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一维两点边值问题的有限元计算流程

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用有限元方法求解两点边值问题

- 1.求方程

$$\begin{cases} -u'' = f, \\ u(0) = u(1) = 0, \end{cases}$$

的解， 其中

$$u \in H_0^1(\Omega), \Omega \triangleq [0,1].$$

证明：此方程的求解等价于变分问题

$$(u', v') = (f, v), \forall v \in H_0^1(\Omega) \tag{1}$$

设$V_h \subset V$为V的有限维子空间,则问题转化为

求$u_h \in V_h$,使得$(u_h, v_h) = (f, v_h)$ 先将$\Omega = [0,1]$四等分，

记$I_i = [x_{i-1}, x_i]$,$x_0 = 0$,$x_4 = 1$,$h = x_i - x_{i-1}$.

因此$dimV_h = 4 \times 2 - 3 = 5$,故其基函数$\{\varphi_i(x)\}_{i=0}^4$.

- 从而

$$V_h = span\{\varphi_0(x), \varphi_1(x), \varphi_2(x), \varphi_3(x), \varphi_4(x)\}$$

$$\varphi_i(x_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

其中φ_i取如下的插值

$$\varphi_0(x) = \begin{cases} \frac{x - x_1}{x_0 - x_1}, & x \in I_1 \\ 0, & x \notin I_1 \end{cases}$$

$$\varphi_1(x) = \begin{cases} \frac{x - x_0}{x_1 - x_0}, & x \in I_1 \\ \frac{x - x_2}{x_1 - x_2}, & x \in I_2 \\ 0, & x \notin I_1 \cup I_2 \end{cases}$$

- $$\varphi_2(x) = \begin{cases} \frac{x-x_1}{x_2-x_1}, & x \in I_2 \\ \frac{x-x_3}{x_2-x_3}, & x \in I_3 \\ 0, & x \notin I_2 \cup I_3 \end{cases}$$

  $$\varphi_3(x) = \begin{cases} \frac{x-x_2}{x_3-x_2}, & x \in I_3 \\ \frac{x-x_4}{x_3-x_4}, & x \in I_4 \\ 0, & x \notin I_3 \cup I_4 \end{cases}$$

  $$\varphi_4(x) = \begin{cases} \frac{x-x_4}{x_5-x_4}, & x \in I_3 \\ 0, & x \notin I_4 \end{cases}$$

- 因为$u_h \in V_h$,则有

$$u_h = \sum_{i=1}^{4} u_i \varphi_i(x) \tag{2}$$

将（2）式代入$(u_h, v_h) = (f, v_h)$并令$v_h = \varphi_i, i = 0, 1, 2, 3, 4.$
则有

$$(\sum_{i=1}^{4} u_i \varphi_i(x), \varphi_i) = (f, \varphi_i)$$

即

$$\sum_{i=1}^{4} u_j(\varphi_j', \varphi_i') = (f, \varphi_i)$$

令

$$U = (u_0, u_1, u_2, u_3, u_4)^T$$

- $$A=(A_{ij})_{5\times5},A_{ij}=(\varphi_j',\varphi_i')=\int_0^1\varphi_j'\varphi_i'dx$$

  $$F=(F_0,F_1,F_2,F_3,F_4)^T,F_i=(f,\varphi_i)=\int_0^1 f\varphi_i dx$$

  则可得有限元方程组为

  $$AU=F$$

  要计算矩阵所对应的元素可通过计算每个剖分单元上对应的值，然后通过映射一一对应过去。

- 令

$$A_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

先考虑I_1单元上,则有

$$(\varphi_0', \varphi_0') = \int_{I_1} \varphi_0'\varphi_0' dx = \int_{I_1} \frac{1}{h^2} dx = \frac{1}{h} \to (0, 0)$$

$$(\varphi_0', \varphi_1') = \int_{I_1} \varphi_0'\varphi_1' dx = \int_{I_1} \frac{1}{-h}\frac{1}{h} dx = \frac{1}{-h} \to (1, 0)$$

$$(\varphi_1', \varphi_1') = \int_{I_1} \varphi_1'\varphi_1' dx = \int_{I_1} \frac{1}{h}\frac{1}{h} dx = \frac{1}{h} \to (1, 1)$$

$$(\varphi_1', \varphi_0') = \int_{I_1} \varphi_1'\varphi_0' dx = \int_{I_1} \frac{1}{h}\frac{1}{-h} dx = \frac{1}{-h} \to (0, 1)$$

- 于是，

$$A_0 \to A_1 = \begin{pmatrix} \frac{1}{h} & \frac{1}{-h} & 0 & 0 & 0 \\ \frac{1}{-h} & \frac{1}{h} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

在I_2单元上,则有

$$(\varphi_1', \varphi_1') = \int_{I_2} \varphi_1' \varphi_1' dx = \int_{I_2} \frac{1}{-h} \frac{1}{-h} dx = \frac{1}{h} \to (1, 1)$$

$$(\varphi_1', \varphi_2') = \int_{I_2} \varphi_1' \varphi_2' dx = \int_{I_2} \frac{1}{-h} \frac{1}{h} dx = \frac{1}{-h} \to (2, 1)$$

$$(\varphi_2', \varphi_1') = \int_{I_2} \varphi_2' \varphi_1' dx = \int_{I_2} \frac{1}{h} \frac{1}{-h} dx = \frac{1}{-h} \to (1, 2)$$

$$(\varphi_2', \varphi_2') = \int_{I_2} \varphi_2' \varphi_2' dx = \int_{I_2} \frac{1}{h} \frac{1}{h} dx = \frac{1}{h} \to (2, 2)$$

- 于是,

$$A_1 \to A_2 = \begin{pmatrix} \frac{1}{h} & \frac{1}{-h} & 0 & 0 & 0 \\ \frac{1}{-h} & \frac{1}{h}+\frac{1}{h} & \frac{1}{-h} & 0 & 0 \\ 0 & \frac{1}{-h} & \frac{1}{h} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

在I_3单元上,则有

$$(\varphi_2', \varphi_2') = \int_{I_3} \varphi_2'\varphi_2' dx = \int_{I_3} \frac{1}{-h}\frac{1}{-h} dx = \frac{1}{h} \to (2,2)$$

$$(\varphi_2', \varphi_3') = \int_{I_3} \varphi_2'\varphi_3' dx = \int_{I_3} \frac{1}{-h}\frac{1}{h} dx = \frac{1}{-h} \to (3,2)$$

$$(\varphi_3', \varphi_2') = \int_{I_3} \varphi_3'\varphi_2' dx = \int_{I_3} \frac{1}{h}\frac{1}{-h} dx = \frac{1}{-h} \to (2,3)$$

$$(\varphi_3', \varphi_3') = \int_{I_3} \varphi_3'\varphi_3' dx = \int_{I_3} \frac{1}{h}\frac{1}{h} dx = \frac{1}{h} \to (3,3)$$

- 于是，

$$A_2 \to A_3 = \begin{pmatrix} \frac{1}{h} & \frac{1}{-h} & 0 & 0 & 0 \\ \frac{1}{-h} & \frac{1}{h}+\frac{1}{h} & \frac{1}{-h} & 0 & 0 \\ 0 & \frac{1}{-h} & \frac{1}{h}+\frac{1}{h} & \frac{1}{-h} & 0 \\ 0 & 0 & \frac{1}{-h} & \frac{1}{h} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

在I_4单元上,则有

$$(\varphi_3', \varphi_3') = \int_{I_4} \varphi_3'\varphi_3' dx = \int_{I_4} \frac{1}{-h}\frac{1}{-h} dx = \frac{1}{h} \to (3,3)$$

$$(\varphi_3', \varphi_4') = \int_{I_4} \varphi_3'\varphi_4' dx = \int_{I_4} \frac{1}{-h}\frac{1}{h} dx = \frac{1}{-h} \to (4,3)$$

$$(\varphi_4', \varphi_3') = \int_{I_4} \varphi_4'\varphi_3' dx = \int_{I_4} \frac{1}{h}\frac{1}{-h} dx = \frac{1}{-h} \to (3,4)$$

$$(\varphi_4', \varphi_4') = \int_{I_4} \varphi_4'\varphi_4' dx = \int_{I_4} \frac{1}{h}\frac{1}{h} dx = \frac{1}{h} \to (4,4)$$

- 于是，

$$A_3 \to A_4 = \begin{pmatrix} \frac{1}{h} & \frac{1}{-h} & 0 & 0 & 0 \\ \frac{1}{-h} & \frac{1}{h}+\frac{1}{h} & \frac{1}{-h} & 0 & 0 \\ 0 & \frac{1}{-h} & \frac{1}{h}+\frac{1}{h} & \frac{1}{-h} & 0 \\ 0 & 0 & \frac{1}{-h} & \frac{1}{h}+\frac{1}{h} & \frac{1}{-h} \\ 0 & 0 & 0 & \frac{1}{-h} & \frac{1}{h} \end{pmatrix}$$

则

$$A = A_4 = \begin{pmatrix} \frac{1}{h} & \frac{1}{-h} & 0 & 0 & 0 \\ \frac{1}{-h} & \frac{1}{h}+\frac{1}{h} & \frac{1}{-h} & 0 & 0 \\ 0 & \frac{1}{-h} & \frac{1}{h}+\frac{1}{h} & \frac{1}{-h} & 0 \\ 0 & 0 & \frac{1}{-h} & \frac{1}{h}+\frac{1}{h} & \frac{1}{-h} \\ 0 & 0 & 0 & \frac{1}{-h} & \frac{1}{h} \end{pmatrix}$$

- 再考虑F,令$f=1.F_0=(0,0,0,0,0)^T$.

在I_1单元上

$$(f,\varphi_0)=\int_{I_1}\varphi_0 dx=\frac{h}{2}\to(0,1)$$

$$(f,\varphi_1)=\int_{I_1}\varphi_1 dx=\frac{h}{2}\to(1,1)$$

于是

$$F_0\to F_1=(\frac{h}{2},\frac{h}{2},0,0,0)^T$$

在I_2单元上

$$(f,\varphi_1)=\int_{I_2}\varphi_1 dx=\frac{h}{2}\to(1,1)$$

$$(f,\varphi_2)=\int_{I_2}\varphi_2 dx=\frac{h}{2}\to(2,1)$$

于是

$$F_1\to F_2=(\frac{h}{2},\frac{h}{2}+\frac{h}{2},\frac{h}{2},0,0)^T$$

- 在I_3单元上

$$(f,\varphi_2)=\int_{I_3}\varphi_2 dx=\frac{h}{2}\to(2,1)$$

$$(f,\varphi_3)=\int_{I_3}\varphi_3 dx=\frac{h}{2}\to(3,1)$$

于是

$$F_2\to F_3=(\frac{h}{2},\frac{h}{2}+\frac{h}{2},\frac{h}{2}+\frac{h}{2},\frac{h}{2},0)^T$$

在I_4单元上

$$(f,\varphi_3)=\int_{I_4}\varphi_3 dx=\frac{h}{2}\to(3,1)$$

$$(f,\varphi_4)=\int_{I_4}\varphi_4 dx=\frac{h}{2}\to(4,1)$$

于是

$$F_3\to F_4=(\frac{h}{2},\frac{h}{2}+\frac{h}{2},\frac{h}{2}+\frac{h}{2},\frac{h}{2}+\frac{h}{2},\frac{h}{2})^T$$

- 于是此问题的有限元方程组具体为

$$
\begin{pmatrix}
\frac{1}{h} & \frac{1}{-h} & 0 & 0 & 0 \\
\frac{1}{-h} & \frac{1}{h}+\frac{1}{h} & \frac{1}{-h} & 0 & 0 \\
0 & \frac{1}{-h} & \frac{1}{h}+\frac{1}{h} & \frac{1}{-h} & 0 \\
0 & 0 & \frac{1}{-h} & \frac{1}{h}+\frac{1}{h} & \frac{1}{-h} \\
0 & 0 & 0 & \frac{1}{-h} & \frac{1}{h}
\end{pmatrix}
\begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}
=
\begin{pmatrix} \frac{h}{2} \\ h \\ h \\ h \\ \frac{h}{2} \end{pmatrix}
$$

下面做边界处理

将A和F中第一行中的对角线上的元素变为1，其余的变为0，即

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
\frac{1}{-h} & \frac{1}{h}+\frac{1}{h} & \frac{1}{-h} & 0 & 0 \\
0 & \frac{1}{-h} & \frac{1}{h}+\frac{1}{h} & \frac{1}{-h} & 0 \\
0 & 0 & \frac{1}{-h} & \frac{1}{h}+\frac{1}{h} & \frac{1}{-h} \\
0 & 0 & 0 & \frac{1}{-h} & \frac{1}{h}
\end{pmatrix}
\begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}
=
\begin{pmatrix} 0 \\ h \\ h \\ h \\ \frac{h}{2} \end{pmatrix}
$$

- 再处理A第一列中的元素除对角线变为1，其余的变为0，即

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{h}+\frac{1}{h} & \frac{1}{-h} & 0 & 0 \\ 0 & \frac{1}{-h} & \frac{1}{h}+\frac{1}{h} & \frac{1}{-h} & 0 \\ 0 & 0 & \frac{1}{-h} & \frac{1}{h}+\frac{1}{h} & \frac{1}{-h} \\ 0 & 0 & 0 & \frac{1}{-h} & \frac{1}{h} \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 0 \\ h+\frac{1}{h}u_0 \\ h \\ h \\ \frac{h}{2} \end{pmatrix}$$

再将A和F中最后一行中的对角线上的元素变为1，其余的变为0，即

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{h}+\frac{1}{h} & \frac{1}{-h} & 0 & 0 \\ 0 & \frac{1}{-h} & \frac{1}{h}+\frac{1}{h} & \frac{1}{-h} & 0 \\ 0 & 0 & \frac{1}{-h} & \frac{1}{h}+\frac{1}{h} & \frac{1}{-h} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 0 \\ h+\frac{1}{h}u_0 \\ h \\ h \\ 0 \end{pmatrix}$$

- 再处理A最后一列中的元素除对角线变为1，其余的变为0，即

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{h}+\frac{1}{h} & \frac{1}{-h} & 0 & 0 \\ 0 & \frac{1}{-h} & \frac{1}{h}+\frac{1}{h} & \frac{1}{-h} & 0 \\ 0 & 0 & \frac{1}{-h} & \frac{1}{h}+\frac{1}{h} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 0 \\ h+\frac{1}{h}u_0 \\ h \\ h+\frac{1}{h}u_4 \\ 0 \end{pmatrix}$$

从上述线性方程组中，解出u_0, u_1, u_2, u_3, u_4.

则得到原方程的数值解为

$$u_h = \sum_{i=1}^{4} u_i \varphi_i(x)$$

此时可以直接删除掉第一行第一列和第n行第n列，然后再去计算