# Functions

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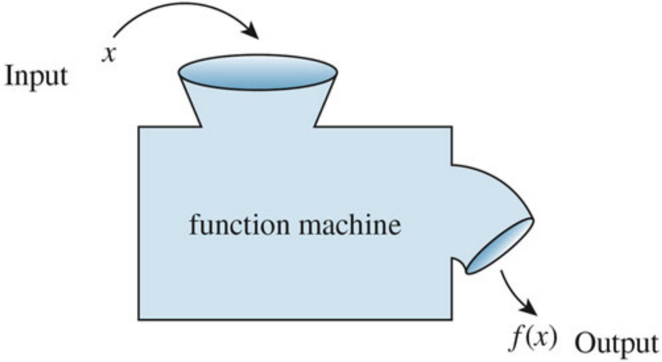
## What is a function?

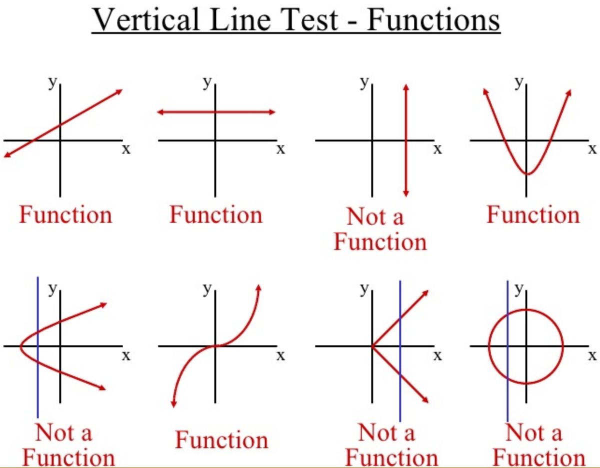
- Anything that takes input(s) and gives one defined output
- Assigns a unique value in its range ($y$ values) for each value in its domain ($x$ values)
- In math, this usually looks something like $f(x) = 3x + 4$
  - $x$ is the **argument** that the function takes.
  - For any $x$, multiply $x$ by 3 and then add 4
  - Alternative but equivalent notation: $y = 3x + 4$
  - $y$ is "a function of" $x$, so $y = f(x)$
- Can describe functions with both equations and graphs

# Function machine



# Function visualization

When graphed, can't draw vertical line through a function. Why?



## Linear functions

- We can make a function that describes a line pretty easily
- $y = mx + b$
  - $m$ is the slope (for every one unit increase in $x$, $y$ increases $m$ units)
  - $b$ is the y-intercept: the value of $y$ when $x = 0$
- More generally, $y = a + bx$
  - $a$ is the intercept and $b$ is the slope

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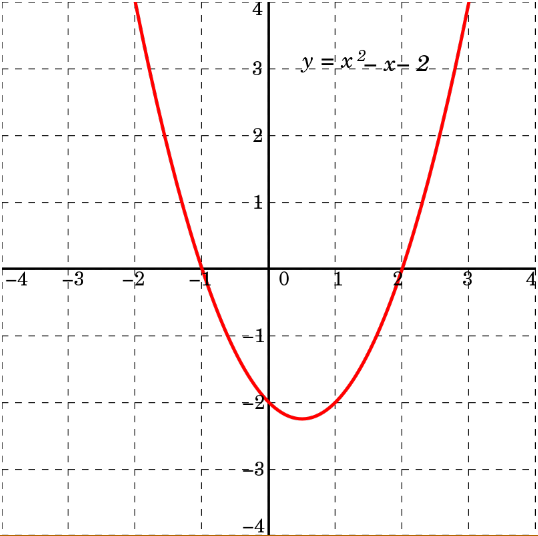
# Quadratic

- These lines have one curve
- $y = ax^2 + bx + c$
- a, b, and c don't have well-defined meanings here
- If a is negative, will open down; if a is positive, will open up
- Note that $x^2$ always returns positive values

# Quadratic



# Cubic

- These lines (generally) have two curves (inflection points)
- $y = ax^3 + bx^2 + cx + d$
- a, b, c, and d don't have well-defined meanings here

# Cubic

$$f(x) = (x^3 + 3x^2 - 6x - 8)/4$$

# Polynomial

- $y = ax^n + bx^{n-1} + ... + c$
- Will have (maximum) $n - 1$ changes in direction (turning points)
- Will have (maximum) $n$ x-intercepts
- Can be made arbitrarily precise

# Exponential

- General form: $y = ab^x$, or $f(x) = ab^x$
- Here our independent variable, or input $(x)$ is the exponent

# Trig functions

- Includes sine, cosine, and tangent
- Interesting, but not usually useful for social science
- Notable exception: the inverse hyperbolic sine transformation (read more here)

# Logarithms

- Opposite (inverse) of exponents
- Logarithms ask how many times you must raise the base to get x
- $log_a(b) = x$ is asking "a raised to what power x gives b?
- $\log_3(81) = 4$ because $3^4 = 81$
- Can be undefined
- Base cannot be 0, 1, or negative

# Logarithms

If

$$log_a x = b$$

then

$$a^{log_a x} = a^b$$

and

$$x = a^b$$

# Basic rules

$$\frac{\log_x n}{\log_x m} = \log_m n$$

$$\log_x(ab) = \log_x a + \log_x b$$

$$\log_x\left(\frac{a}{b}\right) = \log_x a - \log_x b$$

$$\log_x a^b = b \log_x a$$

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## Natural logarithms

- We most often use natural logarithms
- This means $\log_e(x)$, often written $\ln(x)$
- $e \approx 2.7183$
- ln(x) and its exponent opposite, $e^x$, have nice properties when we hit calculus

# Definition and intuition for $e$

- Imagine you invest \$1 in a bank and receive 100% interest for one year, and the bank pays you back once a year:

$$(1+1)^1 = 2$$

- When it pays you twice a year with compound interest:

$$(1+1/2)^2 = 2.25$$

- If it pays you three times a year:

$$(1+1/3)^3 = 2.37...$$

- What will happen when the bank pays you once a month? Once a day?

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$$(1 + \frac{1}{n})^n$$

However, there is limit to what you can get

$$\lim_{n\to\infty}(1 + \frac{1}{n})^n = 2.7183... = e$$

For any interest rate $k$ and number of times the bank pays you $t$:

$$\lim_{n\to\infty}(1 + \frac{k}{n})^{nt} = e^{kt}$$

$e$ is important for defining *exponential growth*. And since $ln(e^x) = x$, the natural logarithm helps us turn exponential functions into linear ones.

# Practice with logarithms

Solve, simplifying as much as you can.

- $log_{10}(1000)$
- $log_2(\frac{8}{32})$
- $10^{log_{10}(300)}$
- $ln(1)$
- $ln(e^2)$
- $ln(5e)$

# Functions of functions

- Functions can take other functions as arguments
- Means that outside function takes output of inside function as its input
- Typically written $f(g(x))$

- Say exterior function f(x)=$x^2$ and interior function g(x)=$x - 3$
- Then if we want f(g(x)), we would subtract 3 from any input, and then square the result
- We write this $(x - 3)^2$, NOT $x^2 - 3$

# PMF, PDF, and CDF

- PMF - probability mass function
  - gives the probability that a discrete random variable is exactly equal to some value
- PDF - probability density function
  - gives the probability that a continuous random variable falls within a particular range of values
- CDF - cumulative distribution function
  - gives the probability that a random variable X takes a value less than or equal to $x$

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