二维有限元方法计算流程

贯晓晓

(湘潭大学数学与信息科学学院)

2015年11月2日

用有限元方法求解两点边值问题

- 1.求解方程

$$\begin{cases} -\triangle u = f, \\ u|\partial\Omega = 0 \end{cases}$$

的解， 其中

$$\Omega = [0,1] \times [0,1], f \in L^2(\Omega).$$

证明：利用Green公式求原方程的等价的变分问题

$$-\triangle u = f, u \in V \triangleq H_0^1(\Omega) \triangleq v \in H(\Omega)', u|\partial\Omega = 0$$

给上式同乘v并积分

$$\int_\Omega -\triangle uvdx = \int_\Omega fvdx$$

- 对上述积分左边用Green公式，有

$$-\int_\Omega -\triangle uv = \int_\Omega \nabla u \nabla v dx - \int_\Omega -uv\nu dx = \int_\Omega \nabla u \nabla v dx$$

$$a(u,v) \triangleq \int_\Omega \nabla u \nabla v dx = \int_\Omega f v dx \triangleq (f,v)$$

则原问题的变分形式为$a(u,v) = (f,v)$,也可以写成

$$(\nabla u, \nabla v) = (f,v)$$

. 对$\Omega = [0,1]\times[0,1]$进行剖分，对$x,y$方向均2 等分，记

$$x_j - x_{j-1} = y_j - y_{j-1} = \frac{1}{2} = h$$

在对节点(x_i,y_j)进行编码,先x方向后y方向,则节点$p((x_i,y_j))$的总体编码 $n_p = (2+1)(j-1)+i,(i,j=1,2,3)$.

- 则原问题的离散化问题为,求u_h使得

$$\int_\Omega \nabla u_h \nabla v_h dx = \int_\Omega f v_h dx, \forall v_h \in V_h$$

即求u_h使得

$$a(u_h, v_h) = (\nabla u_h, \nabla v_h) = (f, v_h)$$

其中数值解

$$u_h = \sum_{i=1}^{9} u_i \varphi_i$$

将上式代入$(\nabla u_h, \nabla v_h) = (f, v_h)$并令$v_h = \varphi_i, i = 1, 2, \ldots, 9.$

▶

$$(\sum_{i=1}^{9} \nabla u_i \varphi_i(x), \varphi_i) = (f, \varphi_i)$$

即

$$\sum_{i=1}^{9} \nabla u_j (\varphi_j', \varphi_i') = (f, \varphi_i)$$

令

$$U = (u_0, u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8)^T$$

$$A = (A_{ij})_{9\times 9}, A_{ij} = (\nabla\varphi_j, \nabla\varphi_i) = \int_\Omega \nabla\varphi_j \nabla\varphi_i dx$$

$$F = (F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9)^T, F_i = (f, \varphi_i) = \int_\Omega f\varphi_i$$

则可得有限元方程组为

- $$AU = F$$

  要计算矩阵所对应的元素可通过计算每个部分单元上对应的值，然后通过映射一一对应过去。令

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- 先考虑l_1单元上,设$\lambda(x,y)=ax+by+c$,将l_1上的三点代入得到方程组

$$\begin{cases} \frac{1}{2}b+c=\lambda_1 \\ c=\lambda_2 \\ \frac{1}{2}a+\frac{1}{2}b+c=\lambda_3 \end{cases}$$

解得

$$\begin{cases} \lambda_1(x,y)=-2x+2y \\ \lambda_2(x,y)=-2y+1 \\ \lambda_3(x,y)=2x \end{cases}$$

则

$$\begin{cases} \nabla\lambda_1(x,y)=(-2,2) \\ \nabla\lambda_2(x,y)=(0,-2) \\ \nabla\lambda_3(x,y)=(2,0) \end{cases}$$

- $$(\nabla\lambda_1, \nabla\lambda_1) = \int_{I_1} \nabla\lambda_1 \nabla\lambda_1 dx = 1$$
$$(\nabla\lambda_1, \nabla\lambda_2) = \int_{I_1} \nabla\lambda_1 \nabla\lambda_2 dx = -\frac{1}{2}$$
$$(\nabla\lambda_1, \nabla\lambda_3) = \int_{I_1} \nabla\lambda_1 \nabla\lambda_3 dx = -\frac{1}{2}$$
$$(\nabla\lambda_2, \nabla\lambda_1) = \int_{I_1} \nabla\lambda_2 \nabla\lambda_1 dx = -\frac{1}{2}$$
$$(\nabla\lambda_2, \nabla\lambda_2) = \int_{I_1} \nabla\lambda_2 \nabla\lambda_2 dx = \frac{1}{2}$$
$$(\nabla\lambda_2, \nabla\lambda_3) = \int_{I_1} \nabla\lambda_2 \nabla\lambda_3 dx = 0$$
$$(\nabla\lambda_3, \nabla\lambda_1) = \int_{I_1} \nabla\lambda_3 \nabla\lambda_1 dx = -\frac{1}{2}$$
$$(\nabla\lambda_3, \nabla\lambda_2) = \int_{I_1} \nabla\lambda_3 \nabla\lambda_2 dx = 0$$
$$(\nabla\lambda_3, \nabla\lambda_3) = \int_{I_1} \nabla\lambda_3 \nabla\lambda_3 dx = \frac{1}{2}$$

- 于是,

$$A_1 \to A_2 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

考虑I_2单元上,设$\lambda(x,y) = ax + by + c$,将$I_2$上的三点代入得到方程组

- $$\begin{cases} \frac{1}{2}a + c = \lambda_1 \\ c = \lambda_2 \\ c = \lambda_3 \end{cases}$$

  解得

  $$\begin{cases} \lambda_1(x,y) = 2x - 2y \\ \lambda_2(x,y) = 2y \\ \lambda_3(x,y) = -2x + 1 \end{cases}$$

  则

  $$\begin{cases} \nabla\lambda_1(x,y) = (2,-2) \\ \nabla\lambda_2(x,y) = (0,2) \\ \nabla\lambda_3(x,y) = (-2,0) \end{cases}$$

-
$$(\nabla\lambda_1, \nabla\lambda_1) = \int_{I_2} \nabla\lambda_1 \nabla\lambda_1 dx = 1$$
$$(\nabla\lambda_2, \nabla\lambda_1) = \int_{I_2} \nabla\lambda_2 \nabla\lambda_1 dx = -\frac{1}{2}$$
$$(\nabla\lambda_3, \nabla\lambda_1) = \int_{I_2} \nabla\lambda_3 \nabla\lambda_1 dx = -\frac{1}{2}$$
$$(\nabla\lambda_1, \nabla\lambda_2) = \int_{I_2} \nabla\lambda_1 \nabla\lambda_2 dx = -\frac{1}{2}$$
$$(\nabla\lambda_2, \nabla\lambda_2) = \int_{I_2} \nabla\lambda_2 \nabla\lambda_2 dx = \frac{1}{2}$$
$$(\nabla\lambda_3, \nabla\lambda_2) = \int_{I_2} \nabla\lambda_3 \nabla\lambda_2 dx = 0$$
$$(\nabla\lambda_1, \nabla\lambda_3) = \int_{I_2} \nabla\lambda_1 \nabla\lambda_3 dx = -\frac{1}{2}$$
$$(\nabla\lambda_2, \nabla\lambda_3) = \int_{I_2} \nabla\lambda_2 \nabla\lambda_3 dx = 0$$
$$(\nabla\lambda_3, \nabla\lambda_3) = \int_{I_2} \nabla\lambda_3 \nabla\lambda_3 dx = \frac{1}{2}$$

- 于是,

$$A_2 \to A_3 = \begin{pmatrix} 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

考虑I_3单元上,设$\lambda(x,y) = ax + by + c$,将$I_3$上的三点代入得到方程组

$$\begin{cases} \frac{1}{2}a + \frac{1}{2}b + c = \lambda_1 \\ \frac{1}{2}a + c = \lambda_2 \\ a + \frac{1}{2}b + c = \lambda_3 \end{cases}$$

解得

$$\begin{cases} \lambda_1(x,y) = -2x + 2y + 1 \\ \lambda_2(x,y) = -2y + 1 \\ \lambda_3(x,y) = 2x - 1 \end{cases}$$

则

$$\begin{cases} \nabla\lambda_1(x,y) = (-2,2) \\ \nabla\lambda_2(x,y) = (0,-2) \\ \nabla\lambda_3(x,y) = (2,0) \end{cases}$$

- $$(\nabla\lambda_1,\nabla\lambda_1)=\int_{I_3}\nabla\lambda_1\nabla\lambda_1 dx=1$$
$$(\nabla\lambda_2,\nabla\lambda_1)=\int_{I_3}\nabla\lambda_2\nabla\lambda_1 dx=-\frac{1}{2}$$
$$(\nabla\lambda_3,\nabla\lambda_1)=\int_{I_3}\nabla\lambda_3\nabla\lambda_1 dx=-\frac{1}{2}$$
$$(\nabla\lambda_1,\nabla\lambda_2)=\int_{I_3}\nabla\lambda_1\nabla\lambda_2 dx=-\frac{1}{2}$$
$$(\nabla\lambda_2,\nabla\lambda_2)=\int_{I_3}\nabla\lambda_2\nabla\lambda_2 dx=\frac{1}{2}$$
$$(\nabla\lambda_3,\nabla\lambda_2)=\int_{I_3}\nabla\lambda_3\nabla\lambda_2 dx=0$$
$$(\nabla\lambda_1,\nabla\lambda_3)=\int_{I_3}\nabla\lambda_1\nabla\lambda_3 dx=-\frac{1}{2}$$
$$(\nabla\lambda_2,\nabla\lambda_3)=\int_{I_3}\nabla\lambda_2\nabla\lambda_3 dx=0$$
$$(\nabla\lambda_3,\nabla\lambda_3)=\int_{I_3}\nabla\lambda_3\nabla\lambda_3 dx=\frac{1}{2}$$

- 于是，

$$A_2 \to A_3 = \begin{pmatrix} 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{3}{2} & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

考虑I_4单元上,设$\lambda(x, y) = ax + by + c$,将$I_4$上的三点代入得到方程组

- $$\begin{cases} a + c = \lambda_1 \\ a + \frac{1}{2}b + c = \lambda_2 \\ \frac{1}{2}a + c = \lambda_3 \end{cases}$$

解得

$$\begin{cases} \lambda_1(x, y) = 2x - 2y - 1 \\ \lambda_2(x, y) = 2y \\ \lambda_3(x, y) = -2x + 2 \end{cases}$$

则

$$\begin{cases} \nabla\lambda_1(x, y) = (2, -2) \\ \nabla\lambda_2(x, y) = (0, 2) \\ \nabla\lambda_3(x, y) = (-2, 0) \end{cases}$$

- $$(\nabla\lambda_1,\nabla\lambda_1)=\int_{I_4}\nabla\lambda_1\nabla\lambda_1 dx=1$$
$$(\nabla\lambda_2,\nabla\lambda_1)=\int_{I_4}\nabla\lambda_2\nabla\lambda_1 dx=-\frac{1}{2}$$
$$(\nabla\lambda_3,\nabla\lambda_1)=\int_{I_4}\nabla\lambda_3\nabla\lambda_1 dx=-\frac{1}{2}$$
$$(\nabla\lambda_1,\nabla\lambda_2)=\int_{I_4}\nabla\lambda_1\nabla\lambda_2 dx=-\frac{1}{2}$$
$$(\nabla\lambda_2,\nabla\lambda_2)=\int_{I_4}\nabla\lambda_2\nabla\lambda_2 dx=\frac{1}{2}$$
$$(\nabla\lambda_3,\nabla\lambda_2)=\int_{I_4}\nabla\lambda_3\nabla\lambda_2 dx=0$$
$$(\nabla\lambda_1,\nabla\lambda_3)=\int_{I_4}\nabla\lambda_1\nabla\lambda_3 dx=-\frac{1}{2}$$
$$(\nabla\lambda_2,\nabla\lambda_3)=\int_{I_4}\nabla\lambda_2\nabla\lambda_3 dx=0$$
$$(\nabla\lambda_3,\nabla\lambda_3)=\int_{I_4}\nabla\lambda_3\nabla\lambda_3 dx=\frac{1}{2}$$

- 于是,

$$A_3 \to A_4 = \begin{pmatrix} 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 2 & -\frac{1}{2} & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

考虑I_5单元上,设$\lambda(x,y) = ax + by + c$,将$I_5$上的三点代入得到方程组

- $$\begin{cases} b+c=\lambda_1 \\ \frac{1}{2}b+c=\lambda_2 \\ \frac{1}{2}a+b+c=\lambda_3 \end{cases}$$

  解得

  $$\begin{cases} \lambda_1(x,y)=-2x+2y-1 \\ \lambda_2(x,y)=-2y-2 \\ \lambda_3(x,y)=2x \end{cases}$$

  则

  $$\begin{cases} \nabla\lambda_1(x,y)=(-2,2) \\ \nabla\lambda_2(x,y)=(0,-2) \\ \nabla\lambda_3(x,y)=(2,0) \end{cases}$$

- $$(\nabla\lambda_1,\nabla\lambda_1)=\int_{I_5}\nabla\lambda_1\nabla\lambda_1 dx=1$$

$$(\nabla\lambda_2,\nabla\lambda_1)=\int_{I_5}\nabla\lambda_2\nabla\lambda_1 dx=-\frac{1}{2}$$

$$(\nabla\lambda_3,\nabla\lambda_1)=\int_{I_5}\nabla\lambda_3\nabla\lambda_1 dx=-\frac{1}{2}$$

$$(\nabla\lambda_1,\nabla\lambda_2)=\int_{I_5}\nabla\lambda_1\nabla\lambda_2 dx=-\frac{1}{2}$$

$$(\nabla\lambda_2,\nabla\lambda_2)=\int_{I_5}\nabla\lambda_2\nabla\lambda_2 dx=\frac{1}{2}$$

$$(\nabla\lambda_3,\nabla\lambda_2)=\int_{I_5}\nabla\lambda_3\nabla\lambda_2 dx=0$$

$$(\nabla\lambda_1,\nabla\lambda_3)=\int_{I_5}\nabla\lambda_1\nabla\lambda_3 dx=-\frac{1}{2}$$

$$(\nabla\lambda_2,\nabla\lambda_3)=\int_{I_5}\nabla\lambda_2\nabla\lambda_3 dx=0$$

$$(\nabla\lambda_3,\nabla\lambda_3)=\int_{I_5}\nabla\lambda_3\nabla\lambda_3 dx=\frac{1}{2}$$

- 于是，

$$A_4 \to A_5 = \begin{pmatrix} 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 2 & -\frac{1}{2} & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{3}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & -1 & 0 & -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

考虑I_6单元上,设$\lambda(x,y) = ax + by + c$,将$I_6$上的三点代入得到方程组

- $$\begin{cases} \frac{1}{2}a + \frac{1}{2}b + c = \lambda_1 \\ \frac{1}{2}a + b + c = \lambda_2 \\ \frac{1}{2}b + c = \lambda_3 \end{cases}$$

  解得

  $$\begin{cases} \lambda_1(x, y) = 2x - 2y + 1 \\ \lambda_2(x, y) = 2y - 1 \\ \lambda_3(x, y) = -2x + 1 \end{cases}$$

  则

  $$\begin{cases} \nabla\lambda_1(x, y) = (2, -2) \\ \nabla\lambda_2(x, y) = (0, 2) \\ \nabla\lambda_3(x, y) = (-2, 0) \end{cases}$$

- $$(\nabla\lambda_1, \nabla\lambda_1) = \int_{I_6} \nabla\lambda_1 \nabla\lambda_1 dx = 1$$

$$(\nabla\lambda_2, \nabla\lambda_1) = \int_{I_6} \nabla\lambda_2 \nabla\lambda_1 dx = -\frac{1}{2}$$

$$(\nabla\lambda_3, \nabla\lambda_1) = \int_{I_6} \nabla\lambda_3 \nabla\lambda_1 dx = -\frac{1}{2}$$

$$(\nabla\lambda_1, \nabla\lambda_2) = \int_{I_6} \nabla\lambda_1 \nabla\lambda_2 dx = -\frac{1}{2}$$

$$(\nabla\lambda_2, \nabla\lambda_2) = \int_{I_6} \nabla\lambda_2 \nabla\lambda_2 dx = \frac{1}{2}$$

$$(\nabla\lambda_3, \nabla\lambda_2) = \int_{I_6} \nabla\lambda_3 \nabla\lambda_2 dx = 0$$

$$(\nabla\lambda_1, \nabla\lambda_3) = \int_{I_6} \nabla\lambda_1 \nabla\lambda_3 dx = -\frac{1}{2}$$

$$(\nabla\lambda_2, \nabla\lambda_3) = \int_{I_6} \nabla\lambda_2 \nabla\lambda_3 dx = 0$$

$$(\nabla\lambda_3, \nabla\lambda_3) = \int_{I_6} \nabla\lambda_3 \nabla\lambda_3 dx = \frac{1}{2}$$

- 于是，

$$A_5 \to A_6 = \begin{pmatrix} 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 2 & -\frac{1}{2} & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 2 & -1 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & -1 & 0 & -1 & \frac{5}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

考虑I_7单元上，设$\lambda(x,y) = ax + by + c$，将$I_7$上的三点代入得到方程组

-

$$\begin{cases} \frac{1}{2}a + b + c = \lambda_1 \\ \frac{1}{2}a + \frac{1}{2}b + c = \lambda_2 \\ a + b + c = \lambda_3 \end{cases}$$

解得

$$\begin{cases} \lambda_1(x,y) = -2x + 2y \\ \lambda_2(x,y) = -2y - 2 \\ \lambda_3(x,y) = 2x - 1 \end{cases}$$

则

$$\begin{cases} \nabla\lambda_1(x,y) = (-2,2) \\ \nabla\lambda_2(x,y) = (0,-2) \\ \nabla\lambda_3(x,y) = (2,0) \end{cases}$$

- $$(\nabla\lambda_1, \nabla\lambda_1) = \int_{I_7} \nabla\lambda_1 \nabla\lambda_1 dx = 1$$

  $$(\nabla\lambda_2, \nabla\lambda_1) = \int_{I_7} \nabla\lambda_2 \nabla\lambda_1 dx = -\frac{1}{2}$$

  $$(\nabla\lambda_3, \nabla\lambda_1) = \int_{I_7} \nabla\lambda_3 \nabla\lambda_1 dx = -\frac{1}{2}$$

  $$(\nabla\lambda_1, \nabla\lambda_2) = \int_{I_7} \nabla\lambda_1 \nabla\lambda_2 dx = -\frac{1}{2}$$

  $$(\nabla\lambda_2, \nabla\lambda_2) = \int_{I_7} \nabla\lambda_2 \nabla\lambda_2 dx = \frac{1}{2}$$

  $$(\nabla\lambda_3, \nabla\lambda_2) = \int_{I_7} \nabla\lambda_3 \nabla\lambda_2 dx = 0$$

  $$(\nabla\lambda_1, \nabla\lambda_3) = \int_{I_7} \nabla\lambda_1 \nabla\lambda_3 dx = -\frac{1}{2}$$

  $$(\nabla\lambda_2, \nabla\lambda_3) = \int_{I_7} \nabla\lambda_2 \nabla\lambda_3 dx = 0$$

  $$(\nabla\lambda_3, \nabla\lambda_3) = \int_{I_7} \nabla\lambda_3 \nabla\lambda_3 dx = \frac{1}{2}$$

- 于是,

$$A_6 \to A_7 = \begin{pmatrix} 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 2 & -\frac{1}{2} & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 2 & -1 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & -1 & 0 & -1 & 3 & -\frac{1}{2} & 0 & -1 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -\frac{1}{2} & 2 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

考虑I_8单元上,设$\lambda(x, y) = ax + by + c$,将$I_8$上的三点代入得到方程组

- $$\begin{cases} a + \frac{1}{2}b + c = \lambda_1 \\ a + b + c = \lambda_2 \\ \frac{1}{2}a + \frac{1}{2}b + c = \lambda_3 \end{cases}$$

  解得

  $$\begin{cases} \lambda_1(x, y) = 2x - 2y \\ \lambda_2(x, y) = 2y - 1 \\ \lambda_3(x, y) = -2x + 2 \end{cases}$$

  则

  $$\begin{cases} \nabla\lambda_1(x, y) = (2, -2) \\ \nabla\lambda_2(x, y) = (0, 2) \\ \nabla\lambda_3(x, y) = (-2, 0) \end{cases}$$

- $$(\nabla\lambda_1, \nabla\lambda_1) = \int_{I_8} \nabla\lambda_1 \nabla\lambda_1 dx = 1$$

$$(\nabla\lambda_2, \nabla\lambda_1) = \int_{I_8} \nabla\lambda_2 \nabla\lambda_1 dx = -\frac{1}{2}$$

$$(\nabla\lambda_3, \nabla\lambda_1) = \int_{I_8} \nabla\lambda_3 \nabla\lambda_1 dx = -\frac{1}{2}$$

$$(\nabla\lambda_1, \nabla\lambda_2) = \int_{I_8} \nabla\lambda_1 \nabla\lambda_2 dx = -\frac{1}{2}$$

$$(\nabla\lambda_2, \nabla\lambda_2) = \int_{I_8} \nabla\lambda_2 \nabla\lambda_2 dx = \frac{1}{2}$$

$$(\nabla\lambda_3, \nabla\lambda_2) = \int_{I_8} \nabla\lambda_3 \nabla\lambda_2 dx = 0$$

$$(\nabla\lambda_1, \nabla\lambda_3) = \int_{I_8} \nabla\lambda_1 \nabla\lambda_3 dx = -\frac{1}{2}$$

$$(\nabla\lambda_2, \nabla\lambda_3) = \int_{I_8} \nabla\lambda_2 \nabla\lambda_3 dx = 0$$

$$(\nabla\lambda_3, \nabla\lambda_3) = \int_{I_8} \nabla\lambda_3 \nabla\lambda_3 dx = \frac{1}{2}$$

- 于是，

$$A_7 \to A_8 = \begin{pmatrix} 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 2 & -\frac{1}{2} & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 2 & -1 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & -1 & 0 & -1 & \frac{7}{2} & -1 & 0 & -1 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & -1 & 2 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -\frac{1}{2} & 2 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 1 \end{pmatrix}$$

则

$$A = A_8$$

- 再考虑F,令$f=1.F_0=(0,0,0,0,0,0,0,0,0)^T$.

$$(f,\varphi_i)=\int_{I_i} f\varphi_i, i=1,2,\ldots,9dx$$

在I_1单元上

$$\int_{I_1} f\lambda_1^{(4)}dx=\int_{I_1}\lambda_1^{(4)}dx=\frac{1}{8}$$

$$\int_{I_1} f\lambda_2^{(1)}dx=\int_{I_1}\lambda_2^{(1)}dx=\frac{1}{8}$$

$$\int_{I_1} f\lambda_3^{(5)}dx=\int_{I_1}\lambda_3^{(5)}dx=\frac{1}{8}$$

于是

$$F_0\rightarrow F_1=(\frac{1}{8},0,0,\frac{1}{8},\frac{1}{8},0,0,0,0)^T$$

▶ 在I_2单元上

$$\int_{I_2} f\lambda_1^{(2)}dx = \int_{I_2} \lambda_1^{(2)}dx = \frac{1}{8}$$

$$\int_{I_2} f\lambda_2^{(5)}dx = \int_{I_2} \lambda_2^{(5)}dx = \frac{1}{8}$$

$$\int_{I_2} f\lambda_3^{(1)}dx = \int_{I_2} \lambda_3^{(1)}dx = \frac{1}{8}$$

于是

$$F_1 \to F_2 = (\frac{2}{8}, \frac{1}{8}, 0, \frac{1}{8}, \frac{1}{4}, 0, 0, 0, 0)^T$$

在I_3单元上

$$\int_{I_3} f\lambda_1^{(5)}dx = \int_{I_2} \lambda_1^{(5)}dx = \frac{1}{8}$$

$$\int_{I_3} f\lambda_2^{(2)}dx = \int_{I_2} \lambda_2^{(2)}dx = \frac{1}{8}$$

- $$\int_{I_3} f\lambda_3^{(6)}dx = \int_{I_3} \lambda_3^{(6)}dx = \frac{1}{8}$$

于是
$$F_2 \to F_3 = (\frac{1}{4}, \frac{1}{4}, 0, \frac{1}{8}, \frac{3}{8}, \frac{1}{8}, 0, 0, 0)^T$$
在I_4单元上
$$\int_{I_4} f\lambda_1^{(3)}dx = \int_{I_4} \lambda_1^{(3)}dx = \frac{1}{8}$$
$$\int_{I_4} f\lambda_2^{(6)}dx = \int_{I_4} \lambda_2^{(6)}dx = \frac{1}{8}$$
$$\int_{I_4} f\lambda_3^{(2)}dx = \int_{I_4} \lambda_3^{(2)}dx = \frac{1}{8}$$
于是
$$F_3 \to F_4 = (\frac{1}{4}, \frac{3}{8}, \frac{1}{8}, \frac{1}{8}, \frac{3}{8}, \frac{1}{4}, 0, 0, 0)^T$$

- 在I_5单元上

$$\int_{I_5} f\lambda_1^{(7)}dx = \int_{I_5} \lambda_1^{(7)}dx = \frac{1}{8}$$

$$\int_{I_5} f\lambda_2^{(4)}dx = \int_{I_5} \lambda_2^{(4)}dx = \frac{1}{8}$$

$$\int_{I_5} f\lambda_3^{(8)}dx = \int_{I_5} \lambda_3^{(8)}dx = \frac{1}{8}$$

于是

$$F_4 \to F_5 = (\frac{1}{4}, \frac{3}{8}, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, 0)^T$$

在I_6单元上

$$\int_{I_6} f\lambda_1^{(5)}dx = \int_{I_6} \lambda_1^{(5)}dx = \frac{1}{8}$$

$$\int_{I_6} f\lambda_2^{(8)}dx = \int_{I_6} \lambda_2^{(8)}dx = \frac{1}{8}$$

- $$\int_{I_6} f\lambda_3^{(4)}dx = \int_{I_6} \lambda_3^{(4)}dx = \frac{1}{8}$$

  于是

  $$F_5 \to F_6 = (\frac{1}{4}, \frac{3}{8}, \frac{1}{8}, \frac{3}{8}, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{4}, 0)^T$$

  在I_7单元上

  $$\int_{I_7} f\lambda_1^{(8)}dx = \int_{I_7} \lambda_1^{(8)}dx = \frac{1}{8}$$

  $$\int_{I_7} f\lambda_2^{(5)}dx = \int_{I_7} \lambda_2^{(5)}dx = \frac{1}{8}$$

  $$\int_{I_7} f\lambda_3^{(9)}dx = \int_{I_7} \lambda_3^{(9)}dx = \frac{1}{8}$$

  于是

  $$F_6 \to F_7 = (\frac{1}{4}, \frac{3}{8}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{1}{4}, \frac{1}{8}, \frac{3}{8}, \frac{1}{8})^T$$

- 在I_8单元上

$$\int_{I_8} f\lambda_1^{(6)}dx = \int_{I_8} \lambda_1^{(6)}dx = \frac{1}{8}$$

$$\int_{I_8} f\lambda_2^{(9)}dx = \int_{I_8} \lambda_2^{(9)}dx = \frac{1}{8}$$

$$\int_{I_8} f\lambda_3^{(5)}dx = \int_{I_8} \lambda_3^{(5)}dx = \frac{1}{8}$$

于是

$$F_7 \to F_8 = (\frac{1}{4}, \frac{3}{8}, \frac{1}{8}, \frac{3}{8}, \frac{3}{4}, \frac{3}{8}, \frac{1}{8}, \frac{3}{8}, \frac{1}{4})^T$$

于是此问题的有限元方程组具体为

- $$\begin{pmatrix} 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 2 & -\frac{1}{2} & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 2 & -1 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & -1 & 0 & -1 & \frac{7}{2} & -1 & 0 & -1 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & -1 & 2 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -\frac{1}{2} & 2 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{3}{8} \\ \frac{1}{8} \\ \frac{3}{8} \\ \frac{3}{4} \\ \frac{3}{8} \\ \frac{1}{8} \\ \frac{3}{8} \\ \frac{1}{4} \end{pmatrix}$$

从上述线性方程组中，解出$u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9$.

则得到原方程的数值解为

$$u_h = \sum_{i=1}^{9} u_i \varphi_i(x)$$