

INTRODUCTION TO LINEAR ELASTICITY

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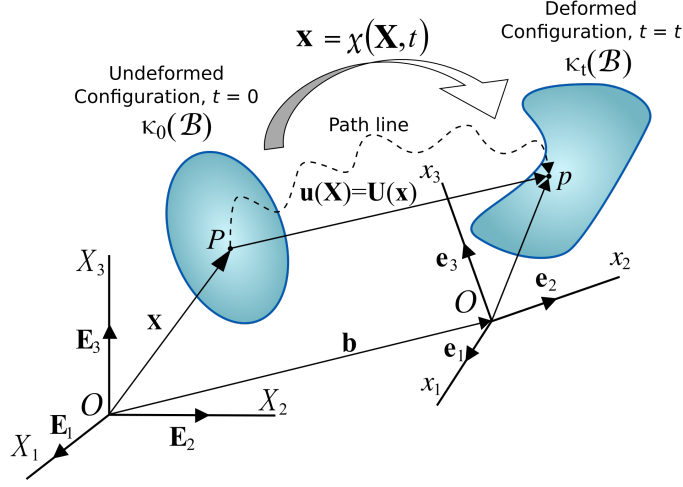
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The theory of elasticity is to study the *deformation* of elastic solid bodies under external forces. A body is *elastic* if, when the external forces are removed, the bodies return to their original (undeformed) shape.

1. INTRODUCTION

How to describe the deformation of a solid body? Let us first denote a solid body by a bounded domain $\Omega \subset \mathbb{R}^3$. Then the deformed domain can be described as the image of a vector-valued function $\Phi : \Omega \rightarrow \mathbb{R}^3$, i.e., $\Phi(\Omega)$, which is called a *configuration* or a *placement* of a body. For most problems of interest, we can require that the map is one-to-one and differentiable. Since we are interested in the deformation, i.e., the change of the domain, we let $\Phi(\mathbf{x}) = \mathbf{x} + \mathbf{u}(\mathbf{x})$ or equivalently $\mathbf{u}(\mathbf{x}) = \Phi(\mathbf{x}) - \mathbf{x}$ and call \mathbf{u} the *displacement*. Here we deal with continuum mechanics, i.e., systems have properties defined at all points in space, ignoring details in atom and molecules level.

FIGURE 1. Motion of a continuum body. *From Wikipedia.*

The displacement is not the deformation. For example, a translation or a rotation, so-called rigid body motions, of Ω will lead to a non-trivial displacement, i.e., $\mathbf{u} \neq 0$, but the shape and volume of Ω does not change at all. How to describe the shape mathematically? Using vectors. For example, a cube can be described by 3 orthogonal vectors. The change of the shape will then be described by a mapping of these vectors, which can be described by a 3×3 matrix. For example, a rigid motion has the form

$$(1) \quad \Phi(\mathbf{x}) = \mathbf{x}_1 + \mathbf{Q}(\mathbf{x} - \mathbf{x}_0),$$

where $\mathbf{x}_0, \mathbf{x}_1$ are fixed material points and \mathbf{Q} is a rotation (unitary) matrix ($\mathbf{Q}^\top \mathbf{Q} = \mathbf{I}$).

Consider a point $\mathbf{x} \in \Omega$ and a vector \mathbf{v} pointing from \mathbf{x} to $\mathbf{x} + \mathbf{v}$. Then the vector \mathbf{v} will be deformed to $\Phi(\mathbf{x} + \mathbf{v}) - \Phi(\mathbf{x}) \approx D\Phi(\mathbf{x})\mathbf{v}$ provided $\|\mathbf{v}\|$ is small. So the *deformation gradient* $\mathbf{F}(\mathbf{x}) := D\Phi(\mathbf{x})$ is a candidate of the mathematical quantity to describe the deformation. Let us compute the change of the squared length. For two points \mathbf{x} and $\mathbf{x} + \mathbf{v}$ in Ω , the squared distance in the deformed domain will be

$$\|\Phi(\mathbf{x} + \mathbf{v}) - \Phi(\mathbf{x})\|^2 \approx \|D\Phi(\mathbf{x})\mathbf{v}\|^2 = \mathbf{v}^\top (\mathbf{F}^\top \mathbf{F})(\mathbf{x}) \mathbf{v}.$$

As Φ is a differential homeomorphism, $\det \mathbf{F} \neq 0$ and $(\mathbf{F}^\top \mathbf{F})(\mathbf{x})$ is symmetric and positive definite which defines a Riemannian metric (geometry) at \mathbf{x} .

Comparing with the squared distance $\|\mathbf{x} + \mathbf{v} - \mathbf{x}\|^2 = \mathbf{v}^\top \mathbf{I} \mathbf{v}$ in the original domain, we then define a symmetric matrix function

$$(2) \quad \mathbf{E}(\mathbf{x}) = \frac{1}{2}[(\mathbf{F}^\top \mathbf{F})(\mathbf{x}) - \mathbf{I}]$$

and call it *strain*. When Φ is a rigid body motion, $\mathbf{F} = \mathbf{Q}$ with $\mathbf{Q}^\top \mathbf{Q} = \mathbf{I}$, and thus $\mathbf{E} = 0$. A nonzero strain \mathbf{E} will describe the change of the geometry from the Euclidean one.

Strain is the geometrical measure of the deformation representing the relative displacement between points in the material body, i.e., a measure of how much a given displacement differs locally from a rigid-body displacement. This is the right quantity to describe the deformation.

What is the relation between the strain \mathbf{E} and the displacement \mathbf{u} ? Substituting $D\Phi = \mathbf{I} + D\mathbf{u}$ into (2), we get

$$(3) \quad \mathbf{E} = \mathbf{E}(\mathbf{u}) = \frac{1}{2}[D\mathbf{u} + (D\mathbf{u})^\top + (D\mathbf{u})^\top D\mathbf{u}].$$

We shall mostly studied the *small deformation* case, i.e. $D\mathbf{u}$ is small. Then we can drop the quadratic part and approximately use

$$(4) \quad \mathbf{E}(\mathbf{u}) \approx \varepsilon(\mathbf{u}) = \nabla^s \mathbf{u} := \frac{1}{2}[D\mathbf{u} + (D\mathbf{u})^\top],$$

where ∇^s is called the symmetric gradient. The strain tensor \mathbf{E} defined in (3) is called the *Green-St. Venant Strain Tensor* or the *Lagrangian Strain Tensor*. Its approximation ε in (4) is called the *infinitesimal strain tensor*.

Remark 1.1. The approximation $\mathbf{E}(\mathbf{u}) \approx \varepsilon(\mathbf{u})$ is valid only for small deformation. Consider the rigid motion (1). The deformation gradient is $\mathbf{F} = \mathbf{Q}$ and thus $\mathbf{E}(\mathbf{u}) = 0$ but $\varepsilon(\mathbf{u}) = (\mathbf{Q} + \mathbf{Q}^\top)/2 - \mathbf{I} \neq 0$. Hence the infinitesimal strain tensor ε does not measure the correct strain when there are large rotations though the strain tensor \mathbf{E} can.

When considering the linear elasticity theory, we also call the kernel of $\varepsilon(\cdot)$ (infinitesimal) rigid body motion and will give its characterization later on.

How to find out the deformation? To cause a deformation, there should be forces applied to the solid body. In the static equilibrium, the deformation should be determined by the balance of forces. We now discuss possible external and internal forces.

There are two types of external forces. *Body forces* are applied forces distributed throughout the body, e.g. the gravity or a magnetic force. *Surface forces* are pressures applied to the surface or outer boundary of the body, such as pressures from applied loads.

There are also internal forces. The deformation of a body will induces resistance forces like the forces produced when we stretch the spring or compress the rubber. How to describe such internal forces? Let us conceptually split the body along a plane through the body. Then we have a new internal surface across which further internal forces are exerted to keep the two parts of the body in equilibrium. These internal forces are called *stresses*.

Mathematically we can define the stress at a point \mathbf{x} as $\lim_{A \rightarrow \{\mathbf{x}\}} \mathbf{t}_A/A$, where \mathbf{t}_A is the internal force applied to the cutting surface A which contains \mathbf{x} and shrinks to \mathbf{x} . The force \mathbf{t}_A is a vector which is not necessarily orthogonal to the surface. The normal part is called the *normal stress* σ_n and the tangential part is *shearing stress* τ_n , where the subscript n represents a normal vector of the cutting surface.

At a fixed point \mathbf{x} of a body there are infinity many different surface with centroid at \mathbf{x} . Therefore the stress can be described as a function $\mathbf{t} : \Omega \times S^2 \rightarrow \mathbb{R}^3$, where S^2 is the unit 2-sphere used to representing a direction. At given a point $\mathbf{x} \in \Omega$, the stress \mathbf{t} maps a unit vector $\mathbf{n} \in S^2$ to another vector (the force). A theorem by Cauchy simplifies the formulation of the stress function, by separating the variables, as

$$\mathbf{t}(\mathbf{x}, \mathbf{n}) = \boldsymbol{\Sigma}(\mathbf{x}) \mathbf{n},$$

where the matrix function $\boldsymbol{\Sigma}(\mathbf{x})$ is called the *stress tensor*.

Suppose \mathbf{f} is the only applied external and body-type force. We can then write out the equilibrium equations

$$(5) \quad \int_V \mathbf{f} \, d\mathbf{x} + \int_{\partial V} \boldsymbol{\Sigma} \mathbf{n} \, dS = 0 \quad \text{for all } V \subset \Omega,$$

$$(6) \quad \int_V \mathbf{f} \times \mathbf{x} \, d\mathbf{x} + \int_{\partial V} (\boldsymbol{\Sigma} \mathbf{n}) \times \mathbf{x} \, dS = 0 \quad \text{for all } V \subset \Omega.$$

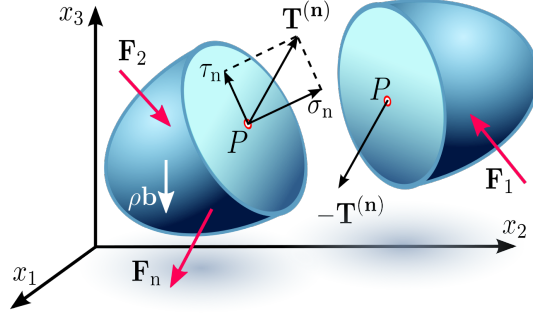


FIGURE 2. Stress vector on an internal surface S with normal vector n . The stress vector can be decomposed into two components: one component normal to the plane, called normal stress σ_n , and another component parallel to this plane, called the shearing stress τ_n . *From Wikipedia.*

Equation (5) is the balance of forces and (6) is the conservation of angular momentum.

It can be derived from (5)-(6) that the stress tensor Σ is symmetric. Otherwise the body would be subjected to unconstrained rotation. Here we give a simple but less rigorous proof in 2D and refer to Section 3.3.3 for a rigorous one. 2D vectors $(x_1, x_2)^\top$ will be extended as a 3D one $(x_1, x_2, 0)^\top$ for which the cross product can be applied and the result can be identified as a scalar. More precisely, $(a_1, a_2)^\top \times (x_1, x_2)^\top = a_1 x_2 - a_2 x_1$. Consider V is a square of size $h \ll 1$ and its center is 0, the origin of the coordinate. Consider σ as a constant function and approximate the boundary integral by one point quadrature at middle points of boundary edges. The volume integral $\int_V \mathbf{f} \times \mathbf{x} \, d\mathbf{x}$ is also approximated by one-point quadrature at the center which is zero. Straight-forward calculation will show $\sigma_{12} = \sigma_{21}$ for the first order approximation of the integral.

Using the Gauss theorem and letting $V \rightarrow \{\mathbf{x}\}$, the equation (5) for the balance of forces becomes

$$(7) \quad \mathbf{f} + \operatorname{div} \Sigma = 0.$$

The 3×3 matrix function Σ has 9 components while (5)-(6) contains only 6 equations. Or the symmetric matrix function Σ has 6 components while (7) contains only 3 equations. We need 3 more equations to determine Σ .

The internal force is a consequence of the deformation. Therefore it is reasonable to assume the stress is a function of the strain and in turn a function of the displacement, i.e.

$$(8) \quad \Sigma = \Sigma(\mathbf{E}) = \Sigma(\mathbf{u}),$$

which is called *constitutive equation*. Then by substituting (8) into (7), it is possible to solve 3 equations to get 3 unknowns (3 components of the vector function \mathbf{u}).

It is then crucial to figure out the relation $\Sigma(\mathbf{E})$ in the constitutive equation (8). First we assume that the stress tensor is intrinsic, i.e., independent of the choice of the coordinate. Second we consider the isotropic material, i.e., the stress vectors do not change if we rotate the non-deformed body. With such hypothesis, we can assume

$$\Sigma = \Sigma(\mathbf{F}^\top \mathbf{F}) = \Sigma(\mathbf{I} + 2\mathbf{E}).$$

By Rivlin-Ericksen theorem [3], we can then expand it as

$$\Sigma = \lambda \operatorname{trace}(\mathbf{E}) \mathbf{I} + 2\mu \mathbf{E} + o(\mathbf{E}),$$

where λ, μ are two positive constants, known as *Lamé constants*. For the theory of *linear elasticity*, we drop the high order term and use the approximation

$$(9) \quad \Sigma \approx \sigma = \lambda \operatorname{trace}(\varepsilon) \mathbf{I} + 2\mu \varepsilon = \lambda \operatorname{div} \mathbf{u} + 2\mu \nabla^s \mathbf{u}.$$

We now summarize the unknowns and equations for *linear elasticity* as the follows.

- $(\sigma, \varepsilon, \mathbf{u})$: (stress, strain, displacement).
 σ, ε are 3×3 symmetric matrix functions and \mathbf{u} is a 3×1 vector function.
- Kinematic equation

$$\varepsilon = \nabla^s \mathbf{u} = \frac{1}{2}(D\mathbf{u} + (D\mathbf{u})^\top).$$

- Constitutive equation

$$\sigma = \lambda \operatorname{trace}(\varepsilon) \mathbf{I} + 2\mu \varepsilon = \lambda \operatorname{div} \mathbf{u} + 2\mu \nabla^s \mathbf{u}.$$

- Balance equation

$$\mathbf{f} + \operatorname{div} \sigma = 0.$$

With appropriate boundary conditions, the kinematic equation, constitutive equation, and balance equation will uniquely determine the deformation described by $(\sigma, \varepsilon, \mathbf{u})$.

Remark 1.2. For different mechanics, the kinematic and balance equations remain the same. The difference lies in different constitutive equations, i.e., how the stress depends on the strain.

2. PHYSICAL INTERPRETATION

In this section, we explain the physical meaning of the stress and strain, and give some physical interpretation of kinematic and constitutive equations.

2.1. Strain. In 1-D, the relation $\varepsilon = u'$ implies that strain can be thought of as a relative displacement per unit distance between two points in a body. In multi-dimensions, besides the change of length, the deformation contains also the change of angles. We thus classify two types of strain:

- *normal strain* defines the amount of elongation along a direction;
- *shear strain* defines the amount of distortion or simply the change of angles.

Mathematically, given three points $\mathbf{x}, \mathbf{x} + \mathbf{v}$ and $\mathbf{x} + \mathbf{w}$ in the undeformed body, the geometry (lengths and angles) of the two vectors \mathbf{v} and \mathbf{w} is characterized by the inner product $(\mathbf{v}, \mathbf{w}) = \mathbf{w}^\top \mathbf{v}$. After the deformation, $\Phi(\mathbf{x} + \mathbf{v}) - \Phi(\mathbf{x}) \approx D\Phi(\mathbf{x})\mathbf{v} = \mathbf{F}\mathbf{v}$ and similarly $\Phi(\mathbf{x} + \mathbf{w}) - \Phi(\mathbf{x}) \approx \mathbf{F}\mathbf{w}$, the inner product is $(\mathbf{F}\mathbf{w}, \mathbf{F}\mathbf{v}) = \mathbf{v}^\top \mathbf{F}^\top \mathbf{F} \mathbf{v}$. Taking $\mathbf{w} = \mathbf{v}$, we get the change of the squared distance, especially

$$\|\delta \mathbf{e}_1\|^2 = \|\mathbf{F}\mathbf{e}_1\|^2 - \|\mathbf{e}_1\|^2 = 2(\mathbf{E}\mathbf{e}_1, \mathbf{e}_1) = 2\varepsilon_{11}.$$

So ε_{ii} represents half of the change of squared length in the i -th coordinate direction

The change in the angle is also encoded in the strain tensor $\mathbf{E} = (\mathbf{F}^\top \mathbf{F} - \mathbf{I})/2$. Choose $\mathbf{w} = \mathbf{e}_1$ and $\mathbf{v} = \mathbf{e}_2$ as the two axes vectors of the coordinate. Then $\mathbf{e}_1^\top \mathbf{E} \mathbf{e}_2 \approx \varepsilon_{12}$. On the other hand, we can calculate the change of angles as follows. The original angle between \mathbf{e}_1 and \mathbf{e}_2 is $\pi/2$. The angle of the deformed vectors is denoted by θ and the change of the angle by $\delta\theta$, i.e., $\theta = \pi/2 - \delta\theta$. Then

$$2(\mathbf{E}\mathbf{e}_1, \mathbf{e}_2) = (\mathbf{F}\mathbf{e}_1, \mathbf{F}\mathbf{e}_2) = \|\mathbf{F}\mathbf{e}_1\| \|\mathbf{F}\mathbf{e}_2\| \cos \theta = (1 + \|\delta \mathbf{e}_1\|)(1 + \|\delta \mathbf{e}_2\|) \sin \delta\theta \approx \delta\theta.$$

In the \approx , we use the linear approximation $\sin \delta\theta \approx \delta\theta$, $1 + |\delta e_i| \approx 1$, and skip the quadratic and higher order terms since the change in length and angle are small. In summary,

$$\epsilon_{12} = \delta\theta/2,$$

i.e., the shear strain ϵ_{ij} represents half of the change of angles.

2.2. Stress. Now we understand the stress. Consider the deformation of a bar-like body. On the imaginary cutting plane, the deformation will introduce two types for stress

- *normal stress* which is normal to the plane;
- *shear stress* which is tangential to the plane.

See Fig. 2.2. The normal stress is considered as positive if it produces tension, and negative if it produces compression. Normal or shear is a relative concept. For a horizontally stretched bar, there is no shear stress if the cutting plane is vertical or no normal stress if the cutting plane is horizontal.

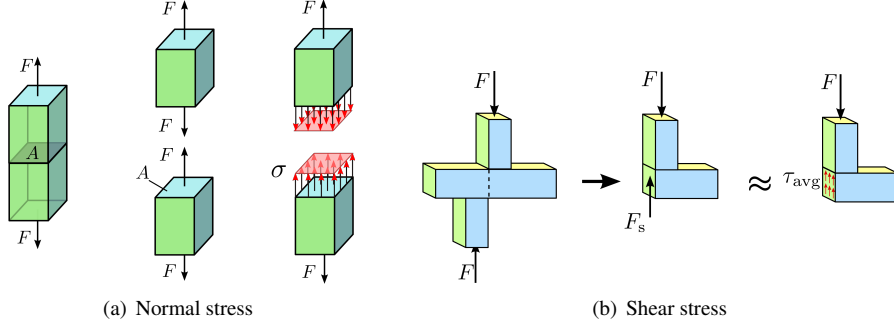


FIGURE 3. Normal and shear stress

For a symmetric and positive definite matrix, we can use eigen-vectors to form an orthonormal basis and in the new coordinate, the stress will be diagonal. The eigen-vectors of the stress tensor will be called *principal directions*, which are functions of locations. Then along the principal directions, we only see the normal stress. Similarly there are principle directions for the strain tensor. Note that strain and stress tensors do not necessarily share the same principal directions.

Both stress and strain can be represented by symmetric matrix functions. We emphasize that as intrinsic quantities of the material, they should not depend on the coordinate. In other words, these matrix representations should follow certain rules when changing the coordinate. For example, if we let $\hat{x} = Qx$, where Q is a rotation, then

$$(10) \quad \hat{\epsilon} = Q\epsilon Q^T, \quad \hat{\sigma} = Q\sigma Q^T$$

are representations in the coordinate \hat{x} .

2.3. Constitutive equation. We first derive a simple relation between the normal stress and the normal strain. Consider a spring elongated in the x -direction. By Hook's law, the stress induced by the elongation will be proportional to the strain, i.e.,

$$\sigma_{11} = \mathbb{E} \epsilon_{11} \text{ or equivalently } \epsilon_{11} = \frac{1}{\mathbb{E}} \sigma_{11}.$$

The positive parameter \mathbb{E} is called the *modulus of elasticity in tension* or *Young's modulus* which is a property of the material. Usually \mathbb{E} is large (tens for solid and hundreds for mental), which means a small strain will lead to a large stress.

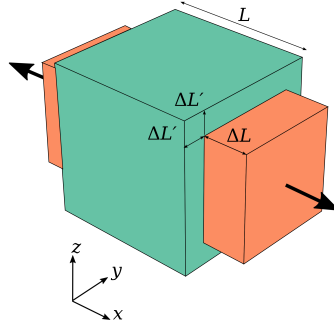


FIGURE 4. A cube with sides of length L of an isotropic linearly elastic material subject to tension along the x axis, with a Poisson's ratio of ν . The green cube is unstrained, the orange is expanded in the x direction by ΔL , and contracted in the y and z directions by $\Delta L'$. The ration $-\Delta L'/\Delta L = \nu$. From Wikipedia.

Stretching a body in one direction will usually have the effect of changing its shape in others – typically decreasing it. The strain in y - and z - directions caused by the stress σ_{11} can be described as

$$\varepsilon_{22} = \varepsilon_{33} = -\nu \varepsilon_{11} = -\frac{\nu}{\mathbb{E}} \sigma_{11},$$

where ν is a property of the material and called *Poisson's ratio*. It can be mathematically shown $\nu \in (0, 0.5)$ and usually takes values in the range $0.25 - 0.3$. See Fig. 5.

Material	Young's modulus (GPa)	Poisson ratio
rubber	0.01–0.1	0.4999
titanium	110.3	0.265 - 0.34
copper	117	0.33
steel	200	0.27 - 0.30
glass	17.2	0.18 - 0.3
concrete	30	0.1 - 0.2

FIGURE 5. Young's modulus \mathbb{E} and Poisson ratio ν

By the superposition, which holds for the linear elasticity, we have the relation

$$\begin{aligned}\varepsilon_{11} &= \frac{1}{\mathbb{E}}(\sigma_{11} - \nu\sigma_{22} - \nu\sigma_{33}), \\ \varepsilon_{22} &= \frac{1}{\mathbb{E}}(-\nu\sigma_{11} + \sigma_{22} - \nu\sigma_{33}), \\ \varepsilon_{33} &= \frac{1}{\mathbb{E}}(-\nu\sigma_{11} - \nu\sigma_{22} + \sigma_{33}).\end{aligned}$$

For the shear stress and the shear strain, we now show

$$(11) \quad \varepsilon_{ij} = \frac{1+\nu}{\mathbb{E}}\sigma_{ij} \quad \text{for } i \neq j.$$

The constant $G = \frac{\mathbb{E}}{2(1+\nu)}$ is called the *shear modulus* or the *modulus of rigidity*. Consider the strain consists of shear strain $\sigma_{12} = \sigma_{21} = 1$ only. We can find a unitary matrix Q and write as

$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = Q^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} Q =: Q^T \hat{\sigma} Q.$$

In the coordinate defined by the principal directions, the strain is diagonal $\hat{\sigma} = \text{diag}(1, -1)$. Therefore by the relation of normal strain and normal stress, we have $\hat{\varepsilon} = \text{diag}(1+\nu, -(1+\nu))/\mathbb{E}$ in the rotated coordinate. Change back to the original coordinate

$$\varepsilon = \frac{1}{\mathbb{E}} \begin{pmatrix} 0 & 1+\nu \\ 1+\nu & 0 \end{pmatrix}, \quad \text{for } \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which implies the relation (11).

In summary, we can write the relation between the strain ε and the stress σ in the matrix form

$$(12) \quad \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{23} \end{pmatrix} = \frac{1}{\mathbb{E}} \begin{pmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{pmatrix},$$

and abbreviate (12) in a simple form

$$(13) \quad \varepsilon = \mathcal{A}\sigma,$$

where \mathcal{A} is called compliance tensor of fourth order. Inverting the matrix, we have the relation

$$(14) \quad \sigma = \mathcal{C}\varepsilon$$

where

$$(15) \quad \mathcal{C} = \frac{\mathbb{E}}{(1+\nu)(1-2\nu)} \begin{pmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-2\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-2\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-2\nu \end{pmatrix}.$$

Comparing (15) with the constitutive equation

$$(16) \quad \sigma = \lambda \text{trace}(\varepsilon)\mathbf{I} + 2\mu\varepsilon,$$

we obtain the relation of the parameters

$$\mathbb{E} = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \quad \nu = \frac{\lambda}{2(\lambda + \mu)},$$

and

$$\lambda = \frac{\mathbb{E}\nu}{(1 + \nu)(1 - 2\nu)}, \quad \mu = \frac{\mathbb{E}}{2(1 + \nu)},$$

which implies $\nu \in (0, 0.5)$ and $\lim_{\nu \rightarrow 0.5} \lambda = \infty$. When ν is close to 0.5, it will cause difficulties in the numerical approximation. Solving ε from (16), we obtain

$$(17) \quad \mathcal{A}\sigma = \frac{1}{2\mu} \left(\sigma - \frac{\lambda}{2\mu + d\lambda} \text{tr}(\sigma) I \right).$$

We have derived the constitutive equation under the assumption: the material is isotropic and the deformation is linear. In general, the constitutive equation can be still described by (14) with a 4-th order tensor \mathcal{C} and $\sigma_{ij} = \mathcal{C}_{ijkl}\varepsilon_{kl}$. Mathematically the 9×9 matrix \mathcal{C} contains 81 parameters. The symmetry of ε and σ reduces the number of independent parameters to 36. For isotropic material, $\mathcal{C} = \mathcal{C}'$, where \mathcal{C}' is the matrix in another coordinate obtained by rotation. By choosing special rotations, one can show that \mathcal{C} depends only on two parameters: the pair (\mathbb{E}, ν) or the Lamé constants (λ, μ) .

3. TENSORS

One difficulty of the elasticity theory roots in the complication of tensor operations. What is a tensor? A second order tensor can be represented by a matrix. But is more than one matrix. Mathematically tensor is a multi-linear mapping while a matrix is just a representation of a tensor in a specific coordinate. The stress and strain tensors are equivalent class of matrix functions. The equivalence is defined by rotation of the coordinates.

3.1. Change of coordinates. Recall that $\Phi : \Omega_R \rightarrow \Phi(\Omega_R)$ is the the deformation mapping. Here we use Ω_R to indicate it is the reference domain. The body force \mathbf{f} is a vector function composed by 3-forms. Therefore

$$\int_V \mathbf{f}(\mathbf{x}) d\mathbf{x} = \int_{V_R} \mathbf{f}_R(\mathbf{x}_R) d\mathbf{x}_R,$$

together with $d\mathbf{x} = \det(D\Phi) d\mathbf{x}_R$, its transformation satisfies

$$\mathbf{f}(\mathbf{x}) = \det(D\Phi)^{-1} \mathbf{f}_R(\mathbf{x}_R).$$

We choose a Cartesian coordinate to describe the deformation. If we change the coordinate, the description will be different. The deformation itself is, however, intrinsic, i.e., independent of the choice of coordinates. Thus different description in different coordinates should be related by some rules. Mathematically speaking, the stress and strain tensors are equivalent class of matrix functions.

Two different Cartesian coordinates can be obtained from each other by a rigid motion. Since the origin of the Cartesian coordinate systems has no bearing on the definition of σ , we assume that the new coordinate system is obtained by rotating the old one about its origin, i.e., $\hat{\mathbf{x}} = \mathbf{Q}\mathbf{x}$, where \mathbf{Q} is unitary.

Recall that σ is a linear mapping of vectors. In the original coordinate, the mapping can be written as $\mathbf{t} = \sigma\mathbf{n}$. Applying \mathbf{Q} both sides, we get

$$\hat{\mathbf{t}} = \mathbf{Q}\mathbf{t} = \mathbf{Q}\sigma\mathbf{Q}^{-1}\mathbf{Q}\mathbf{n} = \mathbf{Q}\sigma\mathbf{Q}^T\hat{\mathbf{n}}.$$

Namely

$$(18) \quad \hat{\sigma} = Q\sigma Q^\top.$$

We can thus think the stress is a linear mapping between linear spaces (of vectors). The stress matrix is one representation of this mapping in a specific coordinate. Different coordinates lead to different representations but the linear mapping remains the same.

For a given linear operator T , the eigenvector v and corresponding eigenvalue λ is defined as $Tv = \lambda v$. By the definition of eigenvalue, it depends only on the linear structure not the representation. Therefore the eigenvalues of σ , and their combination, e.g. $\text{tr}(\sigma) = \lambda_1 + \lambda_2 + \lambda_3$ and $\det(\sigma) = \lambda_1 \lambda_2 \lambda_3$, are invariant in the change of coordinates. Therefore $\text{tr}(\varepsilon) = \text{div } \mathbf{u}$ is invariant in the change of coordinates.

The geometry (length and angle) is also intrinsic. Let $\hat{x} = Qx$. Then

$$\hat{x}^\top \hat{\varepsilon} \hat{y} = x^\top Q^\top \hat{\varepsilon} Q y = x^\top \varepsilon y.$$

Namely

$$(19) \quad \hat{\varepsilon} = Q\varepsilon Q^\top.$$

3.2. Structure of the space of matrices. Let \mathbb{M} be the linear space of $d \times d$ matrices. The symmetric subspace is denoted by \mathbb{S} and the anti-symmetric one by \mathbb{K} .

3.2.1. Matrix-vector and matrix-matrix products. For a matrix A , we write as stack of row vectors \mathbf{a}_i or column vectors \mathbf{a}^j , $i, j = 1, \dots, d$ and present an example for $d = 3$ below:

$$A = \begin{pmatrix} - & \mathbf{a}_1 & - \\ - & \mathbf{a}_2 & - \\ - & \mathbf{a}_3 & - \end{pmatrix} = \begin{pmatrix} | & | & | \\ \mathbf{a}^1 & \mathbf{a}^2 & \mathbf{a}^3 \\ | & | & | \end{pmatrix}.$$

We introduce a Frobenius inner product in \mathbb{M} as

$$(A, B)_F = A : B = A \circ B := \sum_{ij} a_{ij} b_{ij} = \sum_i \mathbf{a}_i \cdot \mathbf{b}_i = \sum_j \mathbf{a}^j \cdot \mathbf{b}^j,$$

which induces the Frobenius norm $\|\cdot\|_F$ of a matrix. Here \circ is the Hadamard product (the entry-wise product) and in MATLAB it is $\cdot *$ operation. We generalize to the sum of the cross product of column vectors

$$A(\cdot \times) B = \sum_i \mathbf{a}^i \times \mathbf{b}^i.$$

The matrix-vector product $A\mathbf{b}$ can be interpret as the linear combination of column vectors $\sum_i b_i \mathbf{a}^i$ or inner product of \mathbf{b} with the row vectors \mathbf{a}_i of A . Define

$$A \cdot \mathbf{b} := A\mathbf{b} = \sum_i b_i \mathbf{a}^i = \begin{pmatrix} \mathbf{a}_1 \cdot \mathbf{b} \\ \mathbf{a}_2 \cdot \mathbf{b} \\ \mathbf{a}_3 \cdot \mathbf{b} \end{pmatrix},$$

and

$$A \times \mathbf{b} := \begin{pmatrix} \mathbf{a}_1 \times \mathbf{b} \\ \mathbf{a}_2 \times \mathbf{b} \\ \mathbf{a}_3 \times \mathbf{b} \end{pmatrix}.$$

When the vector is on the right of the matrix

$$\text{row-wise} \quad A \cdot \mathbf{b}, \quad A \times \mathbf{b},$$

the operation is defined row-wise. Define the dot product and the cross product from the left

$$\text{column-wise} \quad \mathbf{b} \cdot \mathbf{A}, \quad \mathbf{b} \times \mathbf{A},$$

which is applied column-wise to the matrix \mathbf{A} . Namely

$$\mathbf{b} \cdot \mathbf{A} = \mathbf{b}^\top \mathbf{A} = (\mathbf{b} \cdot \mathbf{a}^1 \quad \mathbf{b} \cdot \mathbf{a}^2 \quad \mathbf{b} \cdot \mathbf{a}^3)$$

and

$$\mathbf{b} \times \mathbf{A} = \begin{pmatrix} \left| \begin{smallmatrix} \mathbf{b} \times \mathbf{a}^1 & \mathbf{b} \times \mathbf{a}^2 & \mathbf{b} \times \mathbf{a}^3 \end{smallmatrix} \right| \\ \left| \begin{smallmatrix} \mathbf{b} \times \mathbf{a}^1 & \mathbf{b} \times \mathbf{a}^2 & \mathbf{b} \times \mathbf{a}^3 \end{smallmatrix} \right| \\ \left| \begin{smallmatrix} \mathbf{b} \times \mathbf{a}^1 & \mathbf{b} \times \mathbf{a}^2 & \mathbf{b} \times \mathbf{a}^3 \end{smallmatrix} \right| \end{pmatrix}.$$

Here for cleaner notation, when the vector \mathbf{b} is on the right, it is treated as a row-vector \mathbf{b}^\top while when on the left, it is a column vector.

The ordering of performing the row and column products does not matter which leads to the associative rule of the triple products

$$\mathbf{b} \times \mathbf{A} \times \mathbf{c} := (\mathbf{b} \times \mathbf{A}) \times \mathbf{c} = \mathbf{b} \times (\mathbf{A} \times \mathbf{c}).$$

Similar rules hold for $\mathbf{b} \cdot \mathbf{A} \cdot \mathbf{c}$ and $\mathbf{b} \cdot \mathbf{A} \times \mathbf{c}$ and thus parentheses can be safely skipped. Another benefit is the transpose of products. For the transpose of product of two objects, we take transpose of each one, switch their order, and add a negative sign if it is the cross product. For example, $(\mathbf{b} \times \mathbf{A})^\top = -\mathbf{A}^\top \times \mathbf{b}$.

For two column vectors \mathbf{u}, \mathbf{v} , the tensor product $\mathbf{u} \otimes \mathbf{v} := \mathbf{u}\mathbf{v}^\top$ is a matrix which is also known as the dyadic product $\mathbf{u}\mathbf{v} := \mathbf{u}\mathbf{v}^\top$ with cleaner notation (one $^\top$ is skipped). The row-wise product and column-wise product of $\mathbf{u}\mathbf{v}$ with another vector \mathbf{x} will be applied to the neighboring vector:

$$\begin{aligned} \mathbf{x} \cdot (\mathbf{u}\mathbf{v}) &= (\mathbf{x} \cdot \mathbf{u})\mathbf{v}^\top, & (\mathbf{u}\mathbf{v}) \cdot \mathbf{x} &= \mathbf{u}(\mathbf{v} \cdot \mathbf{x}), \\ \mathbf{x} \times (\mathbf{u}\mathbf{v}) &= (\mathbf{x} \times \mathbf{u})\mathbf{v}, & (\mathbf{u}\mathbf{v}) \times \mathbf{x} &= \mathbf{u}(\mathbf{v} \times \mathbf{x}). \end{aligned}$$

3.2.2. Trace. For a $d \times d$ matrix \mathbf{A} , the trace is defined as the sum of diagonal terms

$$\text{tr}(\mathbf{A}) = \sum_{i=1}^d a_{ii} = \sum_{i=1}^d \lambda_i(\mathbf{A}).$$

The last equality says the trace is equal to the sum of eigenvalues counted with multiplicities which can be easily proved using the characteristic polynomial of \mathbf{A} .

Therefore if two matrices are similar, i.e., if $\mathbf{A} = \mathbf{C}^{-1}\mathbf{B}\mathbf{C}$, then $\text{tr}(\mathbf{A}) = \text{tr}(\mathbf{B})$. As \mathbf{AB} and \mathbf{BA} share the same spectrum (different with zero eigenvalues if they are in different size), we have

$$\text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{BA}).$$

Based on that, we conclude the trace is invariant to the cyclic permutation of product of matrices. For *symmetric* matrices, any permutation is invariant.

The trace operator is not exchangeable to the matrix product but to the tensor product. Namely

$$\text{tr}(\mathbf{AB}) \neq \text{tr}(\mathbf{A})\text{tr}(\mathbf{B}), \quad \text{but} \quad \text{tr}(\mathbf{A} \otimes \mathbf{B}) = \text{tr}(\mathbf{A})\text{tr}(\mathbf{B}).$$

Using the trace operator, the Frobenius inner product in \mathbb{M} is

$$(\mathbf{A}, \mathbf{B})_F = \mathbf{A} : \mathbf{B} = \text{tr}(\mathbf{B}^\top \mathbf{A}) = \text{tr}(\mathbf{A}^\top \mathbf{B}).$$

3.2.3. *Decomposition into traceless tensor and scalar.* Consider the short exact sequence

$$(20) \quad 0 \rightarrow \mathbb{Z} \hookrightarrow \mathbb{M} \xrightarrow{\text{tr}} \mathbb{R}$$

Here \mathbb{Z} is the subspace of traceless (or tracefree) matrices. The sequence (20) is exact as $\text{tr } \mathbb{Z} = 0$. Define a scaled right inverse of tr as follows:

$$\pi : \mathbb{R} \rightarrow \mathbb{S}, \quad \pi(p) = p \mathbf{I}_d.$$

Note that $d^{-1}\pi$ is the right inverse of the trace operator $\text{tr} : \mathbb{M} \rightarrow \mathbb{R}$. Namely $d^{-1}\text{tr} \circ \pi = \text{id}$. Then we have the Helmholtz decomposition from (20)

$$(21) \quad \mathbb{M} = \mathbb{Z} \oplus^\perp \pi(\mathbb{R}),$$

where \oplus^\perp means the decomposition is orthogonal in the $(\cdot, \cdot)_F$ inner product.

Switch the order of composition of tr and π , we get the projection to $\pi(\mathbb{R})$ in the F -product. Define $T_R : \mathbb{M} \rightarrow \pi(\mathbb{R})$ as the composition $T_R = d^{-1}\pi \circ \text{tr}$. It is straightforward to verify that

$$(22) \quad (T_R \sigma, p \mathbf{I}_d)_F = (\sigma, p \mathbf{I}_d)_F, \quad \forall p \in \mathbb{R}.$$

The orthogonal complement $(I - T_R)\sigma \in \mathbb{Z}$ is called the deviation and denoted by σ^D . We summarize this orthogonal decomposition w.r.t. $(\cdot, \cdot)_F$ as

$$\sigma = (I - T_R)\sigma + T_R \sigma = \sigma^D + d^{-1}\text{tr}(\sigma) \mathbf{I}_d.$$

The volumetric stress tensor $T_R \sigma$ tends to change the volume of the stressed body; and the stress deviator tensor σ^D tends to distort it.

3.2.4. *Skew-symmetric matrixess and the cross product.* When $d = 3$, we can define an isomorphism of \mathbb{R}^3 and the space \mathbb{K} of anti-symmetric matrices. Define $[\cdot]_\times : \mathbb{R}^3 \rightarrow \mathbb{K}$ as

$$[\omega]_\times = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}, \quad \text{for any } \omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \in \mathbb{R}^3.$$

Its inverse is denoted by $\text{vskw} : \mathbb{K} \rightarrow \mathbb{R}^3$ satisfying $[\text{vskw}(Z)]_\times = Z$ and $\text{vskw}([\omega]_\times) = \omega$. The map $[\cdot]_\times$ is sometimes denoted by mskw . We prefer $[\cdot]_\times$ to emphasize the relation to the cross product of vectors. For two vectors in \mathbb{R}^3 , one can easily verify the identity

$$u \times v = [u]_\times v,$$

where the right hand side is the regular matrix-vector product. One actually has

$$[u \times v]_\times = [u]_\times [v]_\times - [v]_\times [u]_\times,$$

i.e., the commutator of skew-symmetric 3×3 matrices can be identified with the cross-product of vectors. Therefore $[\cdot]_\times$ also preserves the Lie algebra structure of \mathbb{R}^3 and \mathbb{K} .

When $d = 2$, $\dim \mathbb{K} = 1$. Define $[\cdot]_\times : \mathbb{R} \rightarrow \mathbb{K}$ as

$$[\omega]_\times = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}.$$

We can also embed $\mathbb{R} \rightarrow \mathbb{R}^3$ by $(0, 0, \omega)^\top$ first, then apply $[\cdot]_\times$, and truncate the resulting matrix to 2×2 matrix by deleting the zero row and zero columns.

3.2.5. *Decomposition into symmetric and anti-symmetric matrices.* The decomposition

$$\mathbb{M} = \mathbb{S} \oplus \mathbb{K}$$

is an orthogonal decomposition in $(\cdot, \cdot)_F$ inner product. For any matrix $\mathbf{B} \in \mathbb{M}$, the decomposition can be written as

$$\mathbf{B} = \text{sym}(\mathbf{B}) + \text{skw}(\mathbf{B}) := \frac{1}{2}(\mathbf{B} + \mathbf{B}^\top) + \frac{1}{2}(\mathbf{B} - \mathbf{B}^\top).$$

The skew-symmetric part can be written as:

$$(23) \quad \text{skw}(\mathbf{B}) = \frac{1}{2}[\mathbf{I}(\cdot \times) \mathbf{B}]_\times.$$

For comparison, the trace is obtained by the inner product with the identity matrix

$$\text{tr}(\mathbf{B}) = \mathbf{I} : \mathbf{B}.$$

3.3. Formulae involving differential operators. We consider the function spaces $C^1(\Omega)$, $C^1(\Omega) \otimes \mathbb{R}^d$, and $C^1(\Omega) \otimes \mathbb{M}$, and discuss the relation of the differential operator and the matrix operations. We mainly discuss the three dimensional case $d = 3$. When $d = 2$, we treat $(u_1(x_1, x_2), u_2(x_1, x_2))^\top$ as $(u_1(x_1, x_2), u_2(x_1, x_2), 0)^\top$ as a function defined in \mathbb{R}^3 . We treat the Hamilton operator $\nabla = (\partial_1, \partial_2, \partial_3)^\top$ as a column vector.

3.3.1. *Gradient and symmetric gradient.* For a scalar function $v \in C^1(\Omega)$, $\nabla v = \begin{pmatrix} \partial_1 v \\ \partial_2 v \\ \partial_3 v \end{pmatrix}$

is a column vector. Introduce $Dv = (\nabla v)^\top = (\partial_1 v, \partial_2 v, \partial_3 v)$ as the row vector representation of the gradient. For a vector function $\mathbf{u} = (u_1, u_2, u_3)^\top$, $\text{curl } \mathbf{u} = \nabla \times \mathbf{u}$, and $\text{div } \mathbf{u} = \nabla \cdot \mathbf{u}$ are standard differential operations.

The gradient of a column vector \mathbf{u} is a matrix

$$D\mathbf{u} = (\partial_j u_i) = \begin{pmatrix} Du_1 \\ Du_2 \\ Du_3 \end{pmatrix} = \mathbf{u} \nabla = (\nabla \mathbf{u}^\top)^\top.$$

Recall the dyadic product $\mathbf{u} \mathbf{v} := \mathbf{u} \mathbf{v}^\top$. The symmetric gradient of a vector function \mathbf{u} is defined as

$$\nabla^s \mathbf{u} = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top) = \frac{1}{2}(\mathbf{u} \nabla + \nabla \mathbf{u}).$$

In the last identity, the dyadic product is used to emphasize the symmetry in notation. In the context of elasticity, $\nabla^s \mathbf{u}$ is commonly denoted by $\varepsilon(\mathbf{u})$ or $\text{def}(\mathbf{u})$.

Using the cross product, we have the following identity, cf. (23)

$$(24) \quad \text{skw}(D\mathbf{u}) = \frac{1}{2}[\nabla \times \mathbf{u}]_\times.$$

Consequently we can write the decomposition for the matrix $D\mathbf{u}$ as

$$(25) \quad D\mathbf{u} = \nabla^s \mathbf{u} + \frac{1}{2}[\nabla \times \mathbf{u}]_\times.$$

In 2D, we treat $(u_1(x_1, x_2), u_2(x_1, x_2))^\top$ as $(u_1(x_1, x_2), u_2(x_1, x_2), 0)^\top$. Then $\nabla \times \mathbf{u}$ becomes

$$\text{rot } \mathbf{u} := \partial_1 u_2 - \partial_2 u_1$$

and the decomposition (25) becomes

$$(26) \quad D\mathbf{u} = \nabla^s \mathbf{u} + \frac{1}{2} \text{rot } \mathbf{u} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

As $\ker(\text{grad}) = \mathbf{c}$, $\ker(D) = \mathbf{c} \in \mathbb{R}^3$. We have the following relation between $\ker(\nabla^s)$ and anti-symmetric space \mathbb{K}

$$(27) \quad \ker(\nabla^s) = \{\mathbf{Z}\mathbf{x} + \mathbf{c}, \mathbf{Z} \in \mathbb{K}, \mathbf{c} \in \mathbb{R}^3\} = \{\boldsymbol{\omega} \times \mathbf{x} + \mathbf{c}, \boldsymbol{\omega}, \mathbf{c} \in \mathbb{R}^3\}.$$

A basis of $\ker(\nabla^s)$ can be easily obtained by a basis of \mathbb{R}^3 . For $d = 2$, $\mathbb{K} \cong \mathbb{R}$ and the cross product $\boldsymbol{\omega} \times \mathbf{x}$ is understood as $\boldsymbol{\omega} \mathbf{e}_1 \times (\mathbf{x}; 0) = \boldsymbol{\omega}(-y, x)^\top$, i.e., a rotation of the vector by 90° counter clockwise.

The differential operator div, curl applied to a vector function \mathbf{u} can be written as operators for the matrix function $D\mathbf{u}$

$$\begin{aligned} \text{div } \mathbf{u} &= \nabla \cdot \mathbf{u} = (\mathbf{I} \cdot \nabla) \cdot \mathbf{u} = \text{tr}(D\mathbf{u}) = \text{tr}(\nabla^s \mathbf{u}), \\ \text{curl } \mathbf{u} &= \nabla \times \mathbf{u} = (\mathbf{I} \times \nabla) \mathbf{u} = [\nabla]_\times \mathbf{u} = \text{vskw}(\text{skw}(D\mathbf{u})). \end{aligned}$$

The differential operators ∇^s, grad , and div are related by the identity:

$$\text{tr } \nabla^s(\mathbf{u}) = \text{div } \mathbf{u}, \quad \text{div } \pi p = \text{grad } p.$$

which can be verified by direct computation and summarized in the following diagram

$$\begin{array}{ccc} & \mathbf{H}^1(\Omega, \mathbb{R}^d) & \\ \swarrow \nabla^s & \downarrow \text{div} & \searrow \text{div} \\ L^2(\Omega, \mathbb{S}) & \xrightarrow{\text{tr}} L^2(\Omega) & \mathbf{H}(\text{div}, \Omega, \mathbb{S}) \xleftarrow{\pi} H^1(\Omega) \\ & & \uparrow \text{grad} \end{array}$$

FIGURE 6. Relation of differential operators and trace operator

The following identity

$$(28) \quad 2 \text{div } \nabla^s \mathbf{u} = \Delta \mathbf{u} + \text{grad } \text{div } \mathbf{u}$$

can be proved as follows

$$(\text{div } 2\nabla^s \mathbf{u})_k = \sum_{i=1}^d \partial_i (\partial_i u_k + \partial_k u_i) = \Delta u_k + \partial_k (\text{div } \mathbf{u}).$$

3.3.2. Differentiation of matrix functions. Applying these matrix-vector operations to the Hamilton operator ∇ , we get column-wise differentiations $\nabla \cdot \mathbf{A}, \nabla \times \mathbf{A}$, and row-wise differentiations $\mathbf{A} \cdot \nabla, \mathbf{A} \times \nabla$. Conventionally, the differentiation is applied to the function after the ∇ symbol. So a more conventional notation is

$$\mathbf{A} \cdot \nabla := (\nabla \cdot \mathbf{A}^\top)^\top, \quad \mathbf{A} \times \nabla := -(\nabla \times \mathbf{A}^\top)^\top.$$

By moving the differential operator to the right, the notation is simplified and the transpose rule for matrix-vector products can be formally used. Again the right most column vector is treated as a row vector to make the notation cleaner.

In the literature, differential operators for matrices are usually applied row-wise to tensors. To distinguish with ∇ notation, we define operators in letters as

$$\begin{aligned} \text{grad } \mathbf{u} &:= \mathbf{u} \nabla^\top = (\partial_j u_i) = (\nabla \mathbf{u})^\top, \\ \text{curl } \mathbf{A} &:= -\mathbf{A} \times \nabla = (\nabla \times \mathbf{A}^\top)^\top, \\ \text{div } \mathbf{A} &:= \mathbf{A} \cdot \nabla = (\nabla \cdot \mathbf{A}^\top)^\top. \end{aligned}$$

Note that the transpose operator T is involved for tensors and in particular $\text{grad } \mathbf{u} \neq \nabla \mathbf{u}$, $\text{curl } \mathbf{A} \neq \nabla \times \mathbf{A}$, $\text{curl } \mathbf{A} \neq \mathbf{A} \times \nabla$ and $\text{div } \mathbf{A} \neq \nabla \cdot \mathbf{A}$. For symmetric tensors, $\text{div } \mathbf{A} = (\nabla \cdot \mathbf{A})^\mathsf{T}$, $\text{curl } \mathbf{A} = (\nabla \times \mathbf{A})^\mathsf{T}$.

3.3.3. *Integration by parts.* Assume $L(\cdot)$ is linear. In abstract form, we will have

$$\int_{\partial V} L(\mathbf{n}, \cdot) dS = \int_V L(\nabla, \cdot) d\mathbf{x}.$$

Namely we simply replace the unit outwards normal \mathbf{n} by the Hamilton operator. Its component form is

$$\int_{\partial V} L(n_i, \cdot) dS = \int_V L(\partial_i, \cdot) d\mathbf{x},$$

which is more applicable when mix product of vectors involved.

As a simple example, we verify

$$\int_{\partial V} \boldsymbol{\sigma} \mathbf{n} dS = \int_V \boldsymbol{\sigma} \nabla d\mathbf{x} = \int_V \text{div } \boldsymbol{\sigma} d\mathbf{x},$$

where recall that div operator is applied row-wisely.

As a non-trivial simple, let us derive the symmetry of $\boldsymbol{\sigma}$ from the balance equations

$$(29) \quad \int_V \mathbf{f} d\mathbf{x} + \int_{\partial V} \boldsymbol{\sigma} \mathbf{n} dS = 0 \quad \text{for all } V \subset \Omega,$$

$$(30) \quad \int_V \mathbf{f} \times \mathbf{x} d\mathbf{x} + \int_{\partial V} (\boldsymbol{\sigma} \mathbf{n}) \times \mathbf{x} dS = 0 \quad \text{for all } V \subset \Omega.$$

Using the integral by parts and let $V \rightarrow \{\mathbf{x}\}$, we get from (29) that $\text{div } \boldsymbol{\sigma} + \mathbf{f} = 0$. We write

$$(\boldsymbol{\sigma} \mathbf{n}) \times \mathbf{x} = \left(\sum_i n_i \boldsymbol{\sigma}^i \right) \times \mathbf{x} = \sum_i n_i (\boldsymbol{\sigma}^i \times \mathbf{x}).$$

Integration by parts implies

$$\int_{\partial V} \sum_i n_i (\boldsymbol{\sigma}^i \times \mathbf{x}) dS = \int_V \sum_i \partial_i (\boldsymbol{\sigma}^i \times \mathbf{x}) d\mathbf{x}.$$

By the product rule of differentiation,

$$\partial_i (\boldsymbol{\sigma}^i \times \mathbf{x}) = \partial_i \boldsymbol{\sigma}^i \times \mathbf{x} + \boldsymbol{\sigma}^i \times \partial_i \mathbf{x}.$$

One can easily verify (formally write $\sum_i \partial_i \boldsymbol{\sigma}^i = \boldsymbol{\sigma} \nabla = \text{div } \boldsymbol{\sigma}$)

$$\sum_i \partial_i \boldsymbol{\sigma}^i \times \mathbf{x} = (\text{div } \boldsymbol{\sigma}) \times \mathbf{x}$$

which cancel out with $\mathbf{f} \times \mathbf{x}$. For the second term, note that $\partial_i \mathbf{x} = \mathbf{e}_i$ and

$$\left[\sum_i \boldsymbol{\sigma}^i \times \mathbf{e}_i \right]_{\times} = -[\mathbf{I}(\cdot \times) \boldsymbol{\sigma}]_{\times} = -2 \text{skw}(\boldsymbol{\sigma}).$$

Therefore $\sum_i \boldsymbol{\sigma}^i \times \mathbf{e}_i = 0$ implies $\text{skw}(\boldsymbol{\sigma}) = 0$, i.e., $\boldsymbol{\sigma}$ is symmetric.

As an exercise, the reader is encouraged to prove that: for a symmetric matrix function $\boldsymbol{\sigma}$ and vector function \mathbf{v}

$$(31) \quad - \int_{\Omega} \text{div } \boldsymbol{\sigma} \cdot \mathbf{v} d\mathbf{x} = \int_{\Omega} \boldsymbol{\sigma} : \nabla^s \mathbf{v} d\mathbf{x} - \int_{\partial \Omega} (\boldsymbol{\sigma} \mathbf{n}) \cdot \mathbf{v} dS.$$

4. VARIATIONAL FORM: DISPLACEMENT FORMULATION

We define the energy

$$\mathcal{E}(\mathbf{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}) = \int_{\Omega} \left(\frac{1}{2} \boldsymbol{\varepsilon} : \boldsymbol{\sigma} - \mathbf{f} \cdot \mathbf{u} \right) dx + \int_{\Gamma_N} \mathbf{t}_N \cdot \mathbf{u} \, dx.$$

and consider the optimization problem

$$\inf_{(\mathbf{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma})} \mathcal{E}(\mathbf{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma})$$

subject to the relation

$$\begin{aligned} \boldsymbol{\varepsilon} &= \nabla^s \mathbf{u}, \text{ and} \\ \boldsymbol{\sigma} &= \mathcal{C} \boldsymbol{\varepsilon} = \lambda \operatorname{tr}(\boldsymbol{\varepsilon}) \mathbf{I} + 2\mu \boldsymbol{\varepsilon}, \end{aligned}$$

and boundary condition

$$\mathbf{u}|_{\Gamma_D} = \mathbf{g}, \quad \boldsymbol{\sigma} \mathbf{n}|_{\Gamma_N} = \mathbf{t}_N.$$

Here Γ_N is an open subset of the boundary $\partial\Omega$ and the complement is denoted by Γ_D , i.e., $\Gamma_D \cup \Gamma_N = \partial\Omega$ and Γ_D is closed.

4.1. Displacement Formulation. We eliminate $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ to get the optimization problem

$$(32) \quad \inf_{\mathbf{u} \in \mathbf{H}_{g,D}^1} \mathcal{E}(\mathbf{u}),$$

where $\mathbf{H}_{g,D}^1 = \{\mathbf{v} \in \mathbf{H}^1(\Omega) : \mathbf{v}(x) = \mathbf{g} \text{ for } x \in \Gamma_D\}$, and

$$\begin{aligned} \mathcal{E}(\mathbf{u}) &= \int_{\Omega} \left(\frac{1}{2} \nabla^s \mathbf{u} : \mathcal{C} \nabla^s \mathbf{u} - \mathbf{f} \cdot \mathbf{u} \right) dx + \int_{\Gamma_N} \mathbf{t}_N \cdot \mathbf{u} \, dx \\ &= \int_{\Omega} \left(\frac{\lambda}{2} (\operatorname{div} \mathbf{u})^2 + \mu \nabla^s \mathbf{u} : \nabla^s \mathbf{u} - \mathbf{f} \cdot \mathbf{u} \right) dx + \int_{\Gamma_N} \mathbf{t}_N \cdot \mathbf{u} \, dx. \end{aligned}$$

The strong formulation is

$$(33) \quad \begin{aligned} -2\mu \operatorname{div} \nabla^s \mathbf{u} - \lambda \operatorname{grad} \operatorname{div} \mathbf{u} &= \mathbf{f} \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{g} \quad \text{on } \Gamma_D, \quad \boldsymbol{\sigma}(\mathbf{u}) \cdot \mathbf{n} = \mathbf{t}_N \quad \text{on } \Gamma_N. \end{aligned}$$

The weak formulation is: find $\mathbf{u} \in \mathbf{H}_{g,D}^1(\Omega)$ such that

$$(34) \quad (\mathcal{C} \nabla^s \mathbf{u}, \nabla^s \mathbf{v}) = (\mathbf{f}, \mathbf{v}) + \langle \mathbf{t}_N, \mathbf{v} \rangle_{\Gamma_N} \quad \text{for all } \mathbf{v} \in \mathbf{H}_{0,D}^1(\Omega).$$

4.2. Korn inequalities. To establish the well posedness of (34), it is crucial to have the following Korn's inequality

$$(35) \quad \|\mathbf{D}\mathbf{u}\| \leq C \|\nabla^s \mathbf{u}\|, \quad \forall \mathbf{u} \in \mathbf{H}_{0,D}^1(\Omega).$$

As

$$\|\mathbf{D}\mathbf{u}\|^2 = \|\operatorname{sym}(\mathbf{D}\mathbf{u})\|^2 + \|\operatorname{skw}(\mathbf{D}\mathbf{u})\|^2 = \|\nabla^s \mathbf{u}\|^2 + 2\|\nabla \times \mathbf{u}\|^2,$$

the Korn inequality is non-trivial in the sense that the norm $\|\nabla \times \mathbf{u}\|$ should be controlled by $\|\nabla^s \mathbf{u}\|$. Also (35) cannot be true without any side condition for \mathbf{u} as if $\mathbf{u} \in \ker(\nabla^s)$ but $\mathbf{u} \notin \ker(\mathbf{D})$, then the right hand side is zero while the left is not. In (35), we requires the Lebesgue measure $|\Gamma_D| \neq 0$ so that $\mathbf{H}_{0,D}^1(\Omega) \cap \ker(\nabla^s) = \emptyset$.

We first consider the easy case $\Gamma_D = \partial\Omega$. For $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$, we can freely using integration by parts and skip the boundary term. Therefore multiply (28) by \mathbf{u} and integration by parts to get the identity

$$2\|\nabla^s \mathbf{u}\|^2 = \|\mathbf{D}\mathbf{u}\|^2 - \|\operatorname{div} \mathbf{u}\|^2,$$

which leads to the first Korn inequality.

Lemma 4.1 (First Korn inequality).

$$(36) \quad \|D\mathbf{u}\| \leq \sqrt{2}\|\nabla^s \mathbf{u}\|, \quad \mathbf{u} \in \mathbf{H}_0^1(\Omega).$$

To prove the general case, for Lipschitz domains, we need the following norm equivalence

$$(37) \quad \|\mathbf{v}\|^2 \approx \|\nabla \mathbf{v}\|_{-1}^2 + \|\mathbf{v}\|_{-1}^2 \quad \text{for all } \mathbf{v} \in \mathbf{L}^2(\Omega).$$

Lemma 4.2 (Lion's lemma). *Let $X(\Omega) = \{v \mid v \in H^{-1}(\Omega), \text{grad } v \in (H^{-1}(\Omega))^n\}$ endowed with the norm $\|v\|_X^2 = \|v\|_{-1}^2 + \|\text{grad } v\|_{-1}^2$. Then for Lipschitz domains, $X(\Omega) = L^2(\Omega)$.*

Proof. A proof $\|v\|_X \lesssim \|v\|$, consequently $L^2(\Omega) \subseteq X(\Omega)$, is trivial (using the definition of the dual norm). The non-trivial part is to prove the inequality

$$(38) \quad \|v\|^2 \lesssim \|v\|_{-1}^2 + \|\text{grad } v\|_{-1}^2 = \|v\|_{-1}^2 + \sum_{i=1}^d \|\partial_i v\|_{-1}^2.$$

The difficulty is associated to the non-computable dual norm $\|\cdot\|_{-1}$. We only present a special case $\Omega = \mathbb{R}^n$ by the characterization of H^{-1} norm using Fourier transform. Let $\hat{u}(\xi) = \mathcal{F}(u)$ be the Fourier transform of u . Then

$$\|u\|_{\mathbb{R}^n}^2 = \|\hat{u}\|_{\mathbb{R}^n}^2 = \left\| 1/(\sqrt{1+|\xi|^2})\hat{u} \right\|_{\mathbb{R}^n}^2 + \sum_{i=1}^d \left\| \xi_i/(\sqrt{1+|\xi|^2})\hat{u} \right\|_{\mathbb{R}^n}^2 = \|u\|_X^2.$$

In general, careful extension from $H^{-1}(\Omega)$ to $H^{-1}(\mathbb{R}^d)$ is needed. \square

The following identity for C^2 function can be easily verified by definition of symmetric gradient

$$(39) \quad \partial_{ij}^2 u_k = \partial_j \varepsilon_{ki}(\mathbf{u}) + \partial_i \varepsilon_{jk}(\mathbf{u}) - \partial_k \varepsilon_{ij}(\mathbf{u}).$$

We now use Lemma 4.2 and identity (39) to prove the following Korn's inequality.

Theorem 4.3 (Korn's inequality). *There exists a constant depending only on the geometry of domain Ω s. t.*

$$(40) \quad \|D\mathbf{u}\| \leq C(\|\nabla^s \mathbf{u}\| + \|\mathbf{u}\|), \quad \forall \mathbf{u} \in \mathbf{H}^1(\Omega).$$

Proof. By the norm equivalence and using the identity to switch derivatives (39)

$$\|\partial_i \mathbf{u}\| \lesssim \|\partial_i \mathbf{u}\|_{-1} + \|\nabla \partial_i \mathbf{u}\|_{-1} \lesssim \|\mathbf{u}\| + \sum_j \|\partial_j \nabla^s \mathbf{u}\|_{-1} \lesssim \|\mathbf{u}\| + \|\nabla^s \mathbf{u}\|.$$

\square

We are ready to prove the coercivity of the displacement formulation (34).

Lemma 4.4 (Korn's inequality for non-trivial zero trace). *Assume $|\Gamma_D| \neq 0$. There exists a constant depending only on the geometry of domain Ω s. t.*

$$(41) \quad \|D\mathbf{u}\| \leq C\|\nabla^s \mathbf{u}\|, \quad \forall \mathbf{u} \in \mathbf{H}_{0,D}^1(\Omega).$$

Proof. Assume no such constant exists. Then we can find a sequence $\{u_k\} \subset \mathbf{H}^1(\Omega)$ s.t.

$$\|Du_k\| = 1, \quad \text{and} \quad \|\nabla^s u_k\| \rightarrow 0 \text{ as } k \rightarrow \infty.$$

As $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, we can find a $L^2(\Omega)$ convergent sub-sequence still denoted by $\{u_k\}$ and $u_k \rightarrow u$ in $L^2(\Omega)$. By Korn's inequality (40) $\{u_k\}$ is also a Cauchy sequence in $\mathbf{H}^1(\Omega)$. Therefore $u_k \rightarrow u$ in $\mathbf{H}^1(\Omega)$, and $\|\nabla^s u\| = \lim_{k \rightarrow \infty} \|\nabla^s u_k\| = 0$.

We conclude there exists a $u \in \ker(\nabla^s)$, i.e. $u = \omega \times x + c$. The condition $u|_{\Gamma_D} = 0$ implies $u = 0$. Contradicts with the condition $\|Du\| = 1$. \square

For the completeness, we discuss the second Korn's inequality below. We need to impose conditions to eliminate $\ker(\nabla^s)$. Consider a column vector functional L stacked by $l_i(\cdot)$ for $i = 1, 2, \dots, 6$. We require $\ker(L) \cap \ker(\nabla^s) = \{0\}$: if $u \in \ker(\nabla^s)$, $l_i(u) = 0$ for $i = 1, 2, \dots, 6$, then $u = 0$. Then by modifying the proof of Lemma 4.4, we have the following version of Korn's inequality [4].

Lemma 4.5 (Korn's inequality with functionals). *Assume $\ker(L) \cap \ker(\nabla^s) = \{0\}$. There exists a constant depending only on the geometry of domain Ω s. t.*

$$(42) \quad \|Du\| \leq C\|\nabla^s u\| + \sum_{i=1}^6 |l_i(u)|, \quad u \in \mathbf{H}^1(\Omega).$$

Proof. From the proof of Lemma 4.4, We conclude there exists a $u \in \ker(\nabla^s)$ and $l_i(u) = 0$ for $i = 1, 2, \dots, 6$, and $\|Du\| = 1$. Then $u = 0$ contradicts with fact $\|Du\| = 1$. \square

For special functionals, we can have more direct proofs. Recall that

$$\ker(\nabla^s) = \{\omega \times x + c, \omega, c \in \mathbb{R}^3\}.$$

The constant vector $c \in \ker(\nabla^s)$ can be eliminated by asking $\int_{\Omega} u = 0$. When $u = \omega \times x = [\omega]_{\times} x$, we have $Du = [\omega]_{\times}$ and

$$[\nabla \times u]_{\times} = \text{skw}(Du) = [\omega]_{\times}.$$

Consequently $\omega = \nabla \times u$ if $u = \omega \times x \in \ker(\nabla^s)$ which can be eliminated by the condition $\int_{\Omega} \omega = 0$. We thus introduce the subspace

$$(43) \quad \widehat{\mathbf{H}}^1(\Omega) := \{v \in \mathbf{H}^1(\Omega) : \int_{\Omega} v \, dx = \int_{\Omega} \nabla \times v \, dx = 0\}.$$

Lemma 4.6 (The second Korn's inequality). *There exists a constant depending only on the geometry of domain Ω s. t.*

$$(44) \quad \|Du\| \leq C\|\nabla^s u\|, \quad \forall u \in \widehat{\mathbf{H}}^1(\Omega).$$

Proof. We requires the following result. For a vector function $q \in L_0^2(\Omega)$, there exists a symmetric matrix function $\Phi \in \mathbf{H}_0^1(\Omega; \mathbb{S})$ s.t. $\text{div } \Phi = q$ and $\|\Phi\|_1 \leq C\|q\|$. This is known for Stokes equation and can be generalized to symmetric tensors. As $u \in \widehat{\mathbf{H}}^1(\Omega)$, $q \in L_0^2(\Omega)$. We find such Φ for $q = \nabla \times u$.

We have the following diagram

$$u \xrightarrow{D} Du \oplus^\perp \text{curl } \Phi \xleftarrow{\text{curl}} \Phi \xrightarrow{\text{div}} \nabla \times u.$$

The orthogonality

$$(Du, \text{curl } \Phi) = (\text{curl } Du, \Phi) = 0$$

can be proved using integration by parts. All differential operators are applied row-wisely and no boundary terms as $\Phi \in \mathbf{H}_0^1$. There is one more orthogonality

$$(45) \quad (D\mathbf{u} - \text{curl } \Phi, 2\text{skw}(D\mathbf{u})) = (D\mathbf{u} - \text{curl } \Phi, [\nabla \times \mathbf{u}]_\times) = 0,$$

which can be also proved using integration by parts. Recall that $2\text{skw}(D\mathbf{u}) = D\mathbf{u} - (D\mathbf{u})^\top$ and $(\text{curl } \Phi, D\mathbf{u}) = 0$. So using row-wise and column-wise operations

$$\begin{aligned} (\text{curl } \Phi, 2\text{skw}(D\mathbf{u})) &= -(\Phi \times \nabla, \nabla \mathbf{u}^\top) = (\Phi, \nabla \mathbf{u}^\top \times \nabla) = -(\nabla \cdot \Phi, \mathbf{u}^\top \times \nabla) \\ &= (\text{div } \Phi, \nabla \times \mathbf{u}) = (\nabla \times \mathbf{u}, \nabla \times \mathbf{u}) = \frac{1}{2}([\nabla \times \mathbf{u}]_\times, [\nabla \times \mathbf{u}]_\times) = (D\mathbf{u}, [\nabla \times \mathbf{u}]_\times). \end{aligned}$$

Therefore

$$(D\mathbf{u}, D\mathbf{u}) = (D\mathbf{u}, D\mathbf{u} - \text{curl } \Phi) = (\nabla^s \mathbf{u}, D\mathbf{u} - \text{curl } \Phi).$$

By Cauchy Schwarz inequality

$$\|D\mathbf{u}\|^2 = (\nabla^s \mathbf{u}, D\mathbf{u} - \text{curl } \Phi) \lesssim \|\nabla^s \mathbf{u}\|(\|D\mathbf{u}\| + \|\Phi\|_1) \lesssim \|\nabla^s \mathbf{u}\| \|D\mathbf{u}\|.$$

Canceling one $\|D\mathbf{u}\|$, we obtained the desired Korn inequality. \square

The above proof is a 3D version of the 2D proof in [1, Ch 11]. Tensor calculation in 3D is much more complicated but simplified using our notation system.

4.3. Pure traction boundary condition. Similar to the pure Neumann boundary of the Poisson equation, when $\Gamma_D = \emptyset$ and $\Gamma_N = \partial\Omega$, the differential operator contains non-trivial kernel $\ker(\nabla^s)$. Taking $\mathbf{v} \in \ker(\nabla^s) \cap \mathbf{H}^1$ in (34), we obtain the compatibility condition for the force \mathbf{f} and boundary force \mathbf{t}_N

$$(46) \quad \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x} + \int_{\partial\Omega} \mathbf{t}_N \cdot \mathbf{v} \, dS = 0 \quad \forall \mathbf{v} \in \ker(\nabla^s).$$

The kernel $\ker(\nabla^s)$ can be determined by a dual space of $\ker(\nabla^s)$. Especially when the compatibility condition (46) is satisfied, we can find a unique solution in the space $\widehat{\mathbf{H}}^1(\Omega)$ defined by (43).

4.4. Robustness. When $\lambda \gg 1$ and $\mu \sim 1$, the operator $-\lambda \text{grad div}$ will dominate in the displacement formulation (33) which makes the whole operator is singularly perturbed

$$-\text{grad div} - \epsilon \text{div } \nabla^s, \quad \epsilon = 2\mu/\lambda \ll 1,$$

and the limiting case $\epsilon = 0$ is degenerate as $\text{img}(\text{curl}) \subseteq \ker(\text{div})$. It will cause trouble in both finite element discretization and multigrid solvers.

When $\lambda \gg 1$, the material is called nearly incompressible. The quantity $\text{div } \mathbf{u}$ measures the incompressibility of the material. When $\lambda \gg 1$, $\text{div } \mathbf{u}$ should be also small. Indeed when the domain of interest is smooth or convex, we have the uniform regularity result

$$(47) \quad \|\mathbf{u}\|_2 + \lambda \|\text{div } \mathbf{u}\|_1 \leq C \|\mathbf{f}\|.$$

5. VARIATIONAL FORM: STRESS-DISPLACEMENT FORMULATION

We will use Sobolev spaces and its variants

$$\mathbf{H}(\text{div}, \Omega; \mathbb{S}) := \{\boldsymbol{\tau} \in L^2(\Omega; \mathbb{S}) : \text{div } \boldsymbol{\tau} \in L^2(\Omega)\},$$

endowed with norm

$$\|\boldsymbol{\tau}\|_{\text{div}} = (\|\boldsymbol{\tau}\|^2 + \|\text{div } \boldsymbol{\tau}\|^2)^{1/2},$$

to present a variational form of linear elasticity which is known as Hellinger and Reissner formulation. In applications, when we are mostly interested in computing the stresses, we shall use this mixed formulation.

5.1. Hellinger and Reissner formulation. We keep the displacement and stress $(\mathbf{u}, \boldsymbol{\sigma})$ and eliminate the strain $\boldsymbol{\varepsilon}$ to get the saddle-point problem

$$\inf_{\boldsymbol{\sigma}} \sup_{\mathbf{u}} \mathcal{E}(\mathbf{u}, \boldsymbol{\sigma}) = \inf_{(\mathbf{u}, \boldsymbol{\sigma})} \int_{\Omega} \left(\frac{1}{2} \mathcal{A} \boldsymbol{\sigma} : \boldsymbol{\sigma} - \mathbf{f} \cdot \mathbf{u} \right) dx + \int_{\Gamma_N} \mathbf{t}_N \cdot \mathbf{u} dS,$$

subject to

$$\boldsymbol{\sigma} = \mathcal{C} \nabla^s \mathbf{u} \text{ in } \Omega \quad \boldsymbol{\sigma} \mathbf{n} = \mathbf{t}_N \text{ on } \Gamma_N.$$

Or equivalently

$$\inf_{\boldsymbol{\sigma} \in \mathbf{H}(\operatorname{div}, \Omega, \mathbb{S})} \frac{1}{2} (\mathcal{A} \boldsymbol{\sigma}, \boldsymbol{\sigma}),$$

subject to the constraint

$$-\operatorname{div} \boldsymbol{\sigma} = \mathbf{f}, \text{ in } \Omega, \quad \boldsymbol{\sigma} \mathbf{n} = \mathbf{t}_N, \text{ on } \Gamma_N.$$

The displacement can be mathematically treat as the Lagrange multiplier to impose the constraint $-\operatorname{div} \boldsymbol{\sigma} = \mathbf{f}$.

The strong formulation is

$$\begin{aligned} -\operatorname{div} \boldsymbol{\sigma} &= \mathbf{f}, \quad \boldsymbol{\sigma} = \mathcal{C} \nabla^s \mathbf{u} \text{ in } \Omega, \\ \mathbf{u} &= 0 \text{ on } \Gamma_D, \quad \boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{t}_N \text{ on } \Gamma_N. \end{aligned}$$

Define $\mathbf{H}_{t,N}(\operatorname{div}, \Omega, \mathbb{S}) = \{\boldsymbol{\tau} \in \mathbf{H}(\operatorname{div}, \Omega, \mathbb{S}), \boldsymbol{\tau} \mathbf{n} = \mathbf{t}_N \text{ on } \Gamma_N\}$. The weak formulation is: find $\boldsymbol{\sigma} \in \mathbf{H}_{t,N}(\operatorname{div}, \Omega, \mathbb{S})$ and $u \in L^2(\Omega)$ such that

$$\begin{aligned} (\mathcal{A} \boldsymbol{\sigma}, \boldsymbol{\tau}) + (u, \operatorname{div} \boldsymbol{\tau}) &= 0 \quad \forall \boldsymbol{\tau} \in \mathbf{H}_{0,N}(\operatorname{div}, \Omega, \mathbb{S}), \\ (\operatorname{div} \boldsymbol{\sigma}, v) &= -(\mathbf{f}, v) \quad \forall v \in L^2(\Omega). \end{aligned}$$

Note that the Dirichlet and Neumann boundary conditions are exchanged in the mixed formulation. When $\Gamma_N = \emptyset$, i.e., $\mathbf{u} = 0$ on $\partial\Omega$, the mixed formulation is in the pure Neumann type and solutions are not uniquely. We need to consider the quotient space

$$\widehat{\mathbf{H}}(\operatorname{div}, \Omega, \mathbb{S}) = \{\boldsymbol{\tau} \in \mathbf{H}(\operatorname{div}, \Omega, \mathbb{S}) : \int_{\Omega} \operatorname{tr}(\boldsymbol{\tau}) dx = 0\}.$$

The constraint comes from, for $\boldsymbol{\tau} = \nabla^s(\mathbf{v})$, $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$,

$$\int_{\Omega} \operatorname{tr}(\boldsymbol{\tau}) dx = \int_{\Omega} \operatorname{div} \mathbf{v} dx = \int_{\partial\Omega} \mathbf{v} \cdot \mathbf{n} dS = 0.$$

5.2. inf-sup condition. Denote the space for stress by $\boldsymbol{\Sigma}$ equipped with norm $\|\cdot\|_{\boldsymbol{\Sigma}}$ and the space for displacement by \mathbf{V} with norm $\|\cdot\|_{\mathbf{V}}$. Let us introduce the linear operator $\mathcal{L} : \boldsymbol{\Sigma} \times \mathbf{V} \rightarrow (\boldsymbol{\Sigma} \times \mathbf{V})^*$

$$\langle \mathcal{L}(\boldsymbol{\sigma}, \mathbf{u}), (\boldsymbol{\tau}, \mathbf{v}) \rangle := (\mathcal{A} \boldsymbol{\sigma}, \boldsymbol{\tau}) - (\nabla^s \mathbf{u}, \boldsymbol{\tau}) + (\operatorname{div} \boldsymbol{\sigma}, \mathbf{v}).$$

Define bilinear forms

$$\begin{aligned} a(\boldsymbol{\sigma}, \boldsymbol{\tau}) &:= (\mathcal{A} \boldsymbol{\sigma}, \boldsymbol{\tau}) \\ b(\boldsymbol{\tau}, \mathbf{v}) &:= -(\operatorname{div} \boldsymbol{\tau}, \mathbf{v}). \end{aligned}$$

It is well known that \mathcal{L} is an isomorphism from $\boldsymbol{\Sigma} \times \mathbf{V}$ onto $(\boldsymbol{\Sigma} \times \mathbf{V})^*$ if and only if the following so-called Brezzi conditions hold:

- (1) Continuity of bilinear forms $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$: there exist constants $c_a, c_b > 0$ such that for all $\sigma, \tau \in \Sigma, v \in V$

$$a(\sigma, \tau) \leq c_a \|\sigma\|_{\Sigma} \|\tau\|_{\Sigma}, \quad b(\tau, v) \leq c_b \|\tau\|_{\Sigma} \|v\|_V.$$

- (2) Coercivity of $a(\cdot, \cdot)$ in the kernel space: there exists a constant $\alpha > 0$ such that

$$a(\sigma, \sigma) \geq \alpha \|\sigma\|_{\Sigma}^2 \quad \text{for all } \sigma \in \ker(B),$$

where $\ker(B) = \{\tau \in \Sigma : b(\tau, v) = 0 \text{ for all } v \in V\}$.

- (3) Inf-sup condition of $b(\cdot, \cdot)$: there exists a constant $\beta > 0$ such that

$$\inf_{v \in V, v \neq 0} \sup_{\tau \in \Sigma, \tau \neq 0} \frac{b(\tau, v)}{\|\tau\|_{\Sigma} \|v\|_V} \geq \beta.$$

The continuity condition can be easily verified by choosing appropriate norms. As point out in [2], there is a competition between the coercivity of $a(\cdot, \cdot)$ and the inf-sup condition for $b(\cdot, \cdot)$.

The coercivity of $a(\cdot, \cdot)$ in L^2 -norm is subtle. The L^2 -inner product for two tensor functions involves two inner product structures: one is $(\cdot, \cdot)_F$ among matrices and another is the L^2 -inner product of functions, i.e., $\int_{\Omega} fg \, dx$. We first explore an orthogonal decomposition in the $(\cdot, \cdot)_F$ inner product. The compliance tensor \mathcal{A} is symmetric and positive definite w.r.t. $(\cdot, \cdot)_F$ but the minimal eigenvalue is in the order of $\mathcal{O}(1/\lambda)$.

Lemma 5.1. *Let $T_R \sigma = d^{-1} \operatorname{tr}(\sigma) \mathbf{I}_d$ and $\sigma^D = (I - T_R)\sigma$. Then*

$$\mathcal{A}\sigma : \tau = \frac{1}{2\mu} \sigma^D : \tau^D + \frac{1}{d\lambda + 2\mu} T_R(\sigma) : T_R(\tau),$$

Proof. Using the formulae of \mathcal{A} , c.f. (17), and let $\rho = 2\mu/(d\lambda + 2\mu) \in (0, 1)$, we have

$$2\mu \mathcal{A}\sigma = \sigma + (1 - \rho) T_R \sigma = (I - T_R)\sigma + \rho T_R \sigma = \sigma^D + \rho T_R \sigma.$$

Recall that T_R is the projector in $(\cdot, \cdot)_F$ inner product. Using the property of orthogonal projectors, we expand the product

$$2\mu \mathcal{A}\sigma : \tau = (\sigma^D + \rho T_R \sigma) : (\tau^D + T_R \tau) = \sigma^D : \tau^D + \rho T_R(\sigma) : T_R(\tau),$$

and the identity then follows. \square

In view of the Helmholtz decomposition (21): $\mathbb{M} = \mathbb{Z} \oplus^{\perp} \pi(\mathbb{R})$, the compliance tensor $\mathcal{A} = \operatorname{diag}(1/(2\mu), \dots, 1/(2\mu), 1/(d\lambda + 2\mu))$ which implies the coercivity:

$$(48) \quad a(\sigma, \sigma) \geq \min \left\{ \frac{1}{2\mu}, \frac{1}{d\lambda + 2\mu} \right\} \|\sigma\|^2, \quad \forall \sigma \in \Sigma.$$

The constant α , unfortunately, is in the order of $\mathcal{O}(1/\lambda)$ as $\lambda \rightarrow +\infty$. Namely it is not robust to λ and nearly singular when $\lambda \gg 1$.

Notice that the norm for space Σ is $\|\cdot\|_{\operatorname{div}}$ but restricted to $\ker(\operatorname{div})$ only coercivity in L^2 -norm is needed. If $\ker(B) \neq \ker(\operatorname{div})$, additional care is needed.

5.3. A robust coercivity. We consider the pair $\Sigma \times V = \widehat{H}(\operatorname{div}, \Omega; \mathbb{S}) \times L^2(\Omega; \mathbb{V})$. The inf-sup condition of $b(\cdot, \cdot)$ is relatively easy. Given $v \in L^2(\Omega)$, we consider a simplified displacement formulation: find a $\phi \in H_0^1$ s.t.

$$(\nabla^s \phi, \nabla^s \psi) = (v, \psi), \quad \forall \psi \in H_0^1.$$

By the first Korn inequality, ϕ exists and unique. Let $\tau = \nabla^s \phi$. Then $-\operatorname{div} \tau = v$ and $\|\tau\|_{\operatorname{div}} \lesssim \|v\|$ and with this τ , the inf-sup condition for $b(\cdot, \cdot)$ is verified.

Recall that the coercivity is only needed in the kernel space of div operator and the compliance tensor \mathcal{A} is nearly singular in the subspace $\pi(\mathbb{R})$. We use the following triangle diagram connecting the linear elasticity with the inf-sup stability of Stokes equations to control the norm of the trace by adding $\|\text{div } \sigma\|_{-1}$.

$$\begin{array}{ccc} & & \mathbf{H}_0^1(\Omega, \mathbb{R}^d) \\ & \swarrow \nabla^s & \downarrow \text{div} \\ \hat{\mathbf{H}}(\text{div}, \Omega, \mathbb{S}) & \xrightarrow{\text{tr}} & L^2(\Omega) \end{array}$$

FIGURE 7. Linear elasticity and the inf-sup stability of Stokes equations.

Lemma 5.2. *There exists a constant β depending only the domain Ω s.t.*

$$(49) \quad \|T_R \sigma\|^2 \leq \beta (\|\sigma^D\|^2 + \|\text{div } \sigma\|_{-1}^2), \quad \text{for all } \sigma \in \Sigma.$$

Proof. Let $p = \text{tr } \sigma$. By the inf-sup stability of Stokes equation, we can find $\mathbf{u} \in \mathbf{H}_0^1$ s.t. $\text{div } \mathbf{u} = p$ and $\|\mathbf{u}\|_1 \lesssim \|p\| = \|\text{tr } \sigma\|$.

Using $\text{div } \mathbf{u} = \text{tr } \nabla^s \mathbf{u}$ and $p = \text{tr } \sigma$, we compute

$$\begin{aligned} \|\text{tr}(\sigma)\|^2 &= (\text{div } \mathbf{u}, p) = (\text{tr } \nabla^s \mathbf{u}, \text{tr } \sigma) = d(T_R \nabla^s \mathbf{u}, T_R \sigma) = d(\nabla^s \mathbf{u}, T_R \sigma) \\ &= d(\nabla^s \mathbf{u}, (T_R - I)\sigma) + d(\nabla^s \mathbf{u}, \sigma) = -d(\nabla^s \mathbf{u}, \sigma^D) - d(\mathbf{u}, \text{div } \sigma). \end{aligned}$$

Then using Cauchy-Schwarz inequality and the definition of $\|\cdot\|_{-1}$, we get

$$\|\text{tr}(\sigma)\|^2 \lesssim \|\sigma^D\| \|\nabla^s \mathbf{u}\| + \|\text{div } \sigma\|_{-1} \|D\mathbf{u}\| \lesssim (\|\sigma^D\|^2 + \|\text{div } \sigma\|_{-1}^2) \|\text{tr } \sigma\|.$$

Canceling one $\|\text{tr } \sigma\|$ to get the desired inequality. \square

We then obtain a robust coercivity of $a(\cdot, \cdot)$ restricted to the null space $\ker(\text{div})$.

Theorem 5.3. *There exists a constant α depending on Ω and μ , but independent of λ such that*

$$(50) \quad a(\sigma, \sigma) \geq \alpha \|\sigma\|^2 \quad \text{for all } \sigma \in \Sigma \cap \ker(\text{div}).$$

Proof. By Lemma 5.1, we have

$$2\mu a(\sigma, \sigma) = \|\sigma^D\|^2 + \rho \|T_R \sigma\|^2 \geq \|\sigma^D\|^2,$$

where recall that $\rho = 2\mu/(d\lambda + 2\mu) \rightarrow 0^+$ as $\lambda \rightarrow +\infty$. On the other hand, using (49), we can control

$$\|\sigma\|^2 = \|\sigma^D\|^2 + \|T_R \sigma\|^2 \leq (1 + \beta) \|\sigma^D\|^2.$$

The desired coercivity then follows for $\alpha = (1 + \beta)/(2\mu)$. \square

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