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SUPERCONVERGENCE AND GRADIENT RECOVERY OF LINEAR FINITE ELEMENTS FOR THE LAPLACE–BELTRAMI OPERATOR ON GENERAL SURFACES*

HUAYI WEI†, LONG CHEN‡, AND YUNQING HUANG†

Abstract. Superconvergence results and several gradient recovery methods of finite element methods in flat spaces are generalized to the surface linear finite element method for the Laplace–Beltrami equation on general surfaces with mildly structured triangular meshes. For a large class of practically useful grids, the surface linear finite element solution is proven to be superclose to an interpolant of the exact solution of the Laplace–Beltrami equation, and as a result various postprocessing gradient recovery, including simple and weighted averaging, local and global L^2-projections, and Zienkiewicz and Zhu (Z-Z) schemes are devised and proven to be a better approximation of the true gradient than the gradient of the finite element solution. Numerical experiments are presented to confirm the theoretical results.

Key words. Laplace–Beltrami operator, surface finite element method, superconvergence, gradient recovery method

AMS subject classifications. 65N15, 65N50, 65N30

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1. Introduction.

The Laplace–Beltrami operator is a generalization of the Laplace operator in flat spaces to manifolds. Many partial differential equations (PDEs) on two-dimensional Riemannian manifolds, such as the mean curvature flow [24], surface diffusion flow [33], and Willmore flow [36], etc., are formulated using the Laplace–Beltrami operator. These PDEs are frequently used in diverse applications such as image processing, surface processing, fluid dynamics, weather forecasting, climate modeling, and so on [17, 34, 37]. In this paper, we study the discretization of the Laplace–Beltrami operator by the surface linear finite element method (FEM) [22]. We shall prove a superconvergence result and use it to develop several postprocessing schemes which improve numerical approximations significantly.

Let S be a two-dimensional, compact, and closed C^3-hypersurface in $\mathbb{R}^3$, and $\partial S = \varnothing$. Let f be a given data satisfying $\int_S f \,\mathrm{d}\sigma = 0$, where $\mathrm{d}\sigma$ is the surface measure, and let u be the solution of the Laplace–Beltrami equation

$$-\Delta_S u = f \text{ on } S, \tag{1.1}$$

where Δ_S is the Laplace–Beltrami operator on the surface S. We require $\int_S u \,\mathrm{d}\sigma = 0$ in order to guarantee the uniqueness of the solution to (1.1).

As an effective numerical method for solving PDEs, the FEM plays an important role in modern scientific and engineering computing. In particular, FEMs for PDEs

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†Hunan Key Laboratory for Computation and Simulation in Science and Engineering, School of Mathematics and Computational Science, Xiangtan Uinversity, Xiangtan 411105, Hunan, People's Republic of China (huayiwei1984@gmail.com, huangyq@xtu.edu.cn). The first author was supported in part by the Hunan Provincial Innovation Foundation for Postgraduate CX2009B122 and Hunan Education Department key project 10A117. The third author was supported in part by the NSFC key project 11031006 and Hunan Provincial NSF project 10JJ7001.

‡Department of Mathematics, University of California at Irvine, Irvine, CA 92697 (chenlong@math.uci.edu). This author was supported in part by NSF grant DMS-0811272 and in part by NIH grants P50GM76516 and R01GM75309.

defined on manifolds have been studied in the literatures; see [6, 16, 15, 20, 22, 23, 21, 27, 30, 31] and references therein. We briefly survey results related to our paper below. Dziuk [22] approximated the surface S by a polyhedral surface S_h with triangular faces which avoids a local or global parametrization of S, then discretized (1.1) with the piecewise linear finite element and solved it on S_h. He also proved asymptotic error estimates in [22]. Recently, Demlow [16] defined higher-order analogues to the linear method in [22] and proved a priori error estimates in the L^2, H^1, and corresponding pointwise norms.

Using superconvergence to improve the accuracy of finite element approximations has been an active research topic; see [1, 5, 8, 29, 38, 28]. Again we briefly survey results related to our paper. Bank and Xu [5] developed superconvergence estimates for piecewise linear finite element approximations on meshes in which most pairs of adjacent triangles form $\mathcal{O}(h^2)$ approximate parallelograms. Following the idea in [5], Huang and Xu [28] investigated the superconvergence properties for quadratic triangular elements on such mildly structured grids.

In this paper, we shall generalize theoretical results in [5] to the surface linear finite element. Passing from flat spaces to surfaces, there are two main difficulties. First, the surface S is approximated by a polyhedral surface S_h with a union of triangular faces, and thus an additional error to approximate the geometry is introduced. Second, in general two triangles sharing a common edge are not in the same plane, so we cannot use the techniques developed in [5] directly. To overcome these difficulties, we show that if the surface S is C^3 and the union of the triangular faces of S_h are shape-regular and quasi-uniform of a diameter h, the geometry error from the polyhedral surface approximation is $\mathcal{O}(h^2)$. Then we show that the included angle between the unit outward normal vectors of S_h on any two neighboring triangles is $\mathcal{O}(h)$. Under the $\mathcal{O}(h^{2\sigma})$ irregular grids and C^3 surface assumptions, we show in Theorem 3.5 that

$$\|\nabla_{S_h} u_h - \nabla_{S_h} \bar{u}_I\|_{0,S_h} \le C_1 h^{1+\min\{1,\sigma\}} \left(\|u\|_{3,S} + \|u\|_{2,\infty,S} \right) + C_2 h^2 \|f\|_{0,S}\,, \tag{1.2}$$

where u_h is the finite element approximation of u on S_h and $\bar{u}_I$ is the linear interpolation of $\bar{u}$, the extension of the exact solution u to S_h. In estimate (1.2), we have included with $C_2 h^2 \|f\|_{0,S}$ the effect of the additional approximation error of the geometry.

Gradient recovery is designed to recover a better approximation of the true gradient than the gradient of the finite element solution does [5, 4, 3, 7, 20, 38, 40]. It is used to improve the numerical approximation and supply a posteriori error estimation for adaptive procedure. Bank and Xu [5] introduced a gradient recovery algorithm using the global L^2-projection for $\mathcal{O}(h^2)$ approximated parallelogram meshes and proved the superconvergence of the recovered gradient. In [4], they developed a postprocessing gradient recovery scheme on general unstructured and shape-regular meshes using smoothing operators of the multigrid method. Together with Zheng, they successfully generalized such method to high order FEMs [3]. Xu and Zhang [38] provided a general framework for the analysis of local recovery schemes including simple and weighted averaging, local L^2-projections, and local discrete least-squares fitting (also known as Zienkiewicz and Zhu (Z-Z) recovery [40]). These aforementioned results are for planar meshes. For surfaces meshes, Du and Ju [20] developed a gradient recovery method for a finite volume approximation of linear convection-diffusion equations on spheres and demonstrate numerically the superconvergence of their method.

We shall prove the superconvergence of the simple and weighted averaging gradient recovery methods on polyhedral surface meshes. Due to the curvature of the surface, we cannot use directly the local L^2-projection and local discrete least-squares fitting methods on polyhedral surface meshes. Therefore we project the patch Ω_i onto the tangent plane of S at x_i and modify these local recovery methods accordingly. We show that if the local patch Ω_i of a vertex x_i is $\mathcal{O}(h^2)$-symmetric on S_h, then

$$|(G_h \nabla_{S_h} \bar{u}_I)(x_i) - \nabla_S u(x_i)| \le C h^2 \|u\|_{3,\infty,\mathcal{P}_0(\Omega_i)}, \tag{1.3}$$

where G_h is an appropriate defined recovery operator in section 4 and $\mathcal{P}_0(\Omega_i)$ is the projection of Ω_i on S.

The recovery method using the global L^2-projection Q_h developed by Bank and Xu [5] can be also generalized to the surface linear FEM in a straightforward way. The analysis, however, is not a trivial generalization. Again the difficulty is the error introduced by the surface approximation. Let $w_h = Q_h \nabla_{S_h} u_h$ and $\tilde{w}_h$ be the lift of w_h to the surface. Since the difference between the tangent gradient operator ∇_S on S and ∇_{S_h} on S_h is only $\mathcal{O}(h)$, using the technique in [5], we can get only a first-order approximation, i.e., $\|\nabla_S u - \tilde{w}_h\|_{0,S} \le Ch$. Using a perturbation argument, we develop a new method to prove that if the mesh is $\mathcal{O}(h^{2\sigma})$ irregular,

$$\|\nabla_S u - \tilde{w}_h\|_{0,S} \le C_1 h^{1+\min\{1,\sigma\}} \|u\|_{3,\infty,S} + C_2 h^2 \|f\|_{0,S}. \tag{1.4}$$

Our proof confirms that the symmetry of the element patch of vertices is the source of the superconvergence [35].

By the close relationship between FEM and finite volume methods (FVM), we can use these global and local recovery methods to postprocess the finite volume approximation and get similar superconvergence results. In this way, we give a theoretical justification of superconvergence observed numerically by Du and Ju [20].

We should mention that we assume all vertices of S_h lie on S. In practical applications, the exact surface S is unknown, and S_h is only an approximation of S. We hope that when the distance is $\mathcal{O}(h^2)$ between the vertices of S_h and S, the same results obtained in this paper will still hold, which, of course, deserves further study. Also in the construction of the last four recovery operators, we assume the normal vectors of S is known, which may not be true. In our future work, we will consider how to construct a high order approximation of the normal vectors of S from S_h.

The paper is organized as follows. In section 2, we present some important definitions and preliminaries. In section 3, we discuss the superconvergence between the surface finite element solution and linear interpolation of the true solution. In section 4, we introduce several gradient recovery methods and prove their better approximation property. We also generalize the results to FVM on surfaces. In section 5, we present numerical tests in support of our theoretical results.

We use $x \lesssim y$ to indicate $x \le Cy$ and $x \eqsim y$ for $x \lesssim y$ and $y \lesssim x$.

2. Preliminaries. In this section, at first, we introduce the notation about Sobolev spaces defined on surfaces and then present the weak form of the Laplace–Beltrami equation (1.1) and a regularity result. After that, we discuss the Sobolev spaces on polyhedral surfaces and the relationship between functions defined on the surface and its polyhedral approximation. At the end of this section, we present the surface linear FEM and the corresponding error estimates.

2.1. Sobolev spaces on smooth surfaces. Since S is a closed surface and $\partial S = \varnothing$, it partitions the space $\mathbb{R}^3$ into three distinct sets: points inside the surface, points on the surface, and points outside the surface; denoted by Ω_-, Ω_0, and Ω_+, respectively. For any $x \in \mathbb{R}^3$, let $\operatorname{dist}(x, S) = \min_{y\in S} |x - y|$ be the distance between

x and S, where $|\cdot|$ is the standard Euclidean distance. We can define a strip domain $U := \{x \in \mathbb{R}^3 \,|\, \mathrm{dist}(x,S) < \delta\}$, where $\delta > 0$ and is small enough such that we can define a unique signed distance function $d : U \to \mathbb{R}$, satisfying the following properties: $d \in C^3(U)$; $d(x) < 0$ for all $x \in \Omega_- \cap U$; $d(x) = 0$ for all $x \in \Omega_0 \cap U = S$; $d(x) > 0$ for all $x \in \Omega_+ \cap U$; and $|d(x)| = \mathrm{dist}(x,S)$ for all $x \in U$. We shall view the surface S as the zero level set of the distance function.

Let ∇ be the conventional gradient operator in $\mathbb{R}^3$. So $\nabla d(x) \in \mathbb{R}^3$ is the gradient of $d(x)$ and $\boldsymbol{H}(x) := \nabla^2 d(x) \in \mathbb{R}^{3\times 3}$ is the Hessian matrix of $d(x)$. For any $x \in U$, let y be the closest point of x on S, i.e., $|d(x)| = |x-y|$. Since $d(x)$ is the signed distance function and S is its zero level set, it is easy to show that $\nabla d(x)$ is the unit outward normal vector of S at y, namely, $|\nabla d(x)| = 1$. Then for any $x \in U$, let $\boldsymbol{n}(x) = \nabla d(x)$. We can define the following unique projection $\mathcal{P}_0 : U \to S$:

$$\mathcal{P}_0(x) := x - d(x)\boldsymbol{n}(x). \tag{2.1}$$

For $x \in U$, differentiation on the identity $\nabla d(x) \cdot \nabla d(x) = 1$ implies that

$$\boldsymbol{H}(x)\nabla d(x) = \boldsymbol{H}(x)\boldsymbol{n}(x) = \mathbf{0}.$$

Therefore zero is an eigenvalue of $\boldsymbol{H}(x)$ and $\boldsymbol{n}(x)$ is the corresponding eigenvector. The other two eigenvalues of $\boldsymbol{H}(x)$ are denoted by $\kappa_1(x)$ and $\kappa_2(x)$. When $x \in S$, $\kappa_1(x)$ and $\kappa_2(x)$ are the principal curvatures of S at x.

For $v \in C^1(S)$, since S is C^3, we can extent v to $C^1(U)$ and still denote the extension by v [32]. The tangential gradient of v on S can be written as

$$\nabla_S v = \nabla v - (\nabla v \cdot \boldsymbol{n})\boldsymbol{n} = (\boldsymbol{I} - \boldsymbol{n}\boldsymbol{n}^t)\nabla v = \boldsymbol{P}\nabla v \in \mathbb{R}^3,$$

where $\boldsymbol{P}(x) = (\boldsymbol{I} - \boldsymbol{n}\boldsymbol{n}^t)(x)$ is the projection operator to the tangent plane at a point $x \in S$, and therefore $\boldsymbol{P}^2 = \boldsymbol{P}$. Notice that we use the extension of v to define the surface gradient. However, it can be shown that $\nabla_S v$ depends only on the value of v on S but not on the extension. Namely, ∇_S is an intrinsic operator.

Similarly, for a vector field $\boldsymbol{v} \in (C^1(S))^3$, we can extend it to $(C^1(U))^3$ and define the tangential divergence of $\boldsymbol{v}$ on S as

$$\nabla_S \cdot \boldsymbol{v} = \nabla \cdot \boldsymbol{v} - \boldsymbol{n}^t \nabla \boldsymbol{v} \boldsymbol{n} \in \mathbb{R}.$$

The Laplace–Beltrami operator on S reads as follows:

$$\Delta_S v = \nabla_S \cdot (\nabla_S v) = \Delta v - (\nabla v \cdot \boldsymbol{n})(\nabla \cdot \boldsymbol{n}) - \boldsymbol{n}^t \nabla^2 v \boldsymbol{n} \in \mathbb{R},$$

provided $v \in C^2(S)$ and $\nabla^2 v$ is the Hessian matrix of v (suitably extended as a $C^2(U)$ function).

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}_+^3$ be a vector of nonnegative integers, and $|\alpha| = \sum_{i=1}^3 \alpha_i$. The $|\alpha|$th tangential derivatives of u on S, $D_S^\alpha u$ can be defined recursively from the above definition of the tangential gradient. We introduce the Sobolev spaces

$$W_p^m(S) := \{u \in L^p(S) \,|\, D_S^\alpha u \in L^p(S), |\alpha| \le m\},$$

where $1 \le p \le \infty$ and m is a nonnegative integer. For $1 \le p < \infty$, the space $W_p^m(S)$ is equipped with the norm

$$\|u\|_{m,p,S} := \left(\sum_{|\alpha| \le m} \|D_S^\alpha u\|_{L^p(S)}^p \right)^{1/p}$$

and seminorm

$$|u|_{m,p,S} := \left(\sum_{|\alpha|=m} \|D_S^\alpha u\|^{\,p}_{L^p(S)} \right)^{1/p},$$

with standard modification for $p = \infty$.

For $p = 2$, we denote $W_2^m(S)$ by $H^m(S)$ and the corresponding norm and seminorm by $\|u\|_{m,S} = \|u\|_{m,2,S}$ and $|u|_{m,S} = |u|_{m,2,S}$, respectively.

2.2. The Laplace–Beltrami equation and a regularity result. The variational formulation of the Laplace–Beltrami equation (1.1) is as follows: find $u \in H^1(S)$ such that

$$\int_S \nabla_S u \cdot \nabla_S v \,\mathrm{d}\sigma = \int_S f v \,\mathrm{d}\sigma \quad \text{for all } v \in H^1(S). \tag{2.2}$$

The following well-posedness and regularity results on (2.2) can be found in [16].

LEMMA 2.1. *Let $f \in L^2(S)$ satisfy $\int_S f \,\mathrm{d}\sigma = 0$. Then there exists a unique weak solution u to* (2.2) *satisfying $\int_S u \,\mathrm{d}\sigma = 0$, and*

$$\|u\|_{2,S} \le C \|f\|_{0,S}$$

holds for a constant C depending only on the surface S.

2.3. Sobolev spaces on polyhedral surfaces. Let S be approximated by a polyhedral surface S_h which is a union of triangular faces. We assume that these triangular faces are shape-regular and quasi-uniform of a diameter h and their vertices lie on S. Because S_h is $C^{0,1}$, only $H^1(S_h)$ is well defined [22, 26]. Let $\mathcal{N}_h = \{x_i\}$ be the set of vertices of S_h, $\mathcal{T}_h = \{\tau_h\}$ the set of triangular faces, and $\mathcal{E}_h = \{E\}$ the set of edges of S_h. For any $\tau_h \in \mathcal{T}_h$, let $\boldsymbol{n}_h$ be the unit outward normal vector of S_h on τ_h. For $v_h \in C(S_h)$ and $v_h|_{\tau_h} \in C^1(\tau_h)$, we have

$$\nabla_{S_h} v_h|_{\tau_h} := \nabla v_h - (\nabla v_h \cdot \boldsymbol{n}_h)\boldsymbol{n}_h = (\boldsymbol{I} - \boldsymbol{n}_h\boldsymbol{n}_h^t)\nabla v_h = \boldsymbol{P}_h \nabla v_h \in \mathbb{R}^3,$$

where $\boldsymbol{P}_h = \boldsymbol{I} - \boldsymbol{n}_h\boldsymbol{n}_h^t \in \mathbb{R}^{3\times 3}$. Obviously, $\nabla_{S_h} v_h \in (L^2(S_h))^3$.

Restricting the projection $\mathcal{P}_0 : U \to S$ to S_h, we get a continuous differentiable bijection from S_h to S, still denoted by $\mathcal{P}_0$. For any $\tau_h \in \mathcal{T}_h$, we can get a surface triangle $\tau := \mathcal{P}_0(\tau_h)$ and denote the set consisting of all surface triangles by $\mathcal{T}_S$.

We then establish the relationships between functions defined on S and S_h following [15]. Through the bijection map $\mathcal{P}_0$, a function $v : S \to \mathbb{R}$ induces uniquely a function $\bar{v} : S_h \to \mathbb{R}$, as $\bar{v}(x) = v(\mathcal{P}_0(x))$, for all $x \in S_h$. For any $\tau_h \in \mathcal{T}_h$ and a function $v \in C^1(\mathcal{P}_0(\tau_h))$, we have

$$\nabla_{S_h} \bar{v}(x) = (\boldsymbol{P}_h(\boldsymbol{I} - d\boldsymbol{H})\boldsymbol{P})(x)\nabla_S v(\mathcal{P}_0(x)) \quad \text{for all } x \in \tau_h. \tag{2.3}$$

Conversely, a function $v_h : S_h \to \mathbb{R}$ induces uniquely a function $\tilde{v}_h : S \to \mathbb{R}$, as $\tilde{v}_h(x) = v_h(\mathcal{P}_0^{-1}(x))$, for all $x \in S$. For any $\tau_h \in \mathcal{T}_h$ and a function $v_h \in C^1(\tau_h)$, let $\tau = \mathcal{P}_0(\tau_h)$; we then get

$$\nabla_S \tilde{v}_h(x) = (\boldsymbol{I} - d\boldsymbol{H})^{-1} \left(\boldsymbol{I} - \frac{\boldsymbol{n}_h \boldsymbol{n}^t}{\boldsymbol{n}^t \boldsymbol{n}_h} \right) \nabla_{S_h} v_h(\mathcal{P}_0^{-1}(x)) \quad \text{for all } x \in \tau. \tag{2.4}$$

Let $\mathrm{d}\sigma_h$ and $\mathrm{d}\sigma$ be the surface measures of S_h and S, respectively. They are related by $\mathrm{d}\sigma = J(x)\mathrm{d}\sigma_h$ with (see [15])

$$J(x) = (1 - d(x)\kappa_1(x))(1 - d(x)\kappa_2(x))\boldsymbol{n} \cdot \boldsymbol{n}_h \quad \text{for all } x \in \tau_h.$$

We need the following approximation result [22].

LEMMA 2.2. *For any $\tau_h \in \mathcal{T}_h$, the following estimate holds:*

$$\|d(x)\|_{\infty,\tau_h} + \|1 - J\|_{\infty,\tau_h} + h\,\|\boldsymbol{n} - \boldsymbol{n}_h\|_{\infty,\tau_h} + h\,\|\boldsymbol{P} - \boldsymbol{P}_h\|_{\infty,\tau_h} \lesssim h^2. \tag{2.5}$$

For the relationship between the smoothness of function v defined on S and its extension $\bar{v}$ on S_h, we have the following results [16, 22].

LEMMA 2.3. *Let $\tau_h \in \mathcal{T}_h$ and $\tau = \mathcal{P}_0(\tau_h)$. If $v \in W^{3,\infty}(\tau) \cap H^3(\tau)$, then the following results hold:*

$$\|\bar{v}\|_{0,\tau_h} \lesssim \|v\|_{0,\tau} \lesssim \|\bar{v}\|_{0,\tau_h}, \tag{2.6}$$

$$|\bar{v}|_{1,\tau_h} \lesssim |v|_{1,\tau} \lesssim |\bar{v}|_{1,\tau_h}, \tag{2.7}$$

$$|\bar{v}|_{k,\tau_h} \lesssim \|v\|_{k,\tau}, \quad k = 2, 3, \tag{2.8}$$

$$|\bar{v}|_{k,\infty,\tau_h} \lesssim \|v\|_{k,\infty,\tau}, \quad k = 2, 3. \tag{2.9}$$

2.4. Linear surface FEM. For a triangle $\tau_h \in \mathcal{T}_h$, let $\{\lambda_i\}$ be the barycentric coordinates of τ_h. Let $\mathcal{V}_h$ be the continuous piecewise linear finite element space on S_h, namely, for any $v_h \in \mathcal{V}_h$ and $\tau_h \in \mathcal{T}_h$, v_h is continuous on S_h and $v_h|_{\tau_h} \in \mathrm{span}\{\lambda_1, \lambda_2, \lambda_3\}$. We define the corresponding lifted spaces on S:

$$\tilde{V}_h = \{\tilde{v}_h \,|\, \tilde{v}_h := v_h \circ \mathcal{P}_0^{-1}, \text{ where } v_h \in \mathcal{V}_h\}.$$

Recall that the bijection $\mathcal{P}_0 : S_h \to S$ is defined in (2.1). For $f \in L^2(S)$, let

$$f_h(x) = \bar{f}(x) - \frac{1}{|S_h|}\int_{S_h} \bar{f}\,\mathrm{d}\sigma_h, \tag{2.10}$$

where $|S_h|$ is the area of S_h. Then $\int_{S_h} f_h(x)\,\mathrm{d}\sigma_h = 0$, and therefore there exists a unique finite element solution $u_h \in \mathcal{V}_h$ with $\int_{S_h} u_h\,\mathrm{d}\sigma_h = 0$ to the following equation [22]:

$$\int_{S_h} \nabla_{S_h} u_h \cdot \nabla_{S_h} v_h\,\mathrm{d}\sigma_h = \int_{S_h} f_h v_h\,\mathrm{d}\sigma_h \quad \text{for all } v_h \in \mathcal{V}_h. \tag{2.11}$$

By (2.3) and the fact $\boldsymbol{P}^2 = \boldsymbol{P}$, we can transform (2.11) posed on S_h to the surface S:

$$\int_S \boldsymbol{A}_h \nabla_S \tilde{u}_h \cdot \nabla_S \tilde{v}_h\,\mathrm{d}\sigma = \int_S \frac{1}{J}\tilde{f}_h \tilde{v}_h\,\mathrm{d}\sigma, \tag{2.12}$$

where $\boldsymbol{A}_h = \frac{1}{J}\boldsymbol{P}(\boldsymbol{I} - d\boldsymbol{H})\boldsymbol{P}_h(\boldsymbol{I} - d\boldsymbol{H})\boldsymbol{P}$. Subtracting (2.12) from (2.2), we obtain the error equation

$$\int_S (\nabla_S u - \boldsymbol{A}_h \nabla_S \tilde{u}_h) \cdot \nabla_S \tilde{v}_h\,\mathrm{d}\sigma = \int_S \left(f - \frac{1}{J}\tilde{f}_h\right)\tilde{v}_h\,\mathrm{d}\sigma \quad \text{for all } \tilde{v}_h \in \tilde{\mathcal{V}}_h. \tag{2.13}$$

From (2.13), one can obtain the following estimates [22].

THEOREM 2.4. *When S is C^2, one has the following results:*

$$\left\| f - \frac{1}{J}\tilde{f}_h \right\|_{0,S} \lesssim h^2 \|f\|_{0,S}, \tag{2.14}$$

$$\|(\boldsymbol{A}_h - \boldsymbol{I})\boldsymbol{P}\|_{\infty,\tau_h} \lesssim h^2 \text{ for any } \tau_h \in \mathcal{T}_h \tag{2.15}$$

and consequently the a priori error estimate

$$\|u - \tilde{u}_h\|_{L^2(S)} + h\,|u - \tilde{u}_h|_{H^1(S)} \lesssim h^2 \left(\|u\|_{2,S} + \|f\|_{0,S} \right). \tag{2.16}$$

3. Superconvergence. In this section, we generalize superconvergence results in [5] from planar meshes to surface meshes. We consider the effect of the additional discrete geometry error introduced by using a polyhedral surface S_h as an approximation of S.

Following [11] we introduce the following notations. For each edge $E \in \mathcal{E}_h$, let l_E denote its length, and let Ω_E be the patch of E, consisting of two triangles τ_h and τ_h' sharing the edge E; see Figure 3.1. For the element $\tau_h \in \Omega_E$, θ_E denotes the angle opposite to the edge E, l_{E+1} and l_{E-1} denote the lengths of other two edges, and $\boldsymbol{n}_h$ denotes the outwards normal of S_h on τ_h. All triangles are orientated counterclockwise. The subscript $E+1$ or $E-1$ is used for the next or previous edge in this orientation. Let $\boldsymbol{t}_E$ be the unit tangent vector of E with counterclockwise orientation and $\boldsymbol{n}_E$ be the unit outward normal vector of E in the supporting plane of τ_h. Recall that $\boldsymbol{n}_h$ is the unit outward normal vector of S_h on τ_h; therefore $\boldsymbol{n}_h \perp \boldsymbol{n}_E$ and $\boldsymbol{n}_h \perp \boldsymbol{t}_E$.

An index $'$ will be added for the corresponding quantity in τ_h'. Note that $\boldsymbol{t}_E = -\boldsymbol{t}_E'$ because of the orientation. Unlike the planar domain case, $\boldsymbol{n}_E \neq -\boldsymbol{n}_E'$ in general. By (2.5), however, we have

$$|\boldsymbol{n}_E + \boldsymbol{n}_E'| = |\boldsymbol{n}_h - \boldsymbol{n}_h'| \lesssim |\boldsymbol{n}_h - \boldsymbol{n}| + |\boldsymbol{n}_h' - \boldsymbol{n}| \lesssim h. \tag{3.1}$$

For Ω_E, we introduce the following definition, which is first introduced in [9] and called strong regular conditions there.

DEFINITION 3.1. *The patch Ω_E is an $\mathcal{O}(h^2)$ approximate parallelogram if it satisfies*

$$|\overrightarrow{x_1x_4} - \overrightarrow{x_2x_3}| = \mathcal{O}(h^2), \quad |\overrightarrow{x_1x_2} - \overrightarrow{x_4x_3}| = \mathcal{O}(h^2).$$

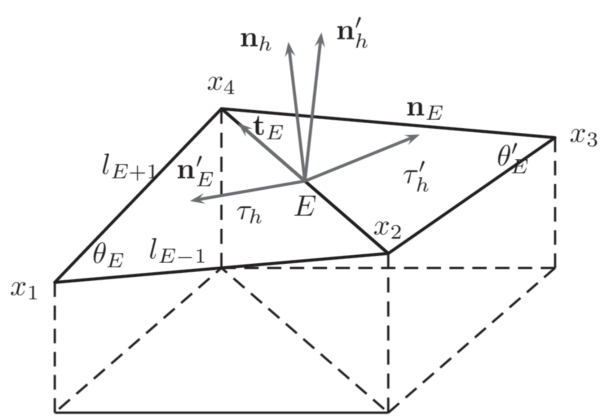


FIG. 3.1. *The patch Ω_E of the edge E.*

Notice that the included angle of $\boldsymbol{n}_h$ and $\boldsymbol{n}'_h$ differ only by $\mathcal{O}(h)$, namely, τ_h and τ'_h are "almost" on the same plane. Therefore we still call Ω_E as an approximate parallelogram following the terminology in [5].

Remark 3.1. The $\mathcal{O}(h^2)$ condition can be relaxed to $\mathcal{O}(h^{1+\rho})$ for $\rho \in (0,1)$ introduced in [38], and the analysis in our paper can be easily adapted to this case.

DEFINITION 3.2 (see [5, 11, 38]). *Let $\mathcal{E}_h = \mathcal{E}_1 \oplus \mathcal{E}_2$ denote the set of edges in $\mathcal{T}_h$. The triangulation $\mathcal{T}_h$ is $\mathcal{O}(h^{2\sigma})$ irregular if for each $E \in \mathcal{E}_1$, Ω_E forms an $\mathcal{O}(h^2)$ approximate parallelogram, while $\sum_{E\in\mathcal{E}_2} |\,\Omega_E| = \mathcal{O}(h^{2\sigma})$.*

When the triangulation $\mathcal{T}_h$ is $\mathcal{O}(h^{2\sigma})$ irregular, we can split $\mathcal{T}_h$ into two parts:

$$\mathcal{T}_h = \mathcal{T}_{1,h} \cup \mathcal{T}_{2,h}, \text{ with } \mathcal{T}_{i,h} = \{\tau \in \mathcal{T}_h, \tau \in \Omega_E, \text{ with } E \in \mathcal{E}_i\}. \tag{3.2}$$

We define the domain $\bar{\Omega}_{i,h} \equiv \bigcup_{\tau_h \in \mathcal{T}_{i,h}} \bar{\tau}_h,\ i = 1, 2$. Then

$$\bar{\Omega}_{1,h} \cup \bar{\Omega}_{2,h} = S_h, \quad \text{and} \quad |\Omega_{2,h}| = \mathcal{O}(h^{2\sigma}). \tag{3.3}$$

For $u \in C(S)$, recall that $\bar{u} = u \circ \mathcal{P}_0 \in C(S_h)$. We define two interpolations of $\bar{u}$ on S_h. First, let $\bar{u}_I \in \mathcal{V}_h$ be the linear interpolation of $\bar{u}$ on S_h, defined by

$$\bar{u}_I(x_i) = \bar{u}(x_i) \quad \text{for all } x_i \in \mathcal{N}_h,$$

where $\mathcal{N}_h$ is the vertex set of S_h. Let W_h be the continuous and piecewise quadratic finite element space on S_h, namely, for any $w_h \in W_h$ and $\tau_h \in \mathcal{T}_h$, w_h is continuous on S_h and $w_h|_{\tau_h} \in \operatorname{span}\{\lambda_1, \lambda_2, \lambda_3, \lambda_1\lambda_2, \lambda_1\lambda_3, \lambda_2\lambda_3\}$. We define $\bar{u}_Q \in W_h$ to be the quadratic interpolation of $\bar{u}$ satisfying

$$\bar{u}_Q(x_i) = \bar{u}(x_i) \text{ for all } x_i \in \mathcal{N}_h, \text{ and } \int_E \bar{u}_Q = \int_E \bar{u} \text{ for all } E \in \mathcal{E}_h.$$

We can lift $\bar{u}_I$ and $\bar{u}_Q$ onto S as $u_I = \bar{u}_I \circ \mathcal{P}_0^{-1} : S \to \mathbb{R}$ and $u_Q = \bar{u}_Q \circ \mathcal{P}_0^{-1} : S \to \mathbb{R}$.

On the flat triangle τ_h, we have the following two important lemmas [11].

LEMMA 3.3. *Let $\bar{u}_I$ and $\bar{u}_Q$ be the linear and quadratic interpolations of $\bar{u}$ defined above. For all $v_h \in \mathcal{V}_h$, we have the local error expansion formula*

$$\int_{\tau_h} \nabla_{S_h}(\bar{u}-\bar{u}_I)\cdot\nabla_{S_h} v_h \,\mathrm{d}\sigma_h = \sum_{E\in\partial\tau_h} \left[\alpha_E \left(\int_E \frac{\partial^2 \bar{u}_Q}{\partial \boldsymbol{t}_E^2}\frac{\partial v_h}{\partial \boldsymbol{t}_E}\right) + \beta_E \left(\int_E \frac{\partial^2 \bar{u}_Q}{\partial \boldsymbol{t}_E \partial \boldsymbol{n}_E}\frac{\partial v_h}{\partial \boldsymbol{t}_E}\right)\right],$$

where

$$\alpha_E = \frac{1}{12}\cot\theta_E (l_{E+1}^2 - l_{E-1}^2),\ \beta_E = \frac{1}{3}\cot\theta_E\, |\tau_h|\,.$$

LEMMA 3.4. *For any edge $E \in \mathcal{E}_h$, we have*

$$|\alpha_E| + |\alpha'_E| = \mathcal{O}(h^2), \quad |\beta_E| + |\beta'_E| = \mathcal{O}(h^2), \tag{3.4}$$

$$|\alpha_E - \alpha'_E| = \mathcal{O}(h^3), \quad |\beta_E - \beta'_E| = \mathcal{O}(h^3) \text{ if } E \in \mathcal{E}_1, \tag{3.5}$$

$$\int_E \frac{\partial^2 \bar{u}}{\partial \boldsymbol{t}_E \partial \boldsymbol{z}_E}\frac{\partial v_h}{\partial \boldsymbol{t}_E} \lesssim h^{-1}\|u\|_{2,\infty,S}\int_{\tau_h} |\nabla_{S_h} v_h|\,, \tag{3.6}$$

$$\int_E \frac{\partial^2 \bar{u}}{\partial \boldsymbol{t}_E \partial \boldsymbol{z}_E}\frac{\partial v_h}{\partial \boldsymbol{t}_E} \lesssim \int_{\tau_h} (h^{-1}\left|\nabla^2_{S_h}\bar{u}\right| + \left|\nabla^3_{S_h}\bar{u}\right|)\,|\nabla_{S_h} v_h|\,, \tag{3.7}$$

$$\int_E \frac{\partial^2 (\bar{u}-\bar{u}_Q)}{\partial \boldsymbol{t}_E \partial \boldsymbol{z}_E}\frac{\partial v_h}{\partial \boldsymbol{t}_E} \lesssim \int_{\tau_h} \left|\nabla^3_{S_h}\bar{u}\right| |\nabla_{S_h} v_h|\,, \tag{3.8}$$

where $\boldsymbol{z}_E$ is $\boldsymbol{n}_E$ or $\boldsymbol{t}_E$.

Our main result is the following superconvergence between the surface finite element solution and the linear interpolation of the extension of the true solution on the polyhedral surface mesh.

THEOREM 3.5. *Suppose the triangulation $\mathcal{T}_h$ is $\mathcal{O}(h^{2\sigma})$ irregular. Let u be the solution of* (1.1) *and u_h the linear finite element solution on S_h. If $u \in H^3(S) \cap W^2_\infty(S)$, then for all $v_h \in \mathcal{V}_h$, we have*

$$\int_{S_h} \nabla_{S_h}(\bar{u} - \bar{u}_I) \cdot \nabla_{S_h} v_h \, \mathrm{d}\sigma_h \lesssim h^{1+\min\{1,\sigma\}} \left(\|u\|_{3,S} + \|u\|_{2,\infty,S} \right) |v_h|_{1,S_h}, \tag{3.9}$$

and

$$\|\nabla_{S_h} u_h - \nabla_{S_h} \bar{u}_I\|_{0,S_h} \lesssim h^{1+\min\{1,\sigma\}} \left(\|u\|_{3,S} + \|u\|_{2,\infty,S} \right) + h^2 \|f\|_{0,S}. \tag{3.10}$$

Proof. Using the basic identity in Lemma 3.3 we get

$$\begin{aligned}(\nabla_{S_h}(\bar{u} - \bar{u}_I), \nabla_{S_h} v_h) &= \sum_{\tau_h \in T_h} \sum_{E \in \partial\tau_h} \left[\alpha_E \left(\int_E \frac{\partial^2 \bar{u}_Q}{\partial \boldsymbol{t}_E^2} \frac{\partial v_h}{\partial \boldsymbol{t}_E} \right) + \beta_E \left(\int_E \frac{\partial^2 \bar{u}_Q}{\partial \boldsymbol{t}_E \partial \boldsymbol{n}_E} \frac{\partial v_h}{\partial \boldsymbol{t}_E} \right) \right] \\ &= I_1 + I_2 + I_3,\end{aligned}$$

where

$$\begin{aligned} I_i &= \sum_{E \in \mathcal{E}_i} \left[(\alpha_E - \alpha'_E) \int_E \frac{\partial^2 \bar{u}_Q}{\partial \boldsymbol{t}_E^2} \frac{\partial v_h}{\partial \boldsymbol{t}_E} + (\beta_E - \beta'_E) \int_E \frac{\partial^2 \bar{u}}{\partial \boldsymbol{t}_E \partial \boldsymbol{n}_E} \frac{\partial v_h}{\partial \boldsymbol{t}_E} \right], \ i = 1, 2, \\ I_3 &= \sum_{E \in \mathcal{E}_h} \left[\beta'_E \int_E \left(\frac{\partial^2 \bar{u}}{\partial \boldsymbol{t}_E \partial \boldsymbol{n}_E} + \frac{\partial^2 \bar{u}}{\partial \boldsymbol{t}_E \partial \boldsymbol{n}'_E} \right) \frac{\partial v_h}{\partial \boldsymbol{t}_E} + \beta_E \int_E \frac{\partial^2 (\bar{u}_Q - \bar{u})}{\partial \boldsymbol{t}_E \partial \boldsymbol{n}_E} \frac{\partial v_h}{\partial \boldsymbol{t}_E} \right. \\ &\qquad \left. + \beta'_E \int_E \frac{\partial^2 (\bar{u}_Q - \bar{u})}{\partial \boldsymbol{t}'_E \partial \boldsymbol{n}'_E} \frac{\partial v_h}{\partial \boldsymbol{t}'_E} \right]. \end{aligned} \tag{3.11}$$

To estimate I_1, we use Lemma 2.3, the estimates (3.5) and (3.7) to get

$$\begin{aligned} |I_1| &\lesssim \sum_{E \in \mathcal{E}_1} \left[|\alpha_E - \alpha'_E| \int_{\tau_h} h^{-1} \left| \nabla^2_{S_h} \bar{u}_Q \right| |\nabla_{S_h} v_h| \right. \\ &\qquad \left. + |\beta_E - \beta'_E| \int_{\tau_h} (h^{-1} \left| \nabla^2_{S_h} \bar{u} \right| + \left| \nabla^3_{S_h} \bar{u} \right|) |\nabla_{S_h} v_h| \right] \\ &\lesssim h^2 \sum_{\tau_h \in \mathcal{T}_{1,h}} \int_{\tau_h} \left(\left| \nabla^2_{S_h} \bar{u} \right| + h \left| \nabla^3_{S_h} \bar{u} \right| \right) |\nabla_{S_h} v_h| \\ &\lesssim h^2 \|u\|_{3,S} |v_h|_{1,S_h}. \end{aligned} \tag{3.12}$$

To estimate I_2, we use Lemma 2.3 and the estimates (3.4) and (3.6) to get

$$|I_2| \lesssim \sum_{E \in \mathcal{E}_2} h \|u\|_{2,\infty,S} \int_{\Omega_E} |\nabla_{S_h} v_h| \lesssim h^{1+\sigma} \|u\|_{2,\infty,S} |v_h|_{1,S_h}.$$

To estimate I_3, we use Lemma 2.3, (3.1), the estimates (3.4), (3.7), and (3.8) to get

$$
\begin{aligned}
|I_3| &\lesssim \sum_{E\in\mathcal{E}_h} |\beta'_E| \int_E \left| \nabla_{S_h} \frac{\partial \bar{u}}{\partial \boldsymbol{t}_E} \cdot (\boldsymbol{n}_E + \boldsymbol{n}'_E) \frac{\partial v_h}{\partial \boldsymbol{t}_E} \right| + \sum_{\tau_h\in\mathcal{T}_h} h^2 \int_{\tau_h} \left|\nabla^3_{S_h}\bar{u}\right| |\nabla_{S_h} v_h| \\
&\lesssim h^2 \sum_{\tau_h\in\mathcal{T}_h} \left(|\bar{u}|_{2,\tau_h} + h\,|\bar{u}|_{3,\tau_h} + |\bar{u}|_{3,\tau_h}\right) |v_h|_{1,\tau_h} \\
&\lesssim h^2 \sum_{\tau\in\mathcal{T}} \|u\|_{3,\tau} |v_h|_{1,S_h} \\
&\lesssim h^2 \|u\|_{3,S} |v_h|_{1,S_h}.
\end{aligned}
$$

The inequality (3.9) then follows. We now prove (3.10) as

$$
\begin{aligned}
&\int_{S_h} \nabla_{S_h}(u_h - \bar{u}_I)\cdot\nabla_{S_h} v_h \,\mathrm{d}\sigma_h \\
&= \int_{S_h} \nabla_{S_h}(u_h - \bar{u})\cdot\nabla_{S_h} v_h \,\mathrm{d}\sigma_h + \int_{S_h} \nabla_{S_h}(\bar{u} - \bar{u}_I)\cdot\nabla_{S_h} v_h \,\mathrm{d}\sigma_h \\
&= \int_S (\boldsymbol{A}_h\nabla_S\tilde{u}_h - \nabla_S u)\cdot\nabla_S\tilde{v}_h \,\mathrm{d}\sigma + \int_S (\nabla_S u - \boldsymbol{A}_h\nabla_S u)\cdot\nabla_S\tilde{v}_h \,\mathrm{d}\sigma \\
&\quad + \int_{S_h} \nabla_{S_h}(\bar{u} - \bar{u}_I)\cdot\nabla_{S_h} v_h \,\mathrm{d}\sigma_h \\
&= I_4 + I_5 + I_6.
\end{aligned} \tag{3.13}
$$

The estimate of I_6 is given by (3.9). Let $C_0 = \frac{\int_S \tilde{v}_h \mathrm{d}\sigma}{|S|}$. From (2.13) and (2.15), we can estimate I_4 as

$$
\begin{aligned}
\int_S (\boldsymbol{A}_h\nabla_S\tilde{u}_h - \nabla_S u)\cdot\nabla_S\tilde{v}_h \,\mathrm{d}\sigma &= \int_S (\boldsymbol{A}_h\nabla_S\tilde{u}_h - \nabla_S u)\cdot\nabla_S(\tilde{v}_h - C_0) \,\mathrm{d}\sigma \\
&= \int_S \left(\frac{1}{J}\tilde{f}_h - f\right)(\tilde{v}_h - C_0)\,\mathrm{d}\sigma \\
&\lesssim h^2 \|f\|_{0,S} |\tilde{v}_h|_{1,S} \lesssim h^2 \|f\|_{0,S} |v_h|_{1,S_h}.
\end{aligned} \tag{3.14}
$$

From (2.7) and (2.15), we can estimate I_5 as

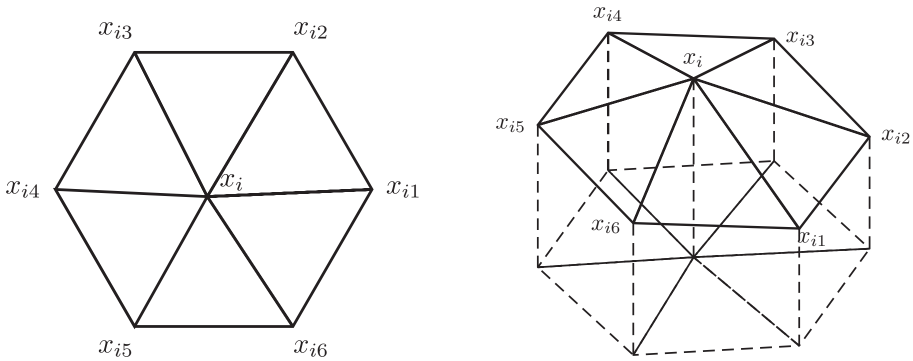
$$
\begin{aligned}
\left|\int_S (\nabla_S u - \boldsymbol{A}_h\nabla_S u)\cdot\nabla_S\tilde{v}_h \,\mathrm{d}\sigma\right| &= \left|\int_S (\boldsymbol{I} - \boldsymbol{A}_h)\,\boldsymbol{P}\nabla_S u\cdot\nabla_S\tilde{v}_h \,\mathrm{d}\sigma\right| \\
&\lesssim \|(\boldsymbol{A}_h - \boldsymbol{I})\boldsymbol{P}\|_{\infty,S} |u|_{1,S} |\tilde{v}_h|_{1,S} \lesssim h^2 |u|_{1,S} |\tilde{v}_h|_{1,S} \\
&\lesssim h^2 |u|_{1,S} |v_h|_{1,S_h}.
\end{aligned} \tag{3.15}
$$

Combining (2.13), (2.15), (3.1), (3.9), and (3.15), we proved (3.10). ∎

Using the equivalence of the H^1 norm of u and $\bar{u}$ (cf. Lemma 2.3), we obtain the supercloseness between $\tilde{u}_h$ and u_I.

COROLLARY 3.6. *Assume the same hypothesis in Theorem* 3.5. *Let $\tilde{u}_h$ and u_I be the lifting of u_h and $\bar{u}_I$ onto S, respectively. Then*

$$
\|\nabla_S\tilde{u}_h - \nabla_S u_I\|_{0,S} \lesssim h^{1+\min\{1,\sigma\}}\left(\|u\|_{3,S} + \|u\|_{2,\infty,S}\right) + h^2 \|f\|_{0,S}. \tag{3.16}
$$



(a) the patch Ω_i of x_i on the plane (b) the patch Ω_i of x_i on a surface

FIG. 4.1. *Local patches on planar meshes and surface meshes.*

4. Gradient recovery schemes. In this section, we discuss how to generalize the gradient recovery methods, including local and global schemes, from planar meshes to polyhedral meshes.

4.1. Local averaging schemes on surface meshes. We first recall the local averaging schemes in the planar case. Give an element patch $\Omega_i \subset \mathbb{R}^2$ around the vertex x_i, namely, $\bar{\Omega}_i = \bigcup_{x_i \in \bar{\tau}_h} \bar{\tau}_h$, let $\{x_{ij}\}_{j=1}^m$ be the boundary vertices of Ω_i which are orientated counterclockwise. Let $\tau_j = \triangle x_i x_{ij} x_{i(j+1)}$, $j = 1, \ldots, m$, and $x_{i(m+1)} = x_{i1}$; see Figure 4.1(a).

We consider the following two gradient recovery operators G_h applied to a finite element function $v_h \in \mathcal{V}_h$:

1. Simple averaging $\quad (G_h \nabla v_h)(x_i) = \dfrac{1}{m} \displaystyle\sum_{j=1}^{m} \nabla v_h|_{\tau_j}(x_i).$
2. Weighted averaging $\quad (G_h \nabla v_h)(x_i) = \displaystyle\sum_{j=1}^{m} \dfrac{|\tau_j|}{|\Omega_i|} \nabla v_h|_{\tau_j}(x_i).$

THEOREM 4.1 (see [38]). *Assume every two adjacent triangles in patch Ω_i form an $\mathcal{O}(h^2)$ approximate parallelogram and $u \in W^3_\infty(\Omega_i)$. Let u_I be the nodal interpolation of u and $(G_h \nabla u_I)(x_i)$ be produced by either the simple averaging or the weighted averaging. Then*

$$|(G_h \nabla u_I)(x_i) - \nabla u(x_i)| \lesssim h^2 \|u\|_{3,\infty,\Omega_i}\,. \tag{4.1}$$

Theorem 4.1 can be found in [38]. In their proof, the authors utilized the result that the barycenter of a triangle is the superconvergence point for the derivative of the linear interpolation, which is not true; see [8, page 285]. We shall extend Theorem 4.1 to polyhedral surface meshes and provide a correct proof.

Give an element patch $\Omega_i \subset S_h$ of the vertex x_i; see Figure 4.1(b). Let $\tilde{\Omega}_i = \mathcal{P}_0(\Omega_i) \subset S$ be the patch lifted to the surface. For a finite element function $v_h \in \mathcal{V}_h(S_h)$, a natural generalization of $(G_h \nabla_{S_h} v_h)(x_i)$ on surface meshes is given below:

1. Simple averaging $\quad (G_h \nabla_{S_h} v_h)(x_i) = \dfrac{1}{m} \displaystyle\sum_{j=1}^{m} \nabla_{S_h} v_h|_{\tau_j}(x_i).$
2. Weighted averaging $\quad (G_h \nabla_{S_h} v_h)(x_i) = \displaystyle\sum_{j=1}^{m} \dfrac{|\tau_j|}{|\Omega_i|} \nabla_{S_h} v_h|_{\tau_j}(x_i).$

Here we simply change the conventional gradient operator ∇ in $\mathbb{R}^2$ to the gradient operator ∇_{S_h} on polyhedral meshes.

In principle, the symmetry of the element patch of a vertex is the source of the superconvergence [35]. We thus introduce the following concept. An element patch Ω_i of a vertex x_i is called $\mathcal{O}(h^2)$-symmetry if for any boundary vertex x_{ij} of Ω_i, there exits another boundary vertex x'_{ij} of Ω_i so that

$$x_{ij} - 2x_i + x'_{ij} = \mathcal{O}(h^2), \tag{4.2}$$

which mean x_{ij} and x'_{ij} are approximately symmetric about x_i. Let $\boldsymbol{n}$ and $\boldsymbol{n}_h|_{\tau_j}$ be the unit outward normal vectors of S at x_i and S_h at the triangle τ_j in the patch Ω_i, respectively. Since the C^3 smoothness of S and the $\mathcal{O}(h^2)$-symmetry of patch Ω_i, for any triangle τ_j in the patch Ω_i, there exists another triangle τ'_j so that their unit outward normal vectors $\boldsymbol{n}_h$ and $\boldsymbol{n}'_h$ at x_i satisfy

$$\boldsymbol{n}_h - 2\boldsymbol{n} + \boldsymbol{n}'_h = \mathcal{O}(h^2) \tag{4.3}$$

and their areas $|\tau_j|$ and $\left|\tau'_j\right|$ satisfy

$$|\tau_j| - \left|\tau'_j\right| = \mathcal{O}(h^3). \tag{4.4}$$

THEOREM 4.2. *Let $u \in W^3_\infty(S)$, $\bar{u} = u \circ \mathcal{P}_0$, and $\bar{u}_I$ be the linear interpolation of $\bar{u}$ on Ω_i; let $(G_h\nabla_{S_h}\bar{u}_I)(x_i)$ be produced by either the simple averaging or the weighted averaging on surface meshes. If Ω_i is $\mathcal{O}(h^2)$-symmetric, then*

$$|(G_h\nabla_{S_h}\bar{u}_I)(x_i) - \nabla_S u(x_i)| \lesssim h^2 \|u\|_{3,\infty,S}\,. \tag{4.5}$$

Proof. For the simple averaging, using the triangle inequality, substituting $x = x_i$ into (2.3), and noting $\boldsymbol{P}^2 = \boldsymbol{P}$, we have

$$\begin{aligned}
&|(G_h\nabla_{S_h}\bar{u}_I)(x_i) - \nabla_S u(x_i)|\\
&\leq \left|\frac{1}{m}\sum_{j=1}^m \left(\nabla_{S_h}\bar{u}_I - \nabla_{S_h}\bar{u}\right)|_{\tau_j}(x_i)\right| + \left|\frac{1}{m}\sum_{j=1}^m \left(\nabla_{S_h}\bar{u}|_{\tau_j} - \boldsymbol{P}\nabla_S u\right)(x_i)\right|\\
&\leq \left|\frac{1}{m}\sum_{j=1}^m \left(\nabla_{S_h}\bar{u}_I|_{\tau_j} - \nabla_{S_h}\bar{u}\right)|_{\tau_j}(x_i)\right| + \left|\frac{1}{m}\sum_{j=1}^m \left((\boldsymbol{P}_h|_{\tau_j} - \boldsymbol{P})\nabla_S u\right)(x_i)\right|\\
&= I_1 + I_2.
\end{aligned}$$

We estimate the two terms as follows. First for I_2, by $\boldsymbol{P}_h = I - \boldsymbol{n}_h\boldsymbol{n}_n^t$, $\boldsymbol{P} = I - \boldsymbol{n}\boldsymbol{n}^t$, and (4.3), we have

$$|I_2| \lesssim \left|\frac{1}{m}\sum_{j=1}^m \left(\boldsymbol{P}_h|_{\tau_j} - \boldsymbol{P}\right)(x_i)\right| \|u\|_{1,\infty,S} \lesssim h^2 \|u\|_{1,\infty,S}\,.$$

For I_1, we use the following identity on the error expansion. For a triangle $\tau_j = \triangle x_i x_{ij} x_{i(j+1)}$ in Ω_i and $\bar{u} \in W^3_\infty(\tau_j)$, we have the following identity [8]:

$$\nabla_{S_h}(\bar{u}_I - \bar{u})(x) = \frac{1}{2}\sum_{k=1}^3 \mathrm{d}_k^2\bar{u}(x)\nabla_{S_h}\lambda_k(x) - \frac{1}{2}\sum_{k=1}^3 \nabla_{S_h}\lambda_k(x)\int_0^1 \mathrm{d}_k^3\bar{u}(\varsigma_k)t^2\mathrm{d}t, \tag{4.6}$$

where x is a point in τ_h^j, $\mathrm{d}_k = (x - x_k)^T \nabla_{S_h}$, $x_1 = x_i$, $x_2 = x_{ij}$, $x_3 = x_{i(j+1)}$, $\varsigma_k = x_k + t(x - x_k)$, and λ_k is the barycenter coordinate of τ_j with respect to the vertex x_k, $k = 1, \dots, 3$. Then by Lemma 2.3 and (4.6), we have

$$\begin{aligned}
&|I_1| \\
&= \left| \frac{1}{m} \sum_{j=1}^{m} \left[\frac{1}{2} \sum_{k=1}^{3} \mathrm{d}_k^2 \bar{u}(x_i) \nabla_{S_h} \lambda_k(x_i) - \frac{1}{2} \sum_{k=1}^{3} \nabla_{S_h} \lambda_k(x_i) \int_0^1 \mathrm{d}_k^3 \bar{u}(x_k + t(x_i - x_k)) t^2 \mathrm{d}t \right] \right| \\
&\lesssim \left| \frac{1}{m} \sum_{j=1}^{m} \left[\frac{1}{2} \sum_{k=1}^{3} \mathrm{d}_k^2 \bar{u}(x_i) \nabla_{S_h} \lambda_k(x_i) \right] \right| + h^2 \|u\|_{3,\infty,S}.
\end{aligned}$$

For any $\mathrm{d}_k^2 \bar{u}(x_i) \nabla_{S_h} \lambda_k(x_i)$ in one element τ_j, by the $\mathcal{O}(h^2)$-symmetry and the gradient formulas of barycentric coordinates, there exists another $\mathrm{d}_{k'}^2 \bar{u}(x_i) \nabla_{S_h} \lambda_{k'}(x_i)$ in another element τ_j' so that

$$\mathrm{d}_k^2 \bar{u}(x_i) \nabla_{S_h} \lambda_k(x_i) + \mathrm{d}_{k'}^2 \bar{u}(x_i) \nabla_{S_h} \lambda_{k'}(x_i) = \mathcal{O}(h^2).$$

Then the conclusion (4.5) follows.

The weighted averaging case is proved similarly by using Lemma 2.3, (4.2), (4.3), (4.4), and (4.6). $\square$

Remark 4.1. When the element patch Ω_i is not $\mathcal{O}(h^2)$-symmetric, since S_h is shape-regular and quasi-uniform, it will be at least $\mathcal{O}(h)$-symmetric. So using the same proof, we can get

$$|(G_h \nabla_{S_h} \bar{u}_I)(x_i) - \nabla_S u(x_i)| \lesssim h \|u\|_{2,\infty,S}. \tag{4.7}$$

4.2. Local least-squares fitting on tangent planes. Let $\Omega_i \subset S_h$ be the element patch around x_i. Let $\boldsymbol{n}_i$ be the unit outward normal vector of S at x_i and M_i be the tangent plane of S at x_i. Along the direction of $\boldsymbol{n}_i$, define the projection

$$\mathcal{P}_1(x) = x + (x_i - x, \boldsymbol{n}_i)\boldsymbol{n}_i \in M_i \text{ for any } x \in \Omega_i. \tag{4.8}$$

Then we can project Ω_i onto M_i and get a planar patch $\bar{\Omega}_i = \mathcal{P}_1(\Omega_i)$. If Ω_i is $\mathcal{O}(h^2)$-symmetric, since S is C^3, it is easy to show that $\bar{\Omega}_i$ has the same property by Taylor expansion. In [20], the authors first projected the patch Ω_i on the tangent plane M_i of S at x_i, then used a simple averaging method to recover the gradient at x_i. We adopt this approach to develop several recovery schemes.

Let $\{\bar{x}_{ij} | \bar{x}_{ij} = \mathcal{P}_1(x_{ij}), j = 1, m\}$ be the boundary vertices of the lifted patch $\bar{\Omega}_i$. Then we construct a local Cartesian coordinate system whose origin is x_i, the direction of z-axis is the direction of $\boldsymbol{n}_z := \boldsymbol{n}_i$, the direction of x-axis is the direction of $\boldsymbol{n}_x := (\bar{x}_{i1} - x_i)/|\bar{x}_{i1} - x_i|$, and the direction of y-axis is the direction of $\boldsymbol{n}_y := \boldsymbol{n}_z \times \boldsymbol{n}_x$. Let $\bar{x}_j' := ((\bar{x}_{ij} - x_i) \cdot \boldsymbol{n}_x, (\bar{x}_{ij} - x_i) \cdot \boldsymbol{n}_y)$, $j = 1, m$. We then get a patch $\bar{\Omega}_i' \subset \mathbb{R}^2$ whose center is $x_O := (0, 0)$ and boundary vertices are $\{\bar{x}_j'\}_{j=1}^m$.

Let v_h be a linear finite element function on Ω_i. We define a linear finite element function $\bar{v}_h'$ on $\bar{\Omega}_i'$ by setting $\bar{v}_h'(x_O) = v_h(x_i)$ and $\bar{v}_h(\bar{x}_j') = v_h(x_{ij})$ for $j = 1, \dots, m$.

We use the the following two least-square fitting schemes:

1. Local L^2-projection. We seek linear functions $p_l \in P_1(\bar{\Omega}_i')$ $(l = 1, 2)$ such that

$$\int_{\Omega_i} [p_l(x) - \partial_l \bar{v}_h'(x)] q \, dx = 0 \quad \text{for all } q \in P_1(\bar{\Omega}_i'),\ l = 1, 2. \tag{4.9}$$

2. Local discrete least-squares fitting proposed by Zienkiewicz and Zhu [40]. Let c_j be the barycenter of $\tau_j \in \bar{\Omega}'_i$. We seek linear functions $p_l \in P_1(\bar{\Omega}'_i)$ $(l = 1, 2)$ such that

$$\sum_{j=1}^{m} [p_l(c_j) - \partial_l \bar{v}'_h(c_j)] q(c_j) = 0 \ \text{ for all } q \in P_1(\bar{\Omega}'_i),\ l = 1, 2. \tag{4.10}$$

Finally, we get $G_h \nabla_{S_h} v_h \in \mathcal{V}_h$, the recovery gradient of v_h, by setting

$$(G_h \nabla_{S_h} v_h)(x_i) = p_1(x_O)\boldsymbol{n}_x + p_2(x_O)\boldsymbol{n}_y.$$

THEOREM 4.3. *Let $u \in W^3_\infty(S)$, $\bar{\Omega}_i = \mathcal{P}_1(\Omega_i)$, and $u^M(\mathcal{P}_1) = u(\mathcal{P}_0)$; let u^M_I be the linear interpolation of u^M on $\bar{\Omega}_i$ and G_h be one of the four gradient recovery operators defined on the tangent plane patches: the simple averaging, the weighted averaging, local L^2-projection* (4.9)*, and local discrete least-squares fitting* (4.10)*. If Ω_i is $\mathcal{O}(h^2)$-symmetric, then*

$$\left|\nabla_S u(x_i) - G_h \nabla_{M_i} u^M_I(x_i)\right| \lesssim h^2 \|u\|_{3,\infty,S}. \tag{4.11}$$

Proof. Since $\nabla_{M_i} u^M(x_i) = \nabla u(x_i) - (\nabla u(x_i) \cdot \boldsymbol{n}_i)\boldsymbol{n}_i = \nabla_S u(x_i)$, we have

$$\left|\nabla_S u(x_i) - G_h \nabla_{M_i} u^M_I(x_i)\right| = \left|\nabla_{M_i} \bar{u}(x_i) - G_h \nabla_{M_i} u^M_I(x_i)\right|,$$

which transforms the surface patch problem into a plane patch problem.

When G_h is the simple or the weighted averaging, (4.11) is a special case of Theorem 4.2. For the local L^2-projection and local discrete least-squares fitting Zienkiewicz and Zhu (ZZ), following the results of the simple and the weighted averaging and the approach of Theorem 3.1 in [38], we can prove (4.11). ∎

Remark 4.2. Again if the patch is not $\mathcal{O}(h^2)$ symmetric, we will have estimate

$$\left|\nabla_S u(x_i) - G_h \nabla_{M_i} u^M_I(x_i)\right| \lesssim h \|u\|_{2,\infty,S}. \tag{4.12}$$

THEOREM 4.4. *Let u be the solution of* (2.2) *and u_h be the solution of* (2.11)*. Let G_h be one of the six recovery operators: the simple averaging, the weighted averaging on the surface patches and on the tangent plane patches, respectively, and the local L^2-projection and the local discrete least-squares fitting (ZZ) defined on the tangent plane patches. Set $w_h = G_h \nabla_{S_h} u_h$. If the triangulation $\mathcal{T}_h$ is $\mathcal{O}(h^{2\sigma})$ irregular and $u \in W^3_\infty(S)$, then*

$$\|\nabla_S u - \tilde{w}_h\|_{0,S} \lesssim h^{1+\min\{1,\sigma\}} \|u\|_{3,\infty,S} + h^2 \|f\|_{0,S}.$$

Proof. Let $\overline{\nabla_S u} = (\nabla_S u) \circ \mathcal{P}_0$. By the norm equivalence (2.7), we need only to estimate $\left\|\overline{\nabla_S u} - G_h \nabla_{S_h} u_h\right\|_{0,S_h}$. We decompose

$$\overline{\nabla_S u} - G_h \nabla_{S_h} u_h = \overline{\nabla_S u} - (\overline{\nabla_S u})_I + (\overline{\nabla_S u})_I - G_h \nabla_{S_h} \bar{u}_I + G_h(\nabla_{S_h} \bar{u}_I - \nabla_{S_h} u_h).$$

For the first term, by the standard approximation theory,

$$\left\|\overline{\nabla_S u} - (\overline{\nabla_S u})_I\right\|_{0,S_h} \lesssim h^2 \|u\|_{3,S}. \tag{4.13}$$

To control the second term, we shall split the $\mathcal{O}(h^{2\sigma})$ irregular $\mathcal{T}_h$ into two parts, slightly different than that in (3.2). We define $\mathcal{N}_{1,h} = \{x_i \in \mathcal{N}_h|$ every two neighboring triangles in Ω_i forms an approximated parallelogram$\}$, $\mathcal{N}_{2,h} = \mathcal{N}_h \setminus \mathcal{N}_{1,h}$, and $\Omega_{i,h} = \cup_{x_k \in \mathcal{N}_{i,h}} \Omega_k$ for $i = 1, 2$.

If we introduce $\mathcal{E}_{x_i} = \{E \in \mathcal{E}_h \,|\, x_i$ is one of the end points of $E\}$, then $x_i \in \mathcal{N}_{2,h}$ is equivalent to that there exists at least one edge $E \in \mathcal{E}_{x_i}$ such that $E \subset \mathcal{E}_2$. By the shape-regularity of the mesh, we still have

$$|\Omega_{2,h}| \lesssim \bigcup_{E \in \mathcal{E}_2} \Omega_E = \mathcal{O}(h^{2\sigma}).$$

By (4.5) and (4.11), in $\Omega_{1,h}$,

$$\begin{aligned} \left\|(\overline{\nabla_S u})_I - G_h \nabla_{S_h} \bar{u}_I\right\|_{0,\Omega_{1,h}} &\le \left(\sum_{\tau_h \in \mathcal{T}_{1,h}} |\tau_h| \sum_{z \in \mathcal{N}_h \cap \bar{\tau}_h} |G_h \nabla_{S_h} \bar{u}_I(z) - \nabla_S u(z)|^2 \right)^{1/2} \\ &\lesssim h^2 \|u\|_{3,\infty,S} |\Omega_{1,h}|^{1/2} \lesssim h^2 \|u\|_{3,\infty,S}. \end{aligned} \tag{4.14}$$

On the other hand, by (4.7) and (4.12), in $\Omega_{2,h}$,

$$\left\|(\overline{\nabla_S u})_I - G_h \nabla_{S_h} \bar{u}_I\right\|_{0,\Omega_{2,h}} \lesssim h \|u\|_{3,\infty,S} |\Omega_{2,h}|^{1/2} \lesssim h^{1+\sigma} \|u\|_{3,\infty,S}. \tag{4.15}$$

Combining (4.14) with (4.15), we have

$$\left\|(\overline{\nabla_S u})_I - \nabla_{S_h} \bar{u}_I\right\|_{0,S_h} \lesssim h^{1+\min\{1,\sigma\}} \|u\|_{3,\infty,S}. \tag{4.16}$$

Since G_h is a linear bounded operator in L^2 norm [38], by Theorem 3.5,

$$\begin{aligned} \|G_h(\nabla_{S_h} \bar{u}_I - \nabla_{S_h} u_h)\|_{0,S_h} &\lesssim \|\nabla_{S_h}(\bar{u}_I - u_h)\|_{0,S_h} \\ &\lesssim h^{1+\min\{1,\sigma\}} \left(\|u\|_{3,S} + \|u\|_{2,\infty,S} \right) + h^2 \|f\|_{0,S}. \end{aligned} \tag{4.17}$$

The conclusion then follows by applying (4.13), (4.16), and (4.17). ∎

4.3. Global L^2-projection. We discuss the postprocessing operator using the global L^2-projection $Q_h : L^2(S_h) \mapsto \mathcal{V}_h$:

$$(Q_h v, w_h) = (v, w_h) \quad \text{for all } w_h \in \mathcal{V}_h.$$

The global L^2-projection of a vector function $\boldsymbol{v} \in (L^2(S_h))^3$ is a vector function in $(\mathcal{V}_h)^3$ whose kth component is the global L^2-projection of the kth component of $\boldsymbol{v}$.

For the purpose of analysis, we introduce a perturbation of Q_h. Let $\varphi_i \in \mathcal{V}_h$ be the nodal basis at the vertex x_i. Set $V = (v_1, \ldots, v_N)^t$ with $v_i = (v, \varphi_i)$ and $\boldsymbol{M} = (m_{ij})$, with $m_{ij} = (\varphi_i, \varphi_j)$. The matrix $\boldsymbol{M}$ is known as the mass matrix. Then the matrix realization of Q_h will be

$$Q_h v = (\varphi_1, \ldots, \varphi_N) \boldsymbol{M}^{-1} V.$$

For piecewise continuous function v, we then define $V' = (v'_1, \ldots, v'_N)^t$ with $v'_i = \frac{1}{3} \sum_{j=1}^m v|_{\tau_j}(x_i)|\tau_j|$. Note that v'_i is an approximation of v_i, as $v'_i = \int_{\Omega_i} v(x_i) \varphi_i \, \mathrm{d}\sigma_h$. We define Q'_h, a perturbation of the global L^2-projection, as

$$Q'_h v = (\varphi_i, \ldots, \varphi_N) \boldsymbol{M}^{-1} V'.$$

Note that if v is piecewise constant on $\mathcal{T}_h$, then $Q_h v = Q'_h v$.

We first consider the error introduced by the inexact evaluation of the integral.

LEMMA 4.5. *Suppose $v \in W^2_\infty(S)$ and the triangulation $\mathcal{T}_h$ is $\mathcal{O}(h^{2\sigma})$ irregular. Let $\bar{v} = v \circ \mathcal{P}_0$; then*

$$\|Q_h\bar{v} - Q'_h\bar{v}\|_{0,S_h} \lesssim h^{1+\min\{1,\sigma\}} \|v\|_{2,\infty,S} . \tag{4.18}$$

Proof. By the definitions of Q_h and Q'_h, we have

$$\|Q_h\bar{v} - Q'_h\bar{v}\|^2_{0,S_h} = (V - V')^t \boldsymbol{M}^{-1}\boldsymbol{M}\boldsymbol{M}^{-1}(V - V') \lesssim h^{-2}\sum_{i=1}^{N}(\bar{v}_i - \bar{v}'_i)^2. \tag{4.19}$$

We then apply the Taylor expansion to get

(4.20)

$$\begin{aligned}
&|\bar{v}_i - \bar{v}'_i|\\
&= \left|\sum_{j=1}^{m}\int_{\tau_j}(\bar{v} - \bar{v}(x_i))\varphi_i \,\mathrm{d}\sigma_h\right|\\
&= \left|\sum_{j=1}^{m}\int_{\tau_j}\left[\nabla_{S_h}\bar{v}|_{\tau_j}(x_i)\cdot(x - x_i) + \frac{1}{2}(x - x_i)\cdot\nabla^2_{S_h}\bar{v}|_{\tau_j}(x_i + t(x - x_i))(x - x_i)\right]\varphi_i \,\mathrm{d}\sigma_h\right|\\
&\le \left|\sum_{j=1}^{m}\int_{\tau_j}\nabla_S v(x_i)\cdot(x - x_i)\varphi_i \,\mathrm{d}\sigma_h\right| + \left|\sum_{j=1}^{m}\int_{\tau_j}(\nabla_{S_h}\bar{v}|_{\tau_j} - \nabla_S v)(x_i)\cdot(x - x_i)\varphi_i \,\mathrm{d}\sigma_h\right|\\
&\quad + \left|\sum_{j=1}^{m}\int_{\tau_j}\frac{1}{2}(x - x_i)\cdot\nabla^2_{S_h}\bar{v}|_{\tau_j}(x_i + t(x - x_i))(x - x_i)\varphi_i \,\mathrm{d}\sigma_h\right|\\
&= I_1 + I_2 + I_3.
\end{aligned}$$

We estimate the three terms as follows. First for I_2, by (2.5) in Lemma 2.2, we have

$$I_2 = \left|\sum_{j=1}^{m}\int_{\tau_j}(\boldsymbol{P}_h|_{\tau_j} - \boldsymbol{P})\nabla_S v(x_i)\cdot(x - x_i)\varphi_i \,\mathrm{d}\sigma_h\right| \lesssim h^4 \|v\|_{1,\infty,S} . \tag{4.21}$$

For I_3, we have

$$I_3 \lesssim h^4 \|v\|_{2,\infty,\mathcal{P}_0(\Omega_i)} . \tag{4.22}$$

For I_1, since $\nabla_S v(x_i)\cdot(x - x_i)\psi_i$ is a quadratic function, using a three-points (middle points of three edges) numerical integration formula, we have

$$I_1 \lesssim \|v\|_{1,\infty,S}\left||\Omega_i|\sum_{j=1}^{m}\frac{|\tau_j|}{|\Omega_i|}(x_{ij} - x_i)\right|. \tag{4.23}$$

Then if Ω_i is $\mathcal{O}(h^2)$-symmetric, by (4.2), (4.21), (4.22), and (4.23), we have

$$|\bar{v}_i - \bar{v}'_i| \lesssim h^4(\|v\|_{1,\infty,S} + \|v\|_{2,\infty,S}). \tag{4.24}$$

Otherwise we have only

$$|\bar{v}_i - \bar{v}'_i| \lesssim h^3(\|v\|_{1,\infty,S} + \|v\|_{2,\infty,S}). \tag{4.25}$$

We split the vertices of $\mathcal{T}_h$ into two parts, $\mathcal{N}_h = \mathcal{N}_{1,h} \cup \mathcal{N}_{2,h}$ as in Theorem 4.4. Then we have $|\mathcal{N}_{1,h}| = \mathcal{O}(h^{-2})$ and $|\mathcal{N}_{2,h}| = \mathcal{O}(h^{-2+2\sigma})$ by the $\mathcal{O}(h^{2\sigma})$ irregularity of $\mathcal{T}_h$, where $|\cdot|$ denotes the cardinality of a set. Then

$$\begin{aligned}\|Q_h\bar{v} - Q_h'\bar{v}\|_{0,S_h}^2 &\lesssim (h^{-2}\,|\mathcal{N}_{1,h}|\,h^8 + h^{-2}\,|\mathcal{N}_{2,h}|\,h^6)(\|v\|_{1,\infty,S} + \|v\|_{2,\infty,S})^2\\ &\lesssim (h^4 + h^{2+2\sigma})(\|v\|_{1,\infty,S} + \|v\|_{2,\infty,S})^2,\end{aligned}$$

from which (4.18) is proved. ∎

We then consider the error introduced by the geometry approximation.

LEMMA 4.6. *Suppose $u \in W^1_\infty(S)$ and the triangulation $\mathcal{T}_h$ is $\mathcal{O}(h^{2\sigma})$ irregular. Let $\bar{u} = u \circ \mathcal{P}_0$ and $\overline{\nabla_S u} = (\nabla_S u)\circ \mathcal{P}_0$; then*

$$\left\|Q_h'(\overline{\nabla_S u} - \nabla_{S_h}\bar{u})\right\|_{0,S_h} \lesssim h^{1+\min\{1,\sigma\}}\,\|u\|_{1,\infty,S}\,. \tag{4.26}$$

Proof. By the definition of Q_h', the $\mathcal{O}(h^{2\sigma})$ irregular of $\mathcal{T}_h$, $\boldsymbol{P}_h = I - \boldsymbol{n}_h\boldsymbol{n}_n^t$, $\boldsymbol{P} = I - \boldsymbol{n}\boldsymbol{n}^t$, and (4.3), we can follow the same proof pattern of (4.18) in Lemma 4.5 to get

$$\begin{aligned}&\left\|Q_h'(\overline{\nabla_S u} - \nabla_{S_h}\bar{u})\right\|_{0,S_h}^2\\ &\lesssim h^{-2}\sum_{i=1}^N\left(\sum_{j=1}^m|\tau_j|\left(\nabla_S u - (\nabla_{S_h}\bar{u})|_{\tau_j}\right)(x_i)\right)^2\\ &= h^{-2}\sum_{i=1}^N\left(|\Omega_i|\sum_{j=1}^m\frac{|\tau_j|}{|\Omega_i|}\left(\nabla_S u - (\nabla_{S_h}\bar{u})|_{\tau_j}\right)(x_i)\right)^2\\ &= h^{-2}\sum_{i=1}^N\left(|\Omega_i|\sum_{j=1}^m\frac{|\tau_j|}{|\Omega_i|}(\boldsymbol{P} - \boldsymbol{P}_h|_{\tau_j})\nabla_S u(x_i)\right)^2\\ &\lesssim (h^4 + h^{2+2\sigma})\,\|u\|_{1,\infty,S}^2\,.\end{aligned}$$

The conclusion (4.26) then follows. ∎

LEMMA 4.7. *Suppose $u \in W^3_\infty(S)$ and the triangulation $\mathcal{T}_h$ is $\mathcal{O}(h^{2\sigma})$ irregular. Let $\bar{u} = u \circ \mathcal{P}_0$ and $\bar{u}_I$ be the linear interpolation of $\bar{u}$ on S_h; then*

$$\|Q_h'(\nabla_{S_h}\bar{u} - \nabla_{S_h}\bar{u}_I)\| \lesssim h^{1+\min\{1,\sigma\}}\,\|u\|_{3,\infty,S}\,. \tag{4.27}$$

Proof. By the definition of Q_h', the $\mathcal{O}(h^{2\sigma})$ irregular of $\mathcal{T}_h$, and the identity (4.6), we can follow the same proof pattern of (4.18) in Lemma 4.5 to get

$$\begin{aligned}&\|Q_h'(\nabla_{S_h}\bar{u} - \nabla_{S_h}\bar{u}_I)\|_{0,S_h}^2\\ &\lesssim h^{-2}\sum_{i=1}^N\left(\frac{1}{3}\sum_{j=1}^m|\tau_j|\,(\nabla_{S_h}\bar{u} - \nabla_{S_h}\bar{u}_I)\,|_{\tau_j}(x_i)\right)^2\\ &\lesssim h^{-2}\sum_{i=1}^N\left(\frac{1}{3}\,|\Omega_i|\sum_{j=1}^m\frac{|\tau_j|}{|\Omega_i|}\,(\nabla_{S_h}\bar{u} - \nabla_{S_h}\bar{u}_I)\,|_{\tau_j}(x_i)\right)^2\\ &\lesssim (h^4 + h^{2+2\sigma})\,\|u\|_{3,\infty,S}^2\,.\end{aligned} \tag{4.28}$$

The conclusion (4.27) then follows. ∎

Now we are in the position to present our main result on the global L^2-projection.

THEOREM 4.8. *Let $u \in W^3_\infty(S)$ be the solution of* (2.2) *and u_h be the solution of* (2.11). *Set $w_h = Q_h \nabla_{S_h} u_h$. If the triangulation $\mathcal{T}_h$ is $\mathcal{O}(h^{2\sigma})$ irregular, then*

$$\|\nabla_S u - \tilde{w}_h\|_{0,S} \lesssim h^{1+\min\{1,\sigma\}} \|u\|_{3,\infty,S} + h^2 \|f\|_{0,S} . \tag{4.29}$$

Proof. Let $\overline{\nabla_S u} = (\nabla_S u) \circ \mathcal{P}_0$. By the norm equivalence (2.7), we need only to estimate $\left\|\overline{\nabla_S u} - Q_h \nabla_{S_h} u_h\right\|_{0,S_h}$. We decompose

$$\begin{aligned}\overline{\nabla_S u} - Q_h \nabla_{S_h} u_h =& (I - Q_h)\overline{\nabla_S u} + (Q_h - Q_h')\overline{\nabla_S u} + Q_h'(\overline{\nabla_S u} - \nabla_{S_h} \bar{u}) \\ &+ Q_h'(\nabla_{S_h} \bar{u} - \nabla_{S_h} \bar{u}_I) + Q_h'(\nabla_{S_h} \bar{u}_I - \nabla_{S_h} u_h).\end{aligned}$$

We bound the first term using the standard approximation property

$$\left\|\overline{\nabla_S u} - Q_h \overline{\nabla_S u}\right\|_{0,S_h} \lesssim h^2 \|u\|_{3,S} , \tag{4.30}$$

the second term by (4.18) in Lemma 4.5, the third term by (4.26) in Lemma 4.6, and the fourth term by (4.27) in Lemma 4.7. For the last one, by (3.10) in Theorem 3.5 and the boundedness of Q_h, we have

$$\begin{aligned}\|Q_h'(\nabla_{S_h} \bar{u}_I - \nabla_{S_h} u_h)\|_{0,S_h} &= \|Q_h(\nabla_{S_h} \bar{u}_I - \nabla_{S_h} u_h)\|_{0,S_h} \\ &\lesssim \|\nabla_{S_h} \bar{u}_I - \nabla_{S_h} u_h\|_{0,S_h} \\ &\lesssim h^{1+\min\{1,\sigma\}} \left(\|u\|_{3,S} + \|u\|_{2,\infty,S}\right) + h^2 \|f\|_{0,S} .\end{aligned}$$

Finally conclusion (4.29) is followed by the triangle inequality. $\square$

4.4. The recovery schemes of FVMs. We discuss the superconvergence between the approximations from FEM and FVM of the Laplace–Beltrami equation (1.1).

Given a triangulation $\mathcal{T}$, we construct a dual mesh $\mathcal{B}$ as follows: for each triangle $\tau \in \mathcal{T}$, select a point $c_\tau \in \tau$. The point c_τ can coincide with one of the midpoints of edges, but not the vertices of triangles (to avoid the degeneracy of the control volume). In each triangle, we connect c_τ to three midpoints on the edges of τ. This will divide each triangle in $\mathcal{T}$ into three regions. For each vertex x_i of $\mathcal{T}$, we collect all regions containing this vertex and define it as ω_i.

The vertex-centered FVM of (1.1) is as follows: find $u_h^B \in \mathcal{V}_h$ such that

$$-\int_{\partial\omega_i} \nabla_{S_h} u_h^B \cdot \boldsymbol{n}_{\omega_i} \, \mathrm{d}s = \int_{\omega_i} f_h \, \mathrm{d}\sigma_h \quad \text{ for all } \omega_i, \tag{4.31}$$

where $\boldsymbol{n}_{\omega_i}$ is the outward unit normal vector of ω_i on the supporting plane of corresponding triangles.

The following identity can be found in [2, 14, 25, 39]. For completeness, we include a simple proof from [39] here.

LEMMA 4.9. *Let u_h be a linear finite element function. One has*

$$-\int_{\partial\omega_i} \nabla_{S_h} u_h \cdot \boldsymbol{n}_{\omega_i} \, \mathrm{d}s = \sum_{\tau_h \subset \Omega_i} \int_{\tau_h} \nabla_{S_h} u_h \cdot \nabla_{S_h} \varphi_i \, \mathrm{d}\sigma_h, \tag{4.32}$$

where $\varphi_i \in \mathcal{V}_h$ is the hat basis function at x_i. Therefore the stiffness matrix of FVMs is the same as that from FEMs.

Proof. Let us consider a triangle $\tau_h = \triangle x_i x_j x_k$ in Ω_i orientated counterclockwise. Let c be an interior point in τ_h and m_1 and m_2 the midpoints of $x_i x_j$ and $x_i x_k$, respectively. So segments $m_1 c$ and $c m_2$ are the parts of $\partial\omega_i$. By the divergence theorem, the fact $\nabla_{S_h} u_h$ is piecewise constant, and φ_i is linear on edges $x_i x_j$ and $x_i x_k$ and vanished on $x_j x_k$, we have

$$
\begin{aligned}
-\left(\int_{m_1}^{c} + \int_{c}^{m_2}\right) \nabla_{S_h} u_h \cdot \boldsymbol{n}_{\omega_i} \,\mathrm{d}s &= \left(\int_{x_i}^{m_1} + \int_{m_2}^{x_i}\right) \nabla_{S_h} u_h \cdot \boldsymbol{n} \,\mathrm{d}s \\
&= \frac{1}{2}\left(\int_{x_i}^{x_j} + \int_{x_k}^{x_i}\right) \nabla_{S_h} u_h \cdot \boldsymbol{n} \,\mathrm{d}s \\
&= \int_{\partial\tau_h} \varphi_i \nabla_{S_h} u_h \cdot \boldsymbol{n} \,\mathrm{d}s \\
&= \int_{\tau_h} \nabla_{S_h} u_h \cdot \nabla_{S_h} \varphi_i \,\mathrm{d}\sigma_h. \qquad \square
\end{aligned}
$$

THEOREM 4.10. *Let u_h^G be the standard Galerkin approximation and $u_h^{\mathcal{B}}$ the box (finite volume) approximation of the Laplace–Beltrami equation* (1.1). *Assume $f \in H_1(S)$; then*

$$
\left|u_h^G - u_h^{\mathcal{B}}\right|_{1,S_h} \le Ch^2 \left\|f\right\|_{1,S}. \tag{4.33}
$$

Proof. Let $V_{0,\mathcal{B}}$ be the piecewise constant space on $\mathcal{B}$. We can define a mapping $\Pi_h^* : \mathcal{V}_h \longmapsto V_{0,\mathcal{B}}$, as $\Pi_h^* v_h = \sum_{i=1}^N v_h(x_i)\chi_{\omega_i}$, where χ_{ω_i} is the characteristic function about ω_i. Since we use barycenters as the vertices of control volumes, we have

$$
\int_{\tau_h} v_h = \int_{\tau_h} \Pi_h^* v_h.
$$

Let f_c be the L^2-projection of f_h(see (2.10)) into the piecewise constant function space on S_h; then

$$
\begin{aligned}
(f_h, v_h - \Pi^* v_h) = (f_h - f_c, v_h - \Pi^* v_h) &\le \left\|f_h - f_c\right\|_{0,S_h} \left\|v_h - \Pi^* v_h\right\|_{S_h} \\
&\le Ch^2 \left\|f\right\|_{1,S} \left|v_h\right|_{1,S_h}.
\end{aligned}
$$

Here we use average-type Poincaré inequality for f_h and v_h on each τ_h. By Lemma 4.9 and (4.4), we have

$$
\int_{S_h} \nabla_{S_h} u_h^B \cdot \nabla_{S_h} v_h \,\mathrm{d}\sigma_h = -\sum_{\omega_i} \int_{\partial\omega_i} v_h(x_i) \nabla_{S_h} u_h^B \cdot \boldsymbol{n}_{\omega_i} \,\mathrm{d}s = (f_h, \Pi_h^* v_h).
$$

Then for any $v_h \in \mathcal{V}_h$, we have

$$
\begin{aligned}
\int_{S_h} \nabla_{S_h}(u_h^G - u_h^B) \cdot \nabla_{S_h} v_h \,\mathrm{d}\sigma_h &= (f_h, v_h) - \int_{S_h} \nabla_{S_h} u_h^B \cdot \nabla_{S_h} v_h \,\mathrm{d}\sigma_h \\
&= (f_h, v_h - \Pi_h^* v_h) \le Ch^2 \left\|f\right\|_{1,S} \left|v_h\right|_{1,S_h}.
\end{aligned}
$$

The estimate (4.33) is obtained by taking $v_h = u_h^G - u_h^B$. $\square$

So we reach the conclusion that the gradient recovery methods for FEM can also be used for FVM approximations and result superconvergence.

5. Numerical tests. In this section, we present three numerical examples to support our theoretical results. The first two are equations on a two-sphere and a torus for which structured grids can be easily constructed. The third is on a more general surface triangulated into unstructured grids.

Let u be the true solution of the Laplace–Beltrami equation (1.1) on the surface S, $\bar{u} = u \circ \mathcal{P}_0$, and $\overline{\nabla_S u} = (\nabla_S u) \circ \mathcal{P}_0$. Let u_h be the surface linear finite element approximation of u and $\bar{u}_I$ the linear interpolation of $\bar{u}$ on S_h. For the ease of computation, we report the following errors on discrete surface S_h:

$$\begin{aligned} E_I &= \|\nabla_{S_h}\bar{u}_I - \nabla_{S_h}u_h\|_{0,S_h}\,, \\ E_h &= \left\|\overline{\nabla_S u} - Q_h\nabla_{S_h}u_h\right\|_{0,S_h}\,, \\ E_i &= \left\|\overline{\nabla_S u} - G_h^i\nabla_{S_h}u_h\right\|_{0,S_h}\,, \quad i = 1,\ldots,6, \end{aligned}$$

where G_h^1 and G_h^2 represent the directly simple and weighted averaging recovery operators on S_h, respectively, and G_h^3, G_h^4, G_h^5, and G_h^6 represent the corresponding simple averaging, weighted averaging, local L^2-projection, and local discrete least-squares fitting (ZZ) recovery operators on the tangent plane, respectively. All the errors are computed on the polyhedral surface S_h using the ninth order quadrature rules. Our numerical programs are based on the package iFEM [13].

Let $\{\mathcal{T}_h^i\}_{i=1}^k$ be a sequence of meshes of a surface by uniform refinement and $E(\mathcal{T}_h^i)$ be the error associated to $\mathcal{T}_h^i$. Since the mesh size $h_{i-1} \approx 2h_i$, we determine the experimental order of convergence by

$$\ln\frac{E(\mathcal{T}_h^{i-1})}{E(\mathcal{T}_h^i)}/\ln 2, \quad i = 2,\ldots,k.$$

Also instead of using h, we list number of degree of freedoms (dofs) in the error tables.

5.1. Example 1 on the unit two-sphere. We consider the unit sphere whose signed distance function is

$$d(x,y,z) = \sqrt{x^2+y^2+z^2} - 1. \tag{5.1}$$

We choose f such that $u(x,y,z) = xy$ is the true solution of the Laplace–Beltrami equation (1.1) defined on the unit spherical surface $S = \{(x,y,z) \in \mathbb{R}^3 \mid d(x,y,z) = 0\}$. The initial mesh $\mathcal{T}_0$ is the projection of an icosahedron on the unit two-sphere. Then we obtain a sequence of meshes by the regular refinement of $\mathcal{T}_0$, that is, each triangle is divide into four conjugate small triangles by connecting midpoints of edges, and the new nodes are projected to the surface. The corresponding results are given in Table 5.1, and the numerical approximation of $u(x,y,z) = xy$ with 2562 dofs is shown in Figure 5.1.

5.2. Example 2 on a torus. We consider a torus surface whose signed distance function is

$$d(x,y,z) = \sqrt{(4-\sqrt{x^2+y^2})^2+z^2} - 1. \tag{5.2}$$

We choose f such that $u(x,y,z) = x - y$ is the true solution of the Laplace–Beltrami equation (1.1) on this torus. Through the parametrization of the torus, we can get a series of uniform meshes on its parameter space and then map the mesh onto the torus. The corresponding results are given in Table 5.2, and the numerical approximation of $u(x,y,z) = x - y$ with 3200 dofs is shown in Figure 5.2.

TABLE 5.1
Example 1. *Error table for surface linear finite element approximation and several gradient recovery methods on the unit sphere.*

N	E_I	Order	E_h	Order	E_1	Order	E_2	Order
12	2.71e−01	0.00	1.41	0.00	1.50	0.00	1.50	0.00
42	1.14e−01	1.25	4.10e−01	1.78	8.17e−01	0.88	8.18e−01	0.88
162	3.66e−02	1.64	1.06e−01	1.95	2.63e−01	1.64	2.64e−01	1.63
642	1.05e−02	1.80	2.89e−02	1.88	7.20e−02	1.87	7.34e−02	1.85
2562	2.88e−03	1.87	8.19e−03	1.82	1.92e−02	1.91	2.01e−02	1.87
10242	7.75e−04	1.89	2.45e−03	1.74	5.18e−03	1.89	5.67e−03	1.83
N	E_3	Order	E_4	Order	E_5	Order	E_6	Order
12	1.50	0.00	1.50	0.00	1.50	0.00	1.50	0.00
42	6.92e−01	1.12	6.92e−01	1.12	6.92e−01	1.12	6.92e−01	1.12
162	2.08e−01	1.73	2.09e−01	1.73	2.07e−01	1.74	2.07e−01	1.74
642	5.65e−02	1.88	5.81e−02	1.85	5.46e−02	1.92	5.44e−02	1.92
2562	1.52e−02	1.89	1.62e−02	1.84	1.40e−02	1.96	1.39e−02	1.97
10242	4.21e−03	1.86	4.71e−03	1.79	3.61e−03	1.96	3.54e−03	1.98

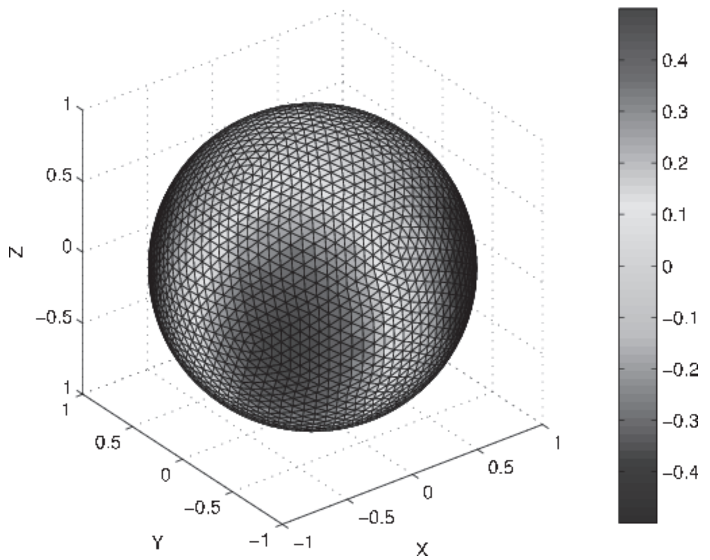


FIG. 5.1. *The numerical approximation of* $u(x, y, z) = xy$ *with* 2562 *dofs on the unit two-sphere.*

TABLE 5.2
Example 2. *Error table for surface linear finite element approximation and several gradient recovery methods on the torus.*

N	E_I	Order	E_h	Order	E_1	Order	E_2	Order
200	1.17	0.00	1.53	0.00	2.56	0.00	2.57	0.00
800	2.93e−01	2.00	3.75e−01	2.03	7.16e−01	1.84	7.20e−01	1.84
3200	7.33e−02	2.00	9.33e−02	2.01	1.84e−01	1.96	1.85e−01	1.96
12800	1.83e−02	2.00	2.33e−02	2.00	4.65e−02	1.99	4.67e−02	1.99
51200	4.58e−03	2.00	5.82e−03	2.00	1.16e−02	2.00	1.17e−02	2.00
N	E_3	Order	E_4	Order	E_5	Order	E_6	Order
200	1.72	0.00	1.72	0.00	1.77	0.00	1.80e	0.00
800	4.47e−01	1.94	4.48e−01	1.94	4.65e−01	1.93	4.72e−01	1.93
3200	1.13e−01	1.99	1.13e−01	1.98	1.18e−01	1.98	1.20e−01	1.98
12800	2.83e−02	2.00	2.84e−02	2.00	2.95e−02	2.00	3.00e−02	2.00
51200	7.07e−03	2.00	7.09e−03	2.00	7.39e−03	2.00	7.50e−03	2.00

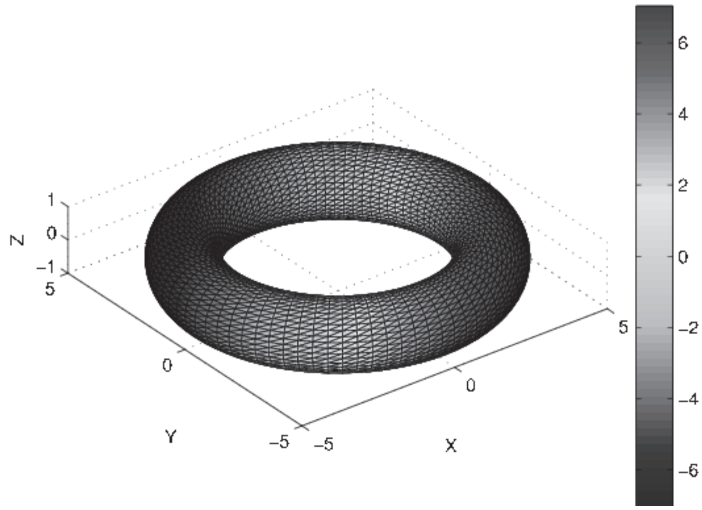


FIG. 5.2. *The numerical approximation of $u(x, y, z) = x - y$ with 3200 dofs on the torus.*

TABLE 5.3
Example 3. Error table for surface linear finite element approximation and several gradient recovery methods on the general surface.

N	E_I	Order	E_h	Order	E_1	Order	E_2	Order
18	5.54e−01	0.00	1.27	0.00	1.56	0.00	1.53	0.00
66	3.55e−01	0.64	6.85e−01	0.90	9.22e−01	0.76	9.43e−01	0.69
258	1.46e−01	1.29	2.51e−01	1.45	3.98e−01	1.21	4.11e−01	1.20
1026	4.47e−02	1.70	7.60e−02	1.72	1.33e−01	1.58	1.42e−01	1.54
4098	1.22e−02	1.87	2.30e−02	1.73	4.00e−02	1.74	4.46e−02	1.67
N	E_3	Order	E_4	Order	E_5	Order	E_6	Order
18	1.62	0.00	1.59	0.00	1.68	0.00	1.74	0.00
66	8.49e−01	0.93	8.79e−01	0.86	8.63e−01	0.96	8.90e−01	0.97
258	3.31e−01	1.36	3.43e−01	1.36	3.30e−01	1.39	3.44e−01	1.37
1026	1.06e−01	1.64	1.15e−01	1.58	1.02e−01	1.69	1.06e−01	1.70
4098	3.21e−02	1.73	3.68e−02	1.64	2.88e−02	1.83	2.97e−02	1.84

5.3. Example 3 on a general surface. We consider the example from [22]. We solve the Laplace–Beltrami equation (1.1) on a general surface whose level set function is

$$\phi(x, y, z) = (x - z^2)^2 + y^2 + z^2 - 1. \tag{5.3}$$

We choose f such that the true solution is $u(x, y, z) = xy$. We start from a crude mesh with 18 nodes to approximate this surface, then refine this mesh and project the new nodes on the surface. We use the first-order projection algorithm in [15]. The corresponding results are given in Table 5.3, and the numerical approximation of $u(x, y, z) = xy$ with 1026 dofs is shown in Figure 5.3.

From the above numerical tests, we can see clearly the existence of superconvergence phenomena in the surface linear FEM. Like the planar case, the mesh quality is a key factor for the superconvergence order. In Example 2, the meshes are almost uniform, so its convergence order is the best possible among the three examples. In future work, we shall develop mesh smoothing schemes on surfaces, such as methods based centroid Voronoi tessellation [18, 19] and optimal Delaunay triangulation [10, 12] to improve the mesh quality.

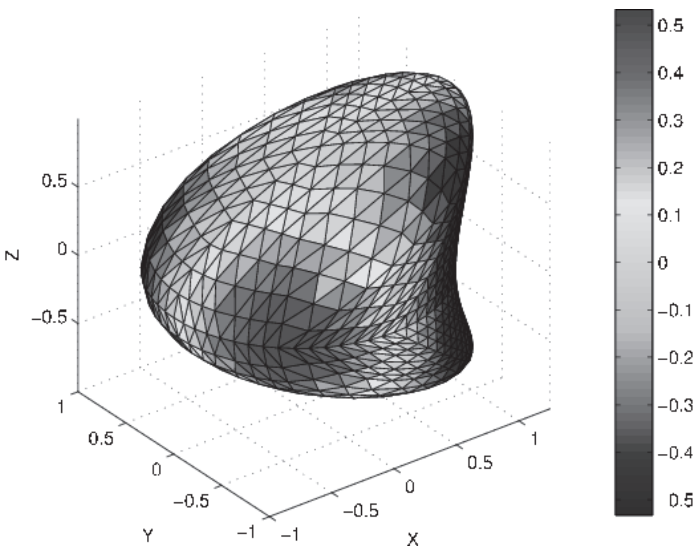


FIG. 5.3. *The numerical approximation of* $u(x, y, z) = xy$ *with* 1026 *dofs on the general surface.*

The error of the global L^2-projection recovery method seems better than local recovery methods on S_h. Although the global L^2-projection requires an inversion of a mass matrix, it can be solved efficiently by preconditioned conjugate gradient method since the mass matrix is well conditioned. It may need more computational time than local methods. Among six local methods, the local recovery methods on the tangent plane are better than local averaging methods on S_h directly. In these three examples, the errors are smaller by using local methods on tangential planes. The reason might be that the outward normal vector of S at every vertex is exact on the tangent plane.

REFERENCES

[1] I. Babuška and T. Strouboulis, *The Finite Element Method and Its Reliability*, Oxford University Press, New York, 2001.
[2] R. E. Bank and D. J. Rose, *Some error estimates for the box method*, SIAM J. Numer. Anal., 24 (1987), pp. 777–787.
[3] R. E. Bank, J. Xu, and B. Zheng, *Superconvergent derivative recovery for Lagrange triangular elements of degree p on unstructured grids*, SIAM J. Numer. Anal., 45 (2007), pp. 2032–2046.
[4] R. E. Bank and J. Xu, *Asymptotically exact a posteriori error estimators, Part* II: *General unstructured grids*, SIAM J. Numer. Anal., 41 (2003), pp. 2313–2332.
[5] R. E. Bank and J. Xu, *Asymptotically exact a posteriori error estimators, Part* I: *Grids with superconvergence*, SIAM J. Numer. Anal., 41 (2003), pp. 2294–2312.
[6] E. Bänsch, P. Morin, and R. H. Nochetto, *A finite element method for surface diffusion: The parametric case*, J. Comput. Phys., 203 (2005), pp. 321–343.
[7] C. Carstensen and S. Bartels, *Each averaging technique yields reliable a posteriori error control in FEM on unstructured grids*. I. *Low order conforming, nonconforming, and mixed FEM.*, Math. Comp., 71 (2002), pp. 945–969.
[8] C. Chen and Y. Huang, *High Accuracy Theory of Finite Element Methods*, Hunan Science and Technology Press, Changsha, 1995 (in Chinese).
[9] C. Chen, *Optimal points of the stresses approximated by triangular linear element in FEM*, J. Xiangtan Univ., 1 (1978), pp. 77–90.
[10] L. Chen and J. Xu, *Optimal Delaunay triangulations*, J. Comput. Math., 22 (2004), pp. 299–308.
[11] L. Chen and J. Xu, *Topics on adaptive finite element methods*, in Adaptive Computations: Theory and Algorithms, T. Tang and J. Xu, eds., Science Press, Beijing, 2007, pp. 1–31.

[12] L. Chen, *Mesh smoothing schemes based on optimal Delaunay triangulations*, in Proceedings of the 13th International Meshing Roundtable, Williamsburg, VA, 2004, Sandia National Laboratories, pp. 109–120.
[13] L. Chen, *iFEM: An Integrated Finite Element Methods Package in MATLAB*, Technical report, University of California at Irvine, Irvine, CA, 2009.
[14] L. Chen, *A new class of high order finite volume methods for second order elliptic equations*, SIAM J. Numer. Anal., 47 (2010), pp. 4021–4043.
[15] A. Demlow and G. Dziuk, *An adaptive finite element method for the Laplace–Beltrami operator on implicitly defined surfaces*, SIAM J. Numer. Anal., 45 (2007), pp. 421–442.
[16] A. Demlow, *Higher-order finite element methods and pointwise error estimates for elliptic problems on surfaces*, SIAM J. Numer. Anal., 47 (2009), pp. 805–827.
[17] M. Desbrun, M. Meyer, P. Schröder, and A. H. Barr, *Implicit fairing of irregular meshes using diffusion and curvature flow*, in SIGGRAPH '99: Proceedings of the 26th Annual Conference on Computer Graphics and Interactive Techniques, ACM Press/Addison-Wesley, New York, pp. 317–324.
[18] Q. Du, M. Gunzburger, and L. Ju, *Constrained centroidal Voronoi tessellations in general surfaces*, SIAM J. Sci. Comput., 24 (2003), pp. 1488–1506.
[19] Q. Du, M. Gunzburger, and L. Ju, *Voronoi-based finite volume methods, optimal Voronoi meshes, and PDEs on the sphere*, Comput. Methods Appl. Mech. Engrg., 192 (2003), pp. 3993–3957.
[20] Q. Du and L. Ju, *Finite volume methods on spheres and spherical centroidal Voronoi meshes*, SIAM J. Numer. Anal., 43 (2005), pp. 1673–1692.
[21] G. Dziuk and C. M. Elliott, *Finite elements on evolving surfaces*, IMA J. Numer. Anal., 27 (2007), pp. 262–292.
[22] G. Dziuk, *Finite Elements for the Beltrami Operator on Arbitrary Surfaces*, Partial Differential Equations Calc. Var. 1357, Springer, Berlin, 1988, pp. 142–155.
[23] G. Dziuk, *An algorithm for evolutionary surfaces*, Numer. Math., 58 (1990), pp. 603–611.
[24] X. Feng and H. Wu, *A posteriori error estimates and an adaptive finite element method for the Allen–Cahn equation and the mean curvature flow*, J. Sci. Comput., 24 (2005), pp. 121–146.
[25] W. Hackbusch, *On first and second order box schemes*, Computing, 41 (1989), pp. 277–296.
[26] E. Hebey, *Sobolev Spaces on Riemannian Manifolds*, Springer, Berlin, 1996.
[27] M. Holst, *Adaptive Numerical Treatment of Elliptic Systems on Manifolds*, Adv. Comput. Math. 15, 2001, pp. 139–191.
[28] Y. Huang and J. Xu, *Superconvergence of quadratic finite elements on mildly structured grids*, Math. Comp., 77 (2008), pp. 1253–1268.
[29] A. M. Lakhany, I. Marek, and J. R. Whiteman, *Superconvergence results on mildly structured triangulations*, Comput. Methods Appl. Mech. Engrg., 189 (2000), pp. 1–75.
[30] U. F. Mayer, *Numerical solutions for the surface diffusion flow in three space dimensions*, Comput. Appl. Math., 20 (2001), pp. 361–379.
[31] K. Mekchay, P. Morin, and R. H. Nochetto, *AFEM for the Laplace-Beltrami Operator on Graphs: Design and Conditional Contraction Property*, Cuadernos de Matemática y Mecnica, Centro Internacional de Métodos Numéricos en Ingenierĺa.
[32] K. Mekchay, *Convergence of Adaptive Finite Element Methods*, Ph.D. thesis, Department of Mathematics, University of Maryland, College Park, MD, 2005.
[33] B. H. Romeny, *Geometry-driven Diffusion in Computer Vision*, Kluwer Academic Publishers, Norwell, MA, 1994.
[34] G. Sapiro, *Geometric Partial Differential Equations and Image Analysis*, Cambridge University Press, Cambridge, UK, 2001.
[35] A. Schatz, I. Sloan, and L. Wahlbin, *Superconvergence in finite element methods and meshes that are locally symmetric with respect to a point*, SIAM J. Numer. Anal., 33 (1996), pp. 505–521.
[36] G. Simonett, *The Willmore flow for near spheres*, Diff. Integral Equ., 14 (2001), pp. 1005–1014.
[37] G. Xu, *Geometric Partial Differential Equation Methods in Computational Geometry*, Science Press, Beijing, People's Republic of China, 2008.
[38] J. Xu and Z. Zhang, *Analysis of recovery type a posteriori error estimators for mildly structured grids*, Math. Comp., 73 (2004), pp. 1139–1152.
[39] J. Xu and Q. Zou, *Analysis of linear and quadratic simplicial finite volume methods for elliptic equations*, Numer. Math., 111 (2009), pp. 469–492.
[40] O. C. Zienkiewicz and J. Zhu, *The superconvergent patch recovery and a posteriori error estimates. Part 1: The recovery technique*, Internat. J. Numer. Methods Engrg., 33 (1992), pp. 1331–1364.