14 & 15 \end{bmatrix} = d \tag{2}$$

# Operators

# Summation

- Recall the summation operator $\sum$, which lets us perform an operation on a sequence of numbers (often but not always a vector)

$$x = \begin{bmatrix} 12 & 7 & -2 & 0 & 1 \end{bmatrix} \tag{3}$$

- we can find

$$\sum_{i=1}^{3} x_i$$

- $12 + 7 + -2 = 17$

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## Product

- Recall the product operator $\prod$, which can also perform operations over a sequence of numbers

$$z = \begin{bmatrix} 5 & -3 & 5 & 1 \end{bmatrix} \tag{4}$$

- we can find

$$\prod_{i=1}^{4} z_i$$

- $5 * -3 * 5 * 1 = -75$

# Product

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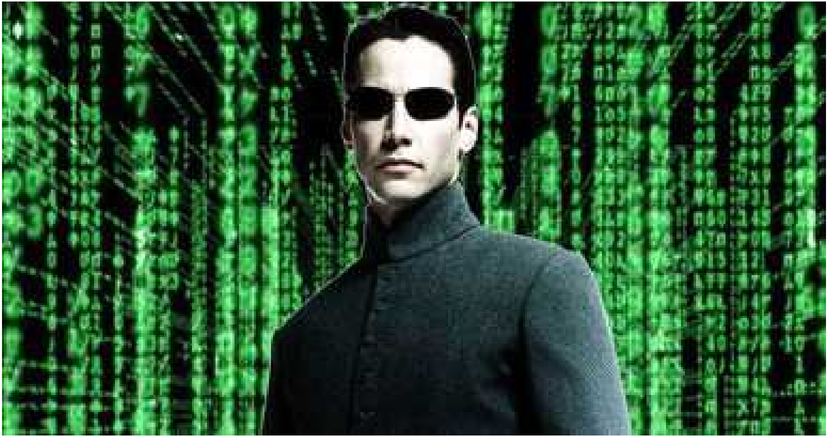
# Matrices

# Exactly.



## Matrices

We can append vectors together to form a matrix

$$\begin{bmatrix} 12 & 14 & 15 \\ 115 & 22 & 127 \\ 193 & 29 & 219 \end{bmatrix} = A \tag{5}$$

# Matrices, continued

- We refer to the dimensions of matrices by *row* x *column*
- So $A$ is a 3$x$3 matrix.
- Note that matrices are usually designated by capital letters
  - And sometimes **bolded** as well

# Product

- How do we refer to specific elements of the matrix???
- Solution: come up with a clever indexing scheme
- Matrix $A$ is an $m \times n$ matrix where $m = n = 3$
- More generally, matrix $B$ is an $m \times n$ matrix where the elements look like this:

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \dots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \dots & b_{2n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ b_{m1} & b_{m2} & b_{m3} & \dots & b_{mn} \end{bmatrix} \tag{6}$$

- Thus $b_{23}$ refers to the second unit down and third across

# Matrix operations

## Addition and subtraction

- Addition and subtraction are logical
- Requirement: Must have *exactly* the same dimensions
- To do the operation, just add or subtract each element with the corresponding element from the other matrix:

$$A \pm B = C$$

$$c_{ij} = a_{ij} \pm b_{ij} \;\forall i, j$$

# Addition and subtraction

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \pm \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \tag{7}$$

$$=$$

$$\begin{bmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & a_{13} \pm b_{13} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & a_{23} \pm b_{23} \\ a_{31} \pm b_{31} & a_{32} \pm b_{32} & a_{33} \pm b_{33} \end{bmatrix} \tag{8}$$

# Addition and subtraction practice

$$A = \begin{bmatrix} 1 & 4 & 2 \\ -2 & -1 & 0 \\ 0 & -1 & 3 \end{bmatrix} \tag{9}$$

$$B = \begin{bmatrix} 5 & 1 & 0 \\ 2 & -1 & 0 \\ 7 & 1 & 2 \end{bmatrix} \tag{10}$$

Find A+B

## Addition and subtraction practice

$$A = \begin{bmatrix} 6 & -2 & 8 & 12 \\ 4 & 42 & 8 & -6 \\ -14 & 5 & 0 & 0 \end{bmatrix} \tag{11}$$

$$B = \begin{bmatrix} 18 & 42 & 3 & 7 \\ 0 & -42 & 15 & 4 \\ -7 & 0 & 21 & -18 \end{bmatrix} \tag{12}$$

Find A-B

## Scalar multiplication

- Recall that a scalar is a single number
- Easy to do–just multiply each value by the scalar

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, cA = \begin{bmatrix} ca_{11} & ca_{12} & ca_{13} \\ ca_{21} & ca_{22} & ca_{23} \\ ca_{31} & ca_{32} & ca_{33} \end{bmatrix} \quad (13)$$

## Scalar multiplication practice

$$A = \begin{bmatrix} 1 & 4 & 2 \\ 8 & -1 & 3 \\ 0 & -2 & 3 \end{bmatrix} \tag{14}$$

$$B = \begin{bmatrix} -15 & 1 & 5 \\ 2 & -42 & 0 \\ 7 & 1 & 6 \end{bmatrix} \tag{15}$$

Find 2*A and -3*B

# Matrix multiplication

- Requirement: the two matrices must be **conformable**
- This means that the number of columns in the first matrix equals the number of rows in the second
- When multiplying **A** x **B**, if **A** is m x n, **B** must have n rows
- The resulting matrix will have the number of rows in the first, and the number of columns in the second!
    - So if **A** is i x k and **B** is k x j, then **A** x **B** will be i x j

# Practice

Which of the following can we multiply? What will be the dimensions of the resulting matrix?

$$b = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix} M = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 4 \\ 2 & 3 & 2 \end{bmatrix} L = \begin{bmatrix} 6 & 5 & -1 \\ 1 & 4 & 3 \end{bmatrix} \tag{16}$$

Note the order. Why can't we multiply in the opposite order?

# Practice

Which of the following can we multiply? What will be the dimensions of the resulting matrix?

$$b = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix} M = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 4 \\ 2 & 3 & 2 \end{bmatrix} L = \begin{bmatrix} 6 & 5 & -1 \\ 1 & 4 & 3 \end{bmatrix} \tag{16}$$

Note the order. Why can't we multiply in the opposite order?

## Matrix multiplication

- How do we actually do this?
- Multiply each row by each column, summing up each pair of multiplied terms
- The element in position ij is the sum of the products of elements in the ith row of the first matrix (**A**) and the corresponding elements in the jth column of the second matrix (**B**).

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

- Examples on board

# Properties of operators

- Addition and subtraction
  - Associative $(A \pm B) \pm C = A \pm (B \pm C)$
  - Communicative $A \pm B = B \pm A$
- Multiplication
  - $AB \neq BA$
  - $A(BC) = (AB)C$
  - $A(B + C) = AB + AC$
  - $(A + B)C = AC + BC$

# Special matrices

# Square matrices

Any $n \times n$ matrix (same number rows and columns)

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 4 \\ 2 & 3 & 2 \end{bmatrix} \tag{17}$$

# Diagonal matrices

A symmetric matrix with zeros everywhere but the main diagonal

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 7 \end{bmatrix} \tag{18}$$

## Scalar matrices

A diagonal matrix with the same number all along the diagonal

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{19}$$

## Identity matrices

A scalar matrix where the diagonal elements are 1.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{20}$$

- You might see diagonal and scalar matrices occasionally
- Identity matrices are common and quite important
- Notation is $I_n$ where $n$ is the number of rows and columns
- Note that $I_n A = A$ and also $A I_n = A$

# Transpose and Inverse

## What is a transpose?

- Switch the rows and columns
- So a $n \times m$ matrix becomes $m \times n$
- Typically denoted $L'$ or $L^T$

## Example

$$A = \begin{bmatrix} 3 & 0 & 2 & -2 \\ 1 & 2 & 1 & 4 \\ 6 & 12 & 2 & 9 \end{bmatrix} A' = \begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & 12 \\ 2 & 1 & 2 \\ -2 & 4 & 9 \end{bmatrix} \tag{21}$$

# Properties of transposition

- Matrix is always conformable for multiplication with its transpose in both directions
- $(A \pm B)' = A' \pm B'$
- $A'' = A$
- $(AB)' = B'A'$
- $(cA)' = cA'$ where $c$ is a scalar
- $AA'$ and $A'A$ will always result in a symmetric matrix
  - A square matrix is equal to its transpose, $A' = A$

# Practice

Find $A'$, $B'$, $A'A$, $AB$, and $BA$

$$A = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 0 & 1 \end{bmatrix} \tag{22}$$

# What is an inverse matrix?

- We use matrix inverses all the time
- If $A$ is an $n \times n$ square matrix:

$$AB = BA = I_n$$

- Then $B$ is said to be the inverse of $A$
  - This is usually denoted $A^{-1}$
  - So $AA^{-1} = I_n = A^{-1}A$
- If B doesn't exist, then the matrix is singular
- In OLS, computers will take inverse of matrices under the hood
  - Will lead to frustrating errors when it tells you a matrix is non-invertible

## Properties of inverse matrices

Let $A$ be $n \times n$ square matrix. If $A^{-1}$ exists:

- $A$ is full rank: $rank(A) = n$
  - Rank is the vector space spanned by rows or columns (think of as number of independent columns and rows)
- $A'$ is also invertible
- $(A^{-1})^{-1} = A$
- $(cA)^{-1} = c^{-1}A^{-1}$ for nonzero scalar $c$
- $(A')^{-1} = (A^{-1})'$

## Matrix inversion

The inverse of a 2 x 2 matrix is below. We will not demonstrate inverting higher-order matrices; computers will do that for you, since it involves recursively taking the inverse of smaller matrices.

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \tag{23}$$

# Matrix derivatives

- $\frac{\partial a'b}{\partial b} = \frac{\partial b'a}{\partial b} = a$ where a and b are Kx1 vectors.
- $\frac{\partial Ab}{\partial b} = A$ where b is n x 1, A is m x n, and A does not depend on b
- $\frac{\partial y'Ab}{\partial b} = y'A$ where y is m x 1, b is n x 1, A is m x n, and A does not depend on b
- $\frac{\partial b'Ab}{\partial b} = b'(A + A')$ where b is n x 1, A is n x n, and A does not depend on b
    - $\frac{\partial b'Ab}{\partial b} = 2Ab = 2b'A$ where A is symmetric
- $\frac{\partial b'A'y}{\partial b} = \frac{\partial b'(A'y)}{\partial b} = A'y$
- $\frac{\partial b'A'Ab}{\partial b} = \frac{\partial b'Xb}{\partial b} = 2Xb = 2A'Ab$

# Practice

# Practice

Consider the following matrices:

$$C = \begin{bmatrix} 1 & 3 \\ 5 & 6 \end{bmatrix} E = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 2 & 1 \\ 1 & 3 \end{bmatrix} F = \begin{bmatrix} 1 & 3 & 7 \\ 2 & 1 & 4 \\ 3 & 1 & 5 \end{bmatrix} G = \begin{bmatrix} 5 & 3 & 2 & 4 & 1 \\ 0 & 2 & 1 & 2 & 0 \end{bmatrix}$$

- Which pairs of matrices are conformable for multiplication?
- Perform the matrix multiplication of all the conformable pairs containing $C$.

# Practice

Consider the following matrices:

$$B = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 4 \end{bmatrix} A = \begin{bmatrix} 2 & 0 & 3 \end{bmatrix} C = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} M = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 4 \\ 2 & 3 & 2 \end{bmatrix} L = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

Find the following:

- $BA$
- $A'B'$
- $A'C$
- $100L$
- $MC$