INTRODUCTION TO CONVERGENCE ANALYSIS OF ADAPTIVE FINITE ELEMENT METHODS

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In this notes, we present a simple convergence analysis of adaptive finite element methods (AFEMs) for second order elliptic partial differential equations and refer to Nochetto, Siebert and Veeser [8] for a detailed introduction to the theory of adaptive finite element methods.

1. Residual type a posterior error estimate

For the easy of presentation, we consider the Poisson equation with homogenous Dirichlet boundary condition

$$-\Delta u = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega. \tag{1}$$

Given a shape regular triangulation $\mathcal{T}$ of Ω, let $\mathcal{V}_{\mathcal{T}} \subset H_0^1(\Omega)$ be the linear finite element space based on $\mathcal{T}$, the linear finite element method of (1) is: find $u_{\mathcal{T}} \in \mathcal{V}_{\mathcal{T}}$ such that

$$(\nabla u_{\mathcal{T}}, \nabla v_{\mathcal{T}}) = (f, v_{\mathcal{T}}), \quad \text{for all } v_{\mathcal{T}} \in \mathcal{V}_{\mathcal{T}}. \tag{2}$$

Here we assume $f \in L^2(\Omega)$ and $(\cdot,\cdot)$ is the L^2-inner product. When the solution $u \in H^2(\Omega)$, we have the *a priori* error analysis

$$|u - u_{\mathcal{T}}|_{1,\Omega} \lesssim h_{\mathcal{T}} \|u\|_{2,\Omega}.$$

Such optimal order of convergence may not hold when u is not in $H^2(\Omega)$. In this section, we shall derive a residual type *a posterior* error estimate of the error $|u - u_{\mathcal{T}}|_{1,\Omega}$.

1.1. **A local and stable quasi-interpolation.**

To define a function in the linear finite element space $\mathcal{V}_{\mathcal{T}}$, we only need to assigned the value at interior vertices. For a vertex $x_i \in \mathcal{N}(\mathcal{T})$, recall that Ω_i consists of all simplexes sharing this vertex and for an element $\Omega_\tau = \cup_{x_i \in \tau} \Omega_i$. Instead of using nodal values of the object function, we can use its integral over Ω_i.

For an interior vertex x_i, we define a constant function on Ω_i by $A_i u = |\Omega_i|^{-1} \int_{\Omega_i} u(x)dx$. To incorporate the boundary condition, when $x_i \in \partial\Omega$, we define $A_i u = 0$. We then define the averaged interpolation $\Pi_{\mathcal{T}} : L^1(\Omega) \mapsto \mathcal{V}_{\mathcal{T}}$ by

$$\Pi_{\mathcal{T}} u = \sum_{x_i \in \mathcal{N}(\mathcal{T})} A_i(u) \varphi_i,$$

where φ_i is the hat function (P_1 basis) at vertex x_i.

Lemma 1.1. *For $u \in H^1(\Omega_\tau)$, we have the error estimate*

$$\|u - \Pi_{\mathcal{T}} u\|_{0,\tau} \lesssim h_\tau |u|_{1,\Omega_\tau}.$$

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Proof. For interior vertices, we use average type Poincaré inequality, to obtain

$$\|u - A_i u\|_{0,\Omega_i} \le C h_\tau |u|_{1,\Omega_i}, \tag{3}$$

and for boundary vertices, we use Poincaré-Friedriches since then $u|_{\partial\Omega_i\cap\partial\Omega} = 0$ and the $\mathbb{R}^{d-1}$ Lebesgue measure of the set $\partial\Omega_i \cap \partial\Omega$ is non-zero. The constant C in the equality (3) is independent with Ω_i since the mesh is shape regular. Then we use the partition of unity $\sum_{i=1}^{d+1} \varphi_i = 1$ restricted to one element τ to write

$$\begin{aligned}\int_\tau |u - \Pi_{\mathcal{T}} u|^2 &= \int_\tau \Big|\sum_{i=1}^{d+1} (u - A_i u)\varphi_i\Big|^2 \,\mathrm{d}x \lesssim \sum_{i=1}^{d+1} \int_{\Omega_i} |u - A_i u|^2 \,\mathrm{d}x \\ &\lesssim h_\tau \sum_{i=1}^{d+1} \int_{\Omega_i} |\nabla u|^2 \,\mathrm{d}x \lesssim C h_\tau^2 \int_{\Omega_\tau} |\nabla u|^2 \,\mathrm{d}x.\end{aligned}$$

□

Exercise 1.2. *Prove $\Pi_{\mathcal{T}}$ is stable in L^2-norm*

$$\|\Pi_{\mathcal{T}} u\|_{0,\tau} \lesssim \|u\|_{0,\Omega_\tau}.$$

We now prove that $\Pi_{\mathcal{T}}$ is stable in H^1 norm. Let us introduce another average operator Q_τ: the L^2 projection to the piecewise constant function spaces on $\mathcal{T}_h$: $(Q_\tau u)|_\tau = |\tau|^{-1} \int_\tau u(x)\,\mathrm{d}x$.

Lemma 1.3. *For $u \in H^1(\Omega_\tau)$, we have the stability*

$$|\Pi_{\mathcal{T}} u|_{1,\tau} \lesssim |u|_{1,\Omega_\tau}.$$

Proof. Using Poincaré inequality it is easy to see

$$\|u - Q_\tau u\|_{0,\tau} \lesssim h_\tau |u|_{1,\tau}.$$

We use the inverse inequality and the first order approximation property of Q_τ and $\Pi_{\mathcal{T}}$ to obtain

$$\begin{aligned}|\Pi_{\mathcal{T}} u|_{1,\tau} &= |\Pi_{\mathcal{T}} u - A_\tau u|_{1,\tau} \le h_\tau^{-1} \|\Pi_{\mathcal{T}} u - Q_\tau u\|_{0,\tau} \\ &\le h_\tau^{-1}\Big(\|u - \Pi_{\mathcal{T}} u\|_{0,\tau} + \|u - Q_\tau u\|_{0,\tau}\Big) \lesssim |u|_{1,\Omega_\tau}.\end{aligned}$$

□

Sum over each element and use the finite overlapping property due to the shape regularity of the mesh, we obtain the stability and approximation property.

Lemma 1.4. *For $u \in H_0^1(\Omega)$, the quasi-interpolant $\Pi_{\mathcal{T}} u$ satisfies the following properties:*

(1) *L^2 and H^1-stable*

$$\|\Pi_{\mathcal{T}} u\| \lesssim \|u\|, \quad |\Pi_{\mathcal{T}} u|_1 \lesssim |u|_1.$$

(2)

$$\sum_{\tau\in\mathcal{T}} \left(\|h^{-1}(u - \Pi_{\mathcal{T}} u)\|_{0,\tau}^2 + \|\nabla(u - \Pi_{\mathcal{T}} u)\|_{0,\tau}^2\right)^{1/2} \lesssim |u|_{1,\Omega}.$$

1.2. **Upper bound.** The equidistribution principle suggested us to equidistribute the quantity $|u|_{2,1,\tau}$; see Introduction to Adaptive Finite Element Methods. It is, however, not computable since u is unknown. One may want to approximate it by $|u_{\mathcal{T}}|_{2,1,\tau}$. For linear finite element function $u_{\mathcal{T}}$, we have $|u_{\mathcal{T}}|_{2,1,\tau} = 0$ and thus no information of $|u|_{2,1,\tau}$ will be obtained in this way.

More rigorously, the derivative of the piecewise constant vector function $\nabla u_{\mathcal{T}}$ will be delta distributions on edges with magnitude as the jump of $\nabla u_{\mathcal{T}}$ across the edge. On the other hand, $\Delta u \in L^2(\Omega)$ implies $\nabla u \in H(\mathrm{div};\Omega)$, i.e., $\nabla u \cdot n_e$ is continuous at an edge e where n_e is a unit norm vector of e. For the finite element approximation $u_{\mathcal{T}} \in \mathcal{V}_{\mathcal{T}}$, it is easy to see $\nabla u_{\mathcal{T}} \cdot n_e$ is not continuous (although the tangential derivative $\nabla u_{\mathcal{T}} \cdot t_e$ is). The discontinuity of norm derivative across edges can be used to measure the error $\nabla u - \nabla u_{\mathcal{T}}$. Therefore the flux jump could be used to bound the error in H^1-seminorm.

In the following, we give a rigorous justification. We provide a posterior error estimate for the Poisson equation with homogenous Dirichlet boundary condition below and refer to [11] for general elliptic equations and mixed boundary conditions.

Before we get into technical details, we emphasize the key consequence of the Galerkin projection.

Lemma 1.5. *Let u be the solution of (1) and $u_{\mathcal{T}} \in \mathcal{V}_{\mathcal{T}}$ be the solution of (2). Then we have the orthogonality*

$$(\nabla u - \nabla u_{\mathcal{T}}, v_{\mathcal{T}}) = 0 \quad \forall v_{\mathcal{T}} \in \mathcal{V}_{\mathcal{T}}. \tag{4}$$

The H^1-norm of the error and the bilinear form is connected through the stability of the method and for Poisson equation which is the identity

$$|u - u_{\mathcal{T}}|_1 = \|\Delta(u - u_{\mathcal{T}})\|_{-1} = \sup_{w \in H_0^1(\Omega)} \frac{a(u - u_{\mathcal{T}}, w)}{|w|_1}, \tag{5}$$

where $a(u, v) = (\nabla u, \nabla v)$ is the bilinear form associated to the Poisson equation.

Let $\mathcal{E}_{\mathcal{T}}$ denote the set of all interior edges, for each interior edge $e \in \mathcal{E}_{\mathcal{T}}$, we fix a unit norm vector n_e. Let τ_1 and τ_2 be two triangles sharing the edge e. The jump of flux across e is defined as

$$[\nabla u_{\mathcal{T}} \cdot n_e] = \nabla u_{\mathcal{T}} \cdot n_e|_{\tau_1} - \nabla u_{\mathcal{T}} \cdot n_e|_{\tau_2}.$$

We define h as a piecewise constant function on $\mathcal{T}$, that is for each element $\tau \in \mathcal{T}$,

$$h|_\tau = h_\tau = |\tau|^{1/2}. \tag{6}$$

We also define a piecewise constant function on $\mathcal{E}_{\mathcal{T}}$ as

$$h|_e = h_e := (h_{\tau_1} + h_{\tau_2})/2, \tag{7}$$

where $e = \tau_1 \cap \tau_2$ is the common edge of two triangles.

Theorem 1.6. *For a given triangulation $\mathcal{T}$, let $u_{\mathcal{T}}$ be the linear finite element approximation of the solution u of the Poisson equation. Then there exists a constant $C_1 > 0$ depending only on the shape regularity of $\mathcal{T}$ such that*

$$|u - u_{\mathcal{T}}|_1 \leq C_1 \Big(\sum_{\tau \in \mathcal{T}} \|hf\|_{0,\tau}^2 + \sum_{e \in \mathcal{E}_{\mathcal{T}}} \|h^{1/2} [\nabla u_{\mathcal{T}} \cdot n_e]\|_{0,e}^2 \Big)^{1/2}. \tag{8}$$

Proof. For any $w \in H_0^1(\Omega)$ and any $w_{\mathcal{T}} \in \mathcal{V}_{\mathcal{T}}$, we have

$$
\begin{aligned}
& a(u-u_{\mathcal{T}}, w) \\
= \; & a(u-u_{\mathcal{T}}, w-w_{\mathcal{T}}) \\
= \; & \sum_{\tau\in\mathcal{T}} \int_\tau \nabla(u-u_{\mathcal{T}})\cdot\nabla(w-w_{\mathcal{T}})\,\mathrm{dx} \\
= \; & \sum_{\tau\in\mathcal{T}} \int_\tau -\Delta(u-u_{\mathcal{T}})(w-w_{\mathcal{T}})\,\mathrm{dx} + \sum_{\tau\in\mathcal{T}} \int_{\partial\tau} \nabla(u-u_{\mathcal{T}})\cdot n(w-w_{\mathcal{T}})\,\mathrm{dS} \\
= \; & \sum_{\tau\in\mathcal{T}} \int_\tau f(w-w_{\mathcal{T}})\,\mathrm{dx} + \sum_{e\in\mathcal{E}_h} \int_e [\nabla u_{\mathcal{T}}\cdot n_e](w-w_{\mathcal{T}})\,\mathrm{dS} \\
\le \; & \sum_{\tau\in\mathcal{T}} \|hf\|_{0,\tau}\|h^{-1}(w-w_{\mathcal{T}})\|_{0,\tau} + \sum_{e\in\mathcal{E}_h} \|h^{1/2}[\nabla u_{\mathcal{T}}\cdot n_e]\|_{0,e}\|h^{-1/2}(w-w_{\mathcal{T}})\|_{0,e} \\
\lesssim \; & \left(\sum_{\tau\in\mathcal{T}} \|hf\|_{0,\tau}^2 + \sum_{e\in\mathcal{E}_{\mathcal{T}}} \|h^{-1/2}[\nabla u_{\mathcal{T}}\cdot n_e]\|_{0,e}^2\right)^{1/2} \\
& \left(\sum_{\tau\in\mathcal{T}} \|h^{-1}(w-w_{\mathcal{T}})\|_{0,\tau}^2 + \|\nabla(w-w_{\mathcal{T}})\|_{0,\tau}^2\right)^{1/2}.
\end{aligned}
$$

In the last step, we have used the trace theorem with the scaling argument to get

$$\|w-w_{\mathcal{T}}\|_{0,e} \lesssim h_\tau^{-1/2}\|w-w_{\mathcal{T}}\|_{0,\tau} + h_\tau^{1/2}\|\nabla(w-w_{\mathcal{T}})\|_{0,\tau}.$$

Now chose $w_{\mathcal{T}} = \Pi_{\mathcal{T}} w$ by the quasi-interpolation operator introduced in Lemma 1.4, we have

$$\left(\sum_{\tau\in\mathcal{T}} \|h^{-1}(w-w_{\mathcal{T}})\|_{0,\tau}^2 + \|\nabla(w-w_{\mathcal{T}})\|_{0,\tau}^2\right)^{1/2} \lesssim |w|_{1,\Omega}. \tag{9}$$

Then we end with

$$|u-u_{\mathcal{T}}|_1 = \sup_{w\in H_0^1(\Omega)} \frac{a(u-u_{\mathcal{T}}, w)}{|w|_1} \lesssim \Big(\sum_{\tau\in\mathcal{T}} \|hf\|_{0,\tau}^2 + \sum_{e\in\mathcal{E}_{\mathcal{T}}} \|h^{1/2}[\nabla u_{\mathcal{T}}\cdot n_e]\|_{0,e}^2\Big)^{1/2}.$$

$\square$

To guide the local refinement, we need to have an element-wise (or edge-wise) error indicator. For any $\tau \in \mathcal{T}$ and any $v_{\mathcal{T}} \in \mathcal{V}_{\mathcal{T}}$, we define

$$\eta(v_{\mathcal{T}}, \tau) = \Big(\|hf\|_{0,\tau}^2 + \sum_{e\in\partial\tau} \|h^{1/2}[\nabla v_{\mathcal{T}}\cdot n_e]\|_{0,e}^2\Big)^{1/2}. \tag{10}$$

For a subset $\mathcal{M}_{\mathcal{T}} \subseteq \mathcal{T}$, we define

$$\eta(v_{\mathcal{T}}, \mathcal{M}_{\mathcal{T}}) = \Big[\sum_\tau \eta^2(v_{\mathcal{T}}, \tau)\Big]^{1/2}.$$

With these notation, the upper bound (8) can be simply written as

$$|u-u_{\mathcal{T}}|_{1,\Omega} \le C_1 \eta(u_{\mathcal{T}}, \mathcal{T}). \tag{11}$$

Remark 1.7. The local version of the upper bound (11)

$$|u-u_{\mathcal{T}}|_{1,\tau} \le C_1 \eta(u_{\mathcal{T}}, \Omega_\tau)$$

does not hold in general as the orthogonality (4) only holds globally.

1.3. **Lower bound.** We shall derive a lower bound of the error estimator η through the following exercises. The technique is developed by Verfürth [11] and known as bubble functions. Let u be the solution of Poisson equation $-\Delta u = f$ with homogeneous Dirichlet boundary condition and $u_{\mathcal{T}}$ be the linear finite element approximation of u based on a shape regular and conforming triangulation $\mathcal{T}$.

Exercise 1.8. (1) For a triangle τ, we denote $V_\tau = \{f_\tau \in L_2(\tau) \,|\, f_\tau = \text{constant}\}$ equipped with L^2 inner product. Let $\lambda_i(x), i = 1, 2, 3$ be the barycenter coordinates of $x \in \tau$, and let $b_\tau = \lambda_1\lambda_2\lambda_3$ be the bubble function on τ. We define $B_\tau f_\tau = f_\tau b_\tau$.

Prove that $B_\tau : V_\tau \mapsto V = H_0^1(\Omega)$ is bounded in L^2 and H^1 norm:

$$\|B_\tau f_\tau\|_{0,\tau} = C\|f_\tau\|_{0,\tau}, \text{ and } \|\nabla(B_\tau f_\tau)\|_{0,\tau} \lesssim h_\tau^{-1}\|f_\tau\|_{0,\tau}.$$

(2) Using (1) to prove that

$$\|hf_\tau\|_{0,\tau} \lesssim |u - u_{\mathcal{T}}|_{1,\tau} + \|h(f - f_\tau)\|_{0,\tau}.$$

(3) For an interior edge e, we define $V_e = \{g_e \in L^2(E) \,|\, g_e = \text{constant}\}$. Suppose e has end points x_i, and x_j, we define $b_e = \lambda_i\lambda_j$ and $B_e : V_e \mapsto V$ by $B_e g_e = g_e b_e$.

Let ω_e denote two triangles sharing e. Prove that

(a) $\|g_e\|_{0,e} = C\|B_e g_e\|_{0,e}$,

(b) $\|B_e g_e\|_{0,\omega_e} \lesssim h_e^{1/2}\|g_e\|_{0,e}$ and,

(c) $\|\nabla(B_e g_e)\|_{0,\omega_e} \lesssim h_e^{-1/2}\|g_e\|_{0,e}$.

(4) Using (3) to prove that

$$\|h^{1/2}[\nabla u_{\mathcal{T}} \cdot n_e]\|_{0,e} \lesssim \|hf\|_{0,\omega_e} + |u - u_{\mathcal{T}}|_{1,\omega_e}.$$

(5) Using (1) and (4) to prove the lower bound of the error estimator. There exists a constant C_2 depending only on the shape regularity of the triangulation such that for any piecewise constant approximation f_τ of $f \in L^2$,

$$C_2\eta^2(u_{\mathcal{T}}, \mathcal{T}) \le |u - u_{\mathcal{T}}|_{1,\Omega}^2 + \sum_{\tau \in \mathcal{T}_h} \|h(f - f_\tau)\|_\tau^2.$$

2. Convergence

Standard adaptive finite element methods (AFEM) based on the local mesh refinement can be written as loops of the form

$$\textbf{SOLVE} \rightarrow \textbf{ESTIMATE} \rightarrow \textbf{MARK} \rightarrow \textbf{REFINE}. \tag{12}$$

Starting from an initial triangulation $\mathcal{T}_0$, to get $\mathcal{T}_{k+1}$ from $\mathcal{T}_k$ we first solve the equation to get u_k based on $\mathcal{T}_k$ (the indices of second order like $u_{\mathcal{T}_k}$ will be contracted as u_k). The error is estimated using u_k and $\mathcal{T}_k$ and used to mark a set of of triangles in $\mathcal{T}_k$. Marked triangles and possible more neighboring triangles are refined in such a way that the triangulation is still shape regular and conforming.

2.1. **Algorithm.** The step **SOLVE** is discussed in Chapter: Iterative method, where efficient iterative methods including multigrid and conjugate gradient methods is studied in detail. Here we assume that the solutions of the finite dimensional problems can be solved to any accuracy efficiently.

The *a posteriori* error estimators are an essential part of the **ESTIMATE** step. We have given one in the previous section and will discuss more in the next section.

The *a posteriori* error estimator is split into local error indicators and they are then employed to make local modifications by dividing the elements whose error indicator is large and possibly coarsening the elements whose error indicator is small. The way we mark these triangles influences the efficiency of the adaptive algorithm. The traditional maximum marking strategy proposed in the pioneering work of Babuška and Vogelius [1] is to mark triangles τ^* such that

$$\eta(u_{\mathcal{T}}, \tau^*) \geq \theta \max_{\tau \in \mathcal{T}} \eta(u_{\mathcal{T}}, \tau), \quad \text{for some } \theta \in (0,1).$$

Such marking strategy is designed to evenly equi-distribute the error. Based our relaxation of the equidistribution principal, we may leave some exceptional elements and focus on the overall amounts of the error. This leads to the bulk criterion firstly proposed by Dörfler [4] in order to prove the convergence of the local refinement strategy. With such strategy, one defines the marking set $\mathcal{M}_{\mathcal{T}} \subset \mathcal{T}$ such that

$$\eta^2(u_{\mathcal{T}}, \mathcal{M}_{\mathcal{T}}) \geq \theta\, \eta^2(u_{\mathcal{T}}, \mathcal{T}), \quad \text{for some } \theta \in (0,1). \tag{13}$$

We shall use Dörfler marking strategy in the proof.

After choosing a set of marked elements, we need to carefully design the rule for dividing the marked triangles such that the mesh obtained by this dividing rule is still conforming and shape regular. Such refinement rules include red and green refinement [2], longest refinement [9], and newest vertex bisection [6]. In addition we also would like to control the number of elements added to ensure the optimality of the refinement. To this end we shall use the newest vertex bisection.

Let us summarize AFEM in the following subroutine:

```
[u_J, T_J] = AFEM (T_1, f, tol, θ)
% AFEM compute an approximation u_J by adaptive finite element methods
% Input: T_1 an initial triangulation; f data; tol <<1 tolerance; θ ∈ (0,1)
% Output: u_J linear finite element approximation; T_J the finest mesh
η = 1, k = 0;
while η ≥ tol
    k = k + 1;
    SOLVE Poisson equation on T_k to get the solution u_k;
    ESTIMATE the error by η = η(u_k, T_k);
    MARK a set M_k ⊂ T_k with minimum number such that η^2(u_k, M_k) ≥ θ η^2(u_k, T_k);
    REFINE τ ∈ M_k and necessary triangles to a conforming triangulation T_{k+1};
end
u_J = u_k; T_J = T_k;
```

2.2. **Contraction of error estimator.** By the orthogonality, one can easily conclude the error is non-increasing, i.e.

$$|u - u_{k+1}|_1 \leq |u - u_k|_1.$$

The equality could hold, i.e. $u_{k+1} = u_k$, if the refinement did not introduce interior nodes for triangles and edges; see Example 3.6 and 3.7 in [7]. A close look reveals that when the solution does not change, the error estimator η will be reduced due to the change of mesh size and the Dörfler marking strategy.

Lemma 2.1. *Given a $\theta \in (0,1)$, let $\mathcal{T}_{k+1}$ be a conforming and shape regular triangulation which is refined from a conforming and shape regular triangulation $\mathcal{T}_k$ using Dörfler marking strategy (13). Let u_k be the solution of (2) in $\mathcal{V}_k$. Then*

$$\eta^2(u_k, \mathcal{T}_{k+1}) \le \rho\, \eta^2(u_k, \mathcal{T}_k)$$

for some $\rho \in (0,1)$ depending only on the shape regularity of $\mathcal{T}_k$ and the parameter θ used in the Dörfler marking strategy.

Proof. We study in detail the change of the error estimator due to the bisection of a triangle. Suppose τ is bisected to τ_1 and τ_2. We first prove element-wise contraction of error indicator: there exists a number $\bar\rho \in (0,1)$ depending only on the shape regularity of $\mathcal{T}_k$ such that

$$\eta^2(u_k, \tau_1) + \eta^2(u_k, \tau_2) \le \bar\rho\, \eta^2(u_k, \tau). \tag{14}$$

To distinguish the different mesh size function, we use h_{k+1} and h_k to denote the mesh size function defined on $\mathcal{T}_k$ and $\mathcal{T}_{k+1}$, respectively. Thanks to our definition, $h^2_{k+1,\tau_i} = |\tau_i| = 1/2|\tau| = 1/2\, h^2_{k,\tau}$. The part involving element residual is reduced by one half:

$$\|h_{k+1} f\|^2_{\tau_1} + \|h_{k+1} f\|^2_{\tau_2} = \frac{1}{2}\|h_k f\|^2_{\tau}.$$

For the jump of gradient on the edges, an important observation is that $[\nabla u_k \cdot n_e] = 0$ for the new created edge inside τ. For other edges on the boundary of τ, h_e is reduced by a factor due to the definition of h_e while the jump $[\nabla u_k \cdot n_e]$ remains unchanged as a constant on the coarse mesh. So $\sum_{e\in\partial\tau} \|h^{1/2}[\nabla v_{\mathcal{T}} \cdot n_e]\|^2_{0,e}$ is also reduced by a factor strictly less than one.

But not all elements are refined. Dörfler marking ensures a portion of the error estimator is reduced which is sufficient. Recall that $\mathcal{M}_k \subseteq \mathcal{T}_k$ is the marked set. We may need to refine more triangles to recover the conformity of the triangulation and thus denote the set of refined triangles by $\overline{\mathcal{M}}_k$. Since $\mathcal{M}_k \subseteq \overline{\mathcal{M}}_k$, we have

$$\eta^2(u_k, \overline{\mathcal{M}}_k) \ge \eta^2(u_k, \mathcal{M}_k) \ge \theta \eta^2(u_k, \mathcal{T}_k).$$

We use notation $\overline{\mathcal{M}}_{k+1} \subseteq \mathcal{T}_{k+1}$ to denote the set of triangles obtained by refinement of that in $\overline{\mathcal{M}}_k$. Then $\mathcal{T}_k \backslash \overline{\mathcal{M}}_k = \mathcal{T}_{k+1} \backslash \overline{\mathcal{M}}_{k+1}$ are untouched triangles. We then have

$$\begin{aligned}
\eta^2(u_k, \mathcal{T}_{k+1}) &= \eta^2(u_k, \mathcal{T}_{k+1}\backslash\overline{\mathcal{M}}_{k+1}) + \eta^2(u_k, \overline{\mathcal{M}}_{k+1}) \\
&\le \eta^2(u_k, \mathcal{T}_k\backslash\overline{\mathcal{M}}_k) + \bar\rho\, \eta^2(u_k, \overline{\mathcal{M}}_k) \\
&= \eta^2(u_k, \mathcal{T}_k) - (1-\bar\rho)\, \eta^2(u_k, \overline{\mathcal{M}}_k) \\
&\le \eta^2(u_k, \mathcal{T}_k) - \theta(1-\bar\rho)\, \eta^2(u_k, \mathcal{T}_k) \\
&= \big[1 - \theta(1-\bar\rho)\big]\eta^2(u_k, \mathcal{T}_k).
\end{aligned}$$

We obtain (4) with $\rho = 1 - \theta(1-\bar\rho) \in (0,1)$. □

2.3. **Contraction between two levels.** We shall prove the convergence of `AFEM` by showing the contraction of the total error between two levels. There exists a positive constant α and a constant $\delta \in (0,1)$ such that for all $k \ge 0$,

$$|u - u_{k+1}|_1^2 + \alpha\, \eta^2(u_{k+1}, \mathcal{T}_{k+1}) \le \delta \left[\, |u - u_k|_1^2 + \alpha\, \eta^2(u_k, \mathcal{T}_k) \,\right]. \tag{15}$$

Recall that we have

$$|u - u_{k+1}|_1^2 \le |u - u_k|_1^2, \quad \eta^2(u_k, \mathcal{T}_{k+1}) \le \rho\, \eta^2(u_k, \mathcal{T}_k).$$

To prove (15), we shall explore more relation between the error and the error estimator in consecutive levels $\mathcal{T}_k$ and $\mathcal{T}_{k+1}$.

Lemma 2.2. *Given a $\theta \in (0,1)$, let $\mathcal{T}_{k+1}$ be a conforming and shape regular triangulation which is refined from a conforming and shape regular triangulation $\mathcal{T}_k$ using Dörfler marking strategy (13). Let u_{k+1} and u_k be solutions of (2) in $\mathcal{V}_{k+1}$ and $\mathcal{V}_k$, respectively. Then we have*

(1) orthogonality:

$$|u - u_{k+1}|_1^2 = |u - u_k|_1^2 - |u_{k+1} - u_k|_1^2;$$

(2) upper bound: $|u - u_k|_1^2 \le C_1 \eta^2(u_k, \mathcal{T}_k)$ for some constant C_1 depending only on the shape regularity of $\mathcal{T}$;

(3) continuity of error estimator: for any $\epsilon > 0$, there exists a constant C_ϵ such that

$$\eta^2(u_{k+1}, \mathcal{T}_{k+1}) \le (1+\epsilon)\eta^2(u_k, \mathcal{T}_{k+1}) + C_\epsilon |u_{k+1} - u_k|_1^2;$$

(4) contraction of error estimator:

$$\eta^2(u_{k+1}, \mathcal{T}_{k+1}) \le \rho(1+\epsilon)\, \eta^2(u_k, \mathcal{T}_k) + C_\epsilon |u_{k+1} - u_k|_1^2$$

for $\rho \in (0,1)$ in Lemma 2.1.

Proof. (1) is trivial since u_{k+1} is the H^1 projection and $u_{k+1} - u_k \in \mathcal{V}_{k+1}$ by the nestness of $\mathcal{T}_k$ and $\mathcal{T}_{k+1}$. (2) has been proved in the previous section.

We now prove (3). The part contains element-wise residual $\|hf\|$ is unchanged since we do not change the triangulation. For each $e \in \mathcal{E}_{\mathcal{T}}$, let $\tau \in \mathcal{T}$ such that $e \in \partial\tau$. From the triangle inequality and the fact $\nabla(u_{k+1} - u_k)$ is piecewise constant, we have

$$\begin{aligned} \|h^{1/2}[\nabla u_{k+1} \cdot n_e]\|_{0,e} &\le \|h^{1/2}[\nabla u_k \cdot n_e]\|_{0,e} + \|h^{1/2}[\nabla(u_{k+1} - u_k) \cdot n_e]\|_{0,e}, \\ &\le \|h^{1/2}[\nabla u_k \cdot n_e]\|_{0,e} + C|u_{k+1} - u_k|_{1,\tau}. \end{aligned}$$

Square both sides, apply the Young's inequality $2ab \le \epsilon a^2 + \epsilon^{-1} b^2$ and sum all edges to get the desired inequality. (4) is a combination of (3) and Lemma 2.1.

□

We are in the position to prove the contraction result. We use the negative term due to the orthogonality of the error to cancel the positive term from the reduction the error estimator.

Theorem 2.3. *Given a $\theta \in (0,1)$, let $\mathcal{T}_{k+1}$ be a conforming and shape regular triangulation which is refined from a conforming and shape regular triangulation $\mathcal{T}_k$ using Dörfler marking strategy (13). Let u_{k+1} and u_k be solutions of (2) in $\mathcal{V}_{k+1}$ and $\mathcal{V}_k$, respectively. Then there exist constants $\delta \in (0,1)$ and α depending only on θ and the shape regularity of $\mathcal{T}_k$ such that*

$$|u - u_{k+1}|_1^2 + \alpha\, \eta^2(u_{k+1}, \mathcal{T}_{k+1}) \le \delta \left[\, |u - u_k|_1^2 + \alpha\, \eta^2(u_k, \mathcal{T}_k) \,\right]. \tag{16}$$

Proof. Let $\rho \in (0,1)$ be the constant in Lemma 2.1. Since $\rho \in (0,1)$, we can choose $\epsilon \in (0,1)$ small enough such that $\rho(1+\epsilon) < 1$. Let $\alpha = C_\epsilon^{-1}$. Adding the two inequalities in Lemma 2.2 (1) and (4) with weight α will imply

$$|u - u_{k+1}|_1^2 + \alpha\, \eta^2(u_{k+1}, \mathcal{T}_{k+1}) \le |u - u_k|_1^2 + \alpha(1+\epsilon)\rho\, \eta^2(u_k, \mathcal{T}_k).$$

Let δ be a number in $(0,1)$ whose value will be clear in a moment. We then have

$$\begin{aligned}
&|u-u_{k+1}|_1^2+\alpha\,\eta^2(u_{k+1},\mathcal{T}_{k+1})\\
\le\,&\delta|u-u_k|_1^2+(1-\delta)|u-u_k|_1^2+\alpha\rho(1+\epsilon)\,\eta^2(u_k,\mathcal{T}_k)\\
\le\,&\delta|u-u_k|_1^2+(1-\delta)C_1\eta^2(u_k,\mathcal{T}_k)+\alpha\rho(1+\epsilon)\,\eta^2(u_k,\mathcal{T}_k)\\
\le\,&\delta\left[|u-u_k|_1^2+\frac{(1-\delta)C_1+\alpha\rho(1+\epsilon)}{\delta}\eta^2(u_k,\mathcal{T}_k)\right].
\end{aligned}$$

This suggests us to choose δ such that

$$\alpha=\frac{(1-\delta)C_1+\alpha\rho(1+\epsilon)}{\delta}.$$

Namely

$$\delta=\frac{C_1+\alpha\rho(1+\epsilon)}{C_1+\alpha}. \tag{17}$$

Recall that we choose ϵ such that $\rho(1+\epsilon)<1$, so $\delta\in(0,1)$. The desired result (16) then follows. □

As a consequence of the contraction of the total error between two levels, we can prove `AFEM` will stop in finite steps for a given tolerance tol and produce a convergent approximation u_J based on an adaptive grid $\mathcal{T}_J$. We refer to [10, 3] for the analysis of complexity which is much more involved.

Theorem 2.4. *Let u_k and $\mathcal{T}_k$ be the solution and triangulation obtained in the k-th loop in the algorithm `AFEM`, then there exist constants $\delta\in(0,1)$ and α depending only on θ and the shape regularity of $\mathcal{T}_0$ such that*

$$|u-u_k|_1^2+\alpha\,\eta^2(u_k,\mathcal{T}_k)\le C_0\delta^k, \tag{18}$$

and thus the algorithm `AFEM` will terminate in finite steps.

2.4. **Alternative convergence proof.** We follow [5] to present an alternative convergence proof of the error estimator.

Theorem 2.5. *Let u_k and $\mathcal{T}_k$ be the solution and triangulation obtained in the k-th loop in the algorithm `AFEM`. Then there exist constants $0<\varrho<1$ and $C>0$ such that: for all positive integers ℓ,m*

$$\eta^2(u_{\ell+m},\mathcal{T}_{\ell+m})\le C\varrho^m\eta^2(u_\ell,\mathcal{T}_\ell). \tag{19}$$

Proof. We recall the contraction of the error estimator

$$\eta^2(u_{i+1},\mathcal{T}_{i+1})\le\rho\eta^2(u_i,\mathcal{T}_i)+C_\theta|u_{i+1}-u_i|_1^2. \tag{20}$$

Therefore, for any $N\ge l+1$, it holds

$$\begin{aligned}
\sum_{i=\ell+1}^{N}\eta^2(u_i,\mathcal{T}_i)&\le\sum_{i=\ell+1}^{N}\left[\rho\eta^2(u_{i-1},\mathcal{T}_{i-1})+C_\theta|u_i-u_{i-1}|_1^2\right]\\
&\le\rho\sum_{i=\ell}^{N-1}\eta^2(u_i,\mathcal{T}_i)+C_\theta|u-u_\ell|_1^2\\
&\le\rho\sum_{i=\ell}^{N-1}\eta^2(u_i,\mathcal{T}_i)+C_\theta C_1^2\eta^2(u_\ell,\mathcal{T}_\ell).
\end{aligned}$$

Here, in the second inequality, we have used the orthogonality to get

$$\sum_{i=\ell+1}^{N} |u_i - u_{i-1}|_1^2 = |u_N - u_\ell|_1^2 = |u - u_\ell|_1^2 - |u - u_N|_1^2 \leq |u - u_\ell|_1^2.$$

Then, rearranging the terms and with the arbitrary choice of N, we obtain

$$\sum_{i=\ell+1}^{\infty} \eta^2(u_i, \mathcal{T}_i) \leq \tilde{C}\eta^2(u_\ell, \mathcal{T}_\ell) \qquad \text{for all positive integer } l,$$

where $\tilde{C} = (\rho + C_\theta C_1^2)/(1-\rho)$. Intuitively we have a positive series $\{a_i\}$ with property $\sum_{i=\ell+1}^{\infty} a_i \leq \tilde{C} a_\ell$, then a_i is geometric decay.

To prove that, we first show the contraction

$$(1 + \tilde{C}^{-1}) \sum_{i=\ell+1}^{\infty} \eta^2(u_i, \mathcal{T}_i) \leq \sum_{i=\ell+1}^{\infty} \eta^2(u_i, \mathcal{T}_i) + \eta^2(u_\ell, \mathcal{T}_\ell) = \sum_{i=\ell}^{\infty} \eta^2(u_i, \mathcal{T}_i).$$

Repeat m times, we have

$$\begin{aligned}\eta^2(u_{\ell+m}, \mathcal{T}_{\ell+m}) &\leq \sum_{i=\ell+m}^{\infty} \eta^2(u_i, \mathcal{T}_i) \leq (1 + \tilde{C}^{-1})^{-m} \sum_{i=\ell}^{\infty} \eta^2(u_i, \mathcal{T}_i) \\ &\leq (1 + \tilde{C})(1 + \tilde{C}^{-1})^{-m} \eta^2(u_\ell, \mathcal{T}_\ell).\end{aligned}$$

Let $C_5^2 = 1 + \tilde{C}$ and $\varrho = (1 + \tilde{C}^{-1})^{-1}$, then the desired result follows. □

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