EQUIVALENCE OF WEAK GALERKIN METHODS AND VIRTUAL ELEMENT METHODS FOR ELLIPTIC EQUATIONS

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ABSTRACT. We propose a modification of the weak Galerkin methods and show its equivalence to a new version of virtual element methods. We also show the original weak Galerkin method is equivalent to the non-conforming virtual element method. As a consequence, ideas and techniques used for one method can be transferred to another. The key of the connection is the degree of freedoms.

1. INTRODUCTION

Recently several numerical methods [3, 5, 6, 2] have been developed for polygon and polyhedral meshes. In this paper, we shall discuss the connection between weak Galerkin (WG) methods and virtual element methods (VEM). We show that, for diffusion equations, a modified WG can be derived from a new version of VEM or equivalently a new version of VEM can be derived from WG. In the same sprit, we will show the original version of WG [7] is equivalent to the non-conforming VEM [4]. The equivalence enables us to apply the convergence theory, as well as computer implementation, of VEM to WG and vice verse. It should also help in giving new insights for each method when applied to other equations.

The key of the connection is the degree of freedoms (d.o.f.). VEM space can be embedded into WG space through the degree of freedoms. Actually WG space contains more d.o.f. but some can be eliminated locally and contribute to the stabilization term only. Essentially WG and VEM share the same degree of freedoms but the associated functions are different. In VEM, the shape functions are determined by a suitable PDE inside each element which is in general non-polynomial. The point-wise information of the shape functions, however, is not needed in the computation which leads to the name 'virtual element' . The resulting space is conforming. In WG, inside each element, the shape functions are simply polynomials. These functions are totally discontinuous across elements. The continuity is imposed by the stabilization through a suitable boundary integral defined on the boundary of elements. Namely, in WG, we know more on the shape function by sacrificing the continuity. The violation of the conformity is under control as we will show WG will also pass the patch test and recast in terms of VEM.

The rest of this paper is organized as follows. We first present the two methods following the presentation in the literature. We then give a modified form for each of the methods and then show they are identical. We will try to stick to the notation used in each method such that the readers who are familiar with one or another can easily follow. In particular, we use letter V for spaces in VEM and W for those in WG.

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2. Function Spaces

We consider a two dimensional domain Ω which is decomposed into a polygon mesh $\mathcal{T}_h$. Each element is a simple polygon and denoted by K. We use two dimensional case for the clear illustration and will talk about the generalization to high dimensions afterwards.

2.1. Spaces in Virtual Element Methods.

We introduce the following space on K

$$V_k(K) := \{v \in H^1(K) : v|_{\partial K} \in \mathbb{B}_k(\partial K), \Delta v \in \mathbb{P}_{k-2}(K)\}, \tag{1}$$

where $\mathbb{P}_k(D)$ is the space of polynomials of degree $\leq k$ on D and conventionally $\mathbb{P}_{-1}(D) = 0$, and the boundary space

$$\mathbb{B}_k(\partial K) := \{v \in C^0(\partial K) : v|_e \in \mathbb{P}_k(e) \text{ for all edges } e \subset \partial K\}.$$

The shape function is defined in (1) but the point-wise value of a function $v \in V_k(K)$ requires solving a PDE inside K and thus considered as implicitly defined not explicitly known. The novelty of VEM is that not the point-wise value but only the d.o.f. is enough to produce an accurate and stable numerical method.

Consider the dual space

$$\mathcal{X}_k(K) = \operatorname{span}\{\boldsymbol{\chi}_v, \boldsymbol{\chi}_e^{k-2}, \boldsymbol{\chi}_K^{k-2}\}, \tag{2}$$

where the functional vectors are

- $\boldsymbol{\chi}_v$: the values of v at the vertices of K;
- $\boldsymbol{\chi}_e^{k-2}$: the moments on edges up to degree $k-2$

$$\chi_e(v) = |e|^{-1}(m, v)_e \quad \forall m \in \mathbb{M}_{k-2}(e), \forall \text{ edge } e \subset \partial K;$$

- $\boldsymbol{\chi}_K^{k-2}$: the moments on element up to degree $k-2$

$$\chi_K(v) = |K|^{-1}(m, v)_K \quad \forall m \in \mathbb{M}_{k-2}(K).$$

Here the scaled monomial $\mathbb{M}_l(D)$ on a d-dimensional domain D is defined as

$$\mathbb{M}_l(D) = \left\{ \left(\frac{\boldsymbol{x} - \boldsymbol{x}_c}{h_D} \right)^{\boldsymbol{s}}, |\boldsymbol{s}| \leq l \right\}$$

with $\boldsymbol{x}_c$ the centroid of D and h_D the diameter of D or $h_D = |D|^{1/d}$.

The verification

$$(V_k(K))' = \mathcal{X}_k(K), \tag{3}$$

is called unisovlence and has been estabilished in [2]. Here we give a simpler proof as follows.

First it is easy to verify the dimensions matches, i.e., $\dim V_k = \dim \mathcal{X}_k$. Therefore it suffices to verify the uniqueness: for $v \in V_k$, if $\chi(v) = 0$ for all $\chi \in \mathcal{X}_k$, then $v = 0$. For $v \in V_k(K) \cap H_0^1(K)$, we apply the integration by parts to conclude

$$(\nabla v, \nabla v) = (v, -\Delta v) = (Q_{k-2} v, -\Delta v),$$

where $Q_{k-2}(K)$ is the L^2-projection of v onto $\mathbb{P}_{k-2}(K)$. The last identity is due to the requirement $\Delta v \in \mathbb{P}_{k-2}(K)$. Now the condition $\chi(v) = 0$ for all $\chi \in \mathcal{X}_k$ implies that $v|_{\partial K} = 0$ and $Q_{k-2}(K)v = 0$. Therefore $v \in H_0^1(K)$ and $\|\nabla v\| = 0$ which implies $v = 0$.

Remark 2.1. The operator Δ used in the definition of VEM space (1) can be replaced by other operators as long as $\mathbb{P}_k(K) \subset V_k(K)$, which ensures the approximation property of $V_k(K)$, and the unisolvence. For example, when K is triangulated into a triangulation $\mathcal{T}_K$, we can chose the standard k-th order Lagrange space on $\mathcal{T}_K$ and impose $\Delta_h v \in \mathbb{P}_{k-2}(K)$ where Δ_h is the standard Galerkin discretization of Δ on this virtual triangulation $\mathcal{T}_K$. □

We relabel the d.o.f. by a single index. Associated to each d.o.f., there exists a basis of $V_k(K)$ such that $\chi_i(\phi_j) = \delta_{ij}$ for $i, j = 1, \ldots, N_k = \dim V_k(K)$. Then every function $v \in V_k(K)$ can be expanded as

$$v = \sum_{i=1}^{N_k} \chi_i(v)\phi_i$$

and in numerical computation it can be identified to the vector $\boldsymbol{v} \in \mathbb{R}^{N_k}$

$$\boldsymbol{v} = (\chi_1(v), \chi_2(v), \ldots, \chi_{N_k}(v)).$$

The isomorphism can be denoted by

$$\boldsymbol{\chi} : V_k(K) \to \mathbb{R}^{N_k}, \quad \boldsymbol{\chi}(v) = (\chi_1(v), \chi_2(v), \ldots, \chi_{N_k}(v)).$$

The inverse of this isomorphism will be denoted by

$$\Phi : \mathbb{R}^{N_k} \to V_k(K), \quad \Phi(\boldsymbol{v}) = \boldsymbol{\phi} \cdot \boldsymbol{v},$$

if we treat the basis $\boldsymbol{\phi} = \{\phi_1, \phi_2, \ldots, \phi_{N_k}\}$ as a vector.

Later on, to compute the L^2-projection of the shape function, the authors of [1] introduce a larger space

$$\widetilde{V}_k(K) = \{v \in H^1(K) : v|_{\partial K} \in \mathbb{B}_k(\partial K), \Delta v \in \mathbb{P}_k(K)\} \tag{4}$$

and a subspace isomporphism to $V_k(K)$

$$V_k^{\mathrm{R}}(K) = \{w \in \widetilde{V}_k(K) : (w - \Pi_k^{\nabla} w, q^*)_K = 0 \quad \forall q^* \in \mathbb{M}_k(K) \backslash \mathbb{M}_{k-2}(K)\}, \tag{5}$$

where the projection Π_k^{∇} will be defined in the next section.

The dual space $(\widetilde{V}_k(K))'$ is enlarged from $(V_k(K))'$ by including volume moments of degree $k-1$ and k. The spaces $V_k(K)$ and $V_k^{\mathrm{R}}(K)$ are different but share the same d.o.f. For the same vector $\boldsymbol{v} \in \mathbb{R}^{N_k}$, we can then have different functions associated to them and in general $\Phi_V(\boldsymbol{v}) \neq \Phi_{V^{\mathrm{R}}}(\boldsymbol{v})$.

Function spaces in each element will be used to design a virtual element space on the whole domain. Given a polygon mesh $\mathcal{T}_h$ of Ω and a given integer $k \geq 1$, we define

$$\begin{aligned} V_h &= \{v \in H^1(\Omega) : v|_K \in V_k(K) \quad \forall K \subset \mathcal{T}_h\}, \\ \widetilde{V}_h &= \{v \in H^1(\Omega) : v|_K \in \widetilde{V}_k(K) \quad \forall K \subset \mathcal{T}_h\}, \\ V_h^{\mathrm{R}} &= \{v \in H^1(\Omega) : v|_K \in V_k^{\mathrm{R}}(K) \quad \forall K \subset \mathcal{T}_h\}. \end{aligned}$$

The d.o.f. can be defined for the global space in the natural way.

For pure diffusion problem, V_h is enough. The introduce of V_h^{R} and $\widetilde{V}_h$ will be helpful to deal with low oder terms, e.g., reaction-diffusion problem, and simplify the implementation in three dimensions.

2.2. **Spaces in Weak Galerkin Methods.** We first introduce the weak function space

$$W(K) := \{v = \{v_0, v_b\} : v_0 \in L^2(K), v_b \in L^2(\partial K)\}. \tag{6}$$

and, for $k \geq 1$, the weak Galerkin finite element space

$$W_k(K) := \{v = \{v_0, v_b\} : v_0 \in \mathbb{P}_k(K), v_b \in \mathbb{B}_k(\partial K)\}.$$

We further split the space as the summation of boundary and interior part

$$W_k(K) = W_0(K) + W_b(K),$$

where

$$W_0(K) = \{v \in W_k(K) \,|\, v = \{v_0, 0\}\},$$
$$W_b(K) = \{v \in W_k(K) \,|\, v = \{0, v_b\}\}.$$

For any function $v = \{v_0, v_b\} \in W_k(K)$, it is convenient to extend the notation of v_0 and v_b so that, without ambiguity, $v_0 \in W_0(K)$ and $v_b \in W_b(K)$.

The space $H^1(K)$ can be embed into the weak space $W(K)$ and project onto $W_k(K)$ by the L^2-projection as

$$Q_h u := \{Q_0 u, Q_b u\},$$

where $Q_0 : L^2(K) \to \mathbb{P}_k(K)$ and $Q_b : L^2(\partial K) \to \mathbb{B}_k(\partial K)$ are L^2-projections. The projection Q_h will preserve $\mathbb{P}_k(K)$ in the sense that

$$Q_h u_k = \{u_k, u_k|_{\partial K}\} \quad \text{for all } u_k \in \mathbb{P}_k(K).$$

We will simply write as $\mathbb{P}_k(K) \subset W_k(K)$.

The global weak Galerkin finite element, for a given polygon mesh $\mathcal{T}_h$ and $k \geq 1$, is defined as

$$W_h := \{v \in W : v = \{v_0, v_b\}, v|_K \in W_k(K)\ \forall K \subset \mathcal{T}_h\}. \tag{7}$$

Remark 2.2. In the original WG space, the boundary part is defined as $v_b|_e \in \mathbb{P}_k(e), \forall e \subset \partial K$ and can be further reduced to $v_b|_e \in \mathbb{P}_{k-1}(e), \forall e \subset \partial K$. That is v_b is discontinuous at vertices. In (7), v_b is continuous on the skeleton of $\mathcal{T}_h$. We will come back to the original version of WG space and show the connection with the non-conforming VEM.

Obviously we can identify the dual space

$$(W_k(K))' = (\widetilde{V}_k(K))' = \operatorname{span}\{\boldsymbol{\chi}_v, \boldsymbol{\chi}_e^{k-2}, \boldsymbol{\chi}_K^k\} = \operatorname{span}\{\boldsymbol{\chi}_b, \boldsymbol{\chi}_0\},$$

which introduces an isomorphism

$$\mathbb{I}_{\widetilde{V} \to W} := \Phi_{\mathrm{W}} \circ \boldsymbol{\chi} : \widetilde{V}_h \to W_h.$$

The space V_h can be embed into $\widetilde{V}_h$ by the zero extension of corresponding vectors. That is for a $v \in V_h$, we can identify it as a function in $\widetilde{V}_h$ with the same d.o.f. except the volume momentum of order $k-1$ and k are zero. Consequently $\mathbb{I}_{\widetilde{V} \to W}|_{V_h}$ leads to an embedding of $\mathbb{I}_{V \to W} : V_h \hookrightarrow W_h$ and will be abbreviated as $\mathbb{I}$. Without ambiguity, the isomorphism $\mathbb{I}_{W \to \tilde{V}}$ will be also abbreviated as $\mathbb{I}$.

In the implementation, the practitioners will use the same vector and the same inner product in Euclidean spaces. In the continuous level, however, the same vector could link to different functions in different spaces and thus leads to different interpretation.

Remark 2.3. The WG space W_h corresponds to a larger VEM space $\widetilde{V}_h$. The space V_h and V_h^{R} will be isomorphism to a reduced space W_h^{R} to be discussed later.

To be consistent to the WG space, we classify the d.o.f. to boundary part $\boldsymbol{\chi}_b$ and interior part $\boldsymbol{\chi}_0$. Note that $\boldsymbol{\chi}_b(v_0) = \boldsymbol{\chi}_b(\{v_0, 0\}) = 0$ but $\boldsymbol{\chi}_b(v_0|_{\partial K}) \neq 0$.

3. METHODS

We use the simplest Poisson equation for a clear illustration. The equivalence of VEM and WG will hold for general diffusion equations. We will also comment on their difference which lies in the treatment of the low order terms.

Consider the Poisson equation with zero Dirichlet boundary condition:

$$-\Delta u = f \text{ in } \Omega, \quad u|_{\partial\Omega} = 0.$$

The weak formulation is: find $u \in H_0^1(\Omega)$ such that

$$a(u,v) := (\nabla u, \nabla v) = (f,v) \quad \forall v \in H_0^1(\Omega).$$

3.1. **Stiffness Matrix.** We can chose a conforming virtual finite element space $V_h^0 := V_h \cap H_0^1(\Omega)$. However, we cannot compute the Galerkin projection to V_h^0 since the traditional way of computing $a(u_h, v_h)$ using numerical quadrature requires point-wise information of functions and/or their gradient inside each element. In virtual element methods, only d.o.f is used to assemble a stiffness matrix having certain approximation property.

Define $\Pi_k^\nabla : V_k(K) \to \mathbb{P}_k(K)$ as the local H^1 projection: given $v \in V_k(K)$, define $\Pi_k^\nabla v \in \mathbb{P}_k(K)$ and $\sum_{i=1}^{N_k} \chi_i(v - \Pi_k^\nabla v) = 0$ such that

$$(\nabla \Pi_k^\nabla v, \nabla p)_K = (\nabla v, \nabla p)_K, \quad \text{for all } p \in \mathbb{P}_k(K).$$

The right hand side can be written as

$$(\nabla v, \nabla p)_K = -(v, \Delta p)_K + \langle v, n \cdot \nabla p \rangle_{\partial K}$$

and can be computed using d.o.f. of v since, for $p \in \mathbb{P}_k(K)$, $\Delta p \in \mathbb{P}_{k-2}(K)$ and $\nabla p \cdot n \in \mathbb{P}_{k-1}(e), e \in \partial K$.

The stiffness matrix of the virtual element method is defined as

$$a^{\mathrm{VEM}}(u,v) := a(\Pi_k^\nabla u, \Pi_k^\nabla v).$$

Remark 3.1. As $\nabla p \cdot n \in \mathbb{P}_{k-1}(e), e \in \partial K$, if we do not enforce the continuity of $v|_{\partial K}$, we could discard the d.o.f. on vertices and use d.o.f. of the edge moments up to order $k-1$ which leads to the non-conforming VEM.

Remark 3.2. The boundary integral $\langle v, n \cdot \nabla p \rangle_e$ can be computed more conveniently if we use Gauss-Lobatto points as the d.o.f. on edges.

We now turn to the weak Galerkin finite element method. Define the weak gradient $\nabla_w^K : W_k(K) \to \nabla \mathbb{P}_k(K)$ as: given $v = \{v_0, v_b\} \in W_k(K)$, define $\nabla_w^K v \in \nabla \mathbb{P}_k(K)$ such that

$$(\nabla_w^K v, \nabla p)_K = -(v_0, \Delta p)_K + \langle v_b, n \cdot \nabla p \rangle_{\partial K} \quad \text{for all } p \in \mathbb{P}_k(K). \tag{8}$$

And the weak gradient $\nabla_w : W_h \to \cup_{K \subset \mathcal{T}_h} \mathbb{P}_k(K)$ as $(\nabla_w v)|_K = \nabla_w^K v$ for all $v \in W_h$.

The stiffness matrix of WG is defined as

$$a^{\mathrm{WG}}(u,v) := (\nabla_w u, \nabla_w v).$$

Remark 3.3. The original weak gradient introduced in [7, 6] is defined as $\nabla_w^K : W_k(K) \to \mathbb{P}_{k-1}^2(K)$ as

$$(\nabla_w^K v, q) = -(v_0, \nabla \cdot q) + \langle v_b, n \cdot q \rangle_{\partial K} \quad \text{for all } q \in \mathbb{P}_{k-1}^2(K), \tag{9}$$

which requires solving a larger local problem for $k \geq 2$. The size of equation (8) is $(k+1)(k+2)/2 - 1$ while the size of (9) is $k(k+1)$. For higher dimensions, the saving is more dramatic. The current modification is motivated by VEM.

We have the following equivalence of the stiffness matrix used in WG and VEM.

Theorem 3.4. *For $u \in V_h$, we have*

$$\nabla \Pi_k^\nabla u = \nabla_w(\mathbb{I}u), \tag{10}$$

and consequently

$$a^{\mathrm{VEM}}(u,v) = a^{\mathrm{WG}}(\mathbb{I}u, \mathbb{I}v), \quad \textit{for all } u, v \in V_h. \tag{11}$$

Proof. The verification (10) is straightforward since u and $\mathbb{I}u$ share the same d.o.f. □

3.2. **Stabilization.** The stiffness matrix alone will not lead to a stable method. The lack of stability can be easily seen from the VEM formulation. Since $\mathbb{P}_k(K) \subset V_k(K)$ and it is a strict subspace except the case K is a triangle, we may have $a^{\mathrm{VEM}}(v,v) = 0$ when $v \in \ker(\Pi_k^\nabla)/\mathbb{R}$. Namely $a^{\mathrm{VEM}}(\cdot,\cdot)$ alone cannot define an inner product on V_h^0. From the WG point of view, the weak gradient operator ∇_w may have non-trivial kernel other than the trivial constant kernel of the gradient operator.

A stabilization term is added to gain the coercivity. To impose the stability while maintain the accuracy, the following assumptions on the element-wise stabilization term $S_K(\cdot,\cdot)$ are imposed in VEM [2].

- k-consistency: for $p_k \in \mathbb{P}_k(K)$
$$S_K(p_k, v) = 0, \quad \forall v \in V_h.$$
- stability:
$$S_K(\tilde{u}, \tilde{u}) \eqsim a(\tilde{u}, \tilde{u}) \quad \text{for all } \tilde{u} \in (I - \Pi_k^\nabla)V_h.$$

We then define

$$a_h^{\mathrm{VEM}}(u,v) = a(\Pi_k^\nabla u, \Pi_k^\nabla v) + \sum_{K \in \mathcal{T}_h} S_K(u,v).$$

The k-consistency will imply the *Patch Test*, i.e., if $u \in \mathbb{P}_k(\Omega)$, then

$$a(u,v) = a_h^{\mathrm{VEM}}(u,v), \quad \text{for all } v \in V_h.$$

The stability will imply

$$a(u,u) \eqsim a_h^{\mathrm{VEM}}(u,u) \quad \text{for all } u \in V_h.$$

An abstract error estimate of VEM with stabilization satisfying k-consistency and stability is given in [2]. So VEM is in fact a family of schemes different in the choice of stabilization terms.

In the continuous level, one such choice is the scaled L^2-inner product

$$h_K^{-2}(u - \Pi_k^\nabla u, v - \Pi_k^\nabla v)_K$$

The k-consistency is obvious as Π_k^∇ preserve k-th polynomial. The stability is from the first order error estimate of $u - \Pi_k^\nabla u$:

$$h_K^{-1}\|(I - \Pi_k^\nabla)u\|_K \lesssim \|\nabla(u - \Pi_k^\nabla u)\|_K.$$

In the implementation, Π_k^∇ is realized as a matrix and the stabilization can be chosen as

$$S_{\boldsymbol{\chi}}(u,v) := \boldsymbol{\chi}((I - \Pi_k^\nabla)u) \cdot \boldsymbol{\chi}((I - \Pi_k^\nabla)v). \tag{12}$$

That is we use the l^2 product of the d.o.f. vector to approximate the L^2-inner product. The scaling is build into the definition of d.o.f. through the scaling of the monomials. The norm equivalence of l^2 and L^2 is due to the well known fact: the mass matrix is spectral equivalent to its diagonal. So far (12) is *the* stabilization used in the current VEM.

We now discuss the stabilization of WG. For $u, v \in W_k(K)$, we can use

$$h_K^{-1}\langle u_b - u_0, v_b - v_0\rangle_{\partial K}.$$

One particular choice is using the d.o.f associated to boundary of elements only

$$S_{\partial K}(u,v) := \boldsymbol{\chi}_b(u_b - u_0)\cdot\boldsymbol{\chi}_b(v_b - v_0). \tag{13}$$

We then define

$$a_h^{\mathrm{WG}}(u,v) = (\nabla_w u, \nabla_w v) + \sum_{K\subset\mathcal{T}_h} S_{\partial K}(u,v).$$

Lemma 3.5. *If $a_h^{\mathrm{WG}}(u,u) = 0$, then both u_0 and u_b are globally constant and $u_0|_{\mathcal{E}_h} = u_b$.*

Proof. $a_h^{\mathrm{WG}}(u,u) = 0$ implies $S_{\partial K}(u,u) = 0$ that is $u_0|_{\partial K} = u_b$. We now prove $\nabla u_0 = 0$ from the fact $\nabla_w u = 0$ as follows

$$\begin{aligned}(\nabla u_0, \nabla u_0)_T &= -(u_0, \nabla\cdot\nabla u_0)_T + \langle u_0, n\cdot\nabla u_0\rangle_{\partial T}\\ &= -(u_0, \nabla\cdot\nabla u_0)_T + \langle u_b, n\cdot\nabla u_0\rangle_{\partial T}\\ &= (\nabla_w u, \nabla u_0)\\ &= 0.\end{aligned}$$

Therefore u_0 is constant in K. So is u_b. They are globally constant since $u_0|_{\partial K} = u_b$. □

The constant kernel can be further eliminated by imposing the boundary condition into the space by defining:

$$W_h^0 = \{v \in W_h : v_b|\partial\Omega = 0\}.$$

When $u \in \mathbb{P}_k(K)$, $u_0|_{\partial K} = u_b$. We thus also have the patch test, i.e., if $u \in \mathbb{P}_k(\Omega)$, then

$$a(u,v) = a_h^{\mathrm{WG}}(u,v) \quad \text{for all } v \in H^1(K).$$

The stability of WG is, however, ensured by a different mechanism: the continuity across the boundary of elements.

3.3. **Right hand side.** In WG, it is straightforward as v_0 represents the function inside element and thus only v_0 contributes to the L^2 inner product. So we simply compute (f, v_0).

In VEM, we have to resort to the d.o.f. For $k \geq 2$, define Q_{k-2}^K as the $L^2(K)$-projection to $\mathbb{P}_{k-2}(K)$. Then the exact inner product (f,v) is approximated by

$$\langle f_h, v\rangle = \sum_{K\subset\mathcal{T}_h} (Q_{k-2}^K f, v)_K,$$

which can be computed using d.o.f. of $v \in V_k(K)$. For $k = 1$, we use

$$(Q_0^K f, \bar{v})_K, \tag{14}$$

where $\bar{v} = \mathrm{mean}(\boldsymbol{\chi}_v(v))$. To achieve optimal order error estimate in L^2-norm, more accurate approximation of the right-hand side (RHS) is needed for $k = 1, 2$.

3.4. **Equivalence.** The VEM method is: find $u_h \in V_h^0$ such that

$$(\nabla \Pi_k^\nabla u_h, \nabla \Pi_k^\nabla v_h) + \sum_{K\in\mathcal{T}_h} S_K(u_h, v_h) = \langle f_h, v_h\rangle \quad \text{for all } v_h \in V_h^0.$$

The modified WG method is: find $u_h \in W_h^0$ such that

$$(\nabla_w u_h, \nabla_w v_h) + \sum_{K\subset\mathcal{T}_h} S_{\partial K}(u_0 - u_b, v_0 - v_b) = (f, v_0), \quad \text{for all } v_h \in W_h^0. \tag{15}$$

We shall propose a new version of VEM on the larger space $\widetilde{V}_h$ and then show that it is exactly WG. The new version of VEM is obtained by using a new projector $\Pi_k : \widetilde{V}_k(K) \to \mathbb{P}_k(K)$ in the stabilization term.

$$\Pi_k : \widetilde{V}_k(K) \to \mathbb{P}_k(K), \quad \boldsymbol{\chi}_K^k(v) = \boldsymbol{\chi}_K^k(\Pi_k v). \tag{16}$$

Note that in $V_k(K)$ only the momentum of oder $k-2$ is defined. To define a function in $\mathbb{P}_k(K)$, we need to work in the larger space $\widetilde{V}_k(K)$. Obviously $\Pi_k p_k = p_k$ if $p_k \in \mathbb{P}_k(K)$.

We propose the following stabilization

$$\widetilde{S}(u, v) := \boldsymbol{\chi}(u - \Pi_k u) \cdot \boldsymbol{\chi}(v - \Pi_k v). \tag{17}$$

Obviously $\widetilde{S}(\cdot,\cdot)$ is k-consistent since Π_k preserves the k-th polynomial. We prove the stability as follows

$$\widetilde{S}(u,u) \eqsim h_K^{-2}\|(I-\Pi_k)u\|^2 = h_K^{-2}\|(I-\Pi_k)(I-\Pi_k^\nabla)u\|^2 \lesssim \|\nabla(I-\Pi_k^\nabla)u\|^2.$$

The corresponding VEM method is: find $u_h \in \widetilde{V}_h^0$ such that

$$(\nabla \Pi_k^\nabla u_h, \nabla \Pi_k^\nabla v_h) + \widetilde{S}(u_h, v_h) = (f, \Pi_k v_h) \quad \text{for all } v_h \in \widetilde{V}_h^0. \tag{18}$$

We now show its equivalence with WG. First of all, by Theorem 3.4, they share the same stiffness matrix. We then look at the stabilization term. By the definition of the projector Π_k, $\boldsymbol{\chi}_K^k(u - \Pi_k u) = 0$. The stabilization (17) can be simplified to boundary terms only

$$\widetilde{S}(u, v) = \boldsymbol{\chi}_b(u - \Pi_k u) \cdot \boldsymbol{\chi}_b(v - \Pi_k v)$$

Given a function $u = \{u_0, u_b\} \in W_k(K)$, we use d.o.f. to identify u as a function $\mathbb{I}u \in \widetilde{V}_k(K)$. Note that $\mathbb{I}u_0 \neq u_0$ since $\mathbb{I}u_0 \in \widetilde{V}_k(K)$ is zero on ∂K and in general non-polynomial while $u_0 \in \mathbb{P}_k(K)$. What is u_0? It is $\Pi_k(\mathbb{I}u)$. We then have the equivalence

$$\boldsymbol{\chi}_b(u_b - u_0|_{\partial K}) \cdot \boldsymbol{\chi}_b(v_b - v_0|_{\partial K}) = \boldsymbol{\chi}_b(\mathbb{I}u - \Pi_k(\mathbb{I}u)) \cdot \boldsymbol{\chi}_b(\mathbb{I}v - \Pi_k(\mathbb{I}v))$$

Finally the right-hand side in (18) is related to WG as

$$(f, \Pi_k \mathbb{I} v) = (f, v_0).$$

We summarize as the following theorem.

Theorem 3.6. *The modified WG method* (15) *is equivalent to the VEM method* (18).

The equivalence is, however, established for W_h and $\widetilde{V}_h$. We can further eliminate some d.o.f. in W_h to show the equivalence to V_h. To this end, for $k \geq 2$, we split the interior part of a weak function $u_0 = u_1 + u_2$ such that $u_1 \in \mathbb{P}_{k-2}(K)$ and $\boldsymbol{\chi}_K^{k-2}(u_0) = \boldsymbol{\chi}_K^{k-2}(u_1)$ for all the volume momentum up to order $k-2$. In other words $u_2 \in^\circ V_k'(K)$, the annihilator of the dual space $V_k'(K)$ and thus in the volume integral $(u_0, \Delta p) = (u_1, \Delta p)$ for all $p \in \mathbb{P}_k(K)$. By the definition of the weak Gradient, we have $\nabla_w(\{u_b, u_0\}) = \nabla_w(\{u_b, u_1\})$.

Namely u_2 does not contribute to the stiffness matrix. We can then determine u_2 by solving the equation of stabilization term only

$$\boldsymbol{\chi}_b(u_b - u_1|_{\partial K} - u_2|_{\partial K}) \cdot \boldsymbol{\chi}_b(v_2|_{\partial K}) = 0, \quad \text{for all } v_2 \in^\circ V_k'(K).$$

If we introduce $D_1 : \mathbb{P}_{k-2} \to \mathbb{R}^{N_b}$ and $D_2 :^\circ V_k'(K) \to \mathbb{R}^{N_b}$ as the matrix representation of $\boldsymbol{\chi}_b$ applied to u_1 and u_2 respectively, then the matrix representation of $S_{\partial K}$ is

$$\begin{pmatrix} I & -D_1 & -D_2 \\ -D_1^T & D_1^T D_1 & D_1^T D_2 \\ -D_2^T & D_2^T D_1 & D_2^T D_2 \end{pmatrix} = \begin{pmatrix} I \\ -D_1^T \\ -D_2^T \end{pmatrix} \begin{pmatrix} I & -D_1 & -D_2 \end{pmatrix}.$$

After the elimination of u_2, the stabilization matrix for the reduced space is

$$\begin{pmatrix} P & -PD_1 \\ -D_1^T P & D_1^T P D_1 \end{pmatrix} = \begin{pmatrix} P \\ -D_1^T P \end{pmatrix} \begin{pmatrix} P & -PD_1^T \end{pmatrix}.$$

where $P = I - D_2(D_2^T D_2)^{-1} D_2^T$ is the projection to the $\ker(D_2^T)$.

Remark 3.7. Here we eliminate u_2 to show the equivalence. In practice, we can chose convenient basis to expand u_0 (e.g. monomials) and eliminate the whole u_0 element-wise. The resulting global system will involve u_b only. Of course, such reduction can be also applied to VEM and known as condensation in finite element methods.

For the right hand side (f, v), for $k \geq 2$, we could write RHS of WG as (f_1, f_2) with $f_i = (f, u_i), i = 1, 2$. To achieve optimal order in H^1 type norm, we can discard f_2 and only keep $(f_1, 0)$. Then the reduced system will have identical RHS as VEM.

3.5. **Difference.** The difference of WG and VEM lies in the treatment of low order terms. We illustrate it by considering the RHS. As we mentioned before, to achieve optimal order in H^1 norm, we only need f_1 and discard f_2. To achieve optimal order in L^2 norm, we do need more. Let us keep both f_1 and f_2 in WG. We then solve

$$\boldsymbol{\chi}_b(u_b - u_1|_{\partial K} - u_2|_{\partial K}) \cdot \boldsymbol{\chi}_b(v_2|_{\partial K}) = f_2, \quad \text{for all } v_2 \in^\circ V_k'(K)$$

to eliminate u_2. After the elimination of u_2, the RHS of the reduced system is modified to $(f_0, \tilde{f}_1)$ with $f_0 = D_2(D_2^T D_2)^{-1} f_2$ and $\tilde{f}_1 = f_1 - D_1^T D_2 (D_2^T D_2)^{-1} f_2$. In VEM, a L^2 projection operator Π_k^0 is introduced [1] and a perturbed RHS is computed using Π_k^0 which is different with the current formulation.

For $k = 1$, the reduced system of WG is in the form

$$(A + P)\boldsymbol{u} = \boldsymbol{f},$$

with $f = D_2(D_2^T D_2)^{-1} f_2$. Since the optimal L^2-error estimate, for $k \geq 1$, has been proved for WG or equivalently can be easily proved for the VEM (18) on $\widetilde{V}_h^0$, we obtain a VEM on V_h^0 with optimal order of convergence in both H^1 and L^2 norms.

In other words, we can view a version of VEM as the Schur complement equation of WG by eliminating high momentum term and apply the convergence theory developed for WG to VEM.

For elliptic equations with low order terms, e.g., the reaction-diffusion equation, the L^2-inner product of functions in WG is computed as

$$(u_0, v_0) = (\Pi_k(\mathbb{I} u_0), \Pi_k(\mathbb{I} v_0)).$$

While in VEM, it is computed as

$$(\Pi_k^0 u, \Pi_k^0 v),$$

where Π_k^0 is defined in [1] and can be computed using Π_k^∇. In WG, however, it is more natural to use u_0.

4. Weak Galerkin Methods and Non-conforming VEM

In this section we review the original version of weak Galerkin methods [7, 6] and show the equivalence to non-conforming VEM [4].

Given a polygon mesh $\mathcal{T}_h$ and an integer $k \geq 1$, we introduce

$$W_h = \{v = \{v_0, v_b\}, v_0|_K \in \mathbb{P}_k(K), v_b|_e \in \mathbb{P}_{k-1}(e)\ \forall e \subset K, \forall K \subset \mathcal{T}_h\},$$
$$W_h^0 = \{v \in W_h, v_b|_{\partial\Omega} = 0\}.$$

As we mentioned before, now v_b is also discontinuous at vertices. The d.o.f. $\boldsymbol{\chi}_v$ can be dropped and the edge d.o.f. is upgraded to the edge momentum to the order $k-1$. The interior part v_0 will be still determined by volume momentum to the order k. We list the d.o.f. below

$$\boldsymbol{\chi} = \{\boldsymbol{\chi}_b, \boldsymbol{\chi}_0\} := \{\boldsymbol{\chi}_e^{k-1}, \boldsymbol{\chi}_K^k\}.$$

The weak gradient ∇_w to $\nabla\mathbb{P}_k$ is defined as before. The stabilization is changed to

$$h_K^{-1}\langle u_b - Q_b u_0, v_b - Q_b v_0\rangle_{\partial K},$$

where recall that Q_b is the L^2-projection from $L^2(e)$ to $\mathbb{P}_{k-1}(e)$. A more convenient formula is

$$\boldsymbol{\chi}_b(u_b - u_0|_{\partial K}) \cdot \boldsymbol{\chi}_b(v_b - v_0|_{\partial K}).$$

The L^2-projection Q_b is simply replaced by edge d.o.f.

The WG finite element is: find $u_h \in W_h^0$

$$a^{\mathrm{WG}}(u_h, v) = (f, v_0) \quad \forall v = \{v_0, v_b\} \in W_h^0. \tag{19}$$

where

$$a^{\mathrm{WG}}(u_h, v_h) := (\nabla_w u_h, \nabla_w v_h) + \sum_{K\subset\mathcal{T}_h} \boldsymbol{\chi}_b(u_b - u_0|_{\partial K}) \cdot \boldsymbol{\chi}_b(v_b - v_0|_{\partial K}).$$

The non-conforming VEM space is defined as follows. First the local space defined as

$$V_k(K) = \{v \in H^1(K) : \nabla v \cdot n \in \mathbb{P}_{k-1}(e), \forall e \subset \partial K, \Delta v \in \mathbb{P}_{k-2}(K)\}.$$

To glue them to get a non-conforming space of H^1, we define

$$H^{1,nc}(\mathcal{T}_h; k) := \{v \in H^1(K), \forall K \subset \mathcal{T}_h, \int_e vp \,\mathrm{d}s \text{ is continuous } \forall p \in \mathbb{P}_{k-1}(e), \forall e \in \mathcal{E}_h\}.$$

The global non-conforming VEM space is

$$V_h = \{v_h \in H^{1,nc}(\mathcal{T}_h; k) : v_h|_K \in V_k(K)\ \forall K \in \mathcal{T}_h\},$$
$$V_h^0 = \{v_h \in V_h, \int_e vp \,\mathrm{d}s = 0 \quad \forall p \in \mathbb{P}_{k-1}(e), \forall e \subset \partial\Omega\}.$$

The d.o.f. of non-conforming VEM space will be the momentum on edges up to order $k-1$ and the volume momentum up to order $k-2$. Again we could enlarge to $\widetilde{V}_h$ by asking $\Delta v \in \mathbb{P}_k$ and including volume momentum to the order k in d.o.f.

The non-conforming VEM is: find $u_h \in V_h^0$ such that

$$a_h^{\mathrm{VEM}}(u_h, v_h) = \langle f_h, v_h\rangle \quad \forall v_h \in V_h^0, \tag{20}$$

where

$$a_h^{\mathrm{VEM}}(u, v) := (\nabla\Pi_k^\nabla u, \nabla\Pi_k^\nabla v) + \sum_{K\in\mathcal{T}_h} \boldsymbol{\chi}(u - \Pi_k^\nabla u) \cdot \boldsymbol{\chi}(v - \Pi_k^\nabla v)$$

and for $k \geq 2$

$$\langle f_h, v_h \rangle = (Q_{k-2} f, Q_{k-2} v_h),$$

with a similar treatment for the lowest order case $k = 1$; see, e.g., (14).

Just as before, we could recast WG as a VEM in the larger space $\widetilde{V}_h$. The additional d.o.f. can be eliminated from the stabilization term. The non-conforming VEM is: find $u_h \in \widetilde{V}_h^0$ such that

$$\bar{a}_h^{\mathrm{VEM}}(u_h, v_h) = (f, \Pi_k v_h) \quad \forall v_h \in \widetilde{V}_h^0, \tag{21}$$

where

$$\bar{a}_h^{\mathrm{VEM}}(u, v) := (\nabla \Pi_k^{\nabla} u, \nabla \Pi_k^{\nabla} v) + \sum_{K \in \mathcal{T}_h} \boldsymbol{\chi}_b(u - \Pi_k u) \cdot \boldsymbol{\chi}_b(v - \Pi_k v).$$

The interpolant Π_k is defined as before.

Theorem 4.1. *The modified non-conforming VEM method* (21) *is exactly the original WG method* (19)*. The original non-conforming VEM* (20) *is the Schur complement of WG* (19) *by eliminating the high momentum part.*

Advantage of the original formulation of WG and non-conform VEM is the generality in aribitary dimension while the conforming version requires the extension of vertices valued d.o.f. to polynomials inside faces which now is a lower dimensional polytope.

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